南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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Assignment #1

Please collect your assignments!

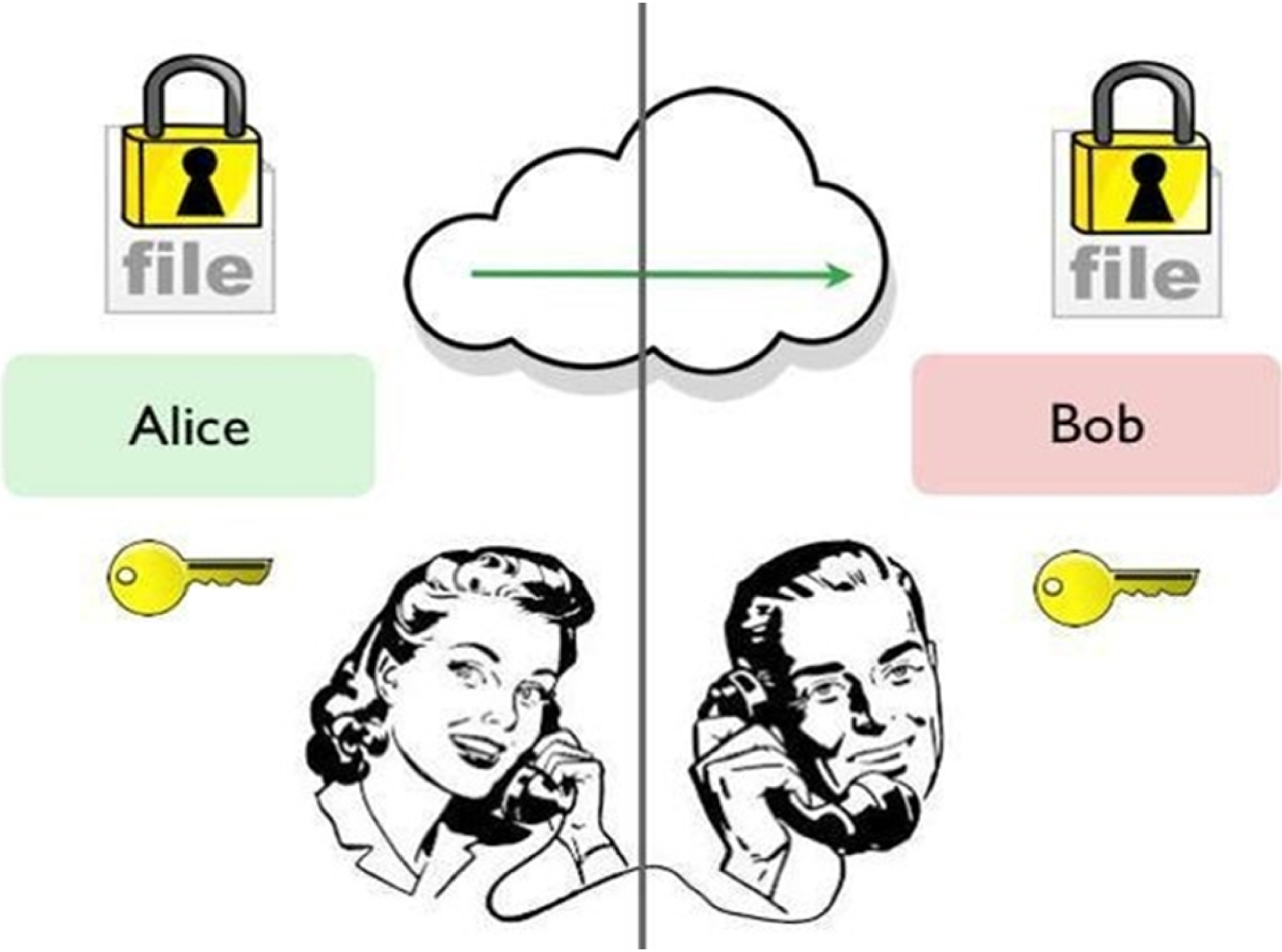
Recall

- **Theorem (Fermat's little theorem)** : Let p be a prime, and let x be an integer such that $x \not\equiv 0 \bmod p$. Then
$$x^{p-1} \equiv 1 \pmod{p}.$$

