DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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Bipartite Graphs and Matchings

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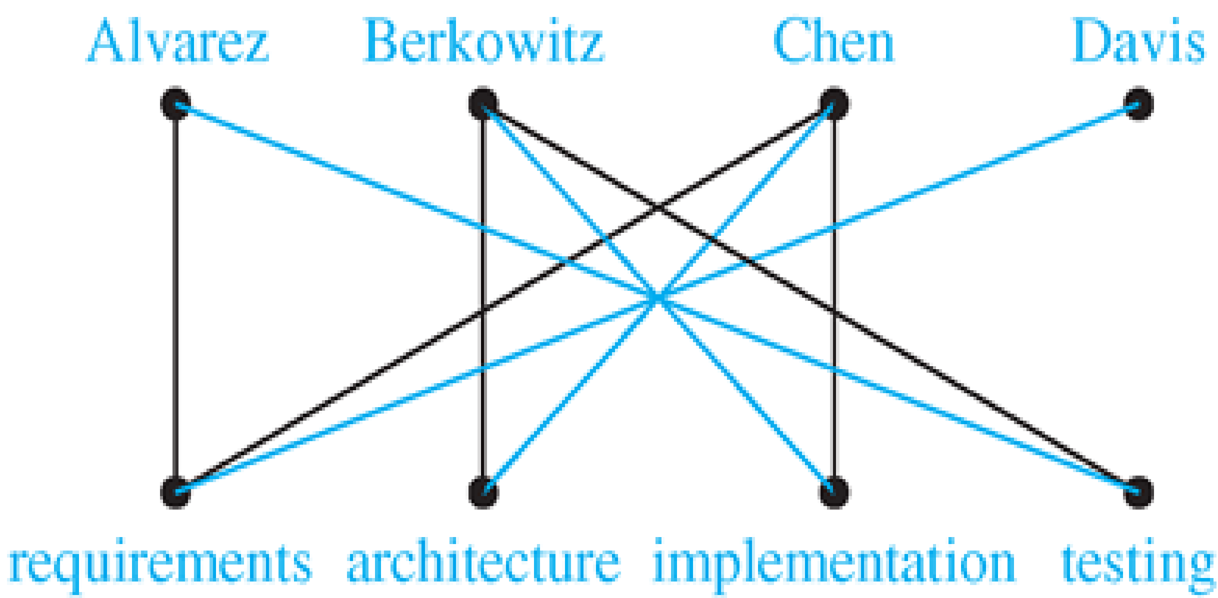

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Then, for every vertex $v \in A$, there is an edge in M connecting v to a vertex in V_2. Thus, there are at least as many vertices in V_2 that are neighbors of vertices in V_1 as there are vertices in V_1.

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Inductive hypothesis: Let k be a positive integer. If $G = (V, E)$ is a bipartite graph with bipartition (V_1, V_2), and $|V_1| = j \leq k$, then there is a complete mathching M from V_1 to V_2 whenever the condition that $|N(A)| \geq |A|$ for all $A \subseteq V_1$ is met.

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Case (i): For all integers j with $1 \leq j \leq k$, the vertices in every set of j elements from W_1 are adjacent to at least $j + 1$ elements of W_2

Case (ii): For some integer j with $1 \leq j \leq k$, there is a subset W_1' of j vertices such that there are exactly j neighbors of these vertices in W_2

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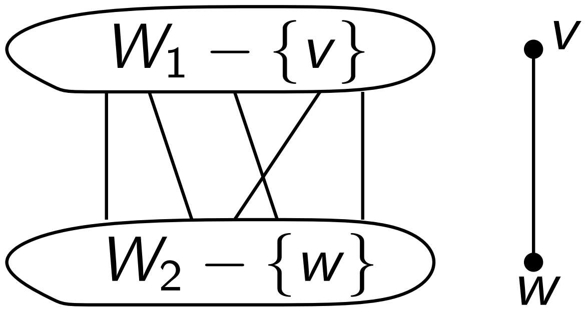

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If not, there is a subset B of t vertices with $1 \leq t \leq k + 1 - j$ s.t. $|N(B)| < t$.

Subgraphs

- **Definition** A *subgraph of a graph* $G = (V, E)$ is a graph (W, F), where $W \subseteq V$ and $F \subseteq E$. A subgraph H of G is a *proper subgraph* of G if $H \neq G$.

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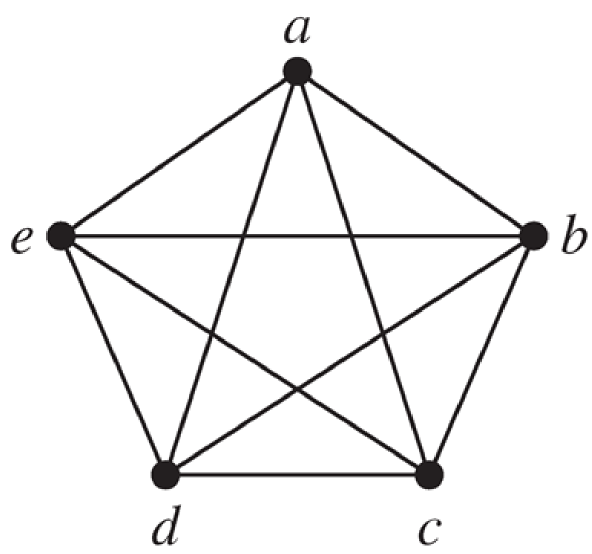

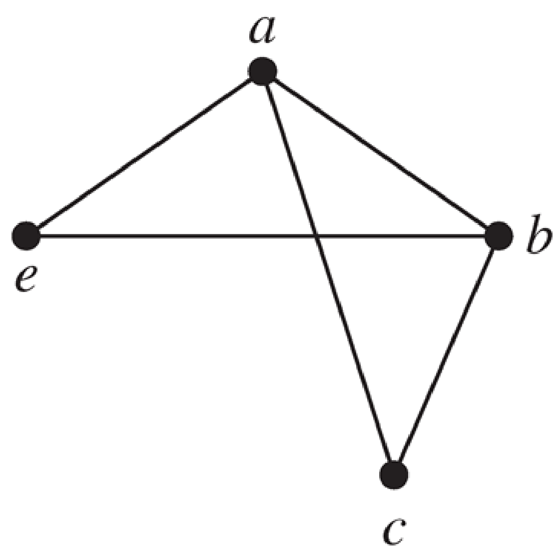

Union of Graphs

- **Definition** The *union of two simple graphs* $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$, denoted by $G_1 \cup G_2$.

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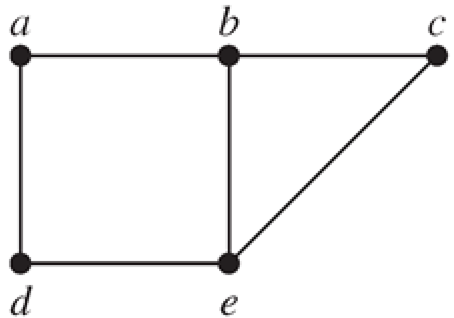
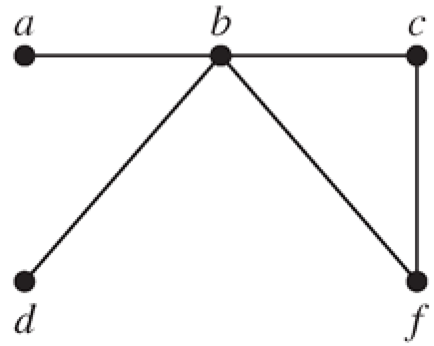


G_1



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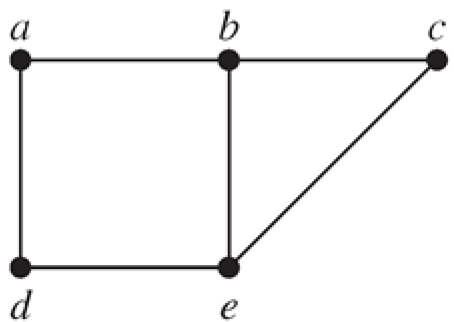

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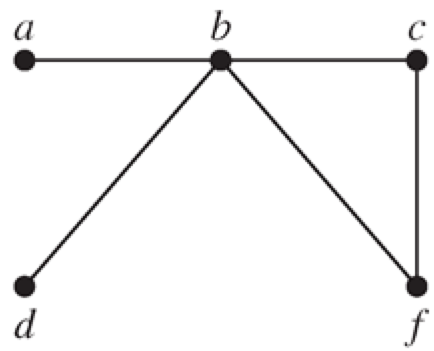
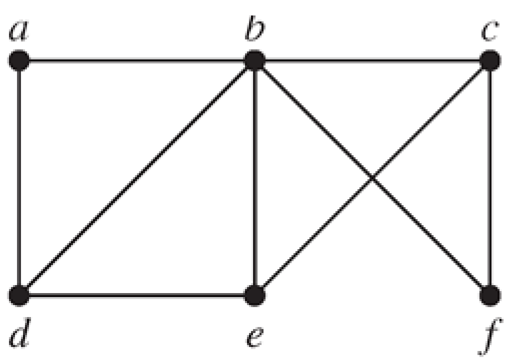


G_2



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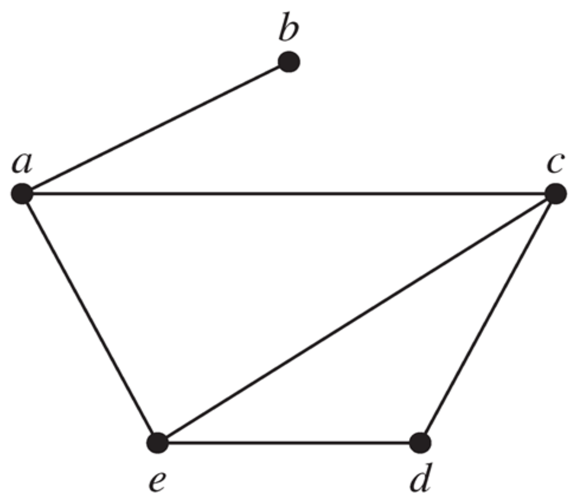

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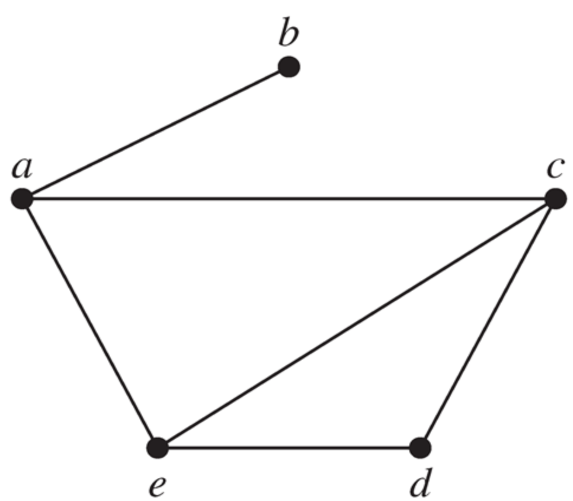

TABLE 1 An Adjacency List for a Simple Graph.

