南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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The Chinese Remainder Theorem

- **Theorem** (*The Chinese Remainder Theorem*) Let $m_1, m_2, \ldots, m_n$ be pairwise relatively prime positive integers greater than 1 and $a_1, a_2, \ldots, a_n$ arbitrary integers. Then the system

$$\begin{aligned} x &\equiv a_1 \pmod{m_1} \\ x &\equiv a_2 \pmod{m_2} \\ &\ldots \\ x &\equiv a_n \pmod{m_n} \end{aligned}$$

has a unique solution modulo $m = m_1 m_2 \cdots m_n$.

The Chinese Remainder Theorem

- **Proof** Let $M_k = m/m_k$ for $k = 1, 2, \ldots, n$ and $m = m_1 m_2 \cdots m_n$. Since $\gcd(m_k, M_k) = 1$, there is an integer y_k, an inverse of M_k modulo m_k such that $M_k y_k \equiv 1 \pmod{m_k}$. Let

$$x = a_1 M_1 y_1 + a_2 M_2 y_2 + \cdots a_n M_n y_n.$$

It is checked that x is a solution to the n congruences.

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$$\begin{aligned} x &\equiv 8 \pmod{15} \\ x &\equiv 2 \pmod{21} \end{aligned}$$

Modular Arithmetic in CS

- Modular arithmetic and congruencies are used in CS:
  - ◇ Pseudorandom number generators
  - ◇ Hash functions
  - ◇ Cryptography

Hash Functions

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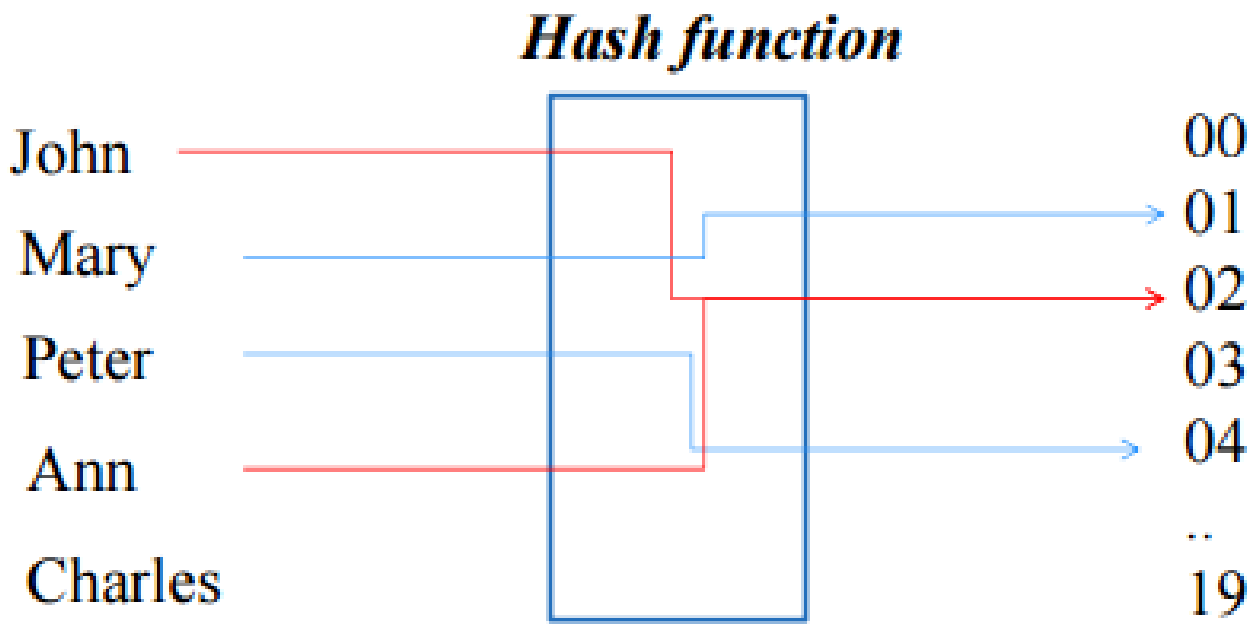

Hash Functions

- **Problem**: Given a large collection of records, how can we store and find a record quickly?

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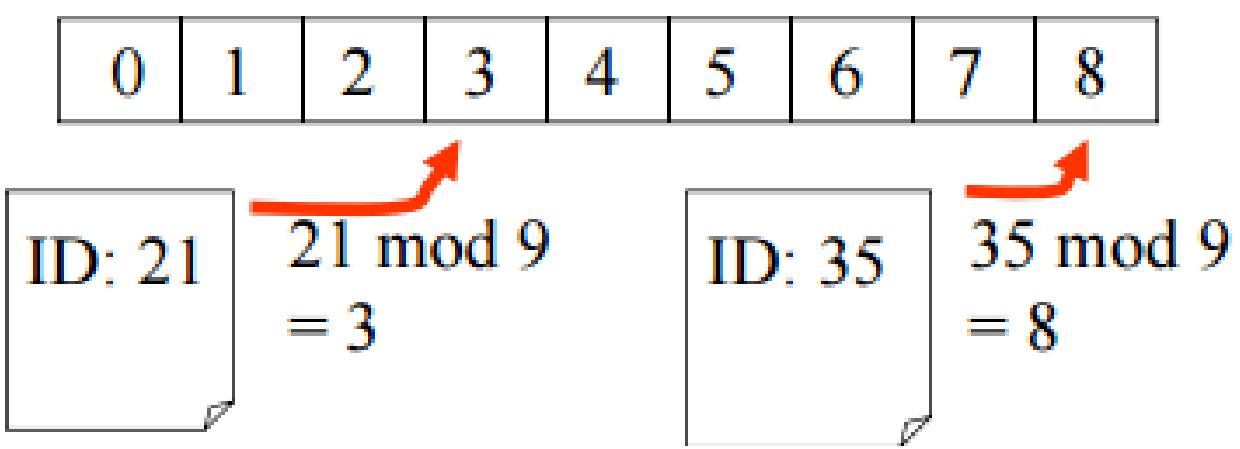
  Solution: Use a hash function, calculate the location of the record based on the record's ID.

