DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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Compound Propositions

- A *proposition* is a declarative statement that is either true or false. More complex propositions can be built from elementary statements using logical connectives.

- Logical connectives:
  - ◇ $\neg p$ (*Negation*)
  - ◇ $p \wedge q$ (*Conjunction*)
  - ◇ $p \vee q$ (*Disjunction*)
  - ◇ $p \oplus q$ (*Exclusive or*)
  - ◇ $p \rightarrow q$ (*Implication*)
  - ◇ $p \leftrightarrow q$ (*Biconditional*)

Applications of Propositional Logic

- Translation of English sentences
  - ◇ use atomic (elementary) propositions
- Inference and reasoning
  - ◇ new true propositions are inferred from existing ones
  - ◇ used in *Artificial Intelligence*:
    - – rule based (expert) systems, automatic theorem provers, ...
- Design of logic circuit

Translation

- If you are older than 13 or you are with your parents then you can watch this movie.

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  Given that p is true.

  With the help of the logic, we can infer the following statement:

  – You can watch this movie.

Inference

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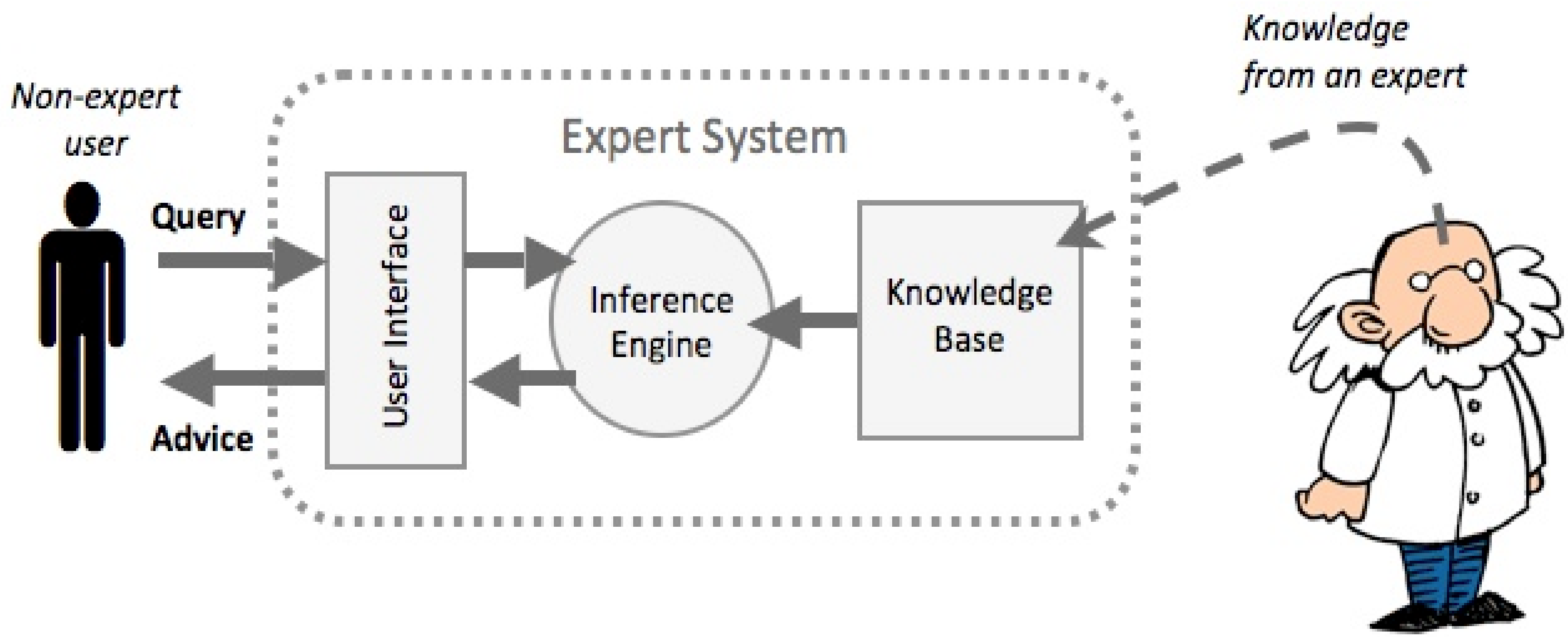
- *Expert systems*
  - ◇ encode knowledge about the world in logic
  - ◇ support inferences where new facts are inferred from existing ones following the semantics of logic

