DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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Planar Graphs

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- **Theorem** (Euler's Formula) Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

The Degree of Regions

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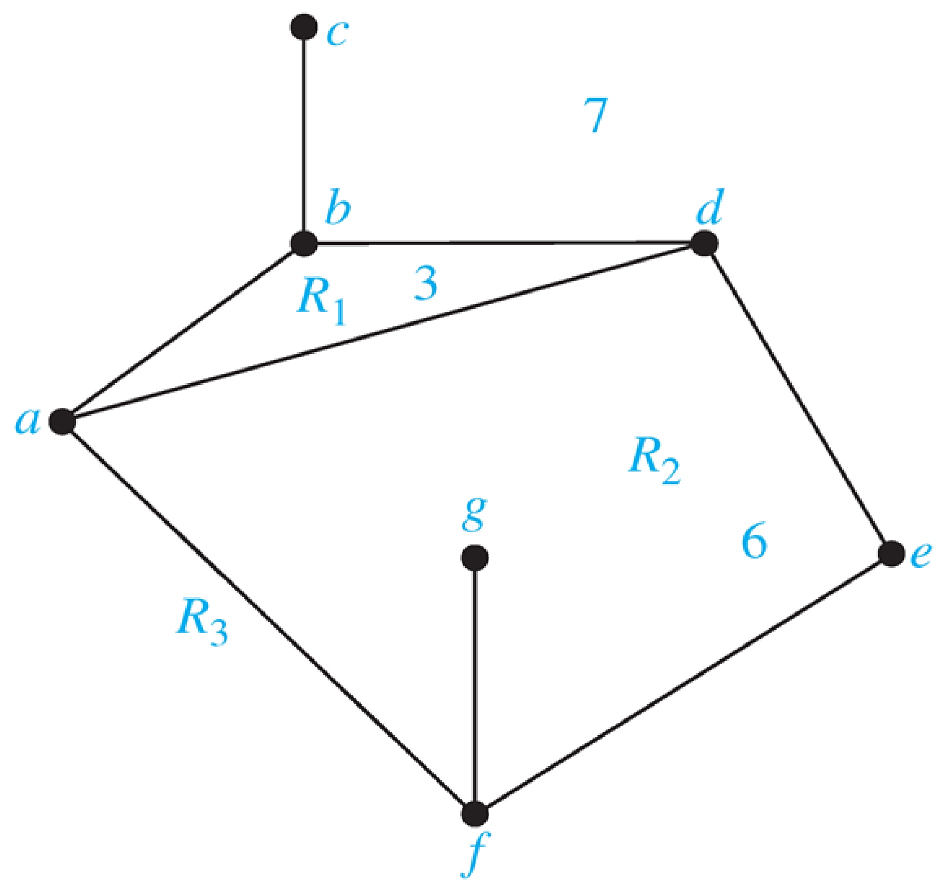

Corollaries

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  Proof The degree of every region is at least 3.

  - G is simple
  - $v \geq 3$

  The sum of the degrees of the regions is exactly twice the number of edges in the graph.

  $$2e = \sum_{\text{all regions } R} \deg(R) \geq 3r$$

  By Euler's formula, the proof is completed.

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  Proof similar to that of Corollary 1.

Examples

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  Corollary 2 is used in the proof of Five Color Theorem.

Kuratowski's Theorem

- **Definition** If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an *elementary subdivision*. The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called *homomorphic* if they can be obtained from the same graph by a sequence of elementary subdivisions.

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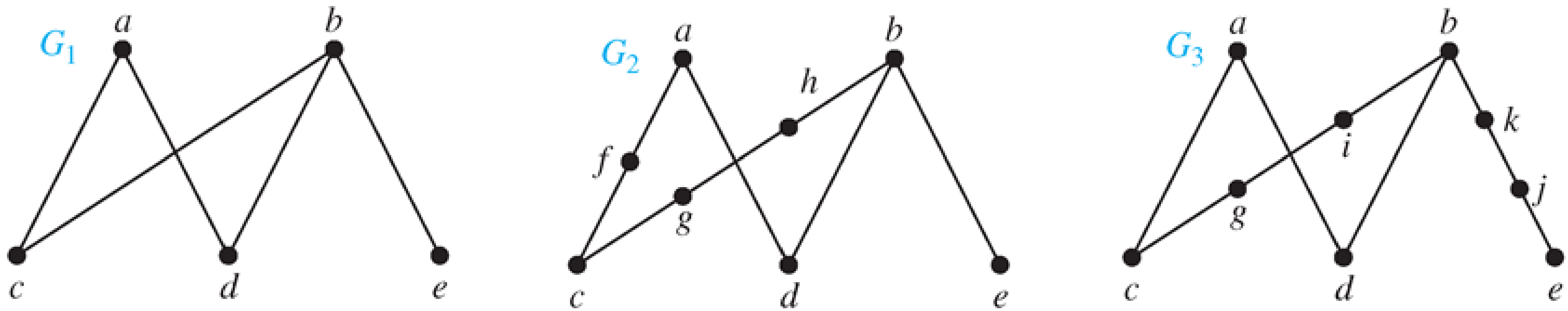

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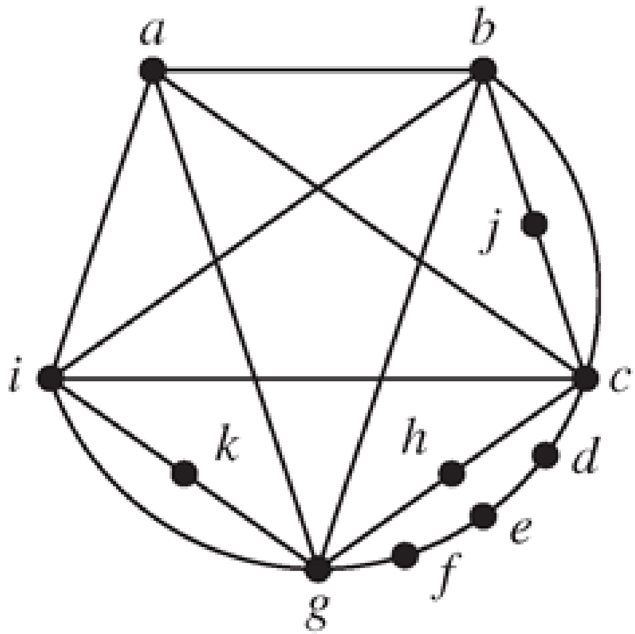
  Theorem A graph is nonplanar if and only if it contains a subgraph homomorphic to $K_{3,3}$ or K_5.

Examples



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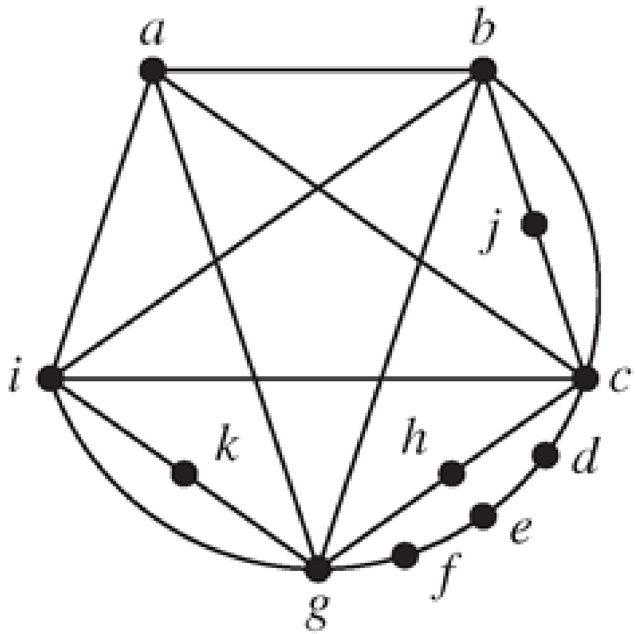

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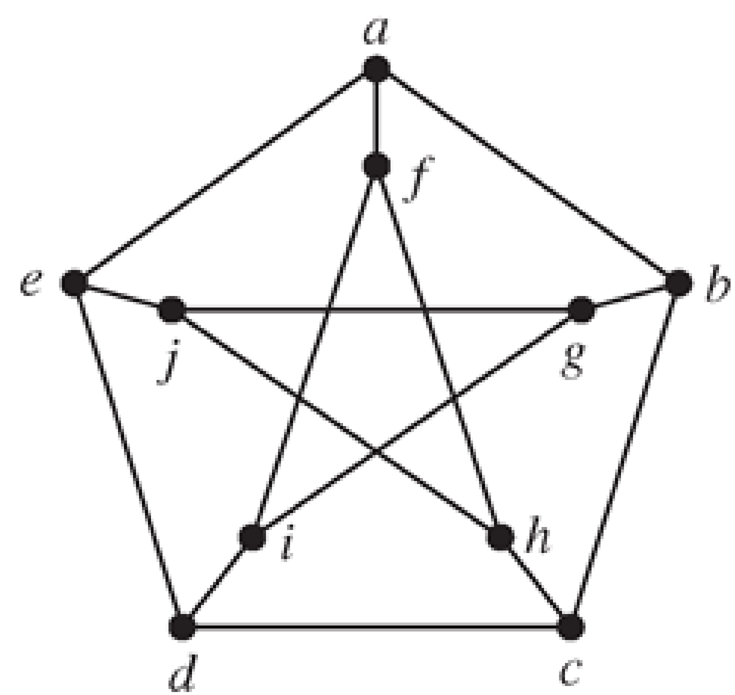