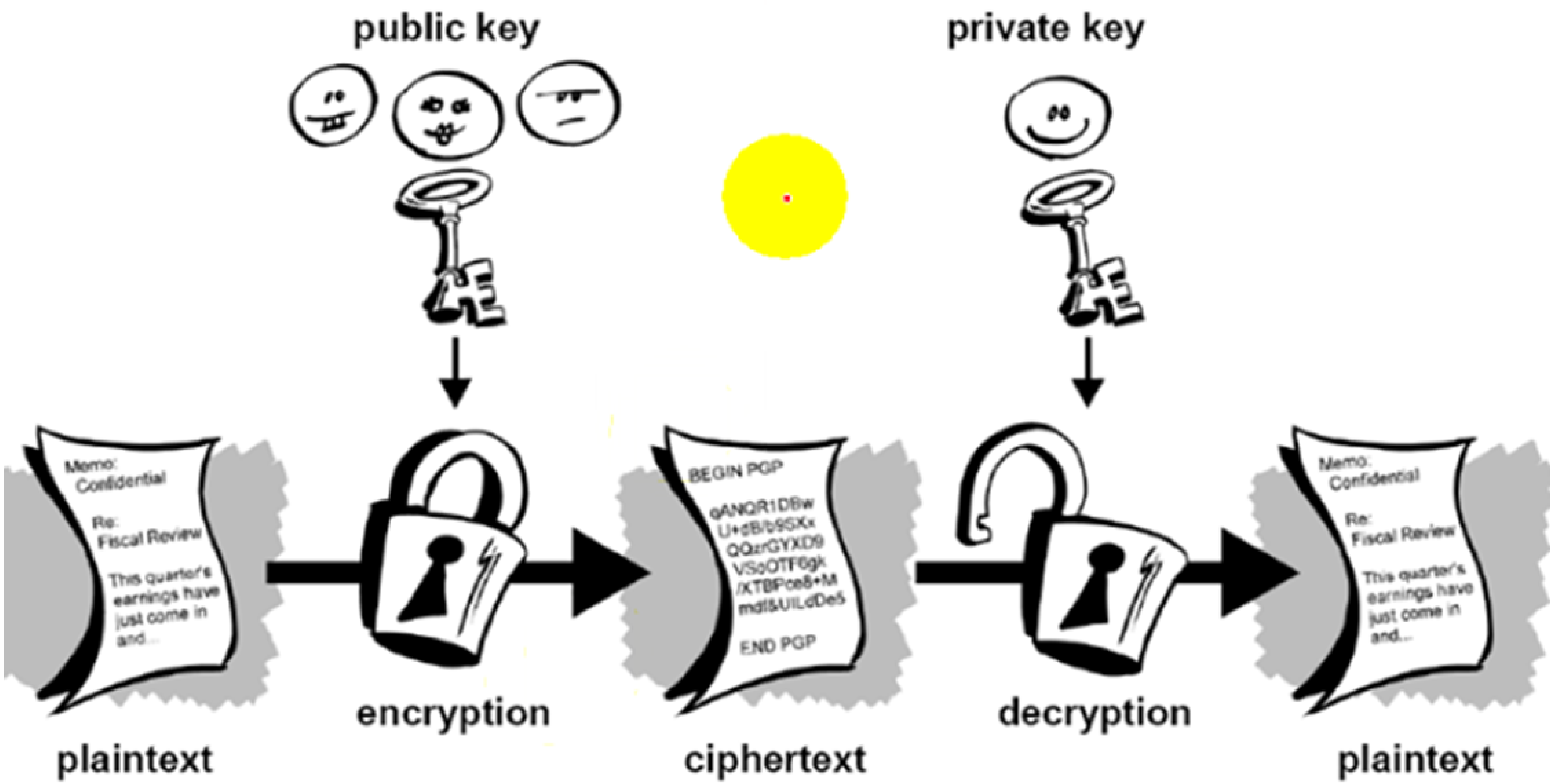
- **Theorem (Euler's theorem)** : Let n be a positive integer, and let x be an integer such that $\gcd(x, n) = 1$. Then
$$x^{\phi(n)} \equiv 1 \pmod{n}.$$

- **Theorem** Let $a \in \mathbb{Z}$ and $n \in \mathbb{N}$ with $\gcd(a, n) = 1$. Then $\mathrm{ord}_n(a)$ exists and divides $\phi(n)$.