- *Theorem provers*
  - ◇ encode existing knowledge (e.g., math) using logic
  - ◇ show that some hypothesis is true

Expert System



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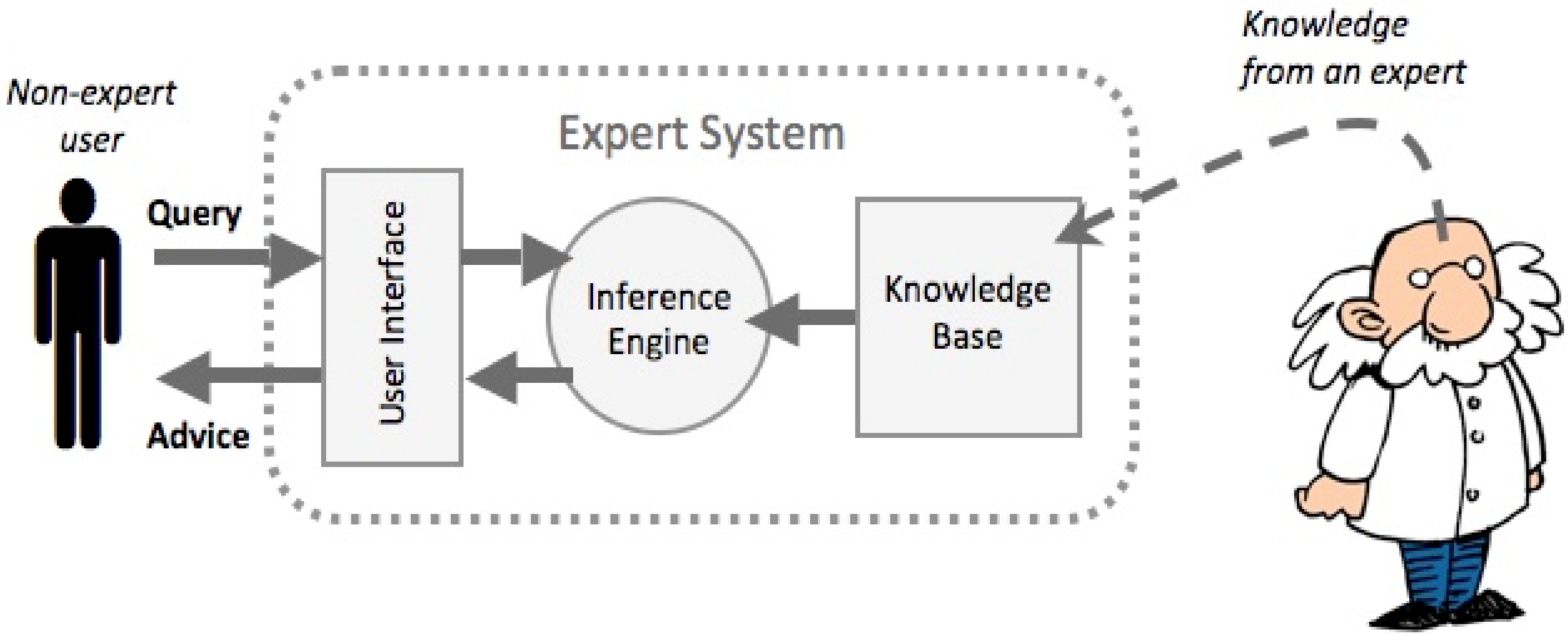

Theorem Prover

Contents lists available at ScienceDirect

Artificial Intelligence

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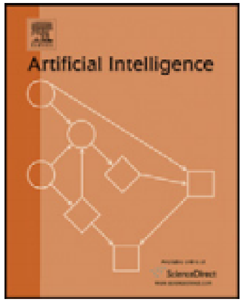

