CS 201: Discrete Math for Computer Science
2017 Fall Semester
Written Assignment # 6

P650**, Ex.** 11**.**

Let G be a simple graph. Show that the relation R on the set of vertices of G such that rRv if and only if there is an edge associated to $\{u, v\}$ is a symmetric, irreflexive relation on G.

Solution:

If uRv, then there is an edge associated with $\{u, v\}$. But $\{u, v\} = \{v, u\}$, so this edge is associated with $\{v, u\}$ and therefore vRu. Thus, by definition, R is a symmetric relation. A simple graph does not allow loops; therefore uRu never holds, and so by definition R is irreflexive.

□

P668**, Ex.** 64**.**

Show that if G is bipartite simple graph with v vertices and e edges, then $e \leq v^2/4$.

Solution:

Suppose that the parts are of sizes k and $v - k$, respectively. Then the maximum number of edges of the graph may have is $k(v - k)$. By algebra, we know that the function $f(k) = k(v-k)$ achieves its maximum value when $k = v/2$, giving $f(k) = v^2/4$. Thus there are at most $v^2/4$ edges.

□

P668**, Ex.** 66**.**

Describe an algorithm to decide whether a graph is bipartite based on the fact that a graph is bipartite if and only if it is possible to color its vertices two different colors so that no two vertices of the same color are adjacent.

Solution:

We start by coloring any vertex red. Then we color all the vertices adjacent to this vertex blue. Then we color all the vertices adjacent to blue vertices red, then color all the vertices adjacent to red vertices blue, and so on. If we ever are in the position of trying to color a vertex with the color opposite to the color it already has, then we stop and know that the graph is not bipartite. If the process terminates (successfully) before all the vertices have been colored, then we color some uncolored vertex red (it will necessarily not be adjacent to any vertices we have already colored) and begin the process again. Eventually we will have either colored all the vertices (producing the bipartition) or stopped and decided that the graph is not bipartite.

□

P676, Ex. 28

What is the sum of the entries in a row of the adjacency matrix for an undirected graph? For a directed graph?

Solution:

For an undirected graph, the sum of the entries in the ith row is the same as the corresponding column sum, namely the number of edges incident to the vertex i, which is the same as the degree of i minus the number of loops at i (since each loop contributes 2 toward the degree count).

For a directed graph, the sum of the entries in the ith row is the number of edges that have i as their initial vertex, i.e., the out-degree of i.

□

P676, Ex. 29

What is the sum of the entries in a column of the adjacency matrix for an undirected graph? For a directed graph?

Solution:

The number of edges incident to the vertex i, which is the same as the degree of i minus the number of loops at i

For a directed graph, the number is the in-degree of i.

□

P676, Ex. 32

Find an adjacency matrix for each of these graphs.

a) K_n b) C_n c) W_n d) $K_{m,n}$ e) Q_n

Solution:

a) The adjacency matrix for K_n is an $n \times n$ matrix that has 0's on the main diagonal and 1's elsewhere, namely

$$\begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{bmatrix}$$

b) We label the vertices so that the cycle goes $v_1, v_2, \dots, v_n, v_1$. The matrix has 1's on the diagonals just above and below the main diagonal and in positions $(1, n)$ and $(n, 1)$, and 0's elsewhere:

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 1 \\ 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

c) The matrix is the same as the that in b), except that we add one more row and column for the vertex in the middle of the wheel; in the matrix below it is the las row and column:

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 1 & 1 \\ 1 & 0 & 1 & \dots & 0 & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 \\ 1 & 0 & 0 & \dots & 1 & 0 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & 0 \end{bmatrix}$$

d) Since the first m vertices are adjacent to none of the first m vertices but all of the last n and vice versa, the matrix splits up into four pieces:

$$\begin{bmatrix} 0 & \dots & 0 & 1 & \dots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & \dots & 1 \\ 1 & \dots & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & 1 & 0 & \dots & 0 \end{bmatrix}$$

e) It is not convenient to show these matrices explicitly. Instead, we will give a recursive definition. Let $\mathbf{Q}_n$ be the adjacency matrix for the graph Q_n. Then

$$\mathbf{Q}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$\mathbf{Q}_{n+1} = \begin{bmatrix} \mathbf{Q}_n & \mathbf{I}_n \\ \mathbf{I}_n & \mathbf{Q}_n \end{bmatrix},$$

where $\mathbf{I}_n$ is the identity matrix (since the corresponding vertices of the two n0cubes are joined by edges in the $(n+1)$-cube).

