Mathematical Stastics Assignment 2

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1 Part I

2.1 Calculate the expectation and variance of the $T \sim t(n)$ via the stochastic representation (SR):

$$T \stackrel{d}{=\!=} \frac{Z}{\sqrt{Y/n}},$$

Where $Z \sim N(0,1), Y \sim \chi^2(n)$ and $Z \,\text{and}\, Y$ are indepented.

Solution.

The density of T is given by

$$f(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\Gamma\left(\frac{n}{2}\right)}\left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}}, -\infty < t < \infty.$$

The expectation is

$$\begin{aligned}
E(T) &= \int_{-\infty}^{\infty} t f(t)\mathrm{dt} \\
&= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\Gamma\left(\frac{n}{2}\right)}\int_{-\infty}^{\infty} t\left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}} dt \\
&= \frac{\Gamma\left(\frac{n+1}{2}\right)}{2\sqrt{\pi n}\Gamma\left(\frac{n}{2}\right)}\int_{-\infty}^{\infty} \left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}} dt^2 \\
&= 0
\end{aligned}$$

The variance is

$$\begin{aligned}
E(T^2) &= \int_{-\infty}^{\infty} t^2 f(t)\mathrm{dt} \\
&= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\Gamma\left(\frac{n}{2}\right)}\int_{-\infty}^{\infty} t^2\left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}} dt \\
&= \frac{\Gamma\left(\frac{n+1}{2}\right)}{2\sqrt{\pi n}\Gamma\left(\frac{n}{2}\right)}\int_{-\infty}^{\infty} t\left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}} dt^2
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(T) &= E(T^2) - E(T)^2 \\
&= \frac{n-1}{n-3}(\text{for}\, n > 3,)
\end{aligned}$$

2.2 Let $X_1, \ldots, X_n$ are iid obey Beta(3,2). Find the sampling distributions of $X_{(1)} = \min\{X_1, \ldots, X_n\}$ and $X_{(n)} = \max\{X_1, \ldots, X_n\}$.

Solution.

The largest order statistic $X_{(n)}$ is

$$\begin{aligned}
G_n(x) &= F^n(x) \\
&= \left\{\frac{B(x;3,2)}{B(3,2)}\right\}^n
\end{aligned}$$

The simallest order statistic $X_{(1)}$

$$\begin{aligned}
G_1(x) &= 1 - \{1 - F(x)\}^n \\
&= 1 - \left\{1 - \frac{B(x;3,2)}{B(3,2)}\right\}^n
\end{aligned}$$

2.3 Let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ be the order statistics of a random sample of size n from the exponential distribution with pdf $f(x) = e^{-x}, x > 0$, zero elsewhere.

a) Show that $Z_1 = \mathrm{n}\mathrm{X}_{(1)}$, $Z_2 = (n-1)[X_{(2)} - X_{(1)}]$, $Z_3 = (n-2)[X_{(3)} - X_{(2)}]$, ..., $Z_n = X_{(n)} - X_{(n-1)}$ are independent and that each Z_i has the exponential distribution.

Proof. $Z_i = (n-(i-1))[X_i - X_{i-1}]$

The joint density of $X_{(i)}$ and $X_{(i-1)}$ is

$$g_{i-1,i}(x_{(i-1)}, x_{(i)}) = c \cdot \exp\{x_{(i-1)}^{1-i} + x_{(i)}^{-i} - (n-i)x_{(i)}\}(1 - e^{-x_{(i-1)}})^{i-2},$$

where $0 \leqslant x_{(i-1)} \leqslant x_{(i)} \leqslant 1$ and

$$c \hat{=} \frac{n!}{(i-2)!(n-i)!}$$

Making the transormation $z = x_{(i)} - x_{(i-1)}$ and $x = x_{(i-1)}$, we have

$$\begin{aligned} J(z, x \to x_{(i-1)}, x_{(i)}) &= \left| \frac{\partial(z, x)}{\partial(x_{(i-1)}, x_{(i)})} \right| \\ &= \det \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} = -1 \end{aligned}$$

