DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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The Principle of Mathematical Induction

- **Principle.** (*the Weak Principle of Mathematical Induction*)
  (a) If the statement $P(b)$ is true
  (b) the statement $P(n-1) \to P(n)$ is true for all $n > b$,
  then $P(n)$ is true for all integers $n \geq b$

  (a) – *Basic Step* *Inductive Hypothesis*

  (b) – *Inductive Step* *Inductive Conclusion*

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  - ◇ Iterating gives us a proof of $P(n)$ for all n

Strong Induction

- **Principle** (*The Strong Principle of Mathematical Induction*)

  (a) If the statement $P(b)$ is true

  (b) for all $n > b$, the statement
  $P(b) \wedge P(b+1) \wedge \cdots \wedge P(n-1) \rightarrow P(n)$ is true.

  then $P(n)$ is true for all integers $n \geq b$.

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  ◇ Then, if n is not a prime power, it is a product of two smaller numbers, each of which is, by the inductive hypothesis, a power a prime power or a product of powers of primes.

  ◇ Thus, by the strong principle of mathematical induction, every positive integer is a power of a prime or a product of powers of primes.

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- In reality, they are equivalent to each other in that the weak form is a special case of the strong form, and the strong form can be derived from the weak form.

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3. We conclude on the basis of the principle of mathematical induction that $P(n)$ is true for all $n \geq b$.

Recursion

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- A classical example of *recursion* is the **Towers of Hanoi** Problem.

Towers of Hanoi

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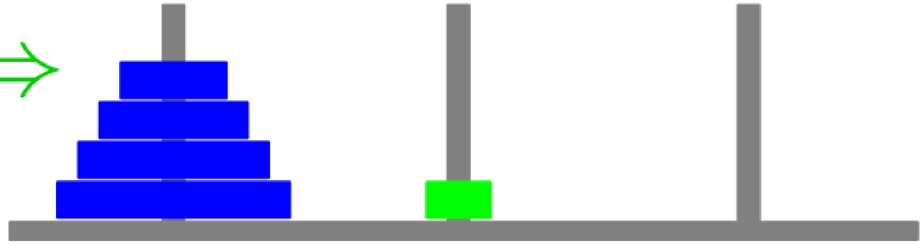
- 3 pegs; n disks of different sizes
- A *legal move* takes a disk from one peg and moves it onto another peg so that it is not on top of a smaller disk
- **Problem**: Find a (efficient) way to move all of the disks from one peg to another

Towers of Hanoi

Towers of Hanoi

legal move

$\Rightarrow$



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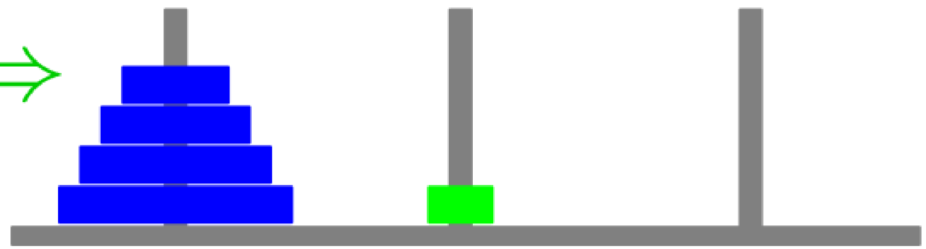

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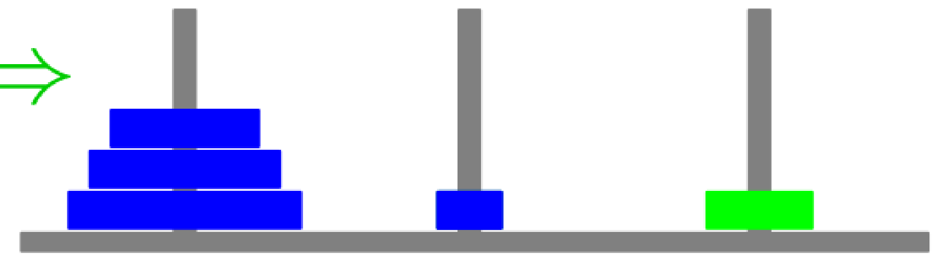

