南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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  Transitive Relation: A relation R on a set A is called *reflexive* if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.

Closures

- **Definition** Let R be a relation on a set A. A relation S on A with property P is called *the closure of R with respect to P* if S is subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

  S is the **minimal set** containing R satisfying the property P.

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- **Definition** Let R be a relation on a set A. The *connectivity relation* R^* consists of all pairs (a, b) s.t. there is a path (of any length) between a and b in R.

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1. If $(a, b) \in R^*$ and $(b, c) \in R^*$, then there are paths from a to b and from b to c in R. Thus, there is a path from a to c in R. This means that $(a, c) \in R^*$.

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We have $S^* \subseteq S$. Thus, $R^* \subseteq S^* \subseteq S$

Simple Transitive Closure Algorithm

- **procedure** transClosure ($\mathbf{M}_R$: zero-one $n \times n$ matrix)
  // computes R^* with zero-one matrices
  $A := B := \mathbf{M}_R$;
  for $i := 2$ to n
  $\quad A := A \odot \mathbf{M}_R$
  $\quad B := B \vee A$
  return B
  // B is the zero-one matrix for R^*

  This algorithm takes $\Theta(n^4)$ time.

Roy-Warshall Algorithm

procedure Warshall ($\mathbf{M}_R$: zero-one $n \times n$ matrix)
// computes R^* with zero-one matrices
$W := \mathbf{M}_R$;
for $k := 1$ to n
  for $i := 1$ to n
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$w_{ij} = 1$ means there is a path from i to j going only through nodes $\leq k$.

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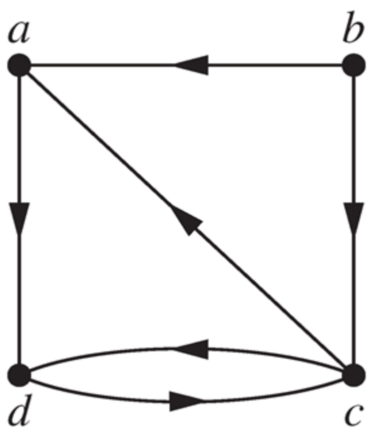
Example

Find the matrices W_0, W_1, W_2, W_3, and W_4. The matrix W_4 is the transitive closure of R.



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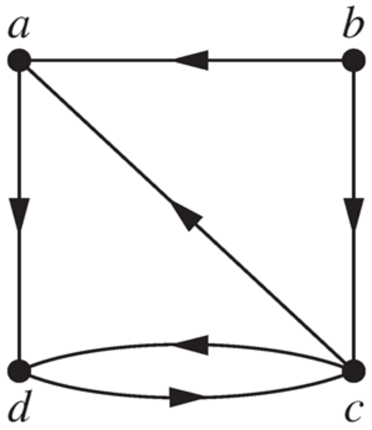

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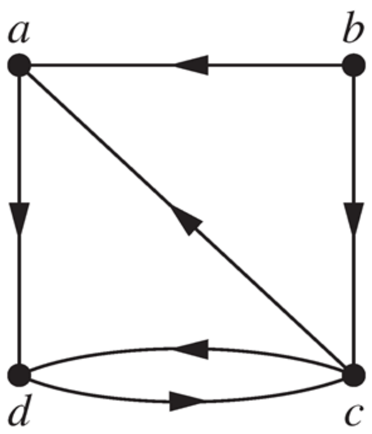

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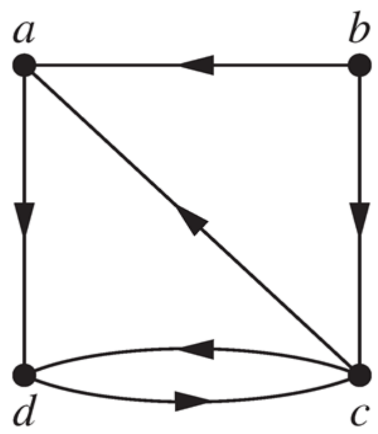

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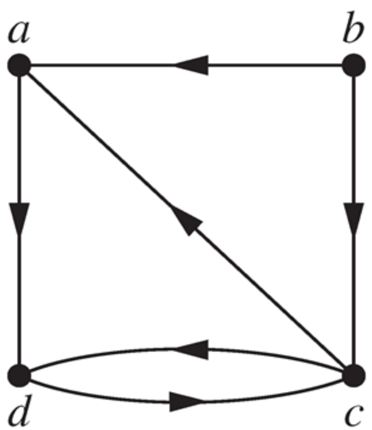

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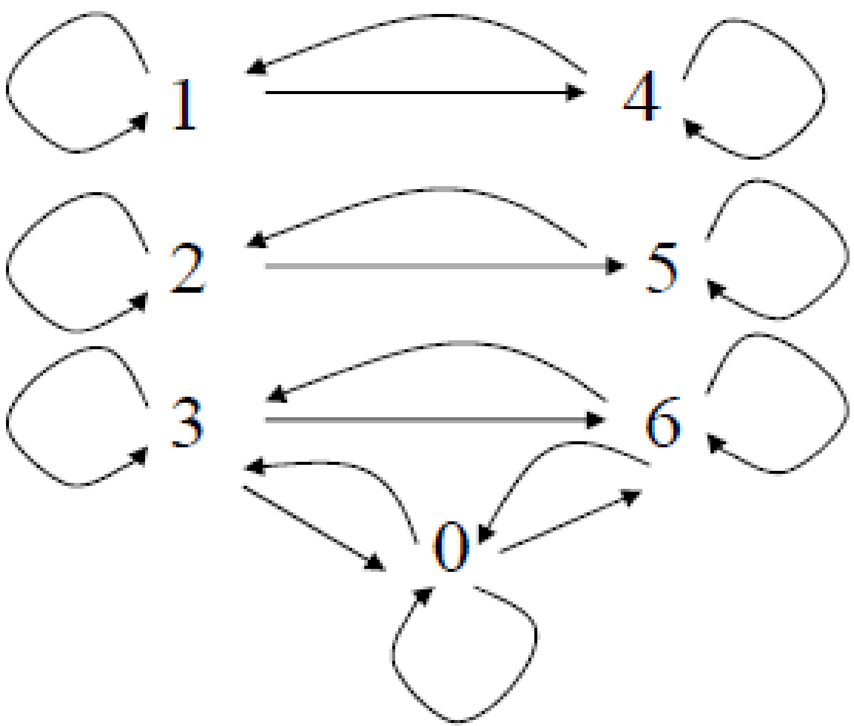

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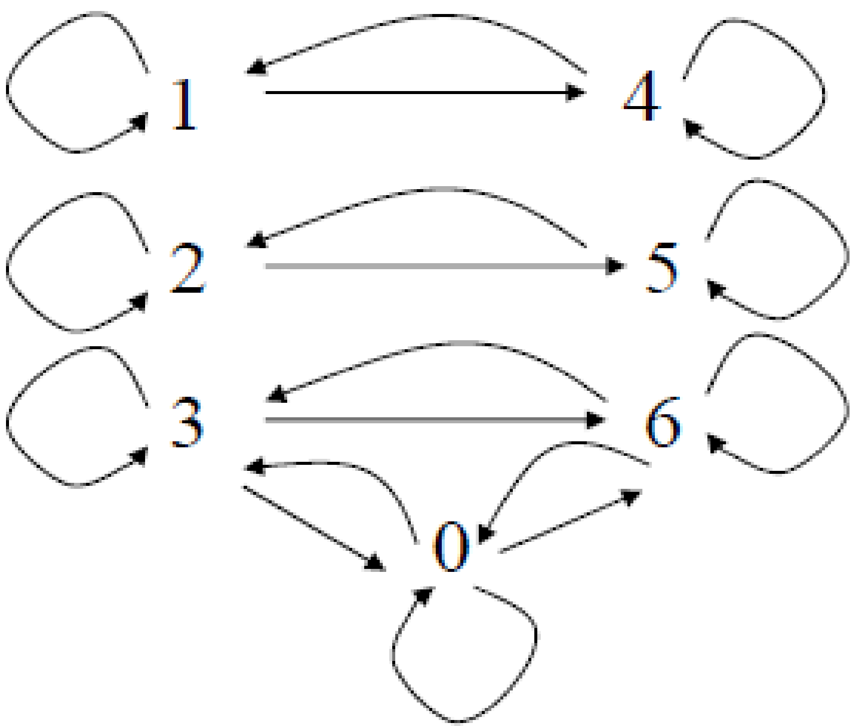

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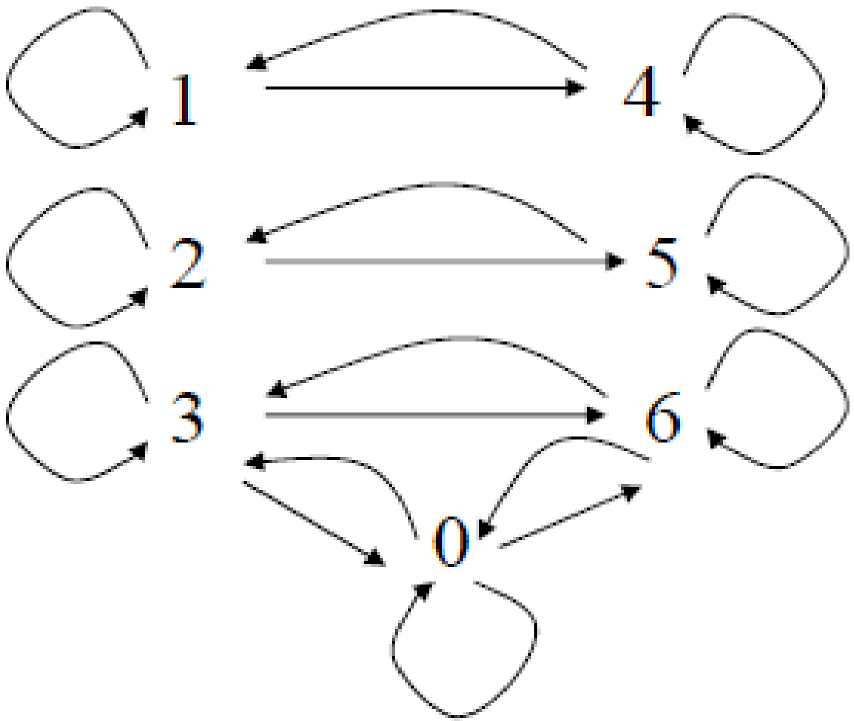

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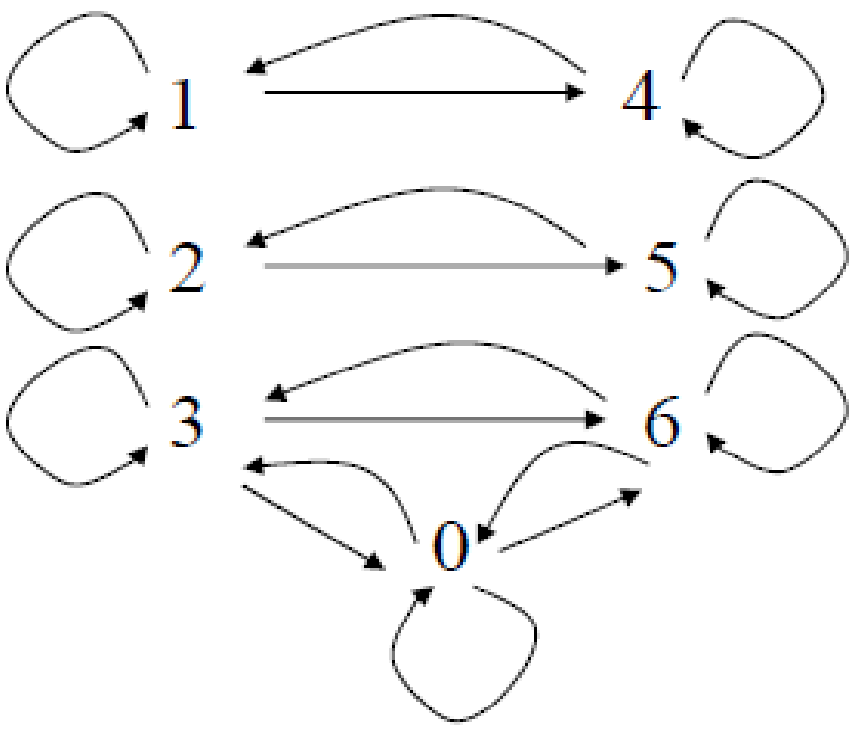

