CS 388C: COMBINATORICS AND GRAPH THEORY
Lecture 20

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We first recall Fisher's inequality which we saw in the previous class and use similar techniques to solve some problems.

Theorem 1 (Fisher's inequality)**.** *$A_1, \ldots, A_m$ be distinct subsets of $\{1, \ldots, n\}$ such that $|A_i \cap A_j| = k$ for some fixed $1 \leq k \leq n$ and every $i \neq j$. Then $m \leq n$.*

1 Even Town

Consider the following problem. A town has n citizens and they form clubs such that

1. Every club has an even number of members
2. Every club shares an even number of members
3. No two clubs have identical membership

We first prove that at least $2^{\frac{n}{2}}$ clubs can be formed which satisfy the above conditions. To see this, let the citizens be numbered $1, \ldots, n$. Consider the collection of all possible unions of the sets $\{1,2\}, \{3,4\}, \ldots, \{n-1, n\}$. These $2^{\frac{n}{2}}$ sets can be seen to satisfy the conditions of even size and even intersection (with each set corresponding to a club). We thus have the following theorem.

Theorem 2. *Under the Even town rules, with n citizens, it is possible to form $m \geq 2^{\frac{n}{2}}$ clubs.*

We now prove an upper bound on the size of the collection.

Theorem 3. *Under the Even town rules, with n citizens, $m \leq 2^{\frac{n}{2}}$, where m is the number of clubs formed.*

Proof. We prove this by contradiction. Suppose if possible there is such a collection of sets C of size more than $2^{\frac{n}{2}}$. We associate an incidence vector in $\mathbb{F}_2^n = \{0,1\}^n$ for each set in the collection and let the corresponding collection of vectors V be $v_1, \ldots, v_m$ ($m > 2^{\frac{n}{2}}$). We also define the inner product $\langle v, u \rangle = \sum_{i=1}^{n} v(i)u(i) \bmod (2)$. Thus any v_i satisfies $\langle v_i, u \rangle = 0$ for $u \in V$ and hence can be seen linear constraints on v_i. Now since V is of size more that $2^{\frac{n}{2}}$, there are at least $\frac{n}{2} + 1$ linearly independent vectors in V. Since each linearly independent vector v' imposes a linearly independent constraint $\langle v, v' \rangle = 0$ on any $v \in V$ it forces V to be confined in some subspace of $\mathbb{F}_2^n$ of dimension at most $\frac{n}{2} - 1$. This is a contradiction since the collection contains at least $\frac{n}{2} + 1$ linearly independent vectors which proves the upper bound of $2^{\frac{n}{2}}$ on the size of the collection. $\square$

2 Odd Town

We look at a close variant of the Even town problem with the following rules.

1. Every club has an odd number of members.

2. The intersection of any two clubs is even

Note that the rule of uniqueness is guaranteed by the rules above.

Theorem 4. *Under the Odd town rules, with n citizens $m \leq n$ where m is the number of clubs formed.*

Proof. As in the upper bound proof of the Even town problem, we associate incidence vectors corresponding to each set in the collection C and use the same definition of inner product. We observe that for any v_i in the collection, $\langle v_i, v_i \rangle = 1$ and $\langle v_i, v_j \rangle = 0$ if $i \neq j$. Consider the matrix $A_{m \times n}$ over $\mathbb{F}_2$, with each row as the incidence vector of a set in the collection C. Now $AA^T = I_{m \times m}$ and hence $rank(A) = m$ (using the following facts from linear algebra : (1) $rank(A) = rank(A^T)$, (2) $rank(AB) \leq min\{rank(A), rank(B)\}$). But since $rank(A) \leq min\{m, n\}$, it follows that $m \leq n$. □

There are many other versions known for this problem.

3 Intersecting Families

Definition 1 (Intersecting family)**.** A family $\mathcal{F} = A_1, \ldots, A_m$ is called intersecting if $A_i \cap A_j \neq \emptyset$, $\forall i \neq j$.

Definition 2 (k-uniform Intersecting family)**.** An intersecting family $\mathcal{F} = A_1, \ldots, A_m$ with each $|A_i| = k$.

Note 1. *We can assume $n \geq 2k$ since otherwise any two k sized subsets will intersect and hence $|\mathcal{F}|$ can be $\binom{n}{k}$.*

Note 2. *For $n \geq 2k$, we can form an intersecting family of size $\binom{n-1}{k-1}$ by fixing one common element. The natural question to ask is if we can do better. The following theorem proves that this in fact is the optimal sized family.*

Theorem 5 (Erdos-Ko-Rado)**.** *If $n \geq 2k$, then a k-uniform intersecting family $\mathcal{F}$ has at most $\binom{n-1}{k-1}$ sets.*

In order to prove this theorem we first prove a simple lemma.

Lemma 1. *Consider a cycle of length n and let $n \geq 2k$. Consider paths P_i starting at i and of length k. Let $\mathcal{H}$ be a family of paths of length k such that any 2 intersect. Then $|\mathcal{H}| \leq k$.*

Proof. Consider any P_0 which is already in $\mathcal{H}$. There are exactly $(2k-2)$ P_i's which intersect P_0. To see this observe that $P_{-(k-1)}, \ldots, P_{-1}, P_1, \ldots, P_{k-1}$ (where the subscripts are taken mod(n)) are distinct and intersect P_0. Further from each pair P_i and P_{i-k}, $1 \leq i \leq k-1$, at most one can appear in $\mathcal{H}$. This proves our claim. □

Now, we provide a proof to Theorem (5).

Proof. Suppose we have a k-intersecting family $\mathcal{F}$. Now, consider a party of n guests, seated on a round table. Suppose there are $\binom{n}{k}$ clubs among them *i.e.* all possible clubs of size k. All those clubs which are in $\mathcal{F}$ are colored Red. Clearly, any two red clubs have a common member.

A club is said to be *honored* if the whole club sits together.

For any fixed seating, exactly n clubs are honored. Now, using Lemma 1, for any fixed seating, we know that atmost k Red clubs can be honored.

There are a total of $n!$ different seating arrangements. Suppose, $n!$ different parties are organised. Then, by symmetry, any club is honored the same number of times overall. Let, $R :=$ the number of times that any club is honored. Then,

$$R = \frac{n!n}{\binom{n}{k}}$$

Counting the number of times any Red club is honored (= R), we get

$$R \leq \frac{n!k}{|\mathcal{F}|}$$

It follows that

$$|\mathcal{F}| \leq \frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$$

□