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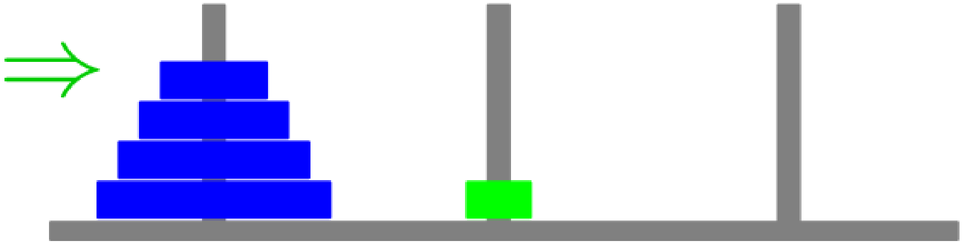

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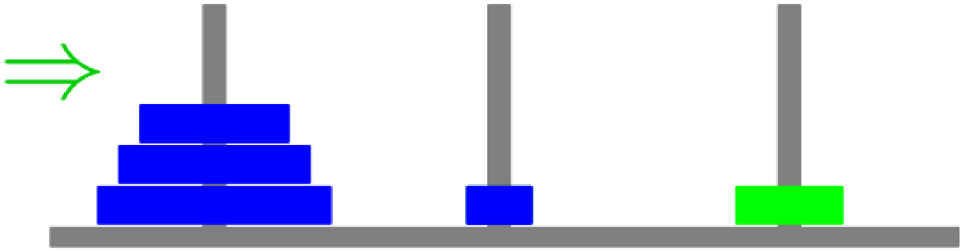

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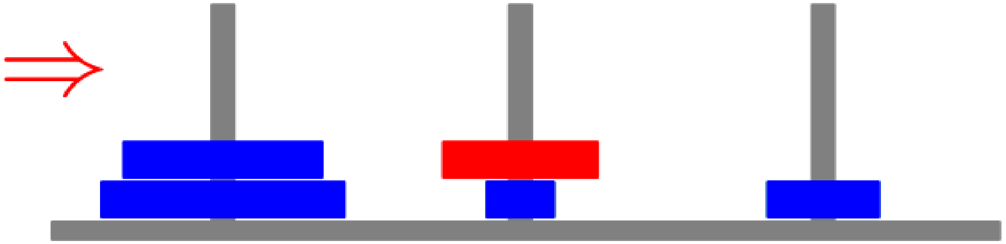

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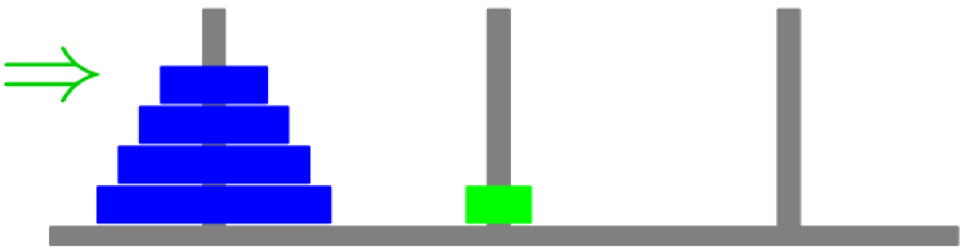

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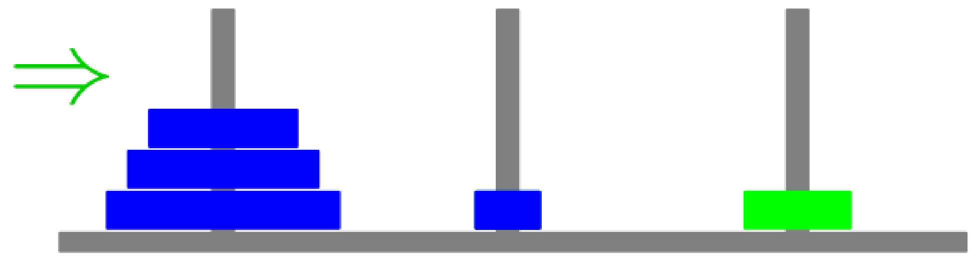

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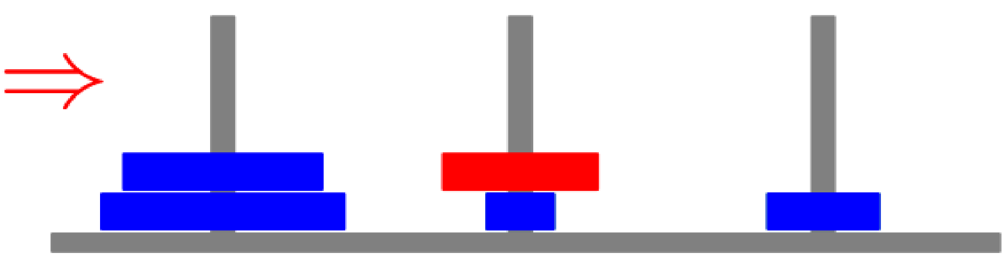