Computer-aided proofs of Arrow's and other impossibility theorems ☆

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A R T I C L E   I N F O

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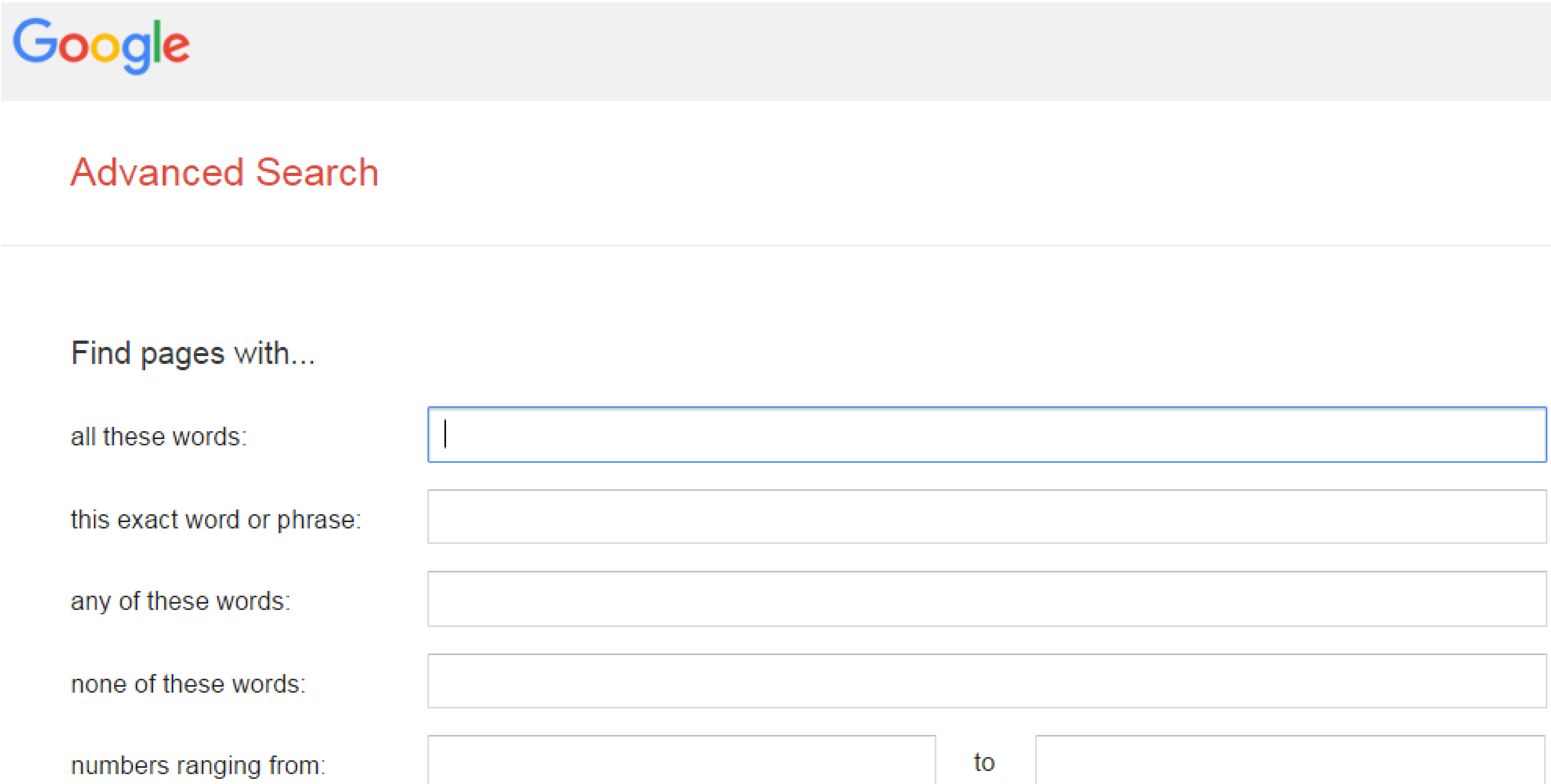
A B S T R A C T

Arrow's impossibility theorem is one of the landmark results in social choice theory. Over the years since the theorem was proved in 1950, quite a few alternative proofs have been put forward. In this paper, we propose yet another alternative proof of the theorem. The basic idea is to use induction to reduce the theorem to the base case with 3 alternatives and 2 agents and then use computers to verify the base case. This turns out to be an effective approach for proving other impossibility theorems such as Muller–Satterthwaite and Sen's theorems as well. Motivated by the insights of the proof, we discover a new theorem with the help of computer programs. We believe this new proof opens an exciting prospect of using computers to discover similar impossibility or even possibility results.

Other Applications



Tautology and Contradiction

- A compound proposition that is always true for all possible truth values is called a *tautology*.
- A compound proposition that is always false for all possible truth values is called a *contradiction*.
- A compound proposition that is neither a tautology nor a contradiction is called a *contingency*.

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p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
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F	T	T	F

Equivalent Propositions

- Two propositions are *equivalent* if they always have the same truth value.

