南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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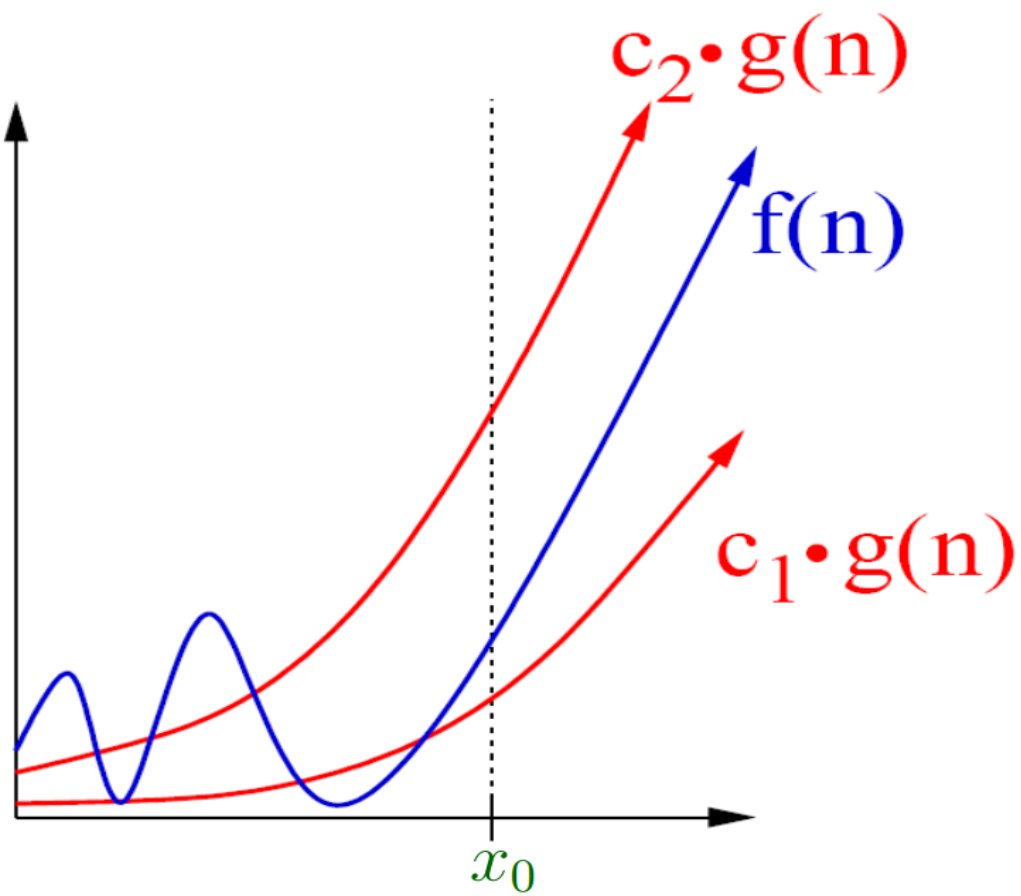
Assignment #1

Please submit your assignments before class!

Big-Theta Notation (Big-O & Big-Omega)

- Two functions $f(n)$, $g(n)$ have the same order growth if $f(n) = O(g(n))$ and $g(n) = O(f(n))$. In this case, we say that $f(n) = \Theta(g(n))$, which is the same as $g(n) = \Theta(f(n))$.



Algorithms

- An *algorithm* is a finite sequence of **precise instructions** for performing a computation or for solving a problem.

  A *computational problem* is a specification of the desired input-output relationship.

  Example (Computational Problem and Algorithm)

  The following procedure is an algorithm for calculating the sum of n given numbers $a_1, a_2, \ldots, a_n$.

  Step 1: set $S = 0$

  Step 2: for $i = 1$ to n, replace S by $S + a_i$

  Step 3: output S

Instance

- An *instance* of a problem is all the inputs needed to compute a solution to the problem.

Example (Instance of Problem)

$$< 8, 3, 6, 7, 1, 2, 9 >$$

- A *correct algorithm* halts with the correct output for **every input instance**. We can then say that the algorithm solves the problem.

Time and Space Complexity

- The number of **machine operations**(addition, multiplication, comparison, replacement, etc) needed in an algorithm is the *time complexity* of the algorithm, and **amount of memory** needed is the *space complexity* of the algorithm.

