南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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Primes

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- **Fundamental Theorem of Arithmetic** Every integer greater than 1 can be written uniquely as a prime or as the product of two or more primes where the prime factors are written in order of nondecreasing size.

Primes and Composites

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Approach 2: test if each prime number $x < n$ divides n.

Approach 3: test if each prime number $x < \sqrt{n}$ divides n.

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Proof.

◇ if n is composite, then it has a positive integer factor a such that $1 < a < n$ by definition. This means that $n = ab$, where b is an integer greater than 1.

◇ assume that $a > \sqrt{n}$ and $b > \sqrt{n}$. Then $ab > n$, contradiction. So either $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

◇ Thus, n has a divisor less than $\sqrt{n}$.

◇ By the Fundamental Theorem of Arithmetic, this divisor is either prime, or is a product of primes. In either case, n has a prime divisor less than $\sqrt{n}$.

Primes

- There are infinitely many primes.

Proof (by contradiction)

Greatest Common Divisor (GCD)

- Let a and b be integers, not both 0. The largest integer d such that $d \mid a$ and $d \mid b$ is called the *greatest common divisor* of a and b, denoted by $\gcd(a, b)$.

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  A systematic way to find the gcd is **factorization**. Let $a = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$ and $b = p_1^{b_1} p_2^{b_2} \cdots p_n^{b_n}$. Then $\gcd(a, b) = p_1^{\min(a_1, b_1)} p_2^{\min(a_2, b_2)} \cdots p_n^{\min(a_k, b_k)}$

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We can also use **factorization** to find the lcm. Let $a = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$ and $b = p_1^{b_1} p_2^{b_2} \cdots p_n^{b_n}$. Then
$\text{gcd}(a, b) = p_1^{\max(a_1, b_1)} p_2^{\max(a_2, b_2)} \cdots p_n^{\max(a_k, b_k)}$

Euclidean Algorithm

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- Luckily, we have an efficient algorithm, called **Euclidean algorithm**. This algorithm has been known since ancient times and named after the ancient Greek mathmaticain Euclid.

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Euclidean Algorithm

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Step 1: $287 = 91 \cdot 3 + 14$

Step 2: $91 = 14 \cdot 6 + 7$

Step 3: $14 = 7 \cdot 2 + 0$

$$\gcd(287, 91) = \gcd(91, 14) = \gcd(14, 7) = 7$$

Euclidean Algorithm

- The Euclidean algorithm in pseudocode

ALGORITHM 1 The Euclidean Algorithm.

procedure $gcd(a, b$: positive integers)
$x := a$
$y := b$
while $y \neq 0$
    $r := x$ **mod** y
    $x := y$
    $y := r$
return $x\{\gcd(a, b)$ is $x\}$

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The number of divisions required to find $\gcd(a, b)$ is $O(\log b)$, where $a \geq b$. (this will be proved later.)

Correctness of Euclidean Algorithm

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◇ Therefore, $\gcd(a, b) = \gcd(b, r)$.

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\begin{array}{llr}
r_0 & = r_1 q_1 + r_2 & 0 \leq r_2 < r_1, \\
r_1 & = r_2 q_2 + r_3 & 0 \leq r_3 < r_2, \\
 & \vdots & \\
r_{n-2} & = r_{n-1} q_{n-1} + r_n & 0 \leq r_n < r_{n-1}, \\
r_{n-1} & = r_n q_n \,. &
\end{array}
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$$

$$\gcd(a,b) = \gcd(r_0, r_1) = \cdots = \gcd(r_{n-1}, r_n) = \gcd(r_n, 0) = r_n$$

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- **Example**: Find the GCD of 286 and 503.

