# DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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# Assignment #4

Please submit your assignments before class!

# The Birthday Paradox

- Suppose that 25 students are in a room. What is the probability that at least two of them share a birthday?

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  We may assume that a year has 365 days and there are no twins in the room.

  This will be very similar to the analysis of hashing $n$ keys into a table of size 365.

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  $\#A_n + \#B_n = 365^n$

# The Birthday Paradox

| $n$ | $A_n$ | $B_n$ | $n$ | $A_n$ | $B_n$ |
|---|---|---|---|---|---|
| 1 | 0.00000000 | 1.00000000 | 16 | 0.28360400 | 0.71639599 |
| 2 | 0.00273972 | 0.99726027 | 17 | 0.31500766 | 0.68499233 |
| 3 | 0.00820416 | 0.99179583 | 18 | 0.34691141 | 0.65308858 |
| 4 | 0.01635591 | 0.98364408 | 19 | 0.37911852 | 0.62088147 |
| 5 | 0.02713557 | 0.97286442 | 20 | 0.41143838 | 0.58856161 |
| 6 | 0.04046248 | 0.95953751 | 21 | 0.44368833 | 0.55631166 |
| 7 | 0.05623570 | 0.94376429 | 22 | 0.47569530 | 0.52430469 |
| 8 | 0.07433529 | 0.92566470 | 23 | 0.50729723 | 0.49270276 |
| 9 | 0.09462383 | 0.90537616 | 24 | 0.53834425 | 0.46165574 |
| 10 | 0.11694817 | 0.88305182 | 25 | 0.56869970 | 0.43130029 |
| 11 | 0.14114137 | 0.85885862 | 26 | 0.59824082 | 0.40175917 |
| 12 | 0.16702478 | 0.83297521 | 27 | 0.62685928 | 0.37314071 |
| 13 | 0.19441027 | 0.80558972 | 28 | 0.65446147 | 0.34553852 |
| 14 | 0.22310251 | 0.77689748 | 29 | 0.68096853 | 0.31903146 |
| 15 | 0.25290131 | 0.74709868 | 30 | 0.70631624 | 0.29368375 |

...

# Euclidean Algorithm

- The Euclidean algorithm in pseudocode

**ALGORITHM 1 The Euclidean Algorithm.**

**procedure** $gcd(a, b$: positive integers)
$x := a$
$y := b$
**while** $y \neq 0$
  $r := x$ **mod** $y$
  $x := y$
  $y := r$
**return** $x\{\gcd(a, b)$ is $x\}$

The number of divisions required to find $\gcd(a, b)$ is $O(\log b)$, where $a \geq b$. (this will be proved later.)

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**Why** ?

# Euclidean Algorithm

- Key steps in the Euclidean algorithm

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\begin{aligned}
r_0 &= r_1 q_1 + r_2 && 0 \le r_2 < r_1, \\
r_1 &= r_2 q_2 + r_3 && 0 \le r_3 < r_2, \\
&\ \vdots \\
r_{n\text{-}2} &= r_{n\text{-}1} q_{n\text{-}1} + r_{\mathrm{n}} && 0 \le r_n < r_{n\text{-}1}, \\
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See [Theorem 1 p. 347].

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# Solving Linear Recurrence Relations

- **Definition** A *linear homogeneous relation of degree $k$* with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

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  By induction, such a recurrence relation is uniquely determined by this recurrence relation, and $k$ initial conditions $a_0, a_1, \ldots, a_{k-1}$.

# Examples of Linear Recurrence Relations

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**Examples**

$$P_n = (1.11)P_{n-1}$$

$$f_n = f_{n-1} + f_{n-2}$$

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$$H_n = 2H_{n-1} + 1$$

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$P_n = (1.11) P_{n-1}$ linear homogeneous recurrence relation of degree 1

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  | $H_n = 2H_{n-1} + 1$ | NOT homogeneous |
  | $B_n = nB_{n-1}$ | coefficients are not constants |

# Solving Linear Recurrence Relations

- **Example** Consider the recurrence relation

  $$a_n = 2a_{n-1} - a_{n-2},$$

  Which of the following are solutions?