Vertex	*Adjacent Vertices*
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

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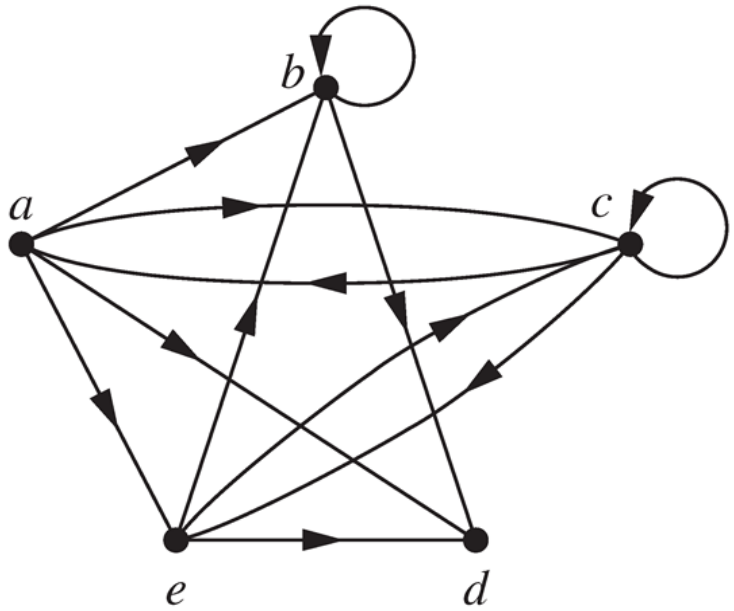

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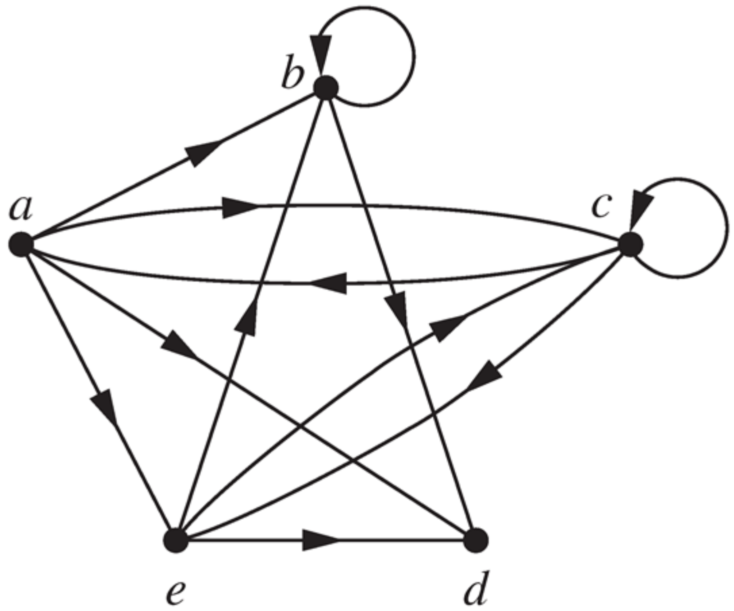

TABLE 2 An Adjacency List for a Directed Graph.

Initial Vertex	*Terminal Vertices*
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

Adjacency Matrices

- **Definition** Suppose that $G = (V, E)$ is a simple graph with $|V| = n$. Arbitrarily list the vertices of G as $v_1, v_2, \ldots, v_n$. The *adjacency matrix* $\mathbf{A}_G$ of G, is the $n \times n$ zero-one matrix with 1 as its (i, j)-th entry when v_i and v_j are adjacent, and 0 as its (i, j)-th entry when they are not adjacent.

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$$\mathbf{A}_G = [a_{ij}]_{n\times n}, \text{ where}$$

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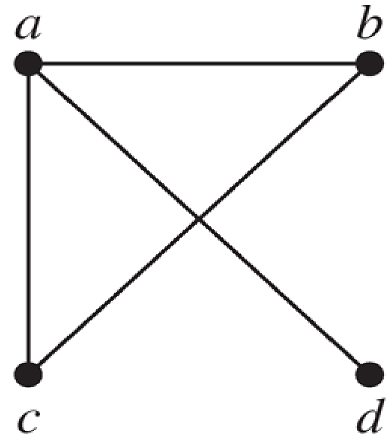

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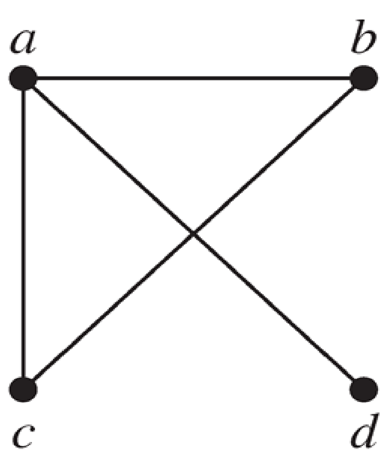

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

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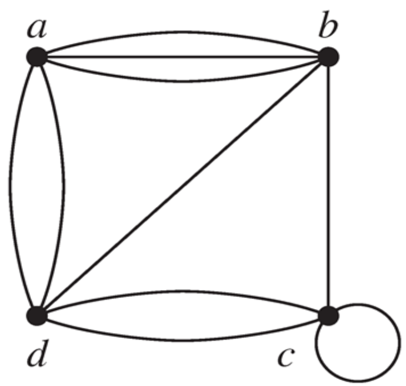

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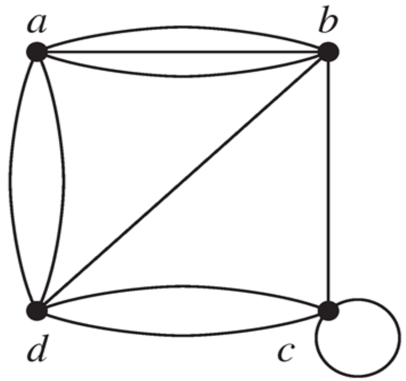

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

Incidence Matrices

- **Definition** Let $G = (V, E)$ be an undirected graph with vertices $v_1, v_2, \ldots, v_n$ and edges $e_1, e_2, \ldots, e_m$. The *incidence matrix* with respect to the ordering of V and E is the $n \times m$ matrix $\mathbf{M} = [m_{ij}]_{n\times m}$, where

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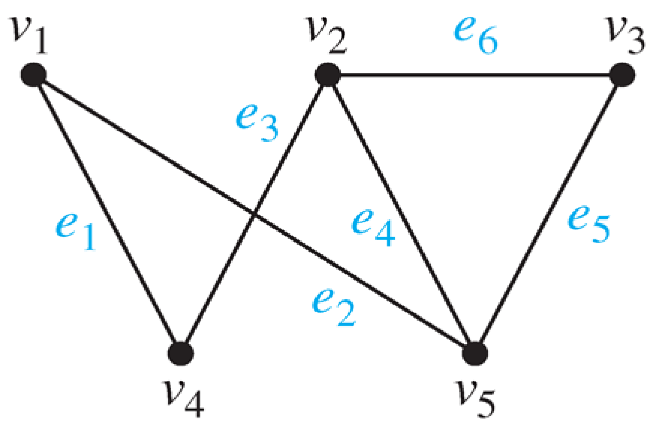

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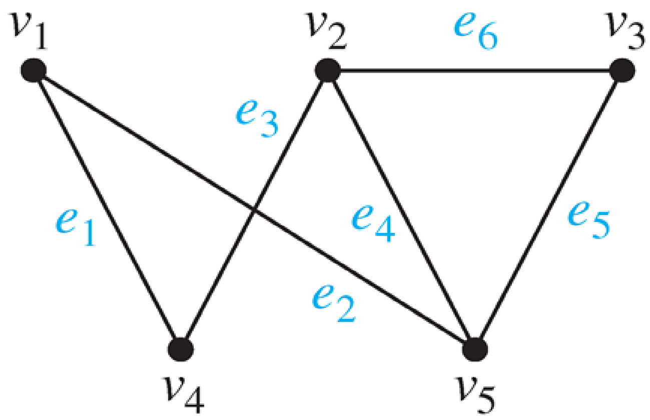

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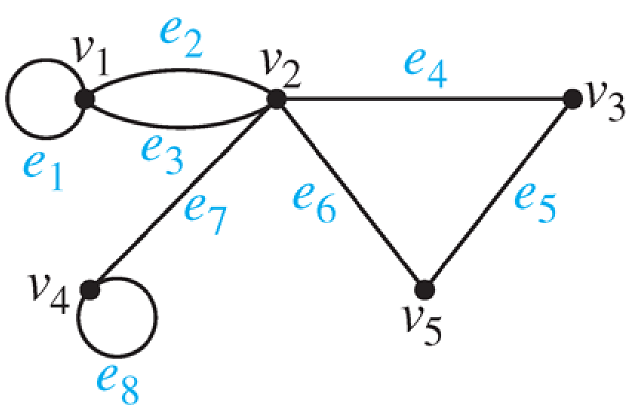

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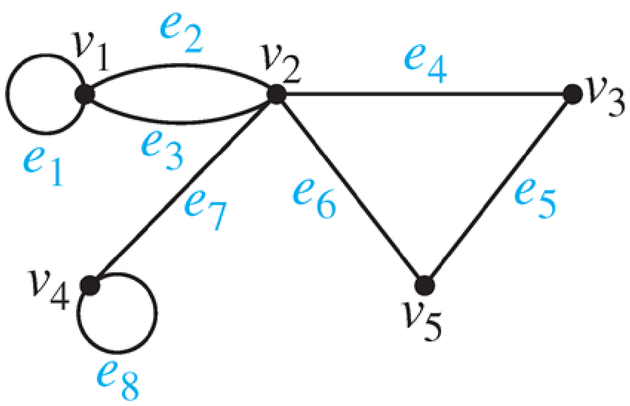

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Isomorphism of Graphs

- **Definition** The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there is a one-to-one and onto function from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function is called an *isomorphism*.

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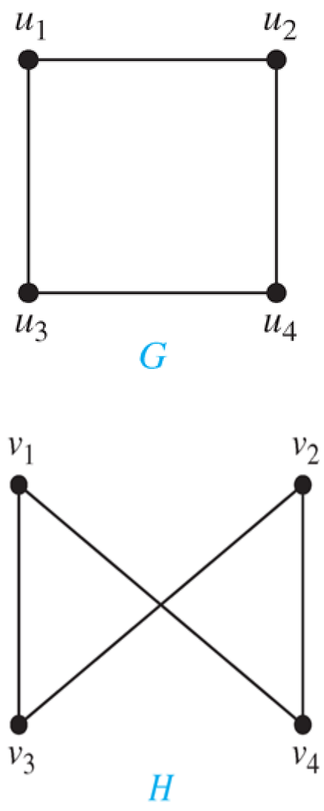

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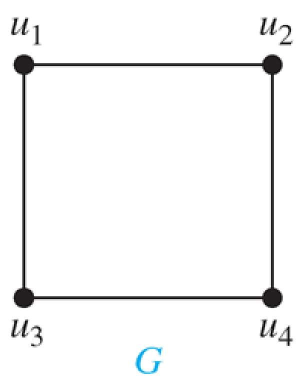

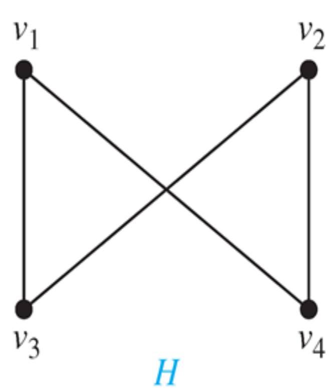

Are the two graphs isomorphic?

Define a one-to-one correspondence:
$f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, and $f(u_4) = v_2$

Isomorphism of Graphs

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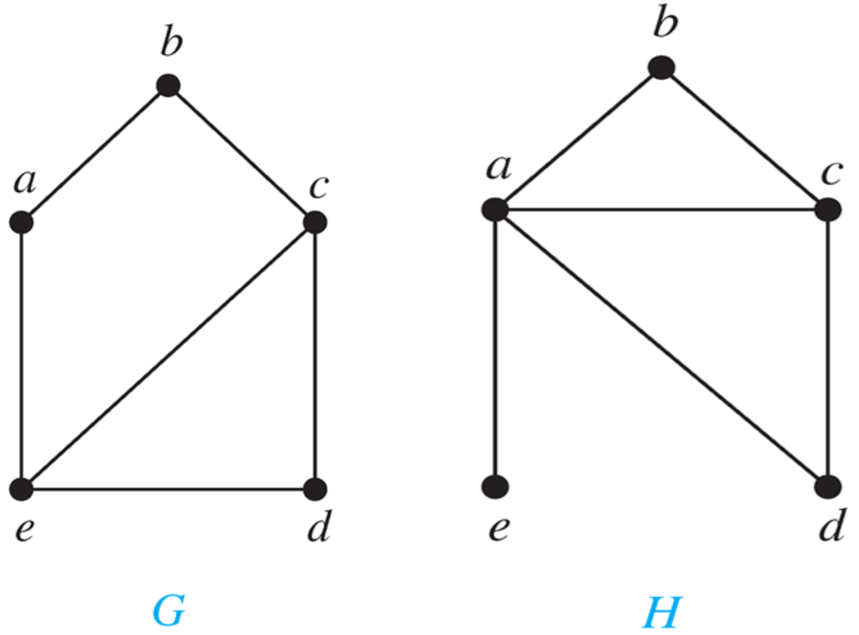
- Useful graph invariants include the number of vertices, number of edges, degree sequence, etc.