Symmetric Cryptography

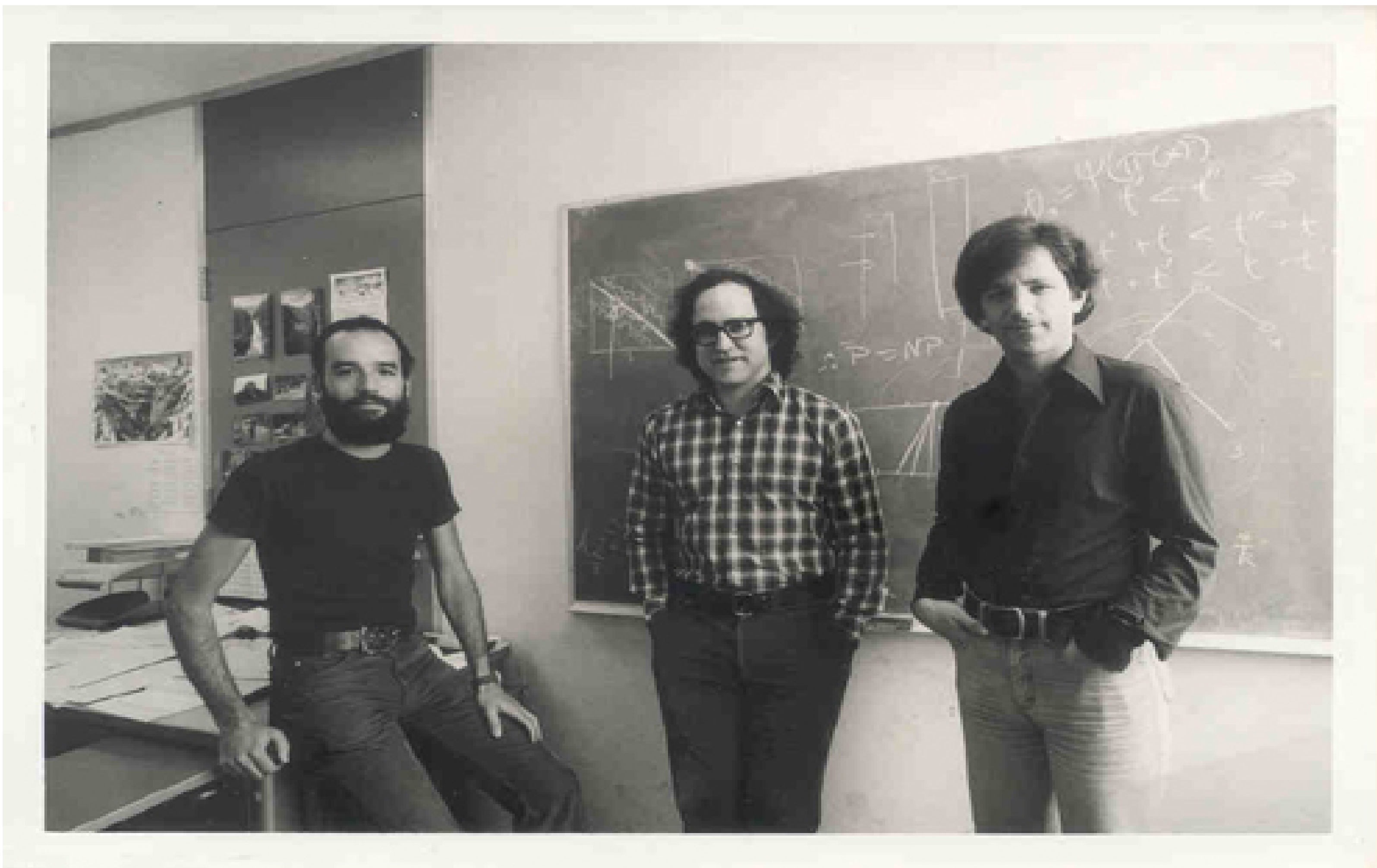


Asymmetric Cryptography



Public-Key Cryptosystems

R. Rivest, A. Shamir, L. Adleman, “A Method for Obtaining Digital Signatures and Public-Key Cryptosystems”, Communications of the ACM, vol. 21-2, pages 120-126, 1978.