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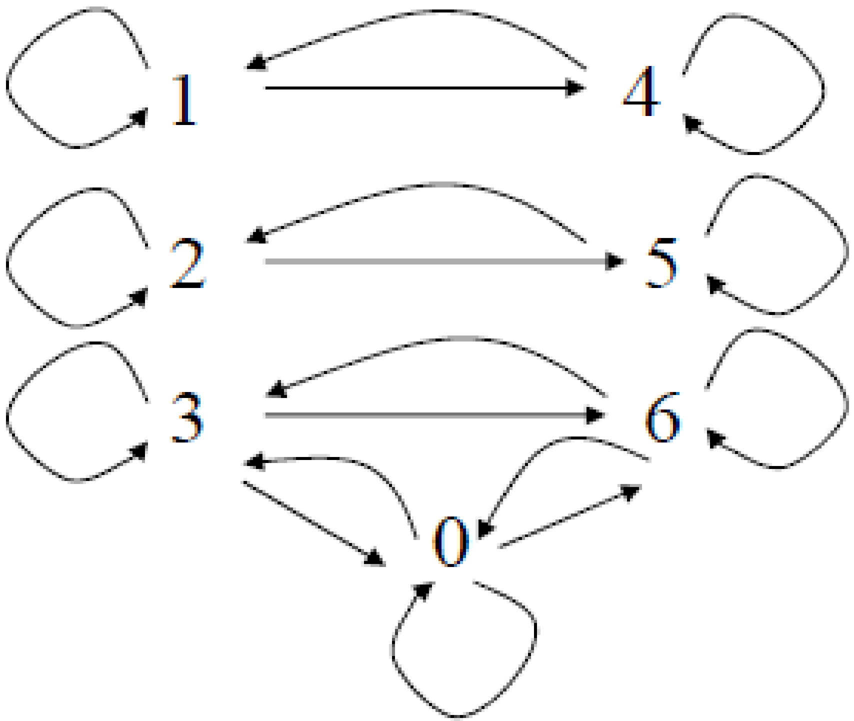

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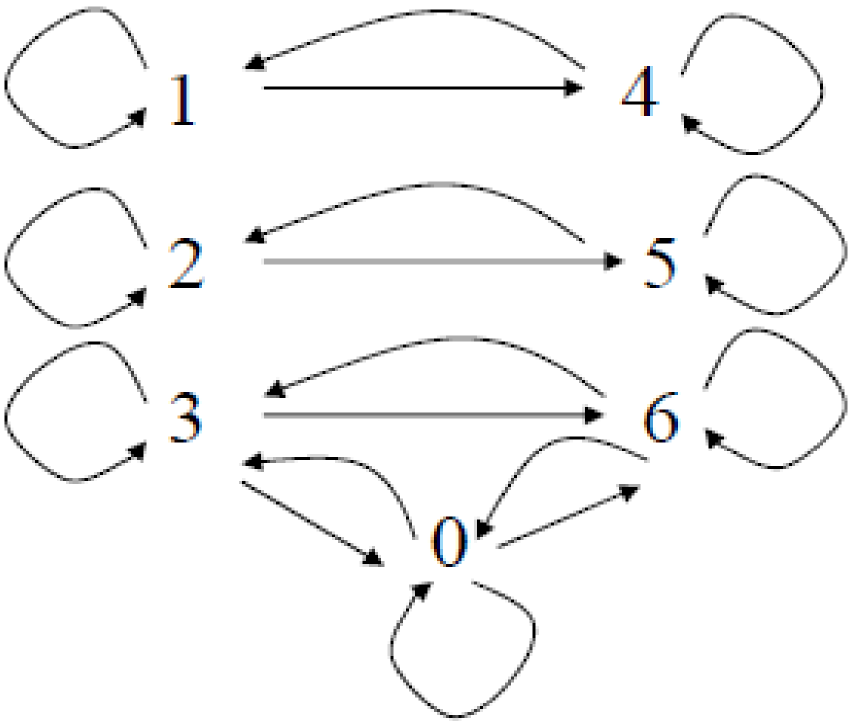

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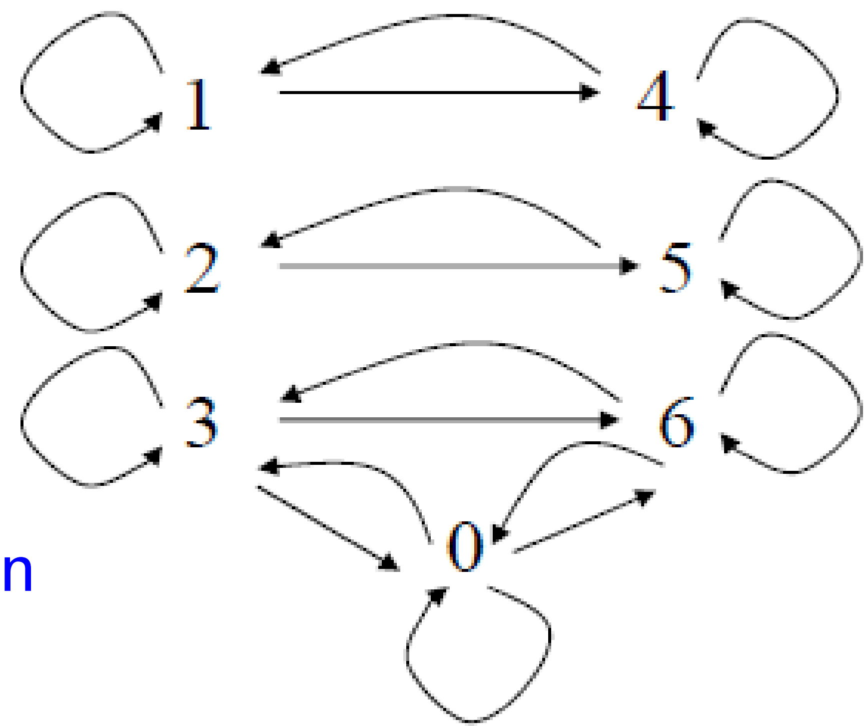

R is an equivalence relation

Examples of Equivalence Relations

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  "Strings a and b are the same length."

  "Integers a and b have the same absolute value."

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  "The relation $\geq$ between real numbers."

  "has a common factor greater than 1 between natural numbers."

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  $[a] =$ the set $\{\ldots, a-2, a-1, a, a+1, a+2, \ldots\}$

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  $(iii) \rightarrow (i)$: there exists a c s.t. $c \in [a]$ and $c \in [b]$

Partition of a Set S

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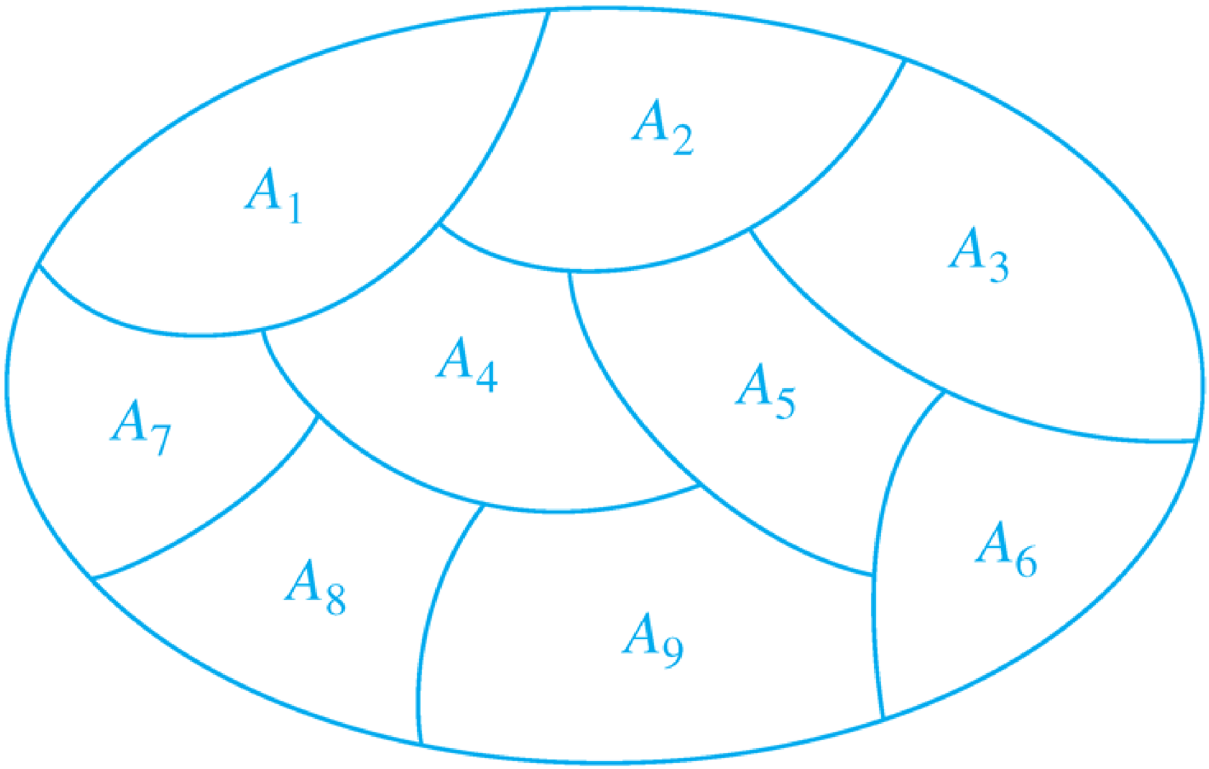

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- **Definition** Let S be a set. A collection of nonempty subsets of S $A_1, A_2, \ldots, A_k$ is called *a partition of S* if:

$$A_i \cap A_j = \emptyset, \; i \neq j \text{ and } S = \bigcup_{i=1}^{k} A_i$$

Example:

$$A = \{0, 1, 2, 3, 4, 5, 6\}$$

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Is A_1, A_2, A_3 a partition of S?

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Theorem Let $\{A_1, A_2, \ldots, A_i, \ldots\}$ be a partition of S. Then there is an equivalence relation R on S, that has the sets A_i as its equivalence classes.

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  $2, 4$ are comparable, $3, 5$ are incomparable.

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- **Definition** Given two posets $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$, the *lexicographic ordering* on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1, b_2), i.e., $(a_1, a_2) \prec (b_1, b_2)$, either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ then $a_2 \prec_2 b_2$.