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Examples:

```
(1) if ((i+j ≤ p+q) && (i ≤ p) &&
    ((j > q) || (List1[i] ≤ List2[j])))
(2)   List3[k] = List1[i]
(3)   i = i+1
(4) else
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Consider the two pieces of codes taken from two different versions of *Mergesort*. Do they do the same thing?

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  $s \sim (i + j \leq p + q)$, $t \sim (i \leq p)$, $u \sim (j > q)$
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(1') $(s \wedge t \wedge u) \vee (s \wedge t \wedge v)$

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(1) $s \wedge t \wedge (u \vee v)$ (1') $(s \wedge t \wedge u) \vee (s \wedge t \wedge v)$

Now set $w \sim (s \wedge t)$

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Truth Tables

(1) $w \wedge (u \vee v)$

w	u	v	$u \vee v$	$w \wedge (u \vee v)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

(1') $(w \wedge u) \vee (w \wedge v)$

w	u	v	$w \wedge u$	$w \wedge v$	$(w \wedge u) \vee (w \wedge v)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	F	F
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Distributive Laws

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  Example $p \rightarrow q \equiv \neg q \rightarrow \neg p$

De Morgan's Laws

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p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Important Logical Equivalences

- *Identity laws*
  - $p \wedge T \equiv p$
  - $p \vee F \equiv p$
- *Domination laws*
  - $p \vee T \equiv T$
  - $p \wedge F \equiv F$
- *Idempotent laws*
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$

Important Logical Equivalences

- *Double negation laws*
  - $\neg(\neg p) \equiv p$
- *Commutative laws*
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- *Associative laws*
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

Important Logical Equivalences

- *Distributive laws*
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- *De Morgan's laws*
  - $\neg(p \vee q) \equiv \neg p \wedge \neg q$
  - $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- *Others*
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$

    Absorption laws
  - $p \vee \neg p \equiv T$
  - $p \wedge \neg p \equiv F$

    Negation laws
  - $p \rightarrow q \equiv \neg p \vee q$

Using Logical Equivalences

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Proof:

$$
\begin{aligned}
(p \wedge q) \rightarrow p &\equiv \neg(p \wedge q) \vee p && \text{Useful} \\
&\equiv (\neg p \vee \neg q) \vee p && \text{De Morgan's} \\
&\equiv (\neg q \vee \neg p) \vee p && \text{Communtative} \\
&\equiv \neg q \vee (\neg p \vee p) && \text{Associative} \\
&\equiv \neg q \vee T && \text{Negation} \\
&\equiv T && \text{Domination}
\end{aligned}
$$

Using Logical Equivalences

- Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences.

- **Example**: Show that $(p \wedge q) \rightarrow p$ is a tautology.

  Proof (alternatively):

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	**T**
T	F	F	**T**
F	T	F	**T**
F	F	F	**T**

Using Logical Equivalences

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Proof:

$$\begin{aligned} \neg q \to \neg p &\equiv \neg(\neg q) \vee (\neg p) && \text{Useful} \\ &\equiv q \vee (\neg p) && \text{Double negation} \\ &\equiv (\neg p) \vee q && \text{Communtative} \\ &\equiv p \to q && \text{Useful} \end{aligned}$$

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  object property

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What is a more natural solution to express the knowledge?

Solution: make statements with variables

- If x is an integer, then $x^2 \geq 0$.
- x is an integer $\rightarrow x^2 \geq 0$.

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  - "Some of the integers are prime."

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  Solutions: make statements with *quantifiers*
  – universal quantifier: the property is satisfied by all members of the group
  – existential quantifier: at least one member of the group satisfies the property

Predicate Logic

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- Remedies the limitations of propositional logic:
  - ◇ explicitly models objects and their properties
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- Basic building blocks of the predicate logic:
  - ◇ *Constant* – models a specific object

    Examples: "1", "SUSTech", ...
  - ◇ *Variable* – represents object of specific type

    Examples: x, y, ... (universe can be people, numbers ...)
  - ◇ *Predicate* – represents properties or relations among objects

    Examples: Red(car23), student(x), married(A, B)...

Predicates

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  - ...

  Is Prime(x) a proposition?

