Mathematical Statistics Assignment 5

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1 Part I

1. In Example 5.5, if $\mathbb{C} = \{1, 7, 3, 8, 4\}$, how to find the Type I error rate $\alpha(0)$ and the Type II error rate $\beta(1)$?

   Solution:

   $\alpha(0) = \Pr(\boldsymbol{x} \in \mathbb{C} | \theta = 0) = 0 + 0.01 + 0.02 + 0.07 + 0.05 = 0.15$

   $\beta(1) = \Pr(\boldsymbol{x} \in \mathbb{C}' | \theta = 1) = 0 + 0.03 + 0.01 + 0.02 + 0.04 = 0.1$

2. Let $Y \sim \text{Binomial}(n, \theta)$. We reject $H_0: \theta = 0.5$ and accept $H_1: \theta > 0.5$ if $Y \geqslant c$. Consider the normal approximation to the binomial distribution, please find n and c to give a power function $p(\theta)$ with $p(0.5) = 0.1$ and $p(2/3) = 0.95$.

   Solution:

$$\begin{cases} p(0.5) = \Pr(Y < c | \theta = 0.5) = \sum_{i=0}^{c-1} \left(\begin{array}{c} n \\ i \end{array} \right) 0.5^n \\ p(2/3) = \Pr(Y < c | \theta = 2/3) = \sum_{i=0}^{c-1} \left(\begin{array}{c} n \\ i \end{array} \right) \left(\frac{1}{3} \right)^{n-i} \left(\frac{2}{3} \right)^i \end{cases}$$

$$\Longrightarrow$$

$$\begin{cases} n = \\ c = \end{cases}$$

3. Let $X_1, \ldots, X_n$ be a random sample from $\text{Gamma}(2, \theta)$ with pdf

$$f(x; \theta) = \begin{cases} \frac{\theta^2}{\Gamma(2)} x e^{-\theta x} & \text{, if } x > 0 \\ 0 & \text{, otherwise,} \end{cases}$$

   where $\theta > 0$.

   a. Find the pdf of $Y = \sum_{i=1}^{n} X_i$.

   b. Find the MPT of size α for testing $H_0: \theta = \theta_0 (=1)$ against $H_1: \theta = \theta_1 (>1)$.

   c. Express the power function as an integral.

   Solution:

   a.

   b.

   c.

4. Let $X_1, ..., X_n$ be a random sample from

$$f(x;\theta) = \begin{cases} \theta(1-x)^{\theta-1} & ,\text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

where $\theta > 0$.

a. Find the MPT of size α for testing $H_0\colon \theta = \theta_0(=1)$ against $H_1\colon \theta = \theta_0(>1)$.

b. Find the LRT for testing $H_0\colon \theta = 1$ against $H_1\colon \theta \neq 1$.

Solution:

5. Let $X_1, ..., X_n \overset{\text{iid}}{\sim} N(\theta, 1)$. Find the UMPT of size α for testing $H_0\colon \theta \geqslant \theta_0$ against $H_1\colon \theta < \theta_0$.

6. Let $X_1, ..., X_n \overset{\text{iid}}{\sim} N(\mu, \sigma^2)$ with unknown mean μ. Find the LRT with size α for testing H_0: $\sigma^2 = \sigma_0^2$ against one of the alternative $\sigma^2 \neq \sigma_0^2$, $\sigma^2 > \sigma_0^2$, or $\sigma^2 < \sigma_0^2$.