南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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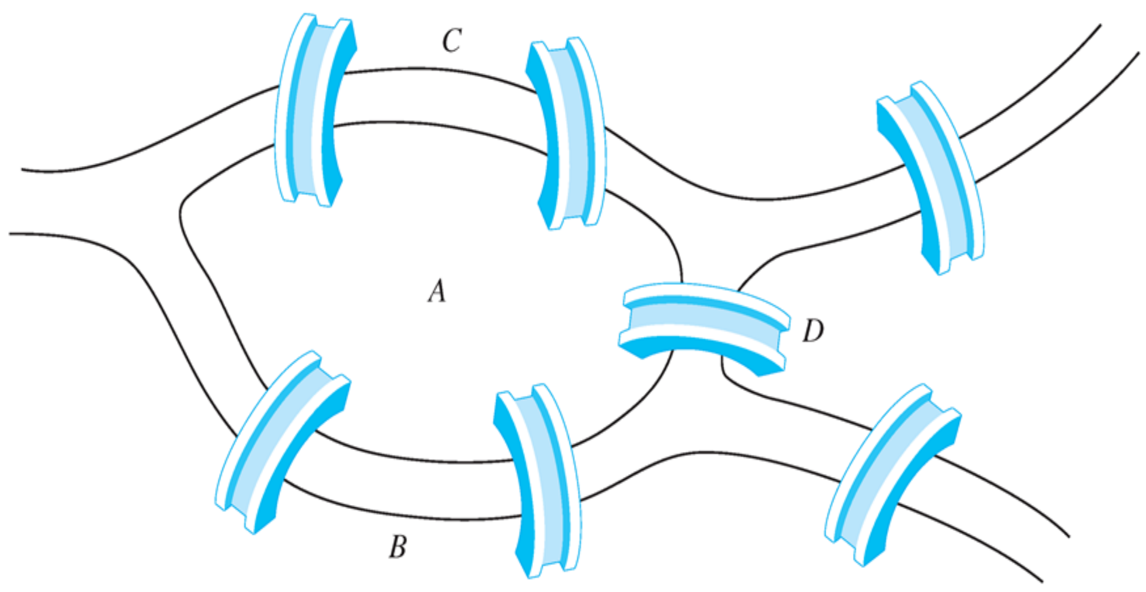
- **Königsberg seven-bridge problem**

  People wondered whether it was possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice, and return to the starting point.



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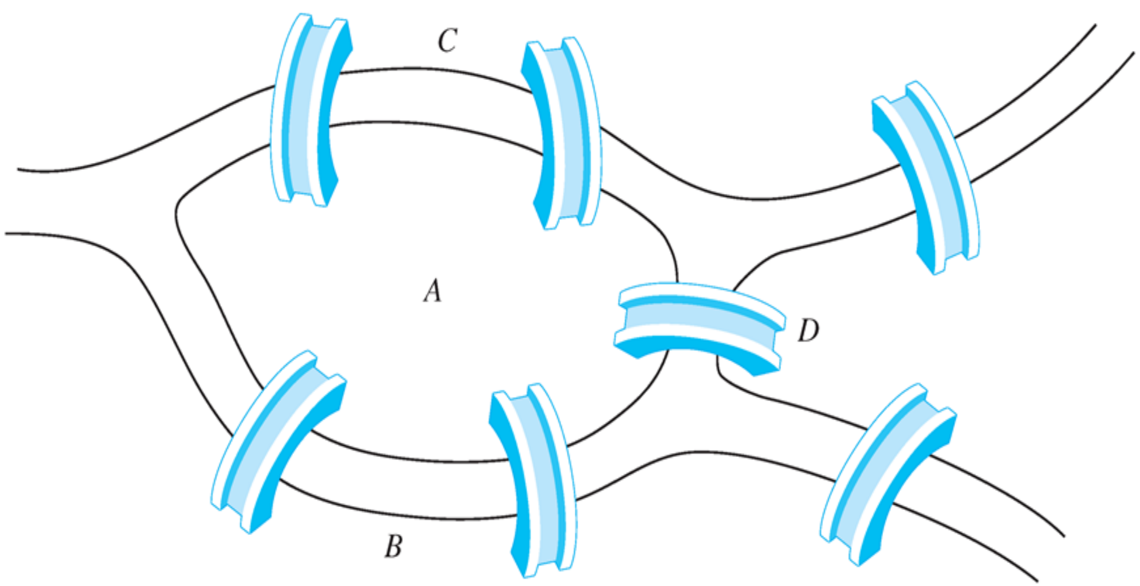

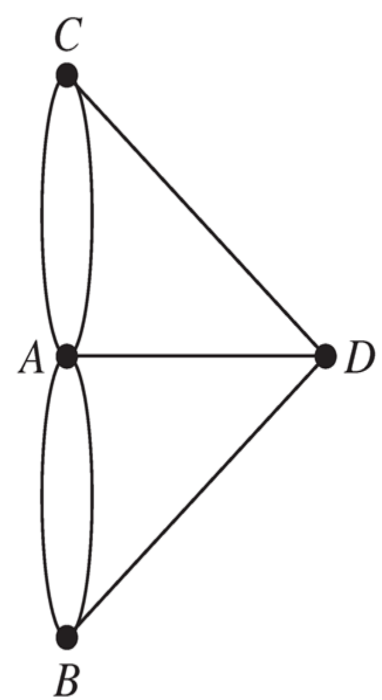

Euler Paths and Circuits

- **Definition** An *Euler circuit* in a graph G is a simple circuit containing every edge of G. An *Euler path* in G is a simple path containing every edge of G.

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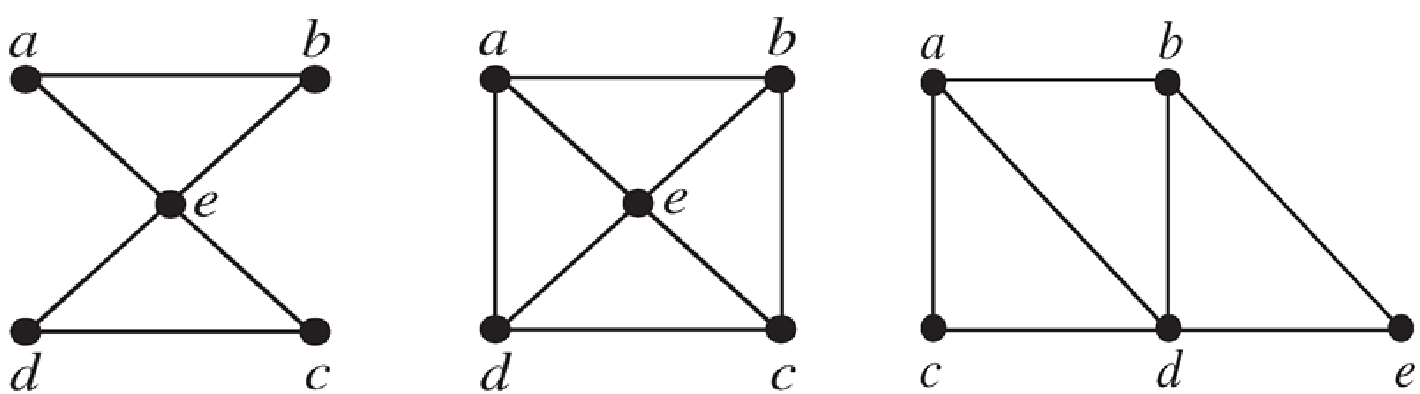

G_1 G_2 G_3

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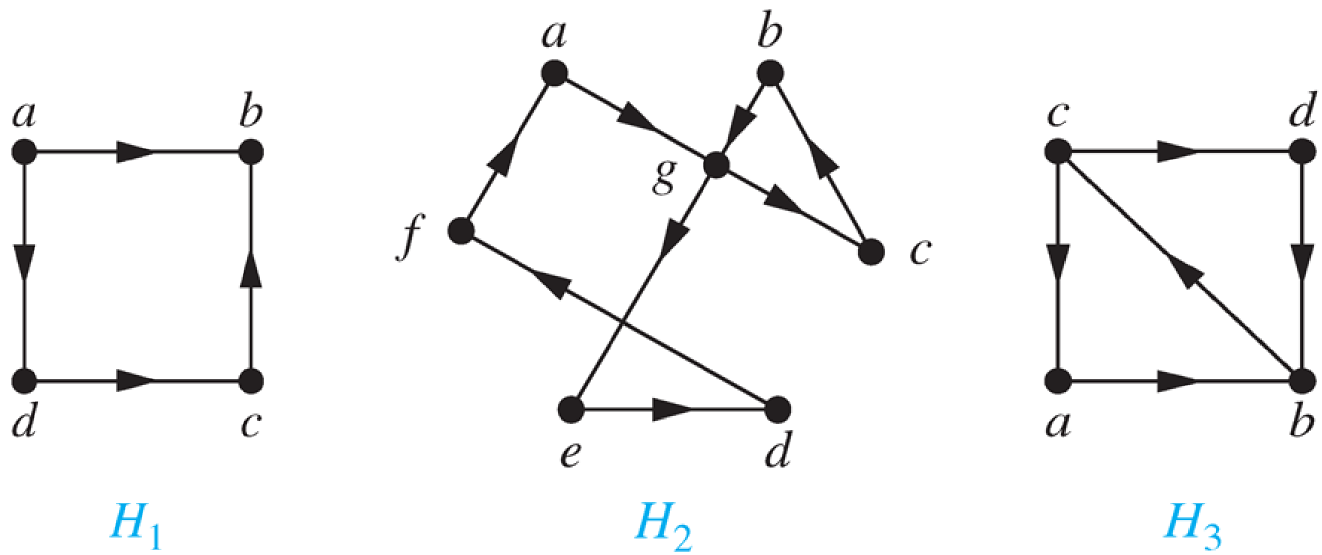

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  - The initial vertex and the final vertex of an Euler path have odd degree.

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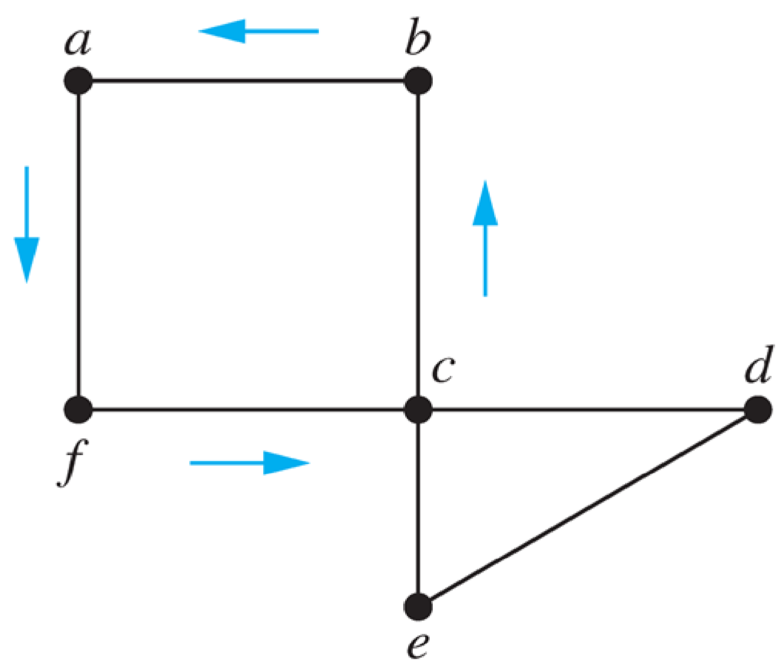

G

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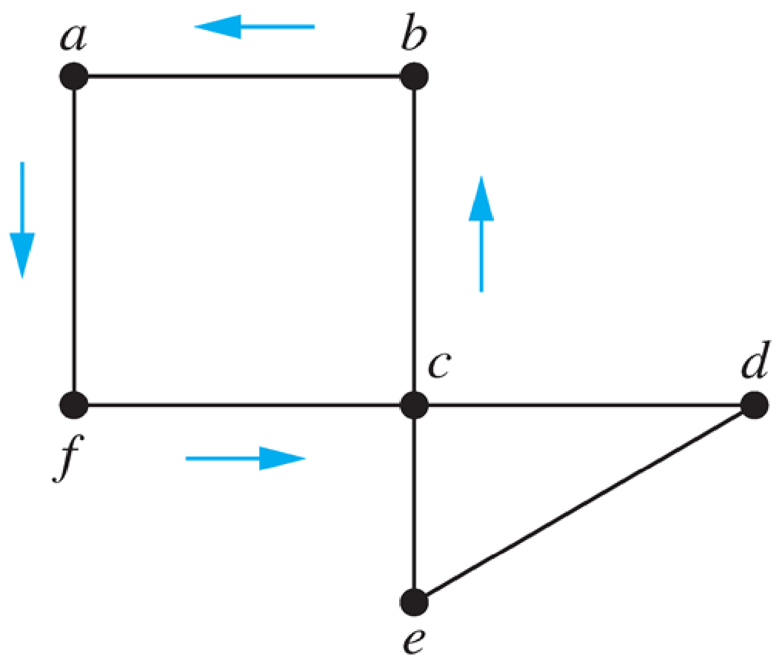

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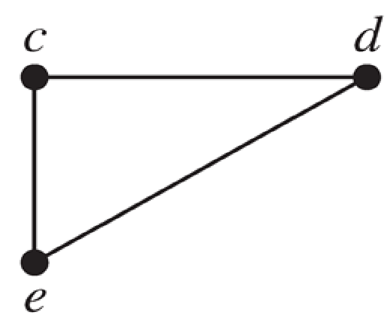

H

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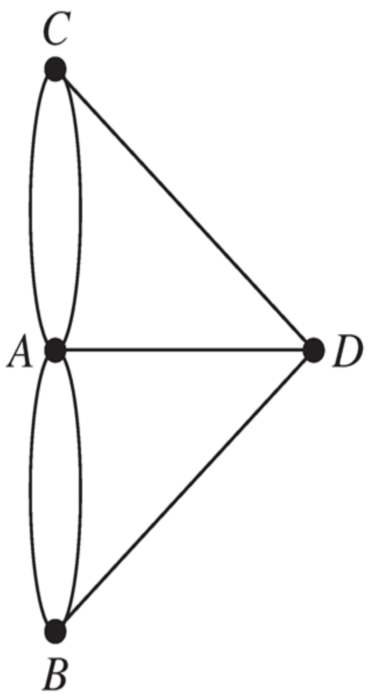

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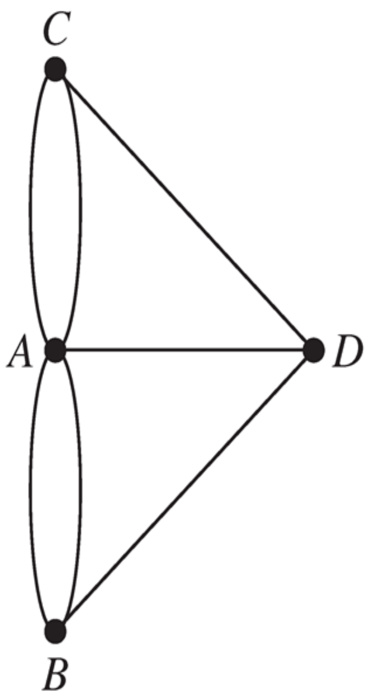