Isomorphism of Graphs

- **Example** Determine whether these two graphs are isomorphic.



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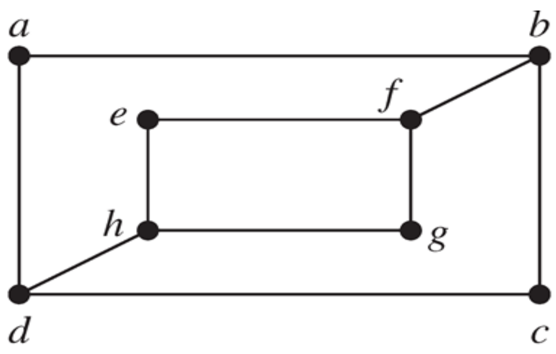

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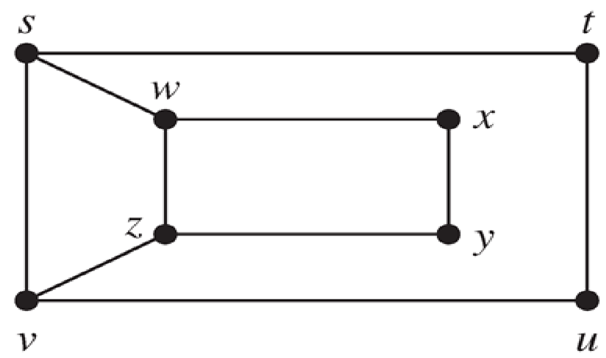

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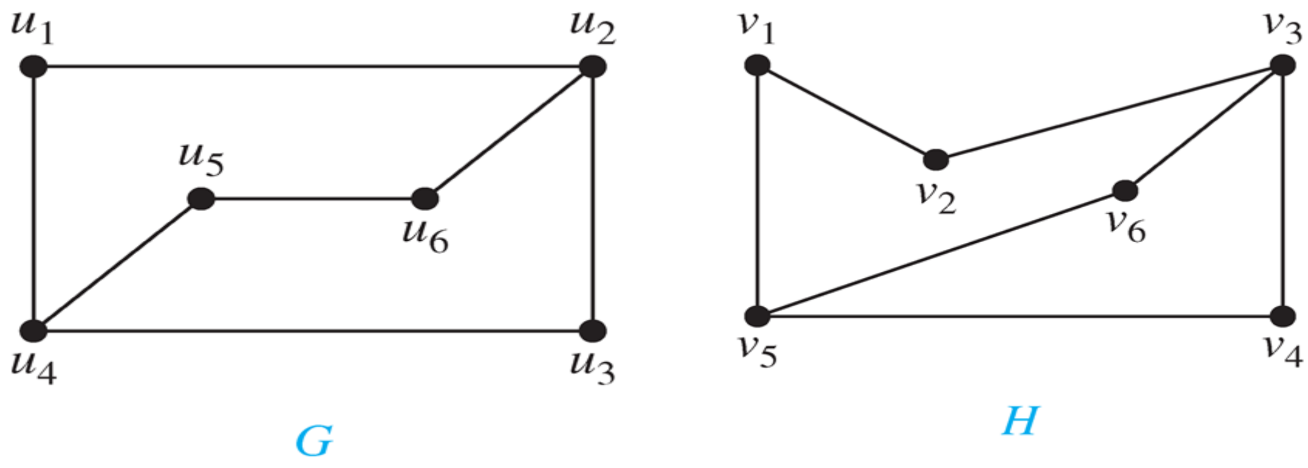

Path

- **Definition** Let n be a nonnegative integer and G an undirected graph. A *path of length n* from u to v in G is a sequence of n edges $e_1, e_2, \ldots, e_n$ of G for which there exists a sequence $x_0 = u, x_1, \ldots, x_{n-1}, x_n = v$ of vertices such that e_i has the endpoints x_{i-1} and x_i for $i = 1, \ldots, n$. The path is a *circuit* if it begins and ends at the same vertex, i.e., if $u = v$ and has length greater than zero. A path or circuit is *simple* if it does not contain the same edge (vertex) more than once.

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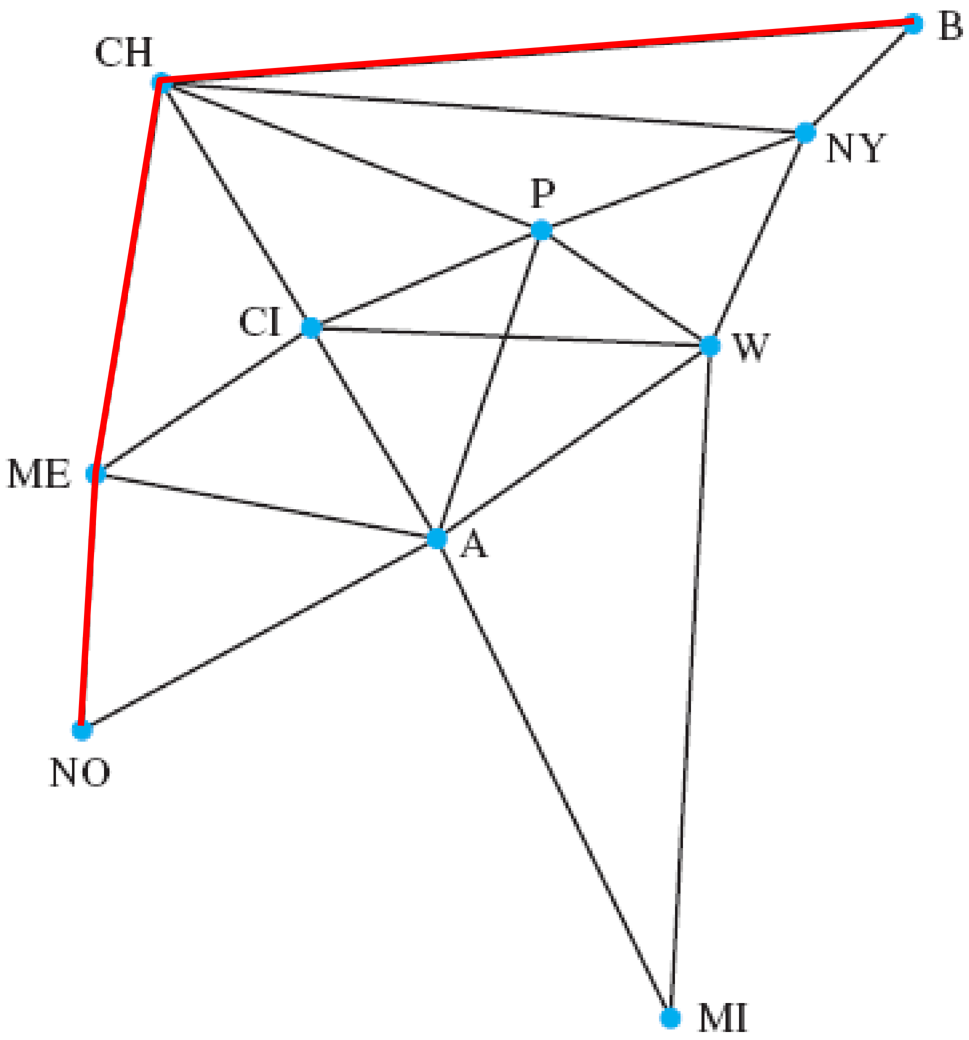
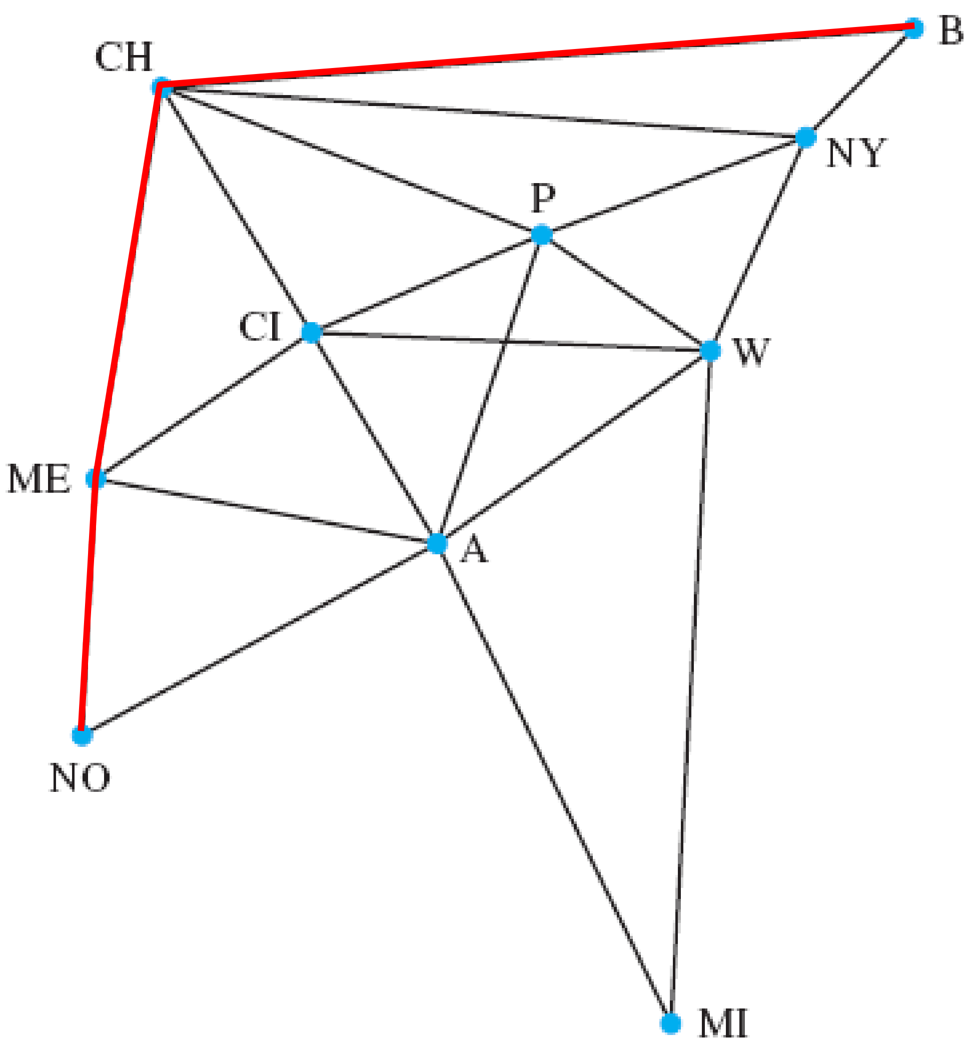
Length of a path $=$ $\#$ of edges on path