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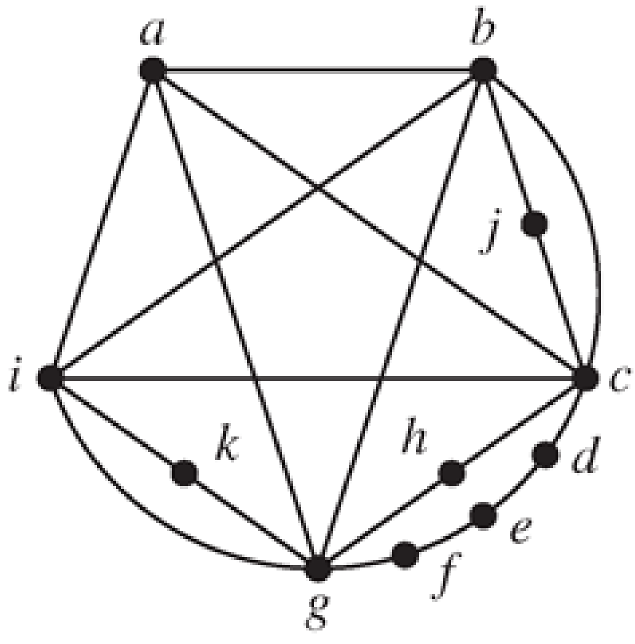

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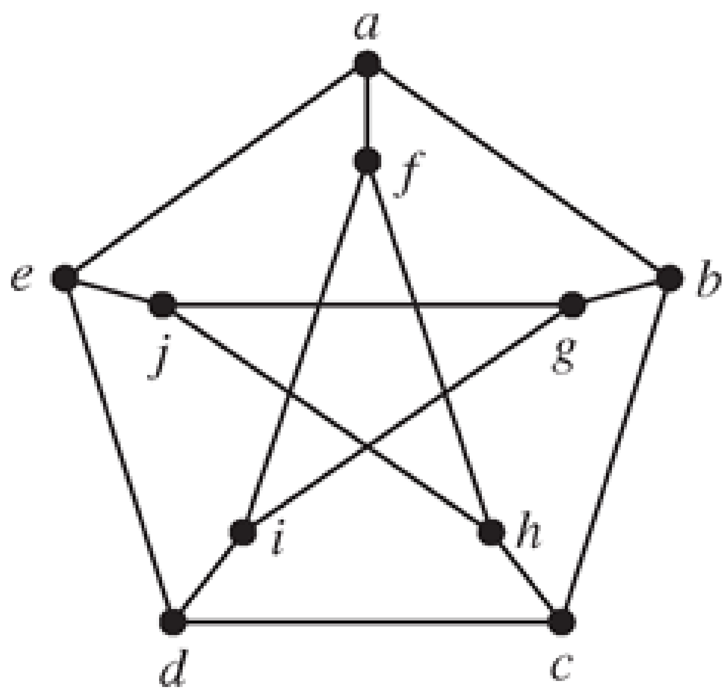
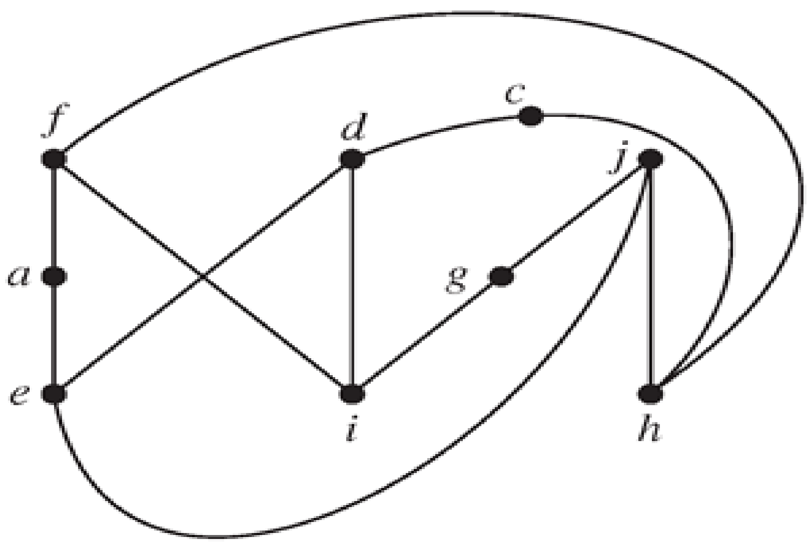
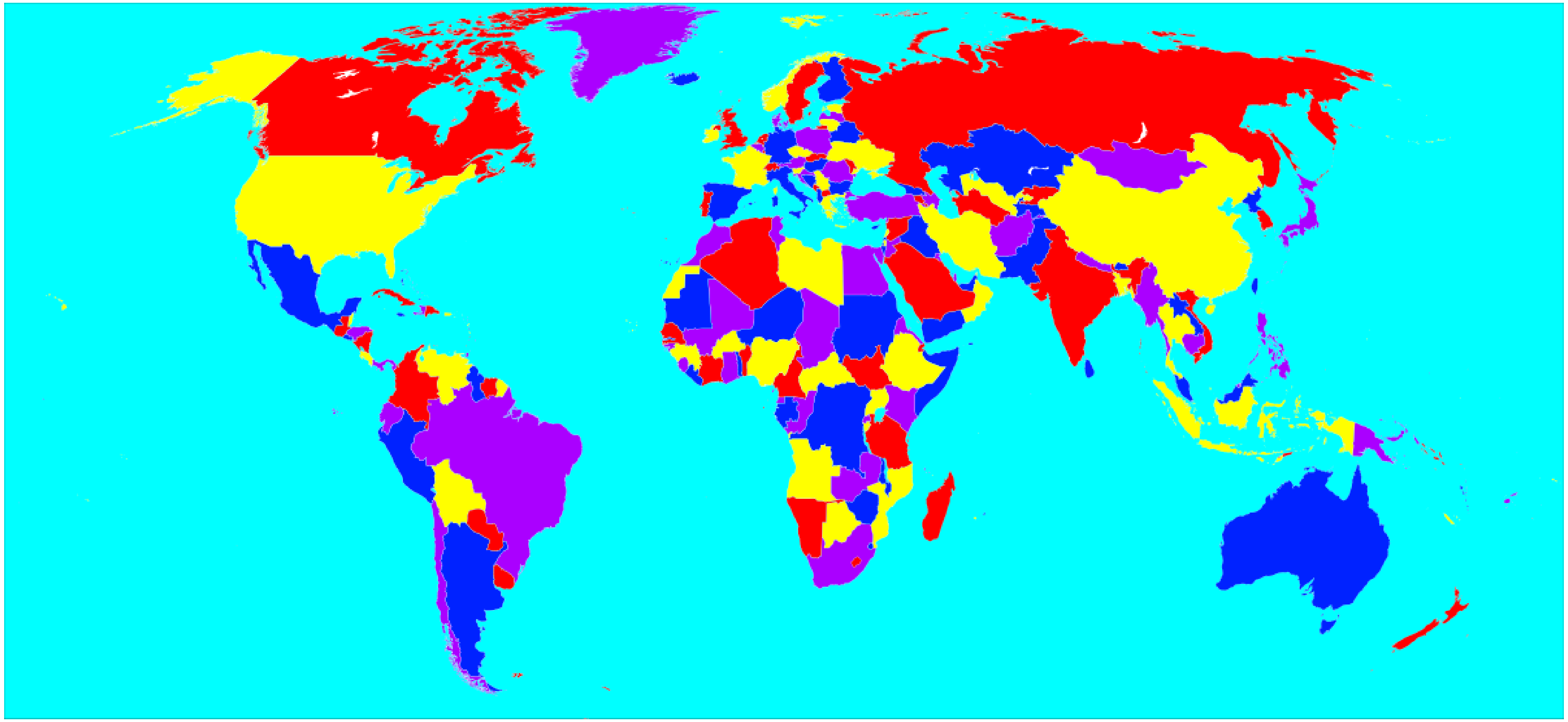




Graph Coloring

- **Four-color theorem** Given any separation of a plane into contiguous regions, producing a figure called a *map*, no more that four colors are required to color the regions of the map so that no two adjacent regions have the same color.