  Example: A common hash function is

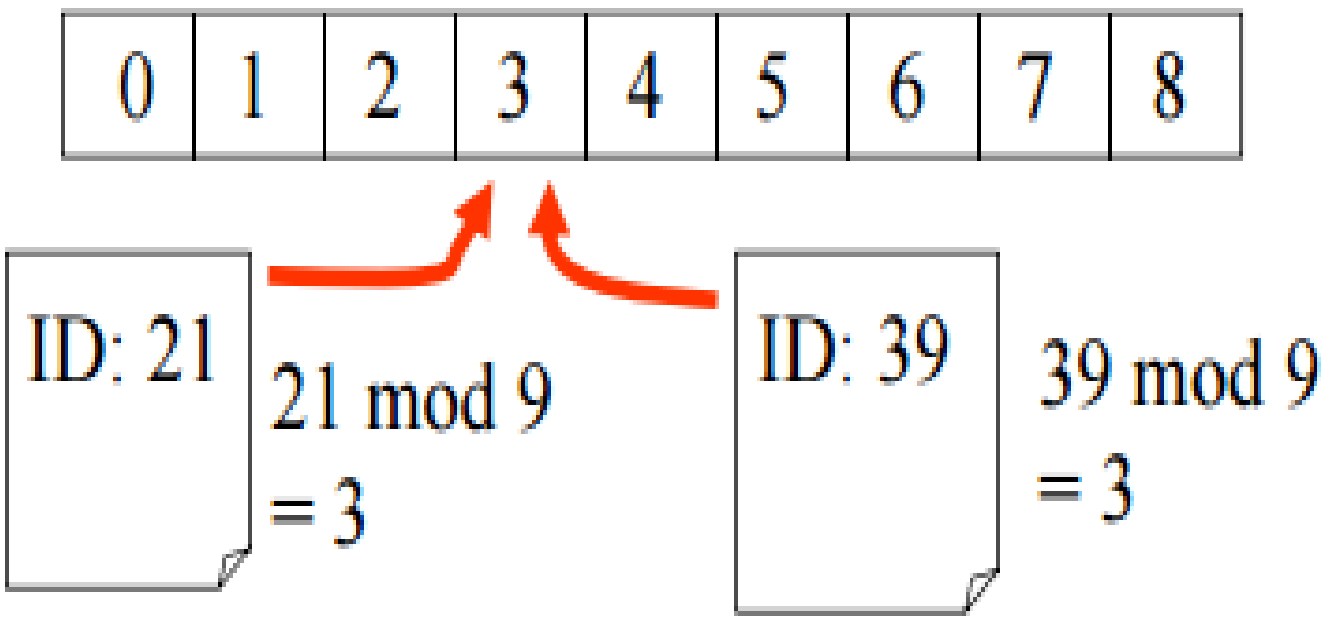
  - $h(k) = k \bmod n$,

  where n is the number of available storage locations.