No Euler circuit

Example

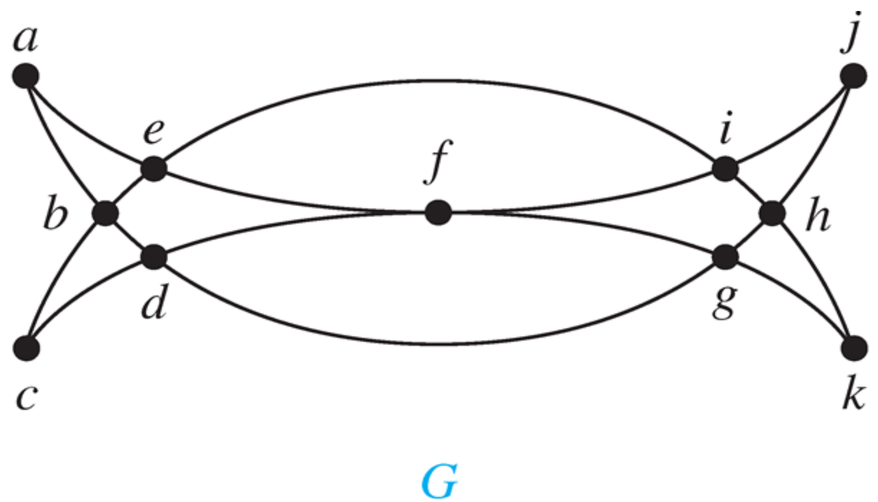

FIGURE 6 **Mohammed's Scimitars.**

Euler Circuits and Paths

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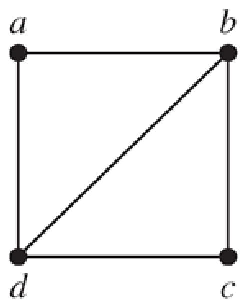

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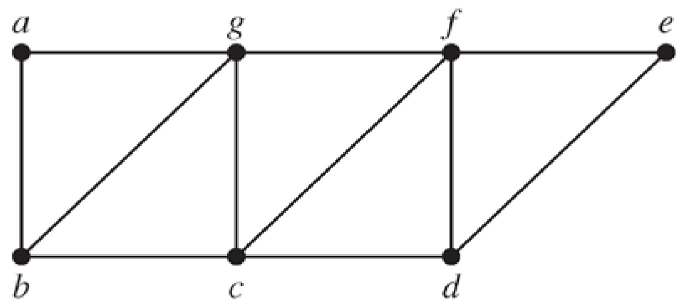

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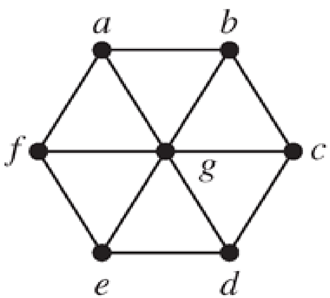

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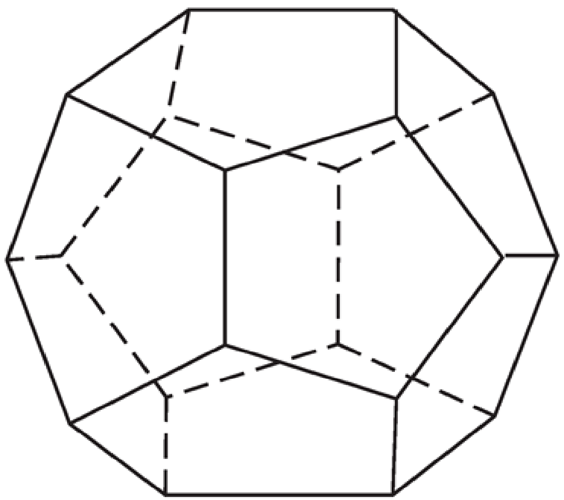

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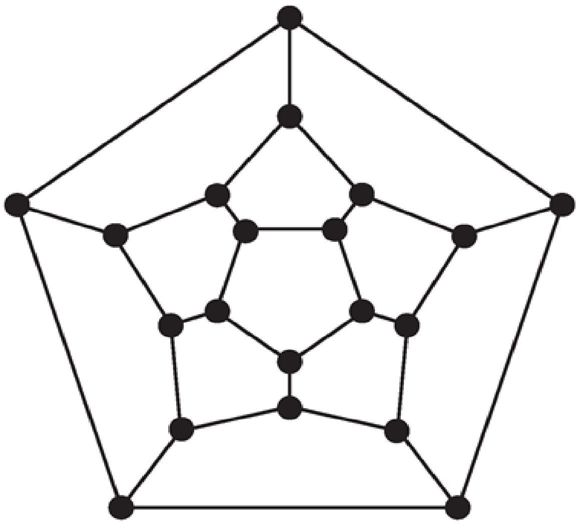

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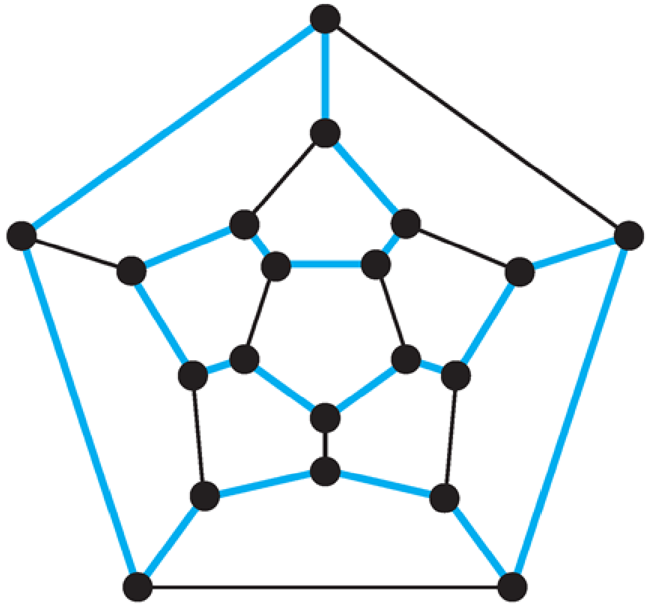

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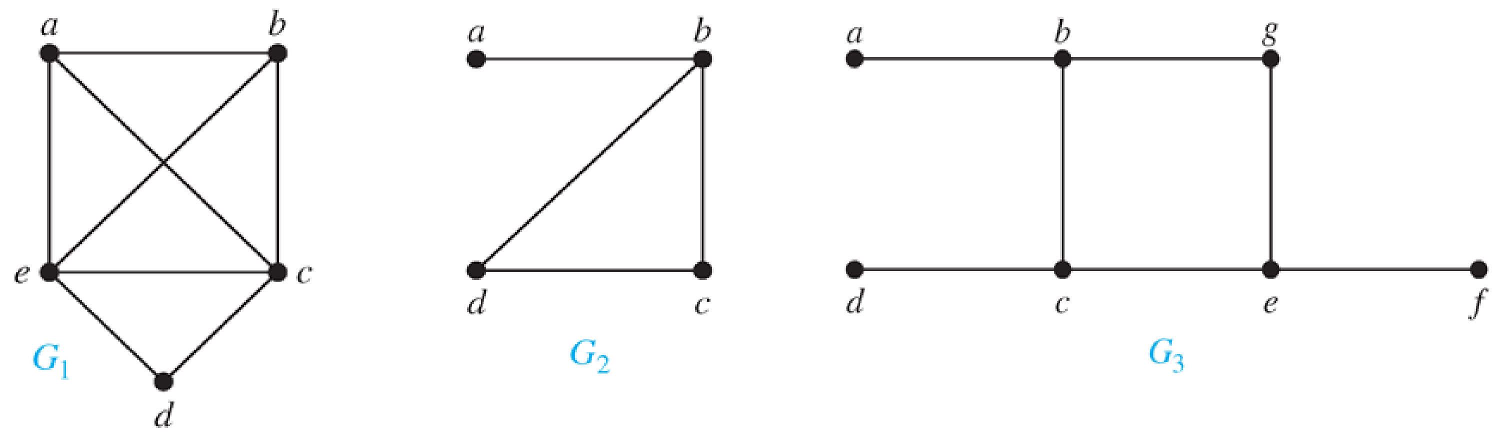

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  the decision version of the TSP $\in$ NPC

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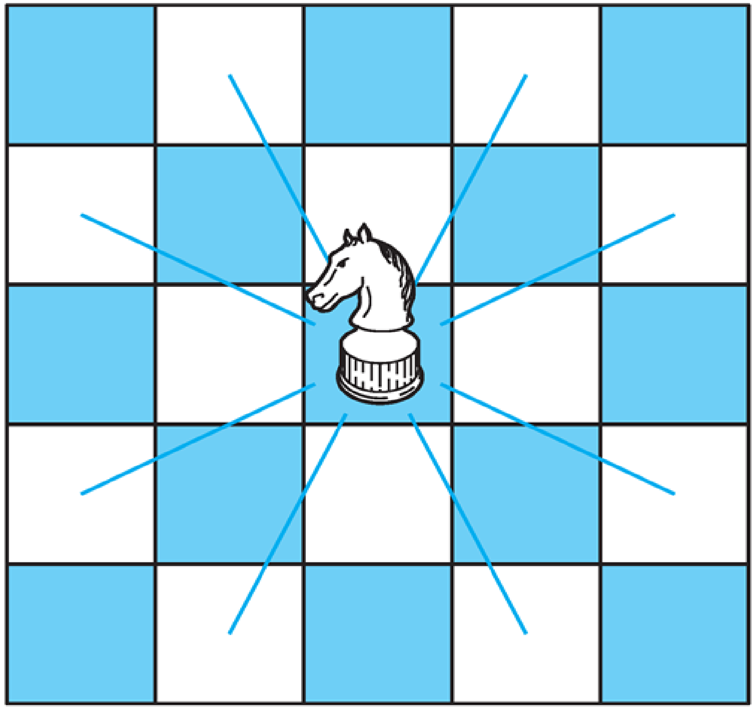

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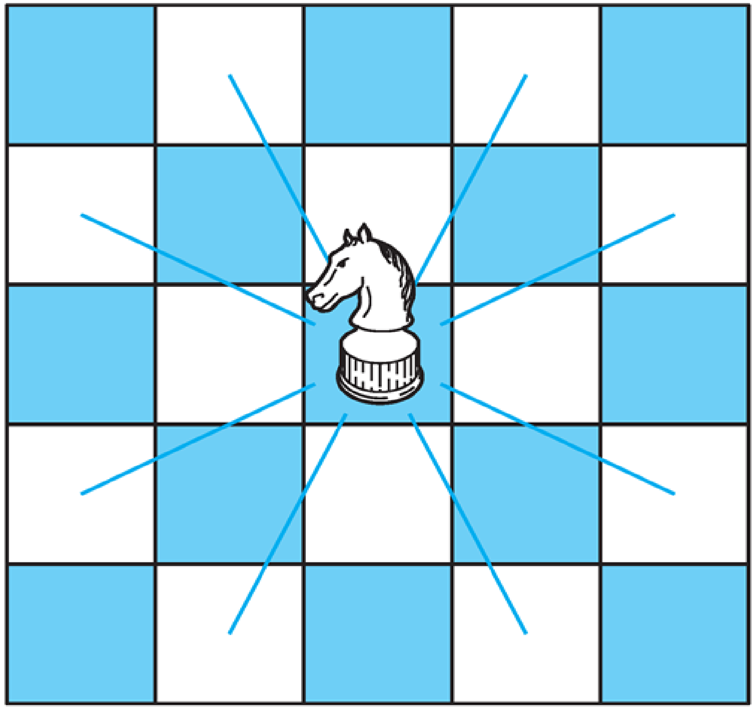

What about in 6×6 chessboard?

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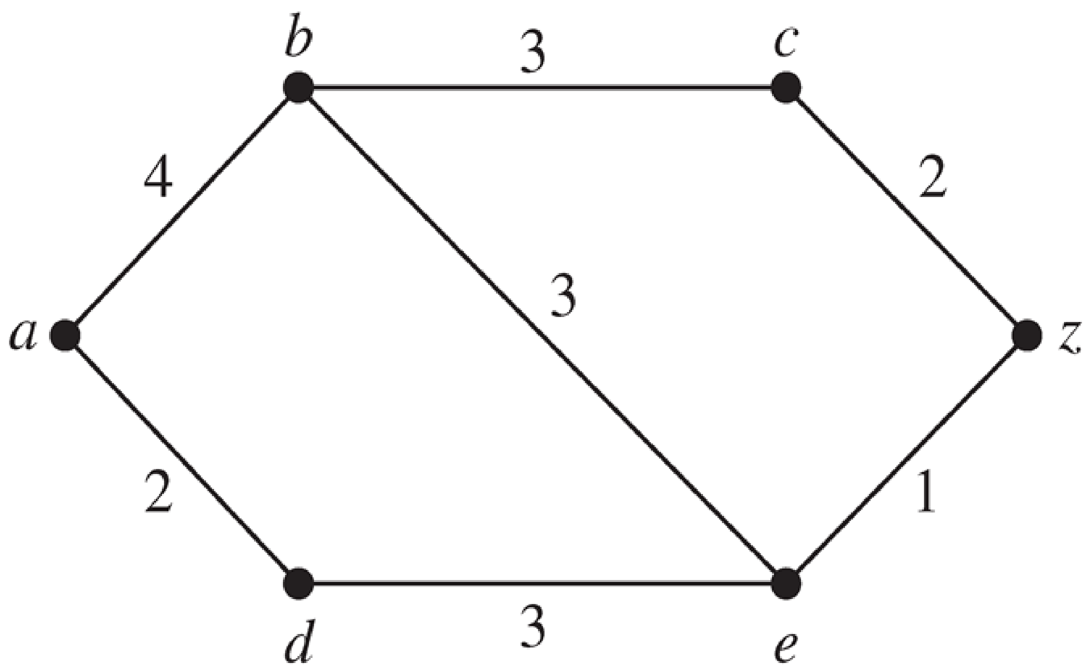

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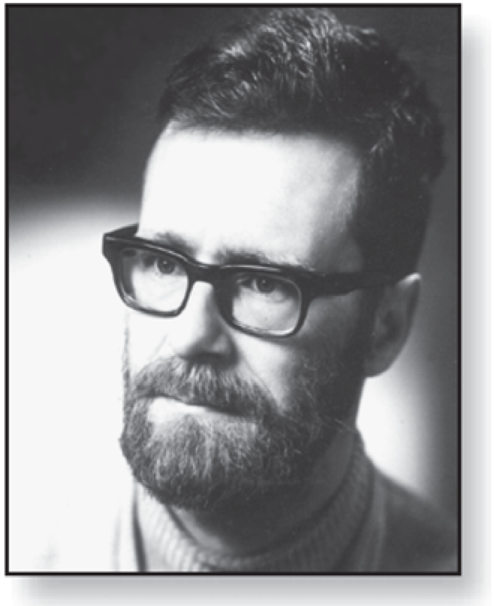

Edsger Wybe Dijkstra

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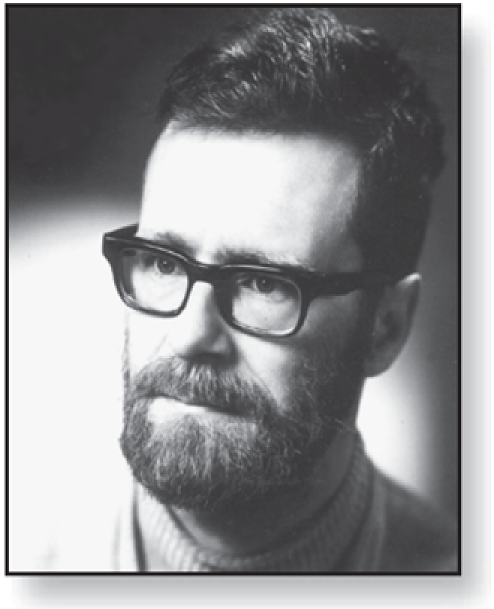

