Probability theory is nothing but common sense reduced to calculation. –Pierre-Simon Laplace

Problems 10-20 on More Properties of the Normal distribution (STAT2802 Tutorial problems for the week of 01-OCT-2012, no class for this week due to public holidays)

Problems 10-20

10. $e^{-n}\left(1+\frac{n}{1!}+\frac{n^2}{2!}+\cdots+\frac{n^n}{n!}\right)\xrightarrow{n\to\infty}\frac{1}{2}$
11. Let $X_1, X_2, \dots$ be independent Cauchy r.v.s, each with p.d.f. $f(x)=\frac{d}{\pi(d^2+x^2)}$. Show that $\frac{X_1+X_2+\cdots+X_n}{n}$ has the same distribution as X_1. Does this contradict the WLLT or the CLT? (*Hint: You may use the fact that the CF for Cauchy distribution in this problem is $\varphi_{X_1}(t)=e^{-d|t|}$. The MGF of a Cauchy distribution is undefined.*)
12. Show that the product of two standard normal random variables which are jointly distributed as the bivariate normal distribution $N\left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}\right)$ has moment generating function $(1-2\rho t-(1-\rho^2)t^2)^{-\frac{1}{2}}$. Then deduce its mean and variance.
13. The r.v. X_i is normally distributed with mean μ_i and variance σ_i^2, for $i=1,2$, and $X_1 \perp\!\!\!\perp X_2$. Find the distribution of $Z=a_1X_1+a_2X_2$, where $a_1,a_2\in\mathbb{R}$.
14. Suppose two univariate normal r.v.s $X\sim N(\mu_X,\sigma_X^2)$ and $Y\sim N(\mu_Y,\sigma_Y^2)$ are also jointly normal with covariance $\begin{bmatrix}\sigma_X^2 & \rho\sigma_X\sigma_Y\\ \rho\sigma_X\sigma_Y & \sigma_Y^2\end{bmatrix}$. Is the conditional density of $Y|X$ also normal? Derive its form.
15. Statisticians are studying relationship between the height of fathers and sons. From the data collected, the average height of the fathers was 1.720 meters; the SD was 0.0696 meters. The average height of the sons was 1.745 meters; the SD was 0.0714 meters. The correlation was 0.501. Assuming normality, True or False and explain: because the sons average 2.5cm taller than the fathers, if the father is 1.830 meters tall, it's 50-50 whether the son is taller than 1.855 meters.
16. Let X and Y be two independent standard normal random variables. Find the p.d.f.s of : (i) $X+Y$; (ii) X^2; (iii) X^2+Y^2.
17. Let X and Y be two independent standard normal random variables. Find the joint p.d.f. of $U=X+Y$ and $V=X-Y$. Show that U and V are independent and write down the marginal distribution for U and V.
18. Let X and Y be two independent standard normal random variables. Let
$$Z=\begin{cases}|Y|, & \text{if } X>0,\\ -|Y|, & \text{if } X<0.\end{cases}$$
By finding $\mathbb{P}(Z\le z)$ for $z<0$ and $z>0$, show that $Z\sim N(0,1)$. Explain briefly why the joint distribution of X and Z is not bivariate normal.
19. Let $X\sim N(\mu,\sigma^2)$ and suppose $h(x)$ is a smooth bounded function (smooth=any-number-of-times differentiable), $x\in\mathbb{R}$. Prove Stein's formula $\mathbb{E}[(X-\mu)h(X)]=\sigma^2\mathbb{E}[h'(X)]$.
20. Let X be a normally distributed r.v. with mean 0 and variance 1. Compute $\mathbb{E}X^r$ for $r=0,1,2,3,4$. Let Y be a normally distributed r.v. with mean μ and variance σ^2. Compute $\mathbb{E}Y^r$ for $r=0,1,2,3,4$.