南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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Assignment #2

Please collect your assignments!

Counting

- Assume we have a set of objects with certain properties

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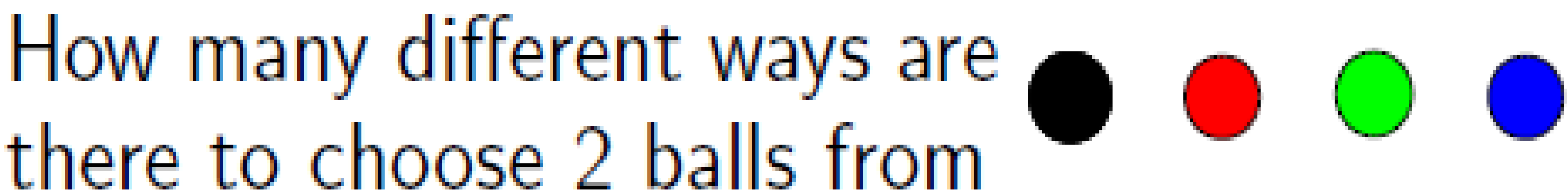

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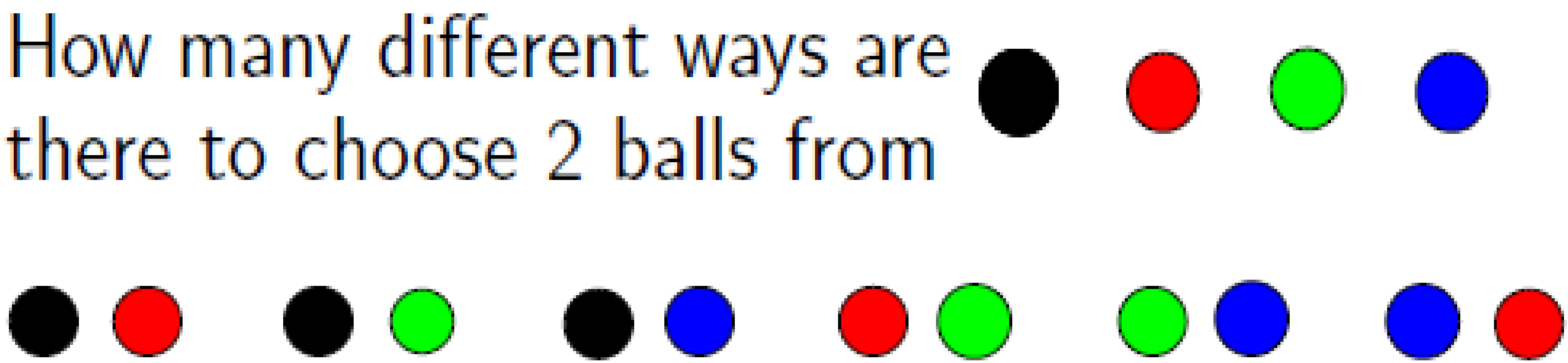

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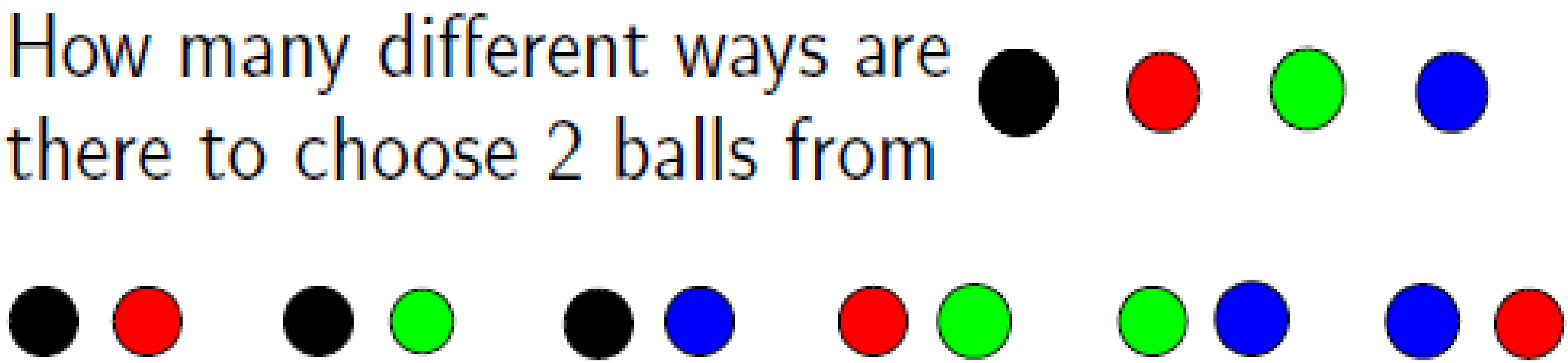

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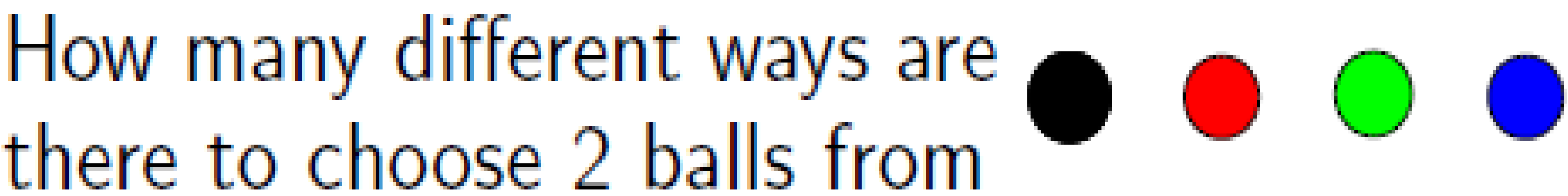

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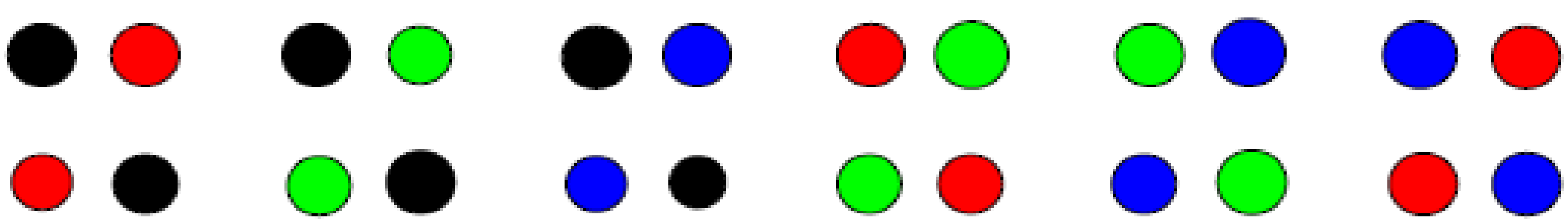

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  – simplify the solution by decomposing the problem

Basic Counting Rules

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Example

In an auditorium, the seats are labeled by a letter and numbers in between 1 to 50 (e.g., $A23$). What is the total number of seats?

We may either list all or use the product rule.

$$26 \times 50 = 1300$$

The Product Rule

- **Product Rule**: If a count of elements can be broken down into a sequence of dependent counts where the first count yields n_1 elements, the second n_2 elements, and kth count n_k elements, then the total number of elements is

$$n = n_1 \cdot n_2 \cdot \cdots \cdot n_k$$

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How many onto functions?

The Product Rule

- The following loop is a part of program computing the prodcut of two matrices.

```
(1) for i = 1 to r
(2)    for j = 1 to m
(3)       S = 0
(4)       for k = 1 to n
(5)          S = S + A[i,k] * B[k,j]
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How many multiplications (in terms of r, m, n) does this program carry out in total among all iterations of line 5?

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We may use the sum rule.

$$12 + 5 + 10$$

The Sum Rule

- **Sum Rule**: If a count of elements can be broken down into a set of independent counts where the first count yields n_1 elements, the second n_2 elements, and kth count n_k elements, then the total number of elements is

$$n = n_1 + n_2 + \cdots + n_k$$

The Sum Rule

- The following loop is from selection sort.

```
(1) for i = 1 to n-1
(2)    for j = i+1 to n
(3)       if (A[i] > A[j])
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How many comparisons (in terms of n) does this program carry out in total among all iterations of line 3?