Graph Coloring

Four-color theorem

◇ first proposed by Francis Guthrie in 1852

◇ his brother Frederick Guthrie told Augustus De Morgan

◇ De Morgan wrote to William Hamilton

◇ Alfred Kempe proved it incorrectly in 1879

◇ Percy Heawood found an error in 1890 and proved the *five-color theorem*

◇ Finally, Kenneth Appel and Wolfgang Haken proved it with case by case analysis by computer in 1976 (*the first computer-aided proof*)

◇ Kempe's incorrect proof serves as a basis

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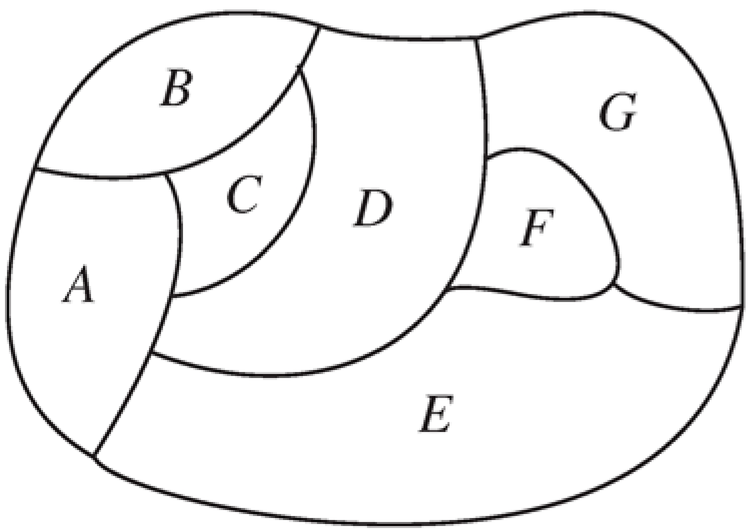

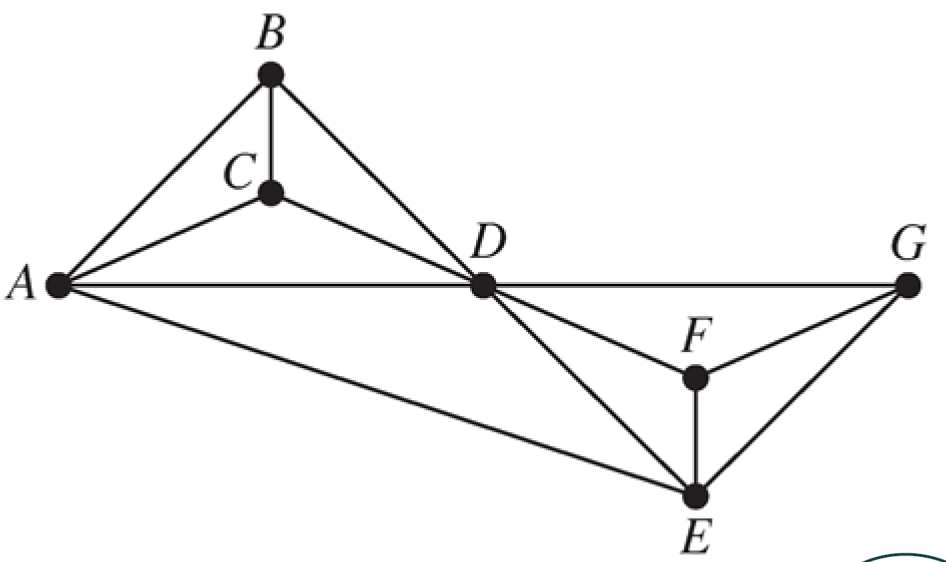

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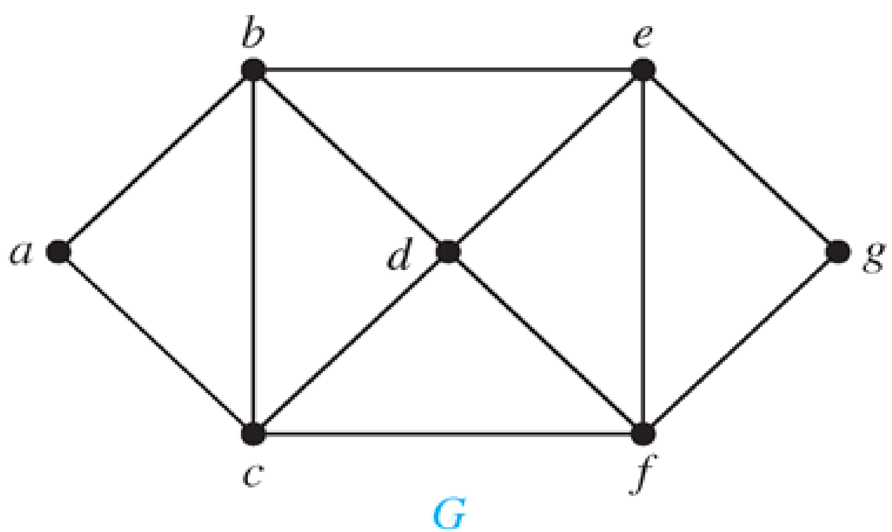

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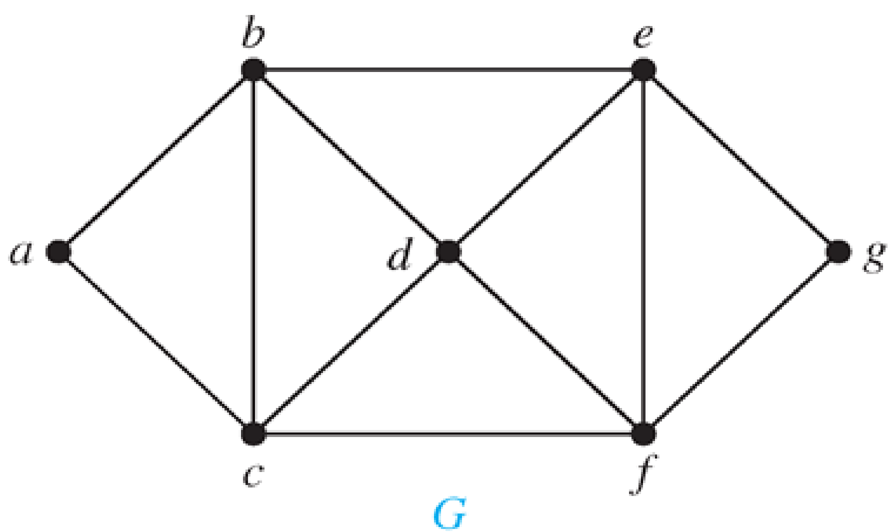

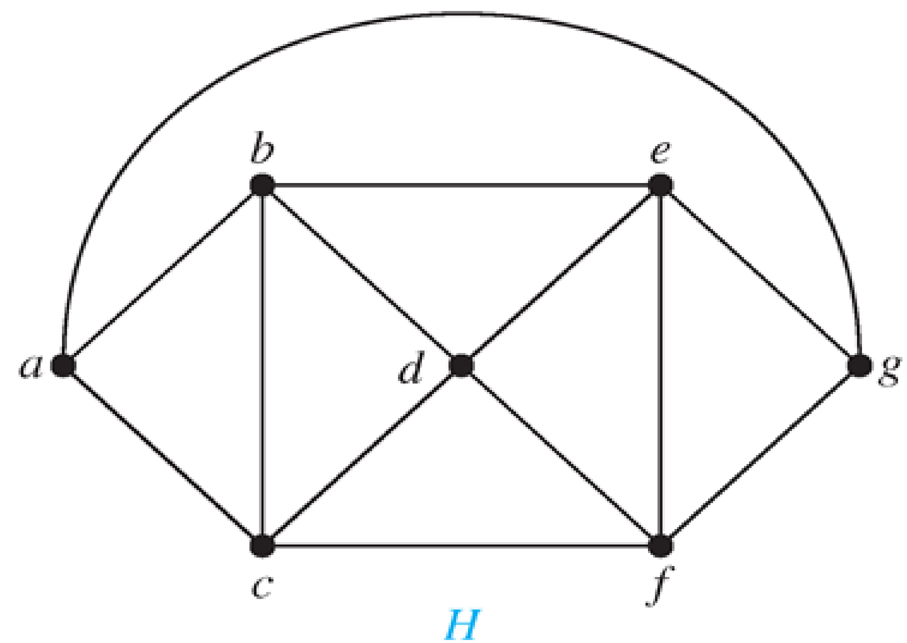

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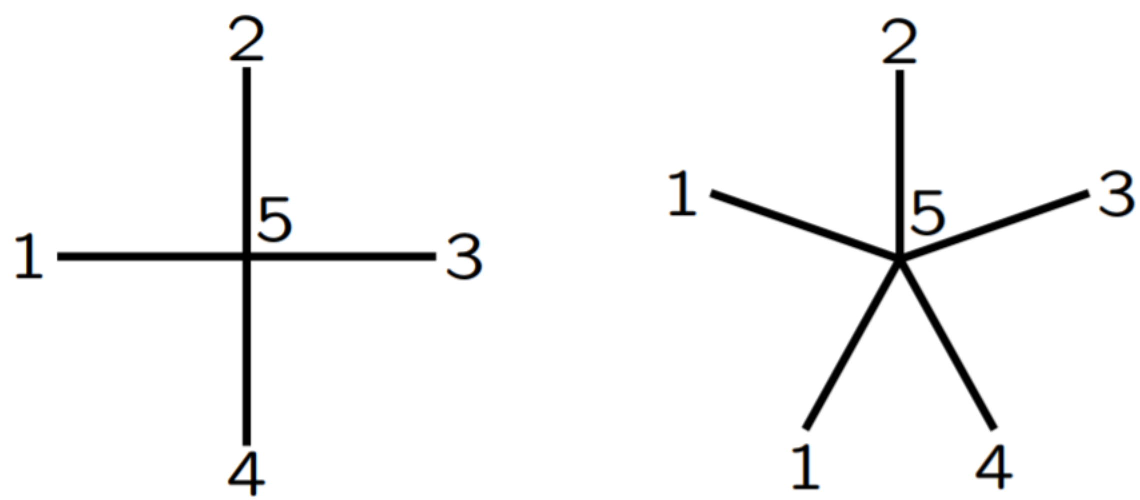
  If the vertex has degree less than 5, or if it has degree 5 and only ≤ 4 colors are used for vertices connected to it, we an pick an available color for it.



