MA204: Mathematical Statistics

Suggested Solutions to Assignment 4

4.1 **Proof**. Define a new random variable $W = S_1^2/n_1 + S_2^2/n_2$. Since

$$\begin{aligned} W &= \frac{\sigma_1^2}{n_1(n_1-1)} \cdot \frac{(n_1-1)S_1^2}{\sigma_1^2} + \frac{\sigma_2^2}{n_2(n_2-1)} \cdot \frac{(n_2-1)S_2^2}{\sigma_2^2} \\ &\hat{=} a_1\chi_1^2 + a_2\chi_2^2 \end{aligned}$$

is a linear combination of two independent chi-squared random variables, where $\chi_k^2 \sim \chi^2(f_k),\ f_k = n_k - 1,\ k = 1, 2$, we could approximate W/g by a chi-squared distribution with f degrees of freedom, i.e.,

$$\frac{W}{g} \dot{\sim} \chi^2(f) \quad \text{or} \quad a_1\chi_1^2 + a_2\chi_2^2 \dot{\sim} g \cdot \chi^2(f). \tag{SA4.1}$$

To determine the g and f, let the corresponding means and variances in both sides of (SA4.1) be equal, i.e.,

$$a_1 f_1 + a_2 f_2 = gf \quad \text{and} \quad a_1^2 \cdot 2f_1 + a_2^2 \cdot 2f_2 = g^2 \cdot 2f. \tag{SA4.2}$$

We obtain

$$g = \frac{a_1^2 f_1 + a_2^2 f_2}{a_1 f_1 + a_2 f_2}$$

and

$$f = \frac{(a_1 f_1 + a_2 f_2)^2}{a_1^2 f_1 + a_2^2 f_2} = \frac{\left(\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}\right)^2}{\left(\dfrac{\sigma_1^2}{n_1}\right)^2 \dfrac{1}{n_1 - 1} + \left(\dfrac{\sigma_2^2}{n_2}\right)^2 \dfrac{1}{n_2 - 1}}. \tag{SA4.3}$$

From the definition of T_{Welch}, we have

$$\begin{aligned} T_{\text{Welch}} &= \frac{(\bar{X}_1 - \bar{X}_2 - \mu_1 + \mu_2)/\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}{\sqrt{W}/\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}} \\ &= \frac{N(0,1)}{\sqrt{W/(a_1 f_1 + a_2 f_2)}} \\ &\overset{\text{(SA4.2)}}{=} \frac{N(0,1)}{\sqrt{\frac{W}{g}/f}} \\ &\doteq \frac{N(0,1)}{\sqrt{\chi^2(f)/f}} \\ &\sim t(f). \end{aligned}$$

Finally, since f is a function of both σ_1^2 and σ_2^2, we replace σ_k^2 in (SA4.3) by S_k^2 $(k = 1, 2)$ and obtain the estimate of f, denoted by

$$\nu = \frac{\left(\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}\right)^2}{\left(\dfrac{S_1^2}{n_1}\right)^2 \dfrac{1}{n_1 - 1} + \left(\dfrac{S_2^2}{n_2}\right)^2 \dfrac{1}{n_2 - 1}}.$$

4.2 Solution. (a) Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Poisson}(\lambda)$. Note that $E(X) = \text{Var}(X) = \lambda$, by the Central Limit Theorem (Theorem 2.9, page 94),

$$\frac{\sqrt{n}(\bar{X}_n - \lambda)}{\sqrt{\lambda}} \overset{\text{L}}{\to} Z \sim N(0,1).$$

Therefore, for large n, we have

$$1 - \alpha = \Pr(|Z| \leqslant z_{\alpha/2}) = \Pr\left\{\left|\frac{\sqrt{n}(\bar{X}_n - \lambda)}{\sqrt{\lambda}}\right| \leqslant z_{\alpha/2}\right\}.$$

Now $n = 100$, $\bar{X}_n = 6.25$, $\alpha = 0.05$, $z_{0.025} = 1.96$, an approximate and equal-tail 95% CI of λ is determined by

$$\left|\frac{10(6.25 - \lambda)}{\sqrt{\lambda}}\right| \leqslant 1.96$$

or

$$\lambda^2 - 12.5384\lambda + 39.0625 \leqslant 0.$$

There are two roots

$$\lambda_1 = \frac{12.5384 - \sqrt{12.5384^2 - 4 \times 39.0625}}{2} = 5.7789$$

and

$$\lambda_2 = \frac{12.5384 + \sqrt{12.5384^2 - 4 \times 39.0625}}{2} = 6.7595.$$

Finally, an approximate and equal-tail 95% CI of α is given by $[5.7789, 6.7595]$.