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&\gcd(503, 286)\\
&= \gcd(286, 217)\\
&= \gcd(217, 69)\\
&= \gcd(69, 10)\\
&= \gcd(10, 9)\\
&= 1
\end{aligned}
$$

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\begin{aligned}
503 &= 1 \cdot 286 + 217\\
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9 &= 9 \cdot 1
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GCD as Linear Combinations

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$$\begin{aligned} 1 &= 10 - 1 \cdot 9 \\ &= 7 \cdot 10 - 1 \cdot 69 \\ &= 7 \cdot 217 - 22 \cdot 69 \\ &= 29 \cdot 217 - 22 \cdot 286 \\ &= 29 \cdot 503 - 51 \cdot 286 \end{aligned}$$

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  Proof. by induction

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Proof. (by contradiction) Suppose that the positive integer n can be written as a product of primes in two distinct ways:

$n = p_1 p_2 \cdots p_s$ and $n = q_1 q_2 \cdots q_t$

Remove all common primes from the factorizations to get

$p_{i_1} p_{i_2} \cdots p_{i_u} = q_{j_1} q_{j_2} \cdots q_{j_v}$

It then follows that p_{i_1} divides q_{j_k} for some k, contradicting the assumption that p_{i_1} and q_{j_k} are distinct primes.

Dividing Congruences by an Integer

- **Theorem** Let m be a positive integer and let a, b, c be integers. If $ac \equiv bc \pmod{m}$ and $\gcd(c, m) = 1$, then $a \equiv b \pmod{m}$.

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Proof. Since $ac \equiv bc \pmod{m}$, we have $m | ac - bc = c(a - b)$. Because $\gcd(c, m) = 1$, it follows that $m | a - b$.

Mersenne Primes

- Prime numbers of the form $2^p - 1$, where p is a prime.

Marin Mersenne

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  ◇ $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 37$, $2^7 - 1 = 127$ are Mersenne primes.

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Largest Known Prime, 49th Known Mersenne Prime Found!

January 7, 2016 — GIMPS celebrated its 20th anniversary with the discovery of the largest known prime number, $2^{74,207,281}$-1.

http://www.mersenne.org/

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- *Twin-prime Conjecture*: There are infinitely many twin primes.

Linear Congruences

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  Systems of linear congruences have been studied since ancient times.

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Modular Inverse

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  One method of solving linear congruences makes use of an inverse $\bar{a}$ if it exists. From $ax \equiv b \pmod{m}$, it follows that $\bar{a}ax \equiv \bar{a}b \pmod{m}$ and then $x \equiv \bar{a}b \pmod{m}$.

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  When does an inverse of a modulo m exist?

Inverse of a modulo m

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  Proof. Since $\gcd(a, m) = 1$, there are integers s and t such that $sa + tm = 1$. Hence $sa + tm \equiv 1 \pmod{m}$. Since $tm \equiv 0 \pmod{m}$, it follows that $sa \equiv 1 \pmod{m}$. This means that s is an inverse of a modulo m.

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  How to prove the uniqueness of the inverse?

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Example. Find an inverse of 101 modulo 4620.

$$4620 = 45 \cdot 101 + 75$$
$$101 = 1 \cdot 75 + 26$$
$$75 = 2 \cdot 26 + 23$$
$$26 = 1 \cdot 23 + 3$$
$$23 = 7 \cdot 3 + 2$$
$$3 = 1 \cdot 2 + 1$$
$$2 = 2 \cdot 1$$

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\end{aligned}
$$

$$
\begin{aligned}
1 &= 3 - 1\cdot 2\\
1 &= 3 - 1\cdot(23 - 7\cdot 3) = -1\cdot 23 + 8\cdot 3\\
1 &= -1\cdot 23 + 8\cdot(26 - 1\cdot 23) = 8\cdot 26 - 9\cdot 23\\
1 &= 8\cdot 26 - 9\cdot(75 - 2\cdot 26) = 26\cdot 26 - 9\cdot 75\\
1 &= 26\cdot(101 - 1\cdot 75) - 9\cdot 75\\
&= 26\cdot 101 - 35\cdot 75\\
1 &= 26\cdot 101 - 35\cdot(4620 - 45\cdot 101)\\
&= -35\cdot 4620 + 1601\cdot 101
\end{aligned}
$$

Using Inverses to Solve Congruences

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  Example. What are the solutions of the congruence $3x \equiv 4 \pmod{7}$?

  Solution: We found that -2 is an inverse of 3 modulo 7. Multiply both sides of the congruence by -2, we have $x \equiv -8 \equiv 6 \pmod{7}$.

