MA204: Mathematical Statistics

Suggested Solutions to Assignment 2

2.1 Solution. (a) We first calculate $E(T)$. From the SR

$$T \stackrel{d}{=} \frac{Z}{\sqrt{Y/n}},$$

we have

$$E(T) = E(Z) \times \sqrt{n} E(Y^{-1/2}) = 0$$

since $Z \sim N(0,1)$ and $Z \perp\!\!\!\perp Y$.

(b) We next calculate $\text{Var}(T) = E(T^2) - [E(T)]^2 = E(T^2)$. The density of $Y \sim \chi^2(n)$ is

$$g(y) = \frac{2^{-n/2}}{\Gamma(n/2)} y^{n/2-1} \mathrm{e}^{-y/2}, \quad y > 0.$$

Hence, we have

$$\begin{aligned}
E(T^2) &= E(Z^2) \times n E(Y^{-1}) = 1 \times n \int_0^\infty y^{-1} g(y)\,\mathrm{d}y \\
&= n \frac{2^{-n/2}}{\Gamma(n/2)} \int_0^\infty y^{(n-2)/2-1} \mathrm{e}^{-y/2}\,\mathrm{d}y \\
&= n \frac{2^{-n/2}}{\Gamma(n/2)} \cdot \frac{\Gamma(\frac{n-2}{2})}{2^{-(n-2)/2}} \\
&= \frac{n}{n-2}, \qquad n > 2,
\end{aligned}$$

where we used the formula $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

2.2 Solution. Let $X \sim \text{Beta}(a,b)$, where $a=3$ and $b=2$. Then, the pdf and cdf of X are given by

$$\begin{aligned} f(x) &= \frac{1}{B(a,b)}\, x^{a-1}(1-x)^{b-1} \cdot I_{(0,1)}(x) \\ &= \frac{\Gamma(3+2)}{\Gamma(3)\Gamma(2)}\, x^2(1-x) \cdot I_{(0,1)}(x) \\ &= 12(x^2-x^3) \cdot I_{(0,1)}(x), \quad \text{and} \end{aligned}$$

$$\begin{aligned} F(x) &= \begin{cases} 0, & \text{if } x \leqslant 0, \\ \displaystyle\int_0^x f(t)\,\mathrm{d}t, & \text{if } 0 < x < 1, \\ 1, & \text{if } x \geqslant 1, \end{cases} \\ &= \begin{cases} 0, & \text{if } x \leqslant 0, \\ 4x^3 - 3x^4, & \text{if } 0 < x < 1, \\ 1, & \text{if } x \geqslant 1. \end{cases} \end{aligned}$$

Thus, the cdf and pdf of $X_{(1)} = \min\{X_1, \ldots, X_n\}$ are given by

$$\begin{aligned} G_1(x) &= 1 - [1 - F(x)]^n \\ &= \begin{cases} 0, & \text{if } x \leqslant 0, \\ 1 - [1 - 4x^3 + 3x^4]^n, & \text{if } 0 < x < 1, \\ 1, & \text{if } x \geqslant 1, \end{cases} \qquad \text{and} \\ g_1(x) &= nf(x)[1 - F(x)]^{n-1} \\ &= 12nx^2(1-x)[1 - 4x^3 + 3x^4]^{n-1} \cdot I_{(0,1)}(x). \end{aligned}$$

Similarly, the cdf and pdf of $X_{(n)} = \max\{X_1, \ldots, X_n\}$ are given by

$$G_n(x) = [F(x)]^n$$

$$= \begin{cases} 0, & \text{if } x \leqslant 0, \\ [4x^3 - 3x^4]^n, & \text{if } 0 < x < 1, \\ 1, & \text{if } x \geqslant 1, \qquad \text{and} \end{cases}$$

$$\begin{aligned} g_n(x) &= nf(x)[F(x)]^{n-1} \\ &= 12nx^2(1-x)[4x^3 - 3x^4]^{n-1} \cdot I_{(0,1)}(x). \end{aligned}$$