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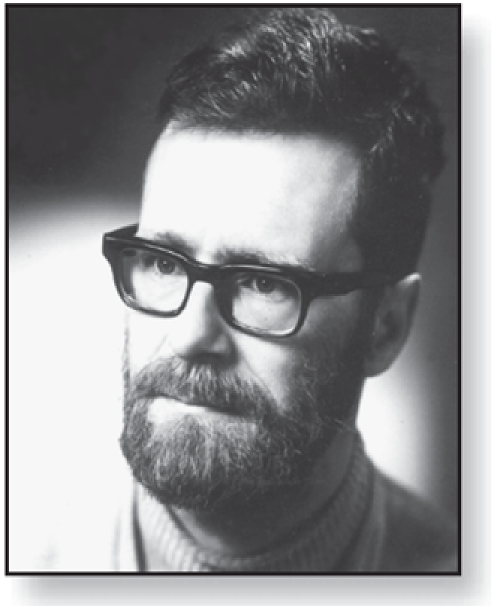

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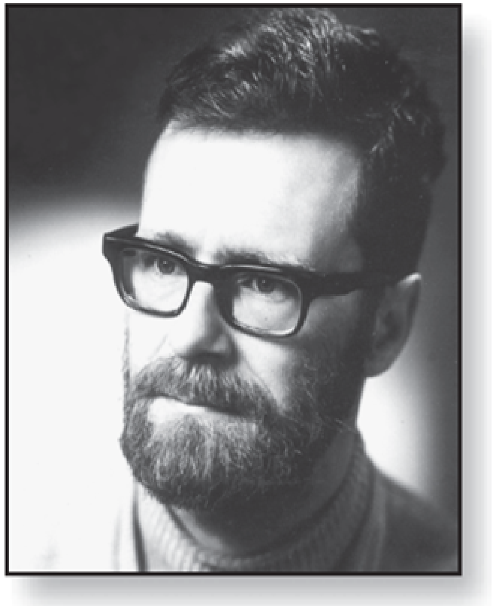

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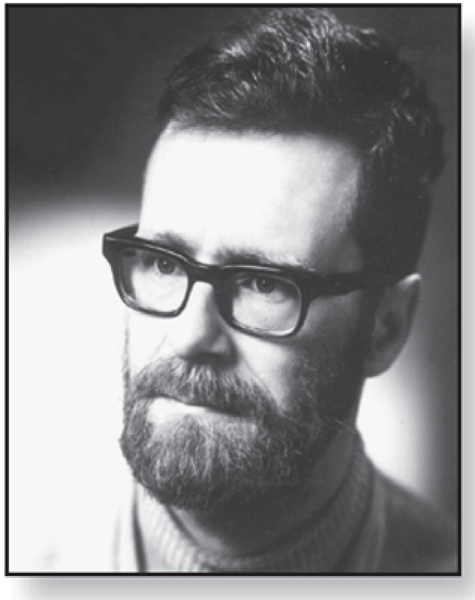

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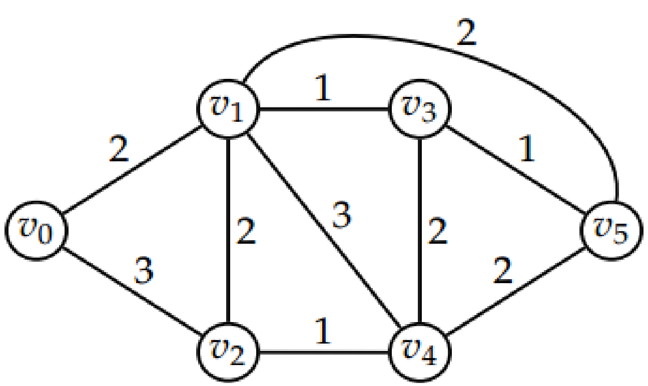

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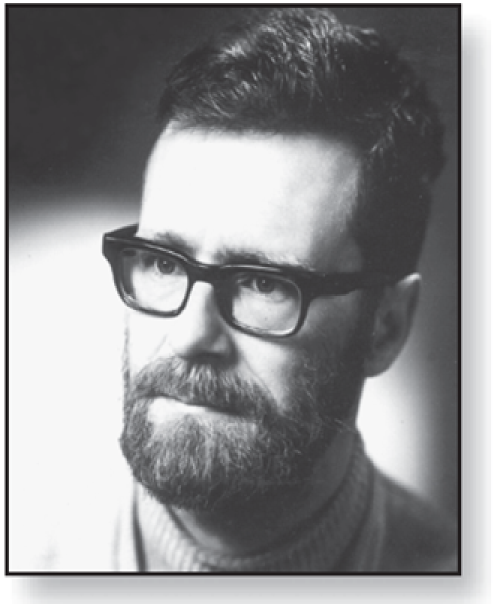

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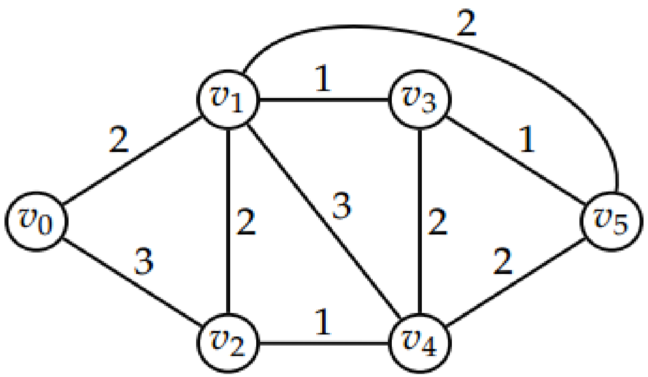

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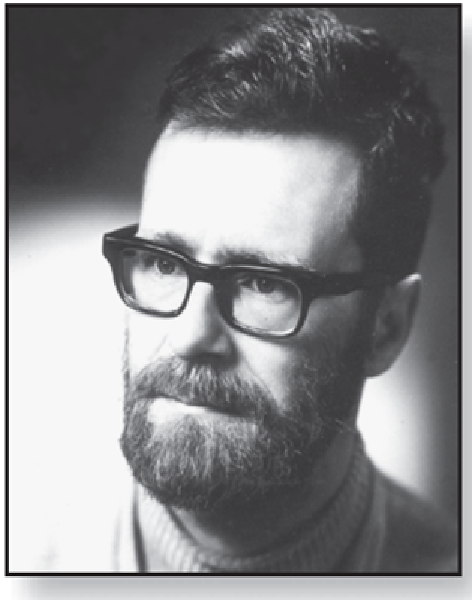

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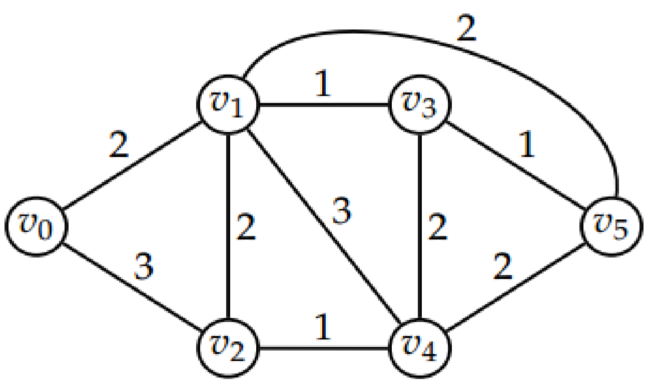

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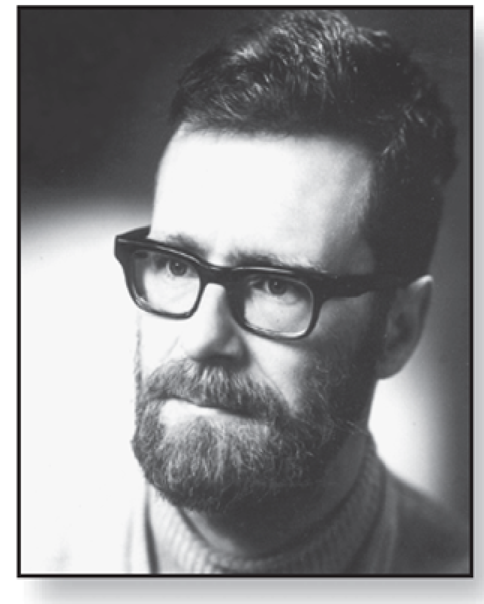

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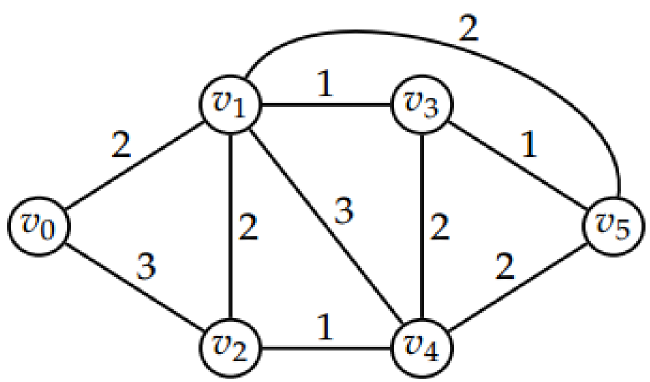

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$i = 0$

$d(v_1) = \min\{\infty, 2\} = 2$, $d(v_2) = \min\{\infty, 3\} = 3$

- (i) Set $d(v_0) = 0$ and $d(v) = \infty$ for all $v \neq v_0$, $S = \emptyset$
  (ii) while $S \neq V$
  let $v \notin S$ be the vertex with the
  least value $d(v)$, $S = S \cup \{v\}$
  for each $u \notin S$, replace $d(u)$ by
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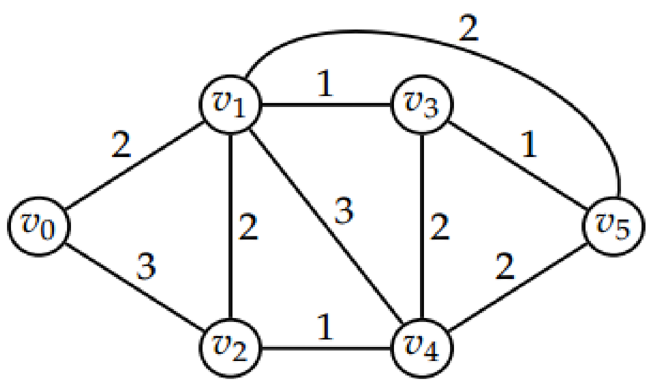
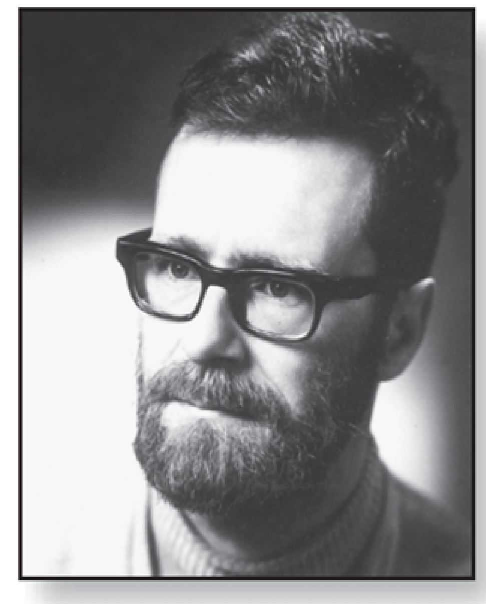
  (iii) return all $d(v)$'s

Example





Edsger Wybe Dijkstra

v_0	v_1	v_2	v_3	v_4	v_5
0	2	3	∞	∞	∞

$i = 0$

$d(v_1) = \min\{\infty, 2\} = 2$, $d(v_2) = \min\{\infty, 3\} = 3$

Dijkstra's Algorithm

- (i) Set $d(v_0) = 0$ and $d(v) = \infty$ for all $v \neq v_0$, $S = \emptyset$

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  let $v \notin S$ be the vertex with the least value $d(v)$, $S = S \cup \{v\}$

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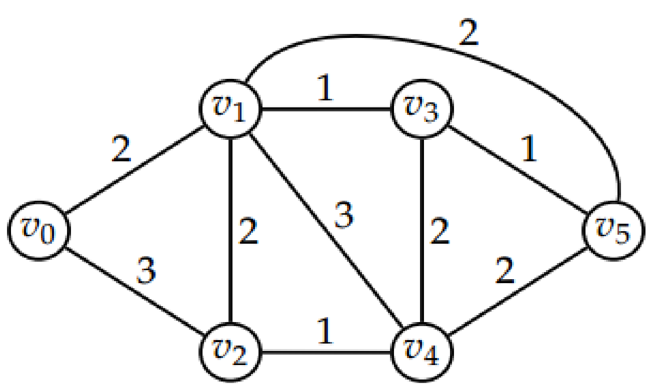
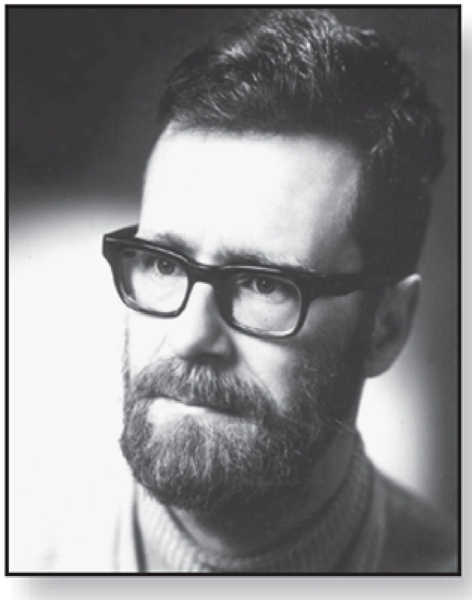
  Example





Edsger Wybe Dijkstra

v_0	v_1	v_2	v_3	v_4	v_5
0	2	3	∞	∞	∞

$i = 1$

$d(v_2) = \min\{3, d(v_1) + \alpha(v_1 v_2)\} = \min\{3, 4\} = 3,$

$d(v_3) = 2 + 1 = 3$, $d(v_4) = 2 + 3 = 5,$

$d(v_5) = 2 + 2 = 4$

- (i) Set $d(v_0) = 0$ and $d(v) = \infty$ for all $v \neq v_0$, $S = \emptyset$
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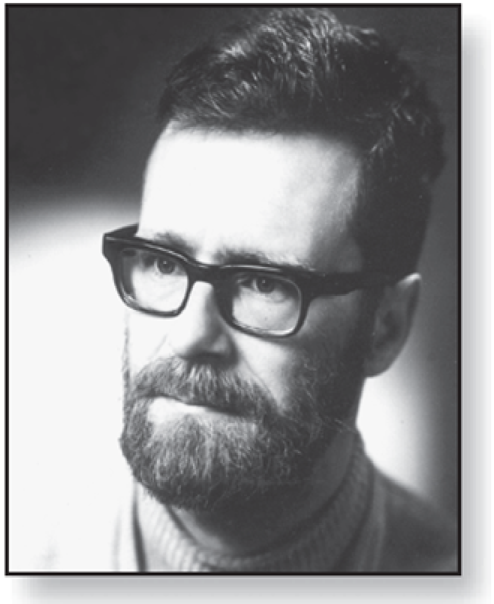