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Step 1 and Step 3 take one operation. Step 2 takes $2n$ operations. Therefore, altogether this algorithm takes $2n + 2$ operations. The time complexity is $O(n)$.

Horner's Algorithm and Its Complexity

- **Example**

  Consider the evaluation of $f(x) = 1 + 2x + 3x^2 + 4x^3$.

  Direct computation takes 3 additions and 6 multiplications.

  Can we do better?

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The final value of S output at Step 3 is the desired value of $a_0 + a_1x + \cdots + a_nx^n$. The number of operations needed in this algorithm is $1 + 3n + 1 = 3n + 2$. So the time complexity of this algorithm is $O(n)$.

Time Complexity

- Determine the time complexity of the following algorithm:

```
for i := 1 to n
    for j := 1 to n
        a := 2 * n + i * j;
    end for
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In the second loop, computing a takes 4 operations (two multiplications, one addition, and one replacement). For each i, it takes $4n$ operations to complete the second loop. So it takes $n \times 4n = 4n^2$ operations to complete the two loops. The time complexity of this algorithm is $O(n^2)$.

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    for $j := 1$ to i

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  Computing S takes 3 operations. For each i, completing the second loop takes $3i$ operations. So altogether it takes

  $$1 + \sum_{i=1}^{n} 3i = 1 + 3\frac{n(n+1)}{2}$$

  operations. So the complexity of this algorithm is $O(n^2)$

More on Time Complexity

- **Example:** (Insertion Sort)

  Input: $A[1 \ldots n]$ is an array of numbers

  for $j := 2$ to n

    $key = A[j]$;

    $i = j - 1$;

    while $i \geq 1$ and $A[i] > key$ do

      $A[i + 1] = A[i]$;

      $i - -$;

    end while

    $A[i + 1] = key$;

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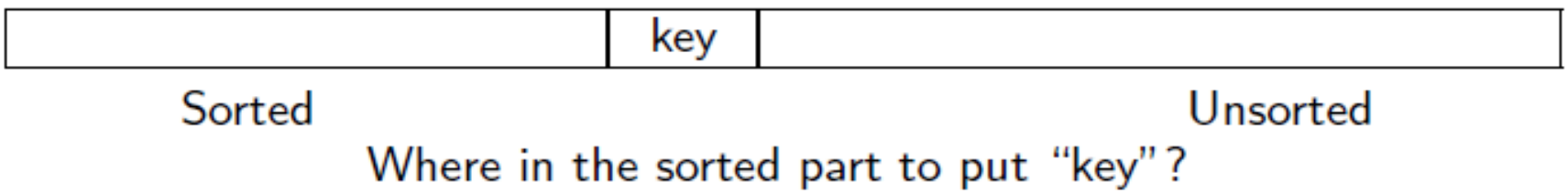

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- Best Case: constraints on the input, other than size, resulting in the fastest possible running time for the given size.

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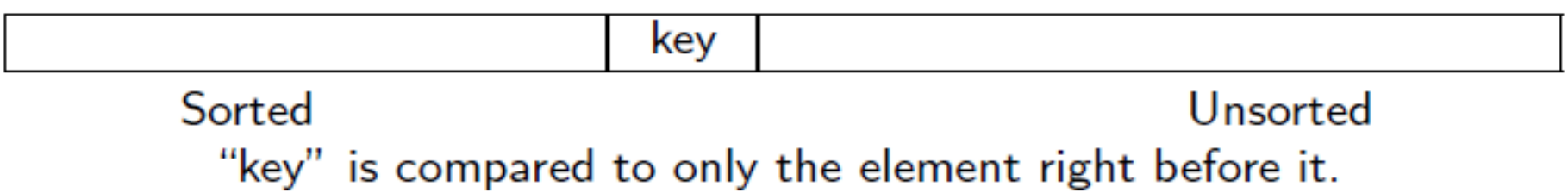
  $$A[1] \leq A[2] \leq A[3] \leq \cdots \leq A[n]$$

  The number of comparisons needed is

  $$\underbrace{1+1+1+\cdots+1}_{n-1} = n-1 = \Theta(n)$$



Three Cases of Analysis: II

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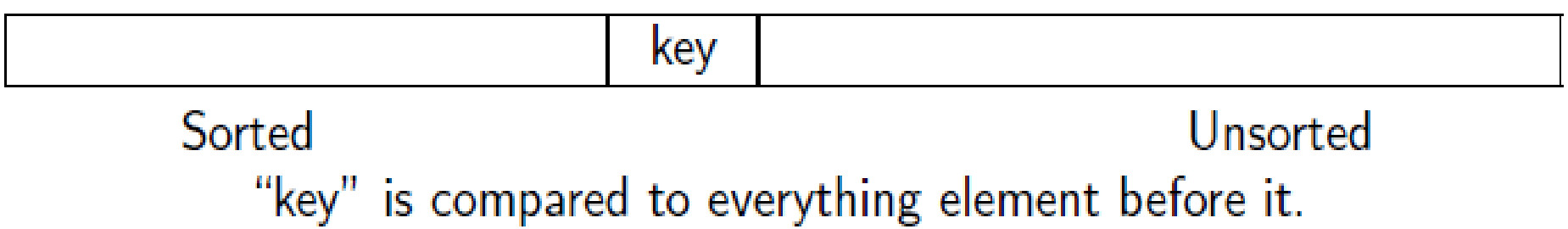
$$1 + 2 + 3 + \cdots + (n-1) = \frac{n(n-1)}{2} = \Theta(n^2)$$



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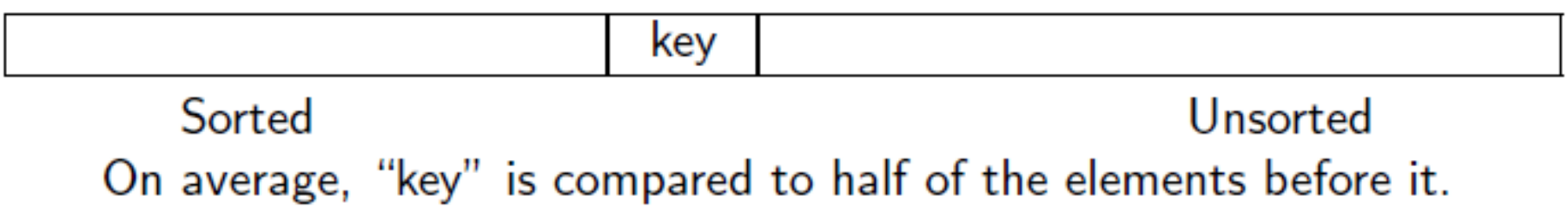
Example: (Insertion Sort)

$\Theta(n^2)$ assuming that each of the $n!$ instances are equally likely