Hash Functions

- Two records mapped to the same location

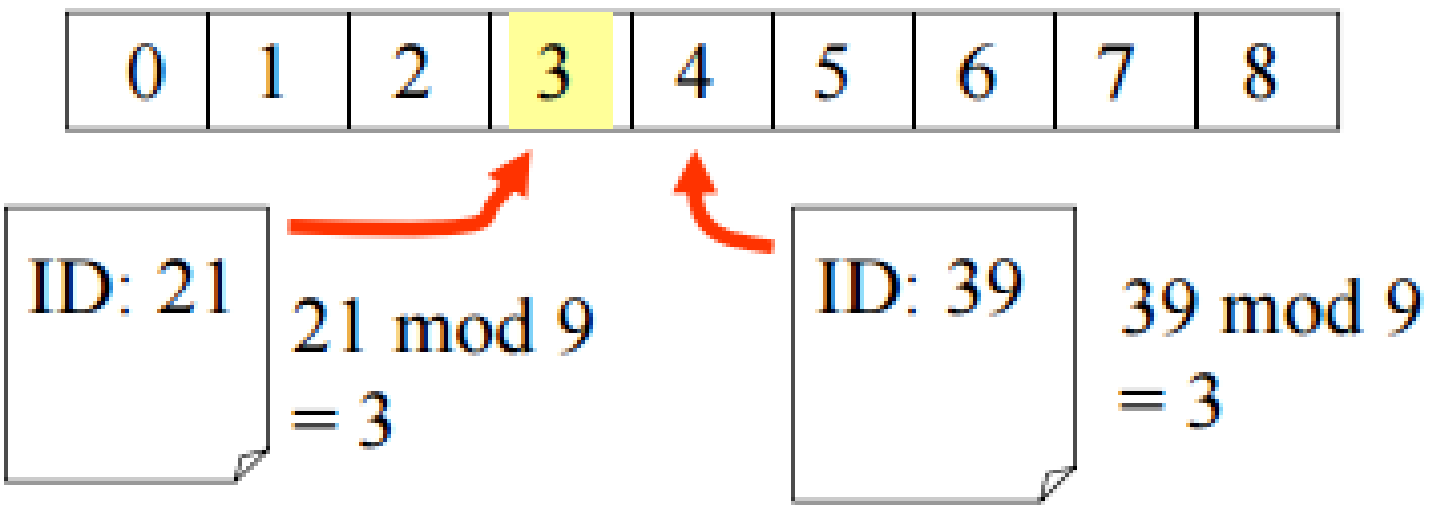


Hash Functions

- **Solution 1**: move to the next available location

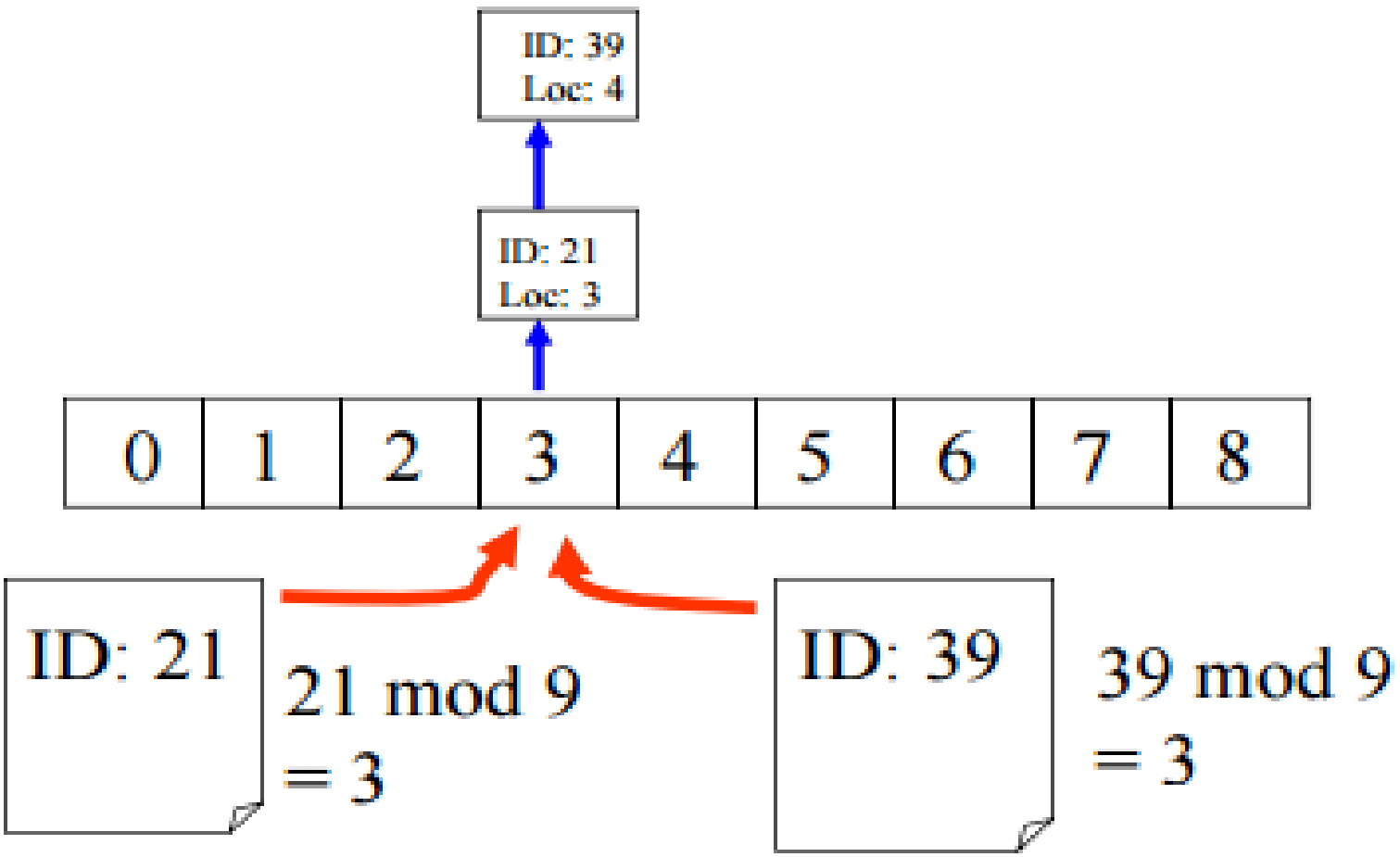
try

$$h_0(k) = k \bmod n$$
$$h_1(k) = (k+1) \bmod n$$
$$\ldots$$
$$h_m(k) = (k+m) \bmod n$$



Hash Functions

- **Solution 2**: remember the exact location in a secondary structure that is searched sequentially



Applications of Number Theory in Cryptography

- Introduction
- Symmetric cryptography
- Asymmetric cryptography
- RSA Cryptosystem
- DLP and El Gamal cryptography
- Diffie-Hellman key exchange protocol
- Cryptocurrency, e.g., bitcoin

Fermat's Little Theorem

- **Theorem (Fermat's little theorem)** : Let p be a prime, and let x be an integer such that $x \not\equiv 0 \bmod p$. Then

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$\mathcal{Q}$: How to prove Fermat's little theorem?

$$\{1, 2, \ldots, p-1\} = \{x, 2x, \ldots, x(p-1) \pmod{p}\}$$

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The Multiplicative Order

- Let $a \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $\gcd(a, n) = 1$, the least $\ell \in \mathbb{N}$ such that $a^{\ell} \equiv 1 \pmod{n}$ is called the *order of a modulo n*, denoted by $\mathrm{ord}_n(a)$.

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Example $n = 7$, then

$$\mathrm{ord}_7(1) = 1,\ \mathrm{ord}_7(2) = 3,\ \mathrm{ord}_7(3) = 6$$
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$\mathcal{Q}$: How to prove this theorem?