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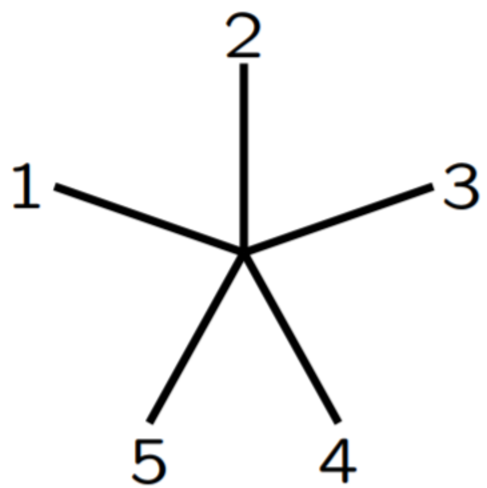
  If the vertex has degree 5, and all 5 colors are connected to it, we label the vertices adjacent to the "special" vertex (degree 5) 1 to 5 (in order).



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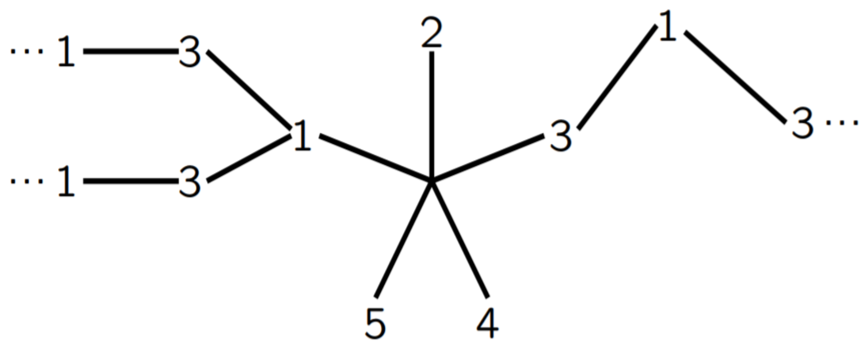

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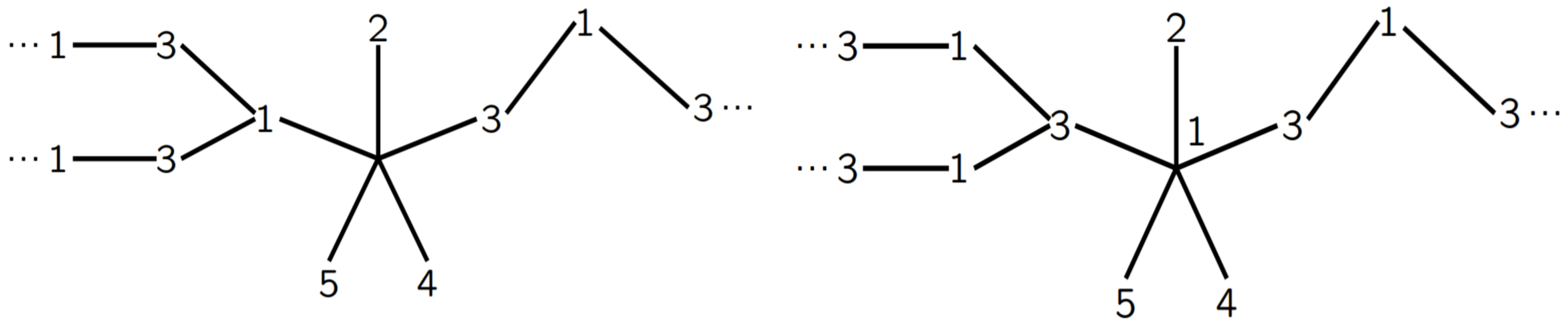

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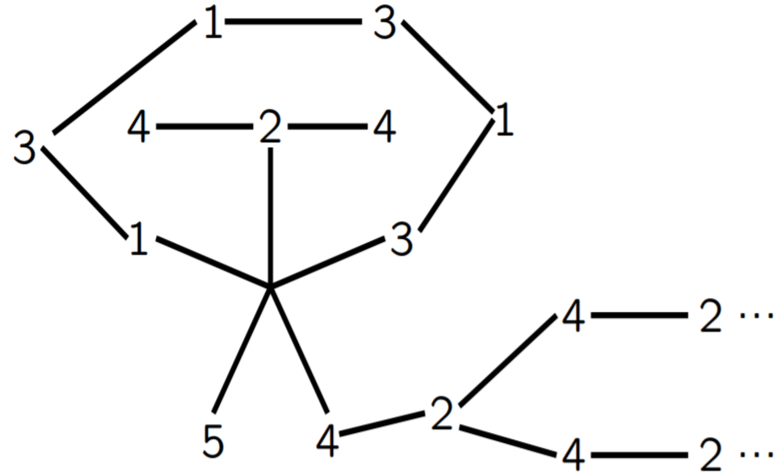

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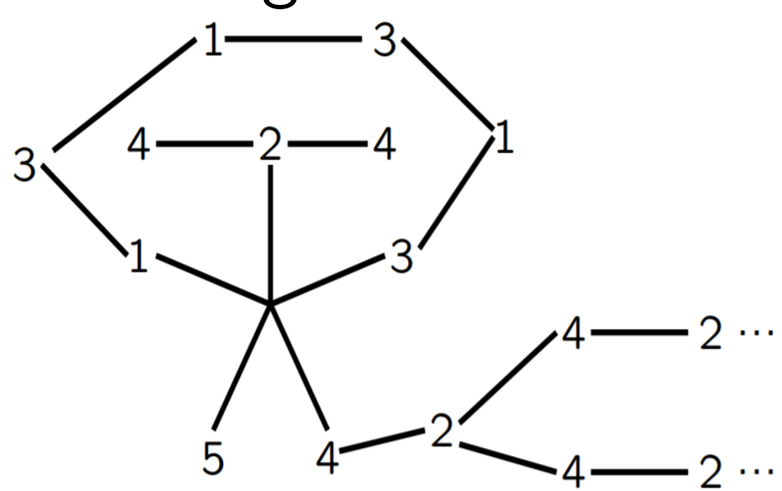

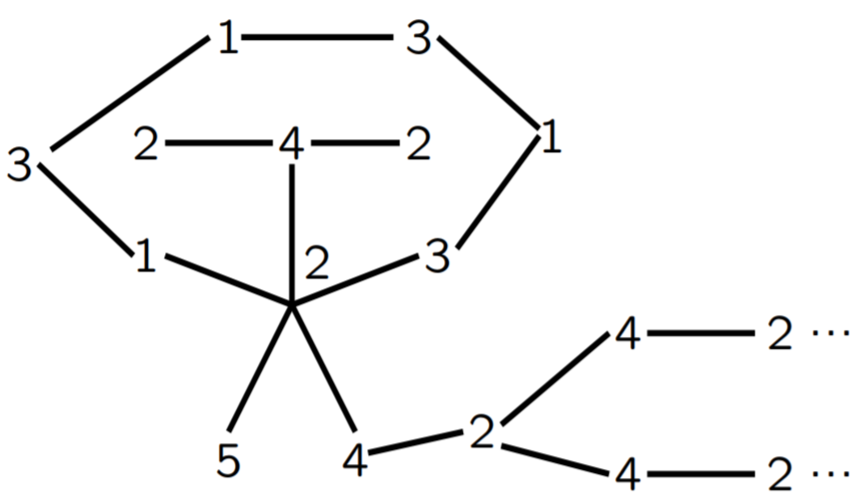

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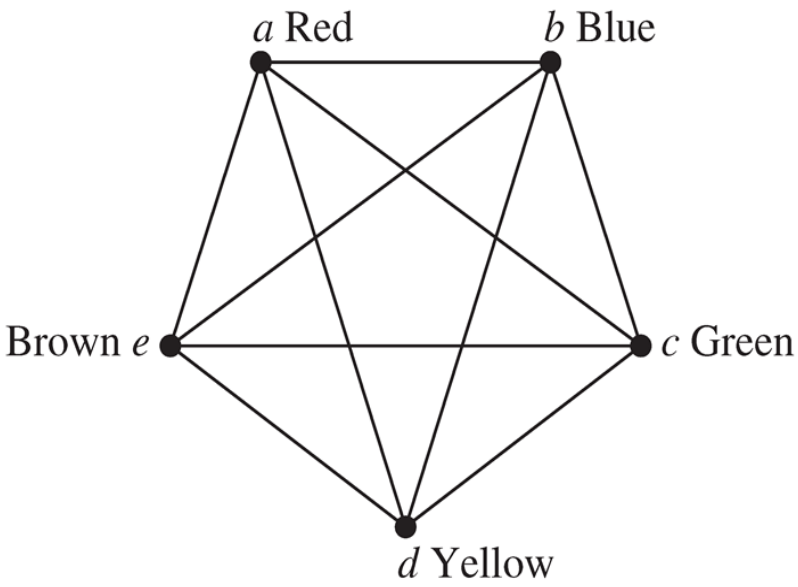