Path



Path

Path from Boston to New Orleans is B, CH, ME, NO



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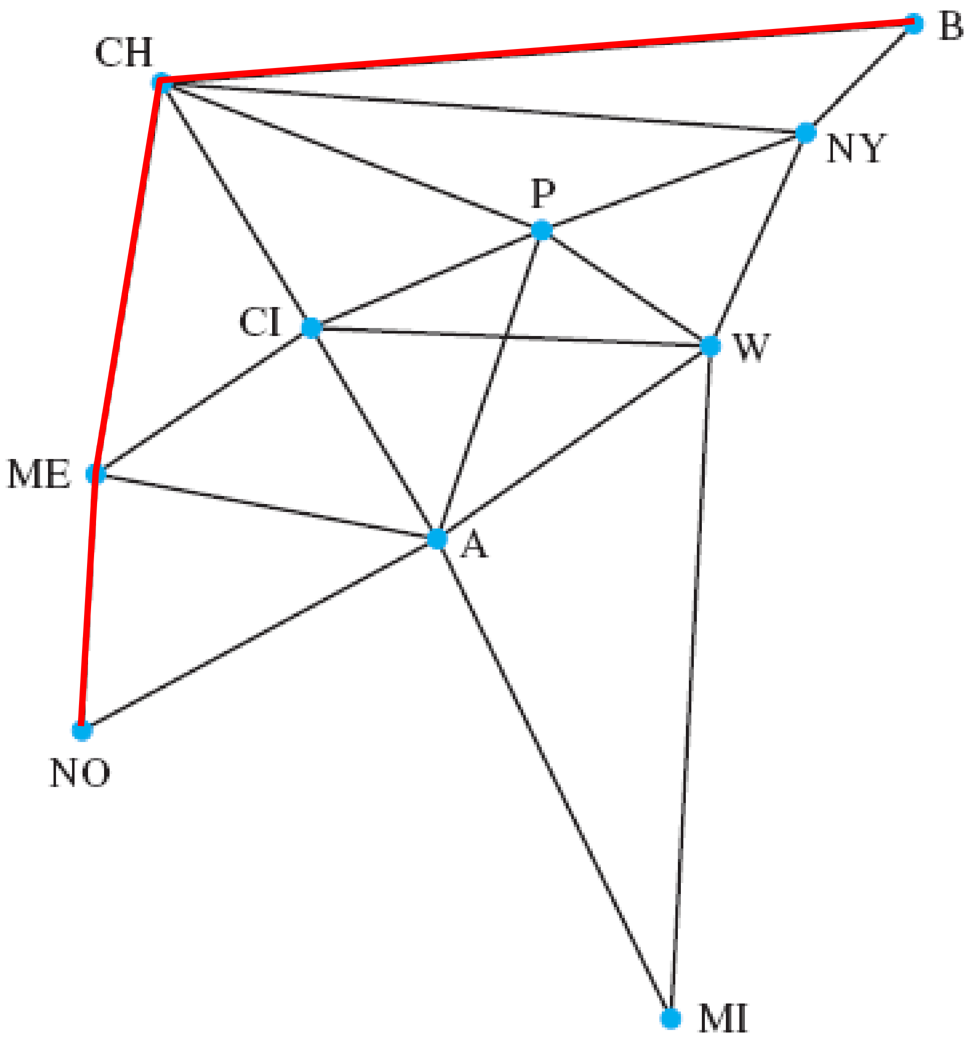
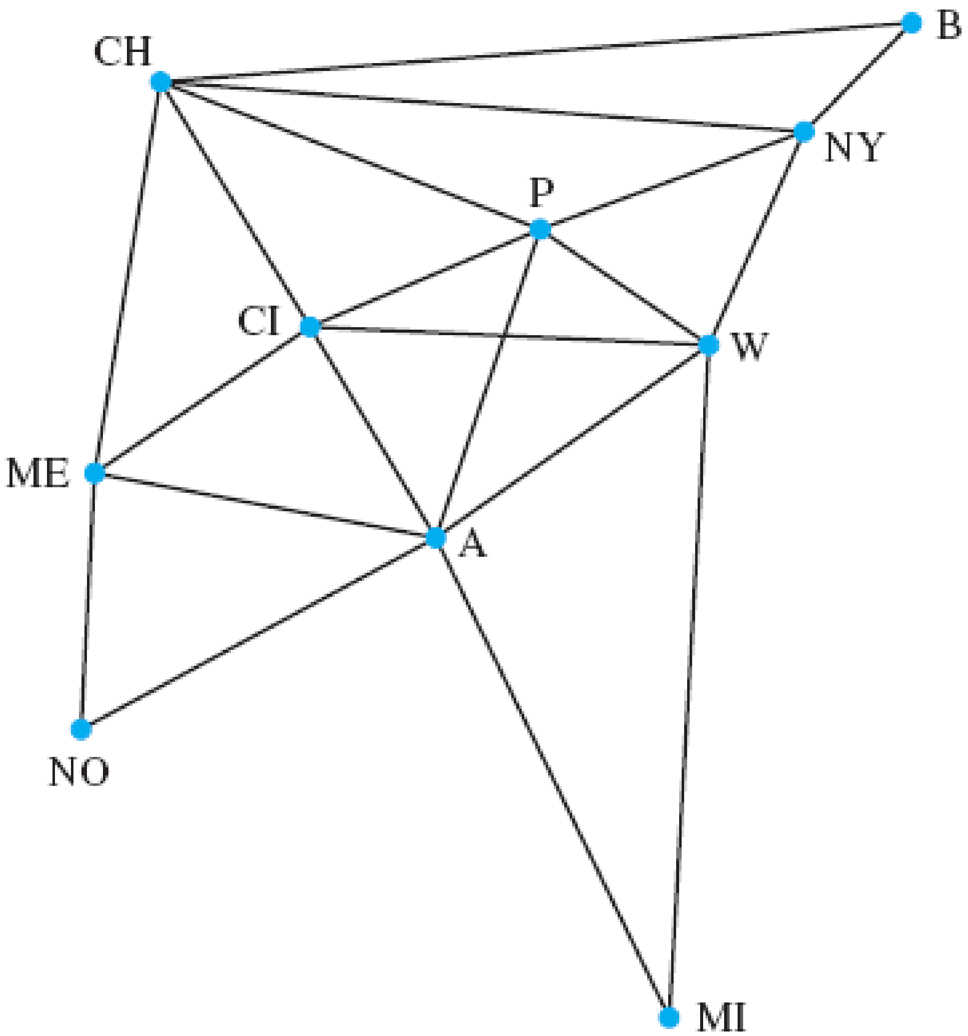
This path has length 3.



Connectivity



Company decides to lease only minimum number of communication lines it needs to be able to send a message from any city to any other city by using any number of intermediate cities.

What is the minimum number of lines it needs to lease?

Connectivity

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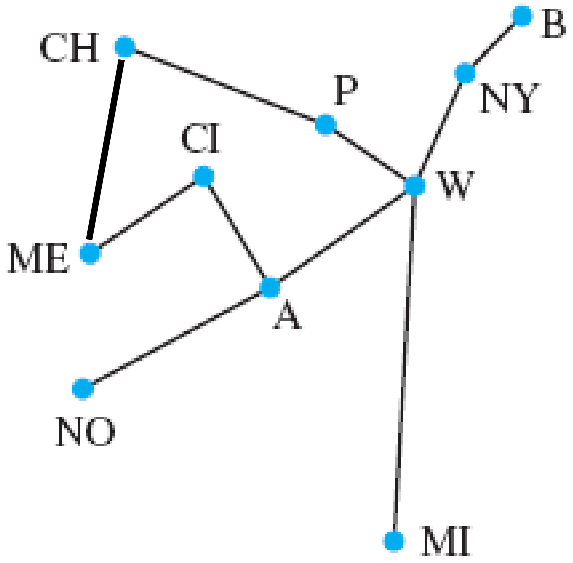

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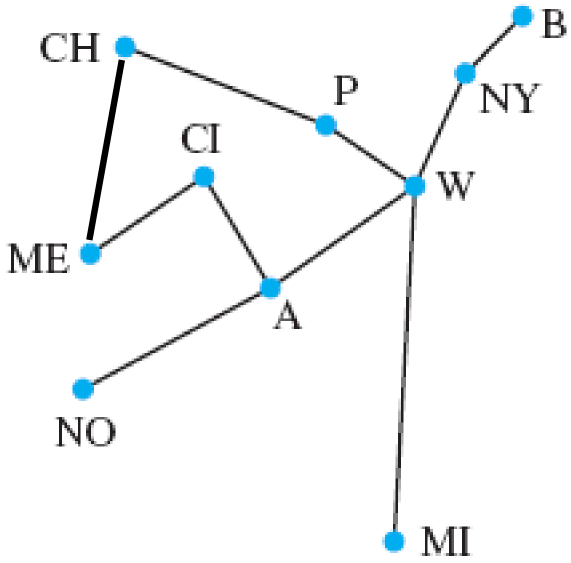

Too many.

Could throw away edge CI, A, and still have a solution.

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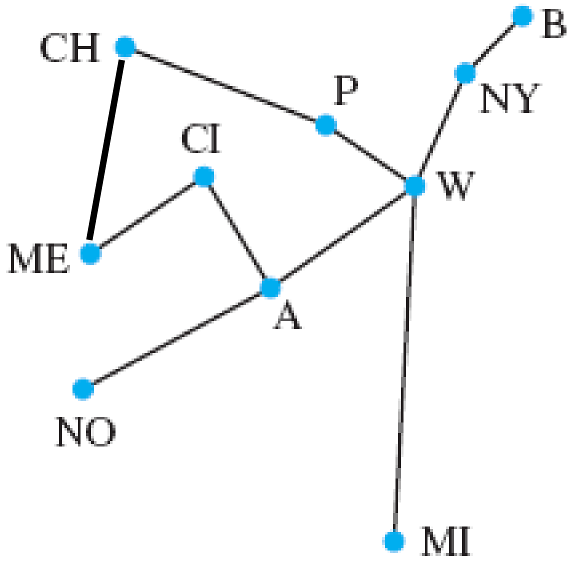

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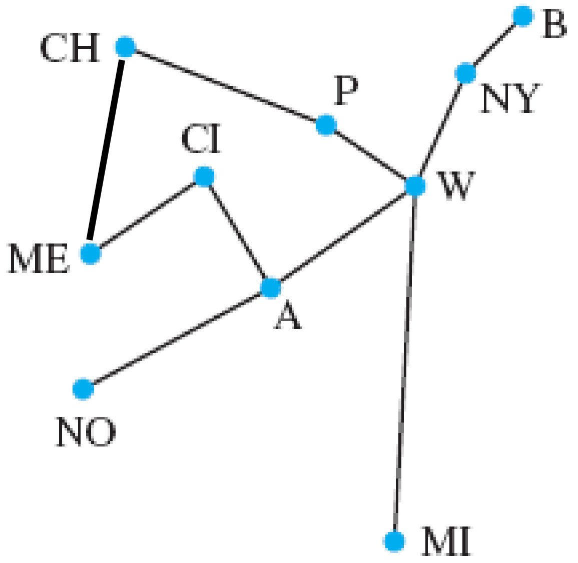

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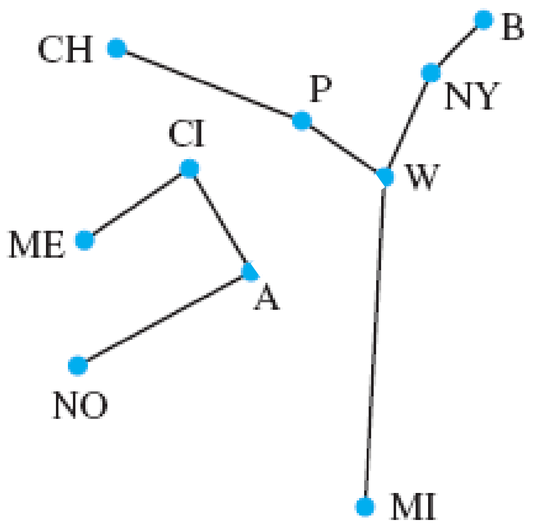

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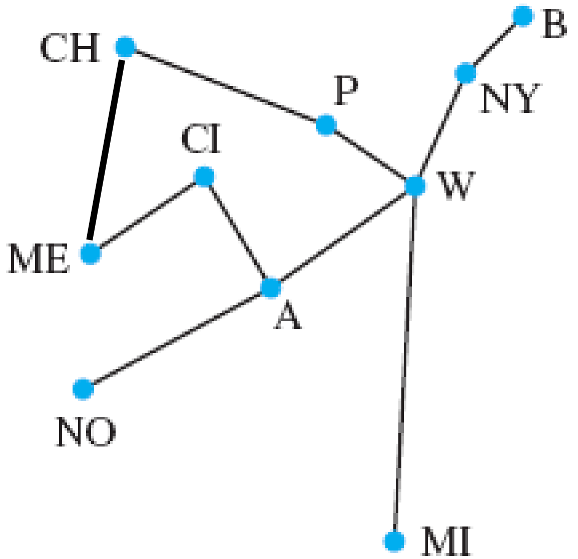

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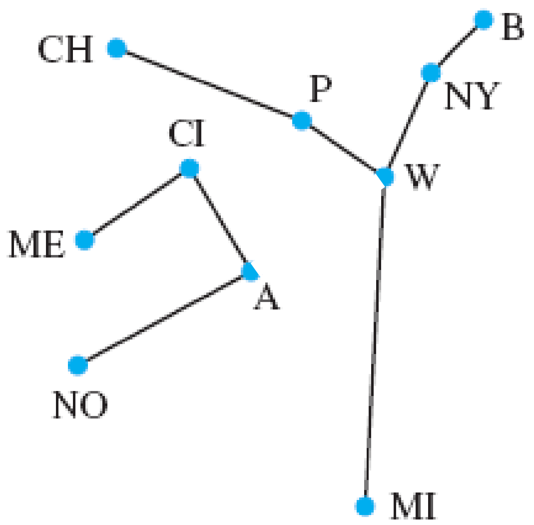

Not enough.