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$$P = P_6 + P_7 + P_8$$

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- Two sets

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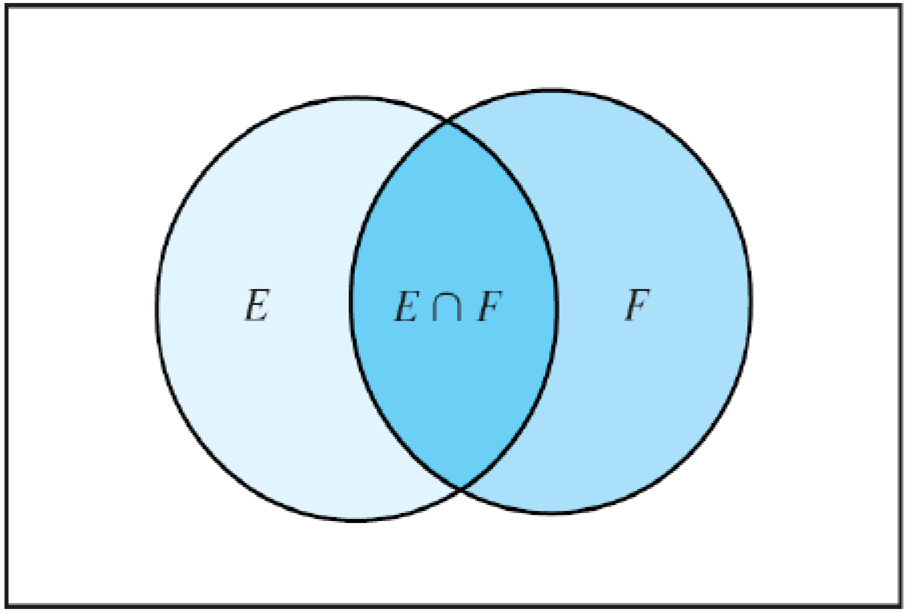

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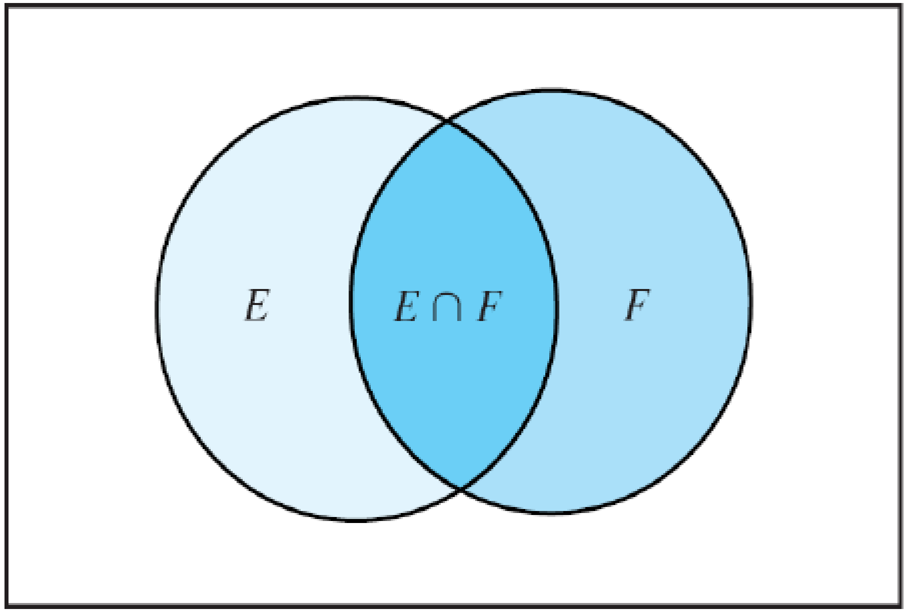
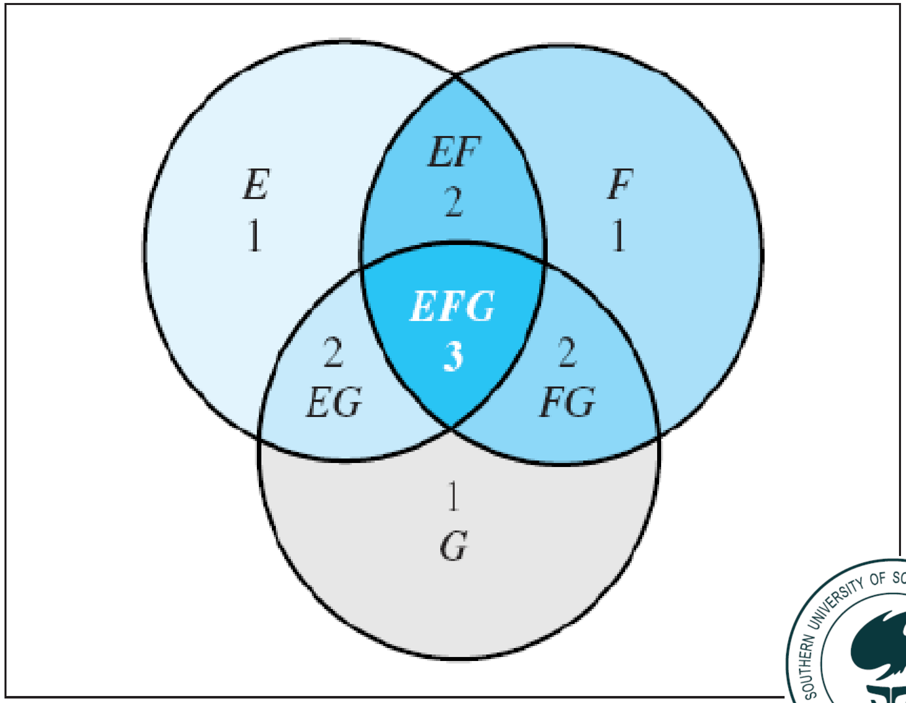


Three sets



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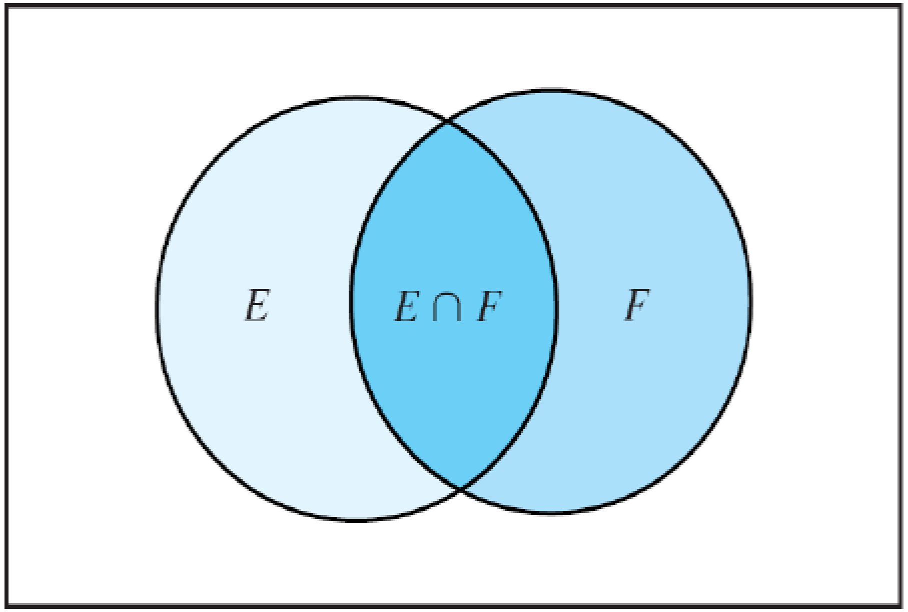
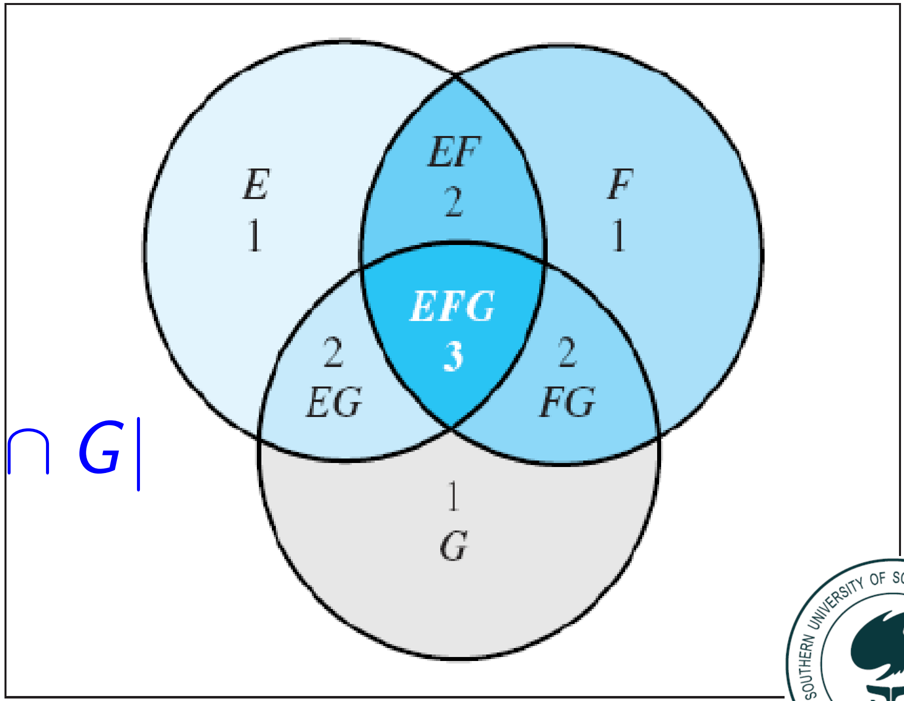


Three sets

$$\begin{aligned} &|E \cup F \cup G| \\ &= |E| + |F| + |G| \\ &\quad -|E \cap F| - |E \cap G| - |F \cap G| \\ &\quad +|E \cap F \cap G| \end{aligned}$$



Inclusion-Exclusion Principle

- $$|\cup_{i=1}^{n} E_i| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$$

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Inductive Hypothesis

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$$|\cup_{i=1}^{n} E_i| = \left|\cup_{i=1}^{n-1} E_i\right| + |E_n| - \left|\left(\cup_{i=1}^{n-1} E_i\right) \cap E_n\right|$$

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  For the third term, by distributive law,

  $$\left|(\cup_{i=1}^{n-1} E_i) \cap E_n\right| = \left|\cup_{i=1}^{n-1} (E_i \cap E_n)\right| = \left|\cup_{i=1}^{n-1} G_i\right|$$

  where $G_i = E_i \cap E_n$.