  - ◇ $a_n = 3n$:
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  - ◇ $a_n = 5$:

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  ◇ Bring $a_n = r^n$ back to the recurrence relation:

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k},$$

i.e.,

$$r^{n-k}(r^k - c_1 r^{k-1} - \cdots - c_k) = 0$$

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  ◇ The solutions to the *characteristic equation* can yield an explicit formula for the sequence.

  $$(r^k - c_1 r^{k-1} - \cdots - c_k) = 0$$

# Recall: Problem IV

- **Fibonacci number**

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  ◇ What is the closed-form expression of $F_n$?

  Consider $x^n = x^{n-1} + x^{n-2}$, with $x \neq 0$. There are two different roots

  $$\phi = \frac{1+\sqrt{5}}{2}, \quad \psi = \frac{1-\sqrt{5}}{2}$$

  Then $F_n$ can be the form of $a\phi^n + b\psi^n$. By $F_0 = 0$ and $F_1 = 1$, we have $a + b = 0$ and $\phi a + \psi b = 1$, leading to $a = \frac{1}{\sqrt{5}}$, $b = -a$. Therefore,

  $$F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$$

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  **Theorem** If this CE has 2 roots $r_1 \neq r_2$, then the sequence $\{a_n\}$ is a solution of the recurrence relation if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n \geq 0$ and constants $\alpha_1, \alpha_2$.

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  **Proof**?

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  Two roots are 2 and $-1$. So, assume that

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  We get $\alpha_1 = 3$ and $\alpha_2 = -1$. Thus,

  $$a_n = 3 \cdot 2^n - (-1)^n$$

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  By the two initial conditions, we have

  $$a_0 = \alpha_1 + \alpha_2 = 2$$
  $$a_1 = 2\alpha_1 + 5\alpha_2 = 1$$

  We get $\alpha_1 = 3$ and $\alpha_2 = -1$. Thus,

  $$a_n = 3 \cdot 2^n - 5^n$$

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  **Example** $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$

# The Case of Degenerate Roots

- **Theorem** If the CE $r^2 - c_1 r - c_2 = 0$ has only 1 root $r_0$, then

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  **Proof**?

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# The Case of Degenerate Roots in General

- **Theorem** [Theorem 4, p.519] Suppose that there are $t$ roots $r_1, \ldots, r_t$ with multiplicities $m_1, \ldots, m_t$. Then

  $$a_n = \sum_{i=1}^{t} \left( \sum_{j=0}^{m_i - 1} \alpha_{i,j} n^j \right) r_i^n,$$

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  **Example**

  $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with $a_0 = 1$, $a_1 = -2$, $a_2 = -1$

# Linear Nonhomogeneous Recurrence Relations

- **Definition** A *linear nonhomogeneous relation* with constant coefficients may contain some terms $F(n)$ that depend only on $n$

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n).$$

The recurrence relation
$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ is called the *associated homogeneous recurrence relation*.

# Linear Nonhomogeneous Recurrence Relations

- **Theorem** If $a_n = p(n)$ is any particular solution to the linear nonhomogeneous relation with constant coefficients,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

Then all its solutions are of the form

$$a_n = p(n) + h(n),$$

where $a_n = h(n)$ is any solution to the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

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  Let $p(n) = cn + d$, then
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  We get $c = 1$ and $d = 3/2$. Thus,
  $p(n) = n + 3/2$

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**Definition** The *generating funciton* for the sequence $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

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$$G(x) = C(m, 0) + \cdots + C(m, m)x^m = (1 + x)^m$$

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  $1, 2, 3, 4, 5, \ldots$

# Operations of Generating Functions

- **Theorem** Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$$

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# Useful Generating Functions

