Mathematical Statistics Assignment 3

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3.1. Let $X_1, ..., X_n \overset{\text{iid}}{\sim} U[\theta_1, \theta_2]$. Find the MLEs of θ_1 and θ_2.

Sovle: The joint desity of $\boldsymbol{x} = (X_1, ..., X_n)^T$ is

$$f(\boldsymbol{x}; \theta_1, \theta_2) = \begin{cases} \dfrac{1}{(\theta_2 - \theta_1)^n} & , \text{if } \theta_1 \leqslant x_i \leqslant \theta_2, i = 1, ..., n, \\ 0 & , \text{elsewhere.} \end{cases}$$

Then the likehood function is

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} f(x_i; \theta_1, \theta_2) = \begin{cases} \displaystyle\prod_{i=1}^{n} \frac{1}{(\theta_2 - \theta_1)} = \frac{1}{(\theta_2 - \theta_1)^n} & , \theta_1 \leqslant x_{(1)} \text{ and } \theta_2 \geqslant x_{(n)} \\ 0 & , \text{elsewhere.} \end{cases}$$

Because $\theta_2 - \theta_1 \geqslant x_{(n)} - x_{(1)}$, and $L(\theta_1, \theta_2)$ is a monotone and decreasing function of $\theta_2 - \theta_1$ when $\theta_2 - \theta_1 \in [x_{(n)} - x_{(1)}, \infty)$. Thus $\hat{\boldsymbol{\theta}} = \widehat{(\theta_2 - \theta_1)} = x_{(n)} - x_{(1)}$ is the MLE of θ_1 and θ_2.

3.2. A sample of size n_1 is drawn from $N(\mu_1, \sigma_1^2)$. A second sample of size n_2 is drawn from $N(\mu_2, \sigma_2^2)$. Assume that the two samples are independent.

a) What is the MLE of $\theta = \mu_1 - \mu_2$?

Solve:

Let

$$X \sim N(\mu_1, \sigma_1^2) \Longrightarrow f_X = \frac{1}{\sqrt{2\pi}\sigma_1} \mathrm{e}^{-(x-\mu_1)/2\sigma_1^2}$$

and

$$Y \sim N(\mu_2, \sigma_2^2) \Longrightarrow f_Y = \frac{1}{\sqrt{2\pi}\sigma_2} e^{-(y-\mu_2)/2\sigma_2^2}$$

Assuming indepence, the likelihood is given by

$$L(\boldsymbol{X}, \boldsymbol{Y}; \mu_1, \mu_2, \sigma_1^2, \sigma_2^2) \ = \ \left(\prod_{i=1}^{m} \right)$$

b) If we assume that the total sample size $n = n_1 + n_2$ is fixed, how should the n observations be approximately divided between the two populations in order to minimize the variance of the $\hat{\theta}$?

Solve:

3.5. Let $X_1, ..., X_n \sim U\left[\mu - \sqrt{3}\sigma, \mu + \sqrt{3}\sigma\right]$, where $\mu \in R$ and $\sigma > 0$.

a) Find the MLEs of μ and σ

Solve:

The joint pmf of $\boldsymbol{x} = (X_1, ..., X_n)^T$ is

$$f(\boldsymbol{x}; \mu, \sigma) = \prod_{i=1}^{n} f(x_i; \mu, \sigma) = \begin{cases} \left(\dfrac{1}{2\sqrt{3}\sigma}\right)^n & \text{, if } \mu - \sqrt{3}\sigma \leqslant x_i \leqslant \mu + \sqrt{3}\sigma \\ 0 & \text{, elsewhere.} \end{cases}$$

Then the likehood function is

$$L(\mu, \sigma) = \prod_{i=1}^{n} f(x_i; \mu, \sigma) = \begin{cases} \left(\dfrac{1}{2\sqrt{3}\sigma}\right)^n & , \mu - \sqrt{3}\sigma \leqslant x_{(1)} \text{ and } \mu + \sqrt{3}\sigma \geqslant x_{(n)} \\ 0 & \text{, else.} \end{cases}$$

Thus, any statistic $\hat{\sigma}$ satisfying

$$\begin{aligned} &\left(\mu - \sqrt{3}\hat{\sigma}\right) - \left(\mu + \sqrt{3}\hat{\sigma}\right) \leqslant x_{(1)} - x_{(n)} \\ \Rightarrow \ &\hat{\sigma} \geqslant \frac{x_{(n)} - x_{(1)}}{2\sqrt{3}} \end{aligned}$$

is an MLE of θ. And any statistic $\hat{\mu}$ satisfying

$$x_{(n)} - \sqrt{3}\hat{\sigma} \leqslant \hat{\mu} \leqslant x_{(1)} + \sqrt{3}\hat{\sigma}$$

is an MLE of μ.

b) Find the moment estimators of μ and σ.

