Mathematical Statistics Assignment 3

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Q3.1 Let $X_1, ..., X_n \overset{\text{iid}}{\sim} U[\theta_1, \theta_2]$. Find the MLEs of θ_1 and θ_2.

Sovle: The joint desity of $\boldsymbol{x} = (X_1, ..., X_n)^T$ is

$$f(\boldsymbol{x}; \theta_1, \theta_2) = \begin{cases} \dfrac{1}{(\theta_2 - \theta_1)^n} & \text{, if } \theta_1 \leqslant x_i \leqslant \theta_2, i = 1, ..., n, \\ 0 & \text{, elsewhere.} \end{cases}$$

Then the likehood function is

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} f(x_i; \theta_1, \theta_2) = \begin{cases} \displaystyle\prod_{i=1}^{n} \frac{1}{(\theta_2 - \theta_1)} = \frac{1}{(\theta_2 - \theta_1)^n} & , \theta_1 \leqslant x_{(1)} \text{ and } \theta_2 \geqslant x_{(n)} \\ 0 & \text{, elsewhere.} \end{cases}$$

Because $\theta_2 - \theta_1 \geqslant x_{(n)} - x_{(1)}$, and $L(\theta_1, \theta_2)$ is a monotone and decreasing function of $\theta_2 - \theta_1$ when $\theta_2 - \theta_1 \in [x_{(n)} - x_{(1)}, \infty)$. Thus $\hat{\boldsymbol{\theta}} = \widehat{(\theta_2 - \theta_1)} = x_{(n)} - x_{(1)}$ is the MLE of θ_1 and θ_2.

Q3.2 A sample of size n_1 is drawn from $N(\mu_1, \sigma_1^2)$. A second sample of size n_2 is drawn from $N(\mu_2, \sigma_2^2)$. Assume that the two samples are independent.

a) What is the MLE of $\theta = \mu_1 - \mu_2$?

   Solve:

b) If we assume that the total sample size $n = n_1 + n_2$ is fixed, how should the n observations be approximately divided between the two populations in order to minimize the variance of the $\hat{\theta}$?

   Solve:

Q3.10 Let $X_1, ..., X_n \overset{\text{iid}}{\sim} N(\mu_0, \theta)$, where μ_0 is known and $\theta > 0$.

a) Find the MLE $\hat{\theta}$ of θ?

   Solve:

   The joint pmf of $\boldsymbol{x} = (X_1, ..., X_n)^T$ is

$$f(\boldsymbol{x}; \theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta}} \text{Exp}\left(-\frac{(x_i - \mu_0)^2}{2\theta}\right)$$

   Then the likehood function is

$$L(\theta) = \prod_{i=1}^{n} f(x_i, \theta) = \left(\frac{1}{\sqrt{2\pi\theta}}\right)^n e^{-n/2\theta} \text{Exp}\left(\sum_{i=1}^{n} (x_i - \mu_0)^2\right)$$

The log likehood function is

$$l(\theta) = -\frac{n}{2}\log(2\pi\theta) - \frac{n}{2\theta} + \sum_{i=1}^{n} (x_i - \mu_0)^2 = -\frac{n}{2}\log(\theta) - \frac{n}{2\theta} - \frac{n}{2}\log(2\pi) + \sum_{i=1}^{n} (x_i - \mu_0)^2$$

b) What is the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$?

Q3.15 Let $X_1, ..., X_n \overset{\text{iid}}{\sim} \text{Bernoulli}(\theta)$. Define $\tau(\theta) = \text{Var}(X) = \theta(1-\theta)$.

a) Find the Cramer-Rao lower bound fro the unbiased estimator of $\tau(\theta)$.

b) Find the unique UMVUE of $\tau(\theta)$ if such exists.

Q3.17. Supposet that X follows a geometeric distribution,

$$\Pr(X = k) = p(1-p)^{k-1}, k = 1, 2. ...$$

where $0 \leqslant p \leqslant 1$, and assume an i.i.d. sample fo size n.

a) Find the moment of estimator of p

b) Find the MLE of p.

c) Let p have a uniform prior distribution on $[0, 1]$. What is the posterior distribution of p? What is the posterior mean?

Q3.18. Let $(X_1, ..., X_n)$ be a random sample from the Pareto distribution with Lebesgus density $\theta a^{\theta} x^{-(\theta+1) I_{(a,\infty)}(x)}$, where $\theta > 0$ and $a > 0$.

i. Find the UMVUE of θ when a is known.

ii. Find the UMVUE of a when θ is known.

iii. Find the UMVUE's of a and θ.

Solution:

i. The joint Lebesgue density of $X_1, ..., X_n$ is

$$f(x_1, ..., x_n) = \theta^n a^{n\theta} \exp\left\{-(\theta+1)\sum_{i=1}^{n} \log x_i\right\} I_{(a,\infty)}(x_{(1)}),$$

where $x_{(1)} = \min_{1 \leqslant i \leqslant n} x_i$. When a is known, $T = \sum_{i=1}^{n} \log X_i$ is complete and sufficient for θ and $T - n\log a$ has a has he gamma distribution with shape parameter n and scale parameter θ^{-1}. Hence, $ET^{-1} = \theta/(n-1)$ and, thus, $(n-1)/T$ is theUMVUE of θ.

ii. When θ is known, $X_{(1)}$ is complete and sufficient for a. Since $X_{(1)}$ has the Lebesgue density $n\theta a^{n\theta} x^{-(n\theta+1)} I_{(a,\infty)}(x)$, $EX_{(1)} = n\theta a / (n\theta - 1)$. Therefore, $(1 - n\theta)X_{(1)} / (n\theta)$ is the UMVUE of a.

iii. When both a and θ are unknoen, $(Y, X_{(1)})$ is complete and sufficient for (a, θ), where $Y = \sum_i (\log X_i - \log X_{(1)})$. Also, Y has the gamma distribution with shape parameter $n-1$ and scale parameter θ^{-1} and $X_{(1)}$ and Y are independent. Since $EY^{-1} = \theta/(n-2)$, $(n-2)/Y$ is the UMVUE of θ. Since

$$
\begin{aligned}
E\left\{\left[1 - \frac{Y}{n(n-1)}\right] X_{(1)}\right\} &= \left[1 - \frac{EY}{n(n-1)}\right] EX_{(1)} \\
&= \left(1 - \frac{1}{n\theta}\right) \frac{n\theta a}{n\theta - 1} \\
&= a,
\end{aligned}
$$

$\left[1 - \frac{Y}{n(n-1)}\right] X_{(1)}$ is the UMVUE of a.