RSA Public-Key Cryptosystem

Pick two large primes, p and q. Let $n = pq$, then $\phi(n) = (p-1)(q-1)$. Encryption and decryption keys e and d are selected such that

- $\gcd(e, \phi(n)) = 1$
- $ed \equiv 1 \pmod{\phi(n)}$

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- $C = M^e \bmod n$ (RSA **encryption)**
- $M = C^d \bmod n$ (RSA **decryption)**

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- **Theorem 12** (Correctness) : Let p and q be two odd primes, and define $n = pq$. Let e be relatively prime to $\phi(n)$ and let d be the multiplicative inverse of e modulo $\phi(n)$. For each integer x such that $0 \leq x < n$,

$$x^{ed} \equiv x \pmod{n}.$$

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$\mathcal{Q}$: How to prove this?

RSA Public-Key Cryptosystem: Example

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	5	11	55	40	7	23

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Encryption: $M = 28, C = M^7 \bmod 55 = 52$

Decryption: $M = C^{23} \bmod 55 = 28$

RSA Public-Key Cryptosystem: Parameters

Parameters: $p \quad q \quad n \quad \phi(n) \quad e \quad d$

Public key: (e, n)

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p, q, $\phi(n)$ must be kept secret!

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Trying all possible private keys.

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Attack: Determine $\phi(n)$ directly.

Attack: Determine d directly.

Comment: It is believed that determining $\phi(n)$ is equivalent to factoring n. Meanwhile, determining d given e and n, appears to be at least as time-consuming as the integer factoring problem.

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$\mathcal{Q}$: Consider the RSA system, where $n = pq$ is the modulus. Let (e, d) be a key pair for the RSA. Define

$$\lambda(n) = \mathrm{lcm}(p - 1, q - 1)$$

and compute $d' = e^{-1} \bmod \lambda(n)$. Will decryption using d' instead of d still work?

Using RSA for Digital Signature

$S = M^d \bmod n$ (RSA **signature)**

$M = S^e \bmod n$ (RSA **verification)**

Why?

The Discrete Logrithm

- **The discrete logarithm** of an integer y to the base b is an integer x, such that

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Discrete Logarithm Problem:

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This is very hard!!!

El Gamal Encryption

- **Setup** Let p be a prime, and g be a generator of $\mathbb{Z}_p$. The private key x is an integer with $1 < x < p - 2$. Let $y = g^x \bmod p$. The public key for *El Gamal encryption* is (p, g, y).

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  El Gamal Encryption: Pick a random integer k from $\mathbb{Z}_{p-1}$,

$$a = g^k \bmod p$$
$$b = My^k \bmod p$$

  The ciphertext C consists of the pair (a, b).

  El Gamal Decryption:

$$M = b(a^x)^{-1} \bmod p$$

Using El Gamal for Digital Signature

$$a = g^k \bmod p$$
$$b = k^{-1}(M - xa) \bmod (p-1)$$

(El Gamal **signature**)

$$y^a a^b \equiv g^M \pmod p$$

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$\mathcal{Q}$: How to verify it?

An Example

Choose $p = 2579$, $g = 2$, and $x = 765$. Hence $y = 2^{765} \bmod 2579 = 949$.

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Encryption: Let $M = 1299$ and choose a random $k = 853$,

$$\begin{aligned}(a, b) &= (g^k \bmod p, My^k \bmod p)\\ &= (2^{853} \bmod 2579, 1299 \cdot 949^{853} \bmod 2579)\\ &= (435, 2396).\end{aligned}$$

Decryption:

$$M = b(a^x)^{-1} \bmod p = 2396 \times (435^{765})^{-1} \bmod 2579 = 1299.$$

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Question 2: Given a ciphertext (a, b), is it feasible to derive the plaintext M?

Attack 1: Use $M = by^{-k}$. However, k is randomly picked.

