problem 19B.

Show that if an $S(3,6,v)$ exists, then $v \equiv 2$ or $6 \pmod{20}$.

proof: Using the notation of Chap. 19 of Van Lint & Wilson,

$$b_2 = \frac{1 \cdot \binom{v-2}{3-2}}{\binom{6-2}{3-2}} = \frac{v-2}{4} \Rightarrow v \equiv 2 \pmod 4.$$

Now

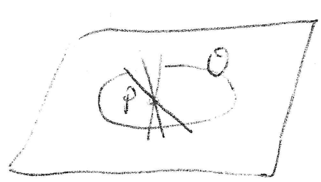
$$b_1 = \frac{1 \cdot \binom{v-1}{3-1}}{\binom{5}{2}} = \frac{(v-1)(v-2)}{5 \cdot 4} \Rightarrow v \equiv 1 \pmod 5 \text{ or } v \equiv 2 \pmod 5$$

Now the Chinese remainder theorem tells us that

$$v \equiv 2 \pmod{20} \text{ or } v \equiv 6 \pmod{20} \qquad \square$$

Problem 19I. Let $\mathcal{O}$ be a subset of the points of a projective plane of order n such that no 3 points of $\mathcal{O}$ are on one line. Show that $|\mathcal{O}| \leq n+1$ if n is odd and that $|\mathcal{O}| \leq n+2$ if n is even.

proof: Let $P \in \mathcal{O}$. Through P there are $(n+1)$ lines. Any point of the projective plane is on one of these lines. Since no 3 points of $\mathcal{O}$ are on a line, we see that each of the $(n+1)$ lines can contribute at most one more point to $\mathcal{O}$. Hence $|\mathcal{O}| \leq n+2$. When n is odd, a parity arguement shows that $|\mathcal{O}| \leq n+1$.



Problem 19E. Let $\mathcal{D}$ be a 2-(v,k,λ) design with b blocks and r blocks through every point. Let B be any block. Show that the number of blocks that meet B is at least

$$k(r-1)^2 / [(k-1)(\lambda-1)+(r-1)].$$

Show that equality holds if & only if any block not disjoint from B meets it in a constant number of pts.

Proof: Let d be the number of blocks which are distinct from but not disjoint from B. Suppose that a_i of these blocks meet B in i points, $0<i\le k$. Counting blocks, we get

$$\sum_{i=1}^{k} a_i = d.$$

Counting pairs, (p, B'), $p \in B \cap B'$, B' is a block, we get

$$\sum_{i=1}^{k} i a_i = k(r-1).$$

Counting triples, (p,q,B'), $p\neq q$, $\{p,q\}\subset B\cap B'$, B' is a block, we get

$$\sum_{i=1}^{k} \binom{i}{2} a_i = \binom{k}{2}(\lambda-1).$$

From the above three equations, we obtain

$$\sum_{i=1}^{k} (i-x)^2 a_i = d x^2 - 2k(r-1)x + k\left((k-1)(\lambda-1)+(r-1)\right).$$

Note that for any $x\in\mathbb{R}$, the left hand side of the above equ. is always nonnegative. Hence the discriminant of the quadratic poly in x in the right hand must be ≤ 0. That is,

$$d \ge \frac{k(r-1)^2}{\left((k-1)(\lambda-1)+(r-1)\right)}.$$

Equality holds $\Leftrightarrow$ the quad. poly. mention above has a unique sol. $x = 1+\frac{(k-1)(\lambda-1)}{r-1}$

$\Leftrightarrow a_i = 0$ for all $i \neq 1+\frac{(k-1)(\lambda-1)}{r-1}$.

4. Let X be a v-set, & let $0 \le t \le k \le v$ be integers. The subset-inclusion matrix $W_{tk}(v)$ is a $(0,1)$-matrix whose rows are indexed by the t-subsets T of X and whose columns are indexed by the k-subsets K of X, with the (T,K) entry being 1 if and only if $T \subseteq K$. Prove that W_{tk} has full rank over $\mathbb{Q}$

proof: First we note that

$$(*) \quad W_{it} W_{tk} = \binom{k-i}{t-i} W_{ik}, \quad \text{for } 0 \le i \le t.$$

This can be proved by noting that the (I,K)-entry of $W_{it} W_{tk}$ is

$$W_{it} W_{tk}(I,K) = \sum_{\substack{T \subset X,\, |T|=t \\ I \subseteq T \subseteq K}} 1 = \binom{k-i}{t-i}, \quad \text{if } I \subseteq K$$