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- So far

$$\left|\cup_{i=1}^{n} E_i\right| = \left|\cup_{i=1}^{n-1} E_i\right| + |E_n| - \left|\cup_{i=1}^{n-1} G_i\right|$$

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Note that (why?)

$$-(-1)^{k+1}|G_{i_1} \cap G_{i_2} \cap \cdots \cap G_{i_k}|$$
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Some discussion:
first summation sums $(-1)^{k+1}|E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$ over all lists $i_1, i_2, \ldots, i_k$ that do not contain n
$|E_n|$ and second summation together sum $(-1)^{k+1}|E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$ over all lists $i_1, i_2, \ldots, i_k$ that do contain n

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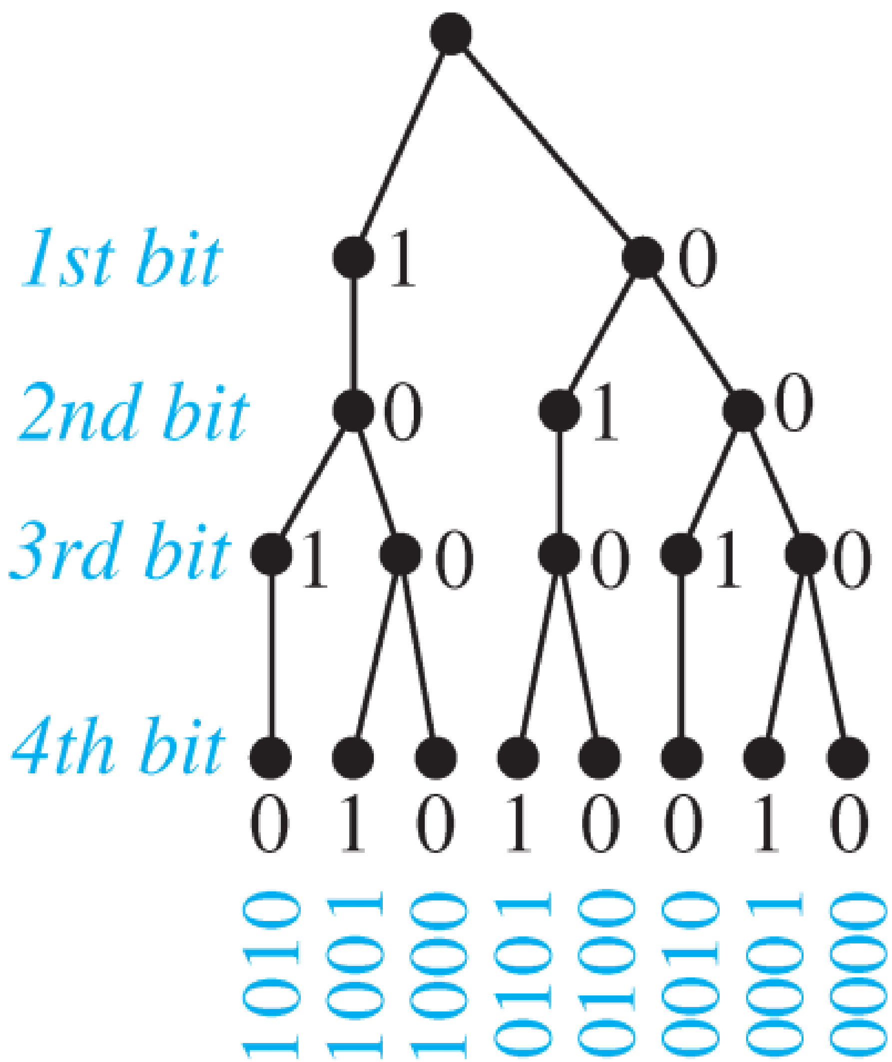

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Tree Diagram

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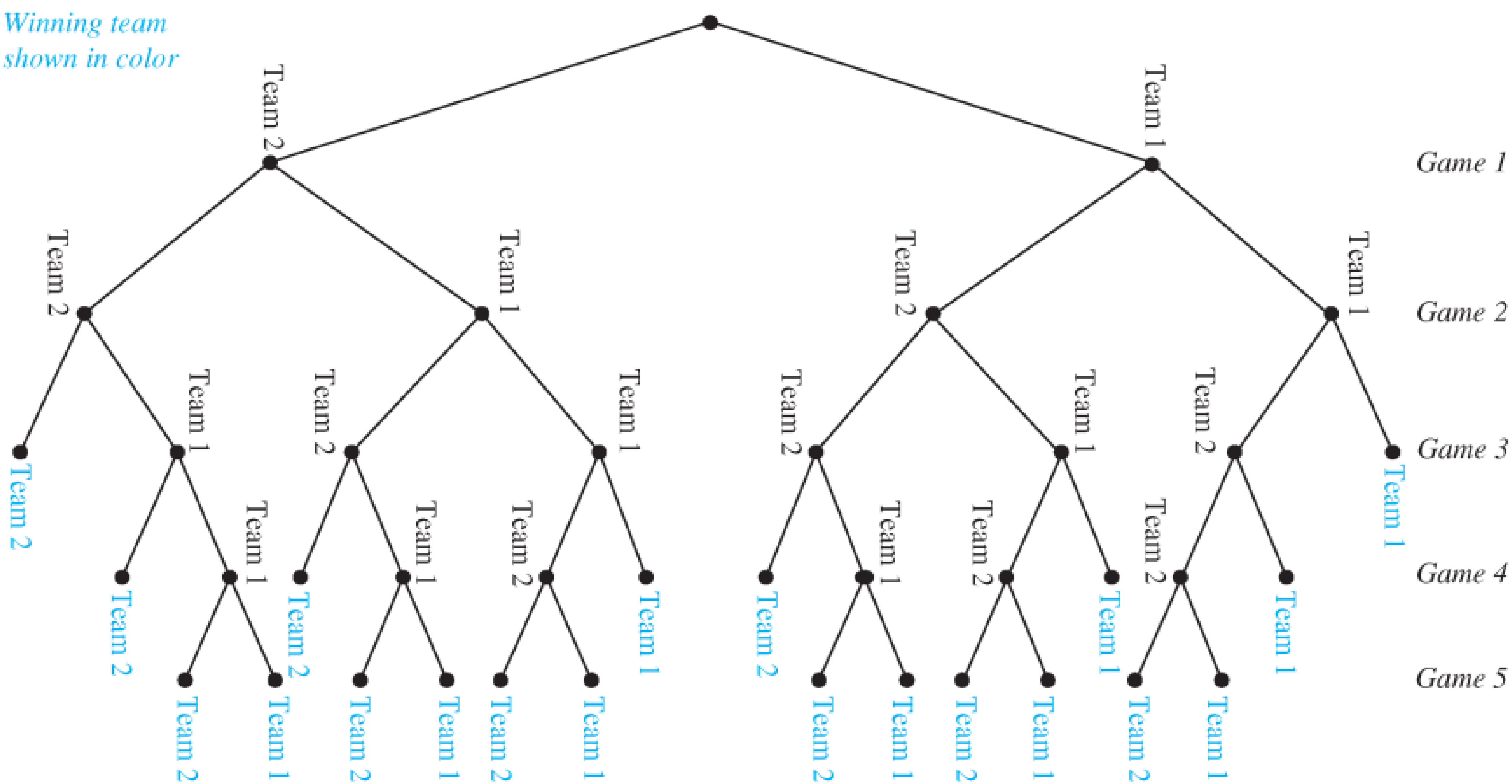