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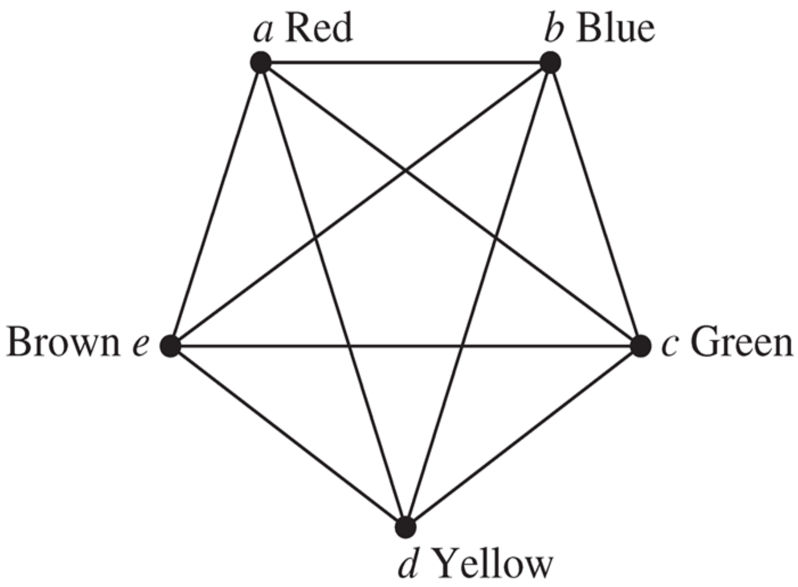

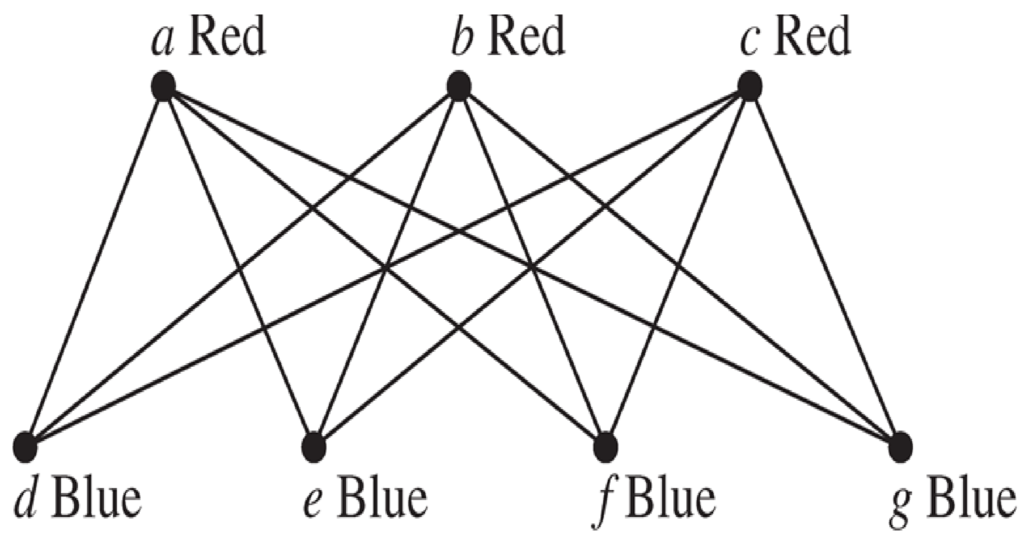

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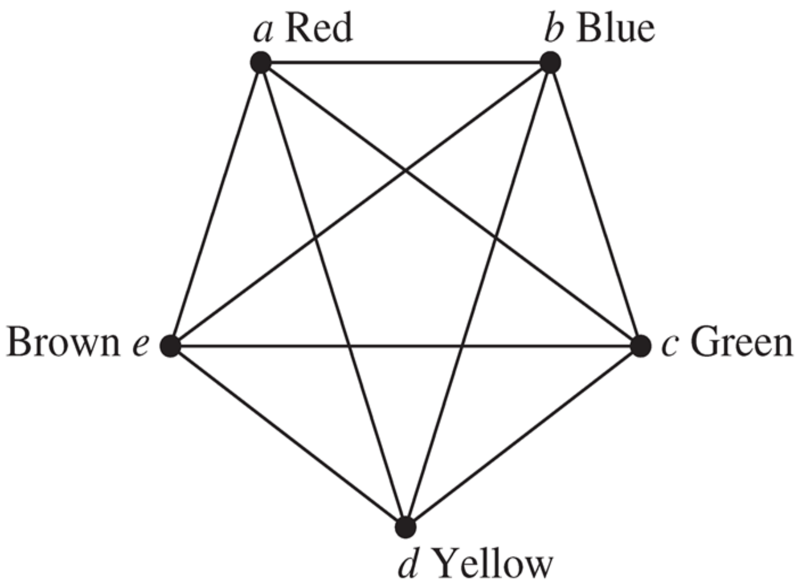