Number of Solutions to Congruences *

- **Theorem** Let $d = \gcd(a, m)$ and $m' = m/d$. The congruence $ax \equiv b \pmod{m}$ has solutions if and only if $d|b$. If $d|b$, then there are exactly d solutions. If x_0 is a solution, then the other solutions are given by $x_0 + m', x_0 + 2m', \ldots, x_0 + (d-1)m'$.

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Proof.
1) "only if": If x_0 is a solution, then $ax_0 - b = km$. Thus, $ax_0 - km = b$. Since d divides $ax_0 - km$, we must have $d|b$.

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2) "if": Suppose that $d|b$. Let $b = kd$. There exist integers s, t such that $d = as + mt$. Multiply both sides by k. Then $b = ask + mtk$. Let $x_0 = sk$. Then $ax_0 \equiv b \pmod{m}$.

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1) "only if": If x_0 is a solution, then $ax_0 - b = km$. Thus, $ax_0 - km = b$. Since d divides $ax_0 - km$, we must have $d|b$.

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3) "$\# = d$": $ax_0 \equiv b \pmod{m}$ $ax_1 \equiv b \pmod{m}$ imply that $m|a(x_1 - x_0)$ and $m'|a'(x_1 - x_0)$. This implies further that $x_1 = x_0 + km'$, where $k = 0, 1, \ldots, d-1$.

The Chinese Remainder Theorem

- In the first century, the Chinese mathematician Sun-Tsu asked:
  "There are certain things whose number is unknown. When divided by 3, the remainder is 2; when divided by 5, the remainder is 3; when divided by 7, the remainder is 2. What will be the number of things?"

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  "There are certain things whose number is unknown. When divided by 3, the remainder is 2; when divided by 5, the remainder is 3; when divided by 7, the remainder is 2. What will be the number of things?"

今有物不知其数 三三数之剩二 五五数之剩三 七七数之剩二 问物几何

$$
\begin{aligned}
x &\equiv 2 \pmod 3 \\
x &\equiv 3 \pmod 5 \\
x &\equiv 2 \pmod 7
\end{aligned}
$$

The Chinese Remainder Theorem

- **Theorem** (*The Chinese Remainder Theorem*) Let $m_1, m_2, \ldots, m_n$ be pairwise relatively prime positive integers greater than 1 and $a_1, a_2, \ldots, a_n$ arbitrary integers. Then the system

$$x \equiv a_1 \pmod{m_1}$$
$$x \equiv a_2 \pmod{m_2}$$
$$\ldots$$
$$x \equiv a_n \pmod{m_n}$$

has a unique solution modulo $m = m_1 m_2 \cdots m_n$.

The Chinese Remainder Theorem

- **Proof** Let $M_k = m/m_k$ for $k = 1, 2, \ldots, n$ and $m = m_1 m_2 \cdots m_n$. Since $\gcd(m_k, M_k) = 1$, there is an integer y_k, an inverse of M_k modulo m_k such that $M_k y_k \equiv 1 \pmod{m_k}$. Let

$$x = a_1 M_1 y_1 + a_2 M_2 y_2 + \cdots a_n M_n y_n.$$

It is checked that x is a solution to the n congruences.

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It is checked that x is a solution to the n congruences.

How to prove the uniqueness of the solution modulo m?

The Chinese Remainder Theorem

- **Example**

$$\begin{aligned} x &\equiv 2 \pmod{3} \\ x &\equiv 3 \pmod{5} \\ x &\equiv 2 \pmod{7} \end{aligned}$$

The Chinese Remainder Theorem

- **Example**

$$x \equiv 2 \pmod 3$$
$$x \equiv 3 \pmod 5$$
$$x \equiv 2 \pmod 7$$

Let $m = 3 \cdot 5 \cdot 7 = 105$, $M_1 = m/3 = 35$, $M_2 = m/5 = 21$, $M_3 = m/7 = 15$.

$$35 \cdot 2 \equiv 1 \pmod 3$$
$$21 \equiv 1 \pmod 5$$
$$15 \equiv 1 \pmod 7$$

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$$x = 2 \cdot 35 \cdot 2 + 3 \cdot 21 \cdot 1 + 2 \cdot 15 \cdot 1 \equiv 233 \equiv 23 \pmod{105}$$

Back Substitution

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- We may also solve systems of linear congruences with pairwise relatively prime moduli by *back substitution*.