(b) The shortest Wilson CI for the parameter λ in a Poisson distribution is constructed as follows. Suppose that we have n random samples $X_1, \ldots, X_n$ from Poisson(λ), and want to construct a $(1-\alpha)100\%$ CI for λ. According to the Central Limit Theorem, we have

$$\frac{\bar{X} - \lambda}{\sqrt{\lambda/n}} \xrightarrow{\text{L}} N(0,1), \qquad \text{as } n \to \infty.$$

Let $\alpha_1 + \alpha_2 = \alpha$ so that $\alpha_2 = \alpha - \alpha_1$. Approximately, we obtain

$$\Pr\left(-z_{\alpha_1} \le \frac{\bar{X} - \lambda}{\sqrt{\lambda/n}} \le z_{\alpha-\alpha_1}\right) = 1 - \alpha.$$

If $-z_{\alpha_1} \le \frac{\bar{X}-\lambda}{\sqrt{\lambda/n}} \le 0$, then $\lambda \ge \bar{X}$ and

$$\bar{X} + \frac{z_{\alpha_1}^2}{2n} - z_{\alpha_1}\sqrt{\frac{\bar{X}}{n} + \frac{z_{\alpha_1}^2}{4n^2}} \le \lambda \le \bar{X} + \frac{z_{\alpha_1}^2}{2n} + z_{\alpha_1}\sqrt{\frac{\bar{X}}{n} + \frac{z_{\alpha_1}^2}{4n^2}}.$$

Taking them together, we have

$$\bar{X} \le \lambda \le \bar{X} + \frac{z_{\alpha_1}^2}{2n} + z_{\alpha_1}\sqrt{\frac{\bar{X}}{n} + \frac{z_{\alpha_1}^2}{4n^2}}.$$

Similarly, if $0 \le \frac{\bar{X}-\lambda}{\sqrt{\lambda/n}} \le z_{\alpha-\alpha_1}$, we have

$$\bar{X} + \frac{z_{\alpha_1}^2}{2n} - z_{\alpha_1}\sqrt{\frac{\bar{X}}{n} + \frac{z_{\alpha_1}^2}{4n^2}} \le \lambda \le \bar{X}.$$

Thus, $-z_{\alpha_1} \le \frac{\bar{X}-\lambda}{\sqrt{\lambda/n}} \le z_{\alpha-\alpha_1}$ if and only if

$$\bar{X} + \frac{z_{\alpha-\alpha_1}^2}{2n} - z_{\alpha-\alpha_1}\sqrt{\frac{\bar{X}}{n} + \frac{z_{\alpha-\alpha_1}^2}{4n^2}} \le \lambda \le \bar{X} + \frac{z_{\alpha_1}^2}{2n} + z_{\alpha_1}\sqrt{\frac{\bar{X}}{n} + \frac{z_{\alpha_1}^2}{4n^2}}.$$

Therefore,

$$\left[\bar{X} + \frac{z_{\alpha-\alpha_1}^2}{2n} - z_{\alpha-\alpha_1}\sqrt{\frac{\bar{X}}{n} + \frac{z_{\alpha-\alpha_1}^2}{4n^2}}, \bar{X} + \frac{z_{\alpha_1}^2}{2n} + z_{\alpha_1}\sqrt{\frac{\bar{X}}{n} + \frac{z_{\alpha_1}^2}{4n^2}}\right]$$

is a $(1-\alpha)100\%$ CI for λ with length

$$l(\alpha_1) = \frac{z_{\alpha_1}^2 - z_{\alpha-\alpha_1}^2}{2n} + z_{\alpha-\alpha_1}\sqrt{\frac{\bar{X}}{n} + \frac{z_{\alpha-\alpha_1}^2}{4n^2}} + z_{\alpha_1}\sqrt{\frac{\bar{X}}{n} + \frac{z_{\alpha_1}^2}{4n^2}}.$$

