MAT8010 Homework #6

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1. (19B) Show that if an $\mathcal{S}(3,6,v)$ exists, then $v \equiv 2$ or $6 (\mathrm{mod}\ 20)$.

   Proof:

   We calculate $b_i, i = 3, 2, 1, 0$;

$$\begin{cases} b_0 &= \dfrac{v(v-1)(v-2)}{120} \\ b_1 &= \dfrac{(v-1)(v-2)}{20} \\ b_2 &= \dfrac{(v-2)}{4} \\ b_3 &= 1 \end{cases}$$

   $\Longrightarrow v \equiv 2 \,\mathrm{or}\, 6 (\mathrm{mod}\, 20)$

   (Calculate b_i, $i = 3, 2, 1, 0$; $b_1 \notin \mathbb{Z}$)

2. (19E) Let $\mathcal{D}$ be a $2-(v,k,1)$ design with b blocks and r blocks through every point. Let B be any block. Show that the number of blocks that meet B is at least

$$k(r-1)^2/[()k-1)(\lambda-1)+(r-1]$$

   Show that equality holds if and only if any block not disjoint from B meets it in a constant number of points.

   Proof:

3. Let $\mathcal{S} = (\mathcal{P}, \mathcal{B})$ be a $t-(v,k,\lambda)$ design, and let $x_1, x_2, \ldots, x_{t+1}$ be points of $\mathcal{P}$. Suppose that μ blocks of $\mathcal{S}$ contain all these points. Use the PIE to show that the number of blocks containing none of points $x_1, x_2, \ldots, x_{t+1}$ is $N + (-1)^{t+1}\mu$, where N depends on t, v, k, λ only. Deduce that if t is even and $v = 2k+1$, then $(\mathcal{P} \cup \{y\}, \{\mathcal{B} \cup \{y\}, \mathcal{P} \langle\backslash\rangle \mathcal{B} | B \in \mathcal{B})$ is a $(t+1)-(v+1, k+1, \lambda)$-design.

   Prove:

4. Let X be a v-set, and let $0 \leq t \leqslant k \leqslant v$ be integers. The subset inclusion matrix $W_{tk}(v)$ is a (0, 1)-matrix whose rows are indexed by the t-subsets T of X and whose columns are indexed by the k-subsets K of X, with the (T,K)-entry being 1 if $T \subseteq K$, and 0 otherwise. Prove that $W_{tk}(v)$ has full rand over $\boldsymbol{Q}$; that is $\mathrm{rank}_{\boldsymbol{Q}}(W_{tk}(v)) = \min\{\binom{v}{t}, \binom{v}{k}\}$.

   Prove:

5. Let O be a subset of the points of a projective plane of order n such that no three points of O are on one line. Show that $|O| \leqslant n+1$ if n is odd and that $|O| \leq n+2$ if n is even. A set of $n+1$ points, no three on a line, is called an oval; a set of $n+2$ points, no three on a line, is a hyperoval. Two constructions of $PG_2(4)$ were given in Example 19.7 and Problem 19F. In each case, construct a hyperoval.

   Prove:

6. Let G be simple graph and let A be the adjacency matrix of G. The eigenvalues of A are called the eigenvalues of G. Also we denote the largest vertex degree of G by $\Delta(G)$. Prove that (i) the eigenvalue of G with the largest absolute value is $\Delta(G)$ if and only if some connected component of G is $\Delta(G)$-regular; (ii) The multipicity of $\Delta(G)$ as an eigenvalue of G is the number of $\Delta(G)$-regular components of G.

   Prove:

   (i)

   (ii)