Hence, the joint density of $Z = X_{(i)} - X_{(i-1)}$ and $X = X_{(i-1)}$ is

$$\begin{aligned} h(z, x) &= g_{i-1,i}(x_{(i-1)}, x_{(i)}) / |J(z, x \to x_{(i-1)}, x_{(i)})| \\ &= c \cdot \exp\{\{x_{(i-1)}^{1-i} + x_{(i)}^{-i} - (n-i)x_{(i)}\}(1 - e^{-x_{(i-1)}})^{i-2} \\ &= c \cdot \exp\{x^{1-i} + (z+x)^{-i} - (n-i)(z+x)\}(1 - e^{-x})^{i-2} \\ &= c \cdot \exp\{x^{1-i}\}(1 - e^{-x})^{i-2} \cdot \exp\{(z+x)^{-i} - (n-i)(z+x)\} \end{aligned}$$

Where $0 \leqslant x \leqslant \infty, 0 \leqslant z \leqslant \infty$, and $0 \leqslant x + z \leqslant \infty$. The marginal density of $Z = X_{(i)} - X_{(i-1)}$ is given by

$$\begin{aligned} h(z) &= \int_0^\infty h(z, x) dx \\ &= c \cdot \int_0^\infty \exp\{x^{1-i} + (z+x)^{-i} - (n-i)(z+x)\}(1 - e^{-x})^{i-2} dx \\ &= c \cdot \end{aligned}$$

□

b) Demostrate that all linear functions of $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$, such as $\sum_{i=1}^n a_i X_{(i)}$, can be expressed as linear functions of independent random variables.

Solve:

2.4 Let $X_i \sim \mathrm{Gamma}(a_i, 1), i = 1, \ldots\ldots n,$ and $X_1, \ldots, X_n$ are mutually independent. Define

$$Y_i = \frac{X_i}{X_1 + \cdots + X_n}, i = 1, \ldots, n-1.$$

a) Find the joint density of $(Y_1, \ldots, Y_{n-1})$.

b) Find the density of $X_1 + \cdots + X_n$.

2.5 Let $X \sim \text{Gamma}(p, 1)$, $Y \sim \text{Beta}(q, p-q)$, and X $\perp\!\!\!\perp$ Y , where $0 < q < p$. Find the distribution of XY.

2.6 Let Z $\sim$ Bernoulli $(1 - \varphi)$, $\boldsymbol{x} = (X_1, ..., X_m)^T$, $X_i \sim \text{Poisson}(\lambda_i)$ for $i = 1, ..., m$, and $(Z, X_1, ..., X_m)$ be mutually independent. Define $\boldsymbol{y} = (Y_1, ..., Y_m)^T = Z\boldsymbol{x}$. Find the joint pmf of **y**.

2.7 Let x_1, x_2 be a random sample form the $N(o, \sigma^2)$ population.

(a) Derive the distribution of the statistic

$\frac{(X_1 - X_2)^2}{(X_1 + \boldsymbol{X}_2)^2}$

(b) Find the constant k, such that

$$P_r\left\{\frac{(X_1 + X_2)^2}{(X_1 + X_2)^2 + (X_1 - X_2)^2} < k\right\} = 00.01$$

Solve:

2.8 Show that if X and Y are independent exponential random variables with $\lambda = 1$, then X/Y follows an F distribution. Also, identify the degrees of freedom.

Proof.

$f(x) = e^{-x}$, $f(y) = e^{-y}$, $f(x, y) = e^{-(x+y)}$

Let $U = X$ and $V = X/Y \Rightarrow X = U \text{ and } Y = U/V$

$|J| = \begin{vmatrix} 1 & U \\ \frac{1}{V} & -\frac{U}{V} \end{vmatrix} = \frac{-U}{V}$

Then

$$\begin{aligned} f_{\text{uv}}(u, v) &= f_{\text{XY}}\left(u, \frac{u}{v}\right)\left|\frac{-U}{V}\right| = e^{-u\left(1+\frac{1}{v}\right)}\left(\frac{u}{v^2}\right) \\ f_v(v) &= \int_0^\infty e^{-u\left(1+\frac{1}{v}\right)}\left(\frac{u}{v^2}\right)du = \frac{1}{v^2}\int_1^\infty e^{-u\left(1+\frac{1}{v}\right)}u\frac{(1+\frac{1}{v})^2}{\Gamma(2)}du\frac{\Gamma(2)}{(1+\frac{1}{v})^2} \\ &= \left(\frac{1}{v}\right)^2\left(1+\frac{1}{v}\right)^{-2} = (1+v)^{-2} \sim F(2, 2) \end{aligned}$$

$\square$