Attack 2: Use $M = b(a^x)^{-1} \bmod p$, but x is secret.

Diffie-Hellman Key Exchange Protocol

User A

Generate random
$X_A < p$
calculate
$Y_A = \alpha^{X_A} \bmod p$

Calculate
$k = (Y_B)^{X_A} \bmod p$

$$\xrightarrow{Y_A}$$

$$\xleftarrow[Y_B]{}$$

User B

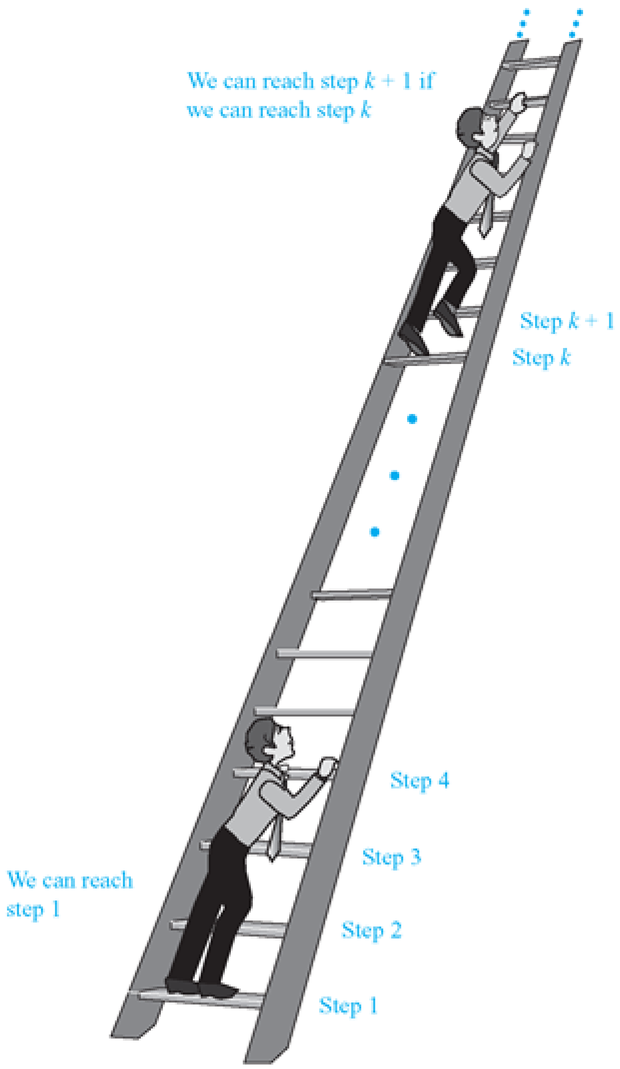
Generate random
$X_B < p$
Calculate
$Y_B = \alpha^{X_B} \bmod p$

Calculate
$k = (Y_A)^{X_B} \bmod p$

Announcements

- **Homework assignment** 3
  - ◇ P245 Ex. 37, 38, 39, P255 Ex. 2, 26, P272 Ex. 11*, 12, P273 Ex. 42, P274 Ex. 50, 55, P284 Ex. 7*, P285 Ex. 22, P286 Ex. 39, P305 Ex. 23, 28, 30
  - ◇ Due on *Nov. 7th, 2017 at the beginning of class*
  - ◇ Please try you best to slove problems marked with $*$
  - ◇ Please write your homeowrk **neatly**, as a courtesy to graders.