2.3 Solution. Define $Y_i = X_{(i)}$ for $i = 1, \ldots, n$. The joint density of $Y_1, \ldots, Y_n$ is given by

$$\begin{aligned} f(y_1, \ldots, y_n) &= n! f(y_1) \cdots f(y_n) \\ &= n! \mathrm{e}^{-\sum_{i=1}^n y_i}, \quad 0 < y_1 < \cdots < y_n. \end{aligned}$$

(a) Taking transformation

$$\begin{cases} z_1 &= ny_1 \\ z_2 &= (n-1)(y_2 - y_1) \\ &\vdots \\ z_n &= y_n - y_{n-1}, \end{cases}$$

we have $z_i > 0$ for $i = 1, \ldots, n$, and the inverse transformation is given by

$$\begin{cases} y_1 &= \dfrac{z_1}{n} \\ y_2 &= \dfrac{z_1}{n} + \dfrac{z_2}{n-1} \\ &\vdots \\ y_n &= \dfrac{z_1}{n} + \dfrac{z_2}{n-1} + \cdots + z_n. \end{cases}$$

Since the Jacobian is

$$J = \left| \frac{\partial(y_1, \ldots, y_n)}{\partial(z_1, \ldots, z_n)} \right| = \begin{vmatrix} \frac{1}{n} & 0 & 0 & \cdots & 0 \\ \frac{1}{n} & \frac{1}{n-1} & 0 & \cdots & 0 \\ \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \cdots & 1 \end{vmatrix} = \frac{1}{n!},$$

the joint density of $Z_1, \ldots, Z_n$ is

$$\begin{aligned} g(z_1, \ldots, z_n) &= f(y_1, \ldots, y_n)|J| \\ &= \mathrm{e}^{-\sum_{i=1}^n z_i}, \quad z_i > 0, \quad i = 1, \ldots, n. \end{aligned}$$

Therefore, the marginal density of Z_i is Exponential(1). Furthermore, note that

$$g(z_1, \ldots, z_n) = g(z_1) \cdots g(z_n),$$

then $Z_1, \ldots, Z_n$ are mutually independent.

(b) We can write

$$\begin{aligned} \sum_{i=1}^n a_i Y_i &= \sum_{i=1}^n a_i \left(\sum_{k=0}^{i-1} \frac{Z_{k+1}}{n-k} \right) \\ &= \sum_{k=0}^{n-1} \left(\sum_{i=k+1}^n a_i \right) \frac{Z_{k+1}}{n-k} \\ &= \sum_{j=1}^n \left(\sum_{i=j}^n a_i \right) \frac{Z_j}{n-j+1}, \end{aligned}$$

which is a linear function of independent random variables $Z_1, \ldots, Z_n$.

2.4 Solution. Let $Y_n = X_1 + \cdots + X_n$. Making transformation

$$\begin{cases} y_1 &= x_1/y_n, \\ & \vdots \\ y_{n-1} &= x_{n-1}/y_n, \\ y_n &= x_1 + \cdots + x_n, \end{cases}$$

we have $y_i \geqslant 0$ for $i = 1, \ldots, n-1$, $y_1 + \cdots + y_{n-1} \leqslant 1$, $y_n \geqslant 0$, and the inverse transformation is given by

$$\begin{cases} x_1 &= y_1 y_n \\ & \vdots \\ x_{n-1} &= y_{n-1} y_n \\ x_n &= (1 - y_1 - \cdots - y_{n-1}) y_n. \end{cases}$$

