Southern University of Science and Technology
Department of Mathematics

MA204: Mathematical Statistics

Tutorial 2: Examples/Solutions

A. Find $E(X^r)$ from the Moment Generating Function of X

The mgf of the random variable X is defined as $M_X(t) = E(\mathrm{e}^{tX})$, and we have

$$\left.\frac{\mathrm{d}^r M_X(t)}{\mathrm{d}\, t^r}\right|_{t=0} = E(X^r).$$

Example T2.1: Let $X \sim \text{Laplace}(\mu, \sigma^2)$, $-\infty < \mu < \infty$, $\sigma^2 > 0$. The density of X is

$$f(x) = \frac{1}{2\sigma} \exp\left\{-\frac{|x-\mu|}{\sigma}\right\}, \quad -\infty < x < \infty.$$

(a) Find the moment generating function of X.

(b) Find $E(X)$ and $\text{Var}(X)$ from the moment generating function.

Solution: (a) Since $M_X(t) = E(\mathrm{e}^{tX})$, we have

$$\begin{aligned}
& M_X(t) \\
= \; & \int_{-\infty}^{\infty} \mathrm{e}^{tx} \cdot \frac{1}{2\sigma} \exp\left(-\frac{|x-\mu|}{\sigma}\right) \,\mathrm{d}x \\
= \; & \frac{1}{2\sigma} \int_{-\infty}^{\infty} \exp\left(tx - \frac{|x-\mu|}{\sigma}\right) \,\mathrm{d}x \\
= \; & \frac{1}{2\sigma} \left[\int_{-\infty}^{\mu} \exp\left(tx + \frac{x-\mu}{\sigma}\right) \,\mathrm{d}x + \int_{\mu}^{\infty} \exp\left(tx - \frac{x-\mu}{\sigma}\right) \,\mathrm{d}x\right] \\
= \; & \frac{1}{2\sigma} \left\{\frac{\sigma}{1+\sigma t}\left[\exp\left(\frac{(1+\sigma t)x - \mu}{\sigma}\right)\right]\Bigg|_{-\infty}^{\mu} - \frac{\sigma}{1-\sigma t}\left[\exp\left(-\frac{(1-\sigma t)x - \mu}{\sigma}\right)\right]\Bigg|_{\mu}^{\infty}\right\} \\
= \; & \frac{1}{2\sigma} \left\{\frac{\sigma}{1+\sigma t} \exp\left[\frac{(1+\sigma t)\mu - \mu}{\sigma}\right] + \frac{\sigma}{1-\sigma t} \exp\left[-\frac{(1-\sigma t)\mu - \mu}{\sigma}\right]\right\} \\
= \; & \frac{\sigma}{2\sigma(1-\sigma^2 t^2)}[(1-\sigma t)\mathrm{e}^{\mu t} + (1+\sigma t)\mathrm{e}^{\mu t}] = \frac{\mathrm{e}^{\mu t}}{1-\sigma^2 t^2}.
\end{aligned}$$

(b) $$E(X) = \left.\frac{\mathrm{d}M_X(t)}{\mathrm{d}t}\right|_{t=0} = \left.\frac{(\mu + 2\sigma^2 t - \mu\sigma^2 t^2)\mathrm{e}^{\mu t}}{(1-\sigma^2 t^2)^2}\right|_{t=0} = \mu,$$

$$E(X^2) = \left.\frac{\mathrm{d}^2 M_X(t)}{\mathrm{d}t^2}\right|_{t=0} = \left.\frac{\mathrm{e}^{\mu t}[(1-\sigma^2t^2)(\mu^2 + 2\sigma^2 - \mu^2\sigma^2t^2) + 4\sigma^2 t(\mu + 2\sigma^2 t - \mu\sigma^2 t^2)]}{(1-\sigma^2t^2)^3}\right|_{t=0} = \mu^2 + 2\sigma^2,$$

$$\begin{aligned}\mathrm{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= \mu^2 + 2\sigma^2 - \mu^2 \\ &= 2\sigma^2.\end{aligned}$$

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B. Inverse Bayes Formulae under the Product Space $\mathcal{S}_{(X,Y)} = \mathcal{S}_X \times \mathcal{S}_Y$