Using Euler's theorem and $\phi(n) = q \cdot \mathrm{ord}_n(a) + r$

Primitive Roots

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  $\mathcal{Q}$: proof? The number of primitive roots? $*$

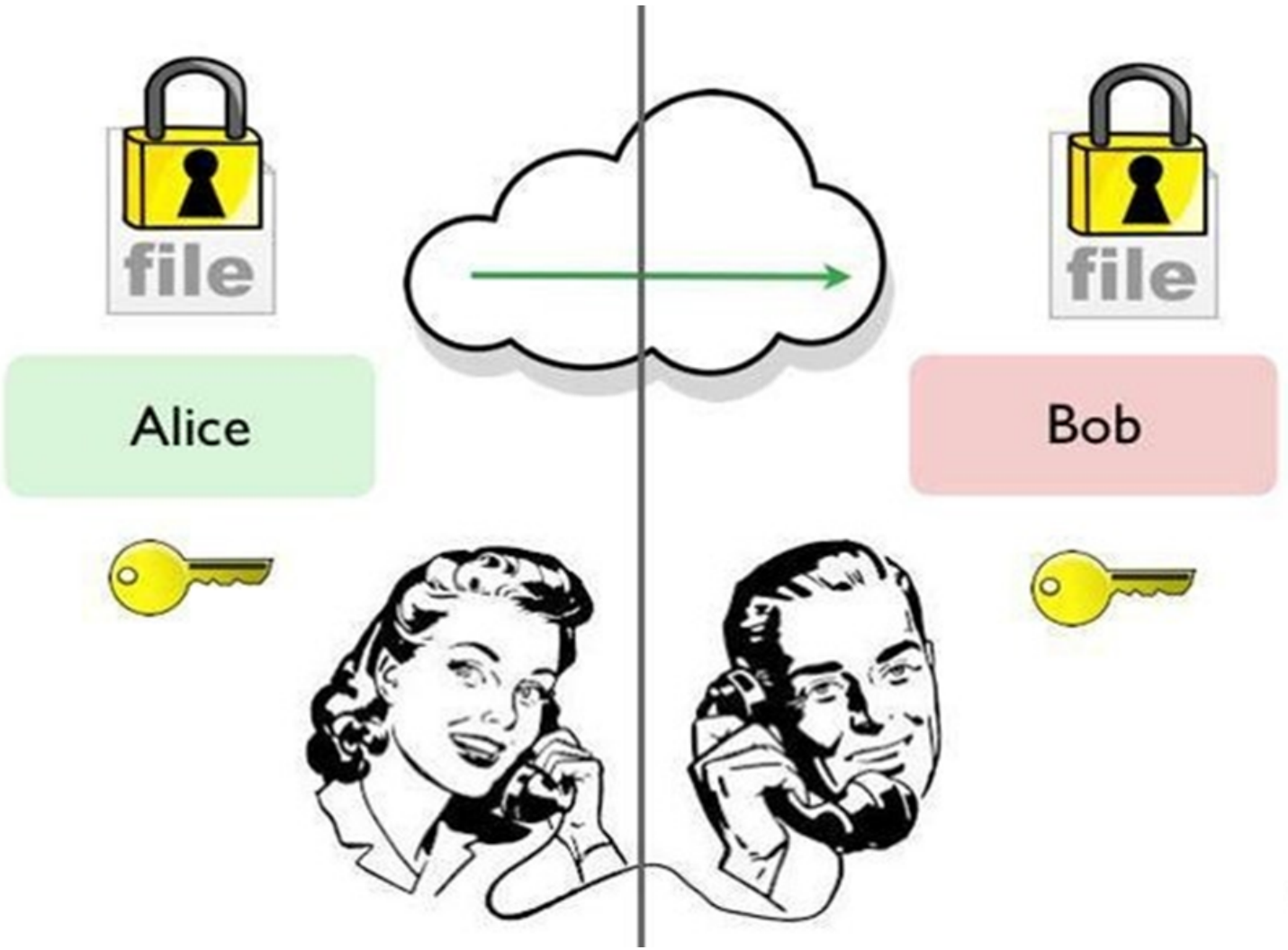
Classical One-Key Ciphers

- A 5-tuple $(\mathcal{M}, \mathcal{C}, \mathcal{K}, E_k, D_k)$, where
  - $\mathcal{M}$, $\mathcal{C}$ and $\mathcal{K}$ denote the plaintext space, ciphertext space, and key space.
  - Any $k \in \mathcal{K}$ could be the encryption and decryption key;
  - E_k and D_k are encryption and decryption transformatons with $D_k(E_k(m)) = m$ for each $m \in \mathcal{M}$.

Classical One-Key Ciphers

- Give a cipher $(\mathcal{M}, \mathcal{C}, \mathcal{K}, E_k, D_k)$:
  - **Encryption:** $c = E_k(m)$, where E_k is usually applied to blocks of the plaintext m.
  - **Decryption:** $m = D_k(c)$, where D_k is usually applied to blocks of the ciphertext c.

Symmetric Cryptography



Attacks on Classical Cryptography

- **Ciphertext-only attack**

  determine the decryption transformation D_k, or k, or m from the ciphertext c.

- **Known-plaintext attack**

  determine the decryption transformation D_k, or k, from a ciphertext-plaintext pair (c, m).

Simple Substitution Ciphers

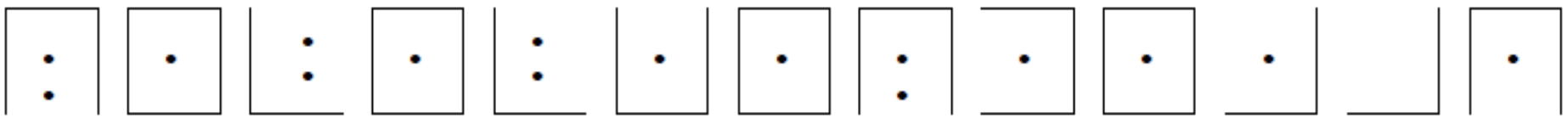
- Let f be a 1-to-1 function from A to B. It is a 5-tuple $(\mathcal{M}, \mathcal{C}, \mathcal{K}, E_k, D_k)$, where
  - $\mathcal{M} = A^*$ and $\mathcal{C} = B^*$
  - $\mathcal{K}$ is the set of all possible f
  - $k = f \in \mathcal{K}$ is the encryption and decryption key
  - For a message $m = m_0 m_1 m_2 \cdots$,