There is no path from, e.g., NO to B.

Connectivity

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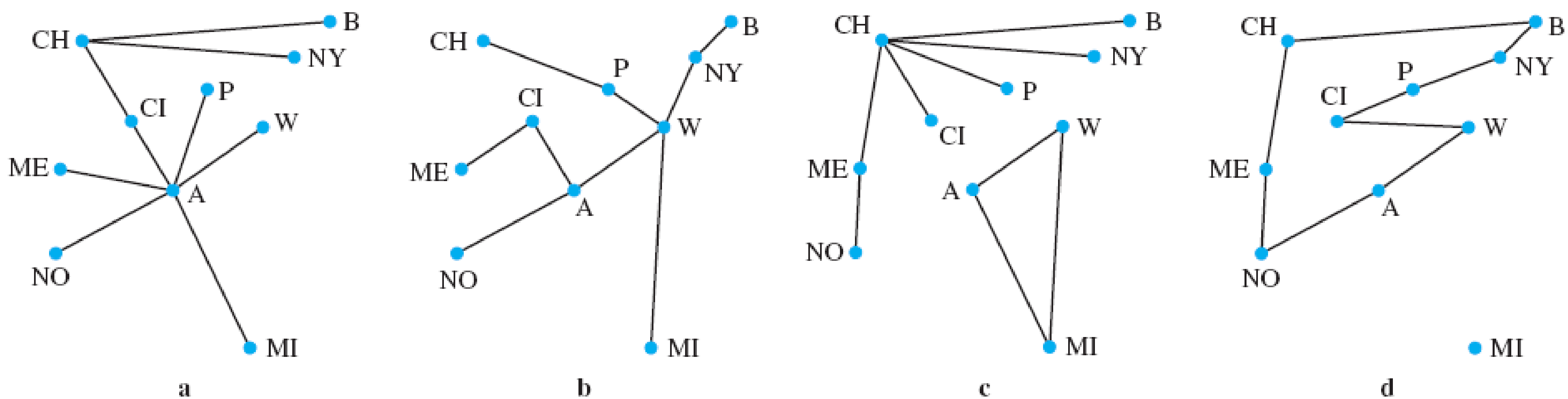

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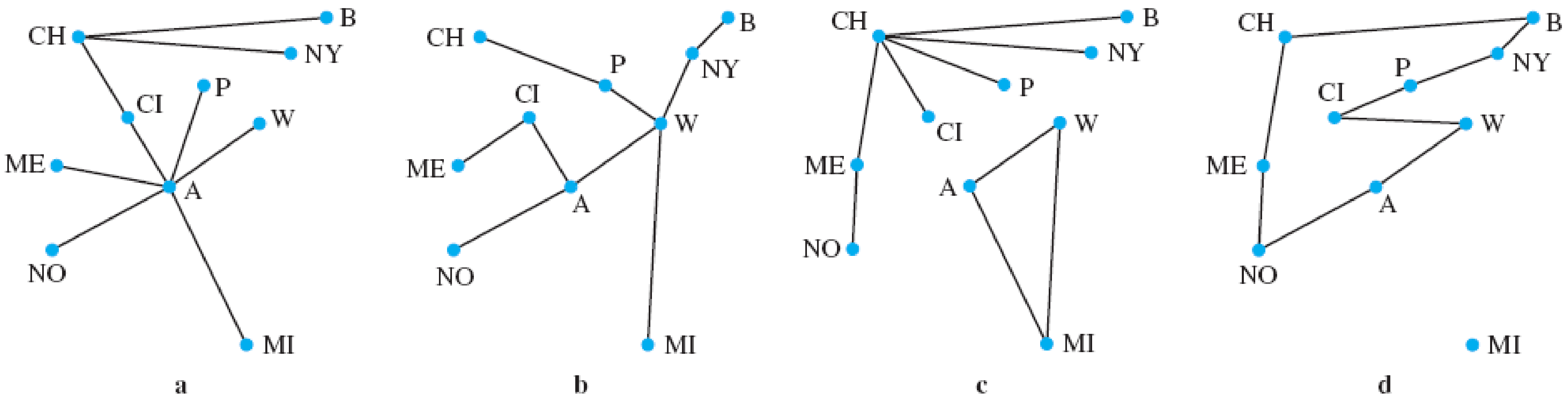

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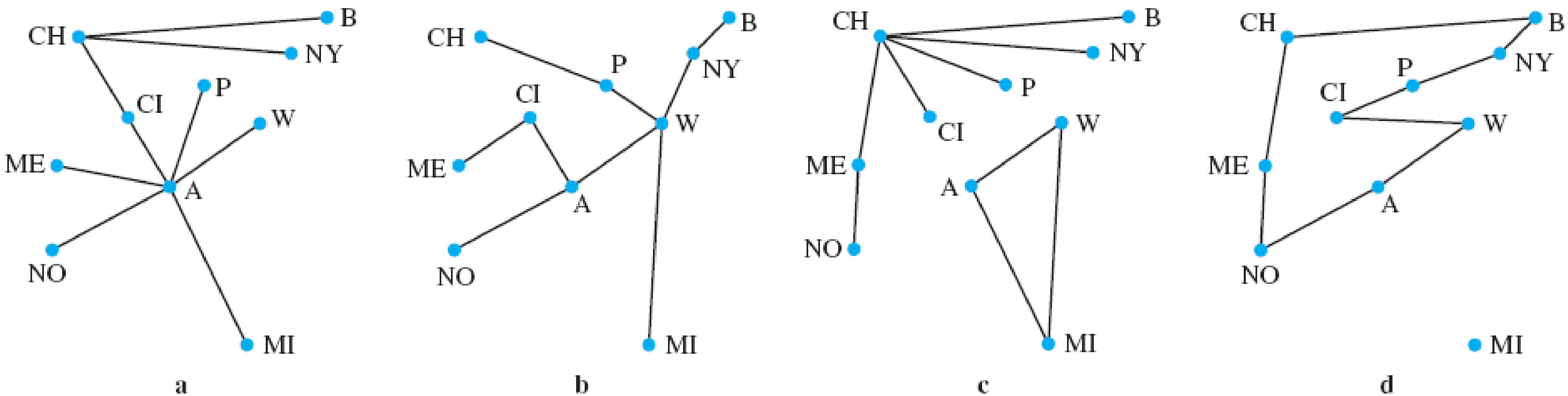

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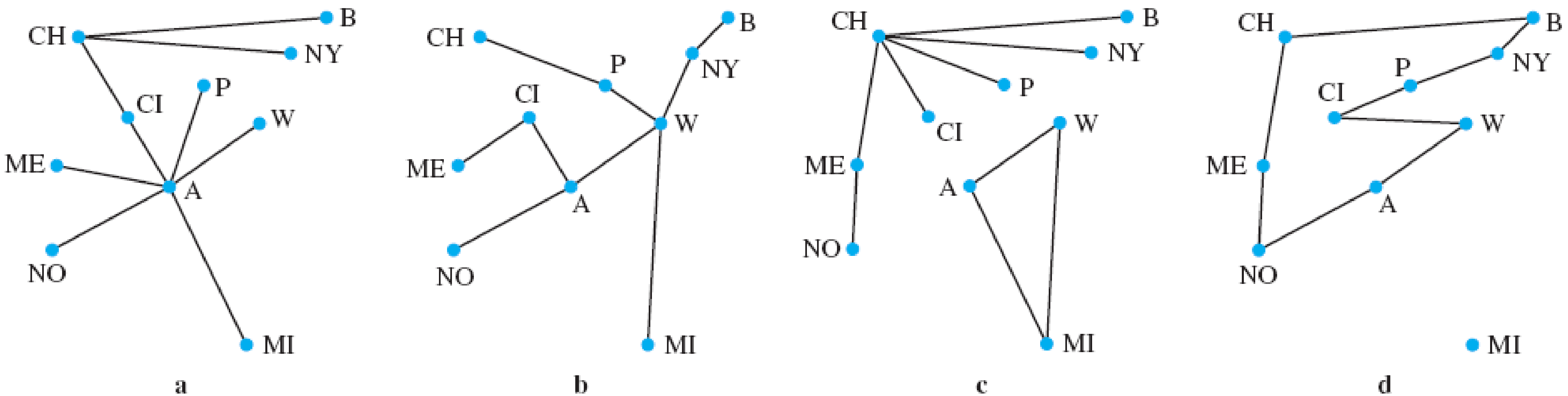

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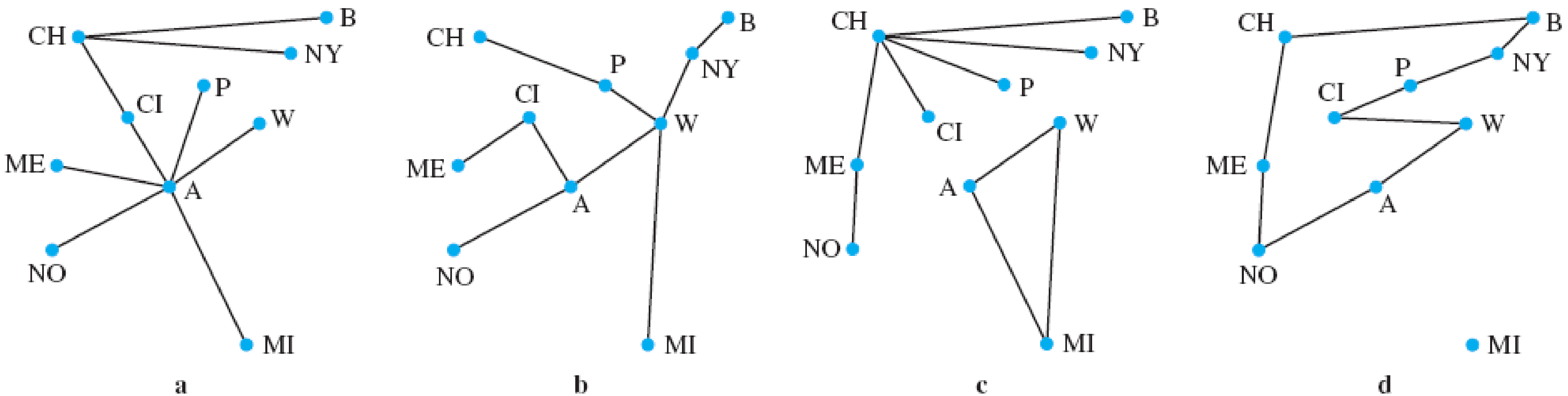

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Example: (a) and (b) are connected, (c) and (d) are disconnected.