Solve:

The moment estimators of μ and σ must satisfy

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n} X_i &= E(X) = \mu \\ \frac{1}{n}\sum_{i=1}^{n} X_i^2 &= E(X^2) = \mu^2 + \sigma^2 \end{aligned}$$

Thus,

$$\hat{\mu}^M = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad \hat{\sigma}^M = \sqrt{\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2}$$

3.7. Let $X \sim \text{Beronulli}(\theta)$. Let $t_1(X) = X$ and $t_2(X) = 1/2$.

a) Are both $t_1(X)$ and $t_2(X)$ unbiased?

Because

$$b_1(\theta) = E(t_1(\boldsymbol{X})) - \theta = E(\boldsymbol{X}) - \theta = 0$$

and

$$b_2(\theta) = E(t_2(\boldsymbol{X})) - \theta = E(1/2) - \theta = 1/2 - \theta \neq 0$$

So, $t_1(X)$ is unbiased and $t_2(X)$ is biased.

b) Compare the MSE of $t_1(X)$ with that of $t_2(X)$.

$$\mathrm{MSE}(t_1(\boldsymbol{X})) = \mathrm{Var}\{\boldsymbol{X}\} + b_1^2(\theta) = \theta(1-\theta) + 0 = \theta - \theta^2$$

$$\mathrm{MSE}(t_2(\boldsymbol{X})) = \mathrm{Var}\{1/2\} + b_2^2(\theta) = 0 + \frac{1}{4} - \theta + \theta^2 = \frac{1}{4} - \theta + \theta^2$$

The MSE of $t_2(X)$ is larger than $t_1(X)$, which mean $t_2(X)$ is biaser than $t_1(X)$.

3.10. Let $X_1, ..., X_n \overset{\text{iid}}{\sim} N(\mu_0, \theta)$, where μ_0 is known and $\theta > 0$.

a) Find the MLE $\hat{\theta}$ of θ?

Solve:

The joint pmf of $\boldsymbol{x} = (X_1, ..., X_n)^T$ is

$$f(\boldsymbol{x}; \theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta}} \mathrm{Exp}\left(-\frac{(x_i - \mu_0)^2}{2\theta}\right)$$

Then the likehood function is

$$L(\theta) = \prod_{i=1}^{n} f(x_i, \theta) = \left(\frac{1}{\sqrt{2\pi\theta}}\right)^n \mathrm{Exp}\left(\frac{\sum_{i=1}^{n} (x_i - \mu_0)^2}{-2\theta}\right)$$

The log likehood function is

$$\ell(\theta) = -\frac{n}{2}\log(2\pi\theta) - \frac{n}{2\theta} + \sum_{i=1}^{n} (x_i - \mu_0)^2 = -\frac{n}{2}\log(\theta) - \frac{n}{2}\log(2\pi) - \frac{\sum_{i=1}^{n} (x_i - \mu_0)^2}{2\theta}$$

By differentiating $\ell(\theta)$ with respect to θ and letting them equal to zeros, we have

$$\frac{\partial \ell(\theta)}{\partial \theta} = \frac{-n}{2\theta} + \frac{\sum_{i=1}^{n} (x_i - \mu_0)^2}{2\theta^2} = 0$$

Therefore,

$$\hat{\theta} = \frac{\sum_{i=1}^{n} (x_i - \mu_0)^2}{n}$$

b) What is the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$?

3.15. Let $X_1, ..., X_n \overset{\text{iid}}{\sim} \mathrm{Bernoulli}(\theta)$. Define $\tau(\theta) = \mathrm{Var}(X) = \theta(1-\theta)$.

a) Find the Cramer-Rao lower bound for the unbiased estimator of $\tau(\theta)$.

b) Find the unique UMVUE of $\tau(\theta)$ if such exists.

3.17. Supposet that X follows a geometeric distribution,

$$\Pr(X = k) = p(1-p)^{k-1}, k = 1, 2. ...$$

where $0 \leqslant p \leqslant 1$, and assume an i.i.d. sample fo size n.

a) Find the moment of estimator of p

Solution: The moment of estimator of p must satisfy

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = E(X) = 1/p$$

Thus,

$$\hat{p}^M = \frac{n}{\sum_{i=1}^{n} X_i}$$

b) Find the MLE of p.

Solution: The parameter space is $\Theta = (0, 1)$. The joint density of x is

$$f(\boldsymbol{x}; p) = \prod_{i=1}^{n} f(x_i; p) = \prod_{i=1}^{n} p(1-p)^{k_i - 1}, k_i = 1, 2, ...$$

so the likehood is given by

$$\begin{aligned} L(p) &= \prod_{i=1}^{n} p(1-p)^{k_i - 1} \\ &= p^n (1-p)^{\sum_{i=1}^{n} k_i - n} \end{aligned}$$

so let the log-likehood function be zeros

$$\ell(p) = n \log(p) + \left(\sum_{i=1}^{n} k_i - n\right) \log(1-p) = 0$$

$$\hat{p} = \frac{1}{n}\sum_{i=1}^{n} k_i$$

c) Let p have a uniform prior distribution on $[0, 1]$. What is the posterior distribution of p? What is the posterior mean?