  No, but after the substitution it becomes one.

Predicates

- A *predicate* is a statement $P(x_1, x_2, \ldots, x_n)$ that contains n variables $x_1, x_2, \ldots, x_n$ and becomes a proposition when specific values are substituted for the variables x_i.

- The *universe* (*domain*) D of the *predicate variables* $(x_1, x_2, \ldots, x_n)$ is the set of all values that may be substituted in place of the variables.

- The *truth set* of $P(x_1, x_2, \ldots, x_n)$ is the set of all values of the predicate variables $(x_1, x_2, \ldots, x_n)$ such that the proposition $P(x_1, x_2, \ldots, x_n)$ is true.

Examples of Predicates

- **Example 1**: (Predicates with one variable)
  Let $P(x)$ be the predicate "$x^2 > x$" with universe of the real numbers.
  - What are the truth values of $P(2)$ and $P(1)$?
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  - ◇ What is the truth set of $P(x)$?

  - ◇ $P(2) = \mathrm{T}$, $P(1) = \mathrm{F}$
  - ◇ $x > 1$ or $x < 0$

Examples of Predicates

- **Example 2**: (Predicates with two variables)
  Let $Q(x, y)$ be the predicate "$x = y + 3$" with universe of the real numbers.
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  - ◇ What is the truth set of $Q(x, y)$?

  - ◇ $Q(1, 2) = F$, $Q(3, 0) = T$
  - ◇ $(a, a - 3)$ for all real numbers a

Compound Statements in Predicate Logic

- Compound statements are obtained via logical connectives.

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  Example:

  ◇ Prime(2) $\wedge$ Prime(3)
  – Translation: "Both 2 and 3 are primes." (T)
  ◇ City(Shenzhen) $\vee$ River(Shenzhen)
  – Translation: "Shenzhen is a city or a river." (T)
  ◇ CS-major(x) $\rightarrow$ Student(x)
  – Translation: "If x is CS-major then x is a student." (not a proposition)

Predicates

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- The statement $P(x)$ is not a proposition since there are more objects that it can be applied to.

- But the difference is:

  ◇ predicate logic allows us to explicitly manipulate and substitute for the objects

  ◇ predicate logic permits quantified sentences where variables are substituted for statements about the group of objects

Quantified Statements

- Two types of quantified statements:
  - *Universal*

    Example: "All CS-major graduates have to pass CS201".
    (This is true for all CS-major graduates.)

  - *Existential*

    Example: "Some CS-major students graduate with honor."
    (This is true for some students.)

Universal Quantifier

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Example:

- ◇ $P(x)$ – "$x > x - 1$".
- ◇ What is the truth value of $\forall x\ P(x)$?

Universal Quantifier

- The *universal quantification* of $P(x)$ is the proposition: "$P(x)$ is true for all values of x in the universe of discourse." denoted by $\forall x\ P(x)$, and is expressed as for every x, $P(x)$.

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  - ◇ $P(x)$ – "$x > x - 1$".
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- ◇ Is $\forall x\ P(x)$ a proposition?

Yes. True if for all x from the unvierse $P(x)$ is ture.

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  Yes. True if there is even an x s.t. $P(x)$ is ture (e.g. 10).

Existential Quantifier

- **Example**:
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  - What is the truth value of $\exists x\ Q(x)$?

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- **Example**:
  - ◇ $Q(x)$ – "$x = x + 2$ where x is a real number".
  - ◇ What is the truth value of $\exists x\ Q(x)$?
  - ◇ $\exists x\ Q(x)$ is false.
  - ◇ $C(x)$ – CS-major(x) $\wedge$ Honor-student(x).
  - ◇ What is the truth value of $\exists x\ C(x)$?
  - ◇ Translation: "There is a person who is a CS-major student and who also graduated with honor." (T)

Summary of Quantified Statements

- When $\forall x\ P(x)$ and $\exists x\ P(x)$ are true and false?

Statement	When true?	When false?
$\forall$x P(x)	P(x) true for all x	There is an x where P(x) is false.
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- Suppose that the elements in the universe can be enumerated as $x_1, x_2, \ldots, x_n$ then:
  - ◇ $\forall x\ P(x)$ is true whenever $P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$ is true
  - ◇ $\exists x\ P(x)$ is true whenever $P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$ is true.

Properties of Quantifiers

- The truth values of $\exists x\ P(x)$ and $\forall x\ P(x)$ depend on both the propositional function $P(x)$ and the universe.

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Example: $P(x)$ – "$x < 2$"

◇ universe: the positive integers

$\exists x\ P(x)$ – T, $\forall x\ P(x)$ – F

◇ universe: the negative integers

$\exists x\ P(x)$ – T, $\forall x\ P(x)$ – T

◇ universe: $\{\ 3,\ 4,\ 5\ \}$

$\exists x\ P(x)$ – F, $\forall x\ P(x)$ – F

