南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

# DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

Dr. QI WANG

Department of Computer Science and Engineering
Office: Room903, Nanshan iPark A7 Building
Email: wangqi@sustc.edu.cn

# Tree Traversal

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- The procedures for systematically visiting every vertex of an ordered tree are called *traversals*.

  The three most commonly used traversals are *preorder traversal*, *inorder traversal*, *postorder traversal*.

# Preorder Traversal

- **Definition** Let $T$ be an ordered rooted tree with root $r$. If $T$ consists only of $r$, then $r$ is the *preorder traversal* of $T$. Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of $r$ from left to right in $T$. The *preorder traversal* begins by visiting $r$, and continues by traversing $T_1$ in preorder, then $T_2$ in preorder, and so on, until $T_n$ is traversed in preorder.

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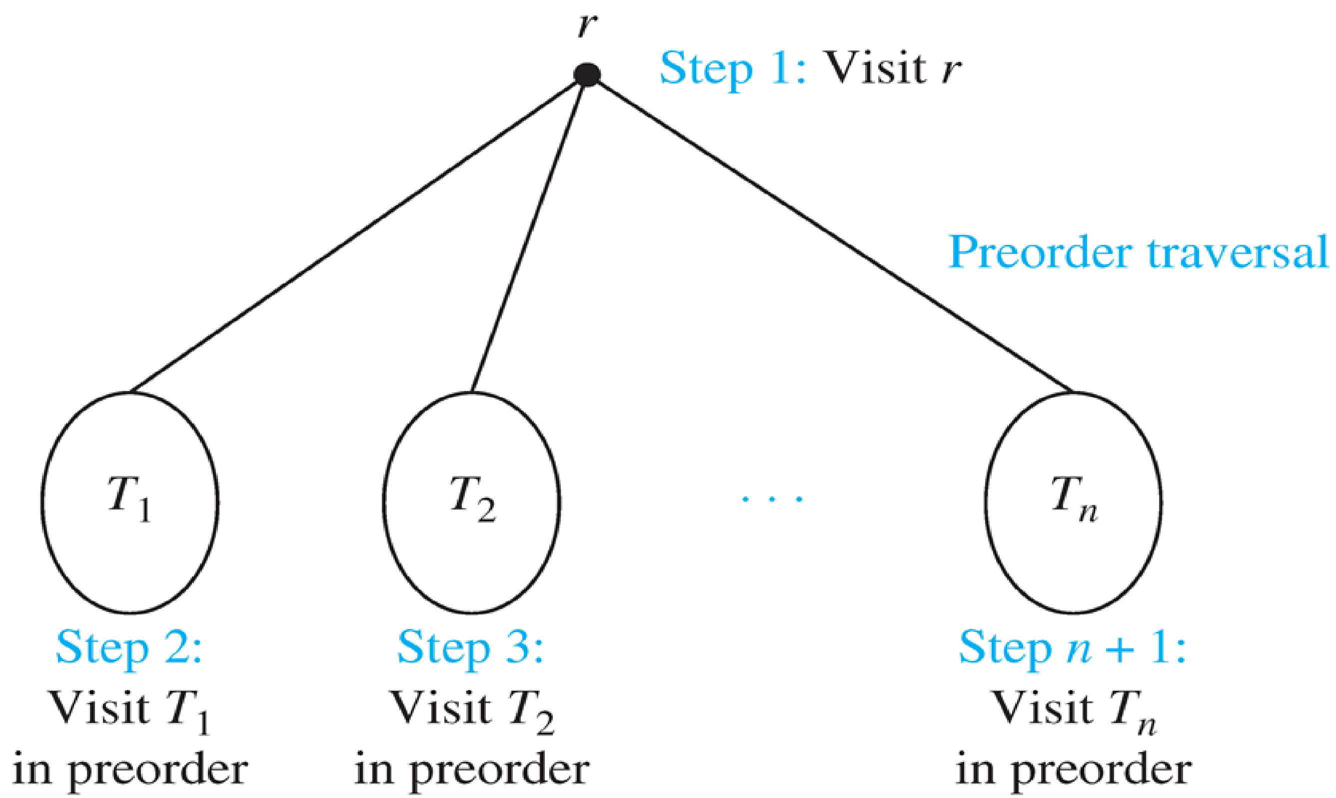
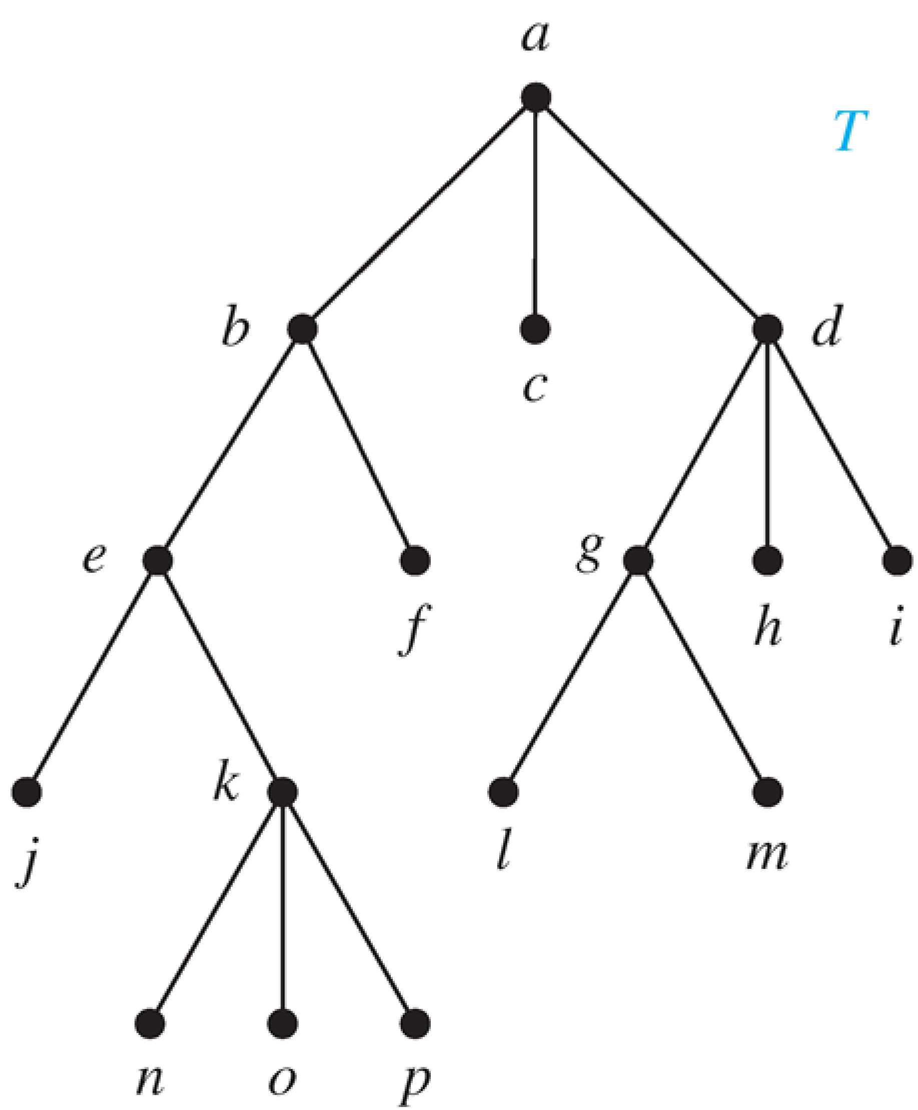


# Preorder Traversal

- **Example**



# Preorder Traversal

**procedure** *preorder* ($T$: ordered rooted tree)
$r$ := root of $T$
list $r$
**for** each child $c$ of $r$ from left to right
    $T(c)$ := subtree with $c$ as root
    *preorder*($T(c)$)

# Inorder Traversal

- **Definition** Let $T$ be an ordered rooted tree with root $r$. If $T$ consists only of $r$, then $r$ is the *inorder traversal* of $T$. Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of $r$ from left to right in $T$. The *inorder traversal* begins by traversing $T_1$ in inorder, then visiting $r$, and continues by traversing $T_2$ in inorder, and so on, until $T_n$ is traversed in inorder.