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- **Algorithm Design**, is mainly about designing algorithms that have small Big-O running time.
- Being able to do good algorithm design lets you identify the hard parts of your problem and deal with them effectively.
- Too often, programmers try to slove problems using brute force techniques and end up with slow complicated code!
- A few hours of abstract thought devoted to algorithm design could have speeded up the solution substantially and simplified it!

Dealing with Hard Problems

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Blame yourself.

I couldn't find a polynomial-time algorithm.
I guess I am too dumb.

Dealing with Hard Problems

- What happens if you can't find an efficient algorithm for a given problem?

Show that no-efficient algorithm exists.

I couldn't find a polynomial-time algorithm, because no such algorithm exists.

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  We will now learn about NP-Complete problems, which provide us with a way to approach this question.

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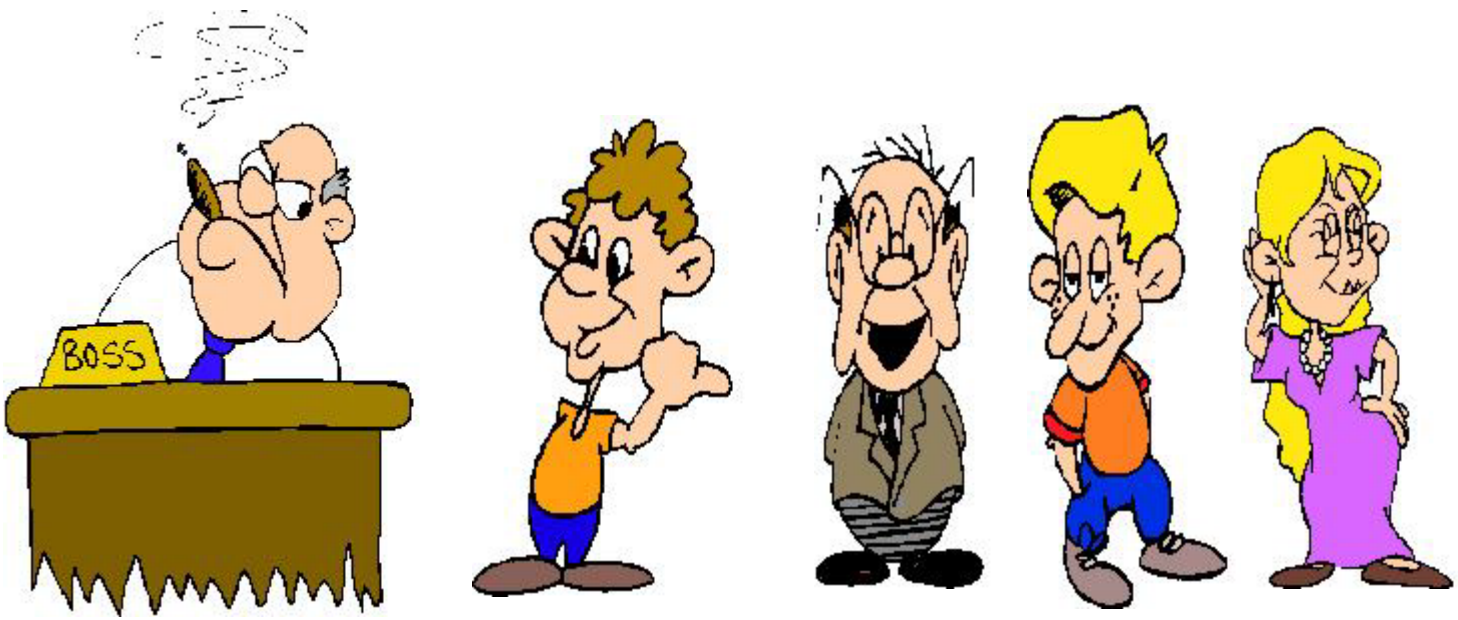
- Researchers have spent innumberable man-years trying to find efficient solutions to these problems but failed.
- So, NP-Complete problems are very likely to be hard.
- What do you do: prove that your problem is NP-Complete.

Introduction

What do you actually do:



I couldn't find a polynomial-time algorithm,
but neither could all these other smart people!

Encoding the Inputs of Problems

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- In order to formally discuss how hard a problem is, we need to be much more formal than before about the input size of a problem.

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- The exact input size s, determined by an optimal encoding method, is hard to compute in most cases.

  However, we do not need to determine s exactly.

  For most problems, it is sufficient to choose some natural, and (usually) simple, encoding and use the size s of this encoding.

Input Size Example: Composite

- **Example:**

  Given a positive integer n, are there integers $j, k > 1$ such that $n = jk$? (i.e., is n a composite number?)

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  Any integer $n > 0$ can be represented in the binary number system as a string $a_0 a_1 \cdots a_k$ of length $\lceil \log_2(n+1) \rceil$.

  Thus, a natural measure of input size is $\lceil \log_2(n+1) \rceil$ (or just $\log_2 n$)

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  Using fixed length encoding, we write a_i as a binary string of length $m = \lceil \log_2 \max(|a_i| + 1) \rceil$.

  This coding gives an input size nm.

Complexity in terms of Input Size

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  This makes $\Theta(n)$ comparisons, so it might seem linear and very efficient.