Example

$$\begin{aligned} x &\equiv 2 \pmod{3} \\ x &\equiv 3 \pmod{5} \\ x &\equiv 2 \pmod{7} \end{aligned}$$

$$\begin{aligned} x &\equiv 8 \pmod{15} \\ x &\equiv 2 \pmod{21} \end{aligned}$$

Modular Arithmetic in CS

- Modular arithmetic and congruencies are used in CS:
  - Pseudorandom number generators
  - Hash functions
  - Cryptography

Pseudorandom Number Generators

- *Linear congruential method*

  We choose four numbers:

  - the modulus m
  - multiplier a
  - increment c
  - seed x_0

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- the modulus m
- multiplier a
- increment c
- seed x_0

We generate a sequence of numbers $x_1, x_2, \ldots, x_n, \ldots$ with $0 \leq x_i < m$ by using the congruence

$$x_{n+1} = (ax_n + c) \pmod{m}$$

Pseudorandom Number Generators

- *Linear congruential method*

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- *Linear congruential method*

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Example:

- Assume : m=9,a=7,c=4, $x_0 = 3$

- x_1= 7*3+4 mod 9=25 mod 9 =7
- x_2 = 53 mod 9 = 8
- x_3 = 60 mod 9 = 6
- x_4= 46 mod 9 =1
- x_5 = 11 mod 9 =2
- x_6 = 18 mod 9 =0
- ....

Hash Functions

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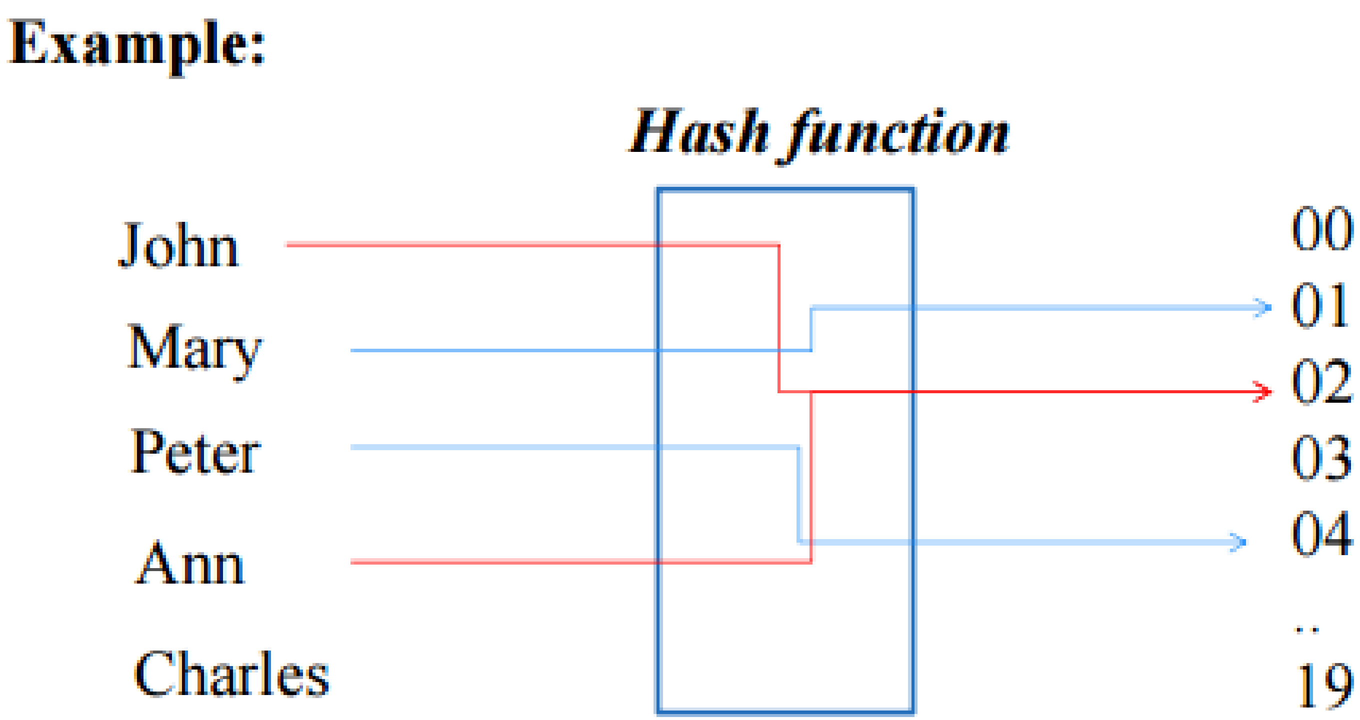

Hash Functions

- **Problem**: Given a large collection of records, how can we store and find a record quickly?