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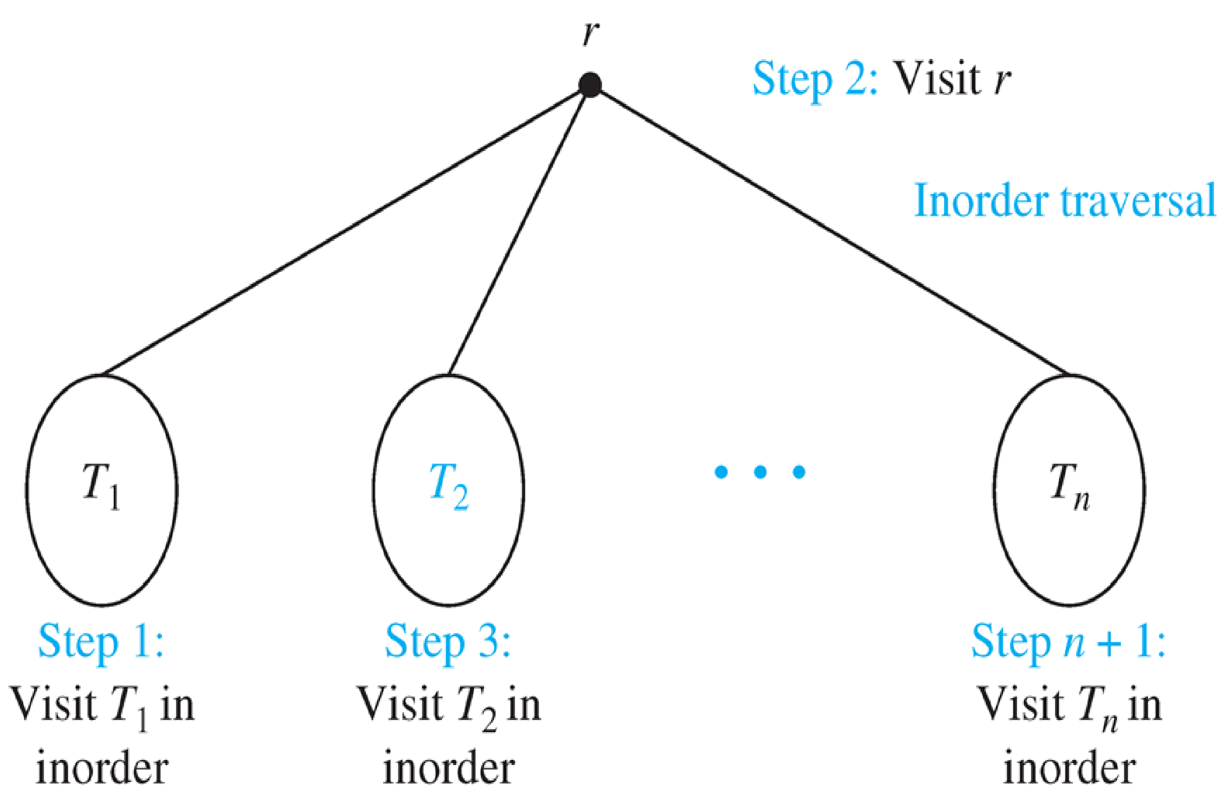

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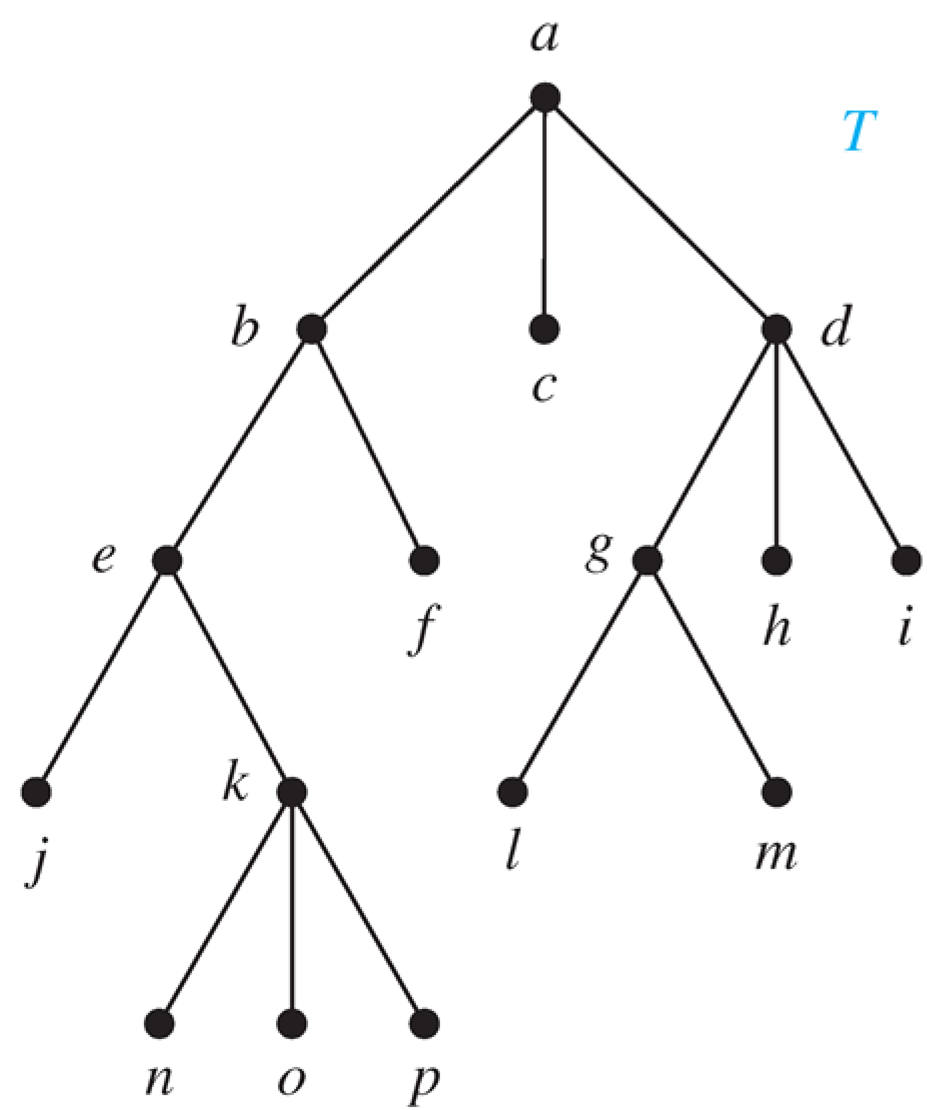

# Inorder Traversal

**procedure** *inorder* ($T$: ordered rooted tree)
$r$ := root of $T$
**if** $r$ is a leaf **then** list $r$
**else**
    $l$ := first child of $r$ from left to right
    $T(l)$ := subtree with $l$ as its root
    *inorder*($T(l)$)
    list($r$)
    **for** each child $c$ of $r$ from left to right
        $T(c)$ := subtree with $c$ as root
        *inorder*($T(c)$)

# Postorder Traversal

- **Definition** Let $T$ be an ordered rooted tree with root $r$. If $T$ consists only of $r$, then $r$ is the *postorder traversal* of $T$. Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of $r$ from left to right in $T$. The *postorder traversal* begins by traversing $T_1$ in postorder, then $T_2$ in postorder, and so on, after $T_n$ is traversed in postorder, $r$ is visited.

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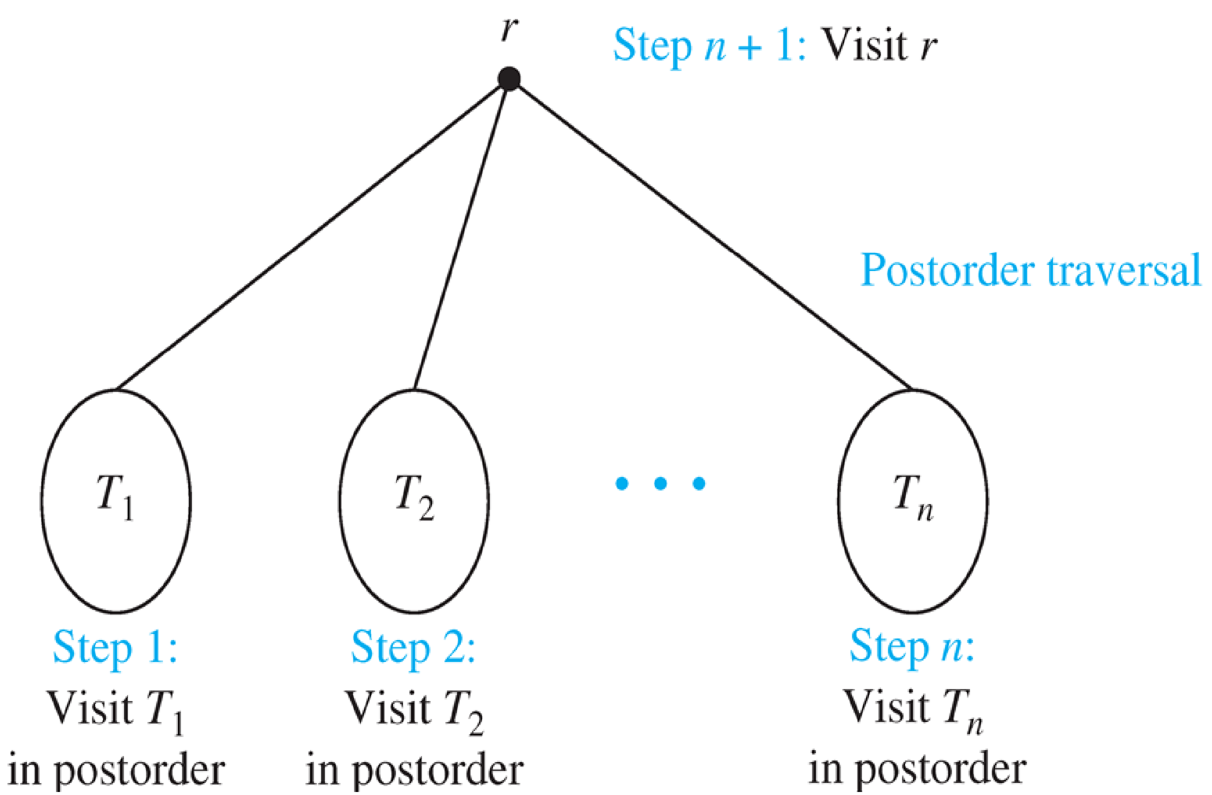