Pigeonhole Principle

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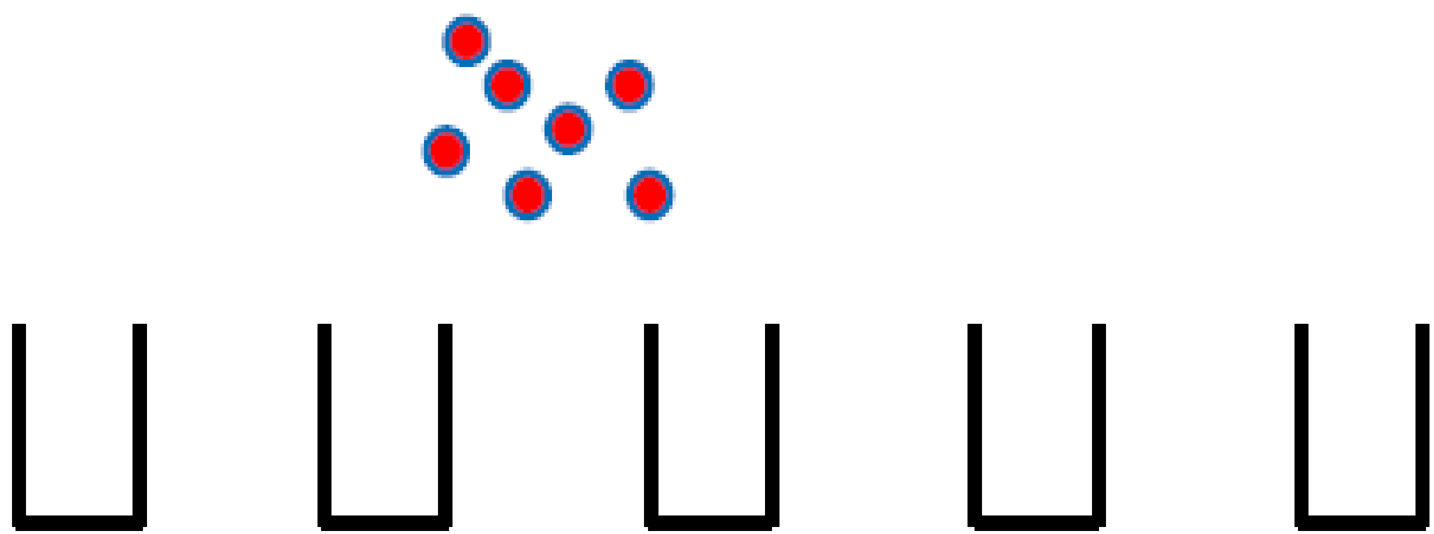

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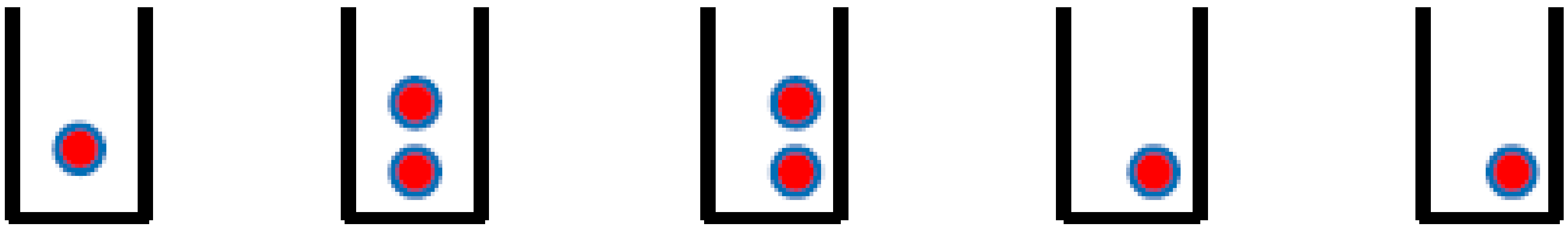

Pigeonhole Principle

- **Theorem** If there are $k+1$ objects and k bins, then there is at least one bin with two or more objects.

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  Example

  Assume that there are 367 students. Are there any two people who has the same birthday?

  There are 5 bins and 12 objects. Then there must be a bin with at least 3 objects. Why?

Generalized Pigeonhole Principle

- If N objects are placed into k bins, then there is at least one bin containing at least $\lceil N/k \rceil$ objects.

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Example

Assume there are 100 students. How many of them were born in the same month?

Bijections and Permutations

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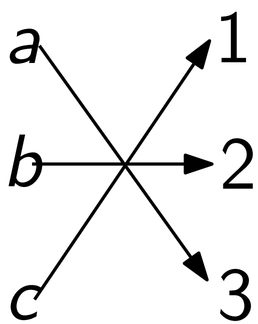

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  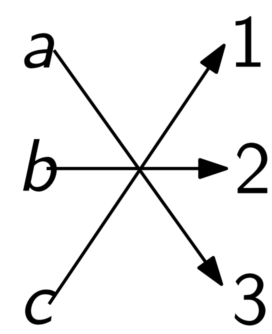

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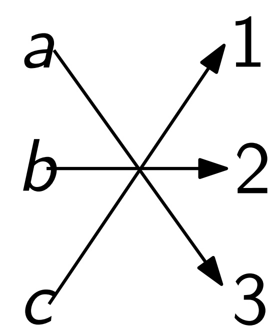

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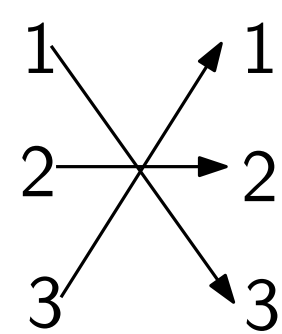

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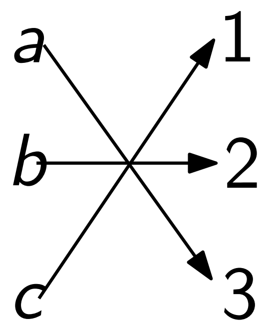

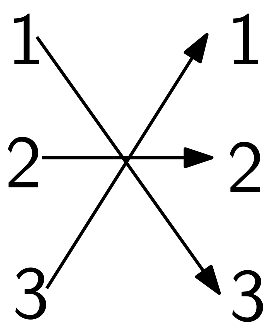

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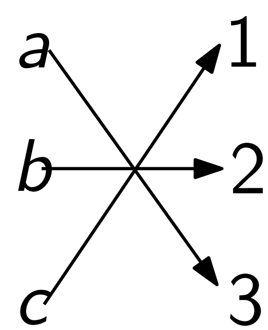
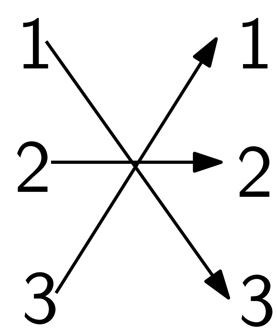
  Thus,
  the left and right sides must have the same size.