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  - ◇ *discreet* $\prec$ *discrete*
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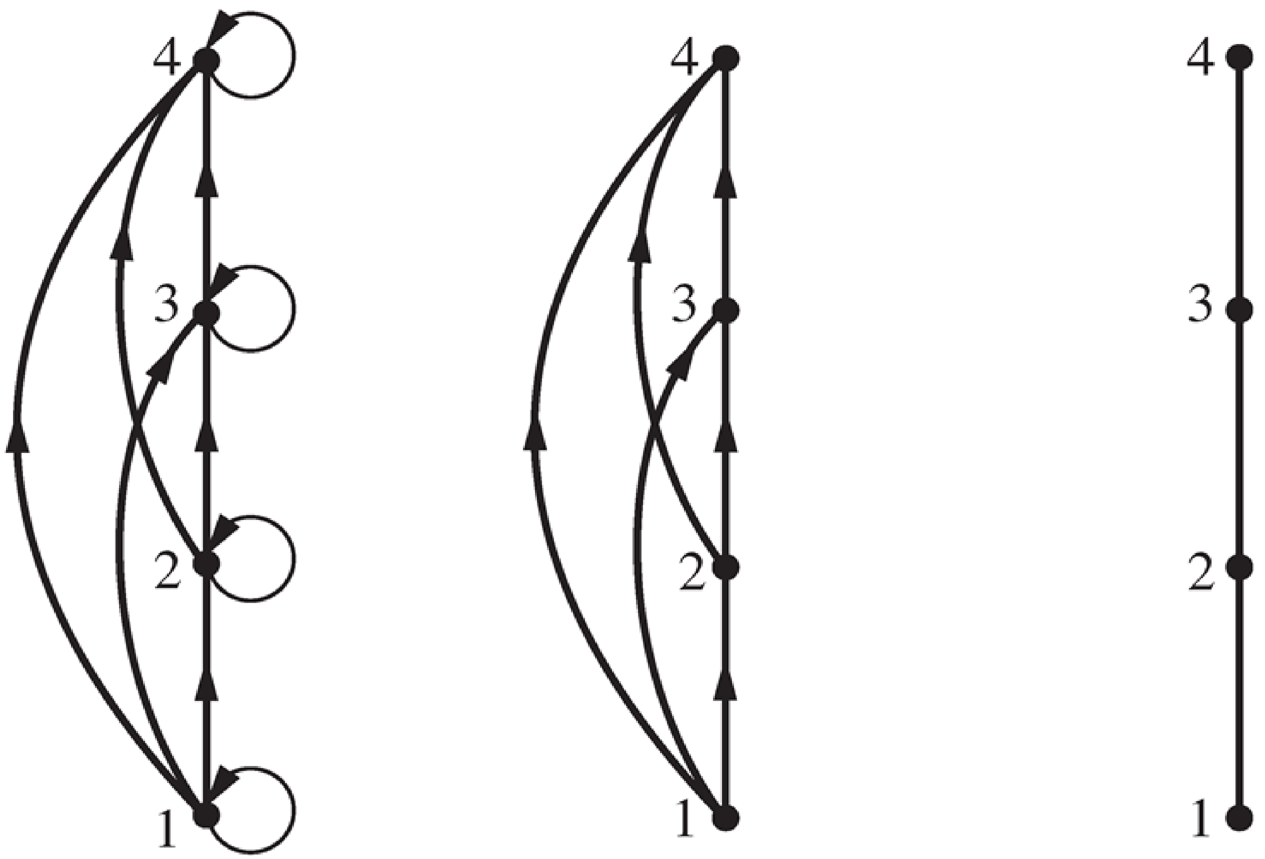
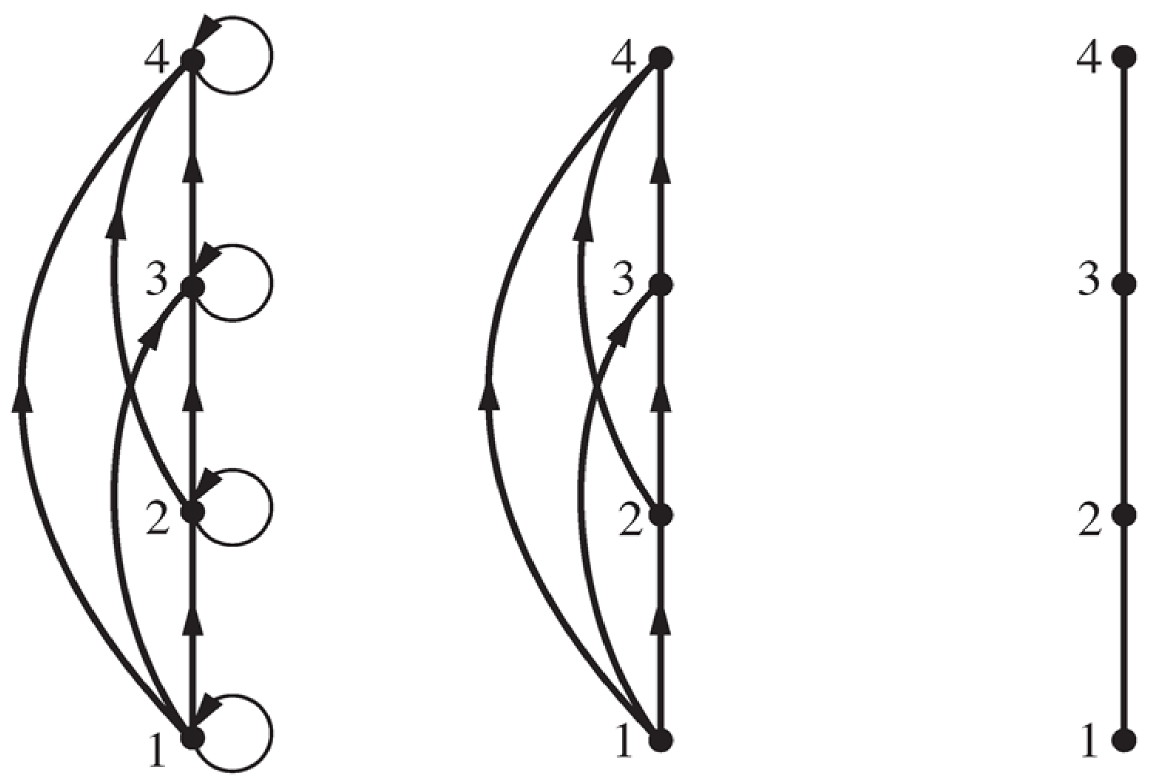


Hasse Diagram

- (a) A partial ordering. The loops are due to the reflexive property

  (b) The edges that must be present due to the transitive property are deleted

  (c) The Hasse diagram for the partial ordering (a)



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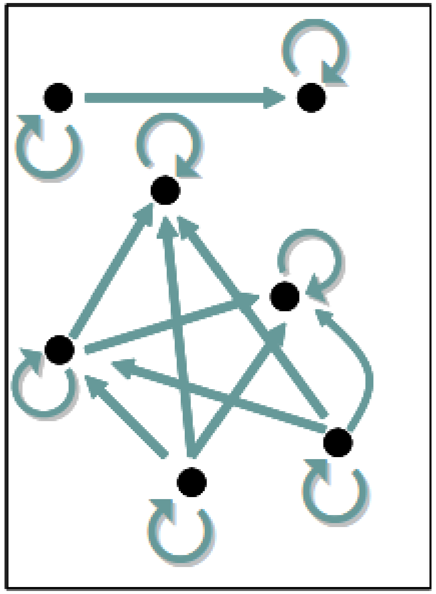
  - ◇ Arrange each edge so that its initial vertex is below the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex.

Hasse Diagram Example



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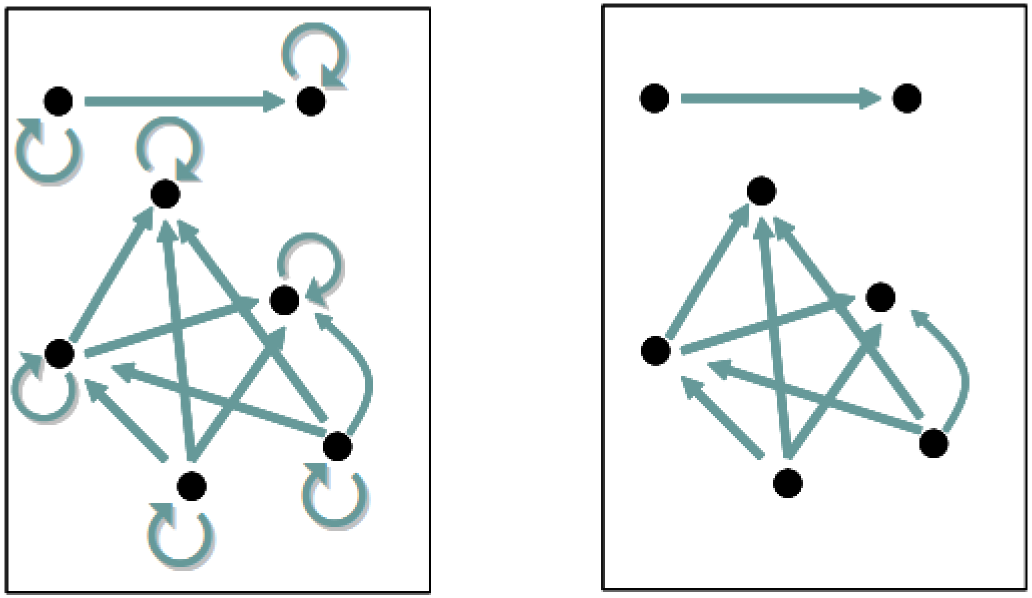

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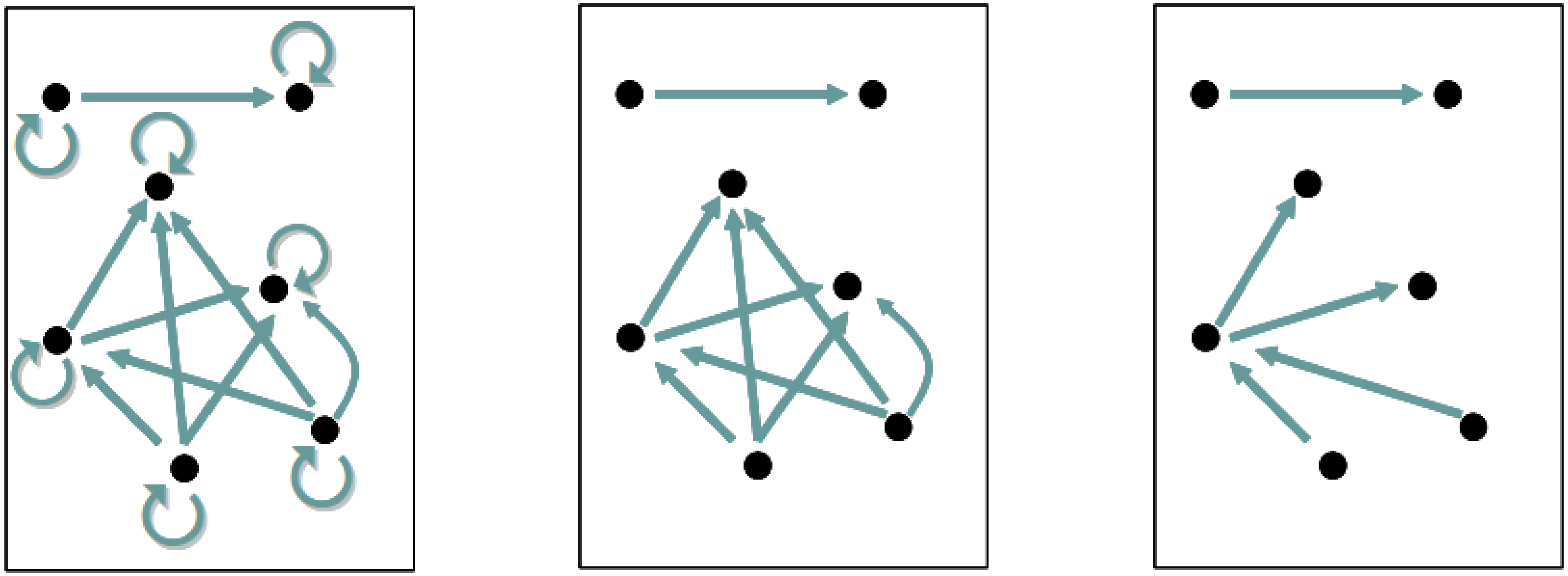

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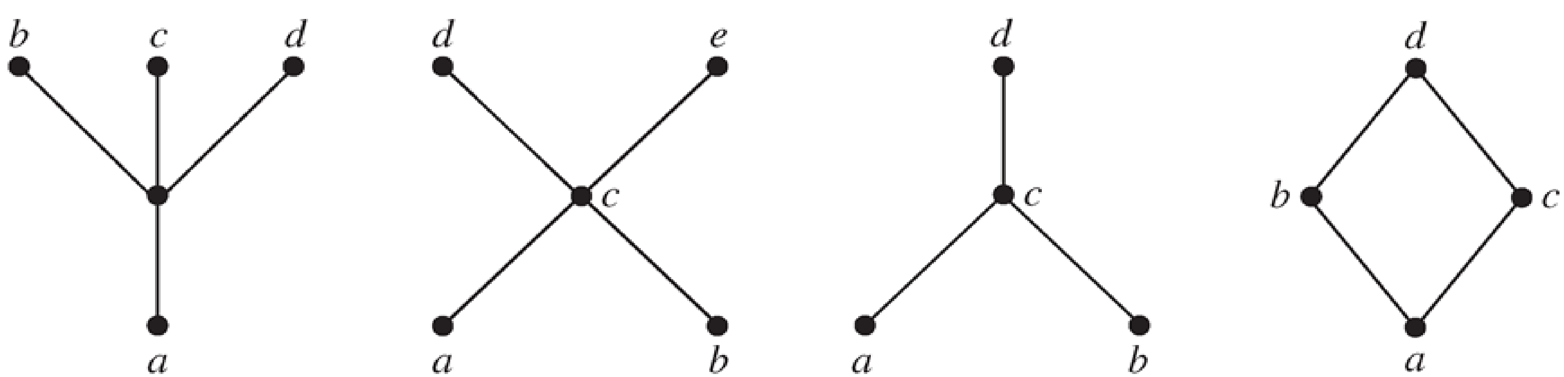

(a) (b) (c) (d)

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Example Find the greatest lower bound and the least upper bound of the sets $\{3, 9, 12\}$ and $\{1, 2, 4, 5, 10\}$, if they exist, in the poset $(\mathbf{Z}^+, |)$.