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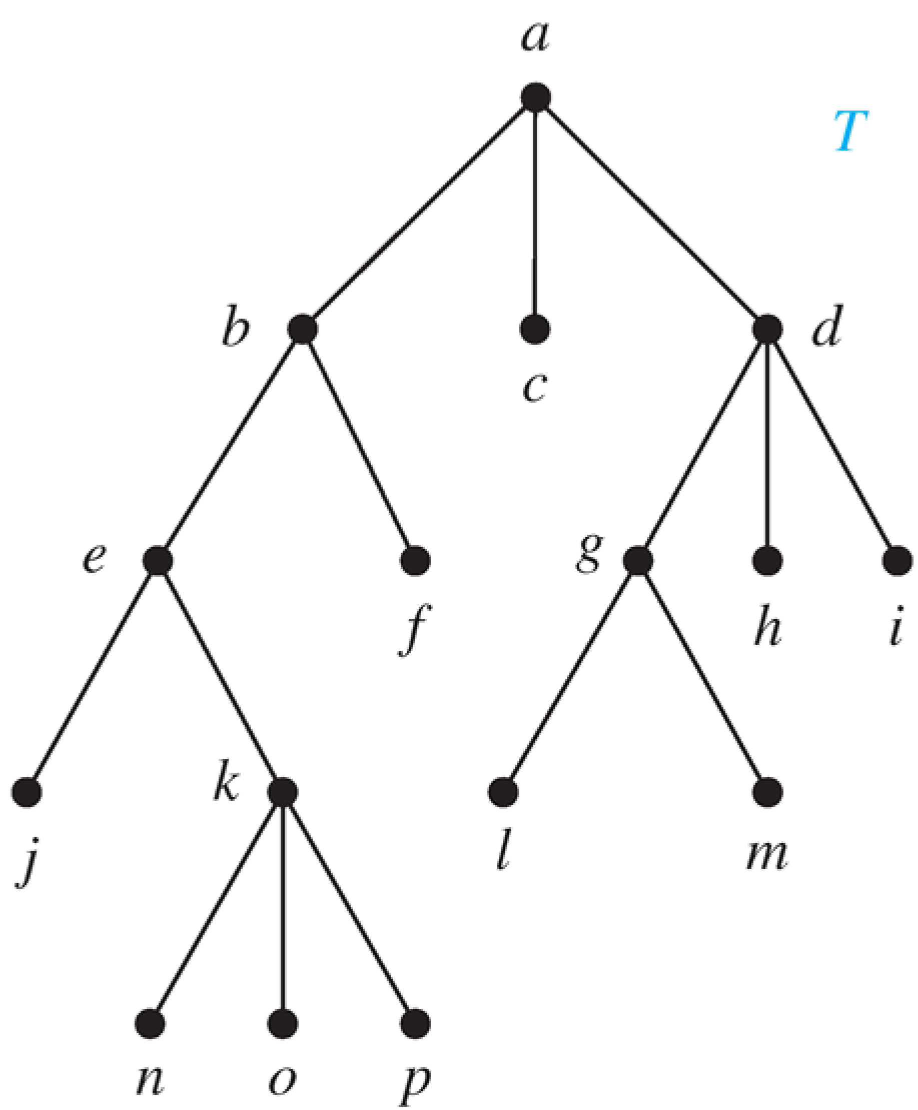
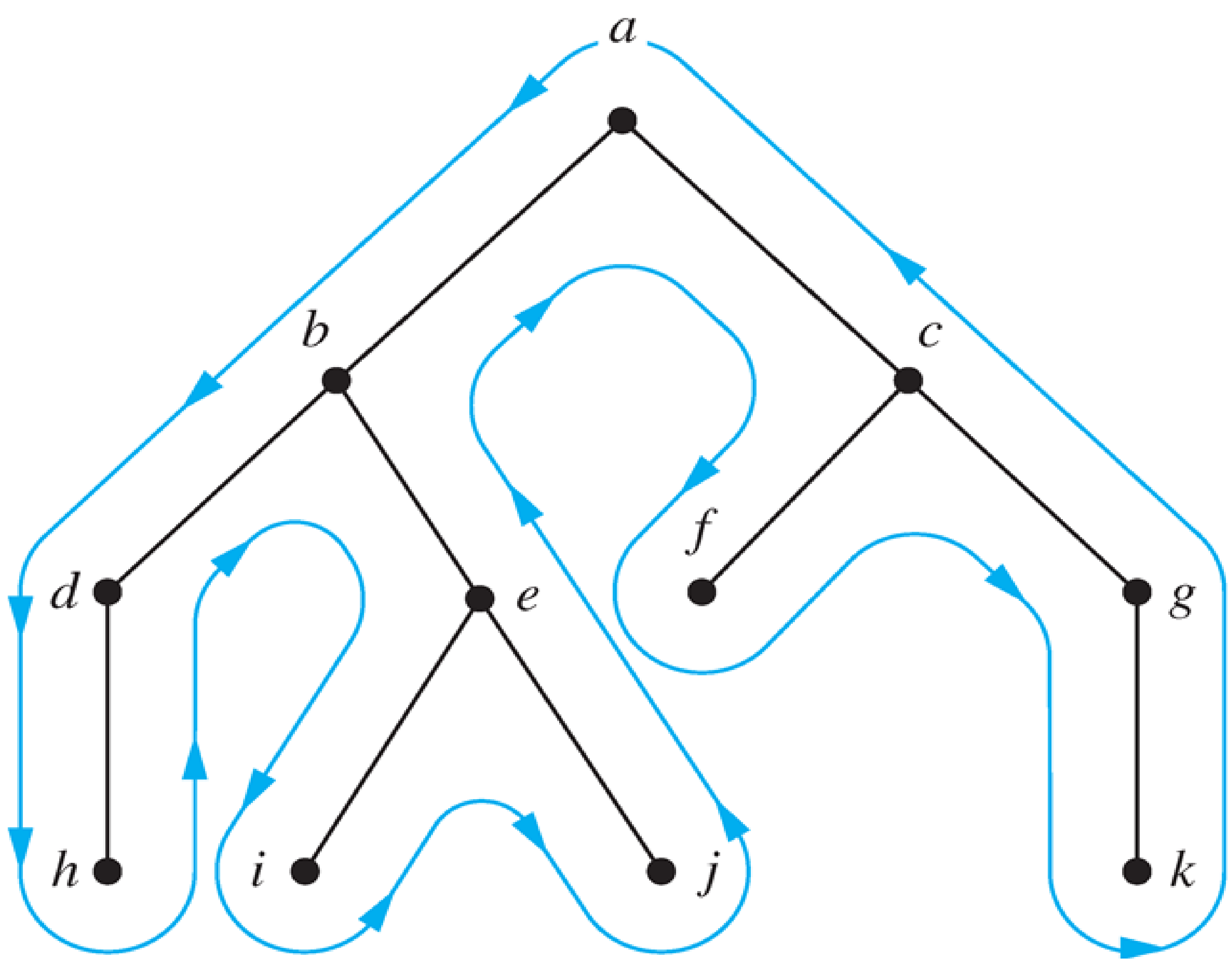


# Postorder Traversal

**procedure** *postordered* ($T$: ordered rooted tree)
$r$ := root of $T$
**for** each child $c$ of $r$ from left to right
    $T(c)$ := subtree with $c$ as root
    postorder($T(c)$)
list $r$

# Preorder, Inorder, Postorder Traversal



# Expression Trees

- Complex expressions can be represented using ordered rooted trees

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**Example**

consider the expression $((x+y)\uparrow 2)+((x-4)/3)$

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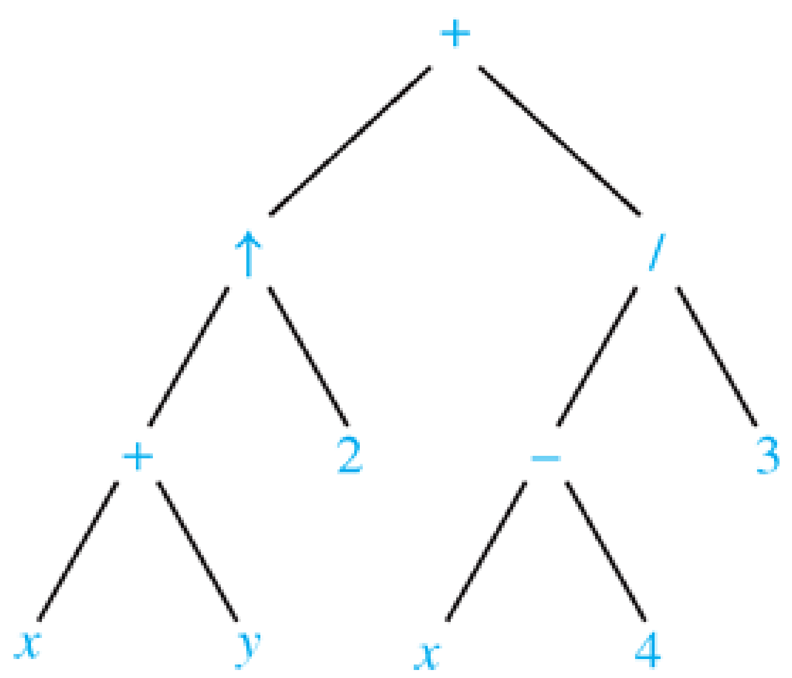