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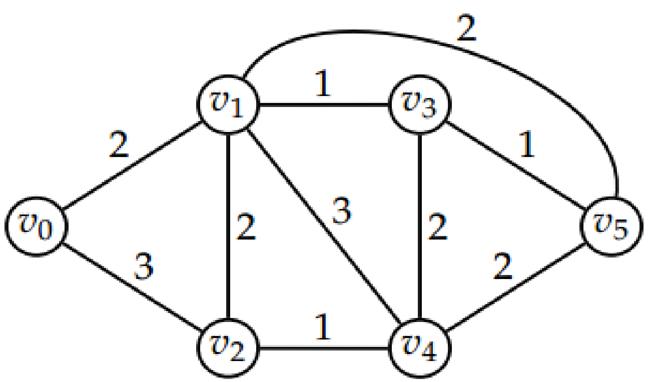

v_0	v_1	v_2	v_3	v_4	v_5
0	2	3	3	5	4

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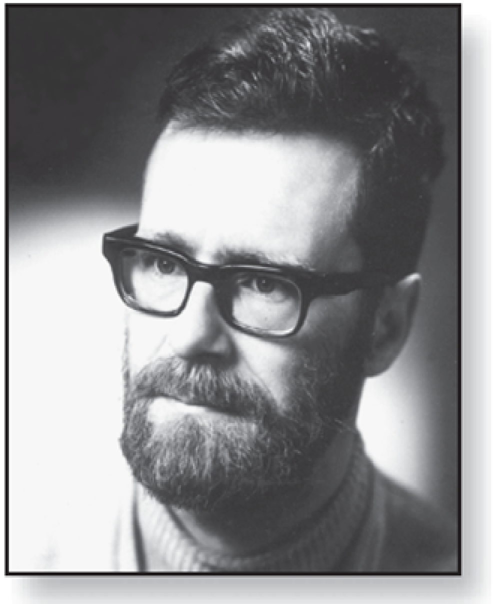

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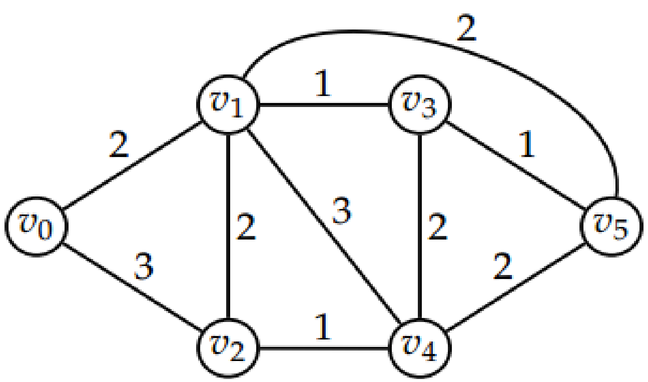

v_0	v_1	v_2	v_3	v_4	v_5
0	2	3	3	5	4

$i = 2$

$d(v_3) = \min\{3, \infty\} = 3,$

$d(v_4) = \min\{5, 3 + 1\} = 4,$

$d(v_5) = \min\{4, \infty\} = 4$

Dijkstra's Algorithm

- (i) Set $d(v_0) = 0$ and $d(v) = \infty$ for all $v \neq v_0$, $S = \emptyset$
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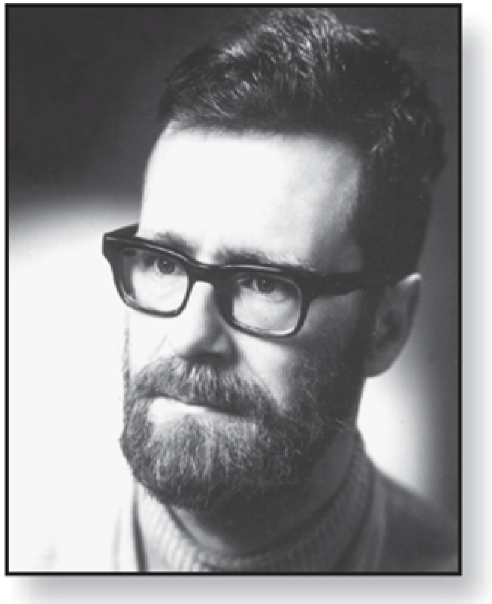

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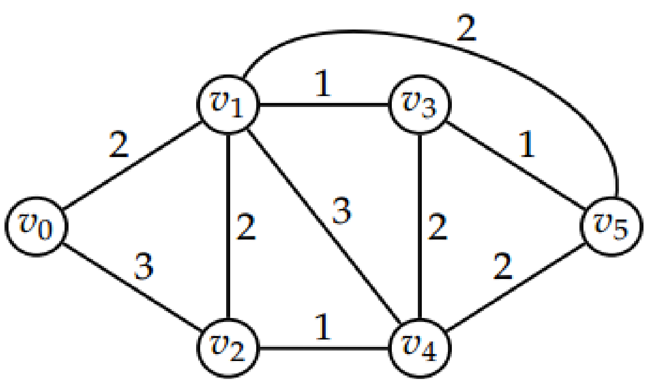

v_0	v_1	v_2	v_3	v_4	v_5
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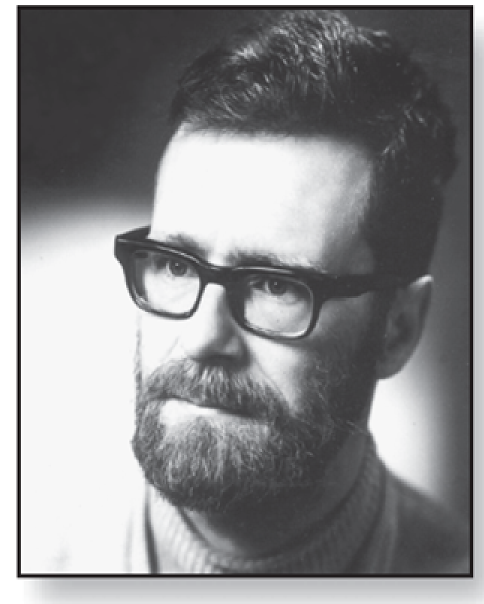

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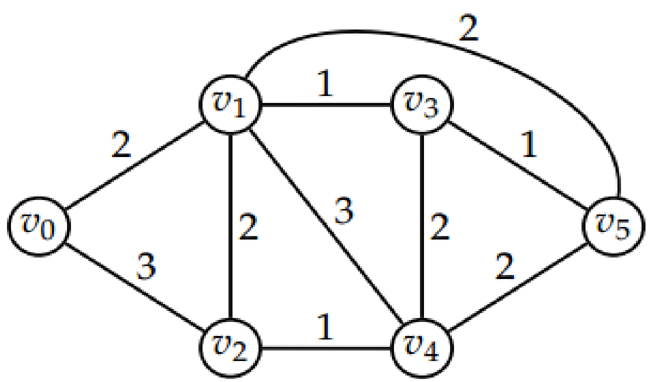

v_0	v_1	v_2	v_3	v_4	v_5
0	2	3	3	4	4

$i = 3$

$d(v_4) = \min\{4, 3 + 2\} = 4,$

$d(v_5) = \min\{4, 3 + 1\} = 4$

Dijkstra's Algorithm

- (i) Set $d(v_0) = 0$ and $d(v) = \infty$ for all $v \neq v_0$, $S = \emptyset$
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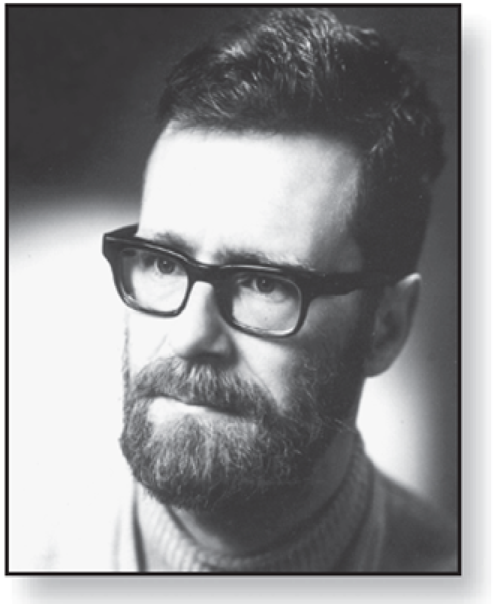

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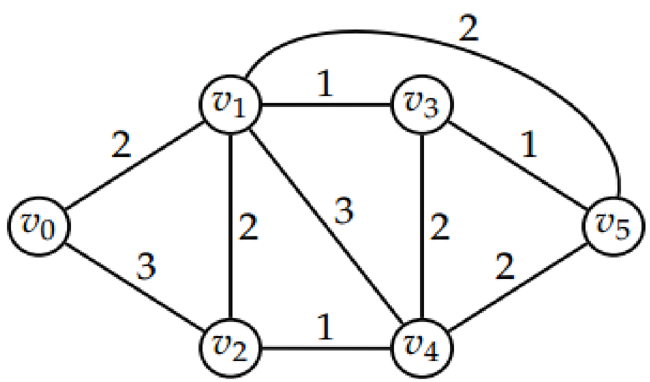
Example



v_0	v_1	v_2	v_3	v_4	v_5
0	2	3	3	4	4

$i = 4$

$d(v_5) = \min\{4, 4 + 2\} = 4$

Dijkstra's Algorithm

- **Theorem** Dijkstra's algorithm finds the length of a shortest path between two vertices in a connnected simple undirected weighted graph.

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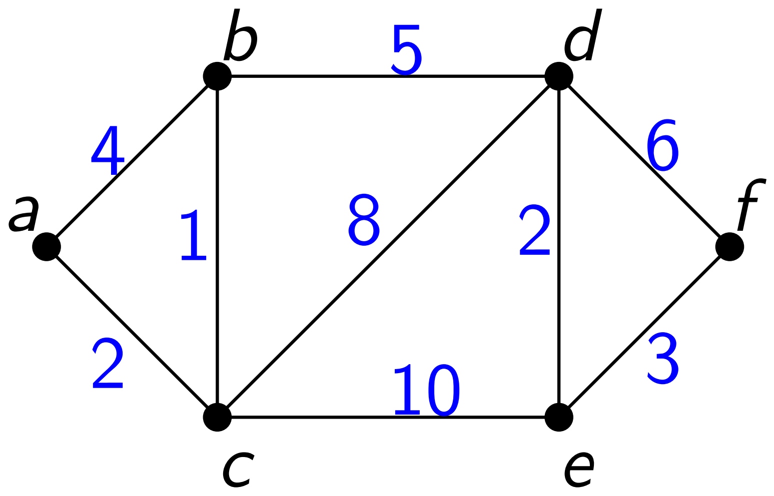
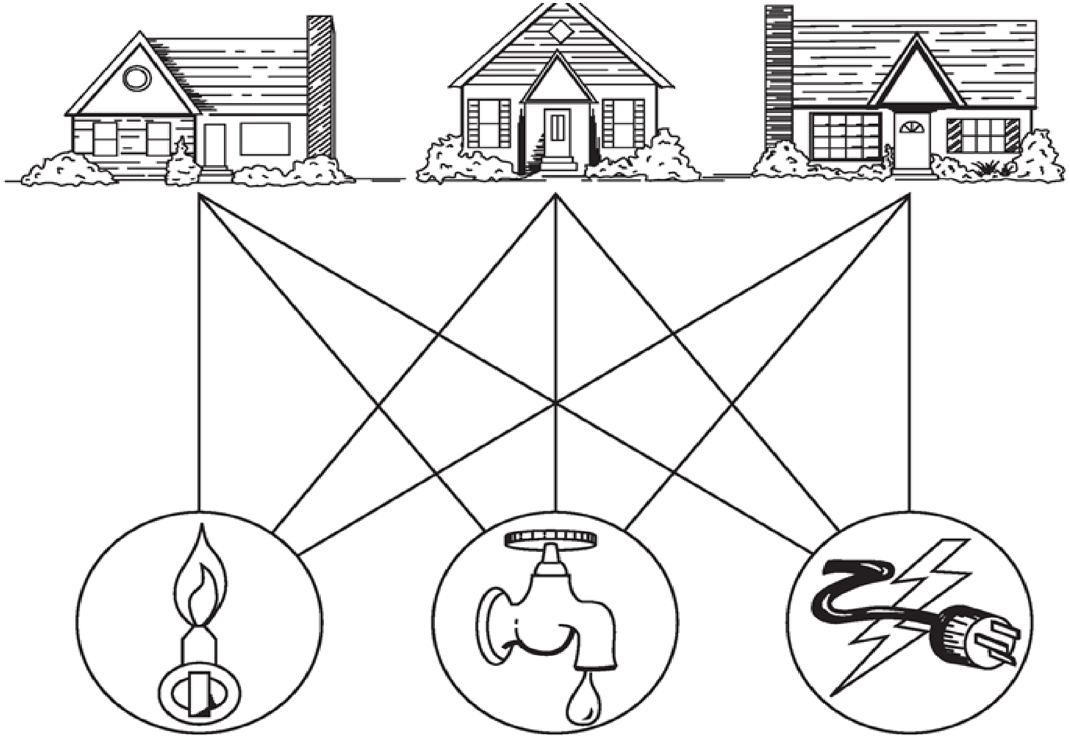
  Complexity

Another Example



Planar Graphs

- Join three houses to each of three seperate utilities.



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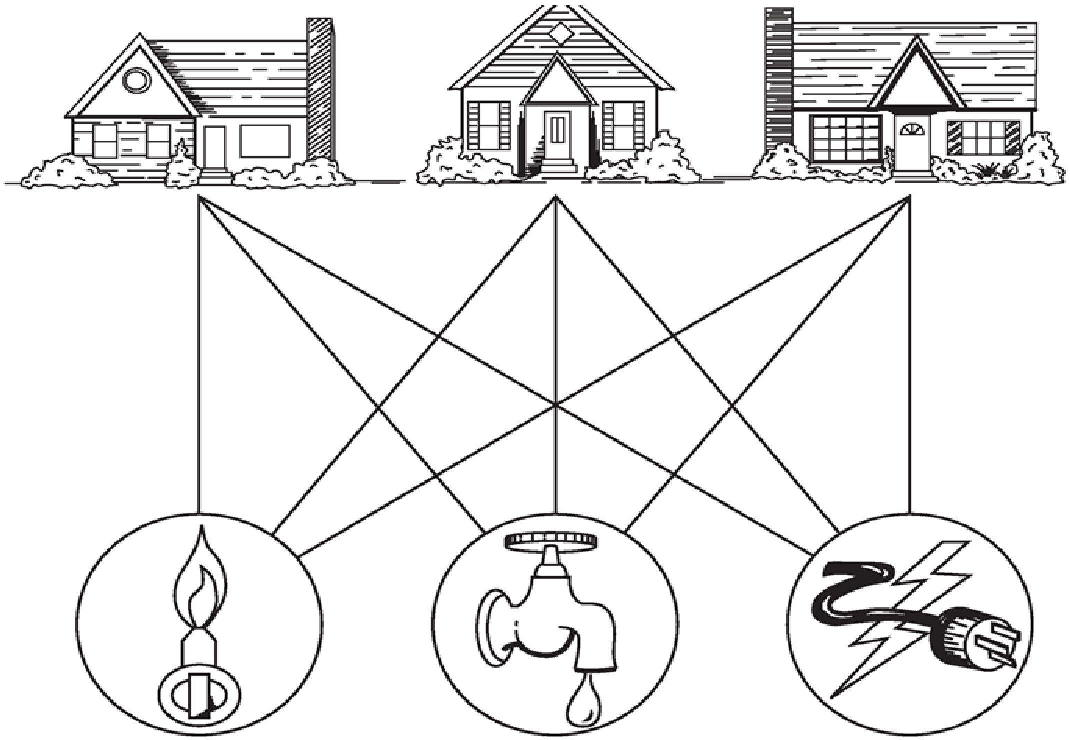

Can this graph be drawn in the plane s.t. no two of its edges cross?