B.1 Continuous Random Variable, for Any $x \in \mathcal{S}_X$

$$\begin{aligned} f_X(x) &= \left\{ \int_{\mathcal{S}_Y} \frac{f_{Y|X}(y \mid x)}{f_{X|Y}(x \mid y)}\,\mathrm{d}y \right\}^{-1} \\ &= \left\{ \int_{\mathcal{S}_X} \frac{f_{X|Y}(x \mid y_0)}{f_{Y|X}(y_0 \mid x)}\,\mathrm{d}x \right\}^{-1} \frac{f_{X|Y}(x \mid y_0)}{f_{Y|X}(y_0 \mid x)} \\ &\propto \frac{f_{X|Y}(x \mid y_0)}{f_{Y|X}(y_0 \mid x)}, \qquad \text{for an arbitrarily fixed } y_0 \in \mathcal{S}_Y. \end{aligned}$$

B.2 Discrete Random Variable, for Any $x_i \in \mathcal{S}_X$

$$\begin{aligned} \Pr(X = x_i) &= \left\{ \sum_j \frac{\Pr(Y = y_j \mid X = x_i)}{\Pr(X = x_i \mid Y = y_j)} \right\}^{-1} \\ &= \left\{ \sum_k \frac{\Pr(X = x_k \mid Y = y_{j0})}{\Pr(Y = y_{j0} \mid X = x_k)} \right\}^{-1} \frac{\Pr(X = x_i \mid Y = y_{j0})}{\Pr(Y = y_{j0} \mid X = x_i)} \\ &\propto \frac{\Pr(X = x_i \mid Y = y_{j0})}{\Pr(Y = y_{j0} \mid X = x_i)}, \qquad \text{for an arbitrarily fixed } y_{j0} \in \mathcal{S}_Y. \end{aligned}$$

Example T2.2: Let X be a discrete random variable with pmf $p_i = \Pr(X = x_i)$ for $i = 1, 2$ and Y be a discrete random variable with pmf $q_j = \Pr(Y = y_j)$ for $j = 1, 2$. Given two conditional distribution matrices

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1/4 & 1/2 \\ 3/4 & 1/2 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 1/3 & 2/3 \\ 3/5 & 2/5 \end{pmatrix},$$

where the (i, j) element of $\mathbf{A}$ is $a_{ij} = \Pr(X = x_i | Y = y_j)$ and the (i, j) element of $\mathbf{B}$ is $b_{ij} = \Pr(Y = y_j | X = x_i)$.

(a) Find the marginal distribution of X.

(b) Find the marginal distribution of Y.

(c) Find the joint distribution of (X, Y).

Solution: Note that $\mathcal{S}_X = \{x_1, x_2\}$ and $\mathcal{S}_Y = \{y_1, y_2\}$. Please mimic the solution to Example 1.2 in Class Exercise 1. By using point-wise IBF, the marginal distribution of X is given by

X	x_1	x_2
$p_i = \Pr\{X = x_i\}$	$3/8$	$5/8$

Similarly, the marginal distribution of Y is given by

Y	y_1	y_2
$q_j = \Pr\{Y = y_j\}$	$1/2$	$1/2$

The joint distribution of (X, Y) is given by

$$\mathbf{P} = \begin{pmatrix} 1/8 & 1/4 \\ 3/8 & 1/4 \end{pmatrix}. \qquad \|$$

C. Distribution of the Function of Random Variables

Let a set of r.v.'s $X_1, \ldots, X_n$ have the joint cdf $F(x_1, \ldots, x_n)$ or the joint pdf $f(x_1, \ldots, x_n)$. We seek the distribution of $Y = h(X_1, \ldots, X_n)$ for some function $h(\cdot)$.

C.1 Cumulative Distribution Function Technique

Step 1: Find the cdf of Y: $F(y) = \Pr\{h(X_1, \ldots, X_n) \leq y\}$;

Step 2: Find the pdf of Y: $f(y) = F'(y)$.

Example T2.3: Let X be a standard normal random variable. Using the cdf technique, find the pdf of $Y = X^2$.

Solution: Let

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty$$

denote the density of $N(0, 1)$. The cdf of Y is

$$\begin{aligned} F(y) &= \Pr(X^2 \leq y) = \Pr(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= \int_{-\sqrt{y}}^{\sqrt{y}} \phi(x) \, dx = 2 \int_0^{\sqrt{y}} \phi(x) \, dx. \end{aligned}$$

Therefore, the density of Y is

$$\begin{aligned} f(y) &= \frac{dF(y)}{dy} = \frac{dF(y)}{dz} \cdot \frac{dz}{dy} \quad \text{(where let } z = \sqrt{y}\text{)} \\ &= \begin{cases} \dfrac{2}{\sqrt{2\pi}} e^{-y/2} \cdot \dfrac{y^{-1/2}}{2} = \dfrac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2}, & 0 < y < \infty, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

C.2 **Monotone Transformation Technique**

(a) Univariate case:

$$g(y) = f(h^{-1}(y))c \times \left| \frac{dh^{-1}(y)}{dy} \right|.$$

(b) Bivariate case:

$$g(y_1, y_2) = f(h_1^{-1}(y_1, y_2), h_2^{-1}(y_1, y_2)) |J(x_1, x_2 \to y_1, y_2)|.$$

Example T2.4: Let X_1 and X_2 be two independently exponentially distributed r.v.'s with a rate parameter λ.