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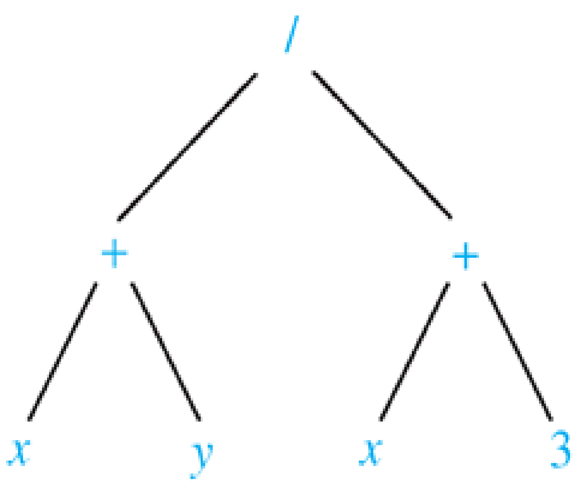

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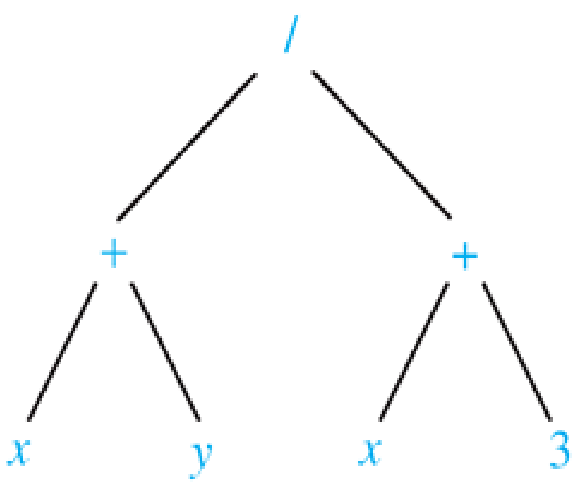


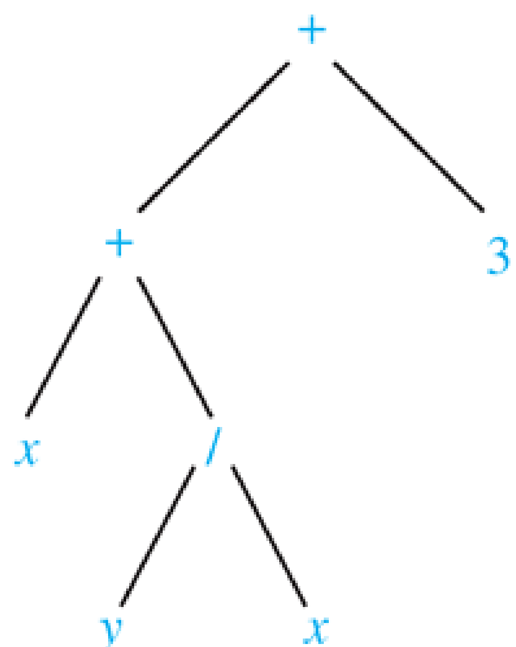

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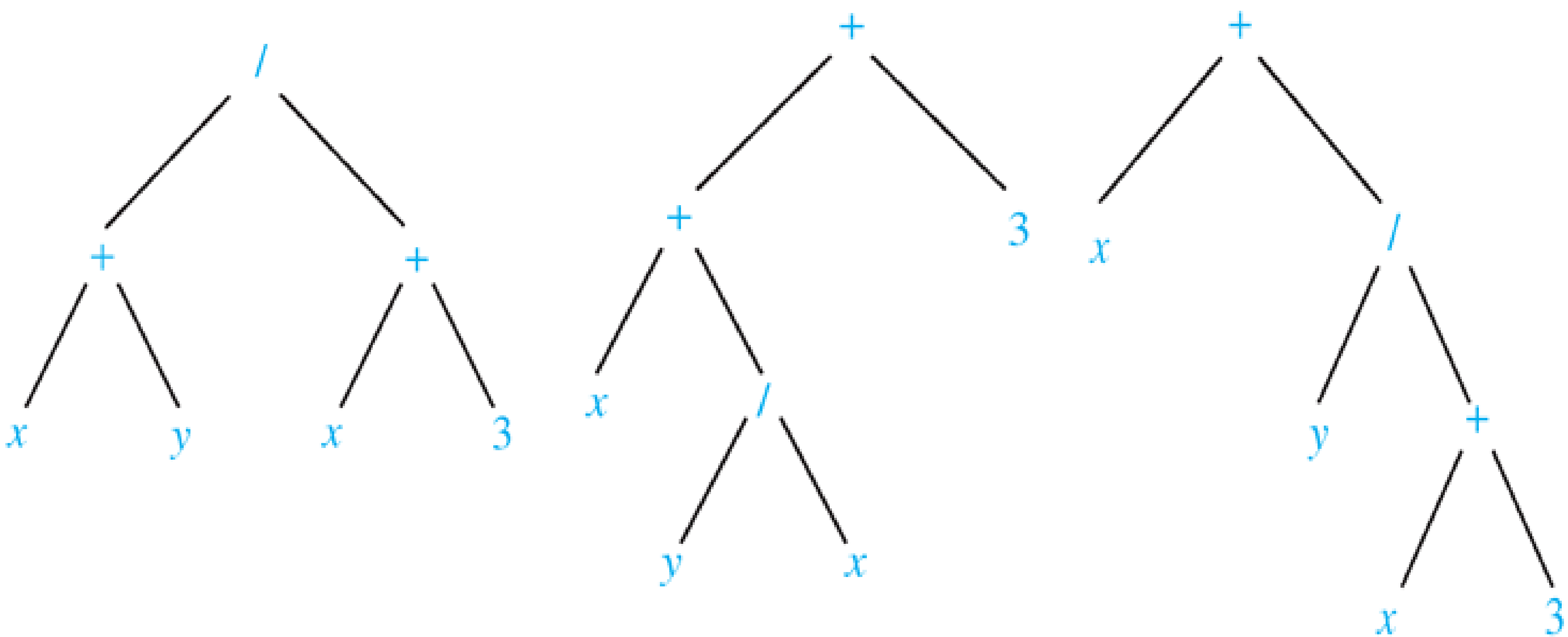

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  Prefix expressions are evaluated by working from right to left. When we encounter an operator, we perform the operation with the two operands to the right.

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- **Example**

$$+ \ - \ * \, 2 \ 3 \ 5 \ / \ \uparrow \ 2 \ 3 \ 4$$

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$$+ \quad - \quad * \quad 2 \quad 3 \quad 5 \quad / \quad \underbrace{\uparrow \quad 2 \quad 3}_{2 \uparrow 3 = 8} \quad 4$$

$$+ \quad - \quad * \quad 2 \quad 3 \quad 5 \quad \underbrace{/ \quad 8 \quad 4}_{8 / 4 = 2}$$

$$+ \quad - \quad \underbrace{* \quad 2 \quad 3}_{2 * 3 = 6} \quad 5 \quad 2$$

$$+ \quad \underbrace{- \quad 6 \quad 5}_{6 - 5 = 1} \quad 2$$

$$\underbrace{+ \quad 1 \quad 2}_{1 + 2 = 3}$$

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  $7\ 2\ 3\ *\ -\ 4\ \uparrow\ 9\ 3\ /\ +$

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$$\underbrace{7\ \ 6\ \ -}_{7-6=1}\ \ 4\ \ \uparrow\ \ 9\ \ 3\ \ /\ \ +$$

$$\underbrace{1\ \ 4\ \ \uparrow}_{1^4=1}\ \ 9\ \ 3\ \ /\ \ +$$

$$1\ \ \underbrace{9\ \ 3\ \ /}_{9/3=3}\ \ +$$

$$\underbrace{1\ \ 3\ \ +}_{1+3=4}$$

# Spanning Trees

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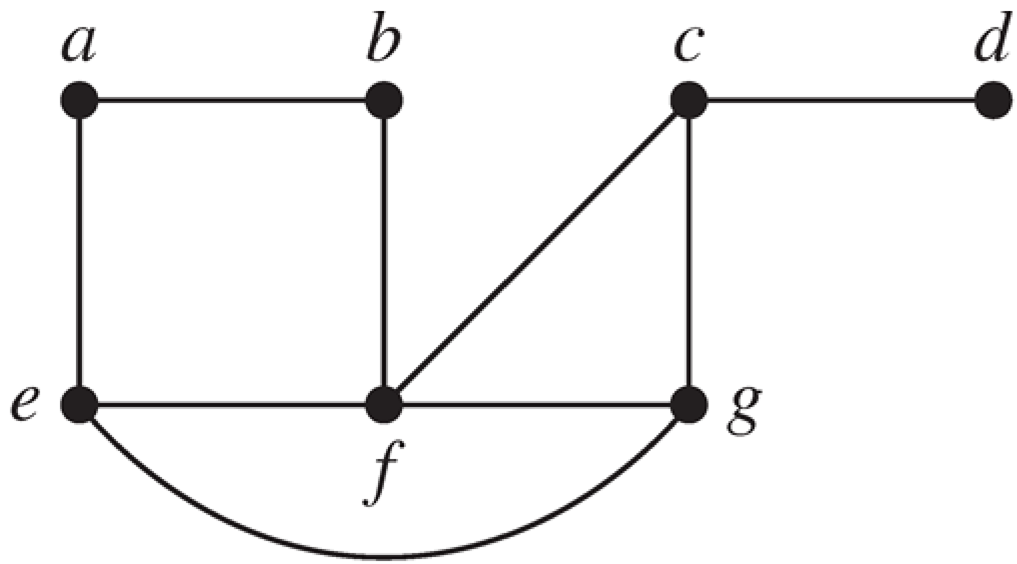