The Bijection Principle

- The following loop is a part of program to determine the number of triangles formed by n points in the plane.

```
triangleCount = 0
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    for j = i+1 to n
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Among all iterations of line 5, what is the total number of times this line checks three points to see if they are collinear?

Counting Triangles

- 3 points form a triangle if and only if they are non collinear

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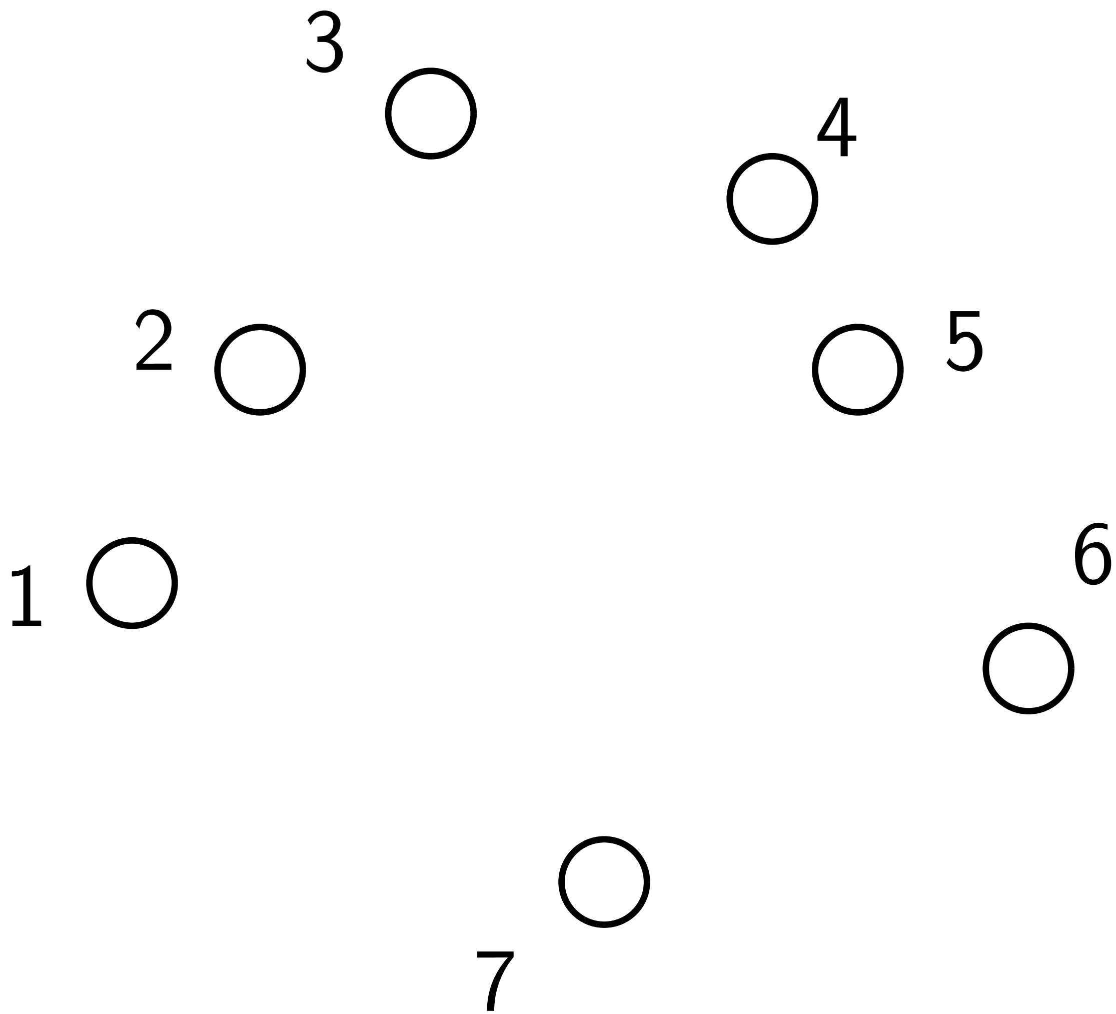

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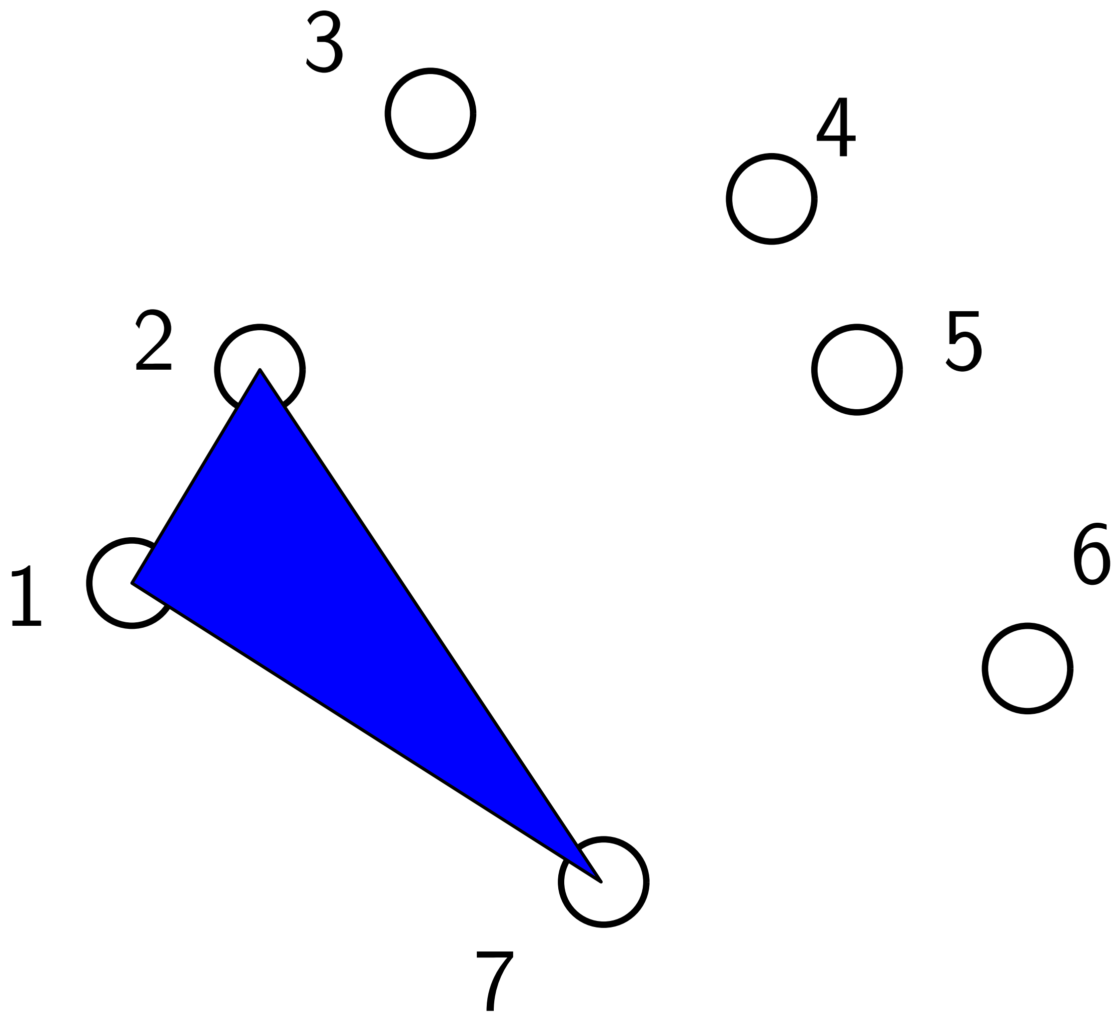

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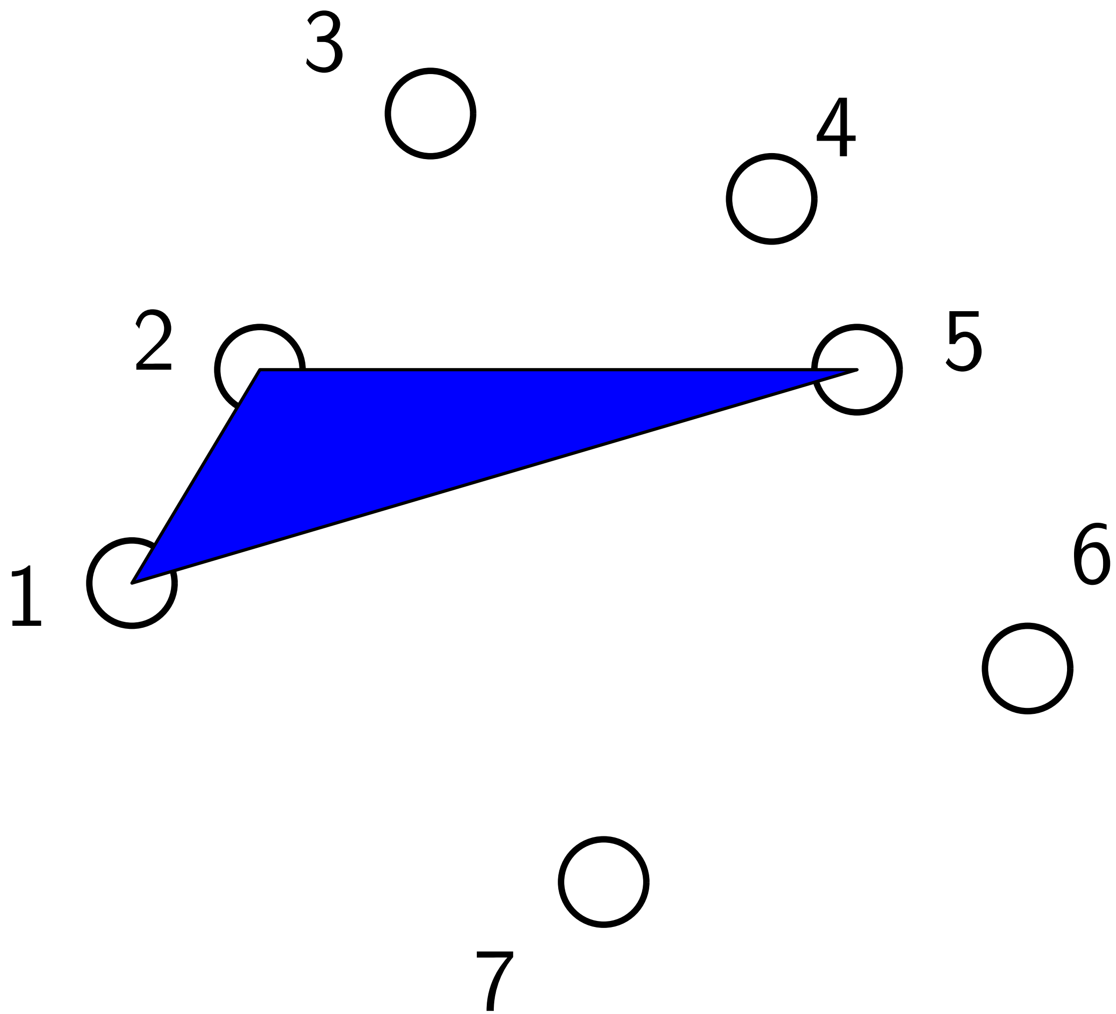

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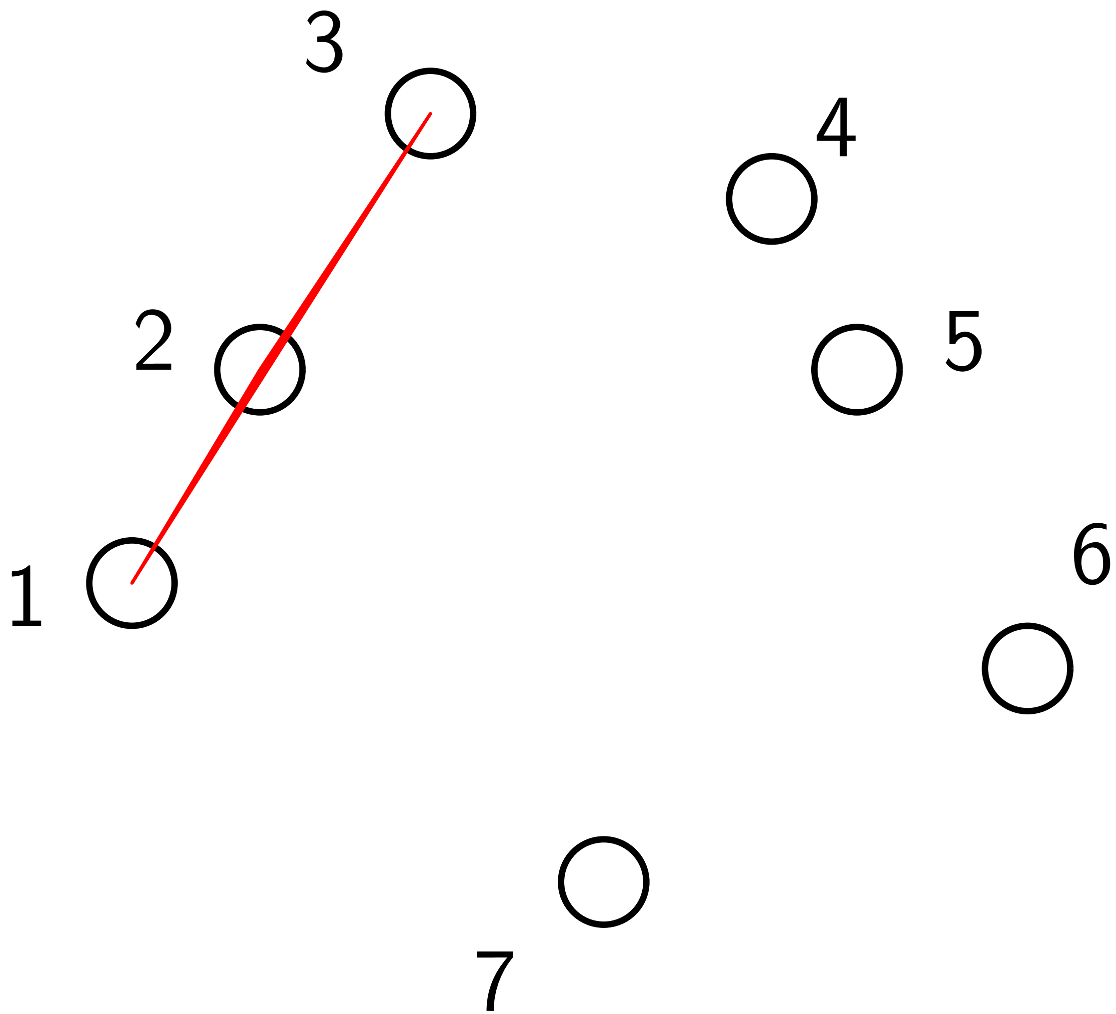

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$1 - 2 - 3$: no

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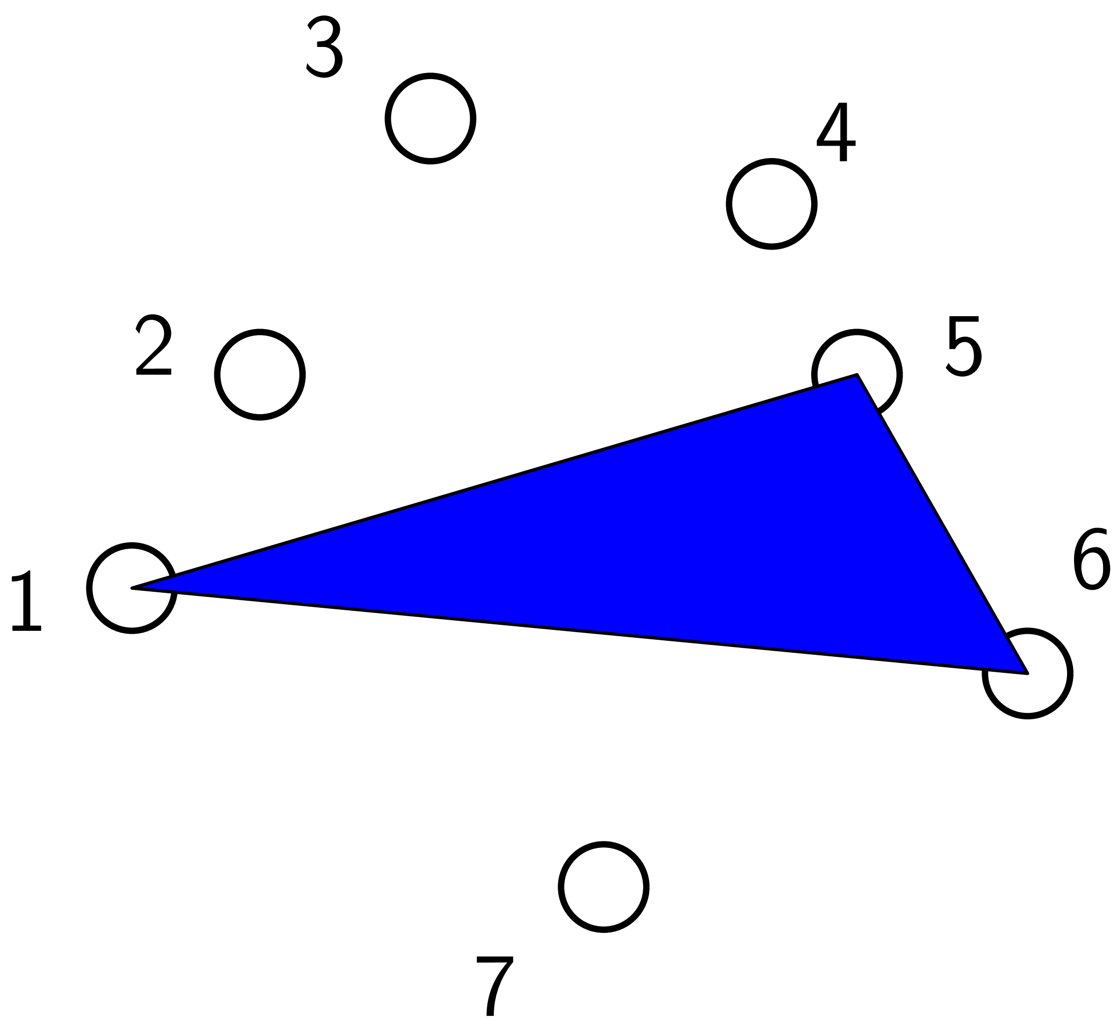

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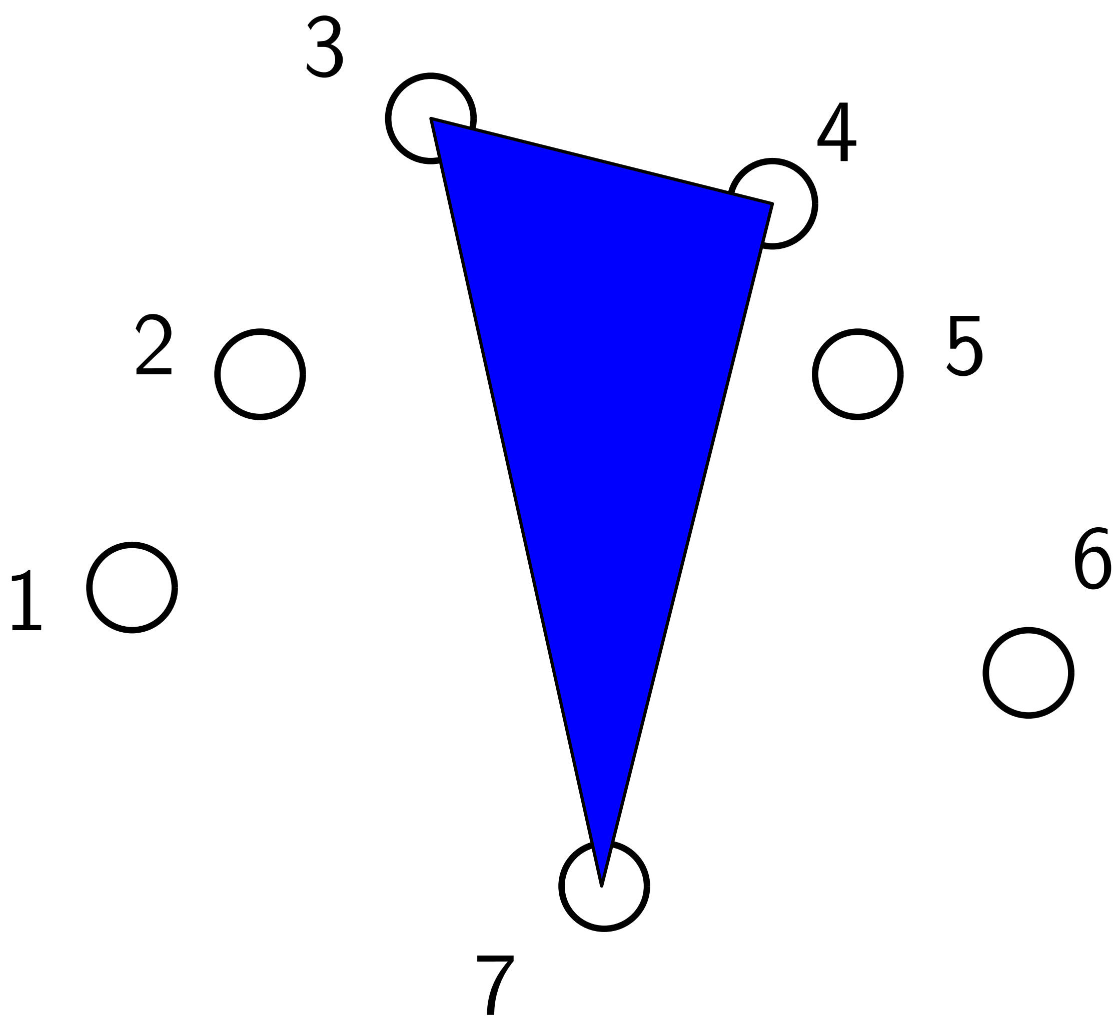

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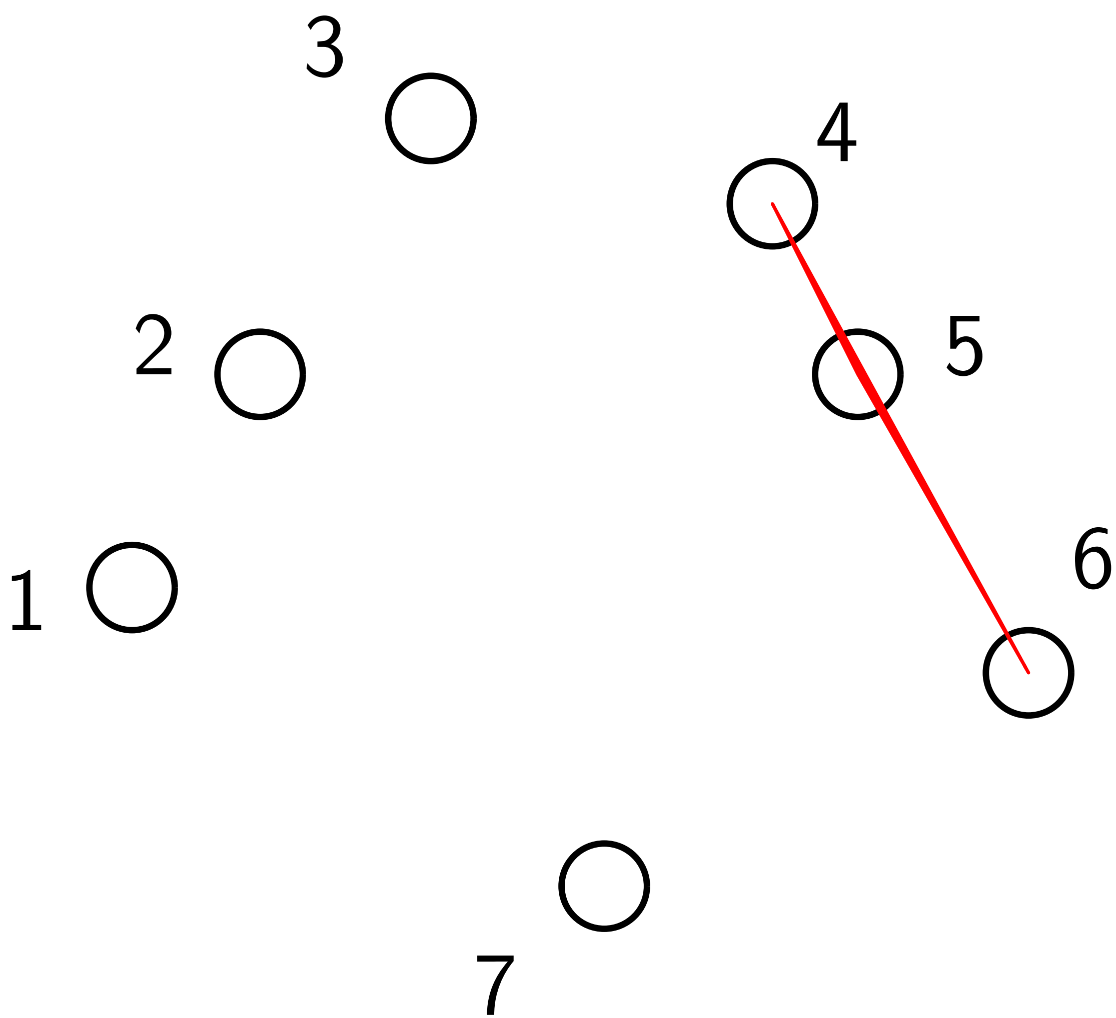

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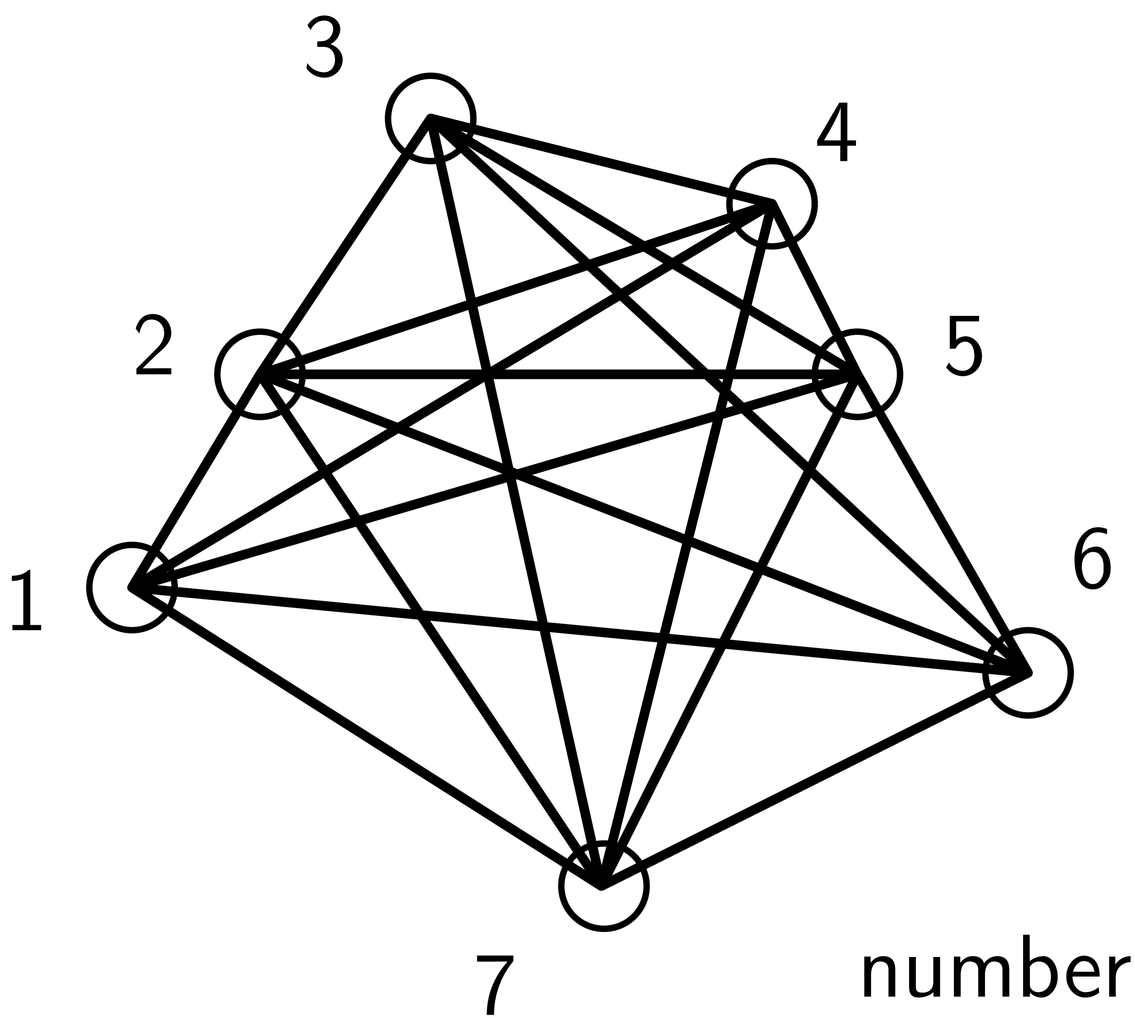

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$3 - 4 - 7$: yes

$4 - 5 - 6$: no

number of triangles: 33

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A loop

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For example, if $n = 4$, then triples (i, j, k) used by algorithm are (1,2,3), (1,2,4), (1,3,4), and (2,3,4).

Counting Triangles

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f is a bijection because
f is one-to-one
if $(i, j, k) \ne (i', j', k') \Rightarrow f(\,(i, j, k)\,) \ne f(\,(i', j', k'))$
f is onto
if γ is a 3-element subset then it can be written as $\gamma = \{i, j, k\}$
where $i < j < k$ so $f(\,(i, j, k)\,) = \gamma$.

Counting Pairs

- We've already seen something very similar.
  The number of
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  We actually already saw that $|X| = |Y| = \binom{n}{2}$

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  Currently, we started with the problem of counting the # of increasing triples and changed it to the problem of counting the # of 3-element sets from $\{1, 2, \ldots, n\}$

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  Note that the case of $k = n$ is special;
  An n-element permutation of a set N of size $|N| = n$ is what we earlier simply called a permutation.

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Ex: When $n = 4$, there are $4 \times 3 \times 2 = 24$
3 -element permutations of $\{1, 2, 3, 4\}$

$$L = \{123, 124, 132, 134, 142, 143, 213, 214, 231, 234, 241, 243$$