Mathematical Induction



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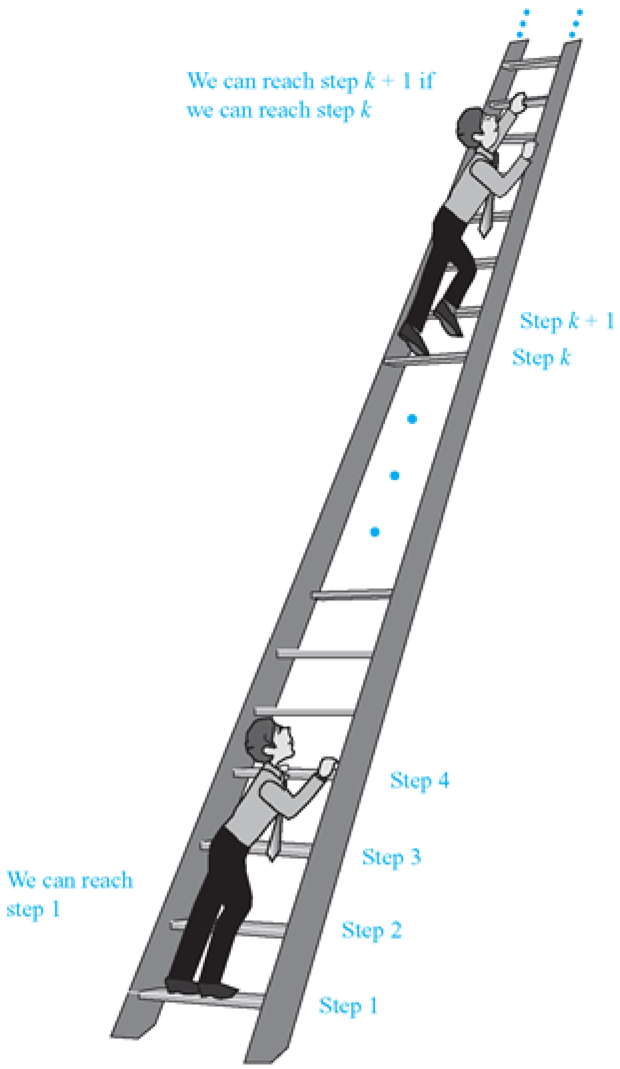

