Chapter 4 Confidence Interval Estimation

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1 Introduction

No matter how point estimators are determined, share the fundamental weakness: They cannot provide the precision of the estimators. An interval is called a *random interval* if at least one of its end points is a random variable. The usual way to quantify the amount of uncertaninty in an estimator is to construct a *confidence interva* (CI). In principle, CIs are ranges of numbers that have a high probability of "containing" the unknown parameter as an interior point. Here are two-sided confidence interval and on-sided CI.

Let $X_1, ..., X_n \overset{\text{iid}}{\sim} f(x;\theta)$ and $\boldsymbol{x} = (X_1, ..., X_n)^T$. $T_1(\boldsymbol{x})$ and $T_2(\boldsymbol{x})$ be two statistics such that $T_1 \leqslant T_2$ and

$$\Pr(T_1 \leqslant \theta \leqslant T_2) = 1 - \alpha$$

Then the random interaval $[T_1, T_2]$ is called a $100(1-\alpha)\%$ CI for $\theta, 1-a$ is called the *confidence coeffcient/level*, and T_1 and T_2 are called the lower and upper confidence limits/bounds, respecitively. A value $[t_1, t_2]$ of the random interval $[T_1, T_2]$ is called a $100(1-\alpha)$ percent CI for θ.

Similarly, we called T_1 as a one-sided lower confidence limit for θ, for which

$$\Pr(T_1 \leqslant \theta) = 1 - \alpha$$

Also, we can define one-sided upper confidence limit for θ.

To find a CI for a parameter, we need to introduce the concept of *pivot(枢轴变量)*. The use of pivots for the construction of CIs or confidence sets can be traced as far back as Fisher(1930), who used the term of *inverse probability*. Based on pivotal quantities, Barnard proposed a so-called *pivotal inference*, which is closely related with the theory of *structural inference* of Fraser. Berger and Wolpert discussed the strengths and weaknesses of these methods. We have following defination.

Definition 4.1(Pivotal quantity 枢轴变量). Assume that $X_1, ..., X_n \overset{\text{iid}}{\sim} f(x;\theta)$ and $T = T(X_1, ..., X_n)$ is a sufficient statstic of θ. Let $P = P(T, \theta)$ be a function(no need be a statistc) of T and θ. If the distribution of P does not depend on θ, then P is called a *pivotal quantity* or *pivot*.

比如说， 正态分布转换成标准正态分布时， 随机变量中还包含未知参数， 但其分布中却不包含任何未知参数。因此标准化后的随机变量是一个枢轴变量。

Pivotal quantities are fundamental to the construction of test stastics, as they allow the stastic to not depend on parameters — for example, Student's t-statistic is for a normal distribution with unknown variance (and mean). 统计量常用于点估计和假设检验， 而枢轴变量常用于区间估计。Here are a general method of constructing a pivotal quantity.

2 The confidence Interval of Normal Mean

When the variance is known

$$[<\boldsymbol{X}> -]$$

$$Z = \frac{(<X_1> - <X_2>) - (\mu_1 - \mu_2)}{\sigma_0} \sim N(0,1)$$

a $100(1-\alpha)\%$ CI for $\mu_1 - \mu_2$ is given by

$$[<X_1> -]$$

Two variances are unknown but equal

a $100(1-\alpha)\%$ CI for $\mu_1 - \mu_2$ is given by

$$<X_1> - <X_2> \pm$$

3 The Confidence Interval of the Difference of Two Normal Means

4 The Confidence Interval of Normal Variance

5 The Confidence Interval of the Ratio of Two Normal Variances

$$\frac{\sigma_2^2 S_1^2}{\sigma_1^2 S_2^2} \sim F(v_1, v_2)$$

$$f(1-\alpha/2, v_1, v_2) = \frac{1}{f(\alpha/2, v_2, v_1)} \qquad \text{(三反公式)}$$

6 Large-Sample Confidence Intervals

In the previous sections, we discussed the construction of *exact* CIs of parameters of interest in independent normal population. We will introduce three methods to construct approximate CIs of parameters in other populations for large sample sizes.

6.1 Method I: Based on the central limit theorem

Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d. from a population with mean μ

The second approximate CI for μ

6.2 Method II: Based on the asymptotic normality of $S(\theta; \boldsymbol{x})$

Let $\{X_n\}_{n=1}^{\infty} \overset{\text{iid}}{\sim} f(x;\theta)$ and $\hat{\theta}_n$

$$\frac{S(\theta; \boldsymbol{x})}{\sqrt{nI(\theta)}} \sim N(0,1)$$

6.3 Method III: Based on the asympotic normality of $\hat{\theta}_n$

$$\{nI(\theta)\}^{1/2}\left(\hat{\theta}_n - \theta\right) \sim N(0,1)$$

7 The shortest Confidence Interval

8 References

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