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Note: This type of "dictionary" ordering of tuples (assuming that we treat numbers the same as letters) is called a *lexicographic ordering* and is used quite often.

k-Element Permutations of a Set

- **Theorem** If N is a positive integer and k is an integer with $1 \leq k \leq n$, then there are
  $P(n, k) = n(n-1)(n-2)\cdots(n-k+1)$
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  $$P(n,3) = 3! \cdot C(n,3)$$

Binomial Coefficient

- **Theorem** For integers n and k with $0 \le k \le n$, the number of k-element subsets of an n-element set is
$$\binom{n}{k} = C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}.$$

  This is the number of k-combinations of a set with n elements.

Some Properties of Binomial Coefficients

- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the number of k-element subsets of an n-element set.

  $\binom{n}{0} = 1$ only one set of size 0.

  $\binom{n}{n} = 1$ only one set of size n.

  $\binom{n}{k} = \binom{n}{n-k}$ Obvious from equation. Can you think of a simple bijection that explains this?

Some Properties of Binomial Coefficients (cont.)

- $$\boxed{\sum_{i=0}^{n} \binom{n}{i} = 2^n}$$

Some Properties of Binomial Coefficients (cont.)

- $$\boxed{\sum_{i=0}^{n} \binom{n}{i} = 2^n}$$

  Use Sum Rule

  Let P = set of all subsets of $\{1,2,\ldots,n\}$

  S_i = set of all i subsets of $\{1,2,\ldots,n\}$

Some Properties of Binomial Coefficients (cont.)

- $$\boxed{\sum_{i=0}^{n} \binom{n}{i} = 2^n}$$

  Use Sum Rule

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  S_i = set of all i subsets of $\{1,2,\ldots,n\}$

  $$\Rightarrow |P| = \sum_{i=0}^{n} |S_i| = \sum_{i=0}^{n} \binom{n}{i}$$

Some Properties of Binomial Coefficients (cont.)

- Let $L = L_1 L_2 \ldots L_n$ be a list of size n from $\{0, 1\}$

  If $\mathcal{L} = \text{set of all such lists} \Rightarrow \; |\mathcal{L}| = 2^n$

  There is a *bijection* between $\mathcal{L}$ and P so
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  If $L \in \mathcal{L}$ then $f(L)$ is the set $S \subseteq \{1, 2, \ldots, n\}$ defined by

$$i \in S \;\Leftrightarrow\; L_i = 1$$

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Ex: $n = 5$

$$f(10101) = \{1, 3, 5\},\ f(11101) = \{1, 2, 3, 5\},\ f(00000) = \emptyset$$

Binomial Coefficients

$n \setminus k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

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Each row increases at first then decreases.

Second half of each row is the reverse of the first half.
Sum of items on n-th row is 2^n

Pascal's Triangle

Take the table

$n \backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
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6	1	6	15	20	15	6	1

and shift each row slightly so that middle element is in middle

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Pascal's Triangle

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Pascal's Triangle