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Question: Why don't we need a *basic step* here?

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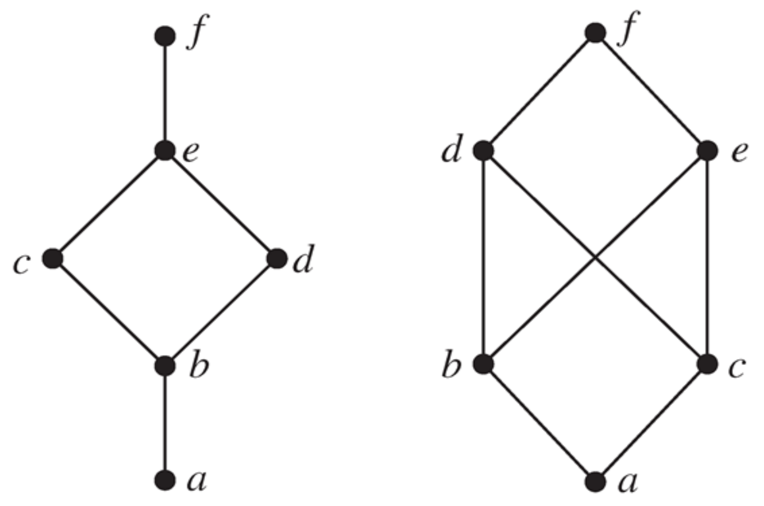

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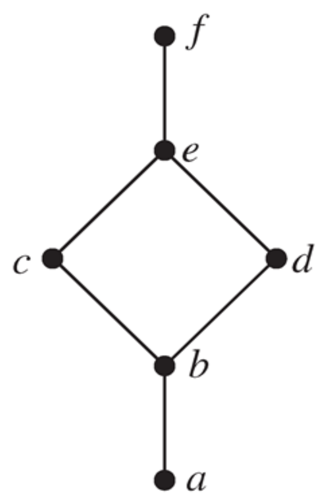

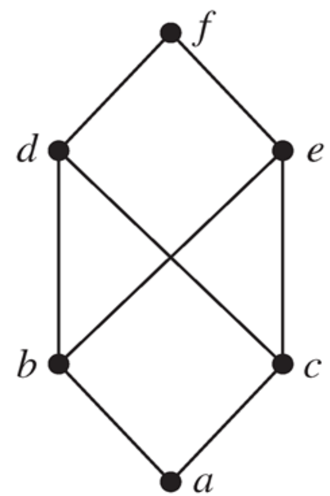

Example Determine whether the posets $(\{1, 2, 3, 4, 5\}, |)$ and $(\{1, 2, 4, 8, 16\}, |)$ are lattices.

Topological Sorting

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Topological sorting: Given a partial ordering R, find a total ordering $\preccurlyeq$ such that $a \preccurlyeq b$ whenever $a\ R\ b$. $\preccurlyeq$ is said *compatible with R*.

Topological Sorting for Finite Posets

procedure topological_sort (S: finite poset)

$k := 1$;
while $S \neq \emptyset$
  $a_k :=$ a minimal element of S
  $S := S \setminus \{a_k\}$
  $k := k + 1$
end while
// $\{a_1, a_2, \ldots, a_n\}$ is a compatible total ordering of S

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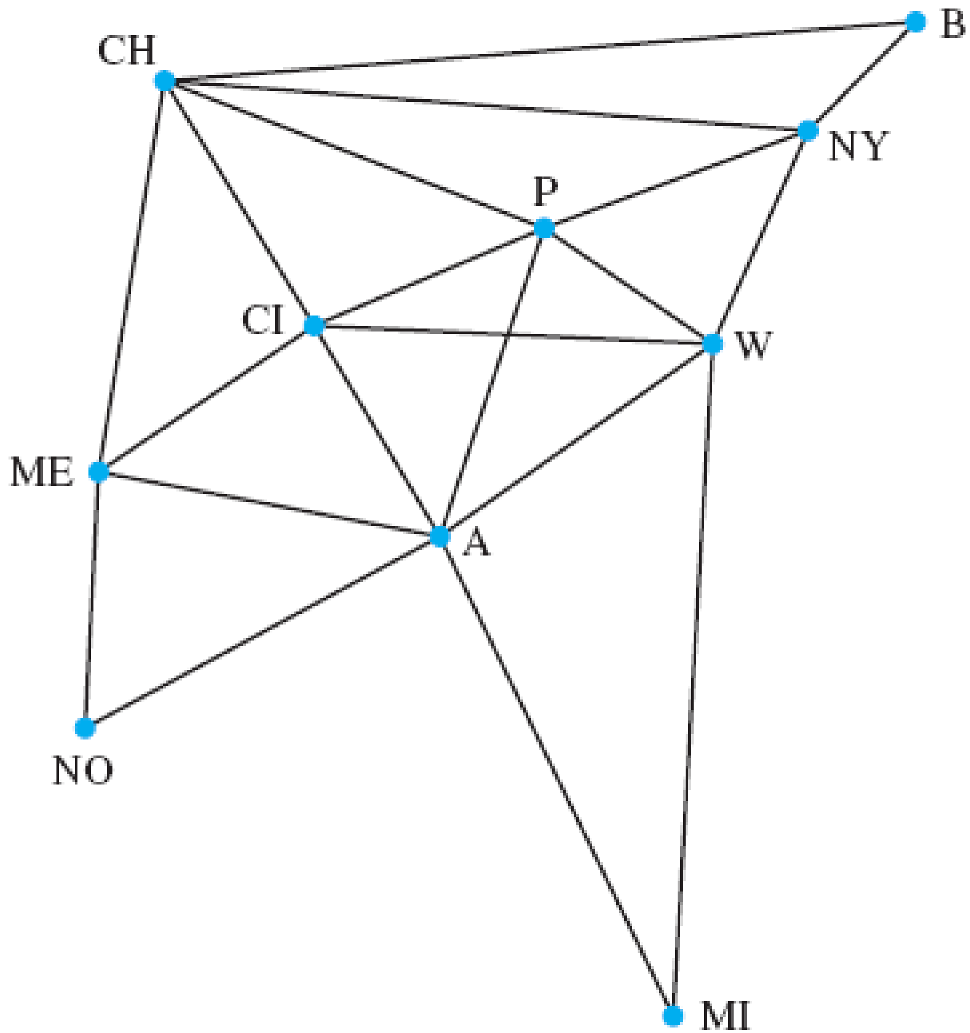

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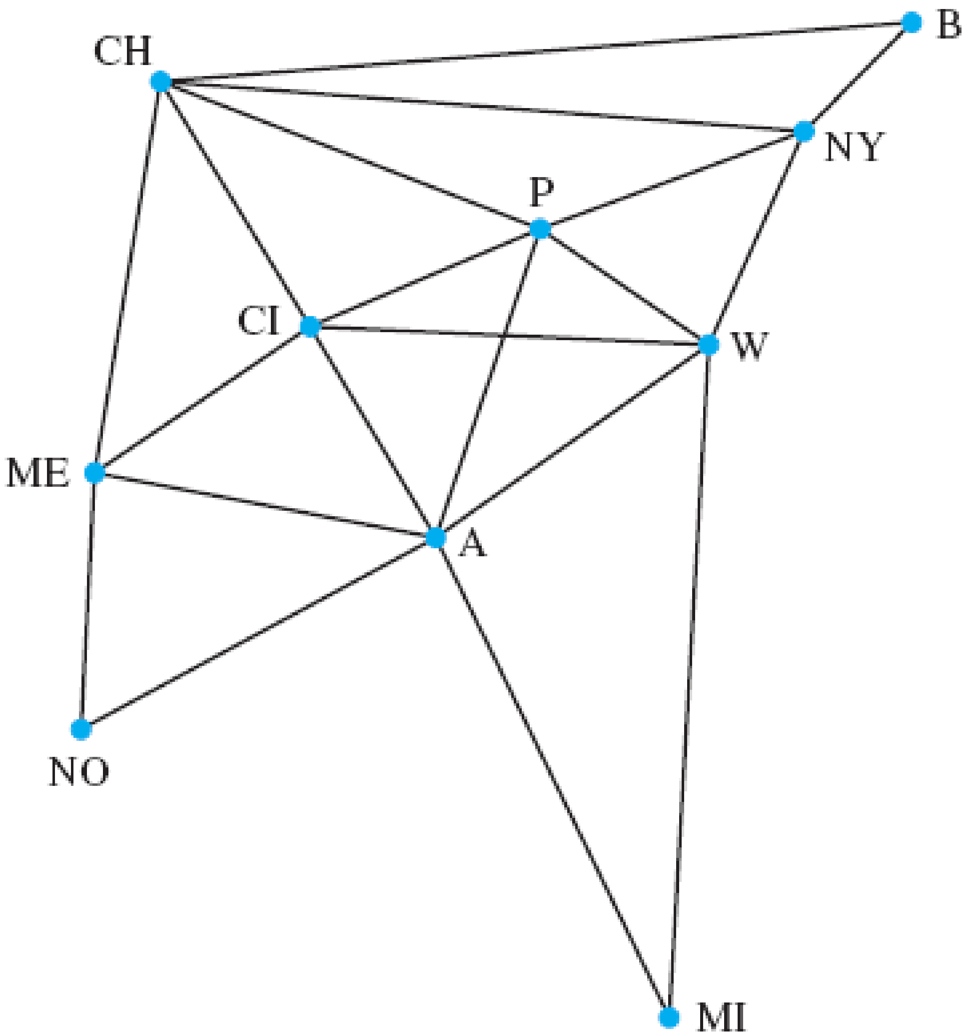

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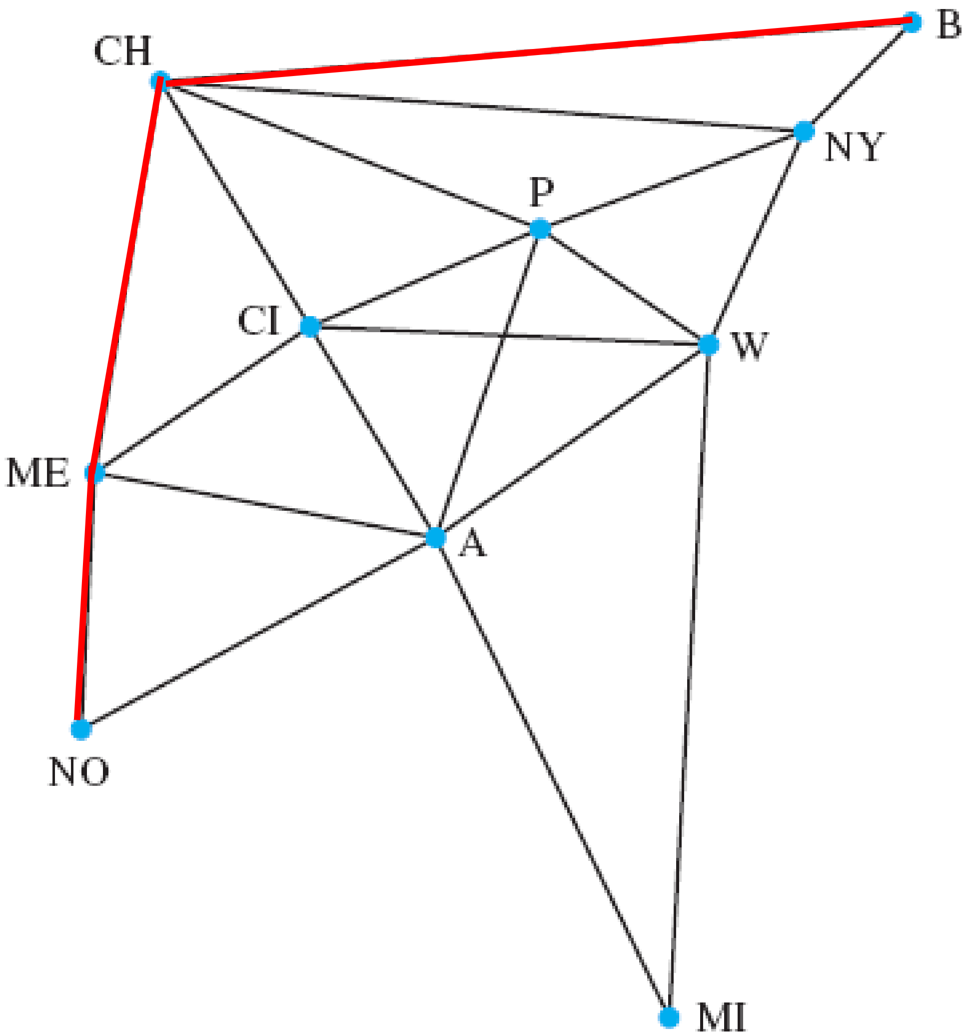