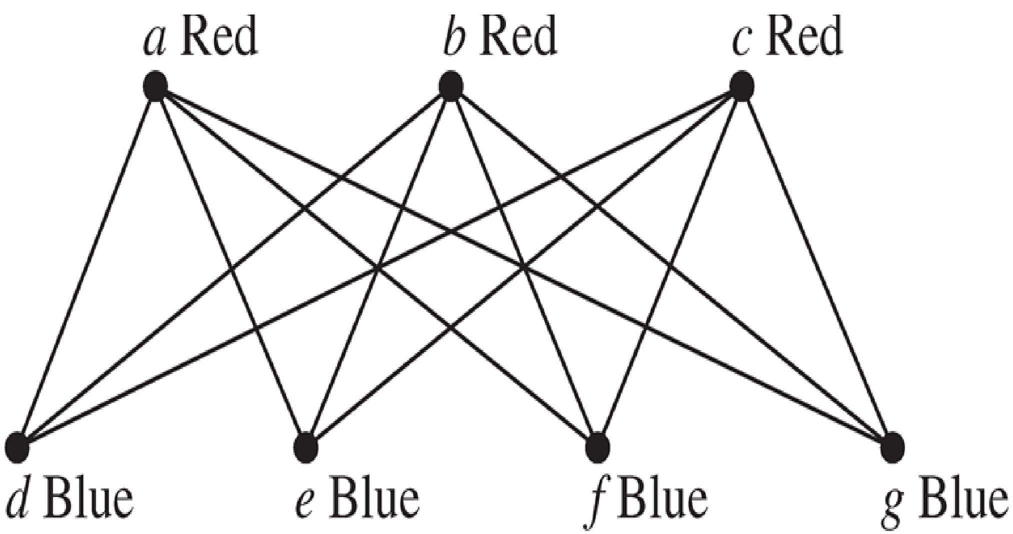

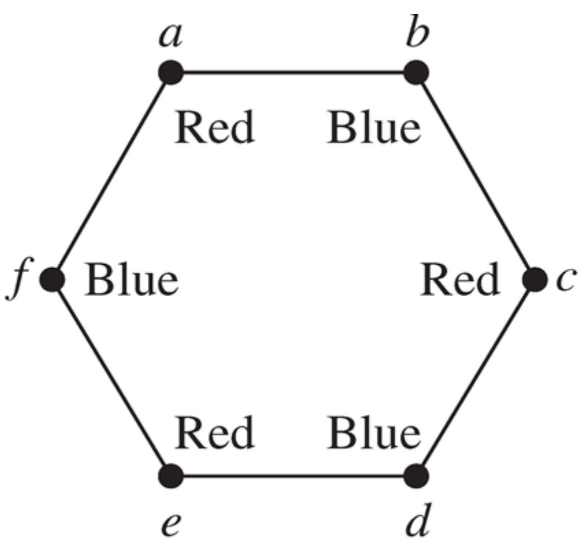

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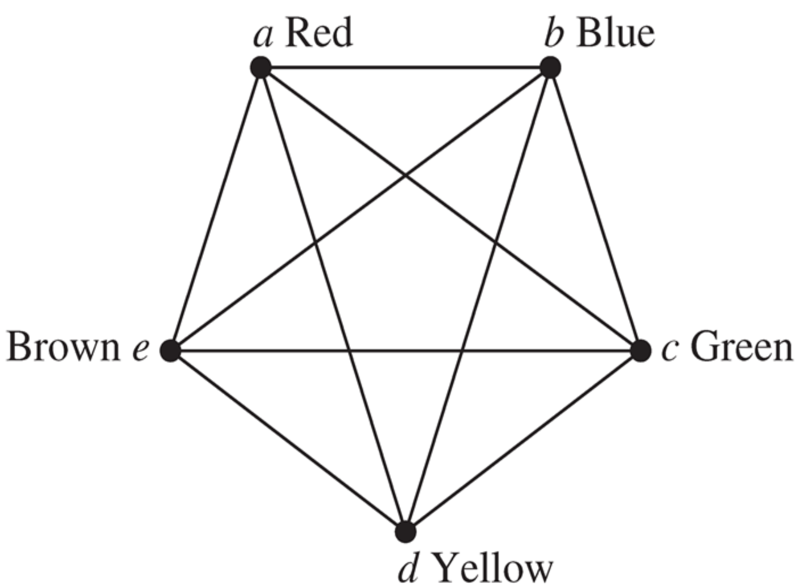
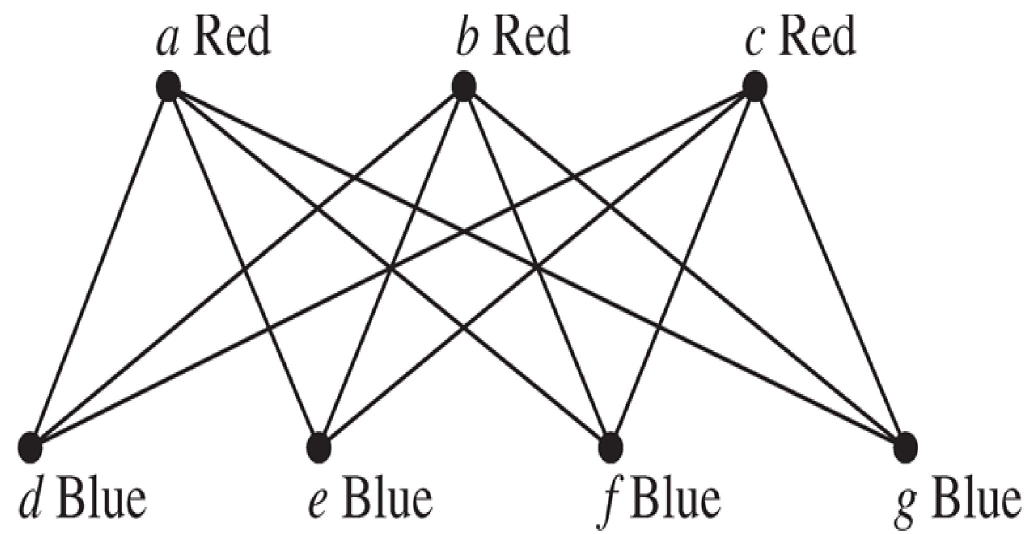
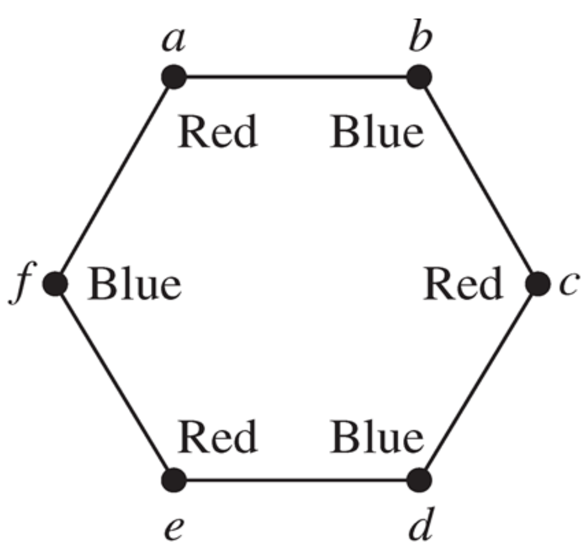
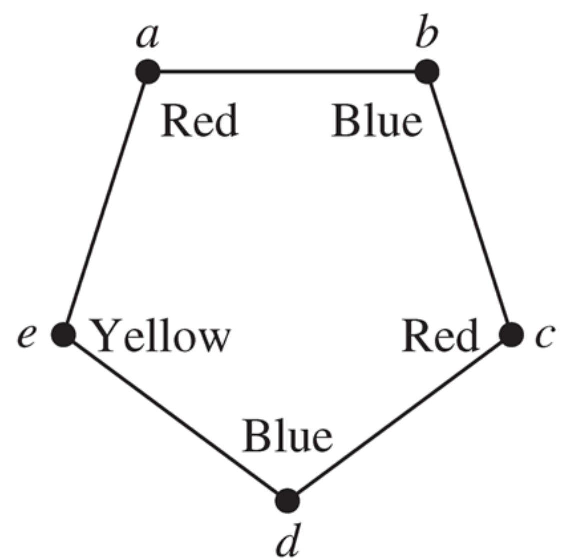




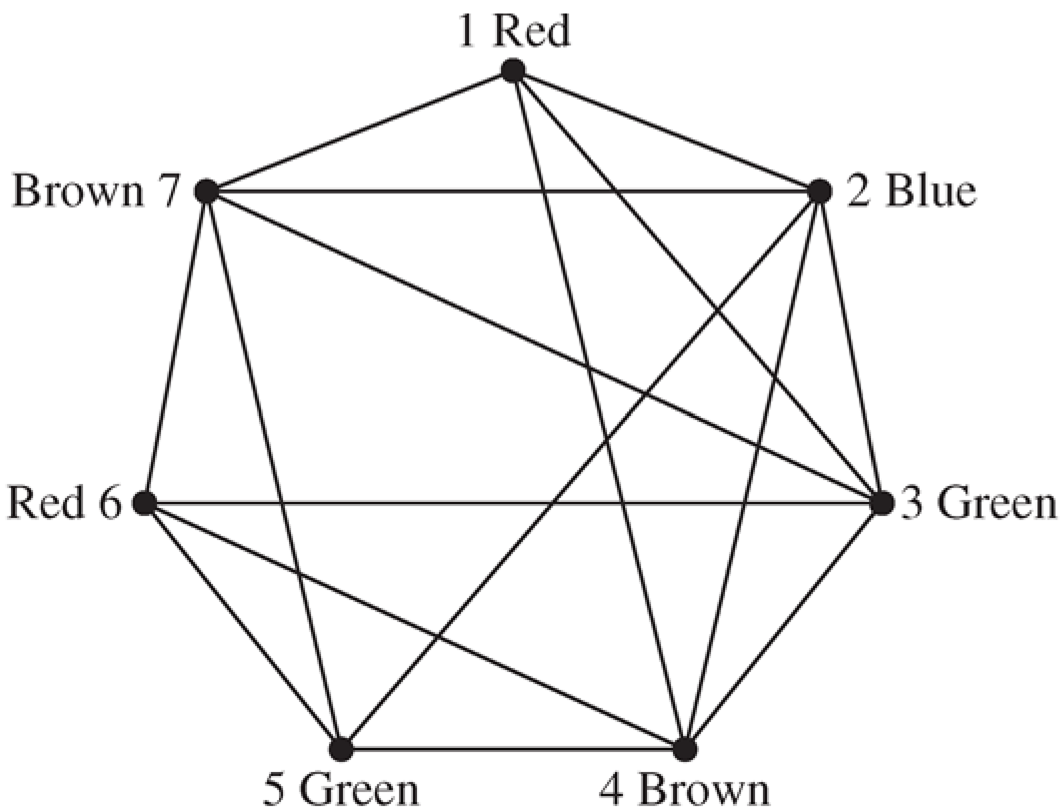




Applications of Graph Coloring

- Scheduling Final Exams

  Vertices represent courses, and there is an edge between two vertices if there is a common student in the courses.



Time Period	Courses
I	1, 6
II	2
III	3, 5
IV	4, 7

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  Televesion channels 2 through 13 are assigned to stations in North America so that no two stations within 150 miles can operate on the same channel . How can the assignment of channels be modeled by graph coloring?

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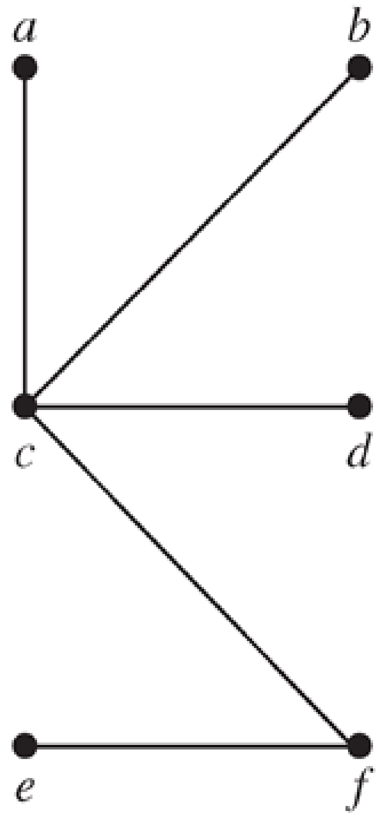

