Review for Midterm Test

by Yuejian Mo

May 27, 2018

1 Probility

i. Distributions

Table 1.

ii. Conditional Exectation

$$E(X) = E\{E(X|Y)\}$$

$$\mathrm{Var}(X) = E(\mathrm{Var}(X|Y)) + \mathrm{Var}(E(X|Y))$$

iii. From conditional densities to marginal densitie

Based on

$$f_X(x) f_{Y|X}(y|x) = f_Y(y) f_{X|Y}(x|y)$$

we have

$$f_X(x) \propto \frac{f_Y(y_0) f_{X|Y}(x|y_0)}{f_{Y|X}(y_0|x)}$$

$$f_X(x) = \left\{\int \frac{f_{Y|X}(y|x)}{f_{X|Y}(x|y|)} dy\right\}^{-1}$$

Samilar,

$$\begin{aligned}
&p_i = \Pr(X = x_i) \propto \frac{\Pr(X = x_i|Y = \mathrm{yj}_0)}{\Pr(Y = y_{j0}|X = X_i)} \hat{=} q_i, \\
&p_i = \frac{q_i}{\sum_{i'} q_i'}, \\
&\Pr(X = x_i) = \left(\sum_j \frac{\Pr(Y = y_j|X = x_i)}{\Pr(X = x_i|Y = y_j)}\right)^{-1}.
\end{aligned}$$

(Why the kernel is known?)

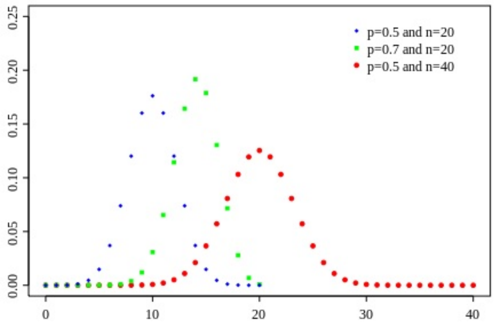
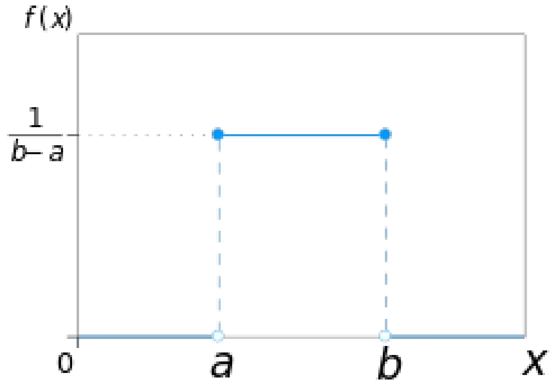
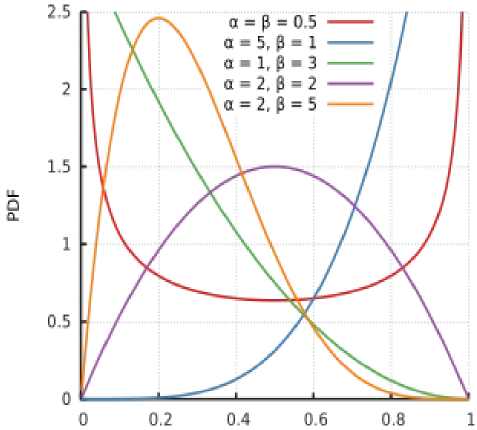
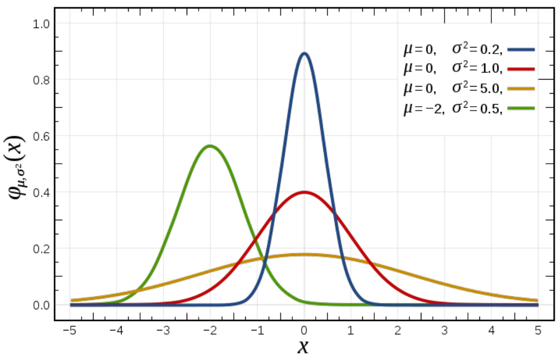
<table>
<tr><th></th><th>pmf/pdf</th><th></th></tr>
<tr><td>Bernoulli(p)</td><td>$f_X(x) = \begin{cases} p & , x = 1 \\ 1 - p & , x \neq 1 \end{cases}$</td><td></td></tr>
<tr><td>Binomial(n,p)</td><td>$f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$</td><td>
</td></tr>
<tr><td>Uniform(a,b)</td><td>$f_X(x) = \begin{cases} \frac{1}{b-a} & , a \leqslant x \leqslant b \\ 0 & , \text{elsewhere} \end{cases}$
$E(X) = \frac{a+b}{2}$
$\mathrm{Var}(X) = \frac{(b-a)^2}{12}$
mgf: $\frac{e^{tb} - e^{ta}}{t(b-a)}$</td><td>
</td></tr>
<tr><td>Beta(a,b)</td><td>$f_X(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}$, $(0 \leqslant x \leqslant 1)$
$a, b > 0$
$E(X) = \frac{a}{a+b}$
$\mathrm{Var}(X) = \frac{ab}{c^2(c+1)}$</td><td>
</td></tr>
<tr><td>$N(\mu, \sigma^2)$</td><td>$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
mgf: $e^{\mu t + 0.5\sigma^2 t^2}$</td><td>
</td></tr>
</table>