$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k$

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$$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$$

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$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

- **Problem 1** How many solutions are there to the equation
$$x_1 + x_2 + x_3 = 17,$$
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This is **equivalent** to the problem of $r$-combinations from a set with $n$ elements when repetition is allowed.

$$C(n + r - 1, n - 1) = C(19, 2)$$

- **Problem 2** Find the number of solutions of

$$x_1 + x_2 + x_3 = 17,$$

where $x_1, x_2, x_3$ are nonnegative integers with $2 \leq x_1 \leq 5$, $3 \leq x_2 \leq 6$, $4 \leq x_3 \leq 7$.

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Using *generating functions*, the number is the coefficient of $x^{17}$ in the expansion of

$$(x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^4 + x^5 + x^6 + x^7)$$

- **Problem 3** In how many ways can eight identical cookies be ditributed among three distinct children if each child receives at least two cookies and no more than four cookies?

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The coefficient of $x^8$ in the expansion

$$(x^2 + x^3 + x^4)^3$$

- **Problem 4** Use generating functions to find the number of $k$-combinations of a set with $n$ elements, $C(n, k)$.

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Each of the $n$ elements in the set contributes the term $(1+x)$ to the generating function $f(x) = \sum_{k=0}^{n} a^k x^k$. Hence, $f(x) = (1+x)^n$.

Then by the binomial theorem, we have $a_k = \binom{n}{k}$.

# Cartesian Product

- Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, the *Cartesian product* $A \times B$ is the set of pairs

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Cartesian product defines a set of all ordered arrangements of elements in the two sets.

# Binary Relation

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  **Example**: Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$
  - ◇ Is $R = \{(a, 1), (b, 2), (c, 2)\}$ a relation from $A$ to $B$?
  - ◇ Is $Q = \{(1, a), (2, b)\}$ a relation from $A$ to $B$?
  - ◇ Is $P = \{(a, a), (b, c), (b, a)\}$ a relation from $A$ to $A$?

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**Example**: Let $A = \{0, 1, 2\}$ and $B = \{u, v\}$, and $R = \{(0, u), (0, v), (1, v), (2, u)\}$. $(R \subseteq A \times B)$

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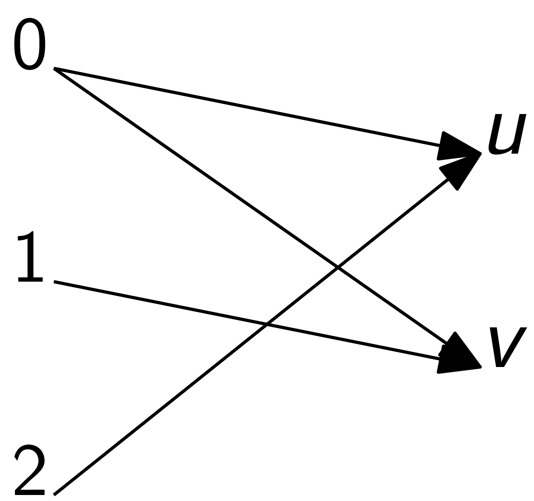
- We can graphically represent a binary relation $R$ as:

  if $a \ R \ b$, then we draw an arrow from $a$ to $b$: $a \rightarrow b$

  **Example**: Let $A = \{0, 1, 2\}$ and $B = \{u, v\}$, and $R = \{(0, u), (0, v), (1, v), (2, u)\}$. $(R \subseteq A \times B)$



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| $R$ | $u$ | $v$ |
|---|---|---|
| 0 | × | × |
| 1 | × | |
| 2 | | × |