  But, note that the input size of this problem is $size(n) = \log_2 n$, so the number of comparisons performed is actually $\Theta(n) = \Theta(2^{size(n)})$, which is exponential.

Input Size of Problems

- **Definition** Two positive functions $f(n)$ and $g(n)$ are of the same type if

$$c_1 g(n^{a_1})^{b_1} \leq f(n) \leq c_2 g(n^{a_2})^{b_2}$$

for all large n, where $a_1, b_1, c_1, a_2, b_2, c_2$ are some positive constants.

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Example:

All polynomials are of the same type, but *polynomials* and *exponentials* are of different types.

Input Size Example: Integer Multiplication

- **Example:** (Integer Multiplication problem)

  Compute $a \times b$.

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  The minimum inpute size is

  $$s = \lceil \log_2(a+1) \rceil + \lceil \log_2(b+1) \rceil.$$

  A natural choice is to use $t = \log_2 \max(a, b)$ since $\frac{s}{2} \leq t \leq s$.

Decision Problems

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  If L is the problem, and x is the input, we will ofter write $x \in L$ to denote a yes answer and $x \notin L$ to denote a no answer.

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  Examples:

  Knapsack vs. Decision Knapsack (DKnapsack)
  Subset Sum vs. Decision Subset Sum (DSubset Sum)

Knapsack vs. DKnapsack

- We have a knapsack of capacity W (a positive integer) and n objects with weights $w_1, \ldots, w_n$ and values $v_1, \ldots, v_n$, where v_i and w_i are positive integers.

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- We have a knapsack of capacity W (a positive integer) and n objects with weights $w_1, \ldots, w_n$ and values $v_1, \ldots, v_n$, where v_i and w_i are positive integers.

  Optimization problem: (Knapsack)

  Find the largest value $\sum_{i \in T} v_i$ of any subset T that fits in the knapsack, i.e., $\sum_{i \in T} w_i \leq W$.

  Decision problem: (DKnapsack)

  Given k, is there a subset of the objects that fits in the knapsack and has total value at least k?

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Optimization problem: (Subset Sum)

Among subsets of the objects with sum at most C, what is the largest subset sum?

Decision problem: (DSubset Sum)

Is there a subset of objects whose values add up to exactly C?

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- Given a subroutine for solving the optimization problem, solving the corresponding decision problem is usually trivial.

  First slove the optimization problem, then check the decision problem. If it does, answer yes, otherwise no.

Thus, if we prove that a given decision problem is hard to solve efficiently, then it is obvious that the optimization problem must be (at least as) hard.

Complexity Classes

- The Theory of Complexity deals with
