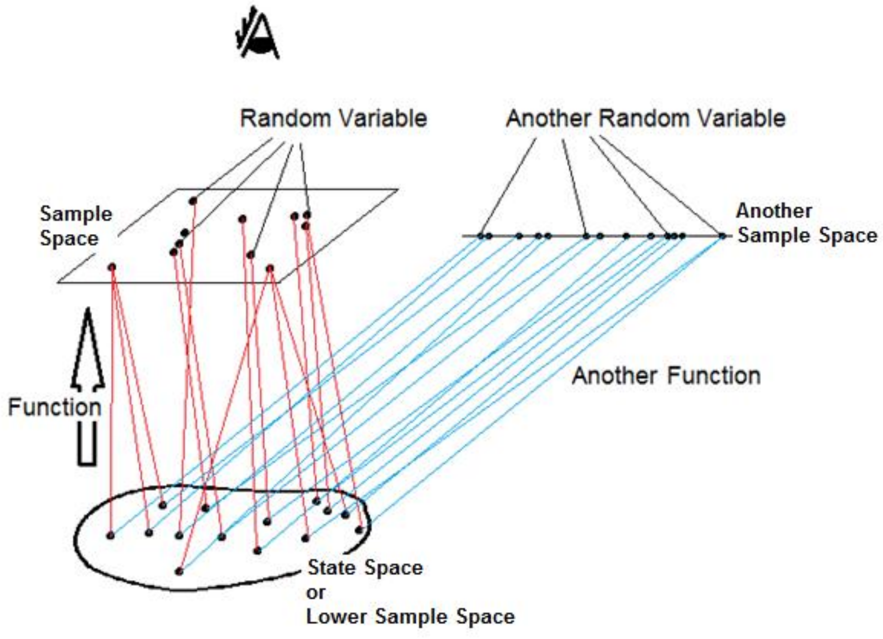
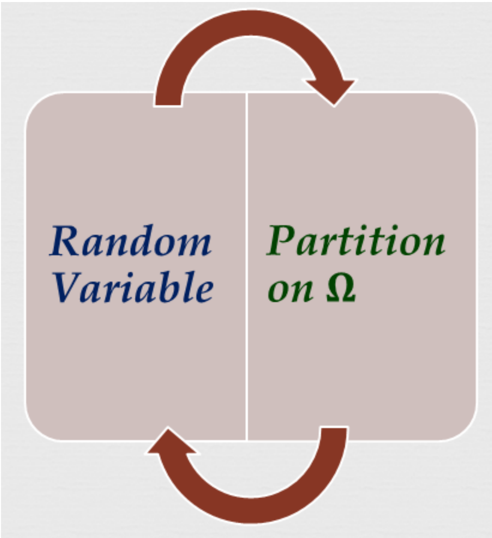
Problems 1-9 on Basic Probability Objects (STAT2802 Tutorial notes for the week of 24-SEP-2012)

The central object of probability theory is the random variable. A random variable is manipulated as a function. It is an asymmetric relation between two structured sets.

As a mathematical function, a random variable maps from its domain into its range. The special name for a random variable's domain is "state space" (Ω); the special name for a random variable's *smallest* range is "sample space" (S).

From the statistical point of view, the sample space (smallest range) is where observations are made. Thus, the random variable is observed as a variable dancing apparently randomly on the sample space. A random variable's state space (domain) can be thought of as the set of all states of our world. The set of all states of our world is too complex to be thoroughly observed or comprehended. A random variable is a simplification instrument on the set of all states of our world. Thus to flip a coin is to physically *perform* a random variable that greatly simplifies the states of our world (e.g., position of every particle in time-space, their electro-magnetic properties, etc.) to two kinds: one that gives rise to a head; the other a tail. Observing a head is to know that bit of information about the world: it is in a head-kind state. A random variable connects the set of all states of our world to our extended perceptions. A random variable is a channel of information on the state of our world.