Since the Jacobian is

$$J = \left| \frac{\partial(x_1, \ldots, x_n)}{\partial(y_1, \ldots, y_n)} \right| = \begin{vmatrix} y_n & 0 & \cdots & 0 & y_1 \\ 0 & y_n & \cdots & 0 & y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & y_n & 0 \\ -y_n & -y_n & \cdots & -y_n & 1 - \sum_{i=1}^{n-1} y_i \end{vmatrix} = y_n^{n-1},$$

the joint density of $Y_1, \ldots, Y_{n-1}, Y_n$ is

$$\begin{aligned} & g(y_1, \ldots, y_{n-1}, y_n) \\ = \; & f(x_1, \ldots, x_n)|J| \\ = \; & \left[\prod_{i=1}^{n} \frac{1}{\Gamma(a_i)} x_i^{a_i - 1} \mathrm{e}^{-x_i} \right] \cdot y_n^{n-1} \\ = \; & \left[\frac{\Gamma(a_+)}{\Gamma(a_1) \cdots \Gamma(a_n)} y_1^{a_1 - 1} \cdots y_{n-1}^{a_{n-1} - 1} \left(1 - \sum_{j=1}^{n-1} y_j \right)^{a_n - 1} \right] \\ & \times \frac{1}{\Gamma(a_+)} y_n^{a_+ - 1} \mathrm{e}^{-y_n}, \end{aligned}$$

where $a_+ = \sum_{i=1}^n a_i$. Therefore,

$$(Y_1, \ldots, Y_{n-1})^\top \sim \text{Dirichlet}(a_1, \ldots, a_{n-1};\ a_n),$$

$Y_n \sim \text{Gamma}(a_+, 1)$, and $(Y_1, \ldots, Y_{n-1})^\top \perp\!\!\!\perp Y_n$.

2.5 Solution. Let $U = \log(X)$ and $V = \log(Y)$. The mgf of U is

$$\begin{aligned} M_U(t) &= E(\mathrm{e}^{tU}) \\ &= \frac{1}{\Gamma(p)} \int_0^\infty \mathrm{e}^{t\log(x)} \cdot x^{p-1}\mathrm{e}^{-x}\,\mathrm{d}x \\ &= \frac{1}{\Gamma(p)} \int_0^\infty x^{p+t-1}\mathrm{e}^{-x}\,\mathrm{d}x \\ &= \frac{\Gamma(p+t)}{\Gamma(p)} \end{aligned}$$

and the mgf of V is

$$\begin{aligned} M_V(t) &= E(\mathrm{e}^{tV}) \\ &= \frac{1}{B(q, p-q)} \int_0^\infty \mathrm{e}^{t\log(y)} \cdot y^{q-1}(1-y)^{p-q-1}\,\mathrm{d}y \\ &= \frac{1}{B(q, p-q)} \int_0^\infty y^{q+t-1}(1-y)^{p-q-1}\,\mathrm{d}y \\ &= \frac{B(q+t, p-q)}{B(q, p-q)} = \frac{\Gamma(q+t)\Gamma(p)}{\Gamma(q)\Gamma(p+t)}. \end{aligned}$$

So the mgf of $\log(XY) = U + V$ is

$$M_{U+V}(t) = M_U(t) \cdot M_V(t) = \frac{\Gamma(q+t)}{\Gamma(q)},$$

which implies that $XY \sim \text{Gamma}(q, 1)$.

2.6 Solution. The joint pmf of $\mathbf{y} = Z\mathbf{x}$ is denoted by

$$f(\boldsymbol{y}|\phi, \boldsymbol{\lambda}) = \Pr(\mathbf{y} = \boldsymbol{y}) = \Pr(ZX_1 = y_1, \ldots, ZX_m = y_m).$$