(a) Find the joint pdf of $Y_1 = X_1/X_2$ and $Y_2 = X_1 + X_2$.

(b) Find the marginal pdf's of Y_1 and Y_2.

NOTE: Let $Z_i \sim \text{Gamma}(\alpha_i, \beta)$ and $Z_1 \perp\!\!\!\perp Z_2$, then $Y = Z_1/Z_2$ is said to follow an inverted beta distribution with parameters α_1 and α_2. Its density is

$$f(y) = \frac{1}{B(\alpha_1, \alpha_2)} \cdot \frac{y^{\alpha_1 - 1}}{(1+y)^{\alpha_1+\alpha_2}}, \quad y > 0.$$

Solution: (a) The transformation is

$$y_1 = \frac{x_1}{x_2} \quad \text{and} \quad y_2 = x_1 + x_2.$$

Hence, $y_1 > 0$ and $y_2 > 0$. The corresponding inverse transformation is

$$x_1 = \frac{y_1 y_2}{1+y_1} \quad \text{and} \quad x_2 = \frac{y_2}{1+y_1}.$$

Hence, the Jacobian determinant is

$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \dfrac{y_2}{(1+y_1)^2} & \dfrac{y_1}{1+y_1} \\ \dfrac{-y_2}{(1+y_1)^2} & \dfrac{1}{1+y_1} \end{vmatrix} = \frac{y_2}{(1+y_1)^2}.$$

The joint pdf of (Y_1, Y_2) is

$$\begin{aligned} f(y_1, y_2) &= f(x_1, x_2) \times |J| \\ &= f(x_1) f(x_2) |J| \\ &= \lambda \exp\left\{-\lambda\left(\frac{y_1 y_2}{1+y_1}\right)\right\} \cdot \lambda \exp\left\{-\lambda\left(\frac{y_2}{1+y_1}\right)\right\} |J| \\ &= \frac{\lambda^2 y_2}{(1+y_1)^2} \mathrm{e}^{-\lambda y_2}, \qquad y_1 > 0,\ y_2 > 0. \end{aligned}$$

(b) The marginal density of Y_1 is

$$\begin{aligned} f(y_1) &= \int_0^\infty \frac{\lambda^2 y_2}{(1+y_1)^2} \mathrm{e}^{-\lambda y_2}\ \mathrm{d}y_2 \\ &= \frac{\lambda}{(1+y_1)^2} \int_0^\infty y_2 \lambda \mathrm{e}^{-\lambda y_2}\ \mathrm{d}y_2 \\ &= \frac{\lambda}{(1+y_1)^2} E(Y_2) = \frac{1}{(1+y_1)^2}, \quad y_1 > 0. \end{aligned}$$

Hence, Y_1 follows the inverted beta distribution with parameters 1 and 1. On the other hand,

$$\begin{aligned} f(y_2) &= \int_0^\infty \frac{\lambda^2 y_2}{(1+y_1)^2} \mathrm{e}^{-\lambda y_2}\, \mathrm{d}y_1 \\ &= \lambda^2 y_2 \mathrm{e}^{-\lambda y_2} \int_0^\infty \frac{\mathrm{d}y_1}{(1+y_1)^2} \\ &= \left. \frac{-1}{1+y_1} \right|_0^\infty = \lambda^2 y_2 \mathrm{e}^{-\lambda y_2}, \quad y_2 > 0, \end{aligned}$$

i.e., $Y_2 \sim \text{Gamma}(2, \lambda)$. ‖

Example T2.5: Let the joint density of (X, Y) be

$$f(x,y) = K \cdot (x+y) I_{(0,1)}(x) I_{(0,1)}(y) I_{(0,1)}(x+y),$$

where K is a positive constant and $I_{(0,1)}(\cdot)$ is the indicator function.

(a) Find the marginal density of X.

(b) Find the joint pdf of $X+Y$ and $Y-X$.

(c) Find the marginal pdf's of $X+Y$ and $Y-X$.