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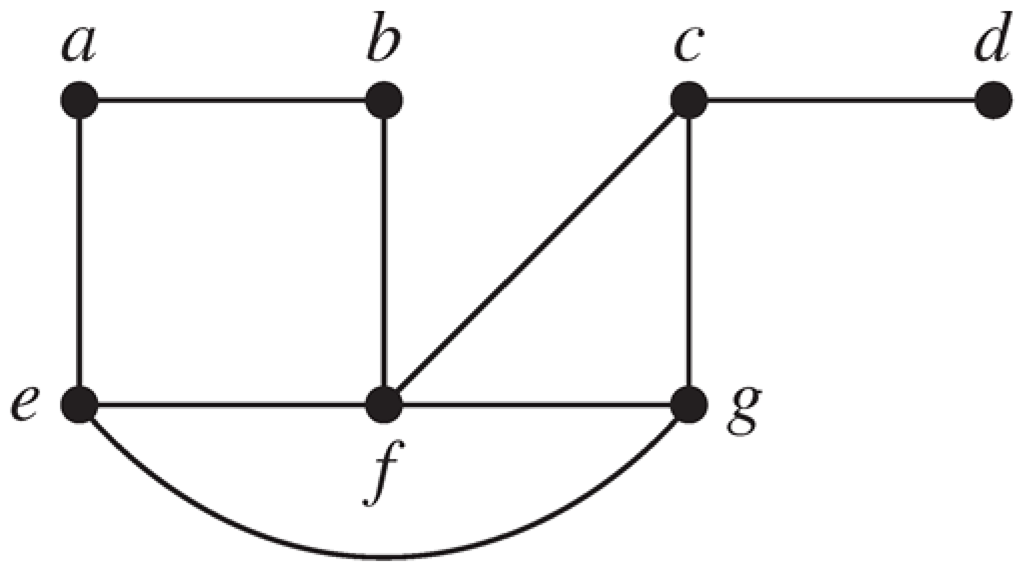


remove edges to avoid circuits

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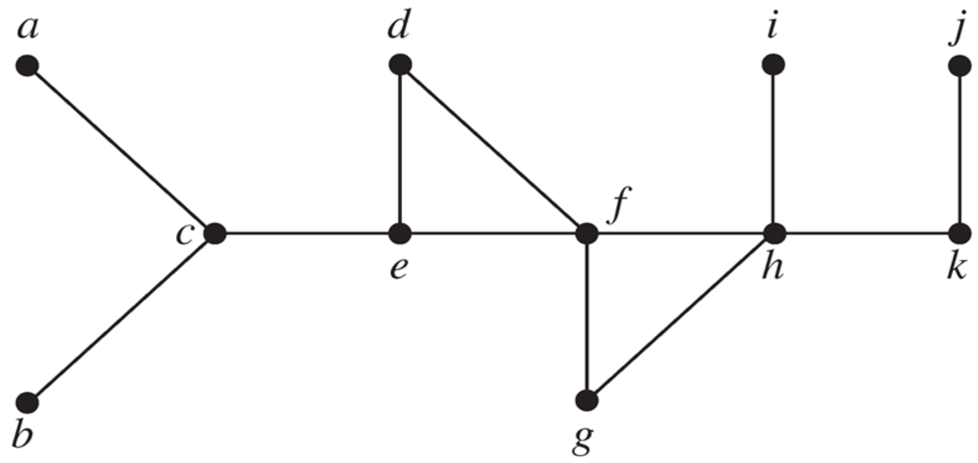
  ◇ If the path goes through all vertices of the graph, the tree is a spanning tree.

  ◇ Otherwise, move back to some vertex to repeat this procedure (*backtracking*)

- **Example**



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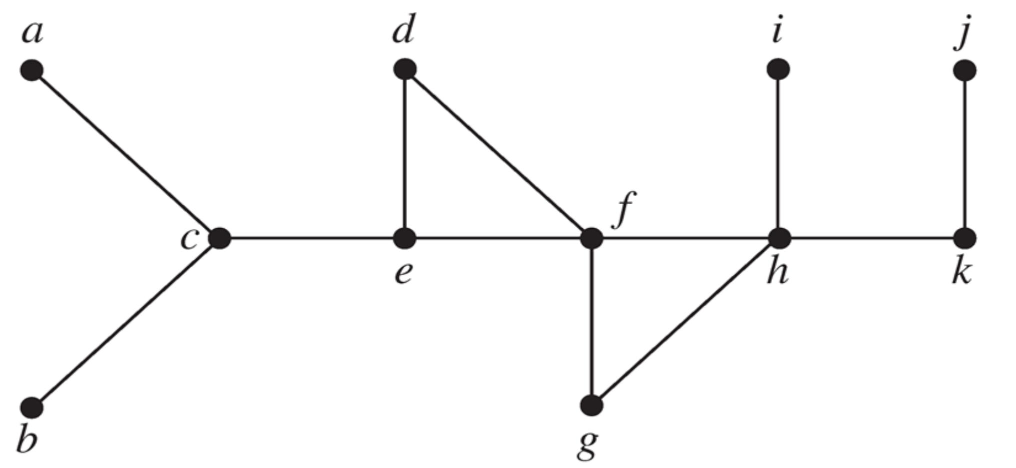


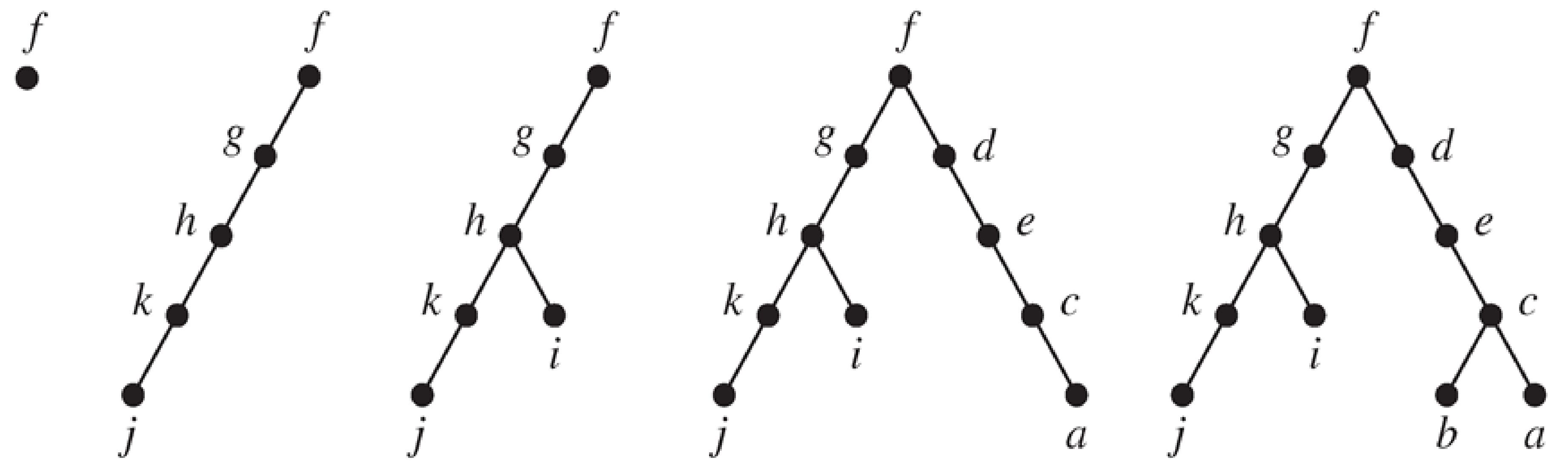

# Depth-First Search Algorithm

**procedure** $DFS(G$: connected graph with vertices $v_1, v_2, \ldots, v_n)$
$T$ := tree consisting only of the vertex $v_1$
$visit(v_1)$

**procedure** $visit(v$: vertex of $G)$
**for** each vertex $w$ adjacent to $v$ and not yet in $T$
    add vertex $w$ and edge $\{v, w\}$ to $T$
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time complexity: $O(e)$

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  ◇ First arbitrarily choose a vertex of the graph as the root.

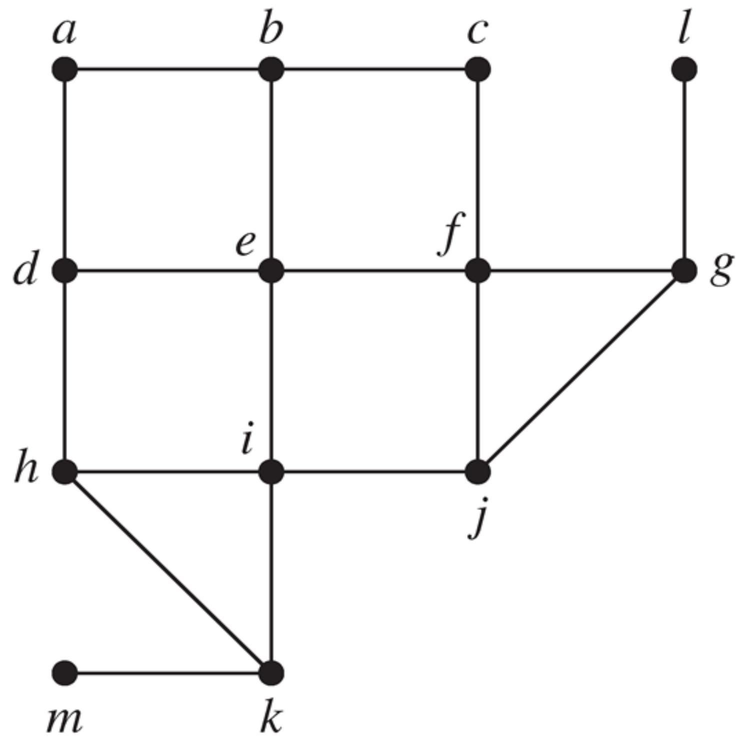
  ◇ Form a path by adding all edges incident to this vertex and the other endpoint of each of these edges

  ◇ For each vertex added at the previous level,add edge incident to this vertex, as long as it does not produce a simple circuit.