# Relations and Functions

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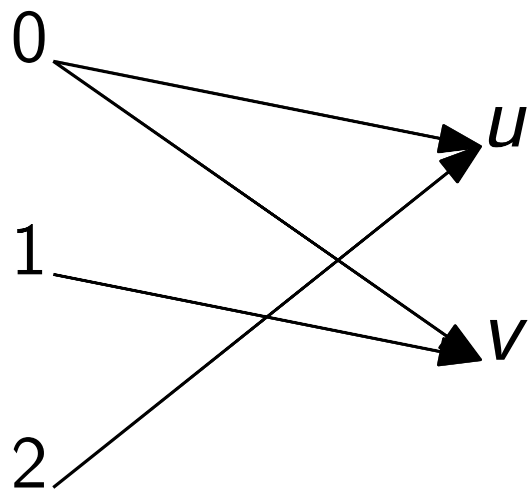

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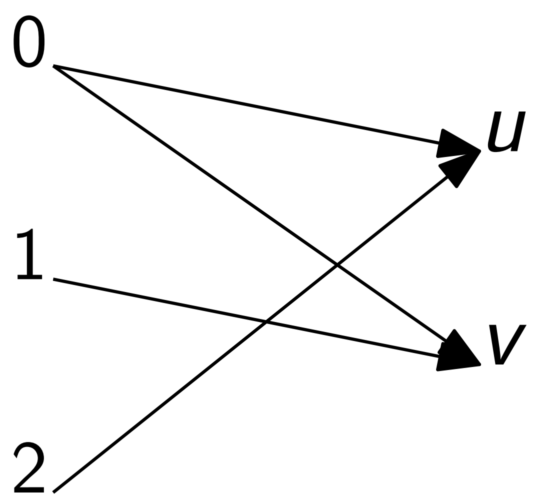


What is the difference between a relation and a function from $A$ to $B$?

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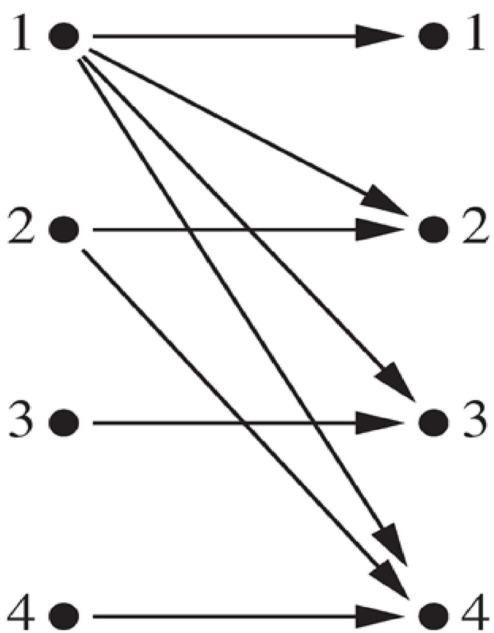

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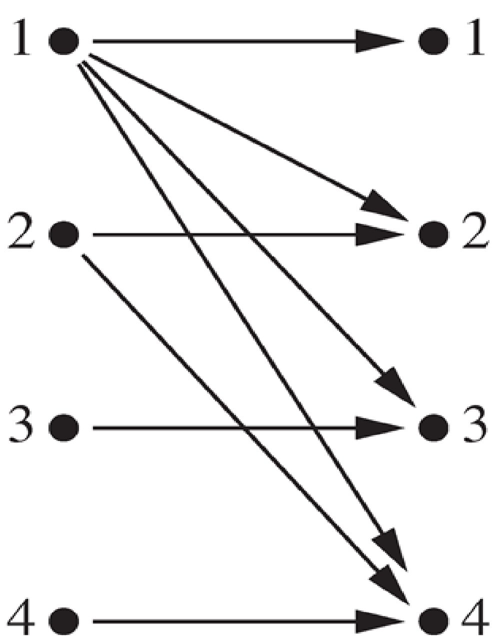


| $R$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | × | × | × | × |
| 2 | | × | | × |
| 3 | | | × | |
| 4 | | | | × |

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  The number of subsets of a set with $k$ elements is $2^k$

# Next Lecture

- relation ...