Southern University of Science and Technology
Department of Mathematics

MA204: Mathematical Statistics

Tutorial 4: Examples/Solutions

A. Maximum Likelihood Estimator (MLE)

Step 1: Calculate the log-likelihood function

$$\ell(\theta) = \sum_{i=1}^{n} \log f(x_i; \theta).$$

Step 2: The MLE of θ is obtained through

$$\hat{\theta} = \arg \max_{\theta \in \Theta} \ell(\theta).$$

Example T4.1 (Unrestricted MLEs of two parameters). Let $X_1, \ldots, X_n$ be a random sample from the distribution function

$$F(x; \theta_1, \theta_2) = \begin{cases} 1 - (\theta_1/x)^{\theta_2}, & \text{if } x \geqslant \theta_1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta_1 > 0$ and $\theta_2 > 0$. Find the MLEs of θ_1 and θ_2.

Solution: The density function is given by

$$f(x; \theta_1, \theta_2) = \frac{\mathrm{d}}{\mathrm{d}x} F(x; \theta_1, \theta_2) = \begin{cases} \theta_1^{\theta_2} \theta_2 x^{-\theta_2 - 1}, & \text{if } x \geqslant \theta_1, \\ 0, & \text{otherwise.} \end{cases}$$

The joint density of $X_1, \ldots, X_n$ is

$$f(x_1, \ldots, x_n; \theta_1, \theta_2) = \begin{cases} \theta_1^{n\theta_2} \theta_2^n (x_1 \cdots x_n)^{-\theta_2 - 1}, & \text{if } x_i \geqslant \theta_1, \forall i = 1, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$

So that the likelihood function is given by

$$L(\theta_1, \theta_2) = \begin{cases} \theta_1^{n\theta_2} \theta_2^n (x_1 \cdots x_n)^{-\theta_2 - 1}, & \text{if } 0 < \theta_1 \leqslant x_{(1)} = \min\{x_1, \ldots, x_n\} \text{ and } \theta_2 > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then the log-likelihood function is

$$\ell(\theta_1,\theta_2)=\begin{cases}(n\theta_2)\log\theta_1+n\log\theta_2-(\theta_2+1)\displaystyle\sum_{i=1}^{n}\log x_i, & \text{if } 0<\theta_1\leqslant x_{(1)} \text{ and } \theta_2>0,\\ 0, & \text{otherwise.}\end{cases}$$

By partially differentiating $\ell(\theta_1,\theta_2)$ with respect to θ_1, we have

$$\frac{\partial\ell(\theta_1,\theta_2)}{\partial\theta_1}=\frac{n\theta_2}{\theta_1}>0,\quad \text{since } n>0,\ \theta_1>0 \text{ and } \theta_2>0.$$

That means $\ell(\theta_1,\theta_2)$ is an increasing function with respect to θ_1 when θ_2 is fixed, and since $0<\theta_1\leqslant x_{(1)}$, $\ell(\theta_1,\theta_2)$ is maximized at $\theta_1=x_{(1)}$. Thus, the MLE of θ_1 is $\hat{\theta}_1=X_{(1)}$. By partially differentiating $\ell(\theta_1,\theta_2)$ with respect to θ_2 and letting it equal zero, i.e.,

$$\frac{\partial\ell(\theta_1,\theta_2)}{\partial\theta_2}=n\log\theta_1+\frac{n}{\theta_2}-\sum_{i=1}^{n}\log x_i=0,$$

we obtain

$$\theta_2=\frac{n}{\sum_{i=1}^{n}\log x_i-n\log\theta_1}.$$

Thus, the MLE of θ_2 is

$$\hat{\theta}_2=\frac{n}{\sum_{i=1}^{n}\log X_i-n\log X_{(1)}}.\qquad\qquad \|$$

Example T4.2 (Restricted MLE of a one-dimensional parameter). Let $X_1,\ldots,X_n$ be a random sample from the Bernoulli distribution

$$f(x;\theta)=\theta^x(1-\theta)^{1-x},\quad x=0,\,1,$$

where $0<\theta\leqslant\frac{1}{2}$, i.e., the parameter space is $\Theta=\{\theta:\ 0<\theta\leqslant\frac{1}{2}\}$. Find the MLE of θ.

Solution: The log-likelihood function is

$$\ell(\theta)=\sum_{i=1}^{n}\log f(x_i;\theta)=\left(\sum_{i=1}^{n}x_i\right)\log\theta+\left(n-\sum_{i=1}^{n}x_i\right)\log(1-\theta),\quad 0<\theta\leqslant\frac{1}{2}.$$

Denote $\bar{x}=\sum_{i=1}^{n}x_i/n$ and we have

$$\ell'(\theta)=\frac{\sum_{i=1}^{n}x_i}{\theta}+\frac{n-\sum_{i=1}^{n}x_i}{\theta-1}=\frac{n(\bar{x}-\theta)}{\theta(1-\theta)},\quad 0<\theta\leqslant\frac{1}{2}.$$

Since x_i $(i = 1, \ldots, n)$ is either 0 or 1, $0 \leqslant \bar{x} \leqslant 1$.

If $0 < \bar{x} \leqslant \frac{1}{2}$, the solution to the equation $\ell'(\theta) = 0$ is $\theta = \bar{x}$.

If $\bar{x} > \frac{1}{2}$, the fact that $\ell'(\theta) > 0$ implies $\ell(\theta)$ is a strictly increasing function of θ. In this case, $\ell(\theta)$ is maximized at $\theta = \frac{1}{2}$.

Thus, the MLE of θ is $\widehat{\theta} = \min\left(\frac{1}{2}, \bar{X}\right)$. ‖

B. Moment Estimator

Equate the sample moments to the corresponding population moments, and then solve the system of equations.

Example T4.3: Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution on the interval $[a, b]$. Find the moment estimators of a and b.

Solution: The pdf of a uniform distribution is

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \leqslant x \leqslant b, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, the first two population moments are

$$E(X) = \int_a^b \frac{x}{b-a} dx = \frac{a+b}{2} \quad \text{and} \quad E(X^2) = \int_a^b \frac{x^2}{b-a} dx = \frac{a^2+ab+b^2}{3}.$$

Denote the first two sample moments as $\hat{\mu}_1$ and $\hat{\mu}_2$ respectively, we have

$$\hat{\mu}_1 = \frac{1}{n}\sum_{i=1}^n X_i \quad \text{and} \quad \hat{\mu}_2 = \frac{1}{n}\sum_{i=1}^n X_i^2.$$

Equating the first two sample moments to the corresponding population moments, we obtain

$$\hat{\mu}_1 = \frac{a+b}{2} \quad \text{and} \quad \hat{\mu}_2 = \frac{a^2+ab+b^2}{3}$$

which, solving for a and b, results in the moment estimators of a and b,

$$\hat{a}^M = \hat{\mu}_1 - \sqrt{3(\hat{\mu}_2 - \hat{\mu}_1^2)} \quad \text{and} \quad \hat{b}^M = \hat{\mu}_1 + \sqrt{3(\hat{\mu}_2 - \hat{\mu}_1^2)}.$$ ‖