  ◇ Continue in this manner until all vertices have been added.

# Breadth-First Search

- **Example**



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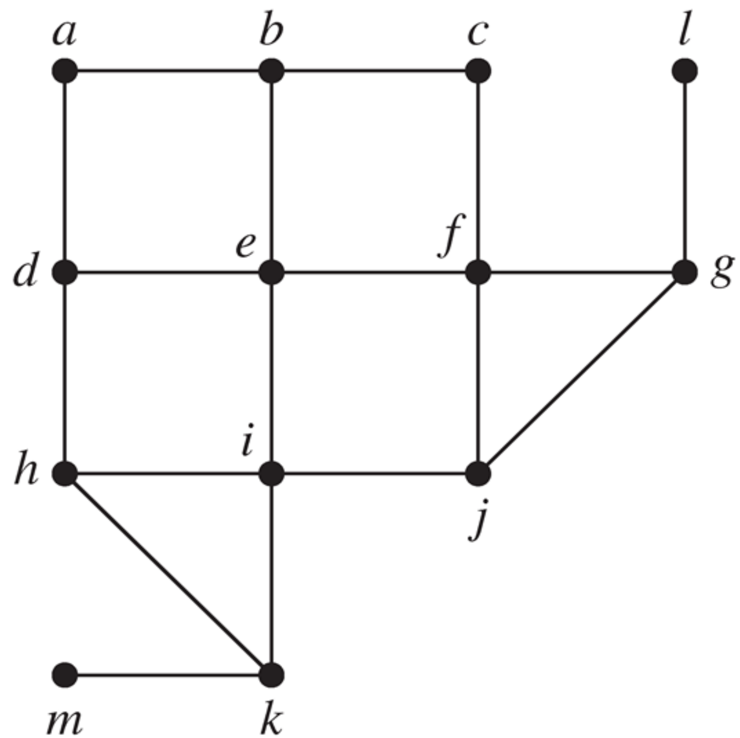


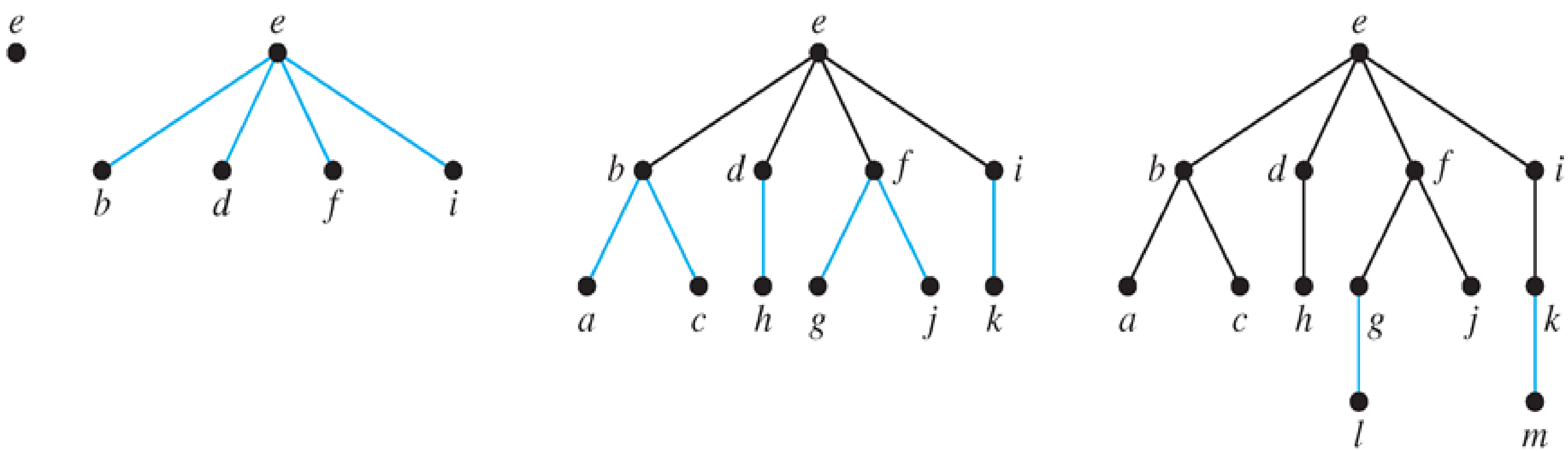

# Breadth-First Search

**procedure** *BFS*($G$: connected graph with vertices $v_1, v_2, \ldots, v_n$)
$T$ := tree consisting only of the vertex $v_1$
$L$ := empty list *visit*($v_1$)
put $v_1$ in the list $L$ of unprocessed vertices
**while** $L$ is not empty
    remove the first vertex, $v$, from $L$
    **for** each neighbor $w$ of $v$
        **if** $w$ is not in $L$ and not in $T$ **then**
            add $w$ to the end of the list $L$
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  graph coloring, sums of subsets, ...

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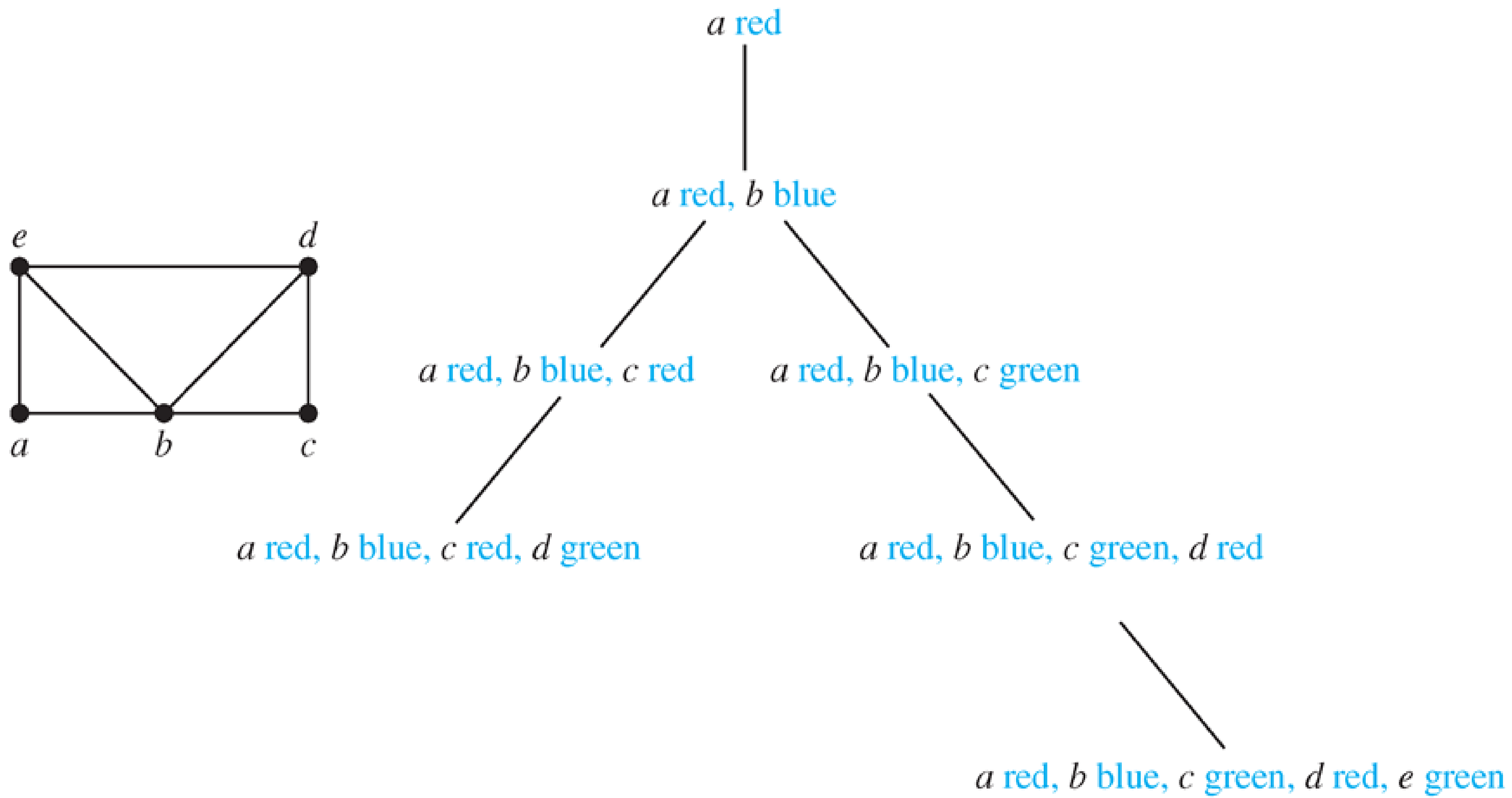
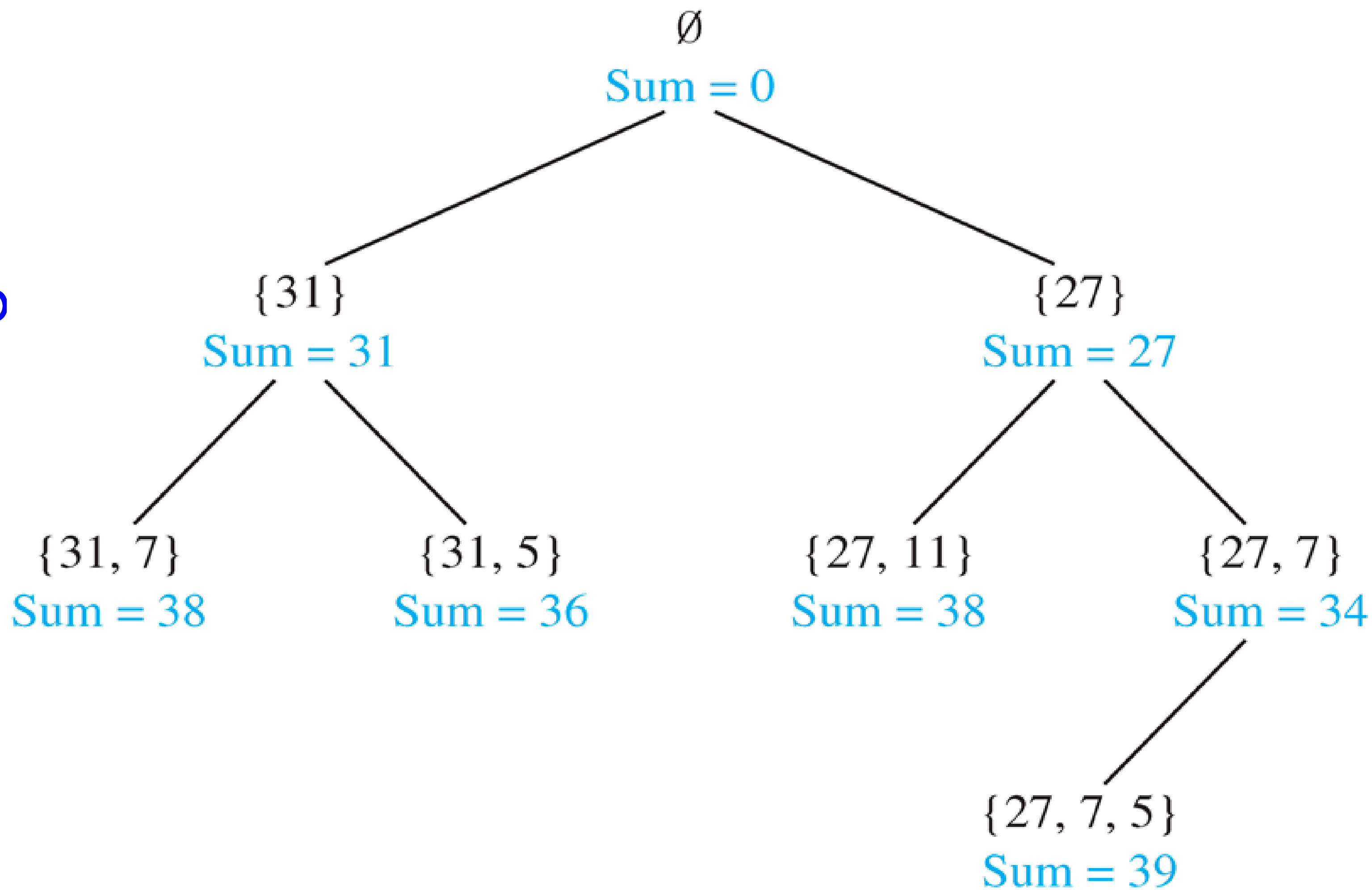