&

$$W_{it} W_{tk}(I,K) = 0 \text{ if } I \not\subseteq K.$$

Secondly, let $\overline{W}_{tk}$ be the $(0,1)$-matrix whose rows are indexed by the t-subsets T of X & whose columns are indexed by the k-subsets K of X, with the (T,K) entry being 1 if & only if $T \cap K = \emptyset$. Then by the PIE we have

$$(**) \quad \overline{W}_{tk} = \sum_{i=0}^{t} (-1)^i W_{it}^T W_{ik}.$$

To see why $(**)$ holds, we compute the (T,K)-entry of the right hand side of $(**)$.

$$\sum_{i=0}^{t} (-1)^i W_{it}^T W_{ik}(T,K) = \sum_{i=0}^{t} (-1)^i \sum_{I,\, |I|=i} W_{it}(I,T)\, W_{ik}(I,K)$$

$$= \sum_{i=0}^{t} (-1)^i \sum_{I \subseteq T \cap K,\, |I|=i} 1$$

$$= \sum_{i=0}^{t} (-1)^i \binom{|T \cap K|}{i} = \begin{cases} 1, & \text{if } T \cap K = \emptyset \\ 0, & \text{otherwise} \end{cases}.$$

Now we can prove the claim of the problem.

Case 1. $k = v - t$.

In this case, W_{tk} is a square matrix, and we can arrange the rows and columns of $\overline{W}_{tk}$ such that $\overline{W}_{tk} = I$. By (*), if

$W_{tk} X = 0$ then $W_{ik} X = 0$ for all $0 \le i \le t$. Now use (**), we

a column vector with entries in $\mathbb{Q}$

have $\overline{W}_{tk} X = 0$; but $\overline{W}_{tk} = I$, we have $X = 0$. This shows that W_{tk} is nonsingular. The claim is proved in this case.

Case 2. $k < v - t$.

By (*), we have $$W_{tk} W_{k,v-t} = \binom{v-2t}{k-t} W_{t,v-t}.$$

Since $W_{t,v-t}$ is nonsingular over $\mathbb{Q}$, we see that W_{tk} has full rank.

Case 3. $k > v - t$.

By (*) again, $$W_{v-k,t} W_{tk} = \binom{2k-v}{k+t-v} W_{v-k,k}.$$

Since $W_{v-k,k}$ is invertible, we see that W_{tk} has full rank.

Another proof by Dom de Caen, the Electronic J. Combin. 2001.

I attached the paper by D. de Caen.

Solutions to HW 3

problem

3. Let $D=(X,\mathcal{B})$ be a t-(v,k,λ) design and let $x_1,x_2,\cdots,x_{t+1}$ be points of X. Suppose that μ blocks contain all these points. Use the PIE to show that the # of blocks containing none of the points $x_1,x_2,\cdots,x_{t+1}$ is $N+(-1)^{t+1}\mu$, where N depends on t, v, k & λ only. Deduce that if t is even and $v=2k+1$, then $(X\cup\{y\},\ \{B\cup\{y\},\ X\setminus B \mid B\in\mathcal{B}\})$ is an extension of D, where $y\notin X$.

Pf: Let $\mathcal{B}_i$ = the set of blocks containing x_i, $i=1,2,\cdots,t+1$. Then by the PIE, the # of blocks containing none of $x_1,x_2,\cdots,x_{t+1}$ is

$$|\overline{\mathcal{B}}_1\cap\overline{\mathcal{B}}_2\cap\cdots\cap\overline{\mathcal{B}}_{t+1}|$$

$$=b-|\mathcal{B}_1\cup\mathcal{B}_2\cup\cdots\cup\mathcal{B}_{t+1}|$$

$$=b-\sum_{i=1}^{t+1}|\mathcal{B}_i|+\sum_{i\neq j}|\mathcal{B}_i\cap\mathcal{B}_j|-\cdots$$

$$=b-\binom{t+1}{1}b_1+\binom{t+1}{2}b_2+\cdots+(-1)^t\binom{t+1}{t}b_t+(-1)^{t+1}\mu,\ \text{where}$$

$$b_i=\lambda\binom{v-i}{t-i}\Big/\binom{k-i}{t-i}\quad\text{for } 1\le i\le t,\ \text{and } b_0=b.$$

Let $N=\sum_{i=0}^{t}\binom{t+1}{i}(-1)^i b_i$. Then N is independent of the choices of the pts $x_1,x_2,\cdots,x_{t+1}$ and depends only on t, v, k, and λ.

Now we show that if t is even, $v = 2k+1$, then

$$D' = \left(X \cup \{y\},\ \{ B \cup \{y\},\ X \setminus B \mid B \in \mathcal{B} \} \right) \text{ is an}$$

extension of D, where $y \notin X$.