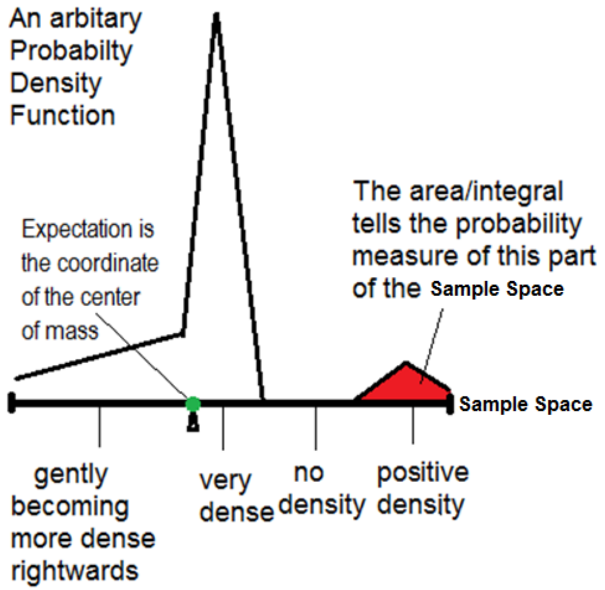


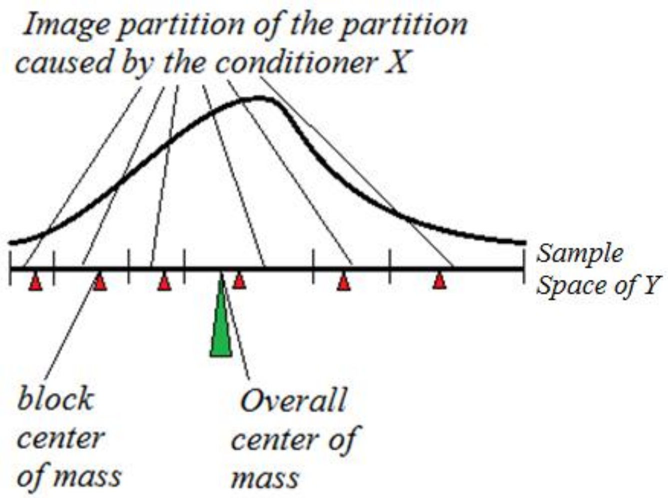


The sample space of a sophisticated random variable can be unnecessarily complicated for certain statistical derivations. Statisticians thus construct functions on that sample space to make further simplifications. Such a function, artificially constructed by statisticians, mapping from the sample space of some random variable into a higher-level sample space is called a statistic. Mathematically there is no difference between a random variable and a statistic because both of them are just functions. From now on, we will make the name "random variable" most general and refer it to any function, or composition of functions, in a probabilistic setting.

Although a random variable dances apparently randomly on the sample space, when its footprints are recorded and counted, these footprints together stabilize to a frequency distribution. Such a pattern that manifests itself only collectively but not individually is the character of statistics. Statistical thinking hinges on the stabilization of such frequency distribution and statisticians investigate logical implications of such stabilization using probability and other mathematical tools. An important target is to infer the probability distribution on the state space. The statistician has control on the sample-space end of the random variable, and he wants to know what it is like on the other end; usually the concrete object he wants to know is the probability distribution there. (To be sure, we rule out the possibility to directly observe that remote probability distribution on the state space. The only way to *know* it is to *infer* it from what we control on the sample-space end.)

A probability is a measure of event intensity. An event is mathematically manipulated as a measurable set, that is, any event must have a legally defined intensity expressed as its probability. The sure event on the sample space is mathematically manipulated as the entire sample space. The almost sure event on the sample space is mathematically manipulated as the entire sample space removed of any number of zero-intensity events. Both sure event and almost sure event has probability 1. The probability of a list of disjoint events unionized equals their individual probabilities added up.





Two *independent* random variables do not interfere with each other while jointly channeling their own piece of information. Mathematically, independence means that the probabilities of two events are multiplied with each other when they jointly occur.

Conditional Expectation is yet a further simplification of the information channeled through the random variable. Conditional Expectation is a random variable by nature and it reduces the sample space to a set of points each corresponding to the center of mass of a block on the sample space. The (unconditional) expectation is a property of the sample space. It is the location of the center of mass of the whole sample space. One can view expectation as a random variable itself because it takes the whole information channeled through the random variable and it evaluates to a single point within the sample space—it is the ultimate simplification of that information because the result is a single point. Expectation is computed mathematically as integration. The expectation of a random variable has (roughly) the same meaning as the Riemann-Stieltjes integral of a function. The probability of an event can be computed as the expectation of the event's indicator.

Problems 1-9

1. Write down the Law of Iterated Expectations and prove it. Then write down the Law of Total Probability and prove it.
2. Write down Markov's Inequality and prove it. Then write down Chebyshev's Inequality and prove it.
3. Write down the Weak Law of Large Numbers and prove it.
4. What is the Moment Generating Function of a distribution? What is the Characteristic Function of a distribution? What are the Moment Generating Function and the Characteristic Function of a univariate normal distribution?
5. Write down the density of a univariate normal distribution and show that it integrates to 1.
6. Write down the Gamma function $\Gamma(t)$ and show that $\Gamma(n) = (n-1)!$ for any positive integer n.
7. Write down the Central Limit Theorem and prove it.
8. Write down the density of the d-dimensional normal distribution. What are the scalar parameters of a general bivariate normal density? Show that the components of a bivariate normal r.v. are independent of each other iff their correlation coefficient is 0.
9. Show that the correlation coefficient is absolutely bounded by 1.