MA204: Mathematical Statistics

Suggested Solutions to Assignment 1

1.1 Solution. (a) The mgf of X is given by

$$\begin{aligned} M_X(t) &= E(\mathrm{e}^{tX}) = \sum_{x=0}^{n} \mathrm{e}^{tx}\binom{n}{x}p^x(1-p)^{n-x} \\ &= \sum_{x=0}^{n}\binom{n}{x}(p\mathrm{e}^t)^x(1-p)^{n-x} \\ &= (p\mathrm{e}^t+1-p)^n. \end{aligned}$$

(b) Now

$$M_X'(t) = \frac{\mathrm{d}M_X(t)}{\mathrm{d}t} = np\mathrm{e}^t(p\mathrm{e}^t+1-p)^{n-1}$$

and

$$M_X''(t) = n(n-1)(p\mathrm{e}^t)^2(p\mathrm{e}^t+1-p)^{n-2} + np\mathrm{e}^t(p\mathrm{e}^t+1-p)^{n-1};$$

hence $E(X) = M_X'(0) = np$ and

$$\begin{aligned} \mathrm{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= M_X''(0) - (np)^2 \\ &= n(n-1)p^2 + np - (np)^2 = np(1-p). \end{aligned}$$

(c) Let $Z = X + Y$. For any $z = 0, 1, 2, \ldots, +\infty$, we define $m = \min(n, z)$. Then, the pmf of Z is

$$\Pr(Z = z) = \Pr(X + Y = z)$$

$$
\begin{aligned}
&= \sum_{x=0}^{m} \Pr(X = x,\ Y = z - x) \\
&= \sum_{x=0}^{m} \Pr(X = x) \cdot \Pr(Y = z - x) \\
&= \sum_{x=0}^{m} \binom{n}{x} p^x (1-p)^{n-x} \cdot \frac{\lambda^{z-x}}{(z-x)!} \mathrm{e}^{-\lambda} \\
&= (1-p)^n \lambda^z \mathrm{e}^{-\lambda} \sum_{x=0}^{m} \binom{n}{x} \left[\frac{p}{\lambda(1-p)} \right]^x \frac{1}{(z-x)!}.
\end{aligned}
$$

1.2 Solution. (a) The marginal distribution of X is

$$\Pr(X = 1) = \sum_{y=1}^{4} \Pr(X = 1, Y = y) = \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{1}{4};$$

similarly, we have

$$\Pr(X = i) = \frac{1}{4}, \quad i = 2, 3, 4.$$

(b) The pmf of $Z = X + Y$ is

$$
\begin{aligned}
\Pr(Z = 2) &= \Pr(X = 1, Y = 1) = \frac{1}{16}, \\
\Pr(Z = 3) &= \Pr(X = 1, Y = 2) = \frac{1}{16}, \\
\Pr(Z = 4) &= \Pr(X = 1, Y = 3) + \Pr(X = 2, Y = 2) \\
&= \frac{1}{16} + \frac{2}{16} = \frac{3}{16}, \\
\Pr(Z = 5) &= \Pr(X = 1, Y = 4) + \Pr(X = 2, Y = 3) \\
&= \frac{1}{16} + \frac{1}{16} = \frac{2}{16}, \\
\Pr(Z = 6) &= \Pr(X = 2, Y = 4) + \Pr(X = 3, Y = 3) \\
&= \frac{1}{16} + \frac{3}{16} = \frac{4}{16},
\end{aligned}
$$

$$\begin{aligned}\Pr(Z=7) &= \Pr(X=3, Y=4) = \frac{1}{16},\\ \Pr(Z=8) &= \Pr(X=4, Y=4) = \frac{4}{16}.\end{aligned}$$

1.3 Solution. (a) Note that

$$f_{(Y|X)}(y|x) = \frac{x\mathrm{e}^{-xy}}{1-\mathrm{e}^{-bx}}, \quad 0 \leqslant y < b$$

by applying the formula

$$f_X(x) \propto \frac{f_{(X|Y)}(x|y_0)}{f_{(Y|X)}(y_0|x)}, \tag{SA1.1}$$

and setting $y_0 = b/2$, the marginal distribution of X is given by

$$f_X(x) \propto \frac{1-\exp(-bx)}{x} \mathrel{\hat{=}} h(x), \quad 0 \leqslant x < b < +\infty. \tag{SA1.2}$$