    $$c = E_k(m) = f(m_0) f(m_1) f(m_2) \cdots$$

  - For a ciphertext $c = c_0 c_1 c_2 \cdots$,

    $$m = D_k(c) = f^{-1}(c_0) f^{-1}(c_1) f^{-1}(c_2) \cdots$$

Simple Substitution Ciphers: Examples



a	b	c
d	e	f
g	h	i

k	l	m
n	o	p
q	r	s

t	u	v
w	x	y
z	j	

Simple Substitution Ciphers: Caesar Cipher

a	b	c	$\cdots$	y	z
0	1	2	$\cdots$	24	25

$$f(x) = (x + 3) \bmod 26$$

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VENI, VIDI, VICI

YHQL YLGL YLFL

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Affine cipher: Take any (k_0, k_1) with $\gcd(k_0, 26) = 1$ and $0 \leq k_0 \leq 25$, define the 1-to-1 mapping f by

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$\mathcal{Q}$: Why $\gcd(k_0, 26) = 1$?

Simple Substitution Ciphers: Examples

Cryptanalysis: Suppose that we know two pairs of plaintext-ciphertext characters (m_1, c_1) and (m_2, c_2), i.e.,

$$c_1 = (m_1 k_0 + k_1) \bmod 26$$

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It is possible to solve the equations to obtain k_0 and k_1.

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$\mathcal{Q}$: When the set of two equations above has a unique solution (k_0, k_1)?

Other Symmetric Ciphers

DES

3DES

IDEA

Blowfish

Rijndael (AES)

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"Confusion" & "Diffusion"

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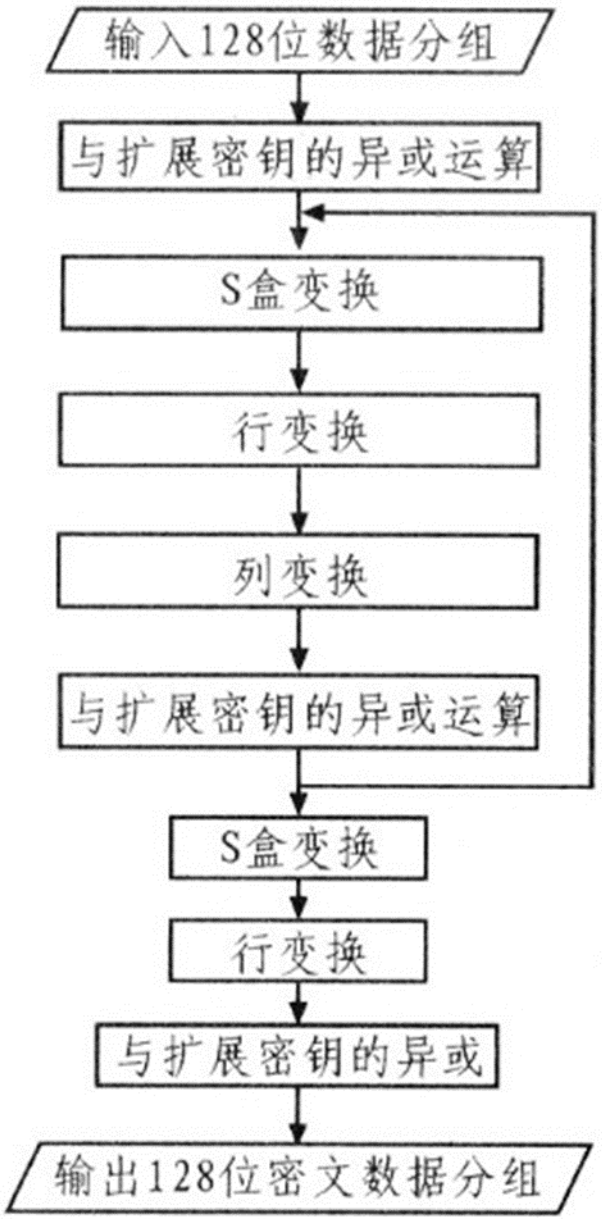

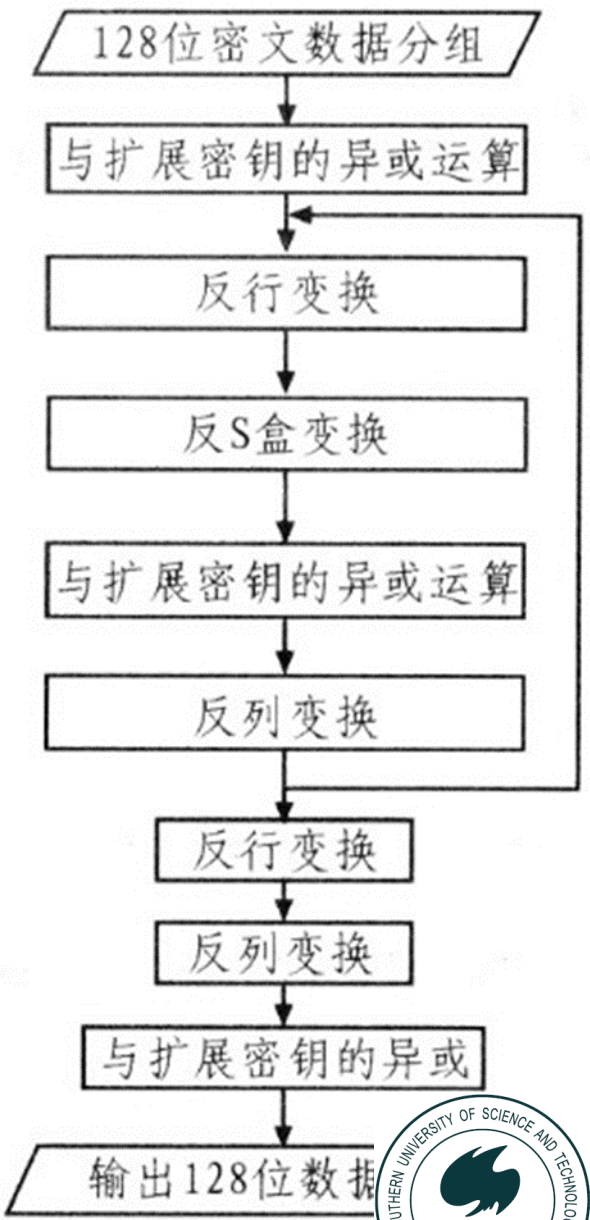

图 1 AES 加密/解密框图

Public-Key Cryptosystems

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Four properties:

1. $D(E(M)) = M$
2. Both E and D are easy to compute
3. It is computationally infeasible to derive D from E
4. $E(D(M)) = M$

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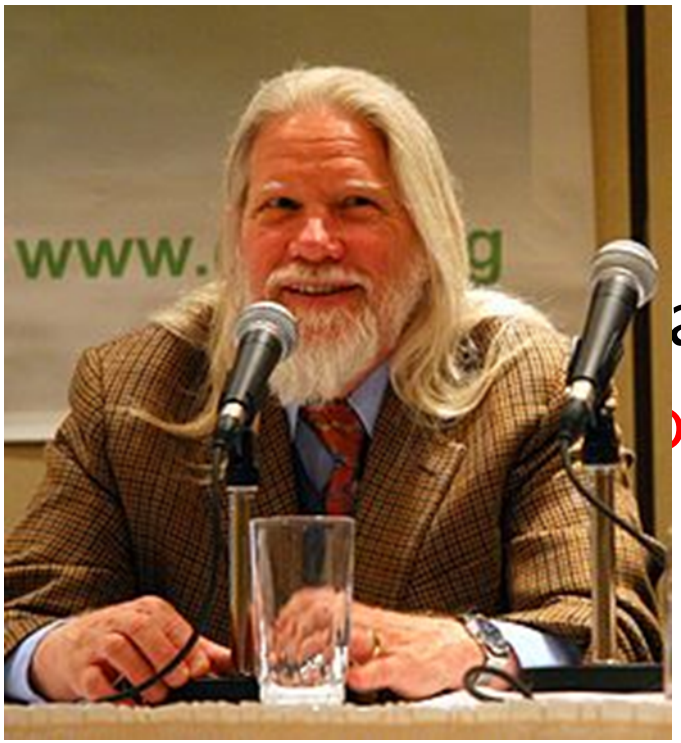