# Applications of DFS, BFS

- find [illegible]es, ...

  find

  grap




find a subset of $\{31, 27, 15, 11, 7, 5\}$ with the sum 39

# Minimum Spanning Trees

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two greedy algorithms: Prim's Algorithm, Kruscal's Algorithm

# Prim's Algorithm

**ALGORITHM 1 Prim's Algorithm.**

**procedure** *Prim*($G$: weighted connected undirected graph with $n$ vertices)
$T$ := a minimum-weight edge
**for** $i := 1$ **to** $n - 2$
  $e$ := an edge of minimum weight incident to a vertex in $T$ and not forming a simple circuit in $T$ if added to $T$
  $T := T$ with $e$ added
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time complexity: $e \log v$

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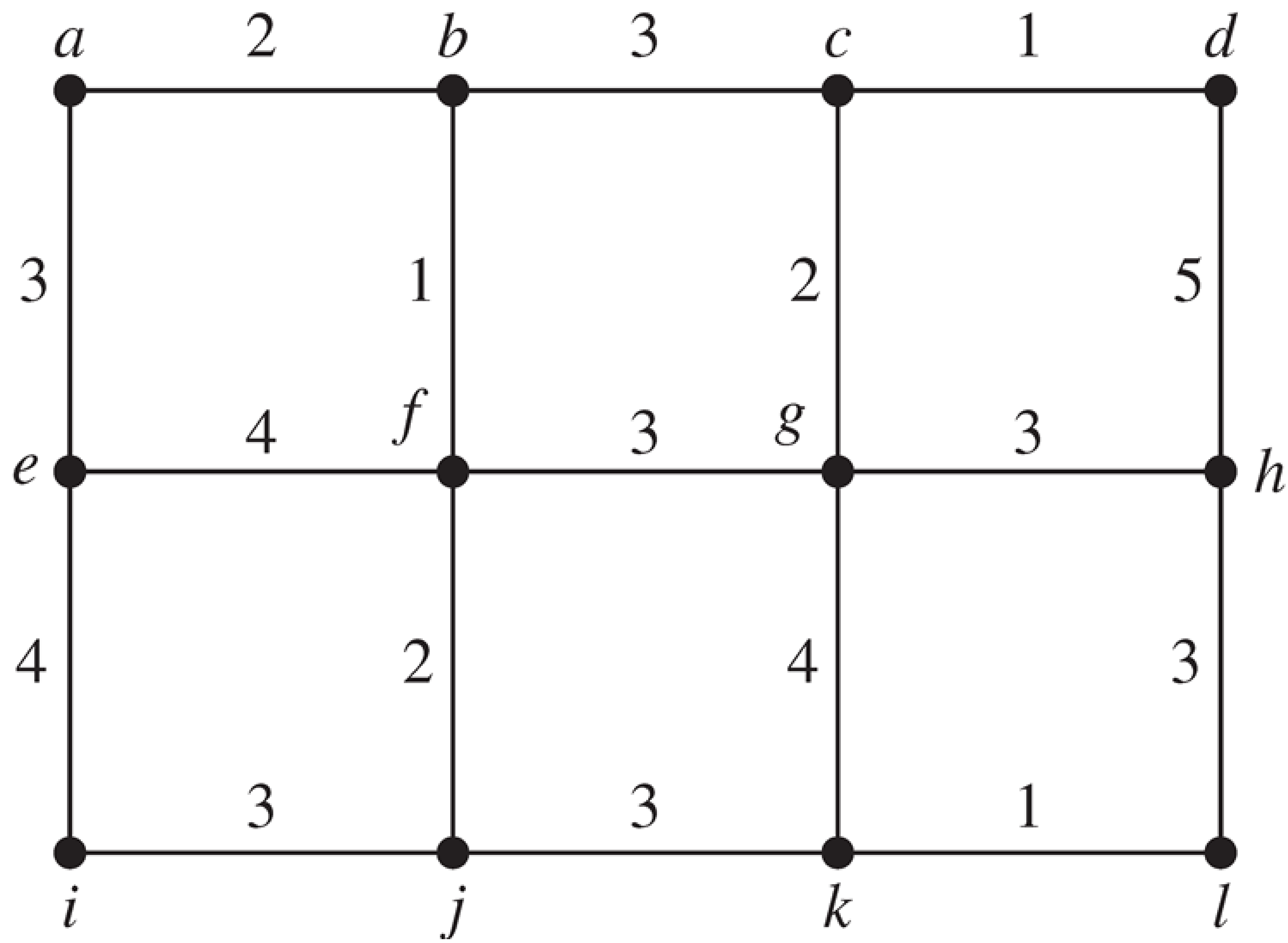
- **Example**



# Kruscal's Algorithm

**ALGORITHM 2 Kruskal's Algorithm.**

**procedure** *Kruskal*($G$: weighted connected undirected graph with $n$ vertices)
$T$ := empty graph
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    $e$ := any edge in $G$ with smallest weight that does not form a simple circuit
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see *CLRS* / *Algorithm Design*, J. Kleinberg, E. Tardos