Hash Functions

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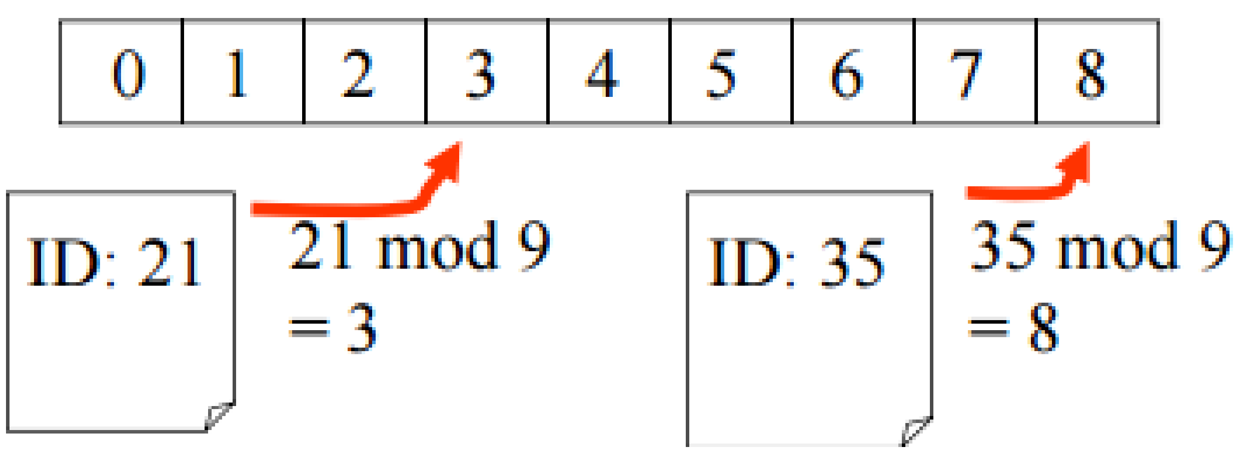
  Solution: Use a hash function, calculate the location of the record based on the record's ID.