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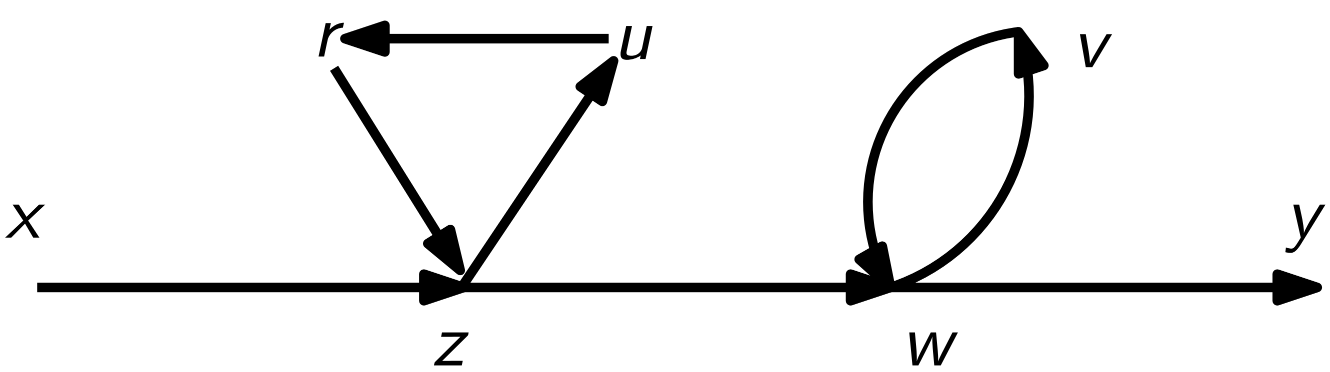

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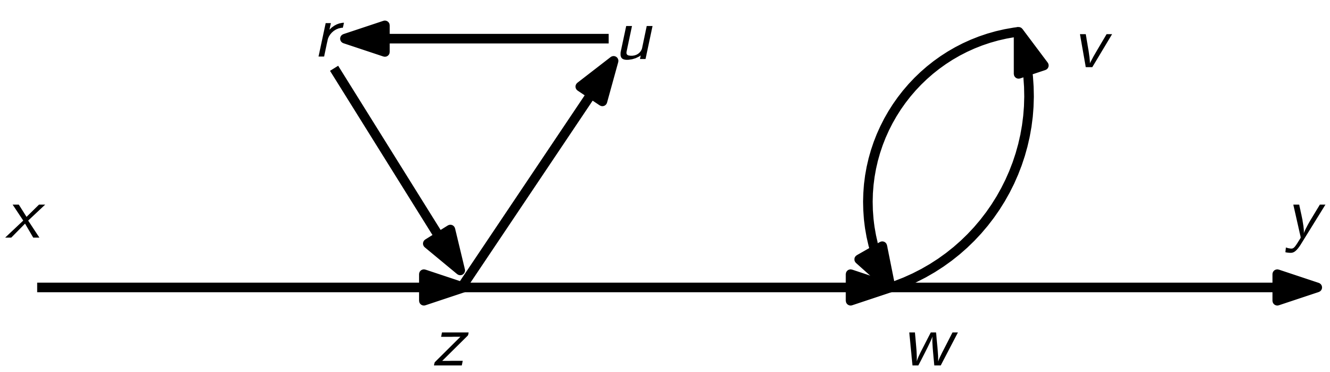

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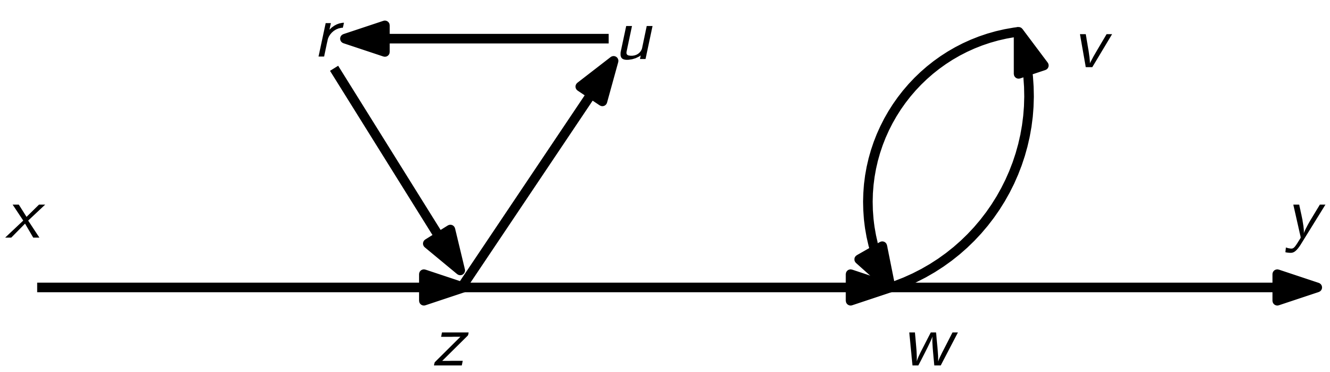

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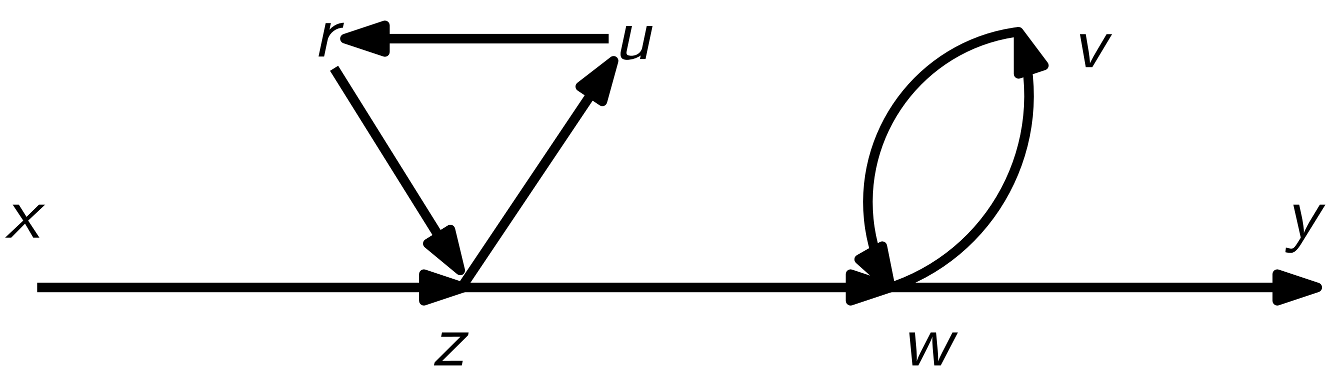

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→

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Theorem There is a simple path between every pair of distinct vertices of a connected undirected graph.

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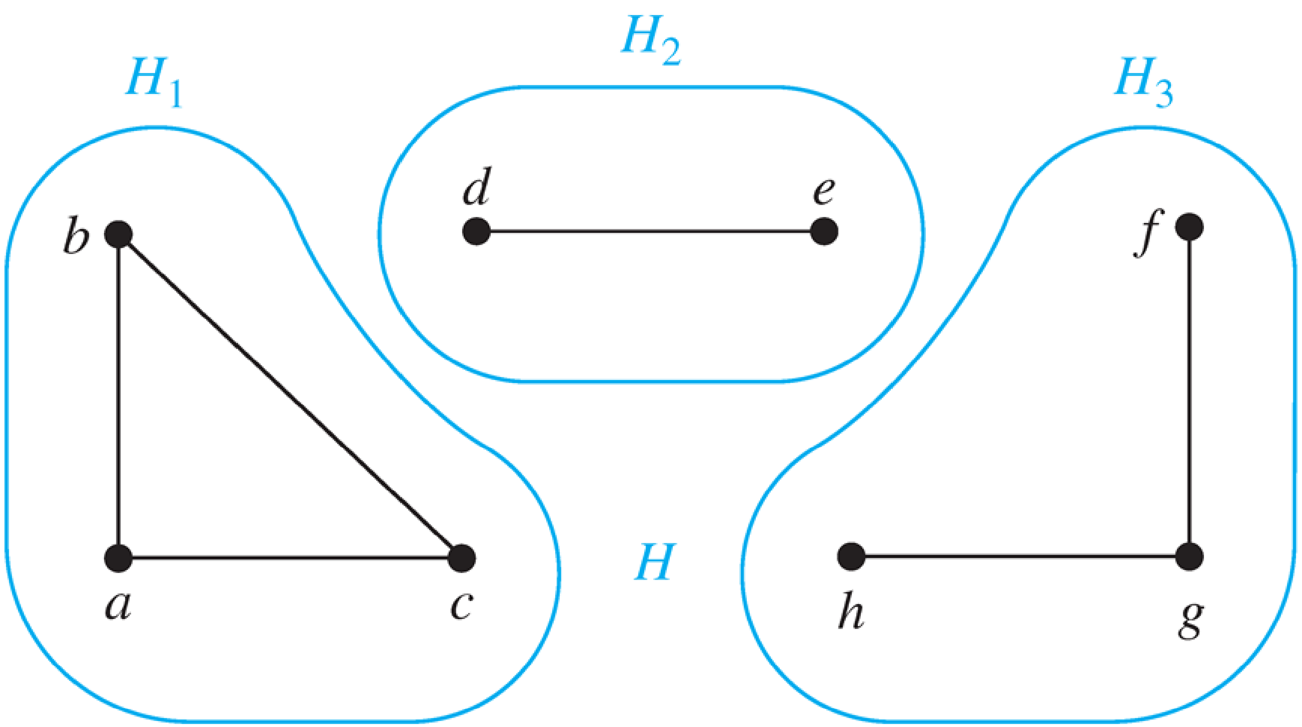

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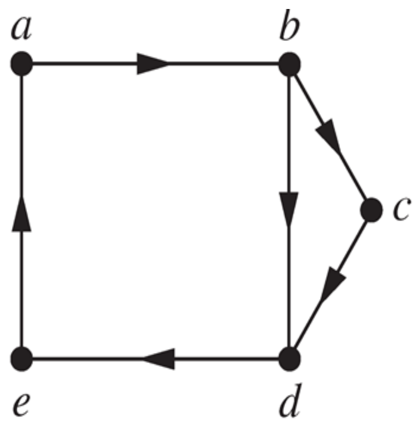

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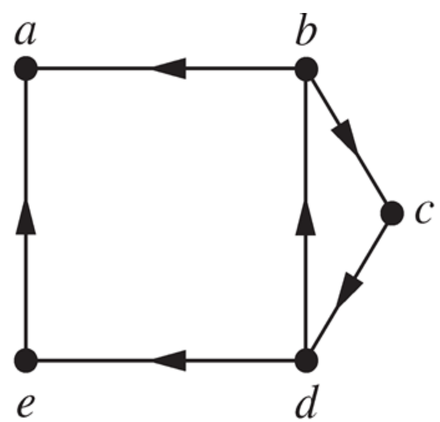

H

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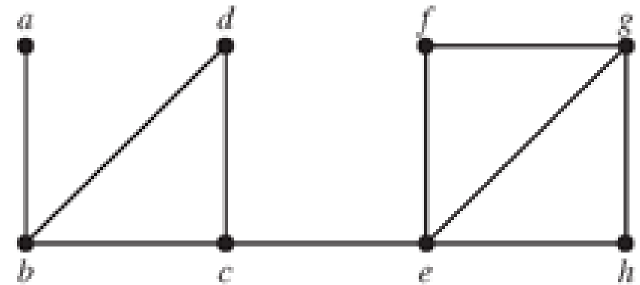

G_1

Paths and Isomorphism

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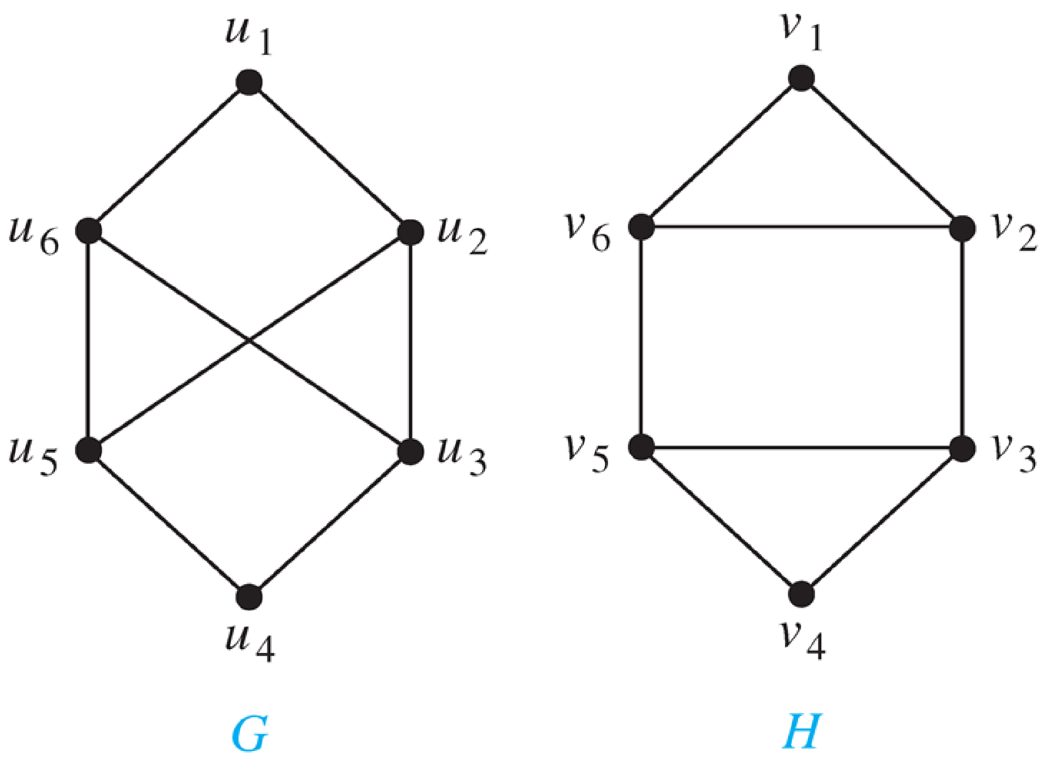