Table 1. The pmf/pdf of some distribution

2 Sample Distribution

Three methods to find the distribution of the Function of Random Variables.

2.1 cdf

2.2 Transormation

Monotone transformation

$$g(y) = f(x) \times |dx/dy|.$$

Piecewise monote transformation(分段)

$$g(y) = \sum_{i=1}^{n} f(h_i^{-1}(y)) \times \left|\frac{dh_i^{-1}(y)}{dy}\right|.(???)$$

Bivariate transormation

$$g(y_1, y_2) = f(x_1, x_2) \times \left|\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)}\right| (\text{need to understand})$$

Multivariate transformation

$$g(y_1, ..., y_n) = f(x_1, ..., x_n) \times \left|\frac{\partial(x_1, ..., x_n)}{\partial(y_1, ..., y_n)}\right|.$$

Momentt generating function

2.3 Order Statistics

The cdf and pdf of $X_{(1)} = \min\{X_1, ..., X_n\}$

$$\begin{aligned} G_1(x) &= 1 - [1 - F(x)]^n, \\ g_1(x) &= n[1 - F(x)]^{n-1} f(x) \end{aligned}$$

The cdf and pdf of $X_{(n)} = \max\{X_1, ..., X_n\}$

$$\begin{aligned} G_n(x) &= [F(x)]^n \\ g_n(x) &= n[F(x)]^{n-1} f(x) \end{aligned}$$

The cdf and pdf of $X_{(r)}$

$$\begin{aligned} G_r(x) &= \sum_{i=r}^{n} \binom{n}{i} F^i(x)[1 - F(x)]^{n-i} \\ g_r(x) &= \frac{n!}{(r-1)!(n-r)!} f(x) F^{r-1}(x)[1 - F(x)]^{n-r} \end{aligned}$$

The joint pdf of $X_{(1)}, ..., X_{(n)}$ is

$$g_{X_{(1)}, ..., X_{(n)}}(x_{(1)}, ..., x_{(n)}) = n! f_X(x_{(1)}) \cdots f_X(x_{(n)})$$

2.4 Central Limit Theorem

Theroem 2.9 Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of i.i.d. r.v.'s with common mean μ and common variance $\sigma^2 > 0$. Let

$$\bar{X}_n = \sum_{i=1}^{n} X_i / n \qquad \text{and} \qquad Y_n = \sqrt{n}\left(\bar{X}_n - \mu\right)/\sigma$$

then $Y_n \xrightarrow{L} Z$, where $Z \sim N(0,1)$

3 Point Estimation

Joint density and likehood function

(b) We can write

$$\begin{aligned}
\sum_{i=1}^{n} a_i Y_i &= \sum_{i=1}^{n} a_i \left(\sum_{k=0}^{i-1} \frac{Z_{k+1}}{n-k} \right) \\
&= \sum_{k=0}^{n-1} \left(\sum_{i=k+1}^{n} a_i \right) \frac{Z_{k+1}}{n-k} \\
&= \sum_{j=1}^{n} \left(\sum_{i=j}^{n} a_i \right) \frac{Z_j}{n-j+1},
\end{aligned}$$

which is a linear function of independent random variables $Z_1, \ldots, Z_n$.

$$\text{Exponential}(1) = \text{Gamma}(1,1) = \frac{1}{2}\text{Gamma}\left(\frac{2}{2}, \frac{1}{2}\right) = \frac{1}{2}\chi^2(2),$$

then, we obtain

$$\frac{X}{Y} \sim \frac{\chi^2(2)/2}{\chi^2(2)/2} = F(2,2).$$

4 Confience Intervation

5 Reference

- https://en.wikipedia.org/wiki/Beta_distribution