  - ◇ the classification of certain "decision problems" into several classes:
    - ◇ the class of "easy" problems
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A. Use polynomial-time algorithms.

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  Example:

  The standard multiplication algorithm has time $O(m_1 m_2)$, where m_1, m_2 denote the number of digits in the two integers, respectively.

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  Question:
  Why?

  In terms of the input size, the complexity is $\Theta(2^n n^2)$.

Polynomial- vs. Nonpolynomial-Time

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  In reality, an $O(n^{20})$ algorithm is not really practical.

Polynomial-Time Solvable Problems

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  Definition (The Class P) The class P consists of all decision problems that are solvable in polynomial time. That is, there exists an algorithm that will decide in polynomial time if any given input is a yes-input or a no-input.

The Class P

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  How to prove that a decision problem is in P?

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  Question:
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  A. You need to prove that there is no polynomial-time algorithm for this problem. (much much harder)

Certificates and Verifying Certificates

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  Verifying a certificate: Given a presumed yes-input and its corresponding certificate, by making use of the given certificate, we verify that the input is actually a yes-input.

The Class NP

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  NP – "nondeterministic polynomial-time"

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Satisfiability (SAT)

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- One of the most important problems in CS is whether P = NP or P ≠ NP?
- Observe that P ⊆ NP.
- Intuitively, NP ⊆ P is doubtful.

  Just being able to verify a certificate in polynomial time does not necessarily mean we can tell whether an input is a yes-input or a no-input in polynomial time.

  However, we are still no closer to solving it and do not know the answer. The search for a solution, though, has provided us with deep insights into what distinguishes an "easy" problem form a "hard" one.

Announcements

- **Homework assignment** 2

  ◇ P126 Ex. 45, P137 Ex. 40, 41, 46*, P155 Ex. 70, 76, 80*, P169 Ex. 35, 38*, 41*, P176 Ex. 16, 17, P203 Ex. 49, P217 Ex. 44, 45, 50, P218 Ex. 71, P229 Ex. 10, P230 Ex. 13 b), 14 b)

  ◇ Due on *Oct. 24th, 2017 at the beginning of class*

  ◇ Please try you best to slove problems marked with $*$

  ◇ Please write your homeowrk **neatly**, as a courtesy to graders.

Next Lecture

- number theory, ...