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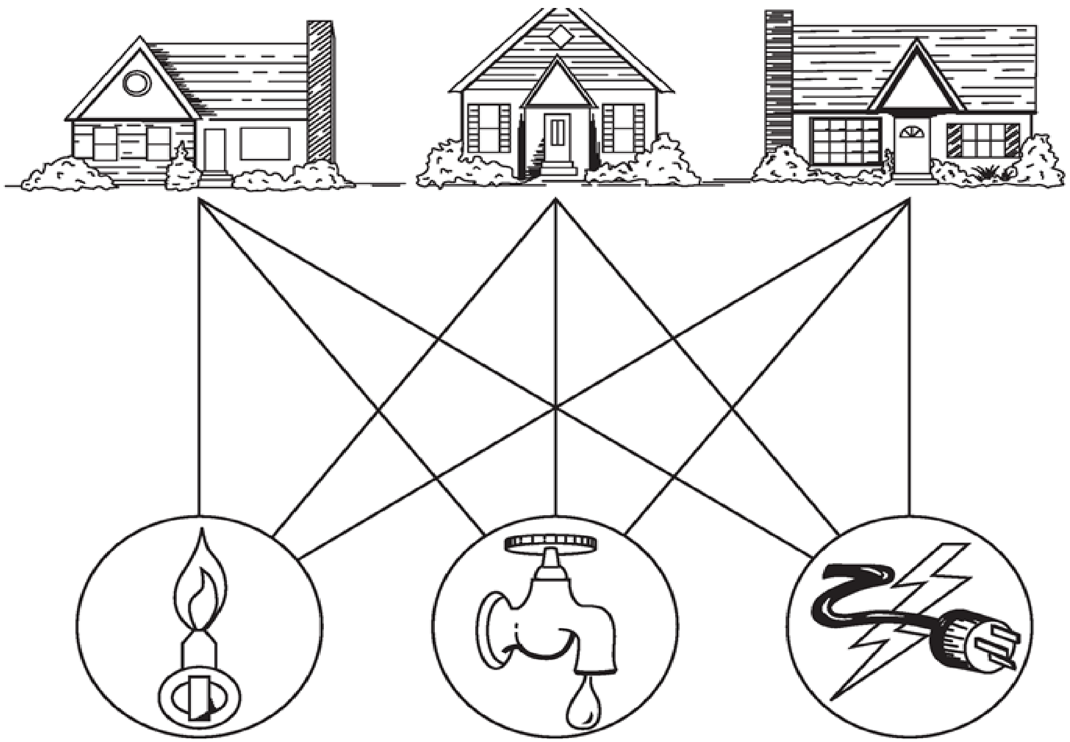

Can this graph be drawn in the plane s.t. no two of its edges cross? $K_{3,3}$

Planar Graphs

- **Definition** A graph is called *planar* if it can be drawn in the plane without any edges crossing. Such a drawing is called a *planar representation* of the graph.

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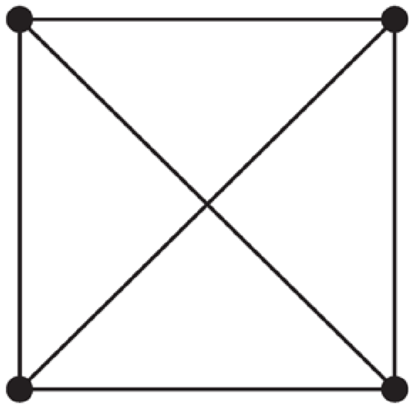

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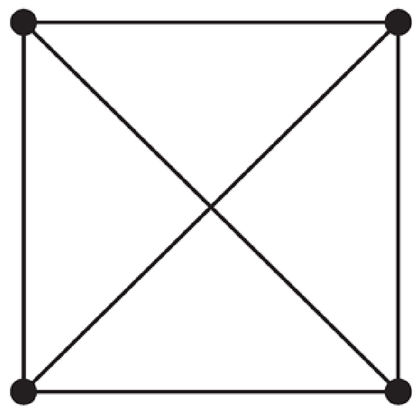

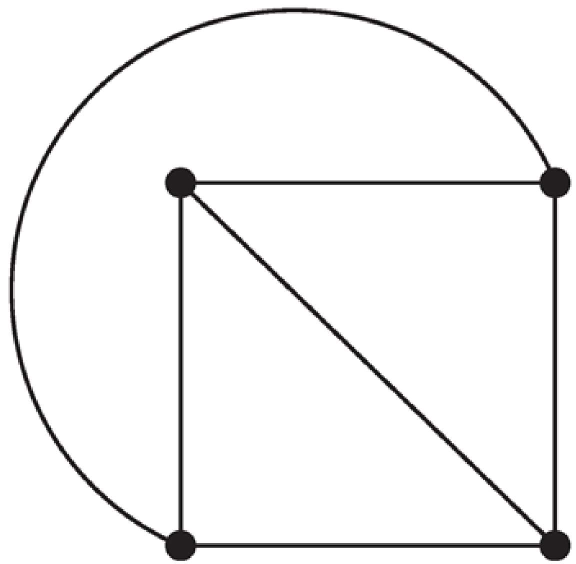

Planar Graphs

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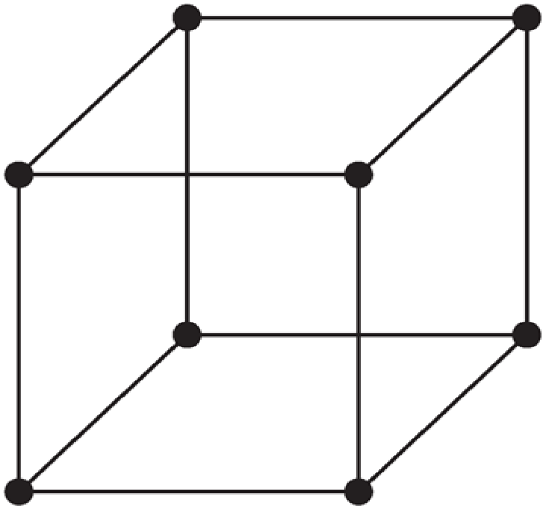

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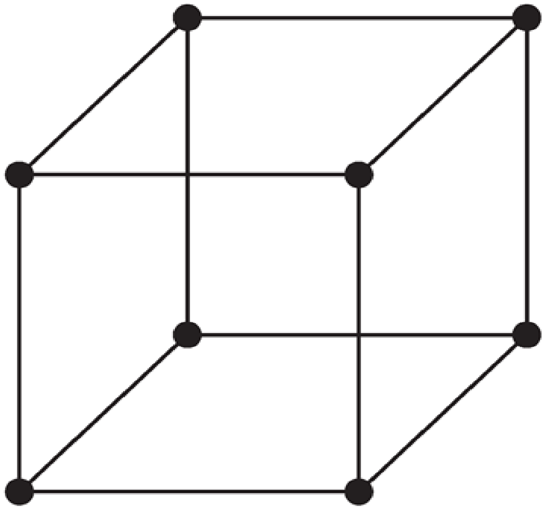

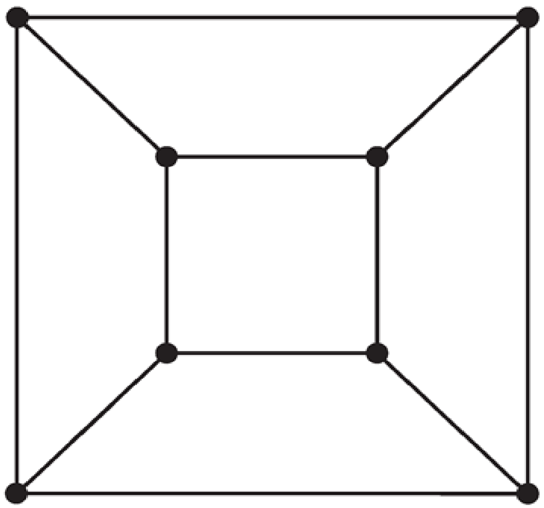

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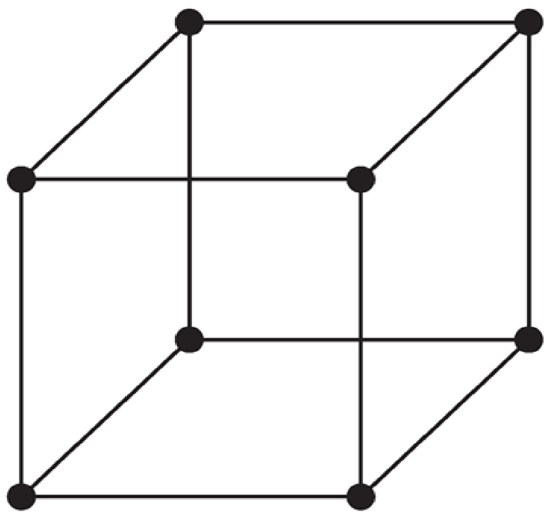
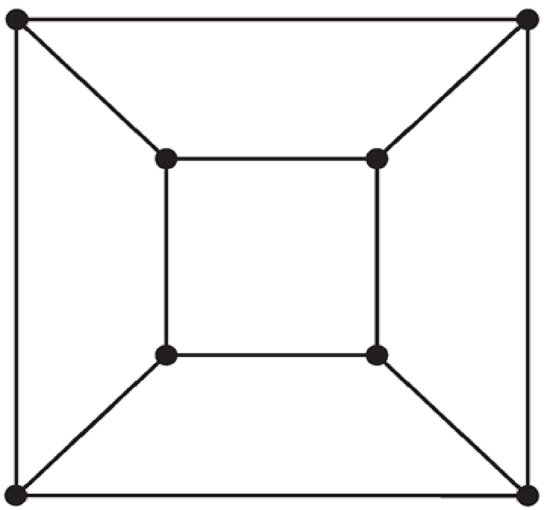
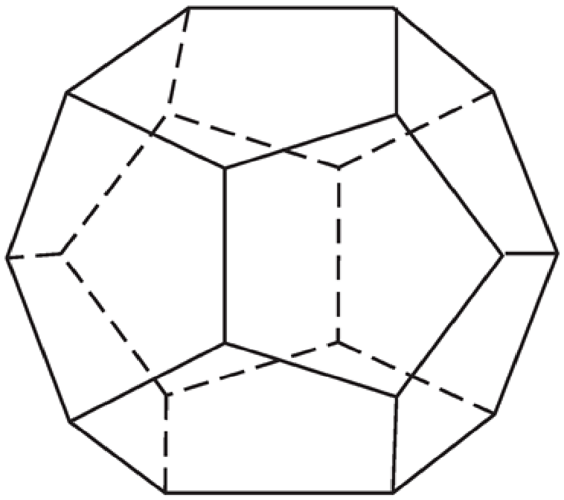


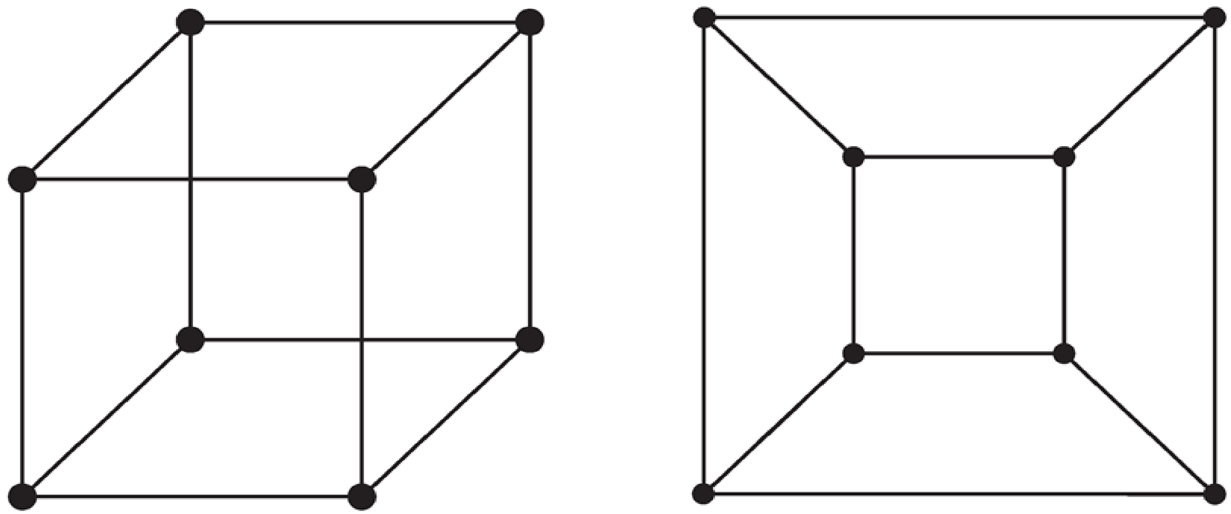


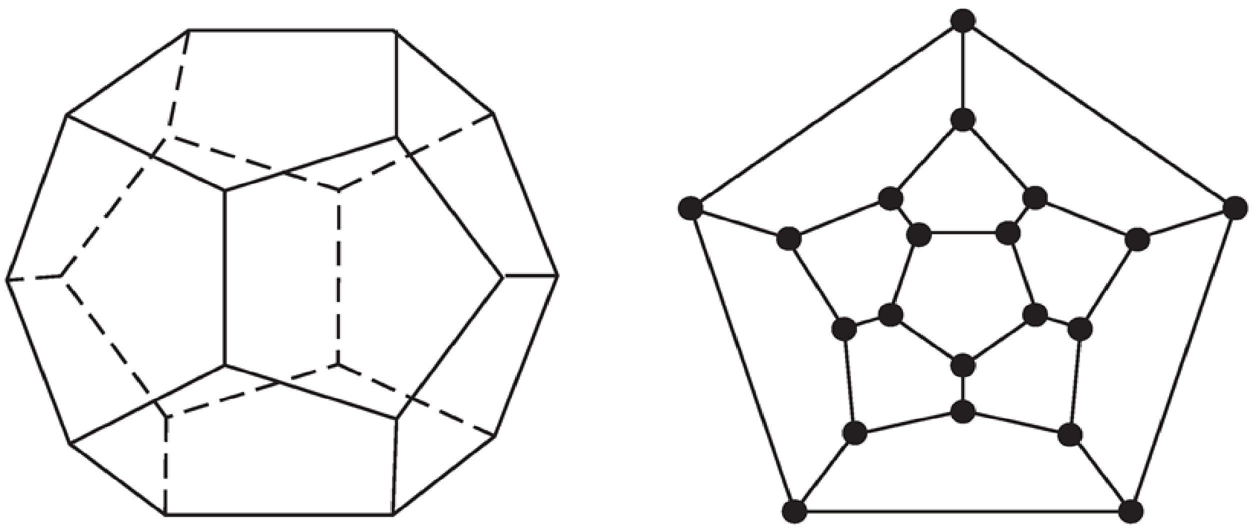


(a)

Planar Graphs

- **Example**





(a)

(b)

Planar Graphs

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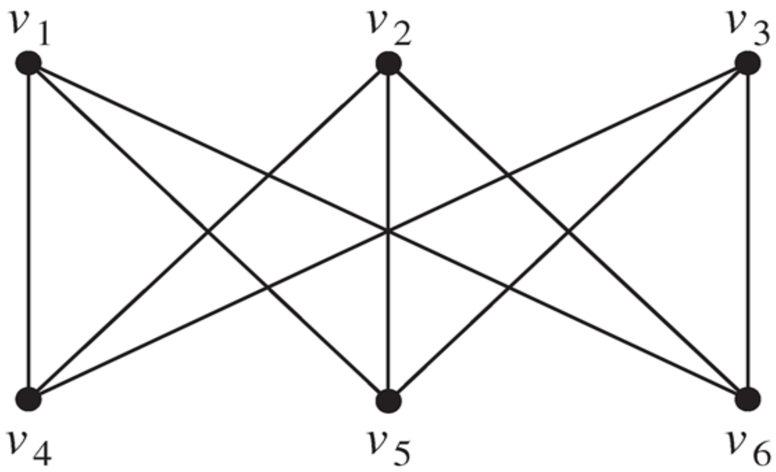