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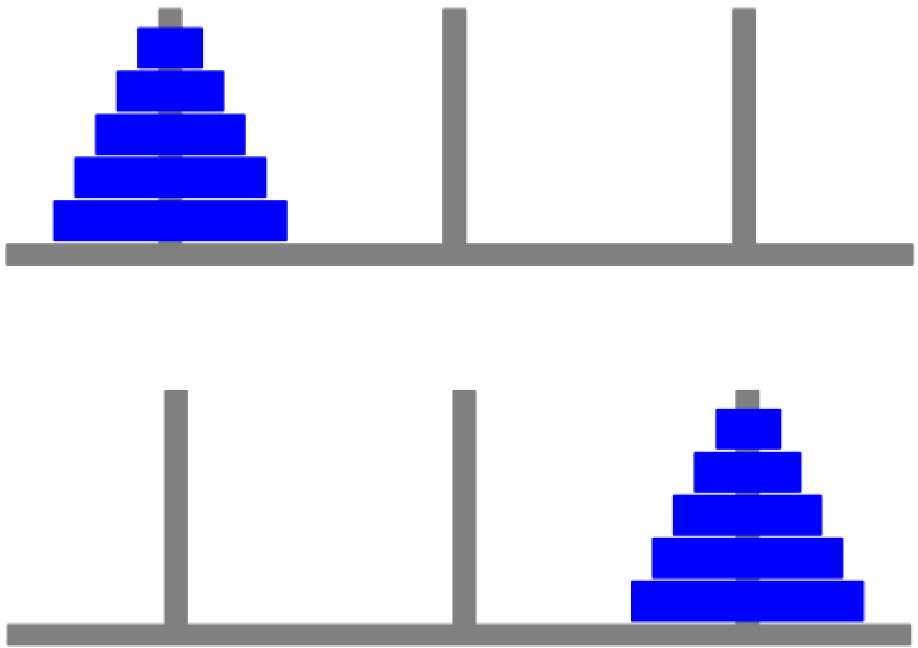

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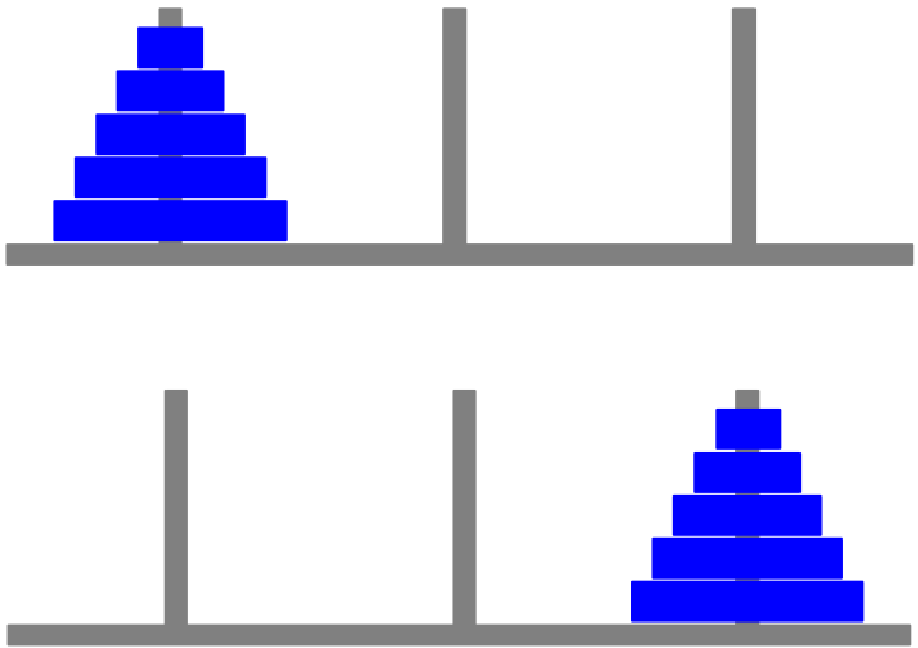

Given $i, j \in \{1,2,3\}$, let
$\overline{\{i,j\}} = \{1,2,3\} - \{i\} - \{j\}$,
i.e., $\overline{\{1,2\}} = \{3\}$, $\overline{\{1,3\}} = \{2\}$,
$\overline{\{2,3\}} = \{1\}$.

