MAT8010 Midterm Exam

You have a total of 115 minutes (from 8am to 9:55am). Show **all** your work. Write your solutions clearly.

1 (10 points). Consider the decimal expansion

$$\frac{1}{9899} = 0.0001010203050813213455\cdots.$$

Why do the Fibonacci numbers $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$ appear?

Solution. Let F_n $(n = 0, 1, 2, \ldots.)$ denote the Fibonacci numbers. That is, $F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_5 = 8, \ldots$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. We know that the generating function of $(F_n)_{n=0}^{\infty}$ is:

$$F_0 + F_1 x + F_2 x^2 + \cdots + F_n x^n + \cdots = \frac{1}{1 - x - x^2}$$

Set $x = 10^{-2}$. We have

$$\frac{10^4}{9899} = F_0 + F_1 \cdot 10^{-2} + F_2 \cdot 10^{-4} + F_3 \cdot 10^{-6} + \cdots$$

Hence

$$\frac{1}{9899} = F_0 \cdot 10^{-4} + F_1 \cdot 10^{-6} + F_2 \cdot 10^{-8} + F_3 \cdot 10^{-10} + \cdots = 0.00010102030508...$$

2. (15 points) State the combinatorial definition of the Stirling numbers $S(n, k)$ of the 2nd kind. Then give formulas for $S(n, 1)$, $S(n, 2)$, $S(n, n)$, $S(n, n-1)$, $S(n, n-2)$ by using the definition you give.

Solution. The Stirling number of the 2nd kind, $S(n, k)$, is the number of partitions of an n-set into k nonempty subsets (blocks). We have $S(n, 1) = 1$, $S(n, 2) = 2^{n-1} - 1$, $S(n, n) = 1$, $S(n, n-1) = \binom{n}{2}$, $S(n, n-2) = \binom{n}{3} + 3\binom{n}{4}$. For the last entry, i.e., $S(n, n-2)$, we give the following explanations: The block sizes of a partition of $[n]$ with $(n-2)$ blocks are either (i) 3 (once) and 1 ($(n-3)$ times); or (ii) 2 (twice) and 1 ($(n-4)$ times). In the first case, there are $\binom{n}{3}$ ways to choose the 3-element block. In the second case, there are $\binom{n}{4}$ ways of choosing the union of the two 2-element blocks, and then 3 ways to choose the blocks themselves.

3. (10 points) Use the expansion of $(1-4x)^{-\frac{1}{2}}$ to prove the following identity

$$\sum_{k=0}^{n} \binom{2k}{k} 2^{2n-2k} = (2n+1)\binom{2n}{n}.$$

Solution. We form the generating function $f(x) = \sum_{k=0}^{n} \binom{2k}{k} 2^{2n-2k} x^n$. Now by changing the order of summation, we have

$$\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \left(\binom{2k}{k} 2^{-2k} \cdot \sum_{n \geq k}^{\infty} (4x)^n \right) \\
&= \sum_{k=0}^{\infty} \binom{2k}{k} 2^{-2k} \cdot \frac{(4x)^k}{(1-4x)} \\
&= \frac{1}{(1-4x)} \sum_{k=0}^{\infty} \binom{2k}{k} x^k \\
&= (1-4x)^{-3/2}
\end{aligned}$$

It follows that $\sum_{k=0}^{n} \binom{2k}{k} 2^{2n-2k} = (-4)^n \binom{-3/2}{n} = (2n+1)\binom{2n}{n}$.

4. (10 points) Prove the following identity

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} \binom{m+k}{2n} = \binom{2m}{2n}$$

Solution. We have

$$(1-x^2)^{-(2n+1)} \cdot (1+x)^{2n+1} = (1-x)^{-(2n+1)} \tag{$*$}$$

Recall that $(1-x)^{-a-1} = \sum_{j=0}^{\infty} \binom{a+j}{j} x^j$. Comparing the coefficients of x^{2m-2n} on both sides of (*), we have

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} \binom{m+k}{2n} = \binom{2m}{2n}$$

5. (20 points) Let p be a prime, and let $\mathbf{F}_p$ be a field of size p (e.g., $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$). So every nonzero element of $\mathbf{F}_p$ has a multiplicative inverse. Let $n \geq 1$, and consider the following equation $\sum_{i=1}^{n} x_i = b$, where $b \in \mathbf{F}_p$.

(i) Prove that the total number of solutions $(x_1, x_2, \ldots, x_n) \in \mathbf{F}_p^n$ (that is, each x_i is in $\mathbf{F}_p$) of the above equation is p^{n-1}.

Solution. For each $(n-1)$-tuple $(x_1, x_2, \ldots, x_{n-1}) \in \mathbf{F}_p^{n-1}$, there is a unique $x_n = b - x_1 - x_2 - \cdots - x_{n-1}$ in $\mathbf{F}_p$ satisfying the equation $\sum_{i=1}^n x_i = b$. Hence there are p^{n-1} solutions to the equation $\sum_{i=1}^n x_i = b$.