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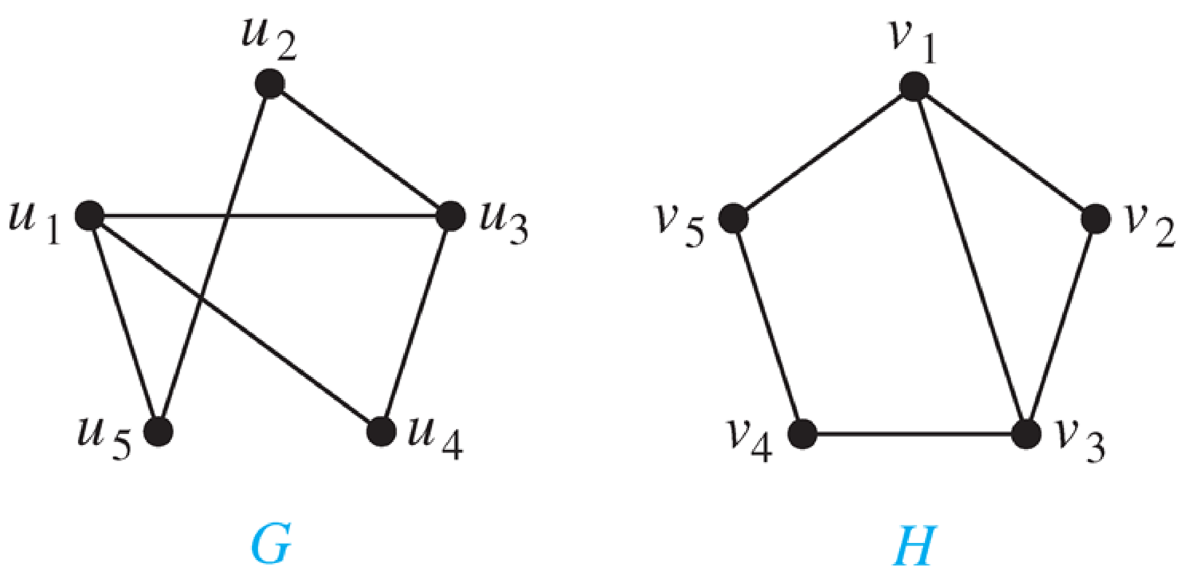

Counting Paths between Vertices

- **Theorem** Let G be a graph with adjacency matrix $\mathbf{A}$ with respect to the ordering $v_1, v_2, \ldots, v_n$ of vertices. The number of different paths of length r from v_i to v_j, where $r > 0$ is positive, equals the (i, j)-th entry of $\mathbf{A}^r$.

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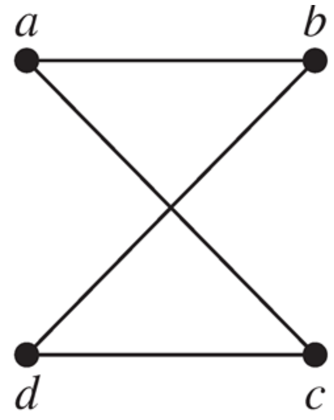
Proof (by **induction**)

$\mathbf{A}^{r+1} = \mathbf{A}^r\mathbf{A}$, the (i, j)-th entry of $\mathbf{A}^{r+1}$ equals $b_{i1}a_{1j} + b_{i2}a_{2j} + \cdots + b_{in}a_{nj}$, where b_{ik} is the (i, k)-th entry of $\mathbf{A}^r$.

Counting Paths between Vertices

- **Example** How many paths of length 4 are there from a to d in the graph G?



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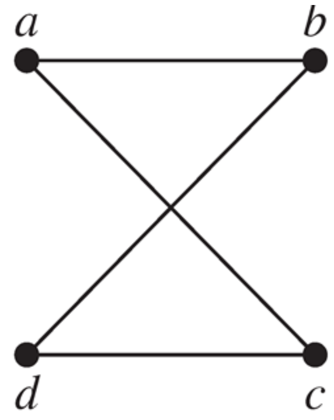

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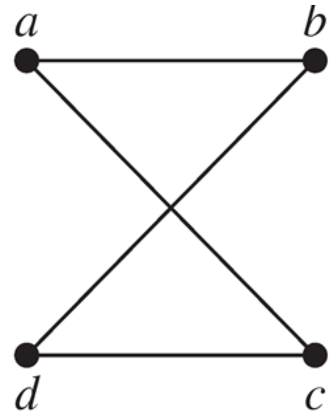
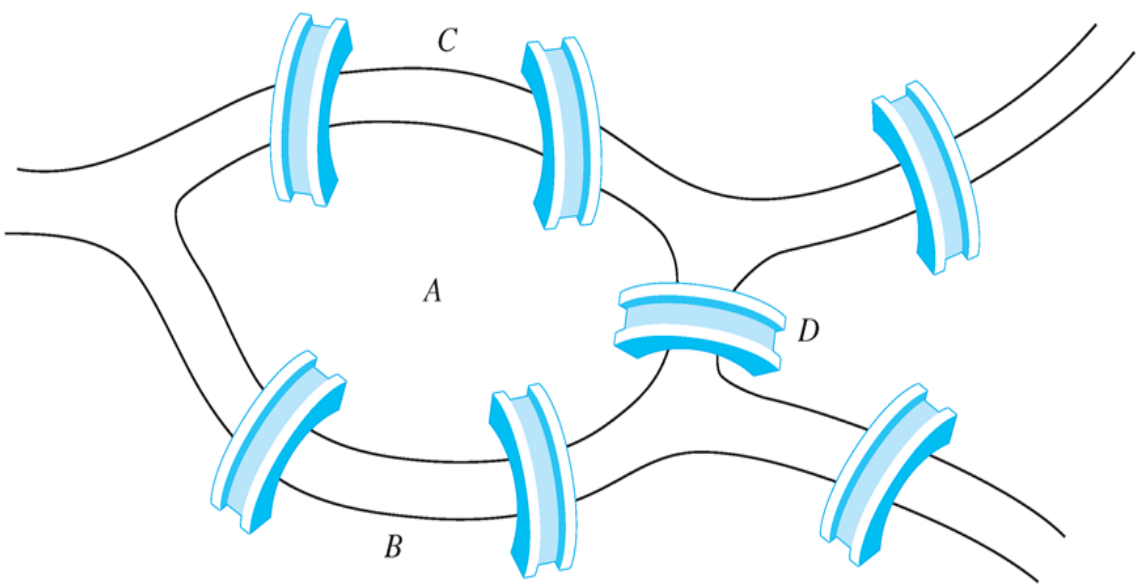


$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

- **Königsberg seven-bridge problem**

  People wondered whether it was possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice, and return to the starting point.



Euler Paths

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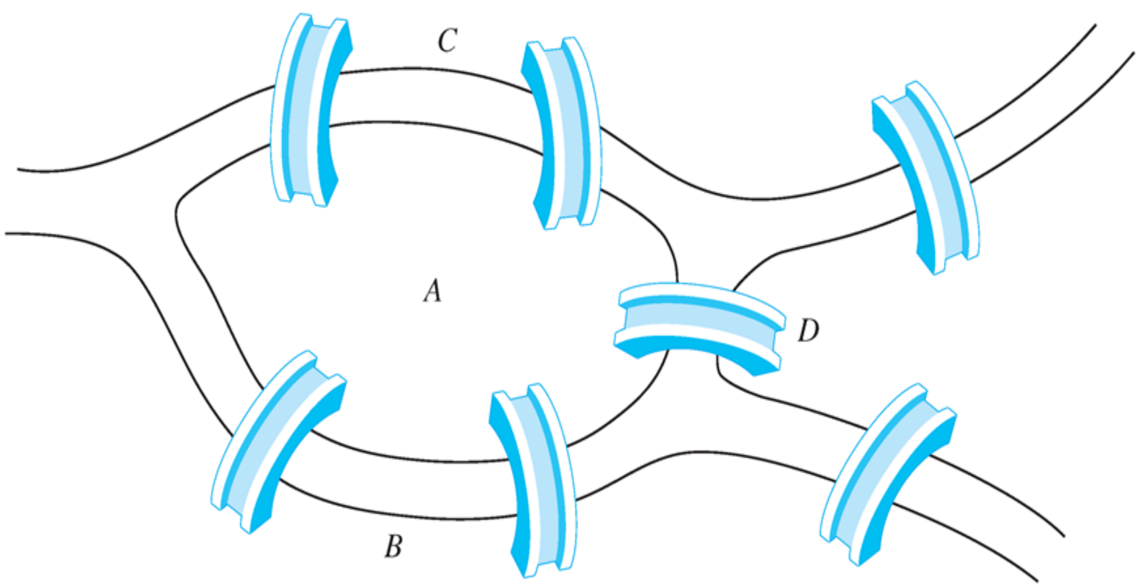

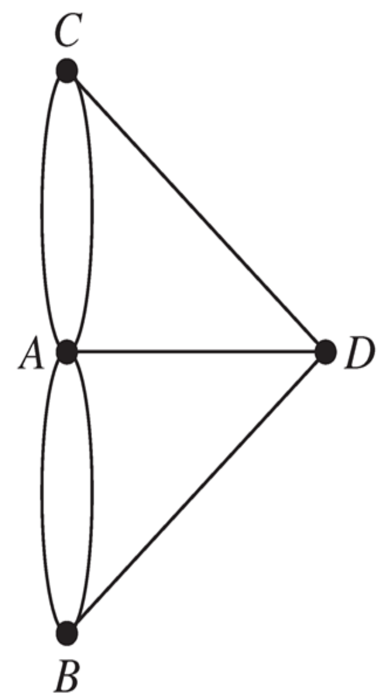

Euler Paths and Circuits

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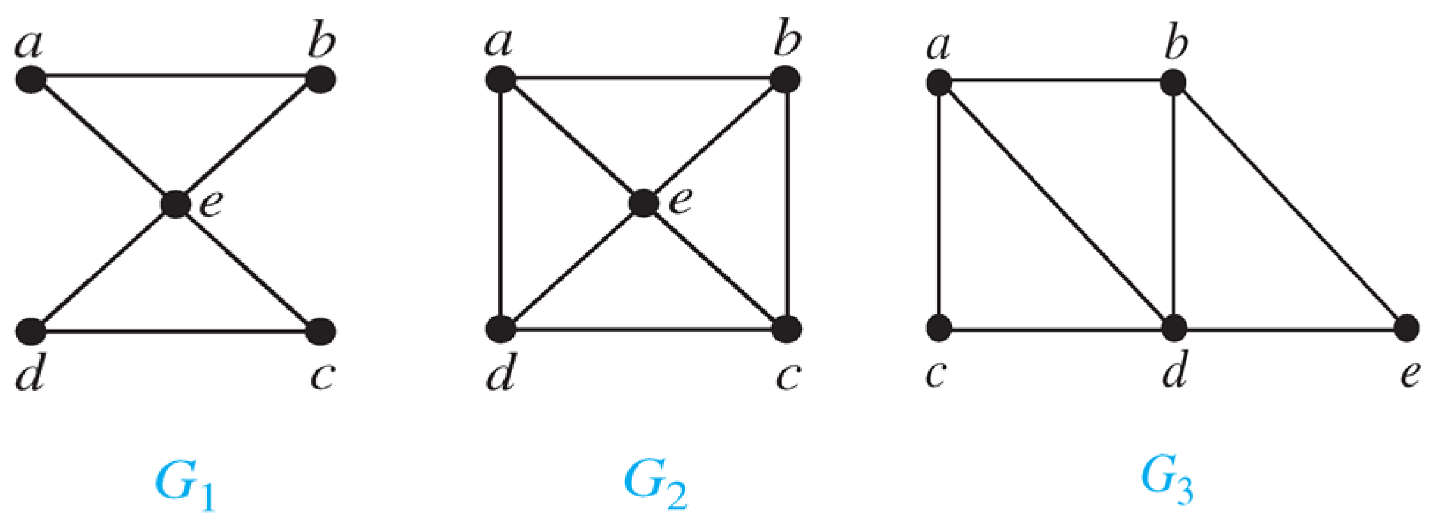