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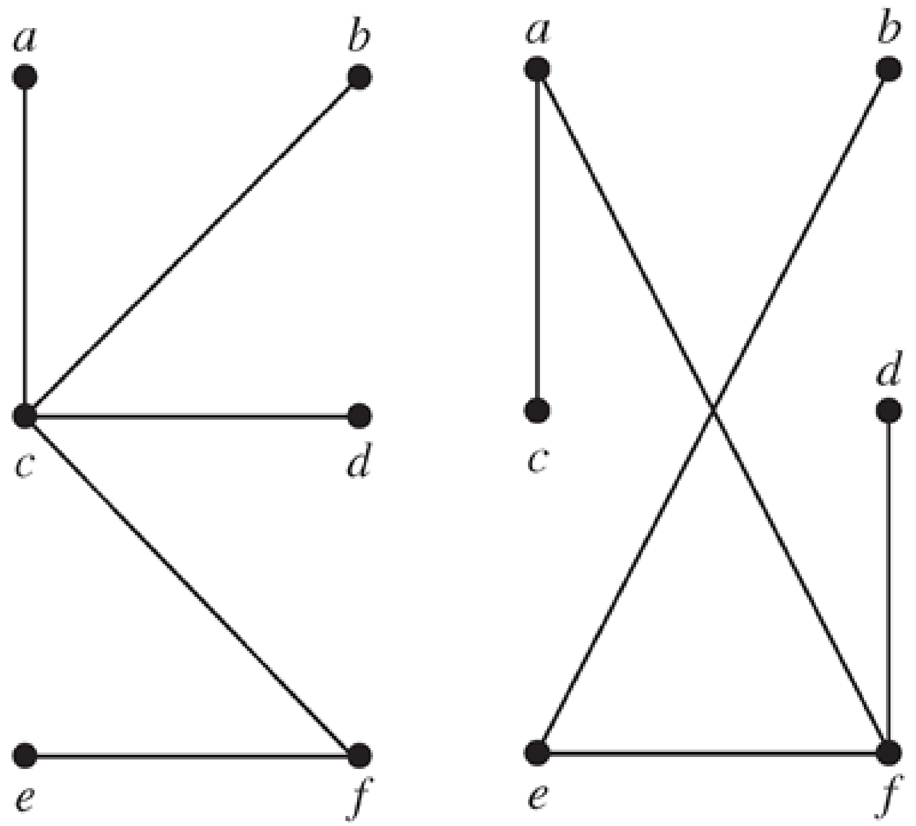

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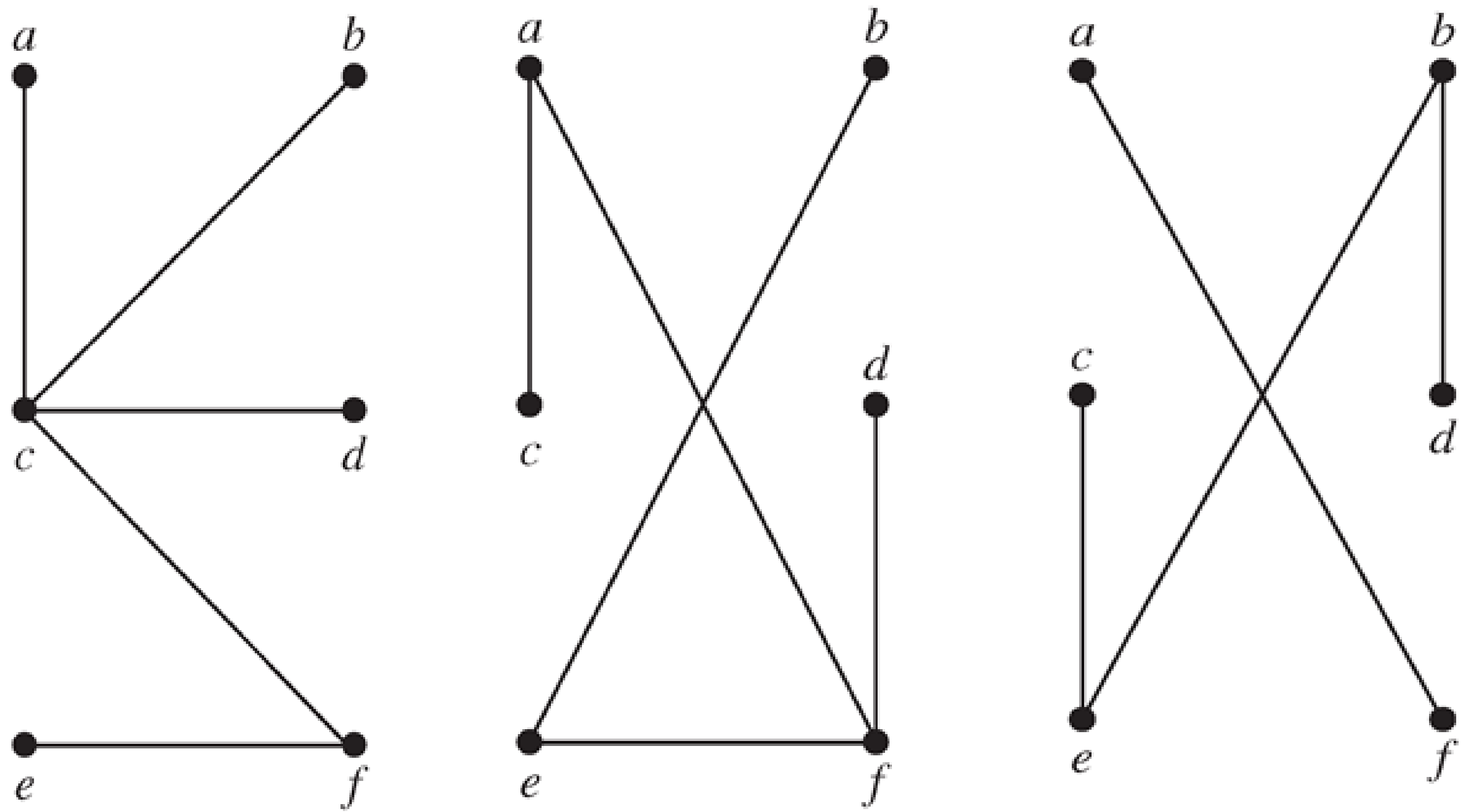

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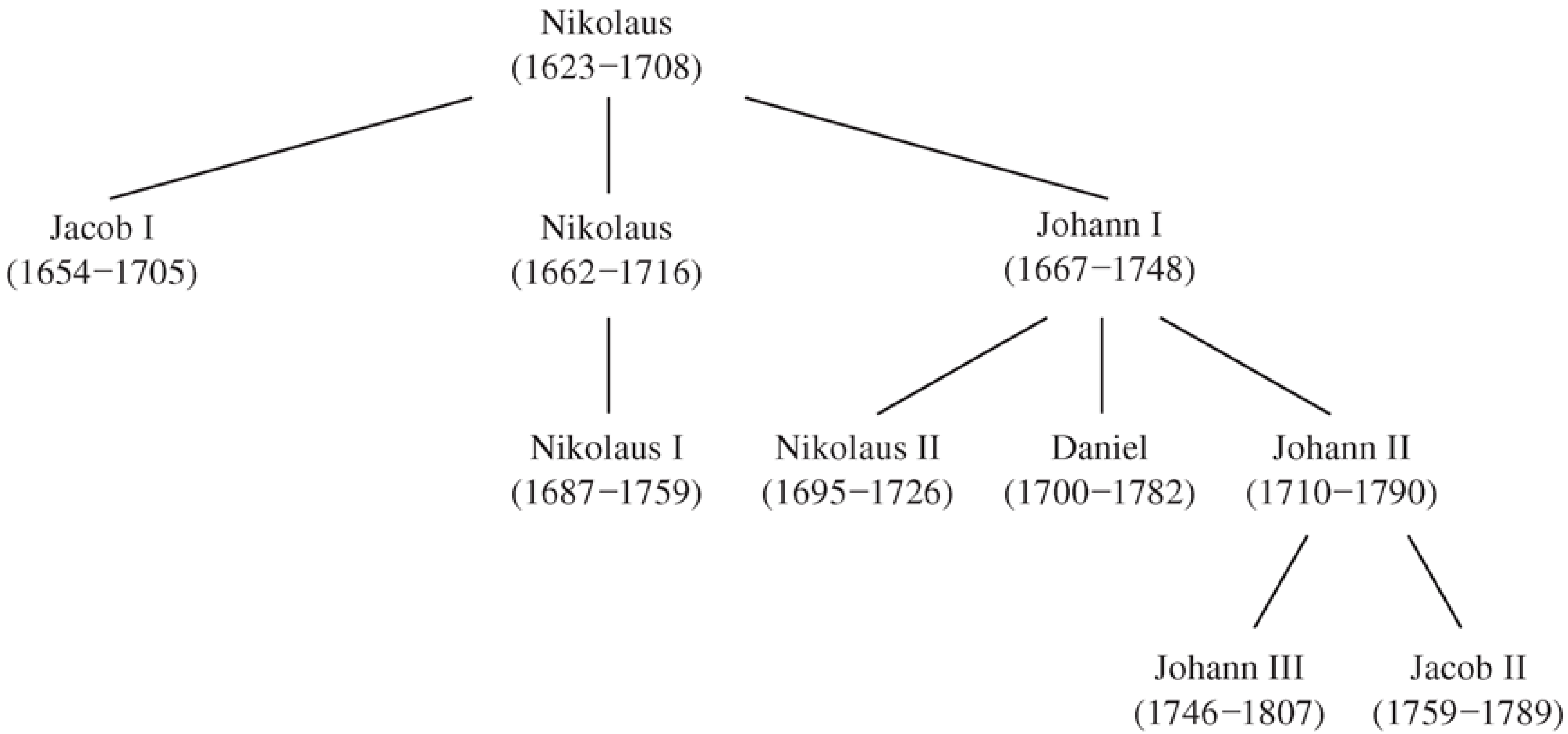

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Two properties of tree: connected, no circuit