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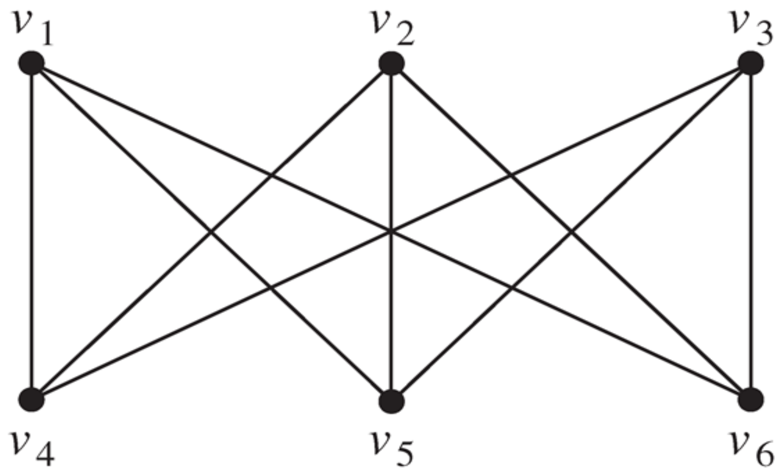

Applications

◇ IC design
◇ degin of road networks

Euler's Formula

- **Theorem** (Euler's Formula) Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

  Proof (by induction)

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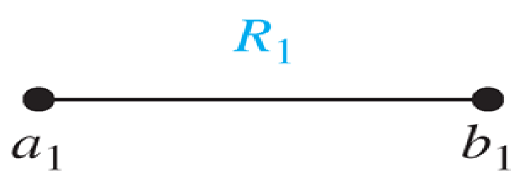

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$$r_k = e_k - v_k + 2$$

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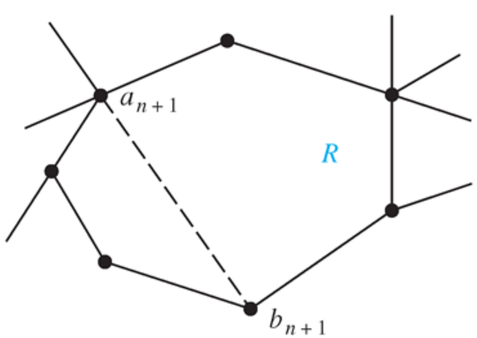

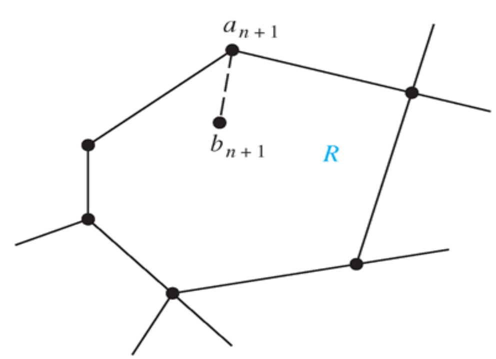

The Degree of Regions

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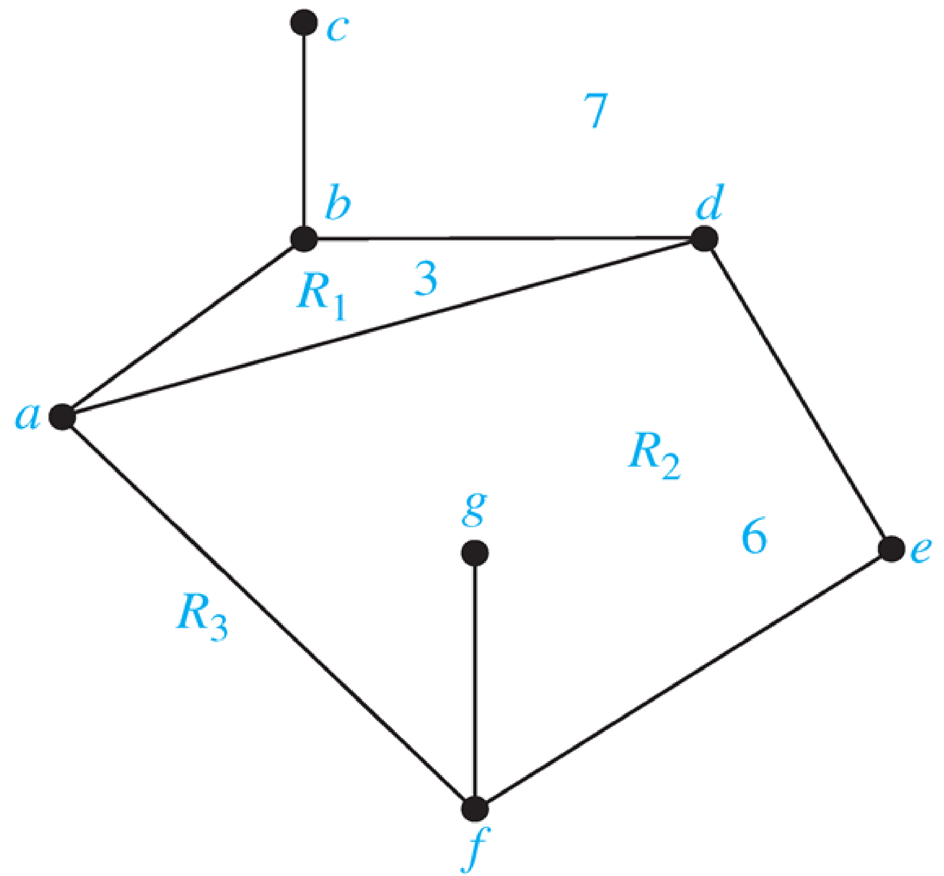

Corollaries

- **Corollary 1** If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

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  By Euler's formula, the proof is completed.

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  Proof similar to that of Corollary 1.

Examples

- Show that K_5 is nonplanar.

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  Show that $K_{3,3}$ is nonplanar.

  Using Corollary 3

  Corollary 2 is used in the proof of Five Color Theorem.

Kuratowski's Theorem

- **Definition** If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an *elementary subdivision*. The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called *homomorphic* if they can be obtained from the same graph by a sequence of elementary subdivisions.

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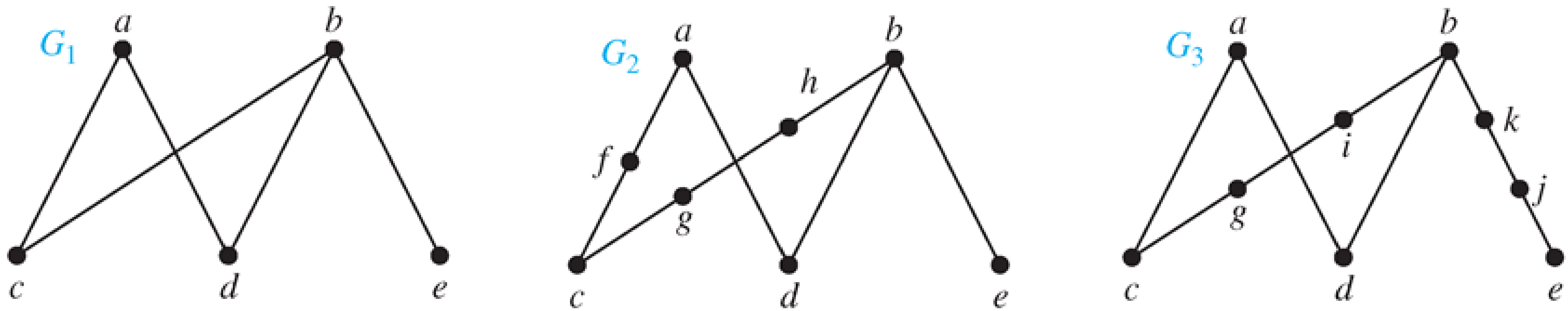

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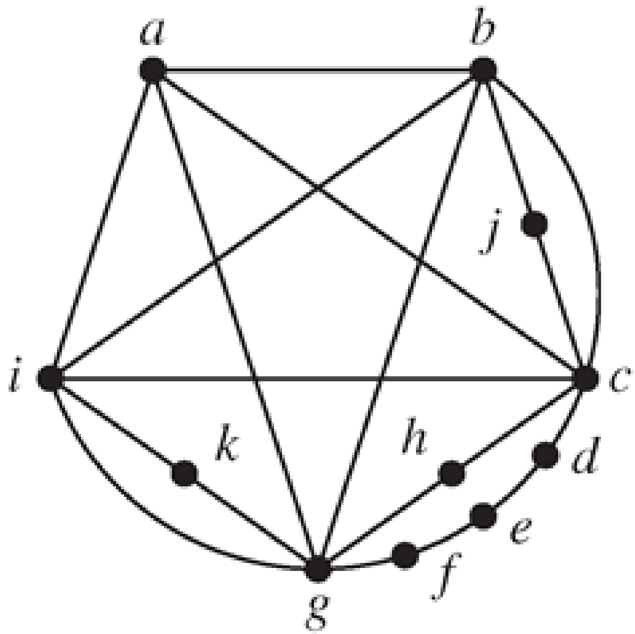
  Theorem A graph is nonplanar if and only if it contains a subgraph homomorphic to $K_{3,3}$ or K_5.

Examples



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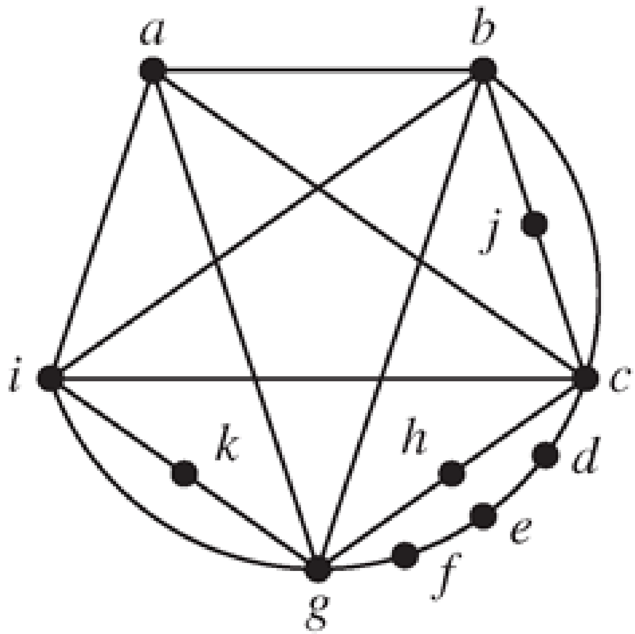
Examples



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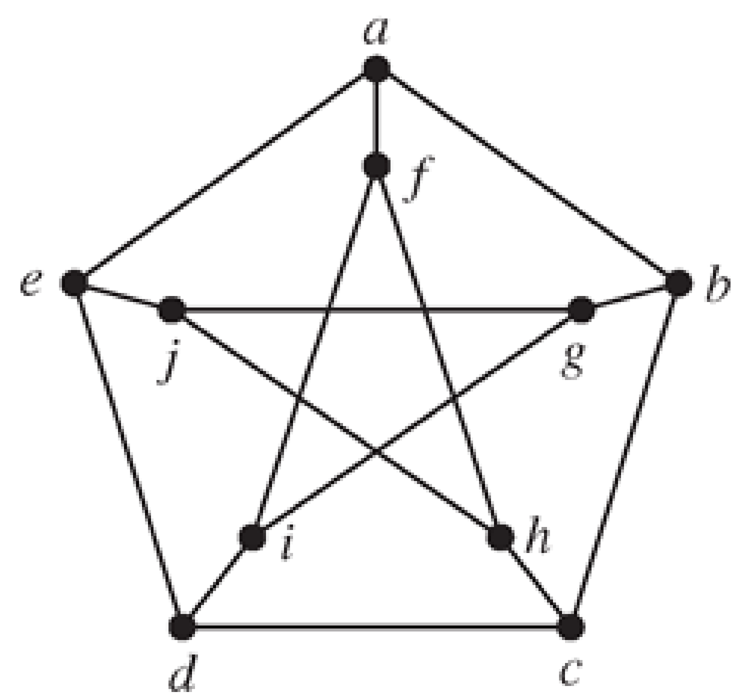

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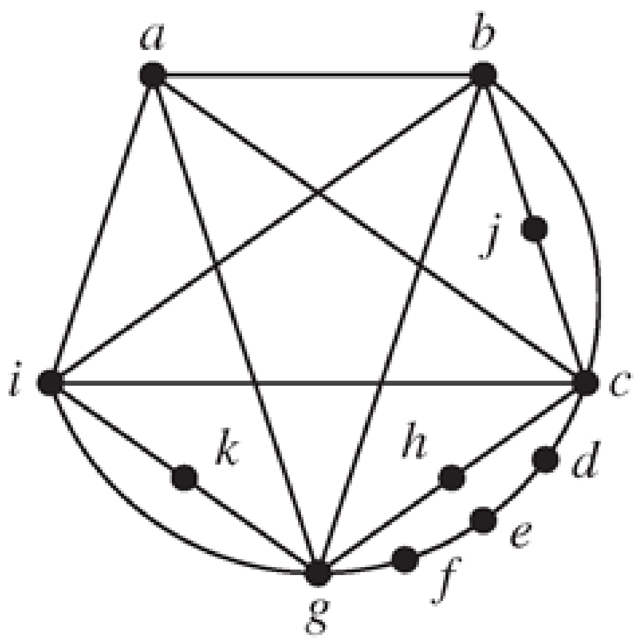