First of all, $|B \cup \{y\}| = k+1$ & $|X \setminus B| = (2k+1) - k = k+1$. So all blocks of D' have the same size. We have to show that for all $(t+1)$-subsets I of $X \cup \{y\}$, the # of blocks of D' containing I is a constant. We consider two cases:

(i) $y \in I$. Let $J = I \setminus \{y\}$. Then $|J| = t$. The blocks of D' containing I are exactly $B \cup \{y\}$, $B \in \mathcal{B}$, and $B \supseteq J$. So in this case, the # of blocks of D' containing I is equal to λ.

(ii) $y \notin I$. So I is a $(t+1)$-subset of X. In this case,

the # of blocks of D' containing I

= (the # of blocks of D containing I) + (the # of blocks of D that are totally disjoint from I)

$$\overset{\text{by part (1) \& } t \text{ is even}}{=} \mu + (N - \mu) = N = \sum_{i=0}^{t} \binom{t+1}{i} (-1)^i b_i .$$

It remains to show that

$$\sum_{i=0}^{t} \binom{t+1}{i} (-1)^i b_i = \lambda, \text{ when } v = 2k+1 \text{ and } t \text{ is even}$$

This can be done by using some identity involving binomial coefficients.

$$N = \frac{\lambda}{\binom{v-t}{k-t}} \sum_{i=0}^{t} (-1)^i \binom{t+1}{i}\binom{v-i}{k-i}$$

$$= \frac{\lambda}{\binom{v-t}{k-t}} \left(\sum_{i=0}^{t+1} (-1)^i \binom{t+1}{i}\binom{v-i}{k-i} + \binom{v-(t+1)}{k-(t+1)} \right)$$

Use the identity $\sum_{i=0}^{n} (-1)^i \binom{n}{i}\binom{m+n-i}{k-i} = \binom{m}{k}$ on page 91 of VanLint & Wilson

$$= \frac{\lambda}{\binom{v-t}{k-t}} \left(\binom{v-(t+1)}{k} + \binom{v-(t+1)}{v-k} \right)$$

$$= \frac{\lambda}{\binom{v-t}{k-t}} \left(\binom{v-t-1}{k} + \binom{v-t-1}{k+1} \right) = \lambda .$$

The proof is now complete. $\square$

6. Let G be a simple graph and let A be the adjacency matrix of G. The eigenvalues of A are called the eigenvalues of G. Also we denote the largest vertex degree of G by $\Delta(G)$. Prove that (i) the eigenvalue of G with largest absolute value is $\Delta(G)$ if and only if some connected component of G is $\Delta(G)$-regular. (ii) The multiplicity of $\Delta(G)$ as an eigenvalue of G is the number of $\Delta(G)$-regular components of G.

proof. Let A be the adj matrix of G. The i^{th} entry of $A\mathbb{1}$ is $d(v_i)$. when G is k-regular, we have $A\mathbb{1} = k\mathbb{1}$, and thus k is an eigenvalue with eigenvector $\mathbb{1}$. More generally, let $\mathbf{x}$ be an eigenvector for eigenvalue λ, and let x_j be a coordinate of largest absol. value among coordinates of $\mathbf{x}$ corresponding to the vertices of some component H of G. For the jth coordinate of $A\mathbf{x}$, we have

$$|\lambda|\,|x_j| = |(A\mathbf{x})_j| = \sum_{v_\ell \in N(v_j)} x_\ell \leq d(v_j)\,|x_j| \leq \Delta(G)\,|x_j|.$$

Hence $|\lambda| \leq \Delta(G)$. Equality requires $d(v_j) = \Delta(G)$ and $x_\ell = x_j$ for all $v_\ell \in N(v_j)$. We can iterate this argument to reach all coordinates corresponding to vertices of H. Hence the eigenvalue associated with $\mathbf{x}$ has absol. value as large as $\Delta(G)$ only if H is $\Delta(G)$-regular.

Thus the eigenvalue associated with an eigenvector $\mathbf{x}$ has absolute value as large as $\Delta(G) \Leftrightarrow$ (i) each component of G contains a vertex where $\mathbf{x}$ is nonzero is $\Delta(G)$-regular, and (ii) $\mathbf{x}$ is constant on the coordinates corresponding to each such component. We can choose the constant independent for each $\Delta(G)$-regular component, so the dimension of the space of eigenvectors associated with $\Delta(G)$ = the # of $\Delta(G)$-regular components of G.

$\square$