$\square$

P677, Ex. 45

Show that isomorphism of simple graphs is an equivalence relation.

Solution:

G is isomorphic to itself by the identity function, so isomorphism is reflexive. Suppose that G is isomorphic to H. Then there exists a one-to-one correspondence f from G to H that preserves adjacency and nonadjacency. It follows that f^{-1} is a one-to-one correspondence from H to G that preserves adjacency and nonadjacency. Hence, isomorphism is symmetric. If G is isomorphic to H and H is isomorphic to K, then there are one-to-one correspondences f and g from G to H and from H to K that preserve adjacency and nonadjacency. It follows that $g \circ f$ is a one-to-one correspondence from G to K that preserves adjacency and nonadjacency. Hence, isomorphism is transitive.

$\square$

P677, Ex. 52

Show that if G is a self-complementary simple graph with v vertices, then $v \equiv 0$ or $1 \pmod 4$.

Solution:

If G is self-complementary, then the number of edges of G must equal the number of edges of $\overline{G}$. But the sum of these two numbers is $n(n-1)/2$, where n is the number of vertices of G, since the union of the two graphs is K_n. Therefore, the number of G must be $n(n-1)/4$. Since this number must be an integer, a look at the four cases shows that n may be congruent to either 0 or 1, but not congruent to either 2 or 3, modulo 4.

□

P690, Ex. 20

Use paths either to show that these graphs are not isomorphic or to find an isomorphism between these graphs.

Solution:

The graph G has a simple circuit containing exactly the vertices of degree 3, namely $u_1u_2u_6u_5u_1$. The graph H has no simple circuit containing exactly the vertices of degree 3. Therefore the two graphs are not isomorphic.

□

P691, Ex. 28

Show that every connected graph with n vertices has at least $n-1$ edges.

Solution:

We show this by induction on n. For $n = 1$ there is nothing to prove. Now assume the inductive hypothesis, and let G be a connected graph with $n+1$ vertices and fewer than n edges, where $n \geq 1$. Since the sum of the degrees of the vertices of G is equal to 2 times the number of edges, we know that the sum of the degrees is less than $2n$, which is less than $2(n+1)$. Therefore, some vertex has degree less than 2. Since G is connected, this vertex is not isolated, so it must have degree 1. Remove this vertex and its edge. Clearly the result is still connected, and it has n vertices and fewer than $n-1$ edges, contradicting the inductive hypothesis. Therefore the statement holds for G, and the proof is complete.

□

P704, Ex. 10

Can someone cross all the bridges shown in this map exactly once and return to the starting point?

Solution:

Represent the map as a graph, and we found that each vertex has an even degree. Thus, there exists an Euler circuit.

□

P716, Ex. 2

Find the length of a shortest path between a and z in the given weighted graph.

Solution:

Using Dijkstra's algorithm, the shortest path between a and z is $abedz$, and the length is 7.

□

P716, Ex. 14

Explain how to find a path with the least number of edges between two vertices in an undirected graph by considering it as a shortest path problem in a weighted graph.

Solution:

We simply assign the weight of 1 to each edge.

□

P725, Ex. 6

Determine whether the graph is planar.

Solution:

It is planar, can be drawn in the plane without any two vertices crossing.

□

P726, Ex. 17

Suppose that a connected planar simple graph with e edges and v vertices contains no simple circuits of length 4 or less. Show that $e \leq (5/3)v - (10/3)$ if $v \geq 4$.

Solution:

As in the argument in the proof of Corollary 1, we have $2e \geq 5r$ and $r = e - v + 2$. Thus $e - v + 2 \leq 2e/5$, which implies that $e \leq (5/3)v - (10/3)$.

□

P756, Ex. 16 Which complete bipartite graphs $K_{m,n}$, where m and n are positive integers, are trees?

Solution:

If both m and n are at least 2, then clearly there is a simple circuit of length 4 in $K_{m,n}$. On the other hand, $K_{m,1}$ is cleraly a tree (as is $K_{1,n}$). Thus we conclude that $K_{m,n}$ is a tree if and only if $m = 1$ or $n = 1$.