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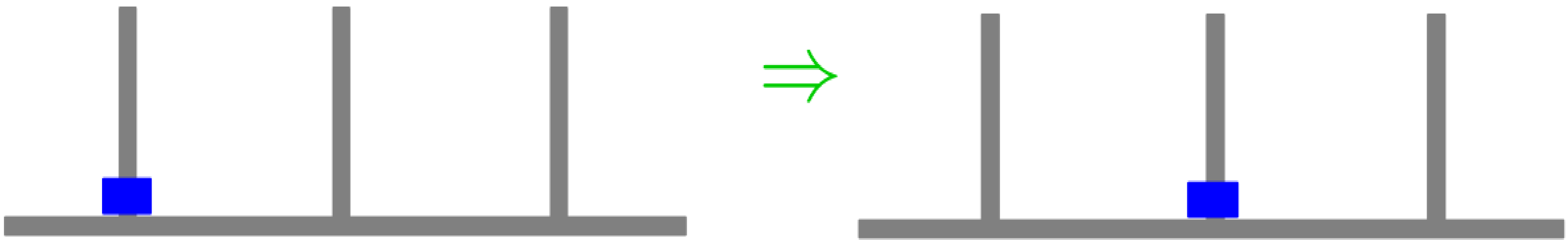

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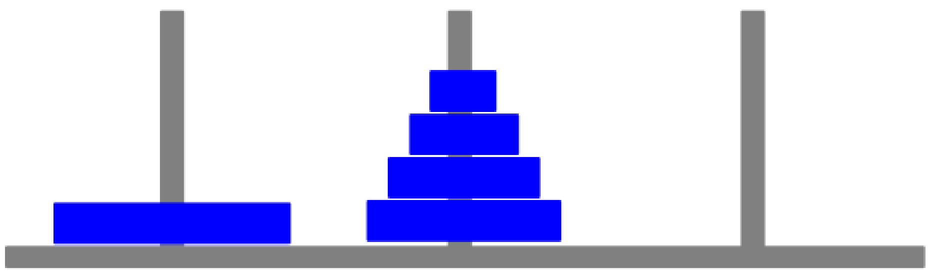

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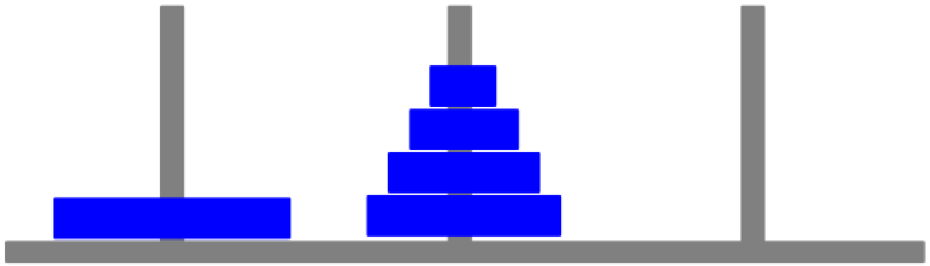

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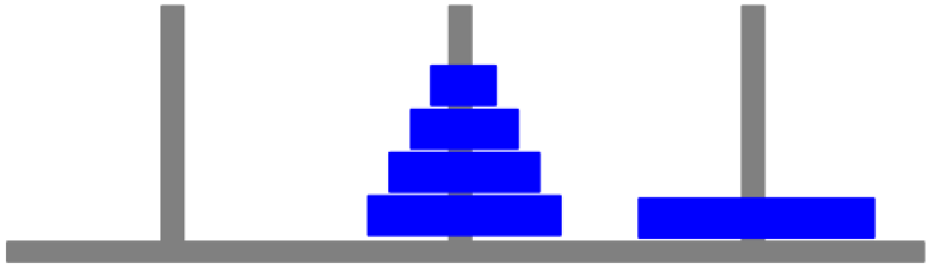