Example: A common hash function is

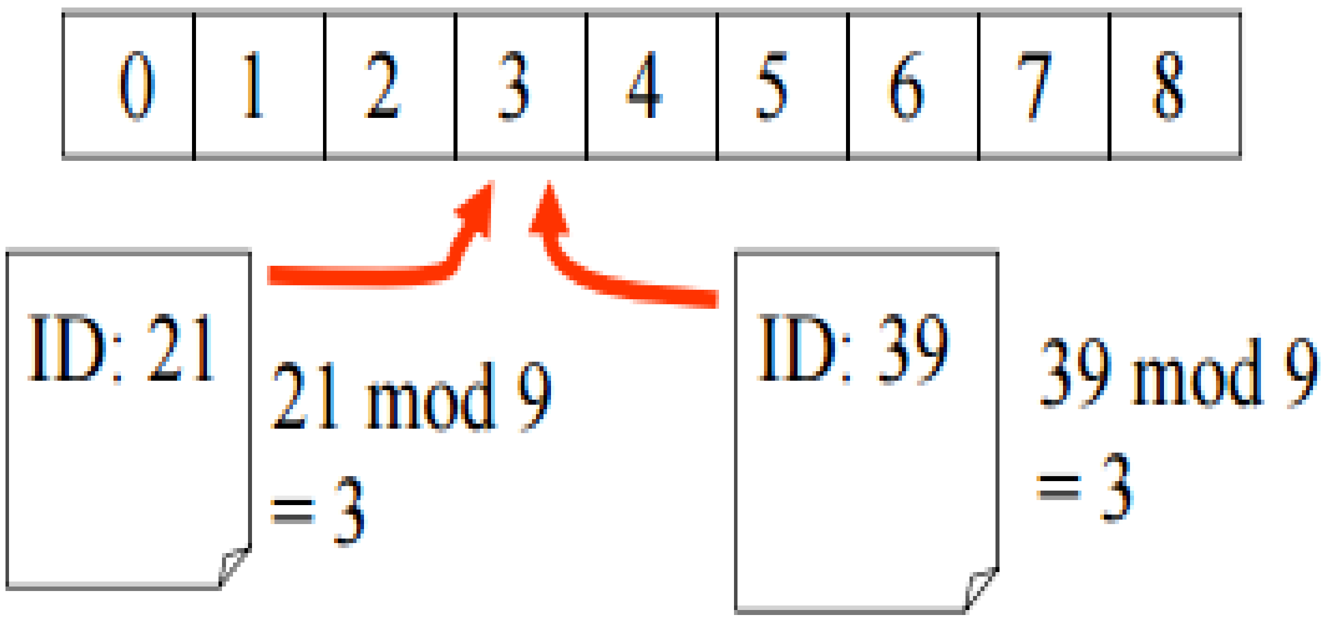
- $h(k) = k \textbf{ mod } n$,

where n is the number of available storage locations.