We first prove

$$h(x) \leqslant b \quad \text{for any } x \in [0, b). \tag{SA1.3}$$

For any continuous and twice differentiable function $g(x)$ with $g''(x) > 0$, the second order Taylor expansion of $g(x)$ around x_0 is

$$\begin{aligned}g(x) &= g(x_0) + (x-x_0)g'(x_0) + \frac{(x-x_0)^2}{2}g''(\xi)\\ &\geqslant g(x_0) + (x-x_0)g'(x_0),\end{aligned}$$

where ξ is a point between x and x_0. Now let $g(x) = \mathrm{e}^{-bx}$ and $x_0 = 0$. Since $g'(x) = -b\mathrm{e}^{-bx}$ and $g''(x) = b^2\mathrm{e}^{-bx} > 0$ for any $x \in [0, b)$, we have

$$\mathrm{e}^{-bx} \geqslant 1 - bx,$$

or

$$b \geqslant \frac{1-\mathrm{e}^{-bx}}{x} = h(x),$$

implying (SA1.3). From (SA1.3), we obtain

$$\int_0^b h(x)\,\mathrm{d}x \leqslant \int_0^b b\,\mathrm{d}x = b^2 < +\infty,$$

which implies $f_X(x)$ exists.

(b) If let $b = +\infty$, then from (SA1.2),

$$f_X(x) \propto 1/x, \quad 0 \leqslant x < +\infty.$$

Obviously, $f_X(x)$ is not a density.

1.4 Solution. Note that $\mathcal{S}_X = \{x_1, x_2, x_3\}$ and $\mathcal{S}_Y = \{y_1, \ldots, y_4\}$. By using point-wise IBF, the marginal distribution of X is given by

X	x_1	x_2	x_3
$p_i = \Pr\{X = x_i\}$	0.24	0.28	0.48

Similarly, the marginal distribution of Y is given by

Y	y_1	y_2	y_3	y_4
$q_j = \Pr\{Y = y_j\}$	0.28	0.16	0.28	0.28

The joint distribution of (X, Y) is given by

$$P = \begin{pmatrix} 0.04 & 0.04 & 0.12 & 0.04 \\ 0.08 & 0.08 & 0.04 & 0.08 \\ 0.16 & 0.04 & 0.12 & 0.16 \end{pmatrix}.$$

1.5 Proof. (a)

$$\begin{aligned} E(|X-b|) &= \int_{-\infty}^{\infty} |x-b| f(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{b} (b-x) f(x)\,\mathrm{d}x + \int_b^{\infty} (x-b) f(x)\,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{m} (b-m+m-x) f(x)\,\mathrm{d}x + \int_{m}^{b} (b-x) f(x)\,\mathrm{d}x \\
&\quad + \int_{m}^{\infty} (x-m+m-b) f(x)\,\mathrm{d}x + \int_{b}^{m} (x-b) f(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{m} (m-x) f(x)\,\mathrm{d}x + (b-m) \int_{-\infty}^{m} f(x)\,\mathrm{d}x \\
&\quad + \int_{m}^{\infty} (x-m) f(x)\,\mathrm{d}x + (m-b) \int_{m}^{\infty} f(x)\,\mathrm{d}x \\
&\quad + 2 \int_{m}^{b} (b-x) f(x)\,\mathrm{d}x \\
&= E(|X-m|) + 2 \int_{m}^{b} (b-x) f(x)\,\mathrm{d}x \\
&\quad + (b-m)\Big[\Pr(X \leqslant m) - \Pr(X \geqslant m)\Big] \\
&= E(|X-m|) + 2 \int_{m}^{b} (b-x) f(x)\,\mathrm{d}x.
\end{aligned}
$$

(b) Since

$$\int_{m}^{b} (b-x) f(x)\,\mathrm{d}x \geqslant 0$$

for all b, $E(|X-b|)$ is minimised if and only if $b = m$.