Precedence of Quantifiers

- The quantifiers $\forall$ and $\exists$ have *higher precedence* than all the logical operators.

  - $\forall x \ P(x) \lor Q(x)$ means $(\forall x \ P(x)) \lor Q(x)$ rather than $\forall x \ (P(x) \lor Q(x))$

Translation with Quantifiers

- Sentence: All SUSTech students are smart.
  - ◇ universe: SUSTech students
    translation: $\forall x\ Smart(x)$

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  - ◇ universe: people

    translation: $\forall x \; (Student(x) \wedge At(x, SUSTech) \rightarrow Smart(x))$

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    (Everything is imperfect.)

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Conclusion: $\neg \exists x \; P(x)$ is equivalent to $\forall x \; \neg P(x)$

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Negation of Quantifiers

- Sentence: Not all horses are white.
  - ◇ translation: $\neg \forall x \ (Horse(x) \to White(x))$
  - ◇ translation: $\exists x \ (Horse(x) \land \neg White(x))$ (There is a horse that is not white.)
  - ◇ logically equivalent to $\exists x \ \neg(Horse(x) \to White(x))$

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Negation of Quantified Statements

- a.k.a. De Morgan laws for quantifiers

Negation	Equivalent Statement	When Is Negation True?	When False?
$\neg \exists x \; P(x)$	$\forall x \; \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg \forall x \; P(x)$	$\exists x \; \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

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  - ◇ a predicate $P(x, y)$ denotes "$x + y = 0$"

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Example 1: "Every real number has its corresponding negative."

- ◇ a real number is denoted by x and its negative as y
- ◇ a predicate $P(x, y)$ denotes "$x + y = 0$"

$$\forall x \exists y \; P(x, y)$$

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- The order of nested quantifiers matters if quantifiers are of different type.

Example: $\forall x \exists y\ L(x,y) \not\equiv \exists y \forall x\ L(x,y)$

- ◇ $L(x,y)$ denotes "x loves y"
- ◇ $\forall x \exists y\ L(x,y)$: Everybody loves somebody.
- ◇ $\exists y \forall x\ L(x,y)$: There is someone who is loved by everyone.

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  - For all x and y, if x is a parent of y then y is a child of x

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- ◇ $\forall x \forall y\ (Parent(x, y) \to Child(y, x))$
- ◇ $\forall y \forall x\ (Parent(x, y) \to Child(y, x))$

Translation Exercise

- Suppose that variables x, y denote people, and $L(x, y)$ denotes x loves y.

  Translate:

  - Everybody loves Raymond.
  - Everybody loves somebody.
  - There is somebody whom everybody loves.
  - There is somebody whom Raymond doesn't love.
  - There is somebody whom no one loves.

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- Suppose that variables x, y denote people, and $L(x, y)$ denotes x loves y.

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  - Everybody loves Raymond. $\forall x\ L(x, Raymond)$
  - Everybody loves somebody. $\forall x \exists y\ L(x, y)$
  - There is somebody whom everybody loves.
    $\exists y \forall x\ L(x, y)$
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    $\exists y\ \neg L(Raymond, y)$
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  - There is exactly one person whom everybody loves.

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    $\exists y\ \neg L(Raymond, y)$
  - There is somebody whom no one loves.
    $\exists y \forall x\ \neg L(x, y)$

  - There is exactly one person whom everybody loves.
    $\exists y(\forall x L(x, y) \wedge \forall z(\forall x L(x, z) \rightarrow z = y))$

Quantifications of Two Variables

Statement	When True?	When False
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	$P(x,y)$ is true for every pair x,y.	There is a pair x, y for which $P(x,y)$ is false.
$\forall x \exists y P(x,y)$	For every x there is a y for which $P(x,y)$ is true.	There is an x such that $P(x,y)$ is false for every y.
$\exists x \forall y P(x,y)$	There is an x for which $P(x,y)$ is true for every y.	For every x there is a y for which $P(x,y)$ is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair x, y for which $P(x,y)$ is true.	$P(x,y)$ is false for every pair x,y

Negating Nested Quantifiers

- Sentence: for every real number x, there exists a real number y such that $xy = 1$.

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  ◇ $\forall x \exists y\ (xy = 1)$

$$
\begin{aligned}
& \neg \forall x \exists y\ (xy = 1) \\
\equiv\ & \exists x \neg \exists y\ (xy = 1) \\
\equiv\ & \exists x \forall y\ \neg (xy = 1) \\
\equiv\ & \exists x \forall y\ (xy \neq 1)
\end{aligned}
$$

Next Lecture

- Proofs, sets ...