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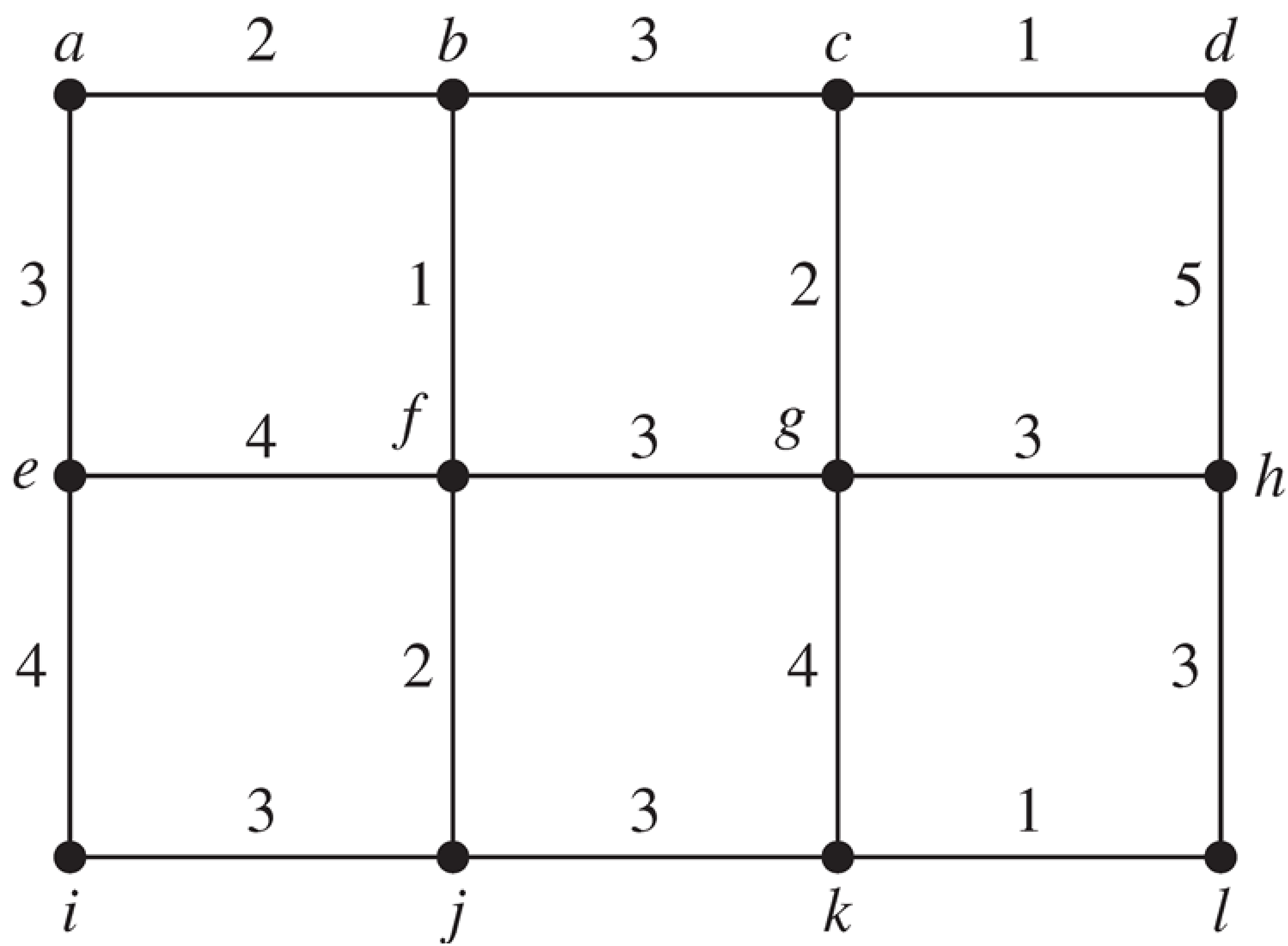

# Review

01. Propositional Logic
02. Predicate Logic
03. Mathematical Proofs
04. Sets
05. Functions
06. Complexity of Algorithms
07. Number Theory
08. Cryptography
09. Mathematical Induction
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Discrete Probability

Groups, Rings and Fields

# Logic

- Logical connectives

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  $\neg p,\ p \vee q,\ p \wedge q,\ p \oplus q,\ p \rightarrow q,\ p \leftrightarrow q$

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  De Morgan's laws, communtative laws, distributive laws, ...

# Logic

- Logical connectives

  $\neg p$, $p \vee q$, $p \wedge q$, $p \oplus q$, $p \rightarrow q$, $p \leftrightarrow q$

- Logical equivalence

  De Morgan's laws, communtative laws, distributive laws, ...

- Predicate logic

  contains variables

# Logic

- Logical connectives

  $\neg p,\ p \vee q,\ p \wedge q,\ p \oplus q,\ p \rightarrow q,\ p \leftrightarrow q$

- Logical equivalence

  De Morgan's laws, communtative laws, distributive laws, ...

- Predicate logic

  contains variables

- Quantified statements

  universal, existential, equivalence

# Methods of Proving Theorems

- Basic methods to prove theorems:
  - ◇ *direct proof*
    - $p \to q$ is proved by showing that if $p$ is true then $q$ follows
  - ◇ *proof by contrapositive*
    - show the contrapositive $\neg q \to \neg p$
  - ◇ *proof by contradiction*
    - show that $(p \wedge \neg q)$ contradicts the assumptions
  - ◇ *proof by cases*
    - give proofs for all possible cases
  - ◇ *proof of equivalence*
    - $p \leftrightarrow q$ is replaced with $(p \to q) \wedge (q \to p)$

# Function

- function?

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- function?

  one-to-one (injective) function?

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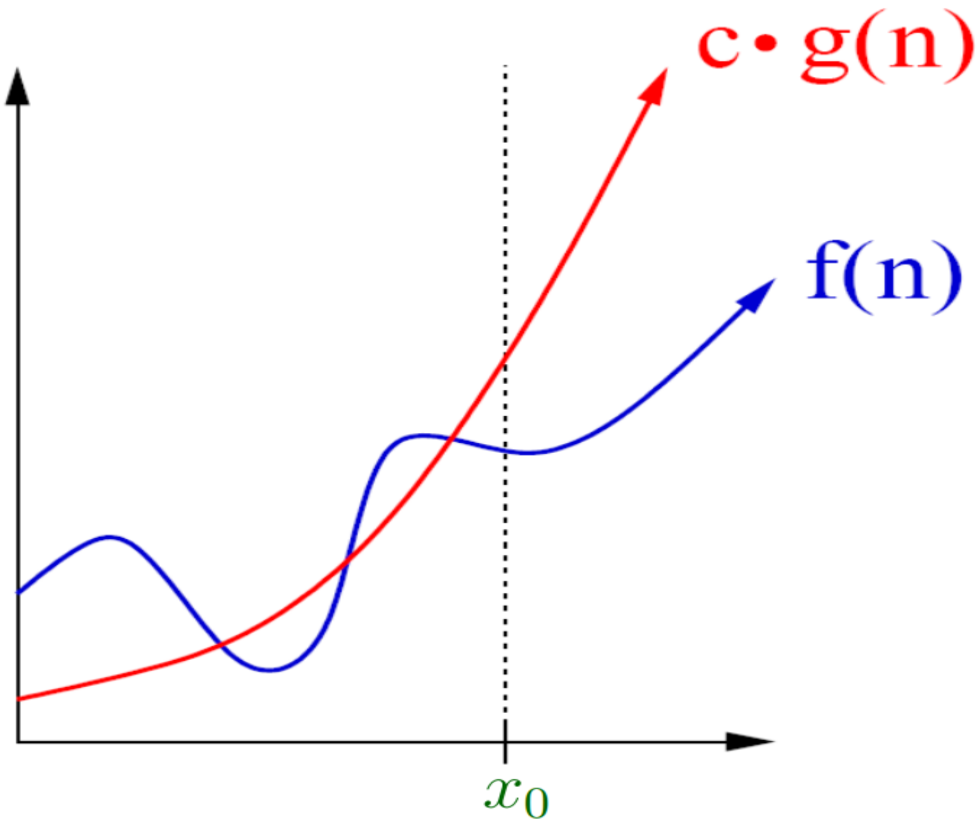
  onto (surjective) function?

  bijective function (one-to-one correspondence)?

- counting the number of such functions?

# Big-$O$ Notation

- Let $f$ and $g$ be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(n) = O(g(n))$ (reads: $f(n)$ is $O$ of $g(n)$), if there exist some positive constants $C$ and $k$ such that $|f(n)| \leq C|g(n)|$, whenever $n > k$.



# Number Theory

- Divisibility

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  - When does an inverse of $a$ modulo $m$ exist?
  - How to find inverses?

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  Back substitution

$$\begin{aligned} x &\equiv 2 \pmod 3 \\ x &\equiv 3 \pmod 5 \\ x &\equiv 2 \pmod 7 \end{aligned}$$

- Fermat's Little Theorem

# Cryptography

- Fermat's Little Theorem

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  Primitive roots, multiplicative order

# Cryptography

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  Euler's Theorem

  Primitive roots, multiplicative order

  RSA cryptosystem

  DLP, Diffie-Hellman protocol

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3. We conclude on the basis of the principle of mathematical induction that $P(n)$ is true for all $n \geq b$.

# Recurrence

- Iterating a recurrence

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  bottom up or top down

# Recurrence

- Iterating a recurrence

  bottom up or top down

  prove by induction, complexity, ...

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  **Theorem** If $N$ is a positive integer and $k$ is an integer with $1 \leq k \leq n$, then there are
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  The Binomial Theorem, Trinomial

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  Partial ordering

  **Definition** A relation $R$ on a set $A$ is called a *partial ordering* if it is reflexive, antisymmetric, and transitive.

- Basic concepts

# Graphs & Trees

- Basic concepts

  connected graph, simple graph, isomophism, chromatic number, Euler circuit, Hamilton circuit, shortest path, bipartite graph, complete graph, special graphs ($K_n$, $K_{m,n}$, $C_n$, $W_n$ ), m-ary tree, tree traversal, spanning tree ...

# Next Lecture

- the last lecture ...