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

What is the next row in the table?

Pascal's Triangle

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

Pascal's Triangle

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

Pascal identity

Each (non-1) entry in Pascal's Triangle is the sum of the two entries directly above it (to left and to right).

Pascal's Triangle

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
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  We will use a combinatorial proof.

A Combinatorial Proof

- $\binom{n}{k}$ is the number of k-element subsets of an n-element set.

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$$\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$$

Therefore, each term (left and right) represents the number of subsets of a particular size chosen from an appropriately sized set.

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  Number of $(k-1)$-subsets of an $(n-1)$-element set.

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  Number of k-subsets of an $(n-1)$-element set.

  Try to use sum principle to explain relationship among these three terms.

  Example: $n = 5$, $k = 2$

$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}.$$

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Consider $S = \{A, B, C, D, E\}$.

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Set S_1 of 2-subsets of S

$$\begin{aligned} S_1 = \{&\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{B, C\}, \\ &\{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\}. \end{aligned}$$

A Combinatorial Proof

$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}.$$

Consider $S = \{A, B, C, D, E\}$.

Set S_1 of 2-subsets of S can be partitioned into 2 disjoint parts.

S_2 the 2-subsets that contain E and

S_3, the set of 2-subsets that do not contain E.

$$S_1 = \{\{A,B\},\{A,C\},\{A,D\},\{A,E\},\{B,C\},\\ \{B,D\},\{B,E\},\{C,D\},\{C,E\},\{D,E\}\}.$$

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Proof: Apply sum rule.

Let S_1 be set of all k-element subsets.

To apply sum rule, partition S_1 into S_2 and S_3.

Let S_2 be set of k-element subsets that contain x_n.

Let S_3 be set of k-element subsets that don't contain x_n.

Blaise Pascal

Born 1623; Died 1662

French Mathematician

A Founder of Probability Theory

Inventor of one of the first mechanical calculating machines

Pascal Programming Language named for him

The Binomial Theorem

$$(x+y) = \binom{1}{0}x + \binom{1}{1}y$$

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$$\begin{aligned}(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\ &= \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3\end{aligned}$$

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Proof?

Application of the Binomial Theorem

- We may use the Binomial Theorem to prove

$$\boxed{\sum_{i=0}^{n} \binom{n}{i} = 2^n}$$