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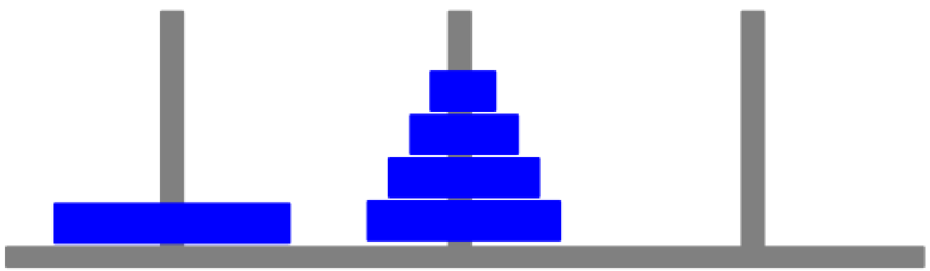

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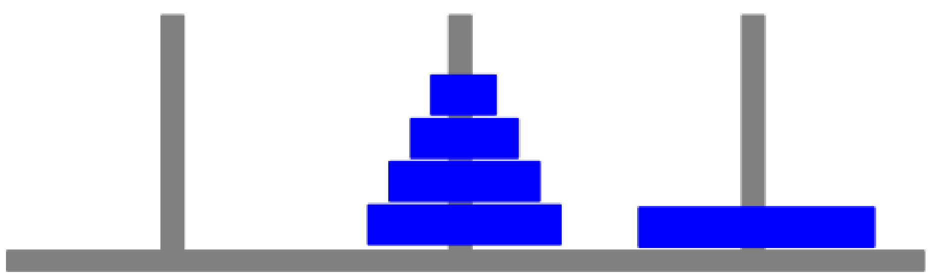

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3)

move top $n-1$ disks from $\overline{\{i,j\}}$ to j

Towers of Hanoi

```java
public class Hanoi
{

    public void move(int n, char a, char b, char c)
    {
        if (n == 1)
            System.out.println("plate " + n + " from " + a + " to " + c);
        else
        {
            move(n-1,a,c,b);
            System.out.println("plate " + n + " from " + a + " to " + c);
            move(n-1,b,a,c);
        }

    }

```

Towers of Hanoi

To move n disks from i to j

i) move top $n-1$ disks from i to $\overline{\{i,j\}}$

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Towers of Hanoi

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- $p(n-1) \to p(n)$ is *recursion* statement that

  if our algorithm works for $n-1$ disks, then we can build a correct solution for n disks

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- Running time

  $M(n)$ is number of disk moves needed for n disks

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$M(1) = 1$

if $n > 1$, then $M(n) = 2M(n-1) + 1$

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  Later, we'll also see how to solve without guessing

Towers of Hanoi

- Formally, given

$$M(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2M(n-1) + 1 & \text{otherwise} \end{cases}$$

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Then $M(n) = 2M(n-1) + 1 = 2(2^{n-1} - 1) + 1 = 2^n - 1$

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$$M(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2M(n-1)+1 & \text{otherwise} \end{cases}$$

- The second time was to derive the closed form solution $M(n) = 2^n - 1$ of the recurrence.

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Towers of Hanoi

Fibonacci Sequence

$$F(n) = \begin{cases} 1 & \text{if } n = 0, 1 \\ F(n-1)+F(n-2) & \text{otherwise} \end{cases}$$

- **Example 2**: Let $S(n)$ be the number of subsets of a set of size n. What is the formula for $S(n)$?

  The empty set, of size $n = 0$ has only one subset (itself), so $S(0) = 1$.

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  We "guess" that $S(n) = 2^n$. But, in order to prove formula, we'll need to think recursively.

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- Consider the eight subsets of $\{1, 2, 3\}$:

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  So, we get a subset of $\{1, 2, 3\}$ either by taking a subset of $\{1, 2\}$ or by adjoining 3 to a subset of $\{1, 2\}$.

  This suggests that the recurrence for the number of subsets of an n-element set $\{1, 2, \ldots, n\}$ is

  $$S(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2S(n-1) & \text{if } n \geq 1 \end{cases}$$

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  Each subset S containing n can be constructed in a unique fashion by adding n to the subset $S - \{n\}$ not containing n.

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Proof by induction is easy.

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  Can we generalize this to find a closed form solution?