The grid-point method can be used to find the shortest $l(\alpha_1)$ on $[0, \alpha]$. The corresponding R code is as follows.

```
function (x, alpha, error = 0.00001)
{
# Shortest.Wilson.CI.for.Poisson(x, alpha, error=0.00001)
# ---------------------------------------------------------
# Let X_1, ..., X_n ~iid Poisson(lambda),
# Aim: To find 100(1-alpha)% shortest Wilson CI for lambda,
# Input:
# x = a sequence of sample values,
# alpha = size, usually 0.05.
# error = increment of searching alpha_1, default is 0.00001
# Output:
# CI is a matrix.
```

```
# CI[1, ]: Lower & upper bounds, and the length of
#          the equal-tail CI (i.e. alpha1=alpha/2)
# CI[2, ]: Lower & upper bounds, and the shortest
#          length of the CI for lambda
# ----------------------------------------------------
n <- length(x)
xbar <- sum(x)/n
alpha1 <- seq(0, alpha, error)
z1 <- qnorm(alpha1)
z2 <- qnorm(1-alpha+alpha1)
LB <- xbar + z2^2/2/n - z2*sqrt(xbar/n+z2^2/(4*n*n))
UB <- xbar + z1^2/2/n - z1*sqrt(xbar/n+z1^2/(4*n*n))
length <- UB - LB
item <- order(length)[1]
length.alpha1 <- length(alpha1)
CI <- matrix(0, 3, 4)
CI[1, ] <- c(alpha1[length.alpha1/2+1], LB[length.alpha1/2+1],
UB[length.alpha1/2 + 1], length[length.alpha1/2 + 1])
CI[2, ] <- c(alpha1[item], LB[item], UB[item], length[item])
# -----------------------------------------------------------
Min <- 0
Max <- alpha
alpha_1 <- (Max + Min)/2
while(Max-Min > error){
    z1 <- qnorm(alpha_1)
    z2 <- qnorm(1-alpha+alpha_1)
    a1 <- (xbar/n+z1^2/4/n^2)^0.25
    a2 <- (xbar/n+z2^2/4/n^2)^0.25
    test <- exp(-z1*z1/2)/(a1-z1/(2*n*a1))^2
    test <- test - exp(-z2*z2/2)/(a2-z2/(2*n*a2))^2
```

```
    if(test<=0) Min <- alpha_1 else Max <- alpha_1
    alpha_1 <- (Max + Min)/2
}
z1 <- qnorm(alpha_1)
z2 <- qnorm(1-alpha+alpha_1)
L_B <- xbar + z2^2/2/n - z2*sqrt(xbar/n+z2^2/(4*n*n))
U_B <- xbar + z1^2/2/n - z1*sqrt(xbar/n+z1^2/(4*n*n))
CI[3, ] <- c(alpha_1, L_B, U_B, U_B - L_B)
dimnames(CI) <- list(c("Equal-tail CI: ",
"Shortest CI (Grid-Point): ", "Shortest CI (Bisection): "),
c("alpha1", "Lower.Bound","Upper.Bound", "UB.minus.LB" ))
return (CI)
}
```

4.3 Solution. (a) When $\sigma = \sigma_0$ is known, from (4.4) of Chapter 4 (page 165), we know that

$$\left[\bar{X} - z_{\alpha/2}\frac{\sigma_0}{\sqrt{n}},\ \bar{X} + z_{\alpha/2}\frac{\sigma_0}{\sqrt{n}}\right] = [-2.0925,\ 2.8425]$$

is a $100(1-\alpha)\%$ confidence interval for the mean μ, where $n = 4$, $\alpha = 0.1$, $z_{\alpha/2} = z_{0.05} = 1.645$, $\sigma_0 = 3$, and

$$\bar{X} = \frac{3.3 - 0.3 - 0.6 - 0.9}{4} = 0.375.$$

(b) When σ is unknown, from (4.6) of Chapter 4 (page 167), we know that

$$\left[\bar{X} - t\Big(\frac{\alpha}{2}, n-1\Big)\frac{S}{\sqrt{n}},\ \bar{X} + t\Big(\frac{\alpha}{2}, n-1\Big)\frac{S}{\sqrt{n}}\right] = [-1.937, 2.687]$$

is a $100(1-\alpha)\%$ confidence interval for the mean μ, where $\bar{X} = 0.375$, $n = 4$, $t(\alpha/2, n-1) = t(0.05, 3) = 2.3534$, and

$$S = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}} = \sqrt{3.863} = 1.965.$$