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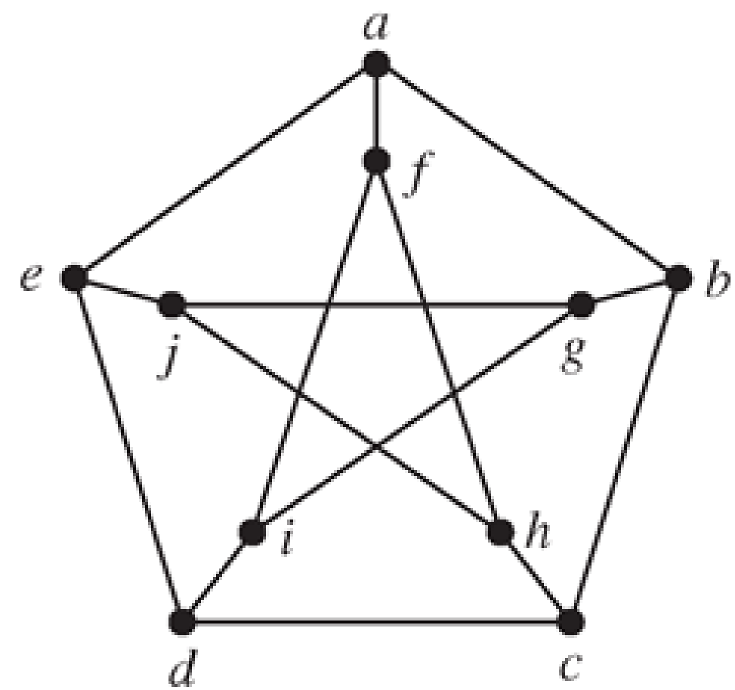
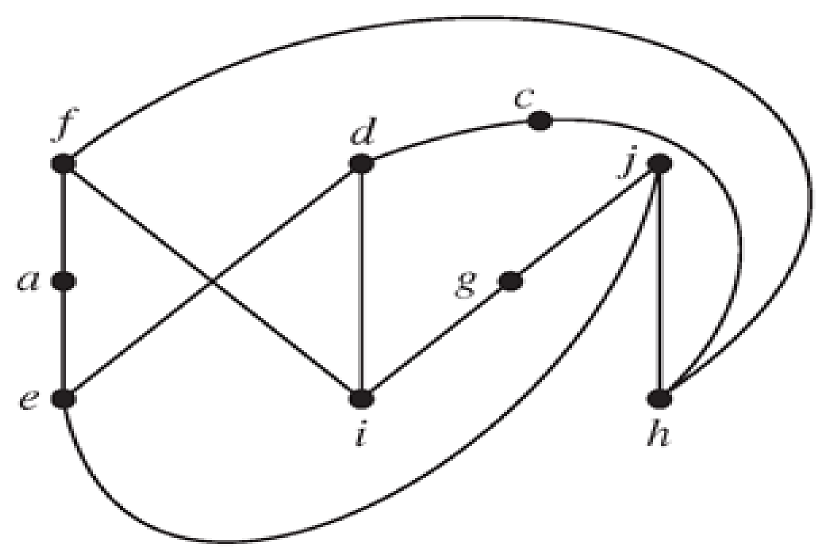
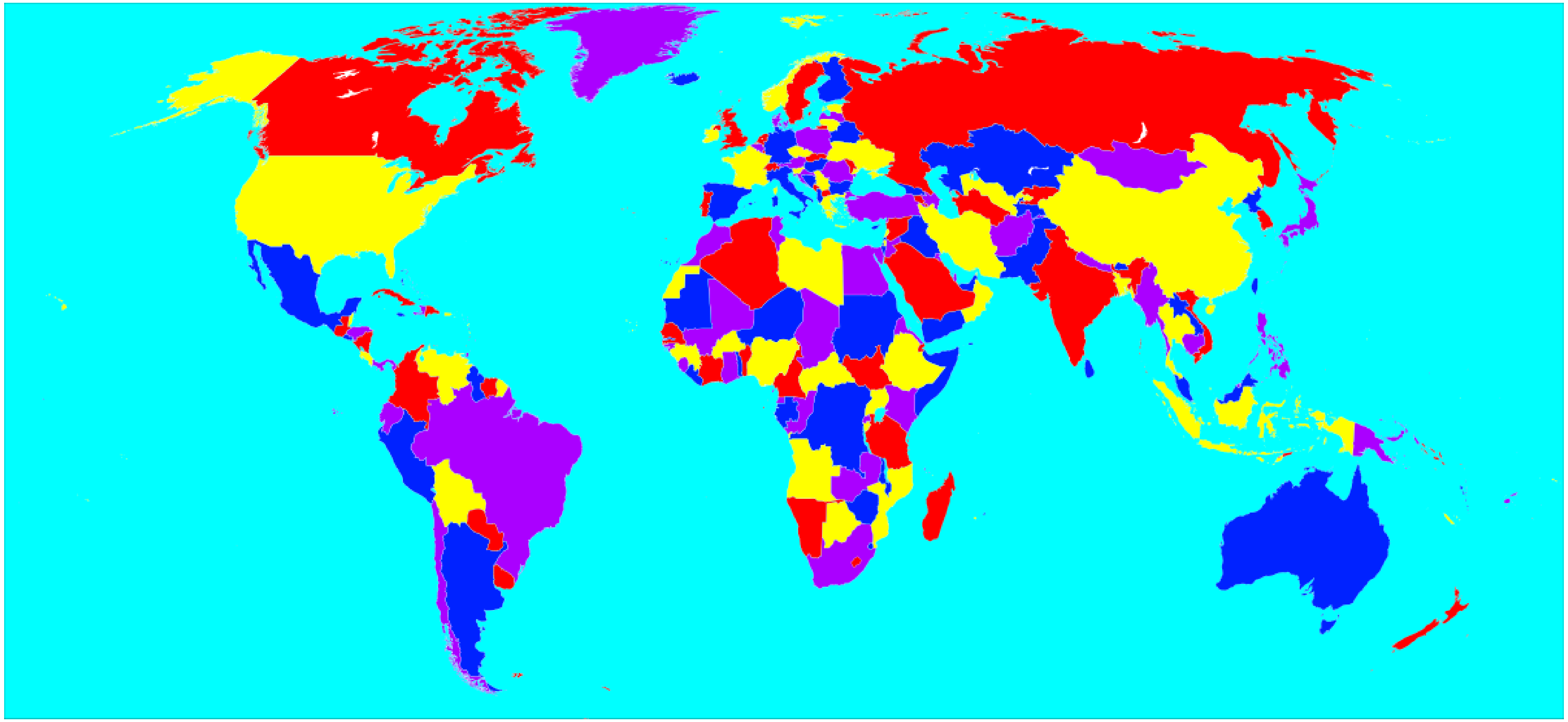




Graph Coloring

- **Four-color theorem** Given any separation of a plane into contiguous regions, producing a figure called a *map*, no more that four colors are required to color the regions of the map so that no two adjacent regions have the same color.



Four-color theorem

◇ first proposed by Francis Guthrie in 1852

◇ his brother Frederick Guthrie told Augustus De Morgan

◇ De Morgan wrote to William Hamilton

◇ Alfred Kempe proved it incorrectly in 1879

◇ Percy Heawood found an error in 1890 and proved the *five-color theorem*

◇ Finally, Kenneth Appel and Wolfgang Haken proved it with case by case analysis by computer in 1976 (*the first computer-aided proof*)

◇ Kempe's incorrect proof serves as a basis

Graph Coloring

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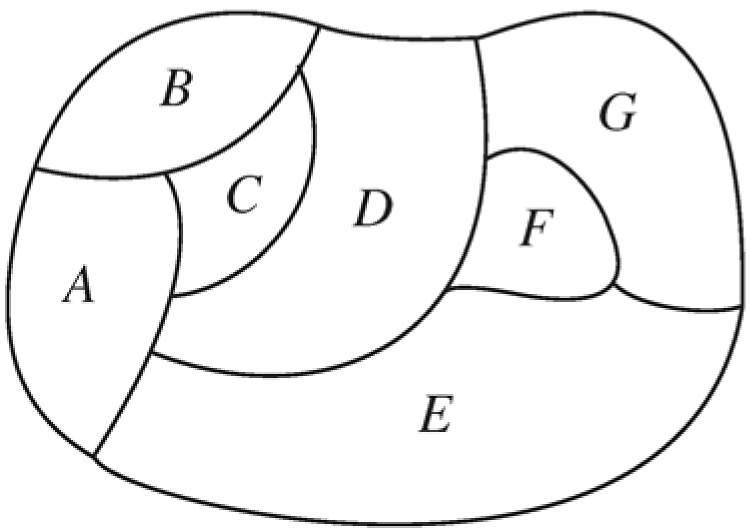

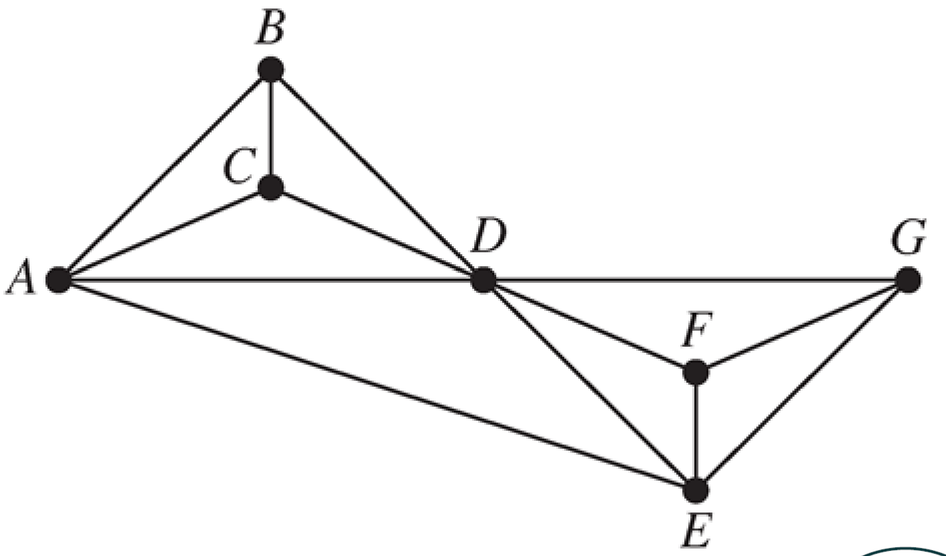

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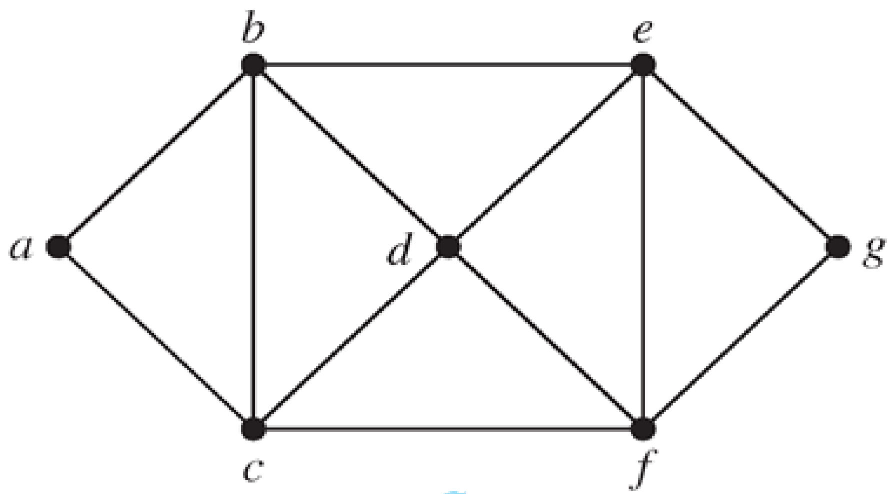

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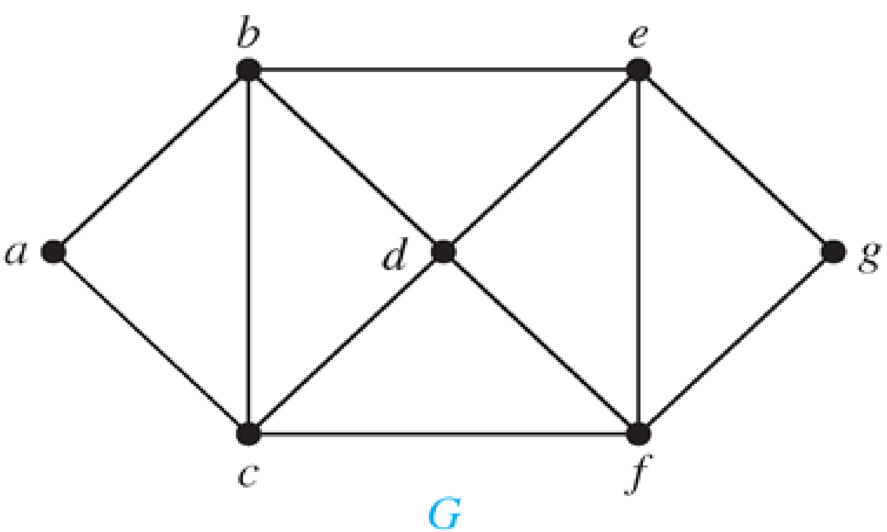

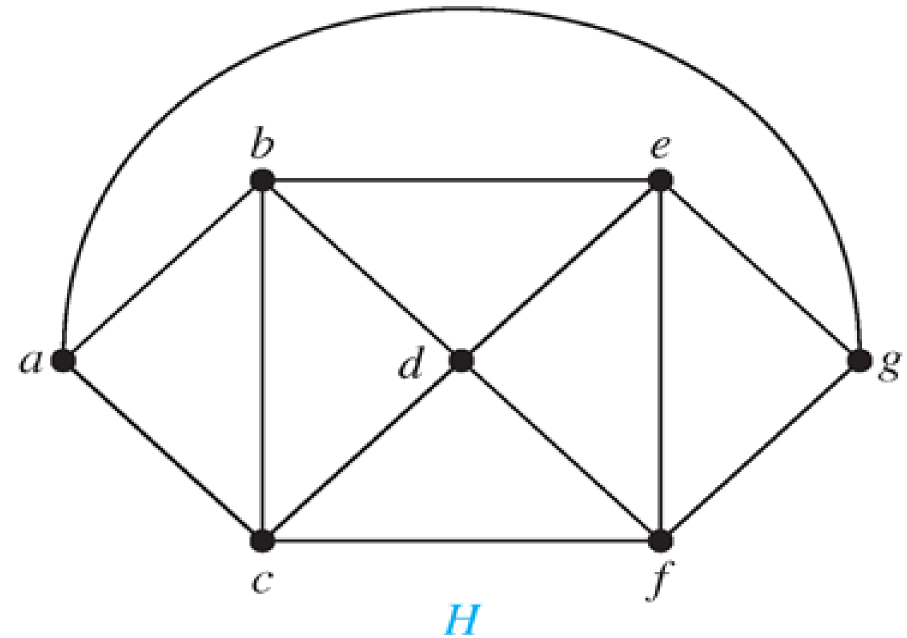

Examples

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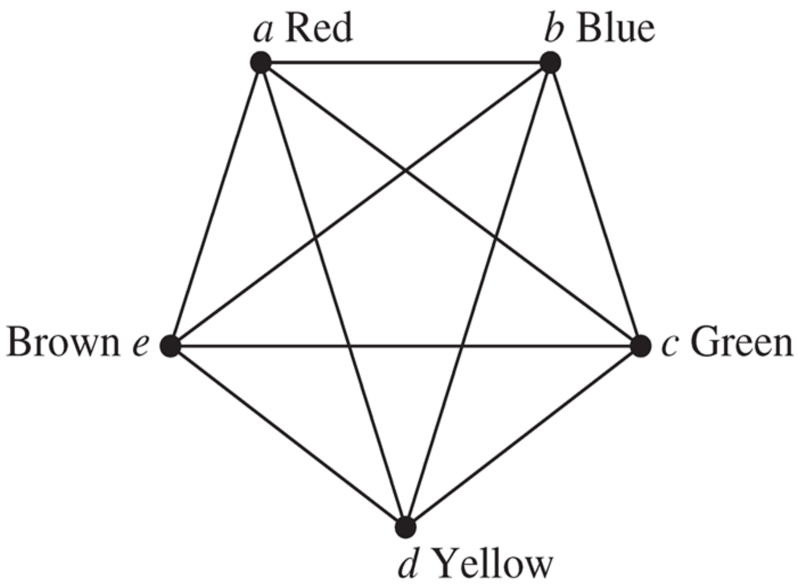

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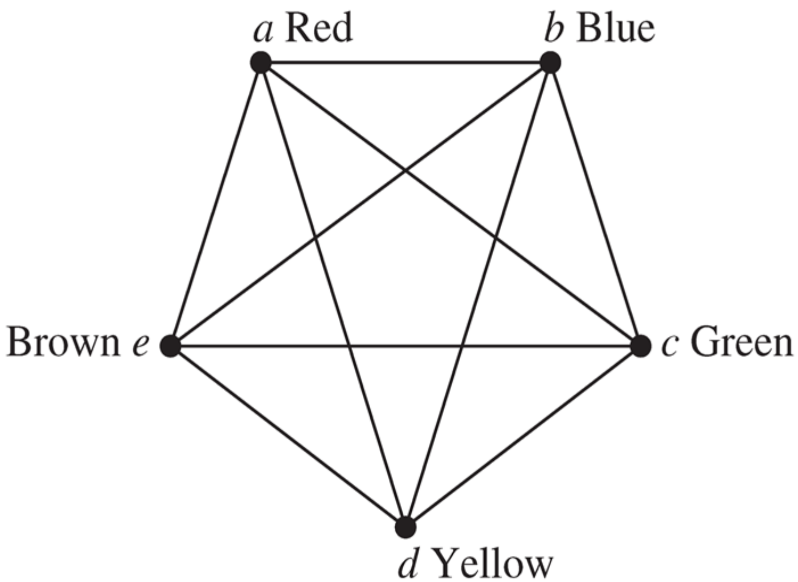