Solution: (a) The marginal density of X is given by

$$\begin{aligned} f(x) &= \int_{-\infty}^\infty K \cdot (x+y) I_{(0,1)}(x) I_{(0,1)}(y) I_{(0,1)}(x+y)\ \mathrm{d}y \\ &= K \cdot I_{(0,1)}(x) \int_0^{1-x} (x+y)\ \mathrm{d}y \\ &= K \cdot I_{(0,1)}(x) \left[xy + \frac{y^2}{2} \right] \Bigg|_0^{1-x} \\ &= \frac{K(1-x^2)}{2} I_{(0,1)}(x). \end{aligned}$$

(b) The transformation is

$$u = x + y \quad \text{and} \quad v = y - x.$$

The corresponding inverse transformation is

$$x = \frac{1}{2}(u - v) \quad \text{and} \quad y = \frac{1}{2}(u + v).$$

Hence, the Jacobian determinant is

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2} & -\dfrac{1}{2} \\ \dfrac{1}{2} & \dfrac{1}{2} \end{vmatrix} = \frac{1}{2}.$$

The joint pdf of (U, V) is

$$\begin{aligned} f(u, v) &= \frac{K}{2}(x + y) I_{(0,1)}(x) I_{(0,1)}(y) I_{(0,1)}(x + y) \\ &= \frac{K}{2} u I_{(0,1)} \left[\frac{1}{2}(u - v)\right] I_{(0,1)} \left[\frac{1}{2}(u + v)\right] I_{(0,1)}(u). \end{aligned}$$

(c) The marginal pdf of U is

$$f(u) = \int_{-u}^{u} \frac{Ku}{2} \, \mathrm{d}v = Ku^2, \quad 0 < u < 1,$$

i.e., $U \sim$ Beta(3,1), so that

$$K = \frac{1}{B(3, 1)} = \frac{\Gamma(4)}{\Gamma(3)\Gamma(1)} = \frac{3!}{2! \cdot 1} = 3.$$

The marginal density of V is

$$\begin{aligned} f(v) &= \begin{cases} \displaystyle\int_{v}^{1} \frac{Ku}{2} \, \mathrm{d}u = \frac{K}{4}(1 - v^2), & 0 \le v < 1, \\ \displaystyle\int_{-v}^{1} \frac{Ku}{2} \, \mathrm{d}u = \frac{K}{4}(1 - v^2), & -1 < v \le 0 \end{cases} \\ &= \frac{K}{4}(1 - v^2), \quad -1 < v < 1. \end{aligned}$$

C.3 Moment Generating Function Technique

Let $Y = \sum_{i=1}^{n} X_i$. If $\{X_i\}_{i=1}^{n}$ are independent r.v.'s, then

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t).$$

We can find the moment generating function of $Y = h(X_1, \ldots, X_n)$ and match it with those of some standard distributions.

Example T2.6: If $X_1, \ldots, X_n$ are independent Gamma r.v.'s with shape parameters α_i, $i = 1, \ldots, n$ and a common rate parameter β. By using the moment generating function technique, find the distribution of $Y = \sum_{i=1}^n X_i$.

Solution: (a) Since

$$f(x_i) = \frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)} x_i^{\alpha_i - 1} \mathrm{e}^{-\beta x_i}, \quad 0 < x_i < \infty, \quad i = 1, \ldots, n,$$

we obtain

$$\begin{aligned} M_{X_i}(t) &= E(\mathrm{e}^{tX_i}) \\ &= \int_0^\infty \mathrm{e}^{tx_i} \frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)} x_i^{\alpha_i - 1} \mathrm{e}^{-\beta x_i} \, \mathrm{d}x_i \\ &= \int_0^\infty \frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)} x_i^{\alpha_i - 1} \mathrm{e}^{-(\beta - t) x_i} \, \mathrm{d}x_i \\ &= = \left(\frac{\beta}{\beta - t}\right)^{\alpha_i} \int_0^\infty \frac{(\beta - t)^{\alpha_i}}{\Gamma(\alpha_i)} x_i^{\alpha_i - 1} \mathrm{e}^{-(\beta - t) x_i} \, \mathrm{d}x_i \\ &= \left(\frac{\beta}{\beta - t}\right)^{\alpha_i}. \end{aligned}$$

(b) In fact,

$$M_Y(t) = \prod_{i=1}^n M_{X_i}(t) = \prod_{i=1}^n \left(\frac{\beta}{\beta - t}\right)^{\alpha_i} = \left(\frac{\beta}{\beta - t}\right)^{\sum_{i=1}^n \alpha_i}.$$

So $Y \sim \text{Gamma}(\sum_{i=1}^n \alpha_i, \beta)$. $\|$