4.4 Solution. Since σ^2 is unknown, from (4.6) of Chapter 4 (page 167), we know that

$$\left[\bar{X} - t\Big(\frac{\alpha}{2}, n-1\Big)\frac{S}{\sqrt{n}},\ \bar{X} + t\Big(\frac{\alpha}{2}, n-1\Big)\frac{S}{\sqrt{n}}\right]$$
$$= \left[\bar{X} - t(0.05, n-1)\frac{S}{\sqrt{n}},\ \bar{X} + t(0.05, n-1)\frac{S}{\sqrt{n}}\right]$$

is a 90% CI for the mean μ. The length of the CI is

$$L = 2t(0.05, n-1)\frac{S}{\sqrt{n}}.$$

Then, we have

$$\begin{aligned}
0.95 &= \Pr(L \leqslant \sigma/5) \\
&= \Pr\left\{2t(0.05, n-1)\frac{S}{\sqrt{n}} \leqslant \frac{\sigma}{5}\right\} \\
&= \Pr\left\{4t^2(0.05, n-1)\frac{S^2}{n} \leqslant \frac{\sigma^2}{25}\right\} \\
&= \Pr\left\{\frac{(n-1)S^2}{\sigma^2} \leqslant \frac{n(n-1)}{100 \times t^2(0.05, n-1)}\right\} \\
&= \Pr\left\{\chi^2(n-1) \leqslant \frac{n(n-1)}{100 \times t^2(0.05, n-1)}\right\}
\end{aligned}$$

or

$$\begin{aligned}
0.05 &= \Pr\left\{\chi^2(n-1) \geqslant \frac{n(n-1)}{100 \times t^2(0.05, n-1)}\right\} \\
&= \Pr\left\{\chi^2(n-1) \geqslant \chi^2(0.05, n-1)\right\}.
\end{aligned}$$

Therefore, the sample size n should satisfy

$$\frac{n(n-1)}{100 \times t^2(0.05, n-1)} = \chi^2(0.05, n-1).$$

When $n = 309.228$, we obtain

$$\left|\frac{n(n-1)}{100 \times t^2(0.05, n-1)} - \chi^2(0.05, n-1)\right| \leqslant 0.00002.$$

Then, $n = 309$.

4.5 Solution. Because

$$\begin{pmatrix} A \\ B \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_A \\ \mu_B \end{pmatrix}, \begin{pmatrix} \sigma_A^2 & \rho\sigma_A\sigma_B \\ \rho\sigma_A\sigma_B & \sigma_B^2 \end{pmatrix} \right),$$

we have

$$D \hat{=} A - B \sim N(\mu_A - \mu_B,\ \sigma^2),$$

where $\sigma^2 \hat{=} \sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B$ is unknown. The objective is to find a 95% CI for $\mu_A - \mu_B$.

Now the random sample of D is: 6, 8, -2, 2, 7, 11, 1, 13. The sample mean $\bar{D} = 5.75$ and

$$S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(D_i - \bar{D})^2} = \sqrt{26.2143} = 5.12.$$

Since σ^2 is unknown, from (4.6) of Chapter 4 (page 167), we know that

$$\begin{aligned}
& \left[\bar{D} - t\Big(\frac{\alpha}{2}, n-1\Big)\frac{S}{\sqrt{n}},\ \bar{D} + t\Big(\frac{\alpha}{2}, n-1\Big)\frac{S}{\sqrt{n}}\right] \\
= & \left[5.75 - t(0.025, 7)\frac{5.12}{\sqrt{8}},\ 5.75 + t(0.025, 7)\frac{5.12}{\sqrt{8}}\right] \\
= & \left[5.75 - 2.3646 \times \frac{5.12}{\sqrt{8}},\ 5.75 + 2.3646 \times \frac{5.12}{\sqrt{8}}\right] \\
= & [1.4696,\ 10.0304].
\end{aligned}$$

is a 95% CI for the difference $\mu_A - \mu_B$.