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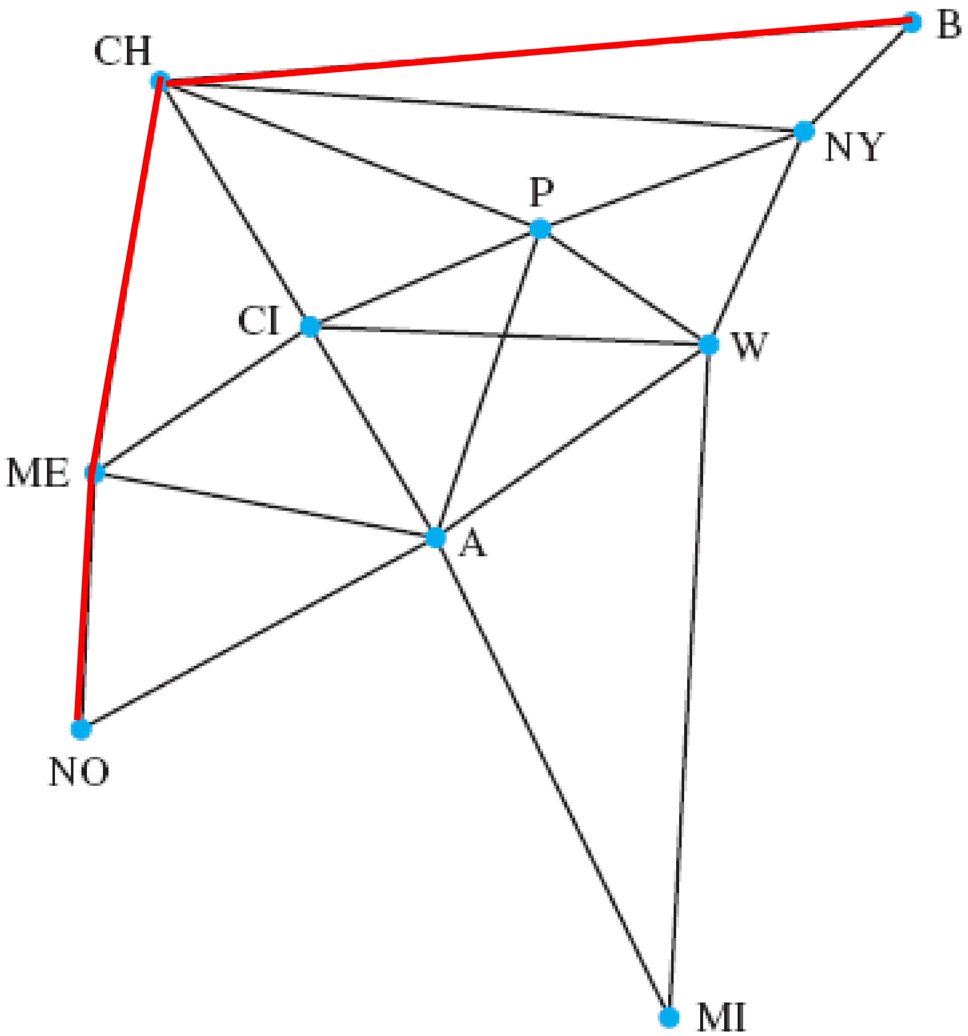

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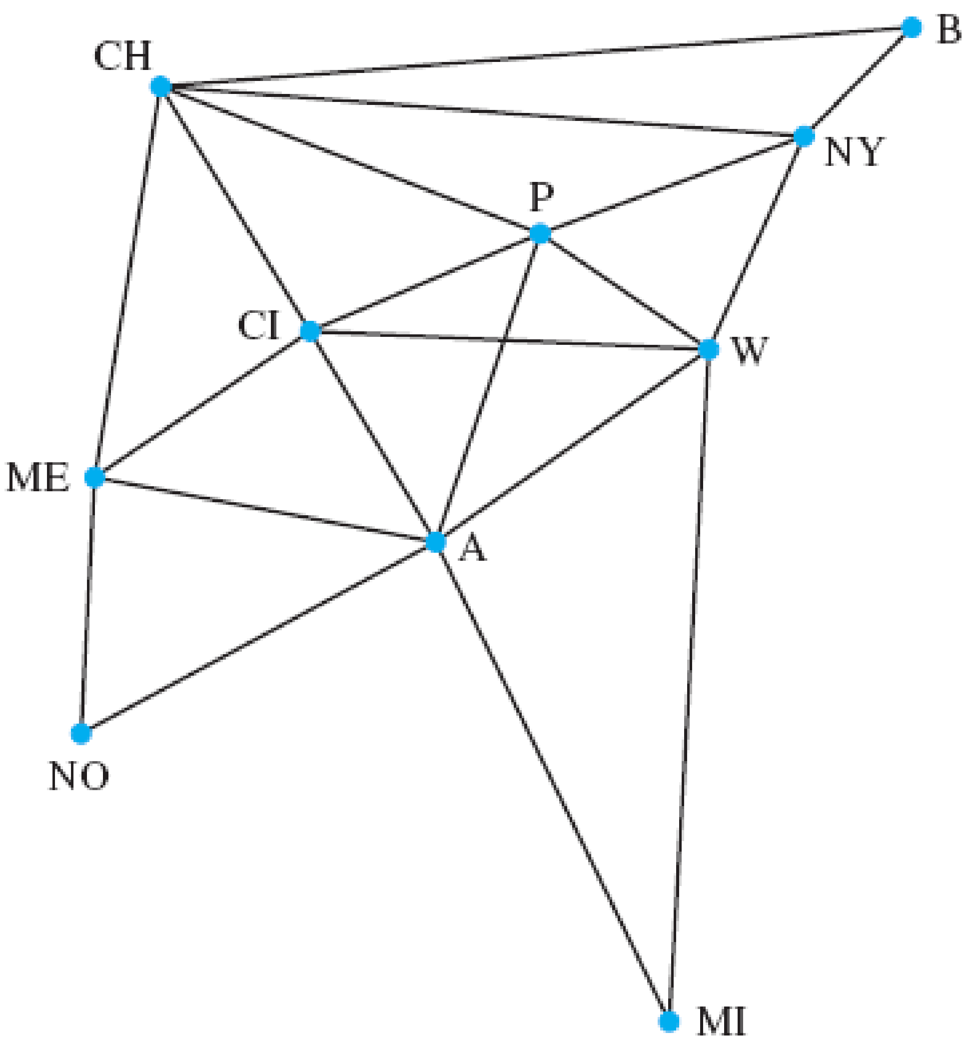

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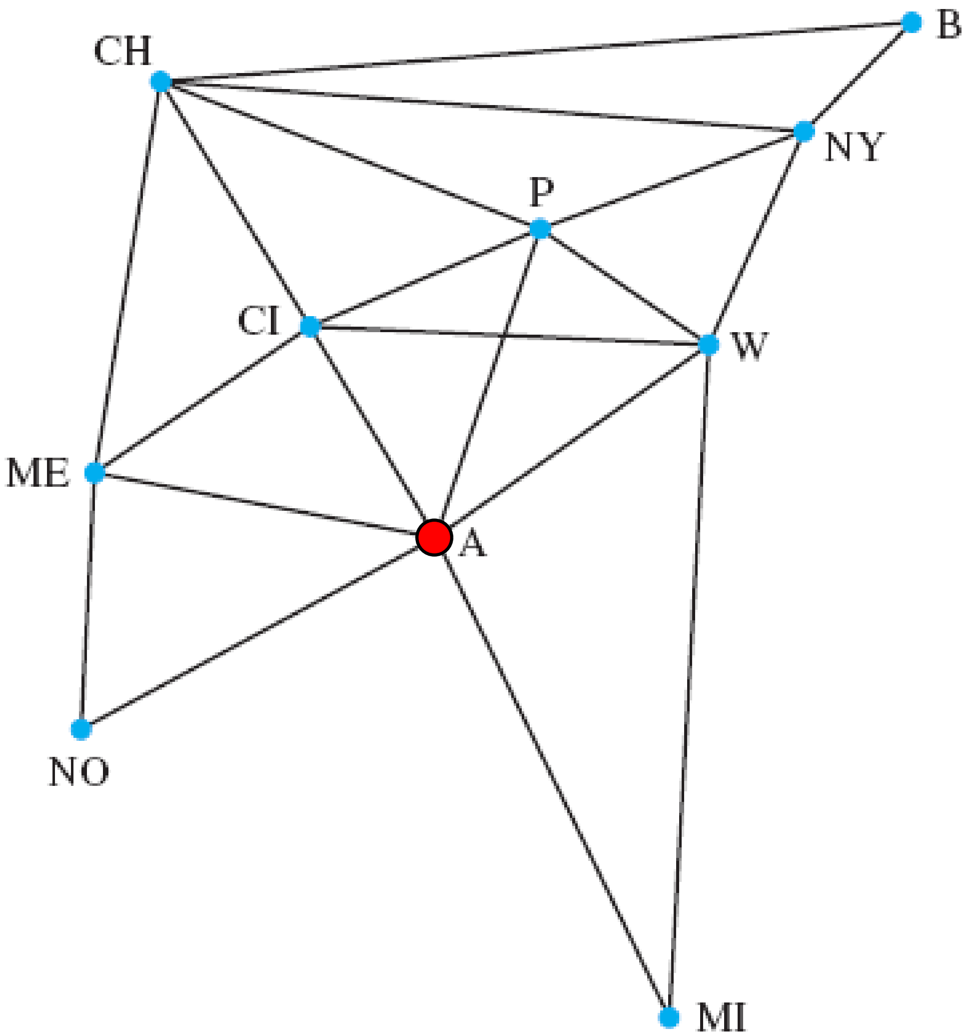

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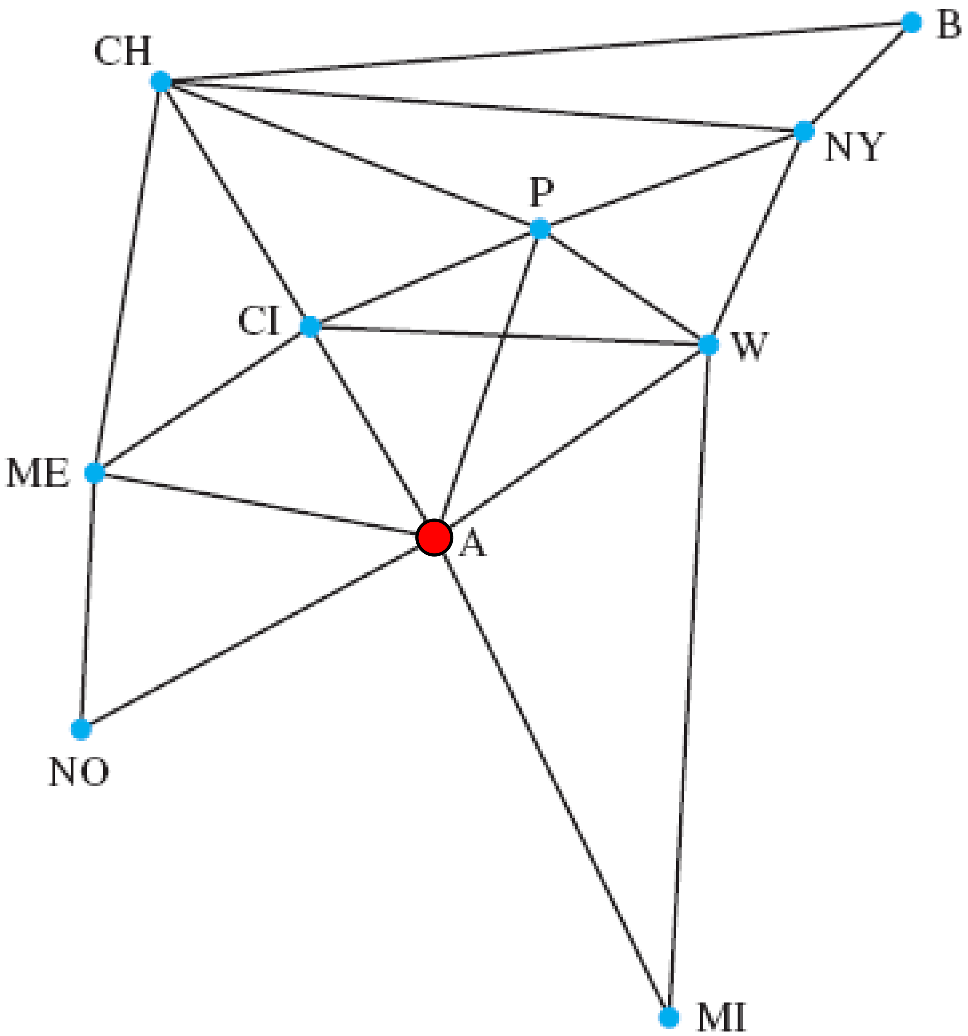