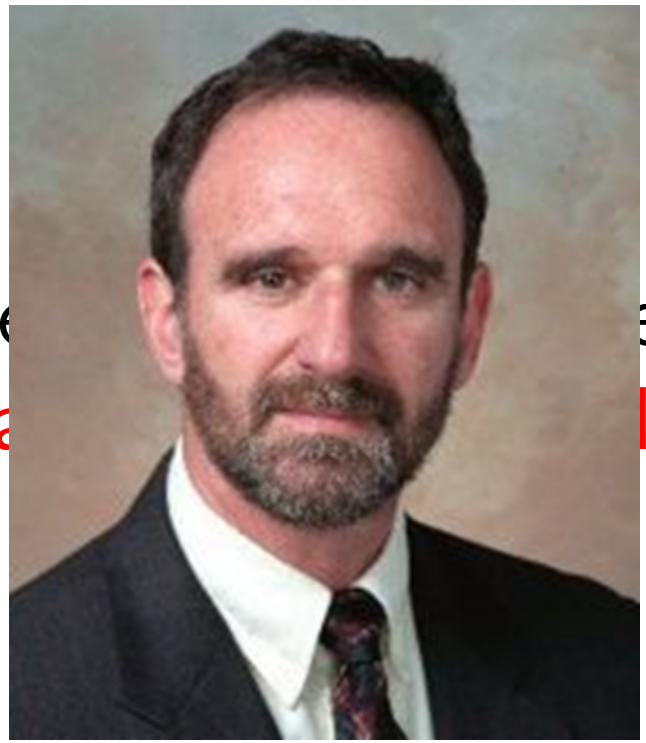

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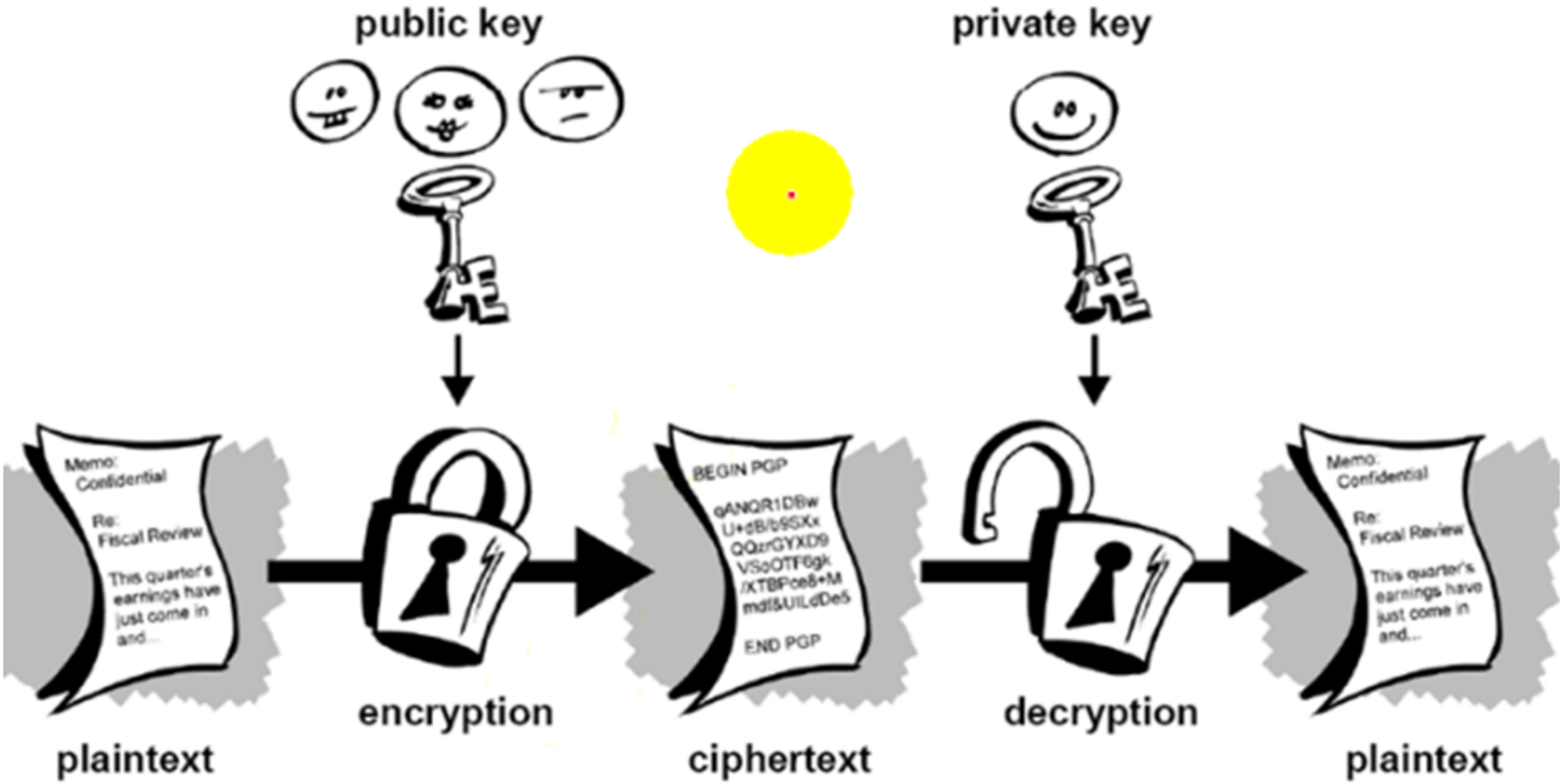
Asymmetric Cryptography



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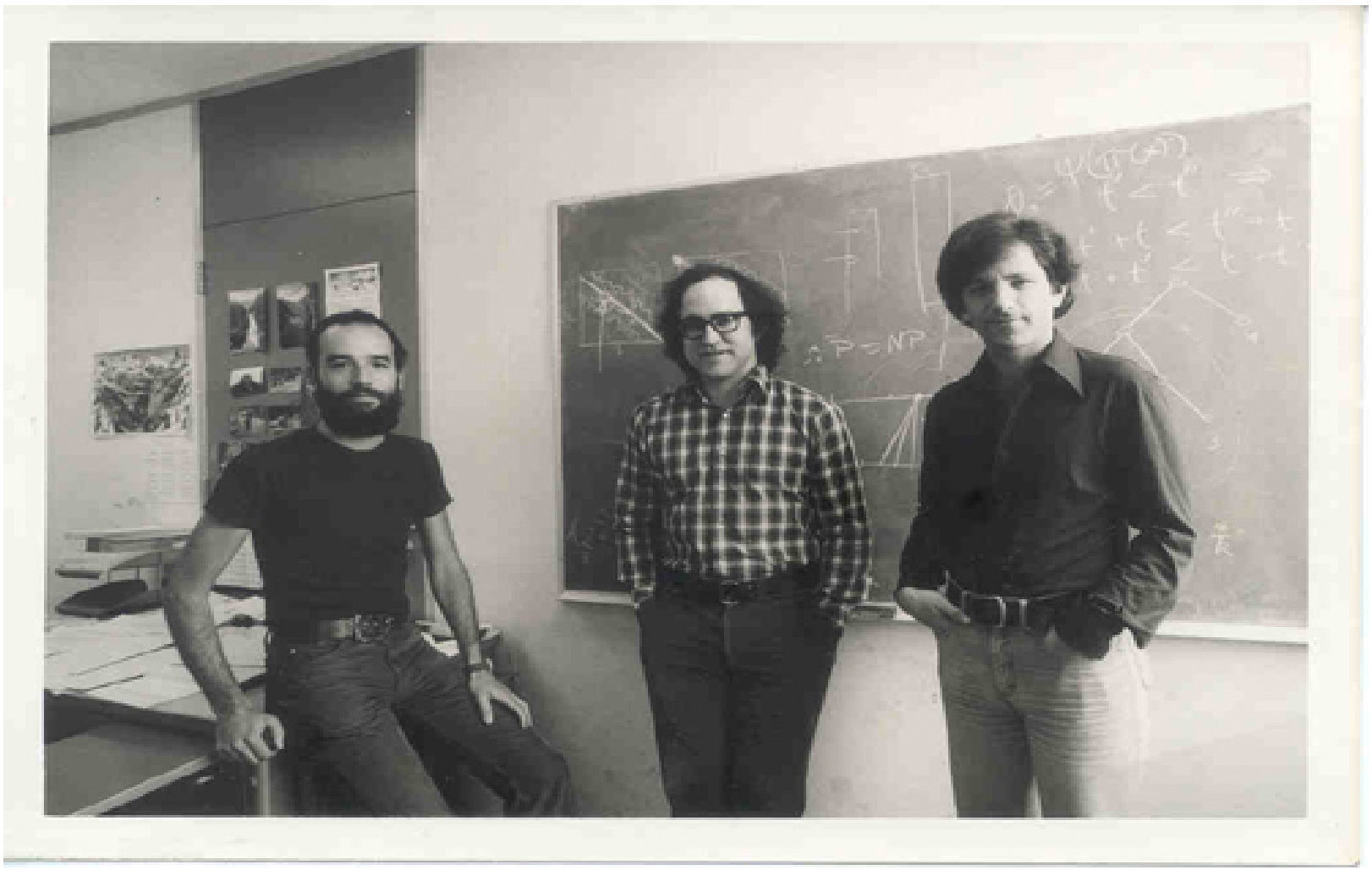
R. Rivest, A. Shamir, L. Adleman, "A Method for Obtaining Digital Signatures and Public-Key Cryptosystems", Communications of the ACM, vol. 21-2, pages 120-126, 1978.