Labelling and Trinomial Coefficients

- Suppose we have k labels of one kind, e.g., red and $n-k$ labels of another, e.g., blue. In how many different ways can we apply these labels to n objects?

Labelling and Trinomial Coefficients

- Suppose we have k labels of one kind, e.g., red and $n-k$ labels of another, e.g., blue. In how many different ways can we apply these labels to n objects?

  Show that if we have k_1 labels of one kind, e.g., red, k_2 labels of a second kind, e.g., blue, and $k_3 = n - k_1 - k_2$ labels of a third kind, then there are $\frac{n!}{k_1!k_2!k_3!}$ ways to apply these labels to n objects

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  What is the coefficient of $x^{k_1}y^{k_2}z^{k_3}$ in $(x+y+z)^n$?

Labelling and Trinomial Coefficients

- There are $\binom{n}{k_1}$ ways to choose the red items There are then $\binom{n-k_1}{k_2}$ ways to choose the blue items from the remaining $n - k_1$. The remaining k_3 items get labelled a third color.

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  Using the *product rule* the total number of labellings is

$$
\begin{aligned}
\binom{n}{k_1}\binom{n-k_1}{k_2} &= \frac{n!}{k_1!(n-k_1)!}\frac{(n-k_1)!}{(k_2)!(n-k_1-k_2)!} \\
&= \frac{n!}{k_1!k_2!(n-k_1-k_2)!} = \frac{n!}{k_1!k_2!k_3!}
\end{aligned}
$$

Labelling and Trinomial Coefficients

- When $k_1 + k_2 + k_3 = n$, we call

$$\frac{n!}{k_1!k_2!k_3!}$$

a *trinomial coefficient* and denote it as

$$\binom{n}{k_1 \quad k_2 \quad k_3}$$

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  This will be very similar to the analysis of hashing n keys into a table of size 365.

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  Sample space: $|S| = 365^n$

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  $\#B_n = 365 \times 364 \times \cdots \times (365 - (n-1))$

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  Sample space: $|S| = 365^n$

  B_n – "there are n students in a room and none of them share a birthday."

  $\#B_n = 365 \times 364 \times \cdots \times (365 - (n-1))$

  $\#A_n + \#B_n = 365^n$

The Birthday Paradox

n	A_n	B_n	n	A_n	B_n
1	0.00000000	1.00000000	16	0.28360400	0.71639599
2	0.00273972	0.99726027	17	0.31500766	0.68499233
3	0.00820416	0.99179583	18	0.34691141	0.65308858
4	0.01635591	0.98364408	19	0.37911852	0.62088147
5	0.02713557	0.97286442	20	0.41143838	0.58856161
6	0.04046248	0.95953751	21	0.44368833	0.55631166
7	0.05623570	0.94376429	22	0.47569530	0.52430469
8	0.07433529	0.92566470	23	0.50729723	0.49270276 . . .
9	0.09462383	0.90537616	24	0.53834425	0.46165574
10	0.11694817	0.88305182	25	0.56869970	0.43130029
11	0.14114137	0.85885862	26	0.59824082	0.40175917
12	0.16702478	0.83297521	27	0.62685928	0.37314071
13	0.19441027	0.80558972	28	0.65446147	0.34553852
14	0.22310251	0.77689748	29	0.68096853	0.31903146
15	0.25290131	0.74709868	30	0.70631624	0.29368375

Next Lecture

- counting II, relation, ...