Hash Functions

- Two records mapped to the same location

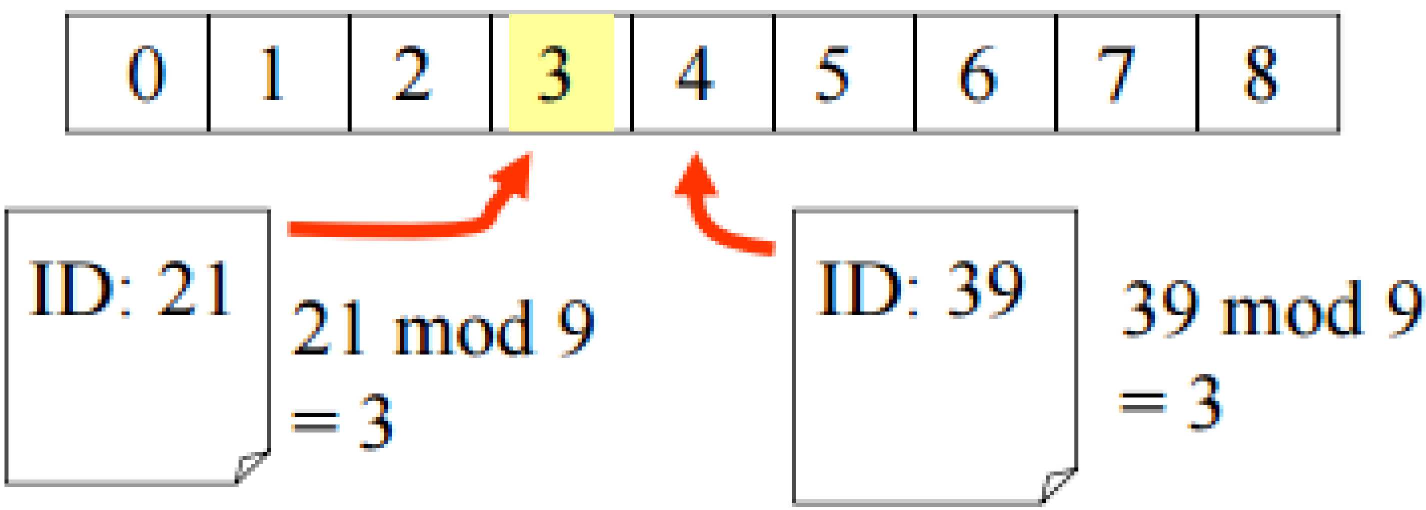


Hash Functions

- **Solution 1**: move to the next available location

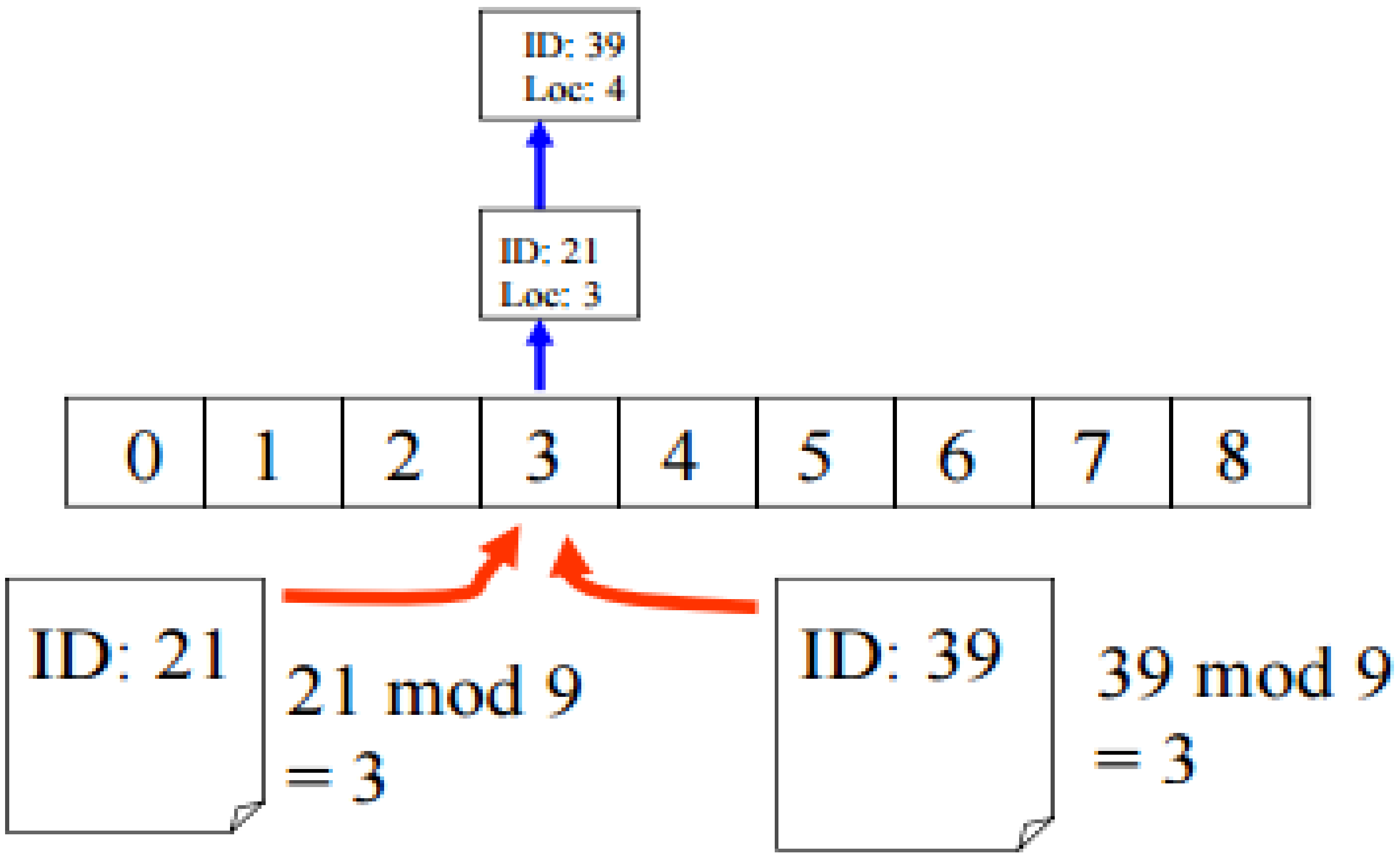
try

$$h_0(k) = k \textbf{ mod } n$$
$$h_1(k) = (k+1) \textbf{ mod } n$$
$$\ldots$$
$$h_m(k) = (k+m) \textbf{ mod } n$$



Hash Functions

- **Solution 2**: remember the exact location in a secondary structure that is searched sequentially



Applications of Number Theory in Cryptography

- Introduction
- Symmetric cryptography
- Asymmetric cryptography
- RSA Cryptosystem
- DLP and El Gamal cryptography
- Diffie-Hellman key exchange protocol
- Crytocurrency, e.g., bitcoin

Next Lecture

- cryptography, ...