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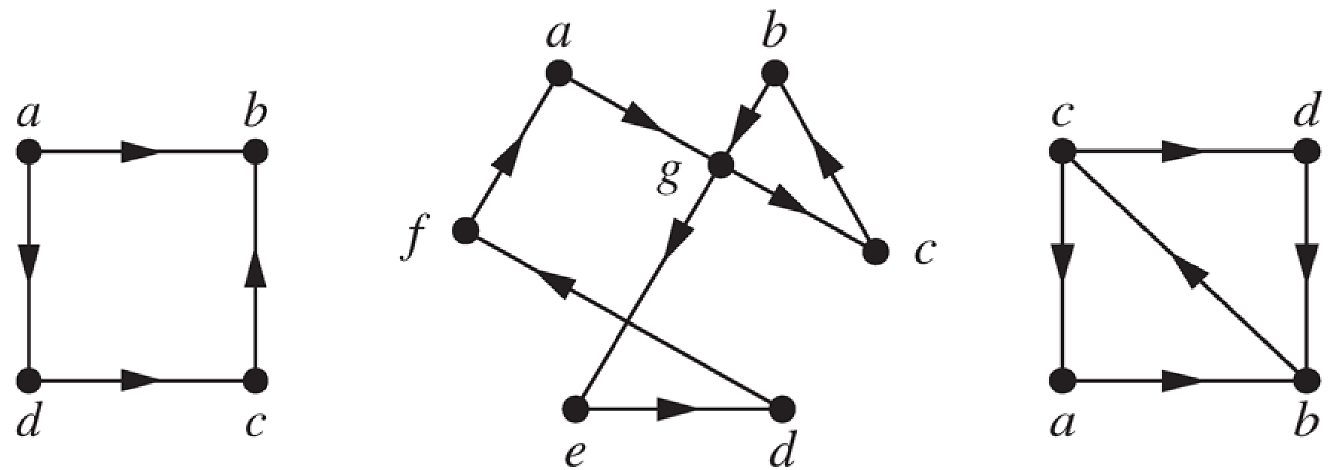

H_1 H_2 H_3

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Euler Path $\Rightarrow$ The graph has exactly two vertices of odd degree

- ◇ The initial vertex and the final vertex of an Euler path have odd degree.

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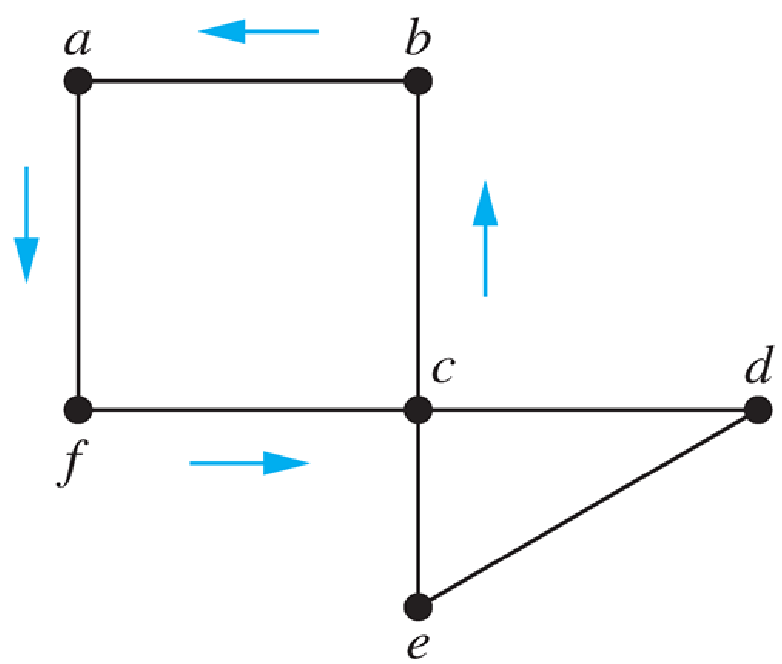

G

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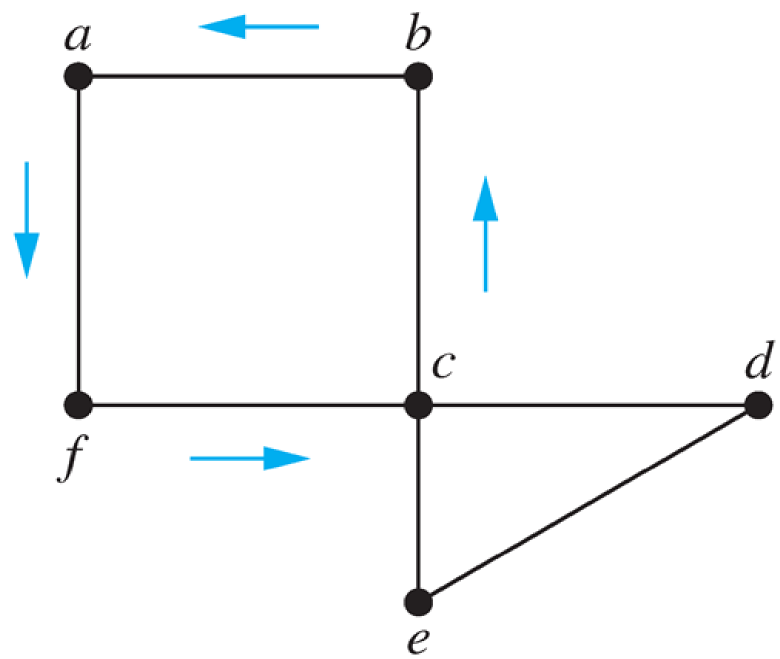

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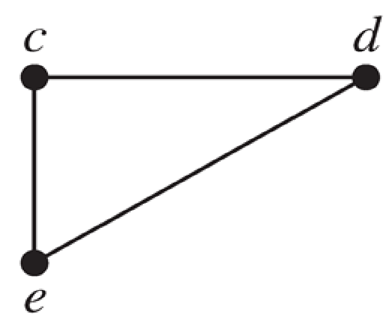

H

Algorithm for Constructing an Euler Circuits

procedure *Euler*(G: connected multigraph with all vertices of even degree)
circuit := a circuit in G beginning at an arbitrarily chosen vertex with edges successively added to form a path that returns to this vertex.
H := G with the edges of this circuit removed
while H has edges
  subciruit := a circuit in H beginning at a vertex in H that also is an endpoint of an edge in circuit.
  H := H with edges of *subciruit* and all isolated vertices removed
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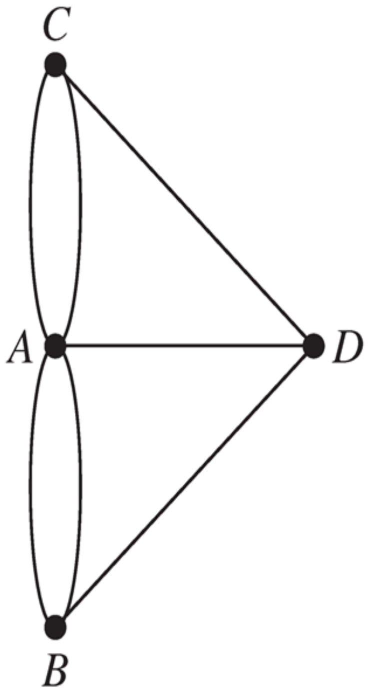

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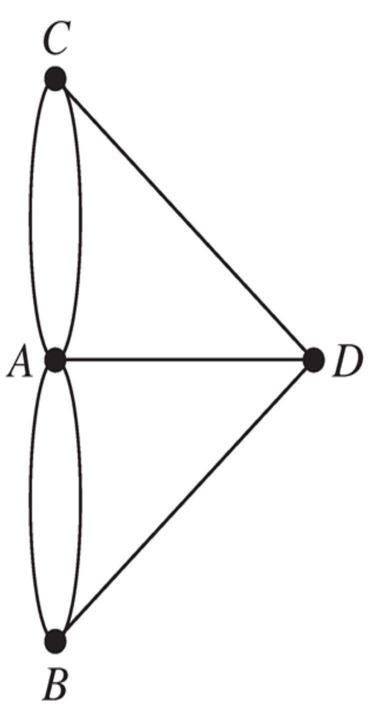


No Euler circuit

Euler Circuits and Paths

- **Example**

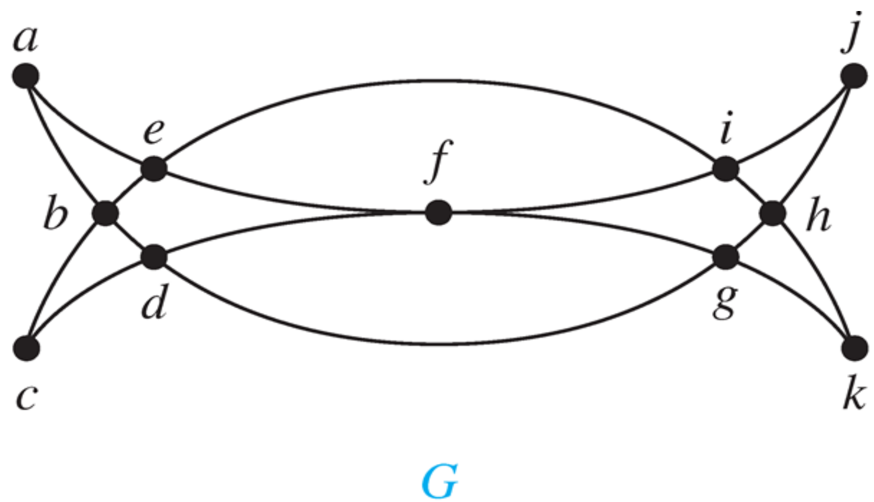
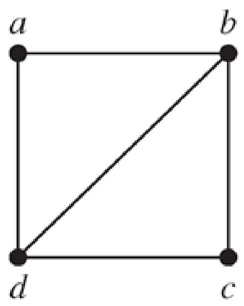
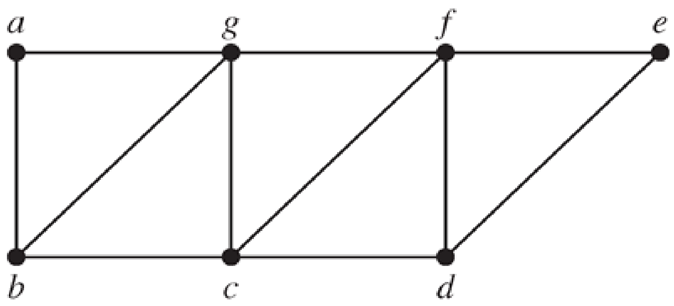


FIGURE 6 **Mohammed's Scimitars.**

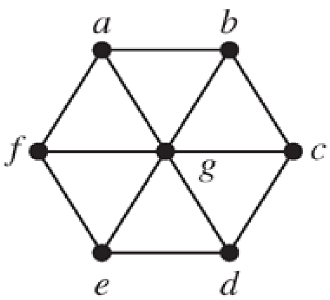
- **Example**



G_1



G_2



G_3

Applications of Euler Paths and Circuits

- Finding a path or circuit that traverses each
  - street in a neighborhood
  - road in a transportation network
  - link in a communication network
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Next Lecture

- graph theory III ...