1.6 Solution. (a) It is easy to obtain

$$
\begin{aligned}
\Pr(1/4 < X < 5/8) &= \int_{1/4}^{5/8} \mathrm{d}F(x) \\
&= F(5/8) - F(1/4) \\
&= 1 - 2(1-5/8)^2 - 2(1/4)^2 \\
&= 19/32.
\end{aligned}
$$

(b) Note that

$$\Pr(X = 3/4) = F(3/4) - F(3/4-)$$

$$\begin{aligned} &= 1 - [1 - 2(1 - 3/4)^2] \\ &= 1/8, \end{aligned}$$

then

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x \, \mathrm{d}F(x) \\ &= \int_0^{1/2} 4x^2 \mathrm{d}x + \int_{1/2}^{3/4} 4x(1-x)\mathrm{d}x \\ &\quad + \frac{3}{4} \Pr(X = 3/4) \\ &= \frac{3}{4}x^3\Big|_0^{1/2} + (2x^2 - \frac{3}{4}x^3)\Big|_{1/2}^{3/4} + \frac{3}{4} \times \frac{1}{8} \\ &= \frac{47}{96}, \end{aligned}$$

and

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} x^2 \, \mathrm{d}F(x) \\ &= \int_0^{1/2} 4x^3 \mathrm{d}x + \int_{1/2}^{3/4} 4x^2(1-x)\mathrm{d}x \\ &\quad + \left(\frac{3}{4}\right)^2 \Pr\left(X = \frac{3}{4}\right) \\ &= \frac{211}{768}. \end{aligned}$$

Therefore,

$$\mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{323}{9216}.$$

1.7 Solution. (a) Let $M_{\mathbf{x}}(t_1, ..., t_d) = E[\exp(t_1 X_1 + \cdots + t_d X_d)]$, then

$$\frac{\partial M_{\mathbf{x}}(t_1, ..., t_d)}{\partial t_i} = E[X_i \exp(t_1 X_1 + \cdots + t_d X_d)]$$

and

$$\frac{\partial M_{\mathbf{x}}(t_1, ..., t_d)}{\partial t_i}\Big|_{t_1 = \cdots = t_d = 0} = E(X_i).$$

(b) Note that

$$\frac{\partial^2 M_{\mathbf{x}}(t_1, ..., t_d)}{\partial t_i \partial t_j} = E[X_i X_j \exp(t_1 X_1 + \cdots + t_d X_d)],$$

we obtain

$$\left.\frac{\partial^2 M_{\mathbf{x}}(t_1, ..., t_d)}{\partial t_i \partial t_j}\right|_{t_1 = \cdots = t_d = 0} = E(X_i X_j).$$

(c) If the joint density of X and Y is

$$f(x,y) = \begin{cases} \mathrm{e}^{-x-y}, & \text{for } x > 0,\ y > 0, \\ 0, & \text{elsewhere,} \end{cases}$$

then, the joint mgf is

$$\begin{aligned} M_{(X,Y)}(t_1, t_2) &= E[\exp(t_1 X + t_2 Y)] \\ &= \int_0^\infty \int_0^\infty \mathrm{e}^{t_1 X + t_2 Y} \mathrm{e}^{-x} \mathrm{e}^{-y} \mathrm{d}x \mathrm{d}y \\ &= \frac{1}{(1-t_1)(1-t_2)}, \qquad t_1 < 1,\ t_2 < 1. \end{aligned}$$

Now

$$\frac{\partial M_{(X,Y)}(t_1, t_2)}{\partial t_1} = \frac{1}{(1-t_1)^2(1-t_2)},$$

then

$$E(X) = \left.\frac{\partial M_{(X,Y)}(t_1, t_2)}{\partial t_1}\right|_{t_1 = t_2 = 0} = 1.$$