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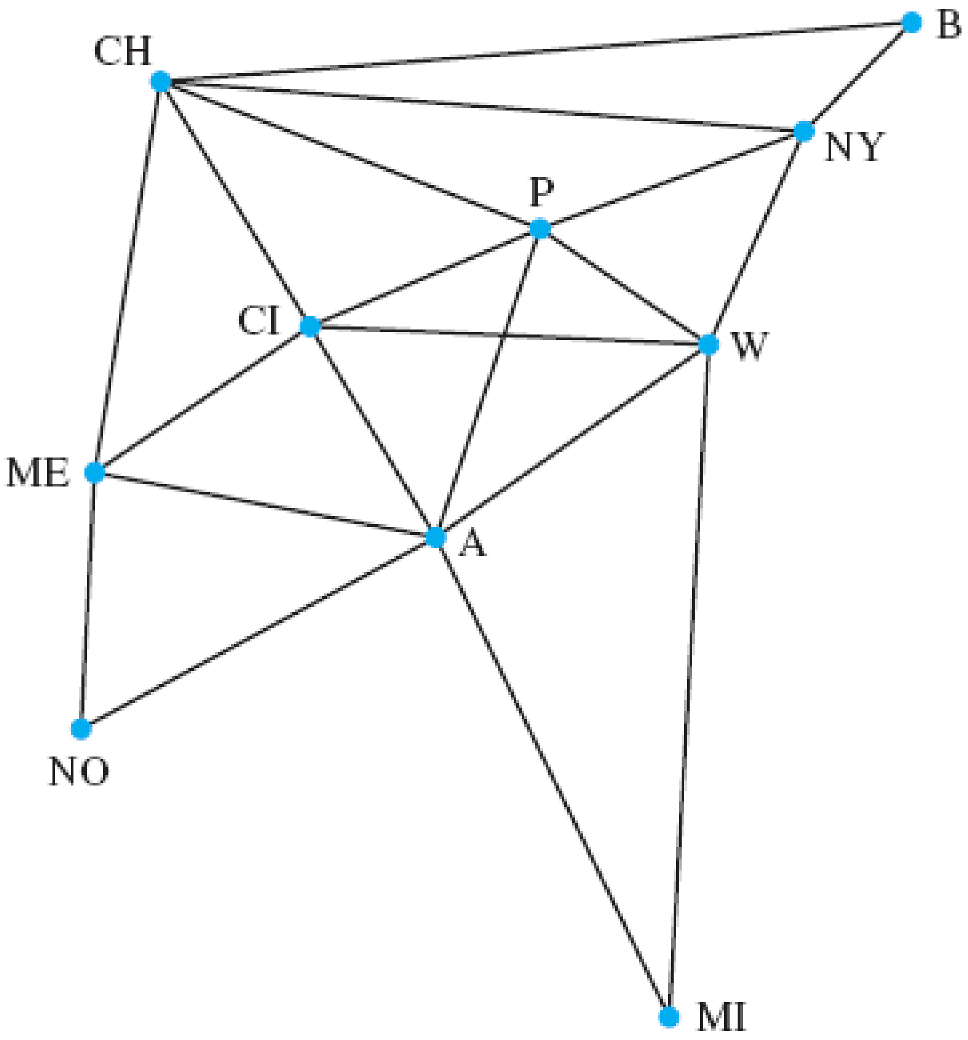

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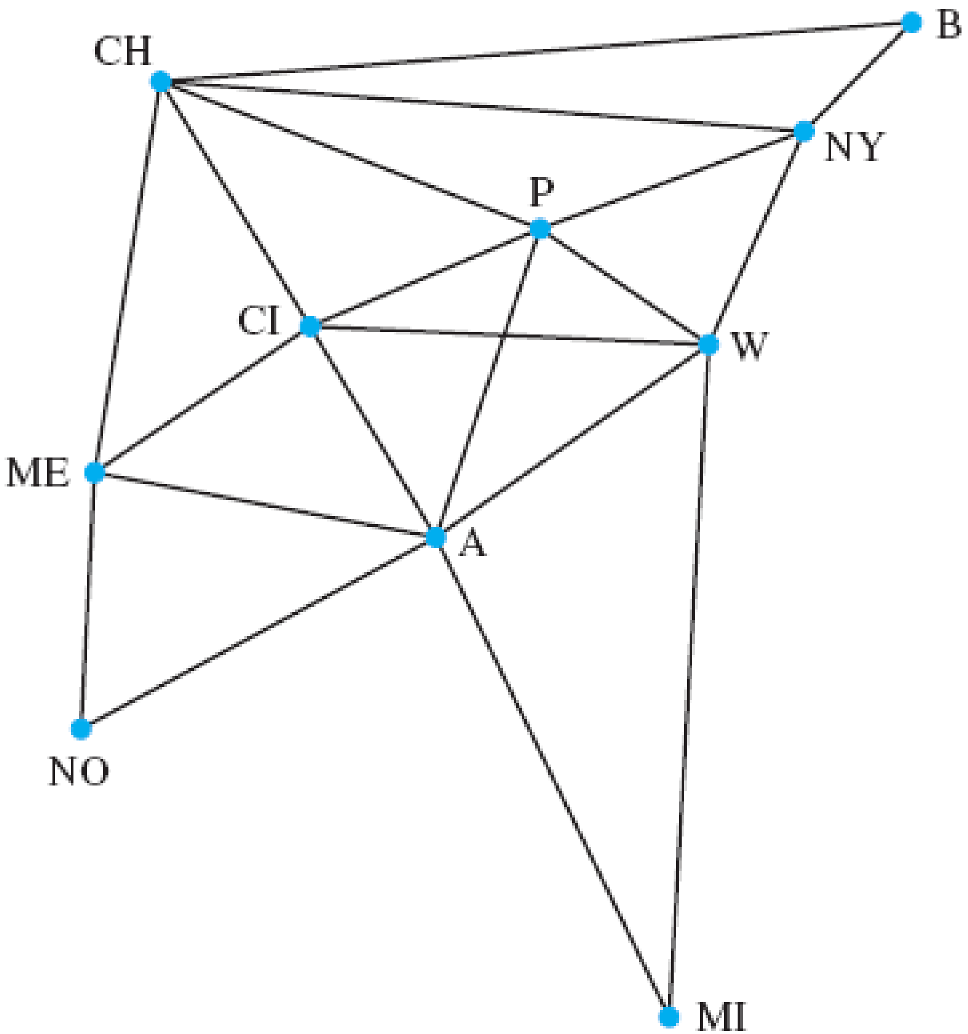

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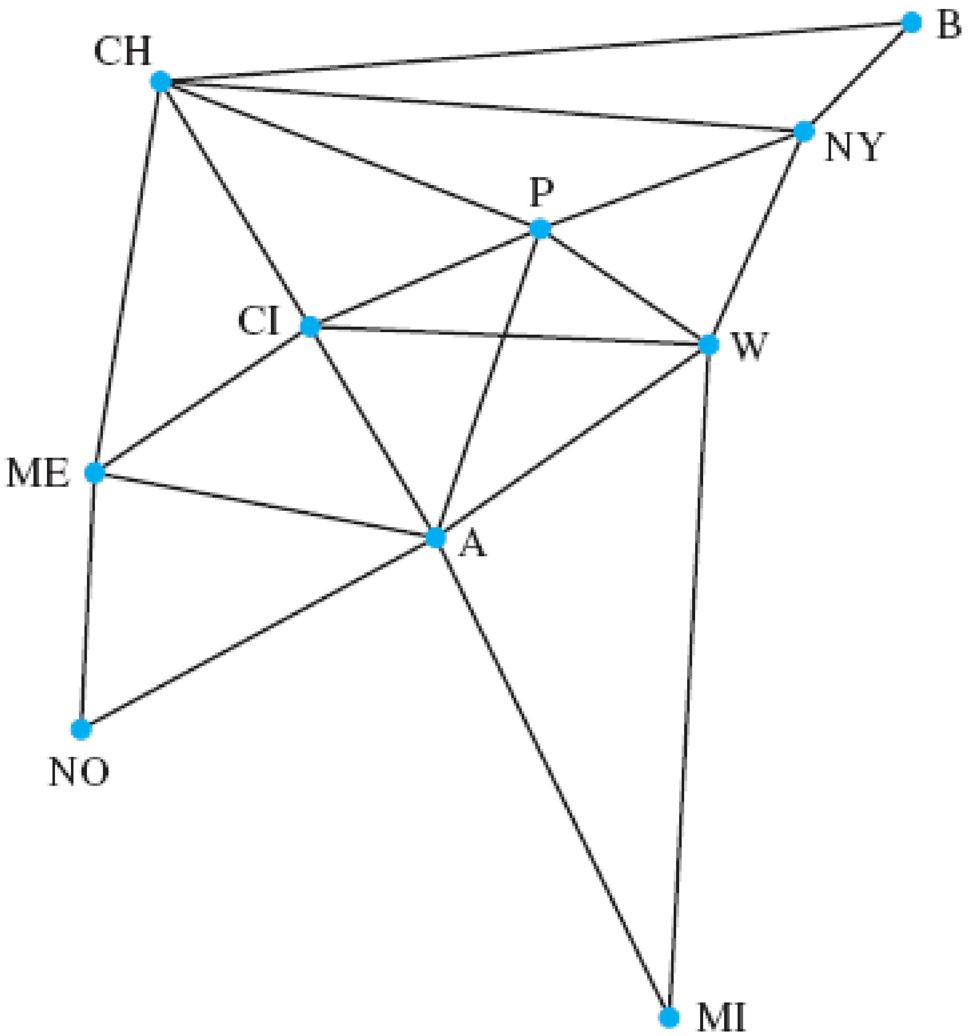

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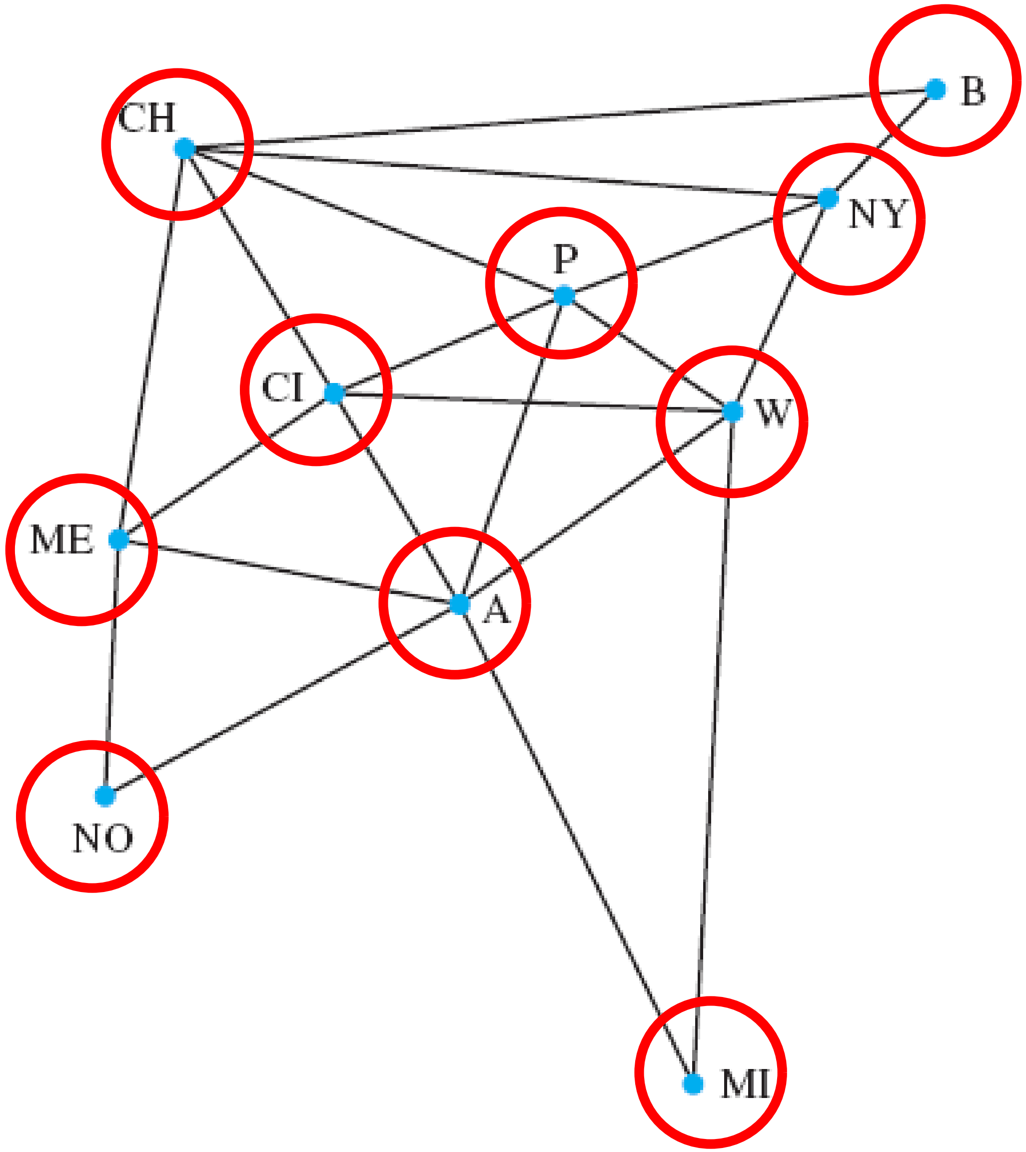