RSA Public-Key Cryptosystem

Pick two large primes, p and q. Let $n = pq$, then $\phi(n) = (p-1)(q-1)$. Encryption and decryption keys e and d are selected such that

- $\gcd(e, \phi(n)) = 1$
- $ed \equiv 1 \pmod{\phi(n)}$

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- **Theorem 12** (Correctness) : Let p and q be two odd primes, and define $n = pq$. Let e be relatively prime to $\phi(n)$ and let d be the multiplicative inverse of e modulo $\phi(n)$. For each integer x such that $0 < x < n$,

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$\mathcal{Q}$: How to prove this?

RSA Public-Key Cryptosystem: Example

Parameters:	p	q	n	$\phi(n)$	e	d
	5	11	55	40	7	23

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Private key: 23

Encryption: $M = 28, C = M^7 \bmod 55 = 52$

Decryption: $M = C^{23} \bmod 55 = 28$

RSA Public-Key Cryptosystem: Parameters

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Public key: (e, n)

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$\mathcal{Q}$: Why?

The Security of the RSA

Brute-force attack:

Trying all possible private keys.

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Attack: Determine $\phi(n)$ directly.

Attack: Determine d directly.

Comment: It is believed that determining $\phi(n)$ is equivalent to factoring n. Meanwhile, determining d given e and n, appears to be at least as time-consuming as the integer factoring problem.

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$\mathcal{Q}$: Consider the RSA system, where $n = pq$ is the modulus. Let (e, d) be a key pair for the RSA. Define

$$\lambda(n) = \text{lcm}(p-1, q-1)$$

and compute $d' = e^{-1} \bmod \lambda(n)$. Will decryption using d' instead of d still work?

Using RSA for Digital Signature

$S = M^d \bmod n$ (RSA **signature)**

$M = S^e \bmod n$ (RSA **verification)**

Why?

The Discrete Logrithm

- **The discrete logarithm** of an integer y to the base b is an integer x, such that

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Given n, b and y, find x.

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Discrete Logarithm Problem:

Given n, b and y, find x.

This is very hard!!!

El Gamal Encryption

- **Setup** Let p be a prime, and g be a generator of $\mathbb{Z}_p$. The private key x is an integer with $1 < x < p - 2$. Let $y = g^x \bmod p$. The public key for *El Gamal encryption* is (p, g, y).

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  El Gamal Encryption: Pick a random integer k from $\mathbb{Z}_{p-1}$,

  $$a = g^k \bmod p$$
  $$b = My^k \bmod p$$

  The ciphertext C consists of the pair (a, b).

  El Gamal Decryption:

  $$M = b(a^x)^{-1} \bmod p$$

Using El Gamal for Digital Signature

$$a = g^k \bmod p$$
$$b = k^{-1}(M - xa) \bmod (p-1)$$

(El Gamal **signature)**

$$y^a a^b \equiv g^M \pmod{p}$$

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$\mathcal{Q}$: How to verify it?

An Example

Choose $p = 2579$, $g = 2$, and $x = 765$. Hence $y = 2^{765} \bmod 2579 = 949$.

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- **(Public key)** $k_e = (p, g, y) = (2579, 2, 949)$
- **(Private key)** $k_d = x = 765$

Encryption: Let $M = 1299$ and choose a random $k = 853$,

$$\begin{aligned}(a, b) &= (g^k \bmod p, My^k \bmod p) \\ &= (2^{853} \bmod 2579, 1299 \cdot 949^{853} \bmod 2579) \\ &= (435, 2396).\end{aligned}$$

Decryption:

$$M = b(a^x)^{-1} \bmod p = 2396 \times (435^{765})^{-1} \bmod 2579 = 1299.$$

Security of the El Gamal Cryptosystem

Question 1: Is it feasible to derive x from (p, g, y)?

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Question 2: Given a ciphertext (a, b), is it feasible to derive the plaintext M?

Attack 1: Use $M = by^{-k}$. However, k is randomly picked.

Attack 2: Use $M = b(a^x)^{-1} \bmod p$, but x is secret.

Diffie-Hellman Key Exchange Protocol

User A		User B
Generate random $X_A < p$		Generate random $X_B < p$
calculate $Y_A = \alpha^{X_A} \bmod p$	$Y_A \rightarrow$	Calculate $Y_B = \alpha^{X_B} \bmod p$
Calculate $k = (Y_B)^{X_A} \bmod p$	$\leftarrow Y_B$	Calculate $k = (Y_A)^{X_B} \bmod p$

Announcements

- **Homework assignment** 3

  ◇ P245 Ex. 37, 38, 39, P255 Ex. 2, 26, P272 Ex. 11*, 12, P273 Ex. 42, P274 Ex. 50, 55, P284 Ex. 7*, P285 Ex. 22, P286 Ex. 39, P305 Ex. 23, 28, 30

  ◇ Due on *Nov. 7th, 2017 at the beginning of class*

  ◇ Please try you best to slove problems marked with $*$

  ◇ Please write your homeowrk **neatly**, as a courtesy to graders.

Next Lecture

- induction, ...