4.6 Solution. (a) When $f(x;\theta) = \theta x^{\theta-1} \cdot I_{(0,1)}(x)$, we have

$$F(x;\theta) = \int_0^x \theta t^{\theta-1}\mathrm{d}t = x^\theta, \qquad 0 < x < 1.$$

From (4.3) of Chapter 4 on page 164, we have

$$-2\sum_{i=1}^{n}\log(X_i^\theta) = -2\theta\sum_{i=1}^{n}\log(X_i) \sim \chi^2(2n).$$

Thus, $-2\theta\sum_{i=1}^n \log(X_i)$ is a pivotal quantity. A $100(1-\alpha)\%$ equal-tail CI of θ can be constructed based on

$$\begin{aligned}
& 1-\alpha \\
= \;& \Pr\left\{\chi^2(1-\alpha/2, 2n) \leqslant -2\theta\sum_{i=1}^n \log(X_i) \leqslant \chi^2(\alpha/2, 2n)\right\} \\
= \;& \Pr\left\{-\frac{\chi^2(\alpha/2, 2n)}{2\sum_{i=1}^n \log(X_i)} \leqslant \theta \leqslant -\frac{\chi^2(1-\alpha/2, 2n)}{2\sum_{i=1}^n \log(X_i)}\right\}.
\end{aligned}$$

(b) Let $\alpha_1+\alpha_2=\alpha$ so that $\alpha_2=\alpha-\alpha_1$. The $100(1-\alpha)\%$ shortest CI of θ can be constructed based on

$$\begin{aligned}
1-\alpha &= \Pr\left\{\chi^2(1-\alpha_2, 2n) \leqslant -2\theta\sum_{i=1}^n \log(X_i) \leqslant \chi^2(\alpha_1, 2n)\right\} \\
&= \Pr\left\{-\frac{\chi^2(\alpha_1, 2n)}{2\sum_{i=1}^n \log(X_i)} \leqslant \theta \leqslant -\frac{\chi^2(1-\alpha_2, 2n)}{2\sum_{i=1}^n \log(X_i)}\right\}.
\end{aligned}$$

The width of this CI is

$$\begin{aligned}
l(\alpha_1) &= -\frac{\chi^2(1-\alpha_2, 2n)}{2\sum_{i=1}^n \log(X_i)} + \frac{\chi^2(\alpha_1, 2n)}{2\sum_{i=1}^n \log(X_i)} \\
&= \frac{\chi^2(\alpha_1, 2n) - \chi^2(1-\alpha+\alpha_1, 2n)}{2\sum_{i=1}^n \log(X_i)}
\end{aligned}$$

Thus, we can find α_1^* numerically such that

$$l(\alpha_1^*) = \min_{\alpha_1\in[0,\alpha]} l(\alpha_1) \quad \text{or} \quad \alpha_1^* = \arg\min_{\alpha_1\in[0,\alpha]} l(\alpha_1).$$

Therefore, The $100(1-\alpha)\%$ shortest CI of θ is

$$\left[-\frac{\chi^2(\alpha_1^*, 2n)}{2\sum_{i=1}^n \log(X_i)},\; -\frac{\chi^2(1-\alpha+\alpha_1^*, 2n)}{2\sum_{i=1}^n \log(X_i)}\right].$$

4.7 Solution. (a) We know from Example 4.1 that $2\theta n\bar{X}$ is a pivotal quantity, and

$$\begin{aligned}[L_p, U_p] &= \left[\frac{\chi^2(1-\alpha/2, 2n)}{2n\bar{X}}, \frac{\chi^2(\alpha/2, 2n)}{2n\bar{X}}\right] \\ &= \left[\frac{9.591}{20 \times 55.087}, \frac{34.170}{20 \times 55.087}\right] = [0.00871, 0.03101]\end{aligned}$$

is an exact 95% equal-tail CI for θ.

(b) An exact 95% equal-tail CI for $1/\theta$ is

$$\left[\frac{2n\bar{X}}{\chi^2(\alpha/2, 2n)}, \frac{2n\bar{X}}{\chi^2(1-\alpha/2, 2n)}\right] = [32.24766, 114.8106].$$

This interval is obviously quite wide, reflecting substantial variability in breakdown times and a small sample size.