□

P757, Ex. 46

The **rooted Fibonacci trees** T_n are defined recursively in the following way. T_1 and T_2 are both the rooted tree consisting of a single vertex, and for $n = 3, 4, \ldots$, the rooted tree T_n is constructed from a root with T_{n-1} as its left subtree and T_{n-2} as its right subtree. How many vertices, leaves, and internal vertices does the rooted Fibonacci tree T_n have, where n is a positive integer? What is its height?

Solution:

The number of vertices in the tree T_n satisfies the recurrence relation $v_n = v_{n-1} + v_{n-2} + 1$ (the "+1" is for the root), with $v_1 = v_2 = 1$. Thus the sequence begins $1, 1, 3, 5, 9, 15, 25, \ldots$. It is easy to prove by induction that $v_n = 2f_n - 1$, where f_n is the n-th Fibonacci number. The number of leaves satisfies the recurrence relation $l_n = l_{n-1} + l_{n-2}$, with $l_1 = l_2 = 1$, so $l_n = f_n$. Since $i_n = v_n - l_n$, we have $i_n = f_n - 1$. Finally, it is clear that the height of the tree T_n is one more than the height of the tree T_{n-1} for $n \geq 3$, with the height of T_2 being 0. Therefore the height of T_n is $n - 2$ for all $n \geq 2$ (and of course the height of T_1 is 0).

□

P784, Ex. 24

What is the value of each of these postfix expressions?

a) $5\ 2\ 1\ -\ -\ 3\ 1\ 4\ +\ +\ *$

b) $9\ 3\ /\ 5\ +\ 7\ 2\ -\ *$

c) $3\ 2\ *\ 2\ \uparrow\ 5\ 3\ -\ 8\ 4\ /\ *\ -$

Solution:

We exhibit the answers by showing with parentheses the operation that is applied next, working from left to right (it always involves the first occurrence of an operator symbol).

a) $5\ (2\ 1\ -)\ -3\ 1\ 4\ +\ +*=(5\ 1\ -)3\ 1\ 4\ +\ +*=4\ 3\ (1\ 4\ +)\ +\ *=4\ (3\ 5\ +)*=(4\ 8\ *)=32$

b) $(9\ 3\ /)\ 5+7\ 2\ -\ *=(3\ 5\ +)\ 7\ 2\ -\ *=8(\ 7\ 2\ -)*=(8\ 5\ *)=40$

c) $(3\ 2\ *)\ 2\ \uparrow\ 5\ 3\ -\ 8\ 4\ /\ *\ -=(6\ 2\uparrow)\ 5\ 3\ -\ 8\ 4\ /\ *\ -=36\ (5\ 3\ -)\ 8\ 4\ /\ *\ -=36\ 2(8\ 4\ /)\ *\ -=36(2\ 2\ *)\ -=(36\ 4\ -)=32$

□

P795, Ex. 11

How many different spanning trees does each of these simple graphs have?
a) K_3 b) K_4 c) $K_{2,2}$ d) C_5

Solution:

a) 3 b) 16 c) 4 d) 5

□

P795, Ex. 12

How many nonisomorphic spanning trees does each of these simple graphs have?

a) K_3 b) K_4 c) K_5

Solution:

a) 1 b) 2 c) 3

□

P802, Ex. 2

Use Prim's algorithm to find a minimum spanning tree for the given weighted graph.

Solution:

We start with the minimum weight edge $\{a, b\}$. The least weight edge incident to the tree constructed so far is edge $\{a, e\}$, with weight 2, so we add it to the tree. Next we add edge $\{d, e\}$, and then edge $\{c, d\}$. This completes the tree, whose total weight is 6.

□

P802, Ex. 6

Use Kruskal's algorithm to find a minimum spanning tree for the weighted graph in Exercise 2.

Solution:

With Kruskal's algorithm, we add at each step the shortest edge and will not complete a simple circuit. Thus we pick edge $\{a, b\}$ first, and then edge $\{c, d\}$ (alphabetical order breaks ties), followed by $\{a, e\}$ and $\{d, e\}$. The total weight is 6.

□