Similarly, we have $E(Y) = 1$. Furthermore, since

$$\frac{\partial^2 M_{(X,Y)}(t_1, t_2)}{\partial t_1 \partial t_2} = \frac{1}{(1-t_1)^2(1-t_2)^2},$$

we have

$$E(XY) = \left.\frac{\partial^2 M_{(X,Y)}(t_1, t_2)}{\partial t_1 \partial t_2}\right|_{t_1 = t_2 = 0} = 1.$$

Note that X and Y are independent, we obtain

$$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0.$$

1.8 Solution. A(a) Since

$$c^{-1} = \sum_{x=1}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} = 1 - e^{-\lambda},$$

we have

$$c = \frac{1}{1 - e^{-\lambda}}.$$

A(b)

$$\begin{cases} E(X) &= c\lambda, \\ E(X^2) &= c(\lambda^2 + \lambda), \\ \mathrm{Var}(X) &= c\lambda[1 + (1-c)\lambda]. \end{cases}$$

A(c) The mgf of X is

$$M_X(t) = E(e^{tX}) = ce^{-\lambda}[\exp(\lambda e^t) - 1].$$

B(d) Let $c_i = 1/(1 - e^{-\lambda_i})$ for $i = 1, 2$. The pmf of $X_1 + X_2$ is

$$\begin{aligned} & \Pr(X_1 + X_2 = x) \\ = & \sum_{i=1}^{x-1} \Pr(X_1 = i, X_2 = x - i) \\ = & \sum_{i=1}^{x-1} \Pr(X_1 = i) \Pr(X_2 = x - i) \\ = & c_1 c_2 \sum_{i=1}^{x-1} \frac{\lambda_1^i e^{-\lambda_1}}{i!} \cdot \frac{\lambda_2^{x-i} e^{-\lambda_2}}{(x-i)!} \\ = & c_1 c_2 \cdot \frac{e^{-(\lambda_1+\lambda_2)}}{x!} \sum_{i=1}^{x-1} \binom{x}{i} \lambda_1^i \lambda_2^{x-i} \\ = & c_1 c_2 \cdot \frac{e^{-(\lambda_1+\lambda_2)}}{x!} [(\lambda_1 + \lambda_2)^x - \lambda_2^x - \lambda_1^x], \quad x = 2, 3, \ldots. \end{aligned}$$

B(e) The conditional distribution of $X_1|(X_1+X_2=x)$ is

$$
\begin{aligned}
& \Pr(X_1 = x_1 | X_1 + X_2 = x) \\
= & \frac{\Pr(X_1 = x_1, X_2 = x - x_1)}{\Pr(X_1 + X_2 = x)} \\
= & \frac{\dfrac{c_1 \lambda_1^{x_1} \mathrm{e}^{-\lambda_1}}{x_1!} \cdot \dfrac{c_2 \lambda_2^{x-x_1} \mathrm{e}^{-\lambda_2}}{(x-x_1)!}}{c_1 c_2 \cdot \dfrac{\mathrm{e}^{-(\lambda_1+\lambda_2)}}{x!} [(\lambda_1+\lambda_2)^x - \lambda_2^x - \lambda_1^x]} \\
= & \frac{\binom{x}{x_1} \lambda_1^{x_1} \lambda_2^{x-x_1}}{(\lambda_1+\lambda_2)^x - \lambda_2^x - \lambda_1^x}, \quad x_1 = 1, 2, \ldots, x-1.
\end{aligned}
$$