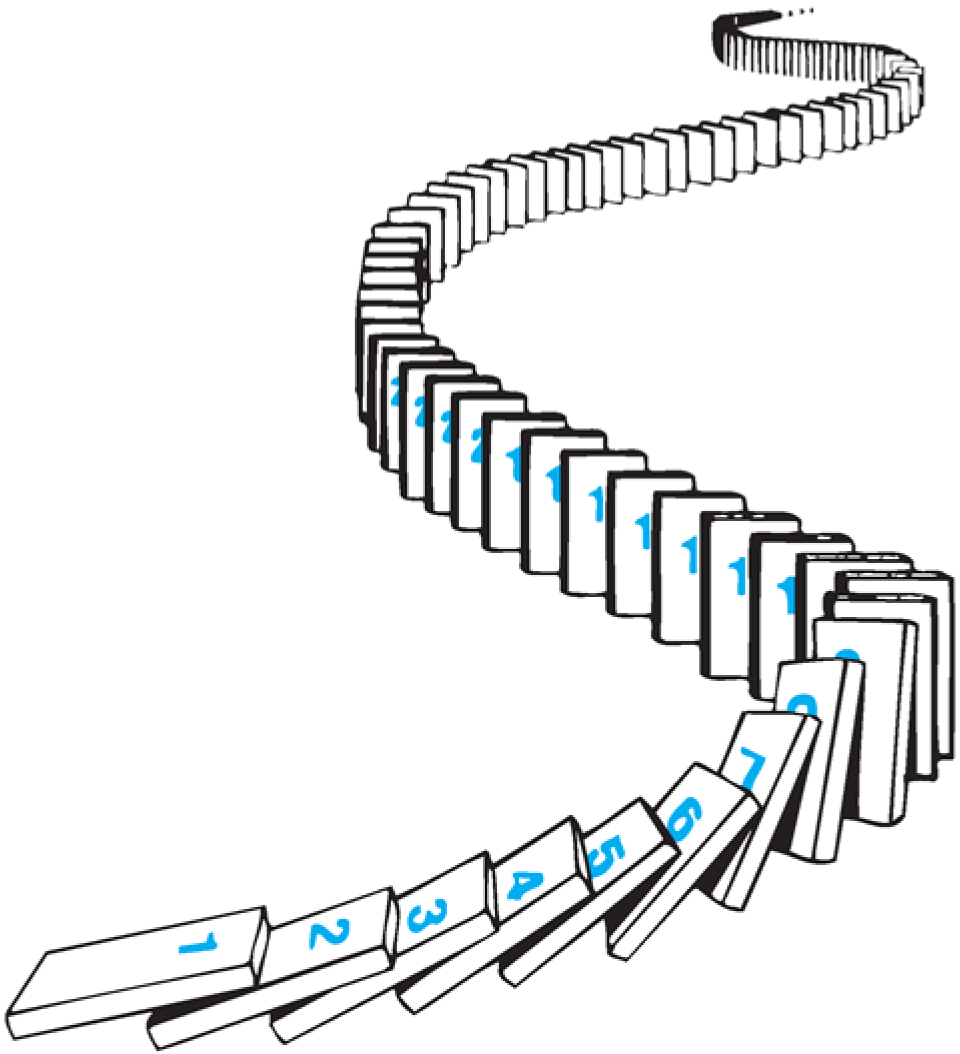

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- We conclude by distinguishing between the *weak principle* of mathematical induction and the *strong principle* of mathematical induction.

  The *strong principle* can actually be derived from the *weak principle*.

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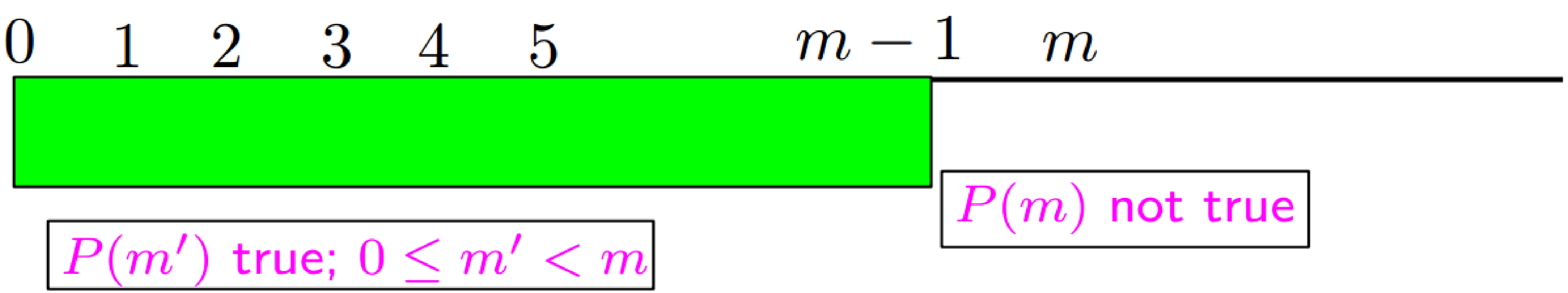

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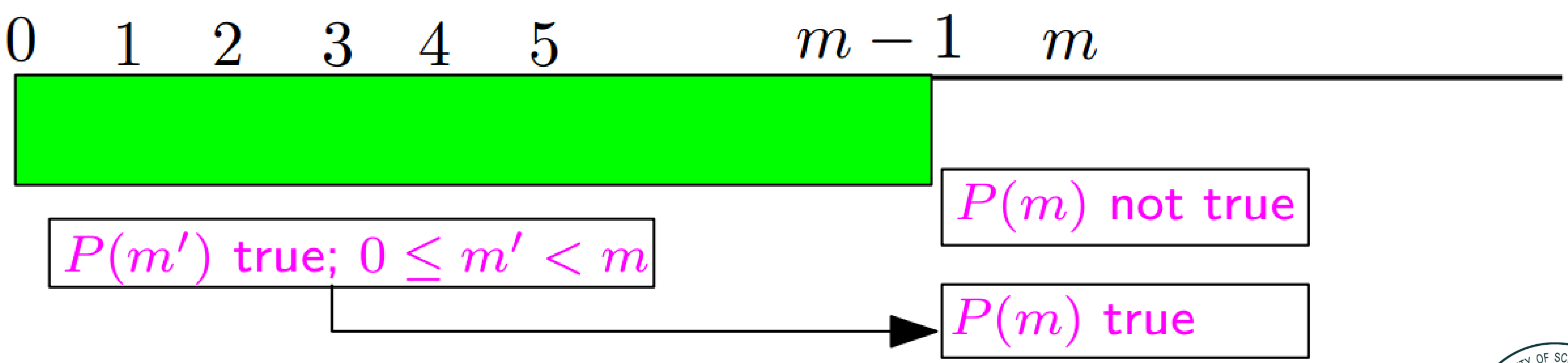