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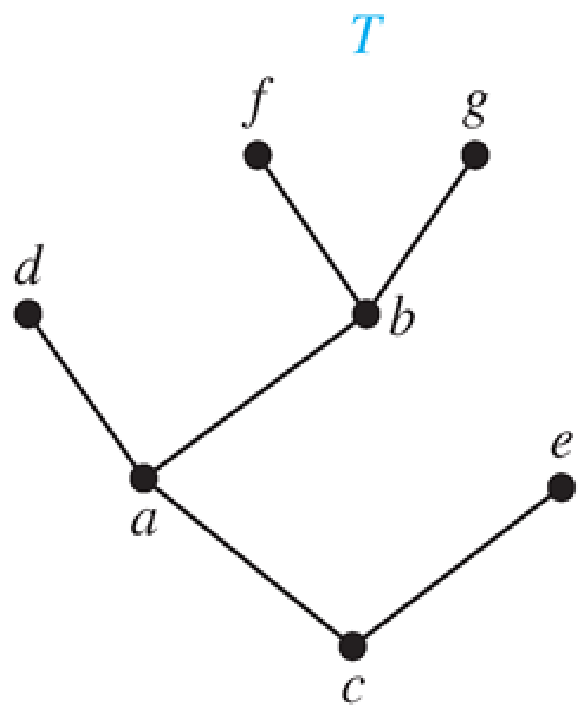

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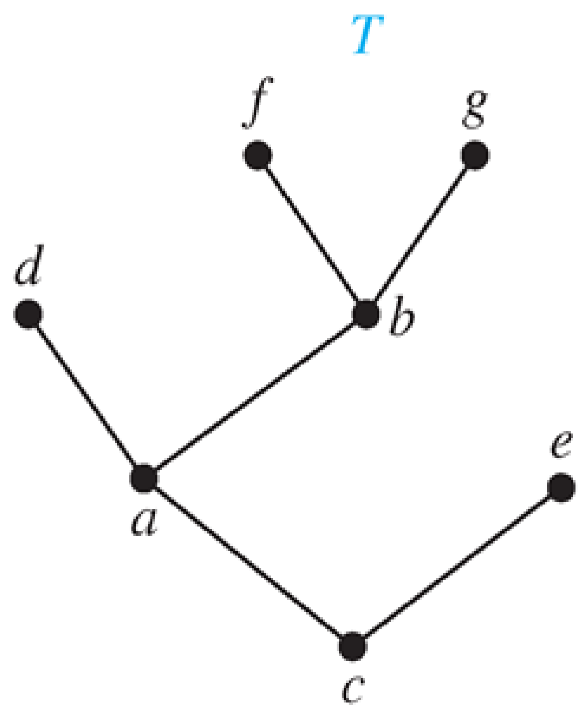

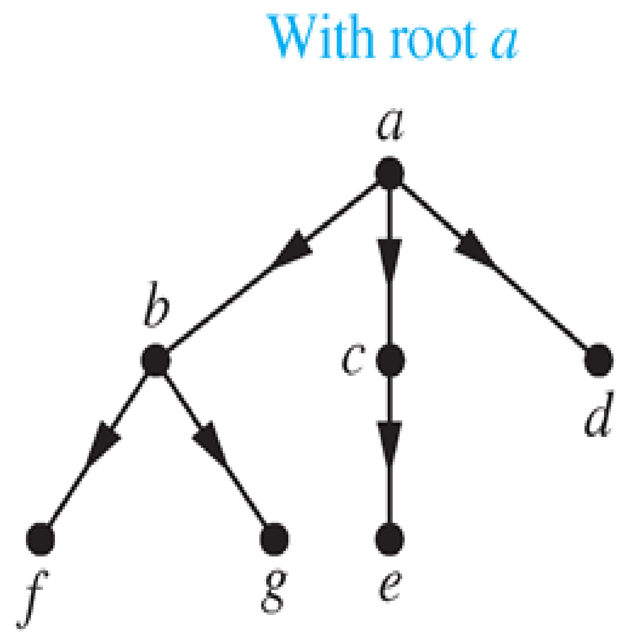

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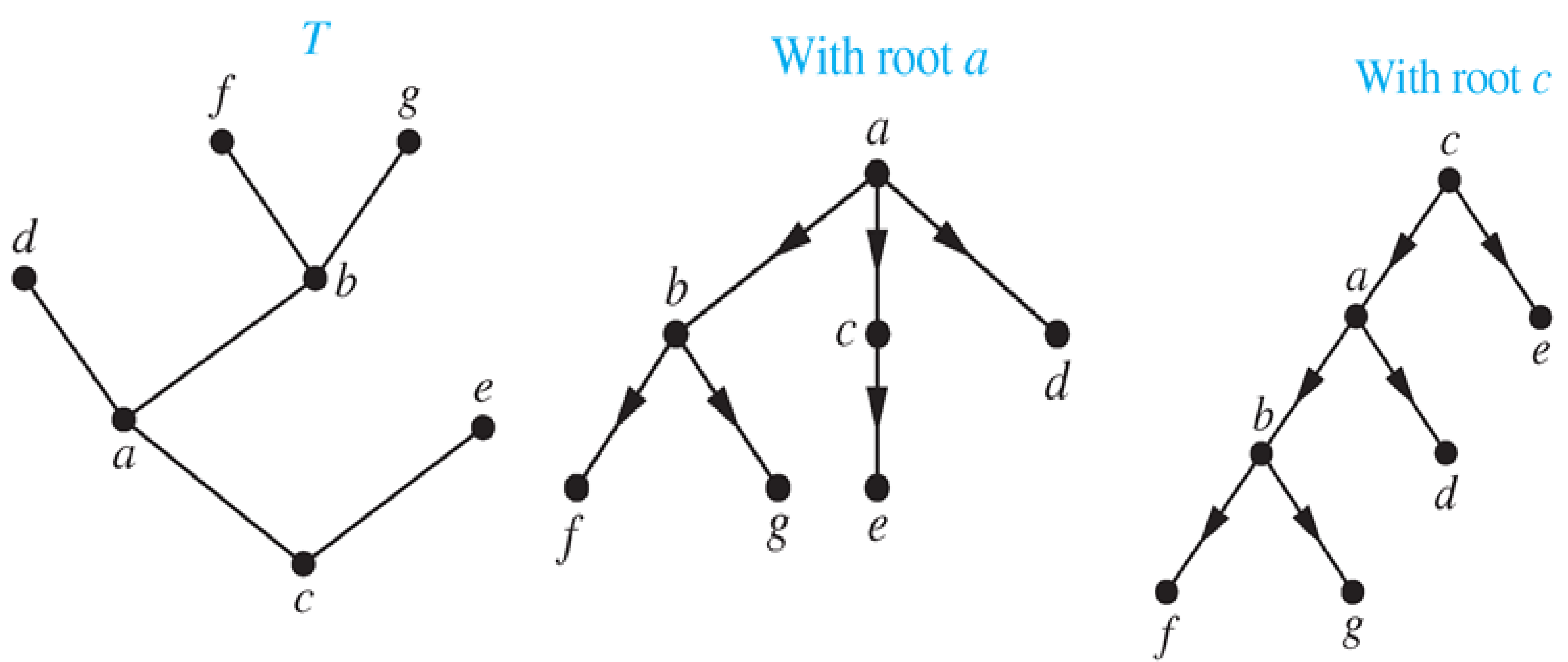

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  subtree with a as its root: consists of a and its descendants and all edges incident to these descendants

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  using $n = mi + 1$ and $n = i + \ell$

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  Definition A rooted m-ary tree of height h is *balanced* if all leaves are at levels h or $h-1$. (differ no greater than 1)

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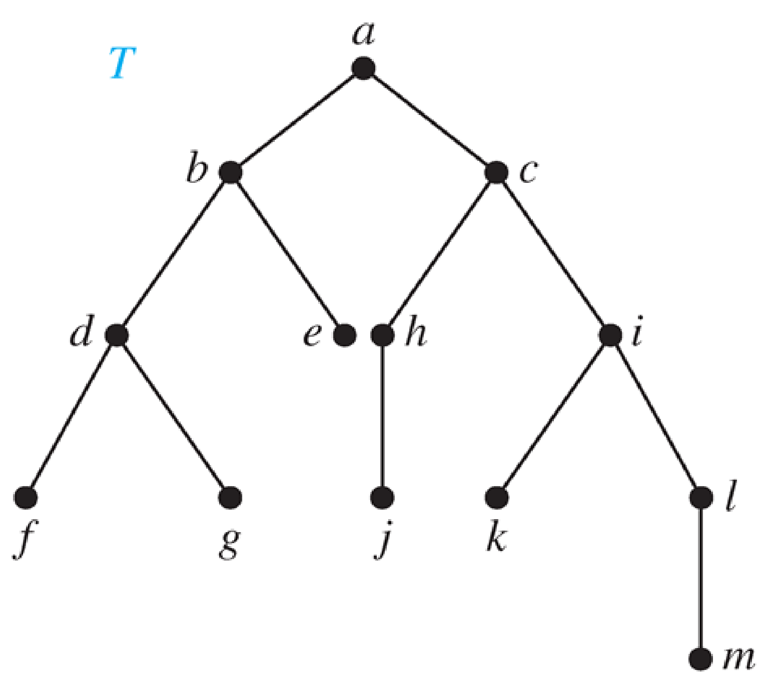

Next Lecture

- tree2 ...