1.9 Solution. (a)

$$
\begin{aligned}
\mathrm{Var}(X) &= \lambda_0 + \lambda, \\
E(Y) &= E(Z) \cdot [E(U) + E(W)] = (1-\phi)(\lambda_0 + \beta\lambda), \\
\mathrm{Var}(Y) &= E[Z^2(U^2 + W^2 + 2UW)] - [E(Y)]^2 \\
&= (1-\phi)[E(U^2) + E(W^2) + 2E(U)E(W)] - [E(Y)]^2 \\
&= (1-\phi)[\lambda_0 + \lambda_0^2 + \beta\lambda + \beta^2\lambda^2 + 2\lambda_0\beta\lambda] \\
&\quad - (1-\phi)^2(\lambda_0 + \beta\lambda)^2 \\
&= (1-\phi)(\lambda_0 + \beta\lambda)[1 + \phi(\lambda_0 + \beta\lambda)], \\
\mathrm{Cov}(X, Y) &= E(XY) - E(X)E(Y) \\
&= E(Z) \cdot E[U^2 + U(W+V) + VW] - E(X)E(Y) \\
&= (1-\phi)[\lambda_0 + \lambda_0^2 + \lambda_0(\beta\lambda + \lambda) + \beta\lambda^2] \\
&\quad - (\lambda_0 + \lambda)(1-\phi)(\lambda_0 + \beta\lambda) \\
&= (1-\phi)\lambda_0.
\end{aligned}
$$

Alternatively

$$\begin{aligned}\mathrm{Cov}(X,Y) &= \mathrm{Cov}(U+V, ZU+ZW) = \mathrm{Cov}(U, ZU)\\ &= E(ZU^2) - E(U)E(ZU) = (1-\phi)\lambda_0.\end{aligned}$$

(b) When $y = 0$, the joint distribution of X and Y is

$$\begin{aligned}&\Pr(X = x, Y = y = 0)\\ =\ &\Pr\{U+V = x, Z(U+W) = 0\}\\ =\ &\Pr(U+V = x, Z = 0) + \Pr(U+V = x, Z = 1, U+W = 0)\\ =\ &\Pr(Z = 0)\Pr(U+V = x)\\ &+\ \Pr(Z = 1)\Pr(U+V = x, U+W = 0)\\ =\ &\phi\Pr(U+V = x) + (1-\phi)\Pr(U = 0, V = x, W = 0)\\ =\ &\phi\frac{(\lambda_0+\lambda)^x \mathrm{e}^{-\lambda_0-\lambda}}{x!} + (1-\phi)\frac{\lambda^x \mathrm{e}^{-\lambda_0-\lambda-\beta\lambda}}{x!}.\end{aligned}$$

When $y > 0$, the joint distribution of X and Y is

$$\begin{aligned}&\Pr(X = x, Y = y)\\ =\ &\Pr\{U+V = x, Z(U+W) = y\}\\ =\ &\Pr(U+V = x, Z = 1, U+W = y)\\ =\ &\Pr(Z = 1)\cdot\Pr(U+V = x, U+W = y)\\ =\ &(1-\phi)\sum_{k=0}^{\min(x,y)}\Pr(U = k, V = x-k, W = y-k)\\ =\ &(1-\phi)\sum_{k=0}^{\min(x,y)}\frac{\lambda_0^k \mathrm{e}^{-\lambda_0}}{k!}\cdot\frac{\lambda^{x-k}\mathrm{e}^{-\lambda}}{(x-k)!}\cdot\frac{(\beta\lambda)^k \mathrm{e}^{-\beta\lambda}}{(y-k)!}\end{aligned}$$

$$= \quad (1-\phi)\mathrm{e}^{-\lambda_0-\lambda-\beta\lambda}\frac{\lambda^x(\beta\lambda)^y}{x!y!}\sum_{k=0}^{\min(x,y)}\binom{x}{k}\binom{y}{k}k!\left(\frac{\lambda_0}{\beta\lambda^2}\right)^k.$$

1.10 Solution. Note that the mgf of $V \sim N(\mu, \sigma^2)$ is

$$M_V(t) = \exp(\mu t + 0.5\sigma^2 t^2),$$

we have $X \sim N(0,1)$ and $Y \sim N(-1,4)$. Hence,

$$W = 3X + 2Y \sim N(-2, 25)$$

since $X \perp\!\!\!\perp Y$.

(a) Let $Z = [W - (-2)]/5$, then $Z \sim N(0,1)$. Thus,

$\Pr(-12 < W < 3) = \Pr(-2 < Z < 1) = \Phi(1) - \Phi(-2) = 0.8185.$

(b) $E(W^2) = \mathrm{Var}(W) + [E(W)]^2 = 25 + 4 = 29.$