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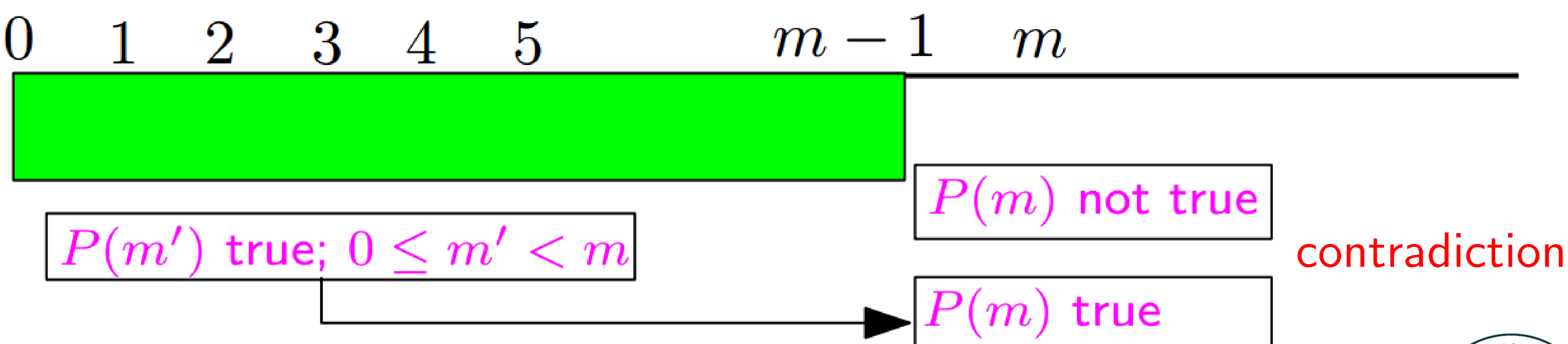

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◇ The smallest counterexample n is larger than 0

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  ◇ Therefore, $(*)$ holds for all positive integers n.

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  The key step was proving that

$$P(n-1) \to P(n)$$

where $P(n)$ is the statement

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

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Let $P(n)$ – $2^{n+1} \geq n^2 + 2$. We start by assuming that the statement

$$\forall n \in N \;\; P(n)$$

is false.

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is false.

When a for all quantifier is false, there must be some n for which it is false. Let n be the smallest nonnegative integer for which $2^{n+1} \not\geq n^2 + 2$.

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  Then setting $i = n - 1$ gives
  $$2^{(n-1)+1} \geq (n-1)^2 + 2.$$

  or
  $$(*) \quad 2^n \geq n^2 - 2n + 1 + 2 = n^2 - 2n + 3$$

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$$2^{n+1} = 2 \cdot 2^n \geq 2 \cdot (n^2 - 2n + 3) = 2n^2 - 4n + 6.$$

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  Thus, we write
  $$\begin{aligned} 2^{n+1} &\geq 2n^2 - 4n + 6 \\ &= (n^2 + 2) + (n^2 - 4n + 4) \\ &= n^2 + 2 + (n - 2)^2 \\ &\geq n^2 + 2. \end{aligned}$$

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Example 2

- Let $P(n)$ – $2^{n+1} \geq n^2 + 2$

  We just showed that

  (a) $P(0)$ is true

  (b) if $n > 0$, then $P(n-1) \rightarrow P(n)$

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  - ◇ Suppose there is some n for which $P(n)$ is false $(*)$
  - ◇ Let n be the smallest counterexample

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  ◇ Thus, $P(n)$ is true for all $n \in N$.

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  Since $P(n-1) \rightarrow P(n)$, we see that

  $P(0)$ implies $P(1)$, $P(1)$ implies $P(2)$, ...

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Let $P(n)$ – $2^{n+1} \geq n^2 + 3$ Base Step

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Inductive Hypothesis

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Inductive Step

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By mathematical induction, $\forall n > 2,\ 2^{n+1} \geq n^2 + 3$.

Inductive Conclusion

Next Lecture

- induction II, ...