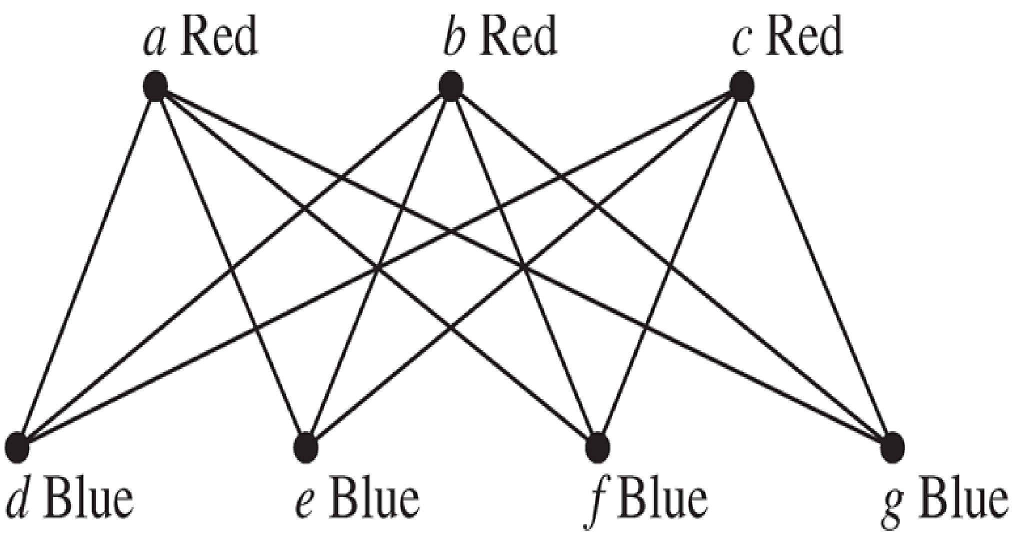

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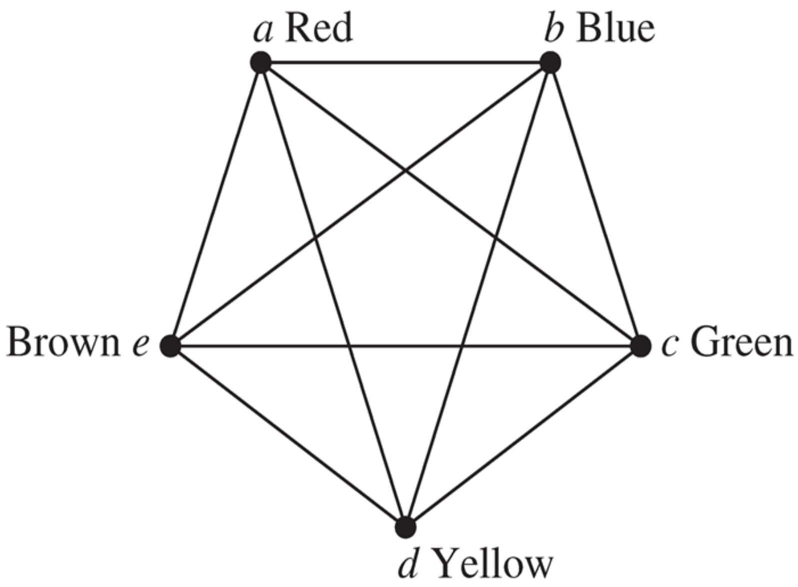

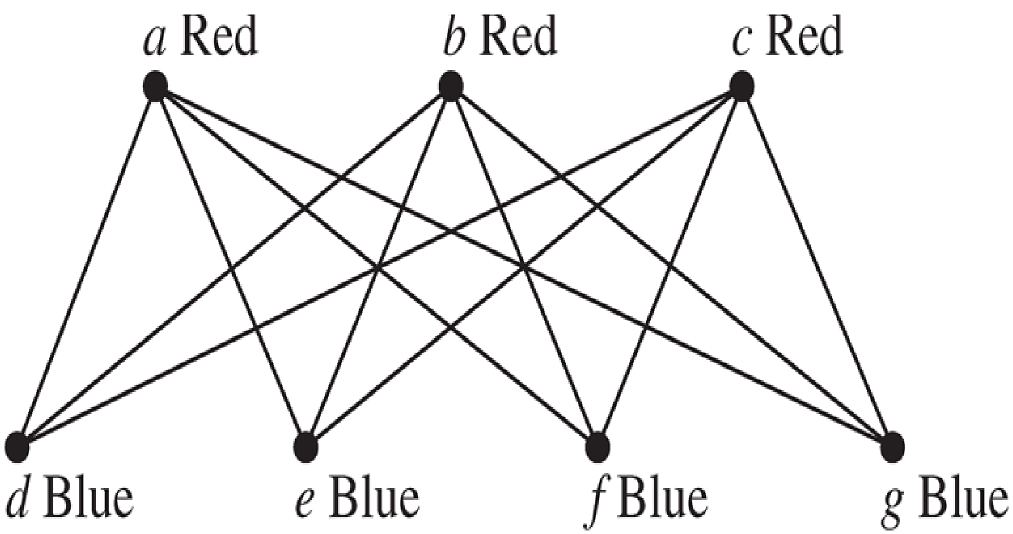

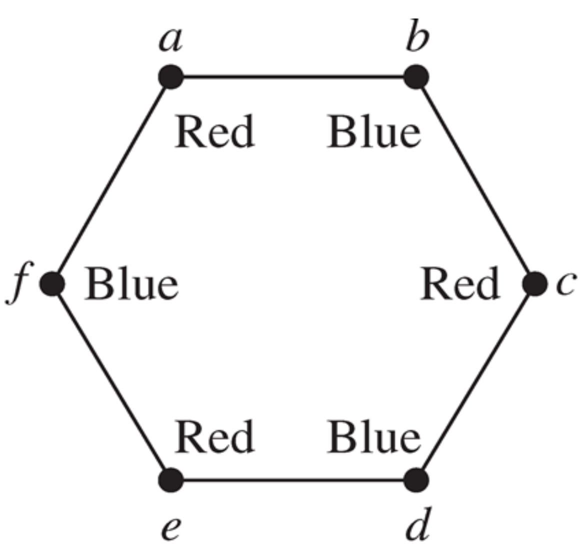

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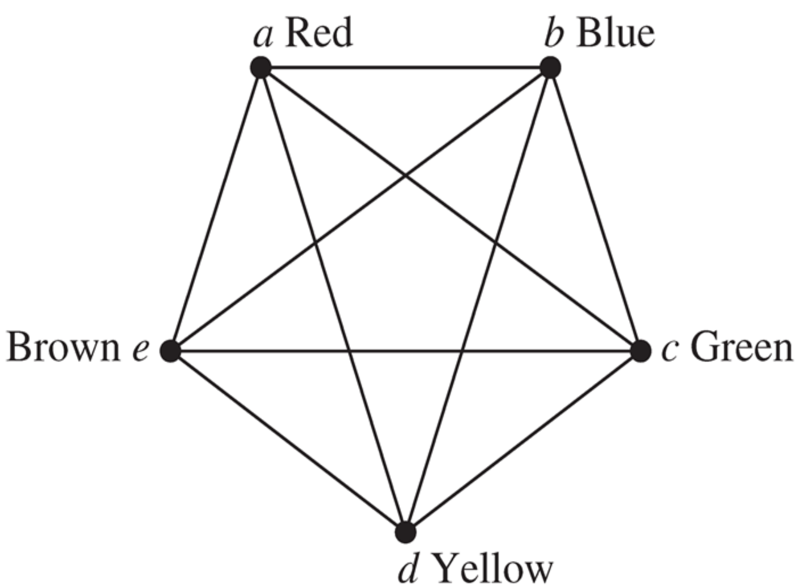
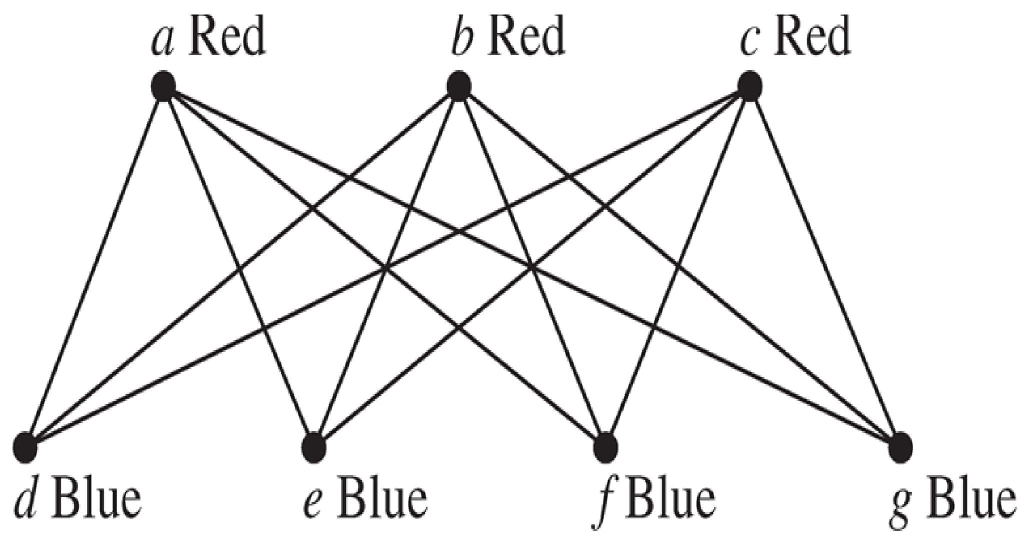
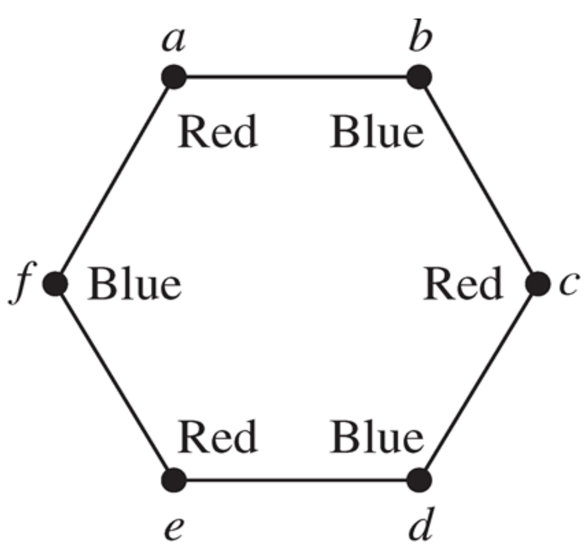
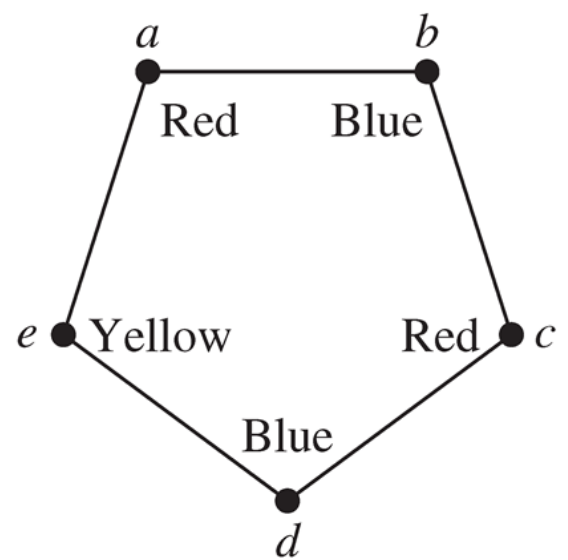




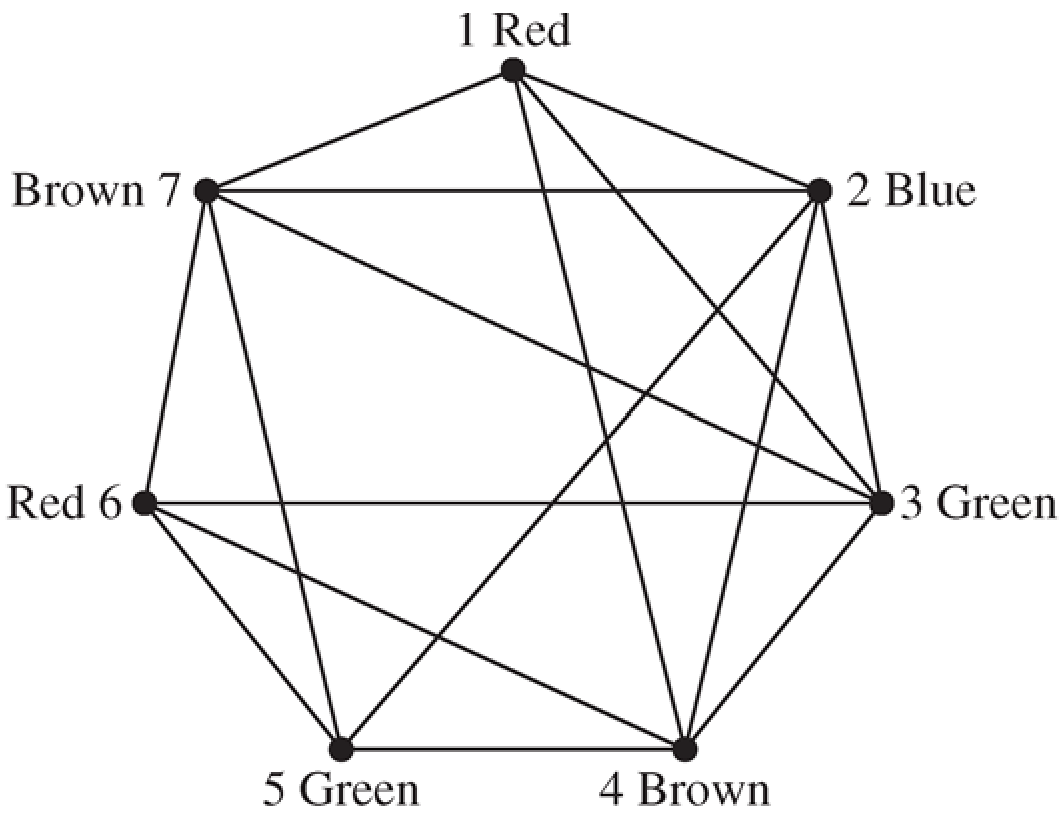




Applications of Graph Coloring

- Scheduling Final Exams

  Vertices represent courses, and there is an edge between two vertices if there is a common student in the courses.



Time Period	Courses
I	1, 6
II	2
III	3, 5
IV	4, 7

Applications of Graph Coloring

- ## Channel Assignments

  Televesion channels 2 through 13 are assigned to stations in North America so that no two stations within 150 miles can operate on the same channel . How can the assignment of channels be modeled by graph coloring?

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Graph Coloring $\in$ NPC

Announcements

- **Homework assignment** 6

  ◇ P650 Ex. 11, P668 Ex. 64, 66, P676 Ex. 28, 29, 32, P677 Ex. 45, 52, P690 Ex. 20, P691 Ex. 28, P704 Ex. 10, P716 Ex.2, 14, P725 Ex. 6, P726 Ex. 17, P756 Ex. 16, P757 Ex. 46, P784 Ex. 24, P795 Ex. 11, 12, P802 Ex. 2, 6

  ◇ Due on *Jan. 2nd, 2018 at the beginning of class*

  ◇ Please write your homeowrk **neatly**, as a courtesy to graders.

Next Lecture

- Trees ...