If $\boldsymbol{y} = \mathbf{0}_m$, we have

$$\begin{aligned} f(\boldsymbol{y}|\phi, \boldsymbol{\lambda}) &= \Pr(ZX_1 = 0, \dots, ZX_m = 0) \\ &= \Pr(Z = 0) + \Pr(Z = 1,\, X_1 = 0, \dots, X_m = 0) \\ &= \phi + (1-\phi)\mathrm{e}^{-\lambda_+}, \end{aligned}$$

where $\lambda_+ = \sum_{i=1}^m \lambda_i$. If $\boldsymbol{y} \neq \mathbf{0}_m$, we have

$$\begin{aligned} f(\boldsymbol{y}|\phi, \boldsymbol{\lambda}) &= \Pr(ZX_1 = y_1, \dots, ZX_m = y_m) \\ &= \Pr(Z = 1,\, X_1 = y_1, \dots, X_m = y_m) \\ &= (1-\phi)\mathrm{e}^{-\lambda_+} \prod_{i=1}^m \frac{\lambda_i^{y_i}}{y_i!}. \end{aligned}$$

Finally, we obtain

$$\begin{aligned} & f(\boldsymbol{y}|\phi, \boldsymbol{\lambda}) = \Pr(\mathbf{y} = \boldsymbol{y}) \\ = \;& [\phi + (1-\phi)\mathrm{e}^{-\lambda_+}] I(\boldsymbol{y} = \mathbf{0}) + \left[(1-\phi)\mathrm{e}^{-\lambda_+} \prod_{i=1}^m \frac{\lambda_i^{y_i}}{y_i!}\right] I(\boldsymbol{y} \neq \mathbf{0}) \\ = \;& \phi \Pr(\boldsymbol{\xi} = \boldsymbol{y}) + (1-\phi)\Pr(\mathbf{x} = \boldsymbol{y}), \end{aligned}$$

where $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m)^\top$ and $\{\xi_i\}_{i=1}^m \stackrel{\text{iid}}{\sim} \text{Degenerate}(0)$.

2.7 Solution. (a) It is easy to know that

$$X_1 + X_2 \sim N(0, 2\sigma^2) \quad \text{and} \quad X_1 - X_2 \sim N(0, 2\sigma^2).$$

Since

$$\begin{aligned} \text{Cov}(X_1 + X_2, X_1 - X_2) &= E[(X_1 + X_2)(X_1 - X_2)] \\ &= E(X_1^2) - E(X_2^2) \\ &= 2\sigma^2 - 2\sigma^2 = 0, \end{aligned}$$

from the result 3) of Theorem 2.1, we have $(X_1 + X_2) \perp\!\!\!\perp (X_1 - X_2)$. Let

$$Z_1 \hat{=} \frac{X_1 + X_2}{\sqrt{2}\sigma} \quad \text{and} \quad Z_2 \hat{=} \frac{X_1 - X_2}{\sqrt{2}\sigma},$$

then $Z_1 \sim N(0,1)$, $Z_2 \sim N(0,1)$ and $Z_1 \perp\!\!\!\perp Z_2$. Therefore,

$$\frac{(X_1 - X_2)^2}{(X_1 + X_2)^2} = \frac{Z_2^2}{Z_1^2} \sim \frac{\chi^2(1)/1}{\chi^2(1)/1} = F(1,1).$$

(b) Since

$$\begin{aligned}
& \Pr\left\{\frac{(X_1 + X_2)^2}{(X_1 + X_2)^2 + (X_1 - X_2)^2} > k\right\} \\
= \; & \Pr\left\{\frac{Z_1^2}{Z_1^2 + Z_2^2} > k\right\} \\
= \; & \Pr\left\{\frac{Z_2^2}{Z_1^2} < \frac{1-k}{k}\right\} = 0.1,
\end{aligned}$$

we obtain $(1-k)/k = 0.02508563$ so that $k = 0.9755283$.

2.8 Solution. Note that

$$\text{Exponential}(1) = \text{Gamma}(1,1) = \frac{1}{2}\text{Gamma}\left(\frac{2}{2}, \frac{1}{2}\right) = \frac{1}{2}\chi^2(2),$$

then, we obtain

$$\frac{X}{Y} \sim \frac{\chi^2(2)/2}{\chi^2(2)/2} = F(2,2).$$