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T(n) &= rT(n-1) + a \\
&= r(rT(n-2) + a) + a \\
&= r^2 T(n-2) + ra + a \\
&= r^2(rT(n-3) + a) + ra + a \\
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&= r^3(rT(n-4) + a) + r^2 a + ra + a \\
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  Guess $T(n) = r^n T(0) + a \sum_{i=0}^{n-1} r^i$

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  T(0) &= b \\
  T(1) &= rT(0) + a = rb + a \\
  T(2) &= rT(1) + a = r(rb + a) + a = r^2b + ra + a \\
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  Another approach is to iterate from the "bottom-up" instead of "top-down".

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  This would lead to the same guess

  $$T(n) = r^n b + a \sum_{i=0}^{n-1} r^i.$$

Formula of Recurrences

- **Theorem** If $T(n) = rT(n-1) + a$, $T(0) = b$, and $r \neq 1$, then

$$T(n) = r^n b + a\frac{1 - r^n}{1 - r}$$

for all nonnegative integers n.

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Proof by induction

The base case:

$$T(0) = r^0 b + a\frac{1 - r^0}{1 - r} = b.$$

So the formula is true when $n = 0$.

Now assume that $n > 0$ and

$$T(n-1) = r^{n-1} b + a\frac{1 - r^{n-1}}{1 - r}.$$

- **Proof by induction**

$$
\begin{aligned}
T(n) &= rT(n-1)+a \\
&= r\left(r^{n-1}b + a\frac{1-r^{n-1}}{1-r}\right)+a \\
&= r^n b + \frac{ar-ar^n}{1-r}+a \\
&= r^n b + \frac{ar-ar^n+a-ar}{1-r} \\
&= r^n b + a\frac{1-r^n}{1-r}.
\end{aligned}
$$

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Plugging $r = 3, a = 2, b = 5$ in the formula, gives

$$T(n) = 3^n \cdot 5 + 2\frac{1-3^n}{1-3} = 3^n \cdot 6 - 1$$

First-Order Linear Recurrences

- A recurrence of the form $T(n) = f(n)T(n-1) + g(n)$ is called a *first-order linear recurrence*.

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  ◇ Linear because $T(n-1)$ only appears to the first power.

  Something like $T(n) = (T(n-1))^2 + 3$ would be a non-linear first-order recurrence relation.

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  When $f(n)$ is a constant, say r, the general solution is almost as easy as we derived before. Iterating the recurrence gives

$$
\begin{aligned}
T(n) &= rT(n-1) + g(n) \\
&= r(rT(n-2) + g(n-1)) + g(n) \\
&= r^2 T(n-2) + rg(n-1) + g(n) \\
&= r^3 T(n-3) + r^2 g(n-2) + rg(n-1) + g(n) \\
&\vdots \\
&= r^n T(0) + \sum_{i=0}^{n-1} r^i g(n-i)
\end{aligned}
$$

First-Order Linear Recurrences

- **Theorem** For any positive constants a and r, and any function g defined on nonnegative integers, the solution to the first-order linear recurrence

$$T(n) = \begin{cases} rT(n-1) + g(n) & \text{if } n > 0 \\ a & \text{if } n = 0 \end{cases}$$

is

$$T(n) = r^n a + \sum_{i=1}^{n} r^{n-i} g(i).$$

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$$
\begin{aligned}
T(n) &= 6 \cdot 4^n + \sum_{i=1}^{n} 4^{n-i} \cdot 2^i \\
&= 6 \cdot 4^n + 4^n \sum_{i=1}^{n} 4^{-i} \cdot 2^i \\
&= 6 \cdot 4^n + 4^n \sum_{i=1}^{n} (\frac{1}{2})^i \\
&= 6 \cdot 4^n + (1 - \frac{1}{2^n}) \cdot 4^n \\
&= 7 \cdot 4^n - 2^n.
\end{aligned}
$$

Examples

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Theorem. For any real number $x \neq 1$,

$$\sum_{i=1}^{n} i x^i = \frac{n x^{n+2} - (n+1) x^{n+1} + x}{(1-x)^2}.$$

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\begin{aligned}
T(n) &= 10 \cdot 3^n + \sum_{i=1}^{n} 3^{n-i} \cdot i \\
&= 10 \cdot 3^n + 3^n \sum_{i=1}^{n} i \cdot 3^{-i} \\
&= 10 \cdot 3^n + 3^n \left(-\frac{3}{2}(n+1)3^{-(n+1)} - \frac{3}{4}3^{-(n+1)} + \frac{3}{4} \right) \\
&= \frac{43}{4}3^n - \frac{n+1}{2} - \frac{1}{4}.
\end{aligned}
$$

Next Lecture

- recurrence, more counting ...