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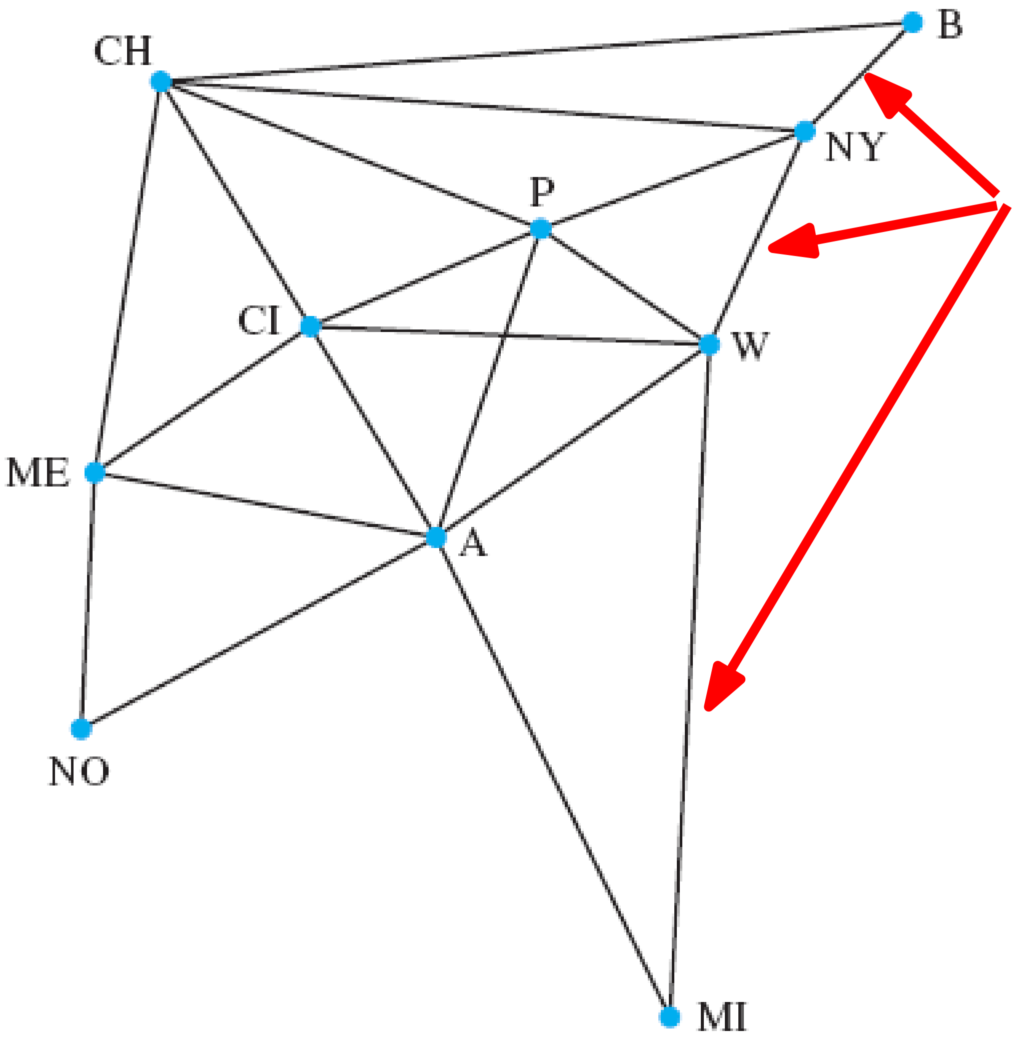

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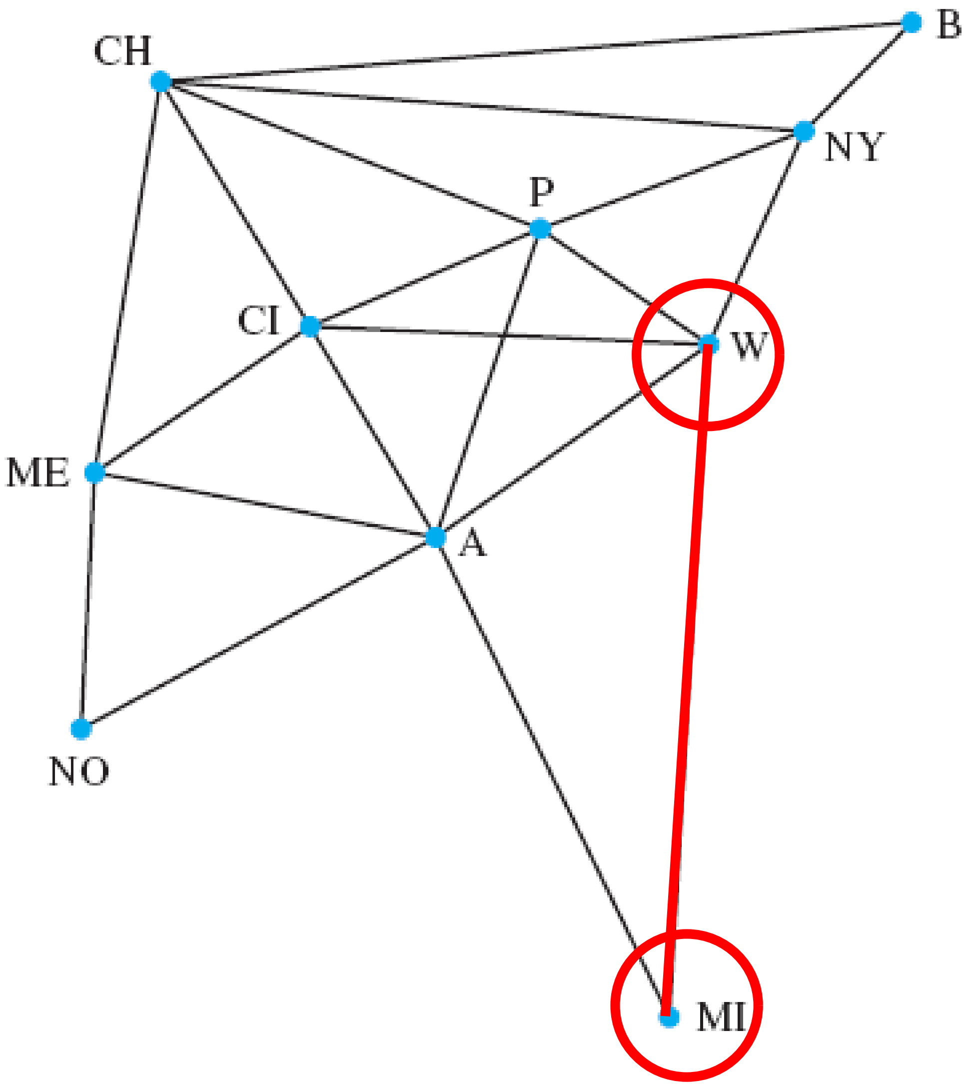
Graph G



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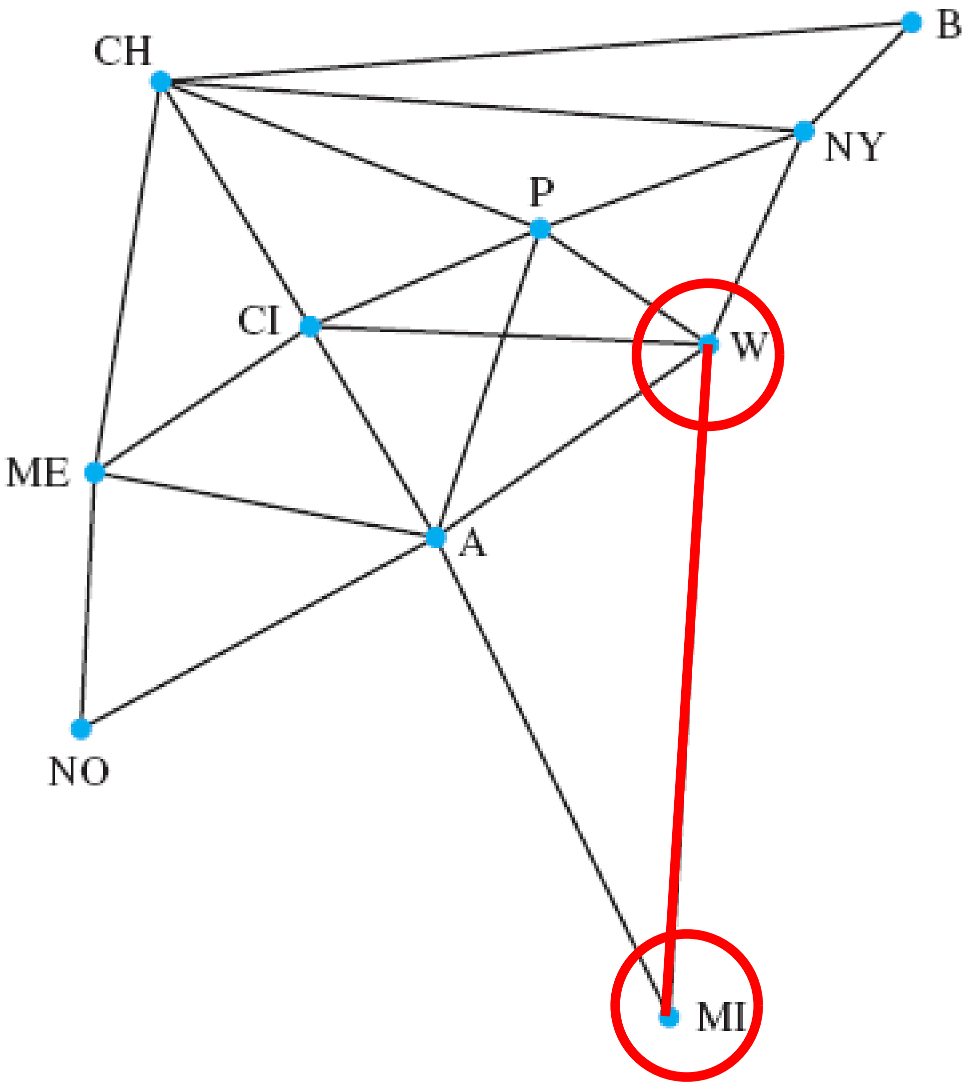