(ii) Prove that the number of solutions of the equation in $(\mathbf{F}_p \setminus \{0\})^n$, i.e., solutions $(x_1, x_2, \ldots, x_n)$ with $x_i \in \mathbf{F}_p$ and $x_i \neq 0$ for all i, is

$$\frac{1}{p}\left((p-1)^n + (-1)^n(p-1)\right), \quad \text{if} \quad b = 0; \quad \text{and}$$

$$\frac{1}{p}\left((p-1)^n + (-1)^{n+1}\right), \quad \text{if} \quad b \neq 0.$$

Solution. We first note that for any nonzero b in $\mathbf{F}_p$,

$$\begin{aligned}&|\{(x_1, x_2, \ldots, x_n) \in (\mathbf{F}_p \setminus \{0\})^n \mid x_1 + x_2 + \cdots + x_n = b\}| = \\ &\quad |\{(x_1, x_2, \ldots, x_n) \in (\mathbf{F}_p \setminus \{0\})^n \mid x_1 + x_2 + \cdots + x_n = 1\}|.\end{aligned}$$

The reason is that there is a bijection $(x_1, x_2, \ldots, x_n) \mapsto (b^{-1}x_1, b^{-1}x_2, \ldots, b^{-1}x_n)$ between the set on the left hand side to the set on the right hand side of the above equality.

Define $a_n = |\{(x_1, x_2, \ldots, x_n) \in (\mathbf{F}_p \setminus \{0\})^n \mid x_1 + x_2 + \cdots + x_n = 0\}|$, and $b_n = |\{(x_1, x_2, \ldots, x_n) \in (\mathbf{F}_p \setminus \{0\})^n \mid x_1 + x_2 + \cdots + x_n = 1\}|$. By the above observation, we have $a_n + b_n(p-1) = (p-1)^n$. Also we have $a_n = (p-1)b_{n-1}$ (why? figure this out). So we have

$$b_n + b_{n-1} = (p-1)^{n-1}.$$

Solving this recurrence, we obtain

$$b_n = (p-1)^{n-1} - (p-1)^{n-2} + (p-1)^{n-3} - \cdots = \frac{1}{p}((p-1)^n + (-1)^{n+1})$$

and $a_n = (p-1)b_{n-1} = \frac{1}{p}\left((p-1)^n + (-1)^n(p-1)\right)$ for $n \geq 1$.

The problem can also be solved by using the PIE.

6. (20 points) Let p be a prime, and let $n = \sum a_i p^i$ and $m = \sum b_i p^i$ be the p-ary expansions of the positive integers m and n, respectively. (So here a_i and b_i are the base-p digits of n and m, respectively, and $0 \leq a_i \leq p-1$ and $0 \leq b_i \leq p-1$ for all i.)

(a). Show that

$$\binom{n}{m} \equiv \binom{a_0}{b_0}\binom{a_1}{b_1}\cdots \pmod p$$

Solution. First note that p divides $\binom{p}{i}$ for all $1 \le i \le p-1$. A consequence of this fact is

$$(1+x)^p \equiv 1 + x^p \pmod p$$

Now we compute

$$\begin{aligned}
(1+x)^n &= (1+x)^{\sum a_i p^i} \\
&\equiv \prod_i (1+x^{p^i})^{a_i} \pmod p \\
&= \prod_i \sum_{j=0}^{a_i} \binom{a_i}{j} x^{jp^i}
\end{aligned}$$

That is,

$$(1+x)^n \equiv \prod_i \sum_{j=0}^{a_i} \binom{a_i}{j} x^{jp^i} \pmod p$$

The coefficient of x^m on the left is $\binom{n}{m}$ and on the right is $\binom{a_0}{b_0}\binom{a_1}{b_1}\cdots$. The result now follows. [This result is called Lucas' theorem, which is very useful.]

(b). Use (a) to determine when $\binom{n}{m}$ is odd. For what positive integers n are $\binom{n}{m}$ odd for all $0 \le m \le n$.

Solution. Write $n = \sum_i a_i 2^i$ and $m = \sum_i b_i 2^i$, where $a_i, b_i \in \{0,1\}$. By Lucas' theorem, we have

$$\binom{n}{m} \equiv \binom{a_0}{b_0}\binom{a_1}{b_1}\cdots \pmod 2$$

Hence $\binom{n}{m} \equiv 1 \pmod 2$ if and only if $\binom{a_0}{b_0}\binom{a_1}{b_1}\cdots = 1$ if and only if $a_i \ge b_i$ for all i (that is, the binary expansion of m is "contained" in the binary expansion of n).

Using what we just proved, we obtain: $\binom{n}{m}$ is odd for all m, $0 \le m \le n$ if and only if $n = 2^k - 1 = 111\cdots 1$ for some positive integer k.

7. (15 points) (a) A graphic sequence is a list of nonnegative numbers that is the degree sequence of some **simple** graph. Which of the following are graphic sequences? Provide a construction or a proof of impossibility for each.

(i) $(5, 5, 4, 3, 2, 2, 2, 1)$; One can show that his sequence is graphic by using the Havel-Hakimi theorem.

(ii) $(5, 5, 5, 3, 2, 2, 1, 1)$. One can show that this sequence is graphic by using the Havel-Hakimi theorem.

(b) For each positive integer k, let

$$n_k = k!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{k!}\right) + 1.$$

We color all edges of the complete graph K_{n_k} with one of k colors. Prove that there will be a triangle with monochromatic edges.

Solution. We prove the result by using induction on k.

When $k = 1$, we have $n_1 = 3$. We color the edges of K_3 with one color; of course we get a triangle with monochromatic edges.

Let v be a vertex of $K_{n_{k+1}}$. The degree of v is $n_{k+1} - 1 > (k+1)n_k$. When we color all edges of the complete graph $K_{n_{k+1}}$ with one of $k+1$ colors, by the pigeonhole principle, there are n_k edges incident with v which receive the same color, say color 1. Assume that these n_k edges are vu_1, $vu_2, \ldots, vu_{n_k}$. If one of the edges u_iu_j $(i \neq j)$ receives color 1, we have found a monochromatics triangle. If none of the edges u_iu_j $(i \neq j)$ receives color 1, then we are coloring the edges of a K_{n_k} with k colors; and the result follows by using induction hypothesis.