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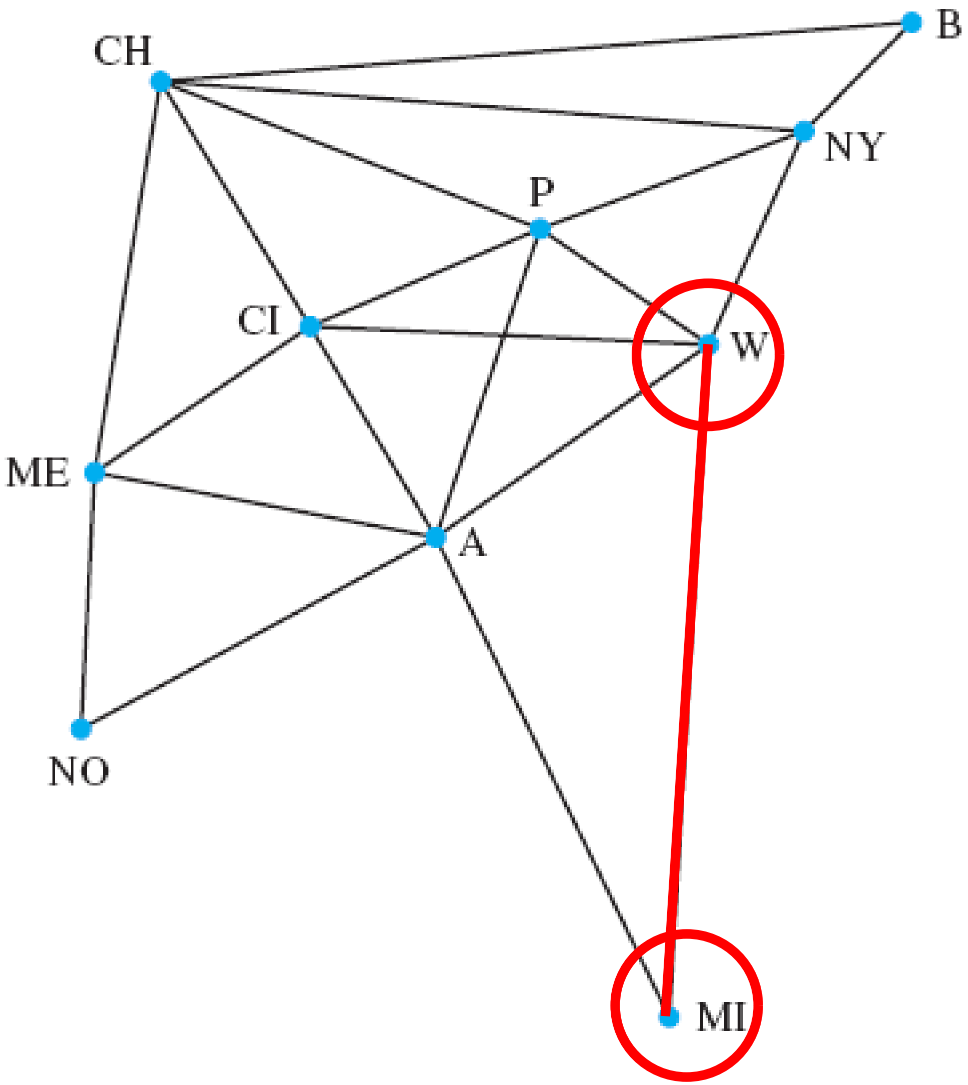

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When a vertex is an endpoint of an edge, we say that the edge and the vertex are incident to each other

More Examples

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  Edges: species have a common ancestor

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How Google models the Internet!

Definition of a Graph

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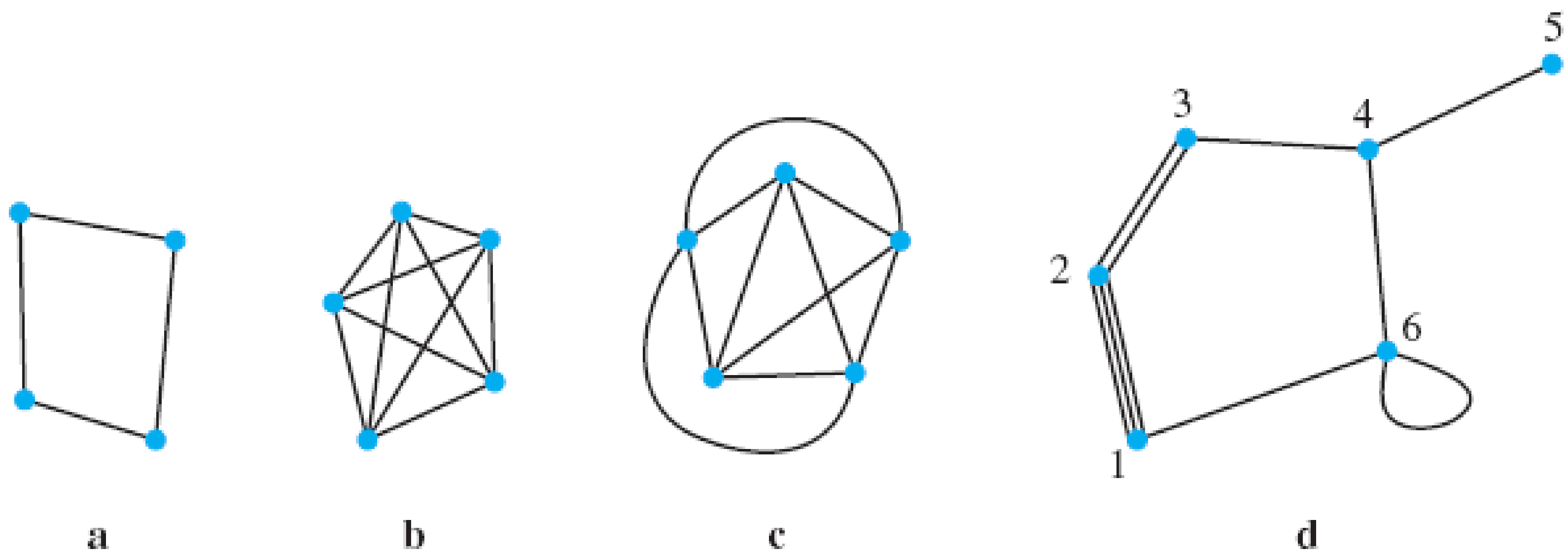

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  Complete graph K_n

  A graph with n vertices that has an edge between each pair of vertices

Graphs

- Graphs and graph theory can be used to model:
  - Computer networks
  - Social networks
  - Communication networks
  - Infromation networks
  - Software design
  - Transportation networks
  - Biological networks

Graph Models

- Computer Networks

  Vertices: computers
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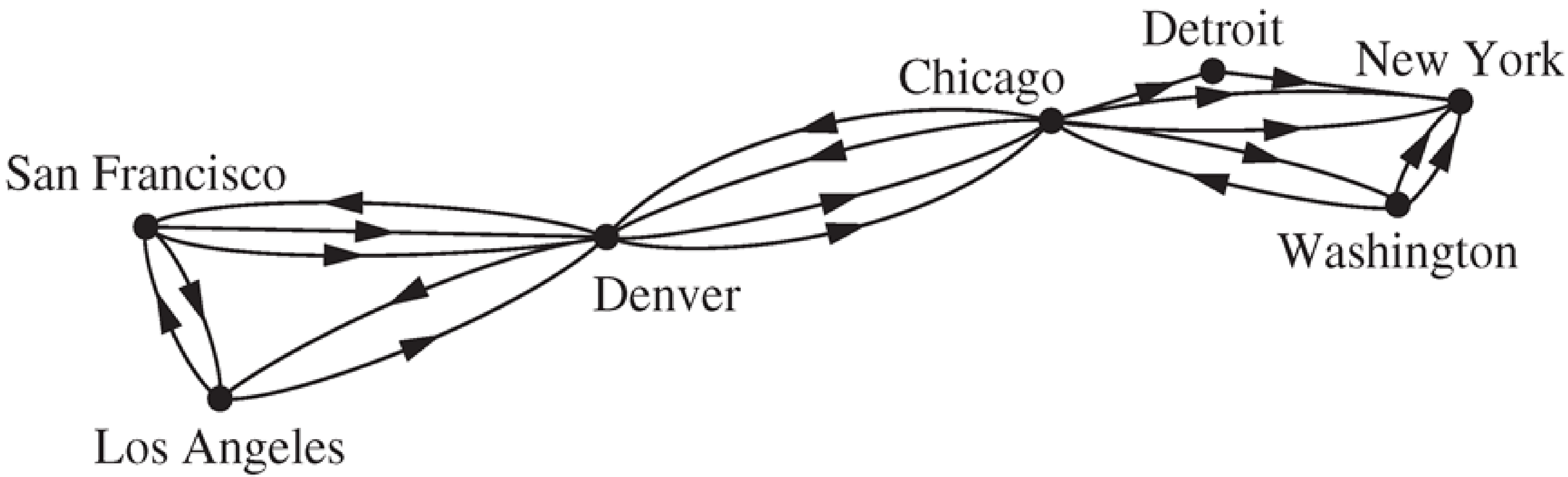

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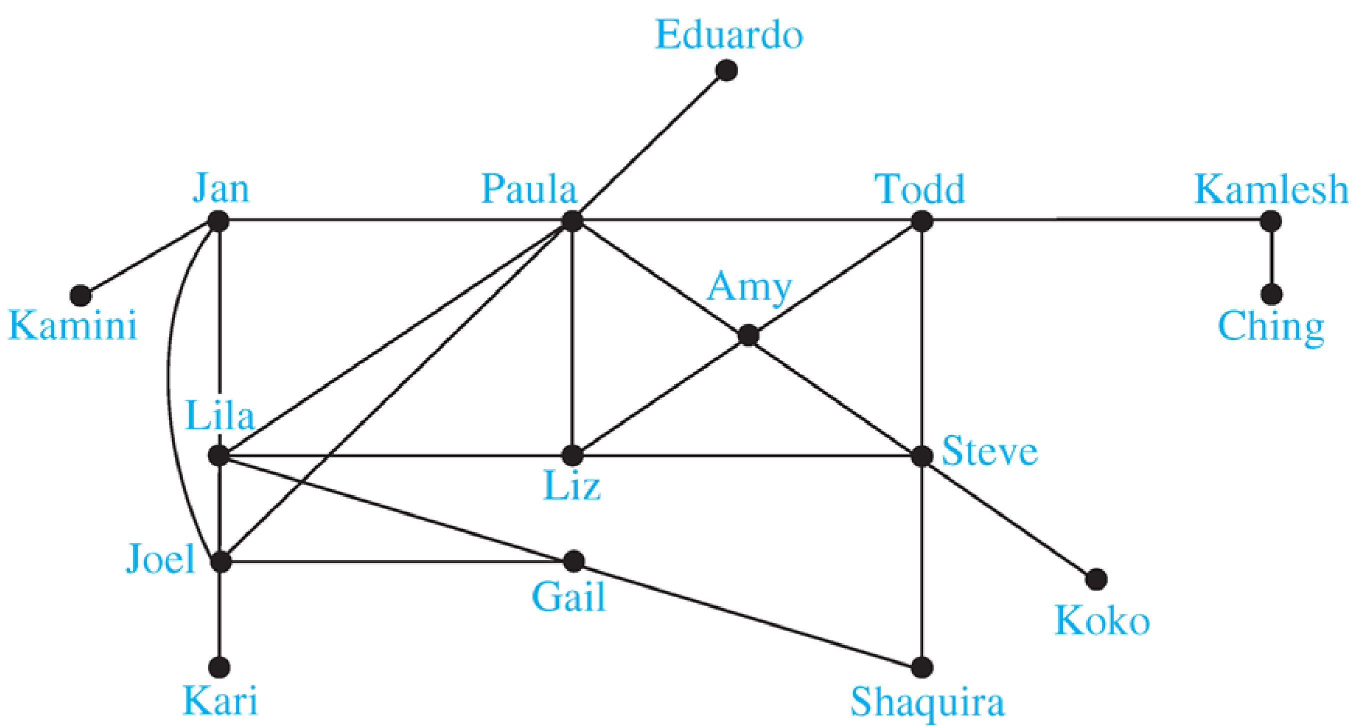

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  Example

  - the Hollywood graph
  - the Erdös number

Announcements

- **Homework assignment** 5
  - ◇ P397 Ex. 37, P398 Ex. 50, 62, 64, P405 Ex. 10, P406 Ex. 40, P413 Ex. 13, P422 Ex. 24, 27, P525 Ex. 12, 28, P526 Ex. 44, P550 Ex. 22, P551 Ex. 42, P583 Ex. 47 (a) (b) (d) (e), P607 Ex. 20, 22, 23, 24, P615 Ex. 16, P616 Ex. 40, P630 Ex. 6, P631 Ex. 32
  - ◇ Due on *Dec. 12th, 2017 at the beginning of Class*
  - ◇ Please write your homeowork **neatly**, as a courtesy to graders.

Next Lecture

- graph theory ...