Combinatorics

Homework #3

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1. Find a closed form expression for each of the following generating functions:

a)

$$\begin{aligned}
\sum_{n=0}^{\infty}(n+2)^2x^n &= \sum_{n=0}^{\infty}(n+1)(n+2)x^n+\sum_{n=0}^{\infty}(n+2)x^n\\
&= \sum_{n=0}^{\infty}(n+1)(n+2)x^n+\sum_{n=0}^{\infty}(n+1)x^n+\sum_{n=0}^{\infty}x^n\\
&= \sum_{n=0}^{\infty}(x^{n+2})''+\sum_{n=0}^{\infty}(x^{n+1})'+\sum_{n=0}^{\infty}x^n\\
&= (\sum_{n=0}^{\infty}x^n\cdot x^2)''+(\sum_{n=0}^{\infty}x^n\cdot x)'+3\sum_{n=0}^{\infty}x^n\\
&= \left(\frac{x^2}{1-x}\right)''+\left(\frac{x}{1-x}\right)'+\frac{1}{1-x}\\
&= \frac{-x^2+x+2}{(1-x)^2}
\end{aligned}$$

b)

$$\begin{aligned}
\sum_{n=0}^{\infty}(n+2)^2\frac{x^n}{n!} &= \sum_{n=0}^{\infty}(n+1)(n+2)\frac{x^n}{n!}+\sum_{n=0}^{\infty}(n+2)\frac{x^n}{n!}\\
&= \sum_{n=0}^{\infty}(n+1)(n+2)\frac{x^n}{n!}+\sum_{n=0}^{\infty}(n+1)\frac{x^n}{n!}+\sum_{n=0}^{\infty}\frac{x^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\frac{x^{n+2}}{n!}\right)''+\sum_{n=0}^{\infty}\left(\frac{x^{n+1}}{n!}\right)'+\sum_{n=0}^{\infty}\frac{x^n}{n!}\\
&= \left(\sum_{n=0}^{\infty}\frac{x^{n+2}}{n!}\right)''+\left(\sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}\right)'+\sum_{n=0}^{\infty}\frac{x^n}{n!}\\
&= (x^2e^x)''+(xe^x)'+e^x\\
&= (-x^2+3x+2)e^x
\end{aligned}$$

c)

$$\begin{aligned}
\sum_{n=0}^{\infty}(n+2)^2\binom{2n}{n}x^n &= \sum_{n=0}^{\infty}(n+1)(n+2)\binom{2n}{n}x^n+\sum_{n=0}^{\infty}(n+1)\binom{2n}{n}x^n+\sum_{n=0}^{\infty}\binom{2n}{n}x^n\\
&=
\end{aligned}$$

2. Problem 14K. Consider walks in the X-Y plane where each step is R : (x, y) $\rightarrow$ (x+1, y) or U : (x, y) $\rightarrow$ (x, y +1). We start at (0,0) and ask in how many ways we can reach (2n, 2n) without passing through one of the points (2i − 1, 2i − 1) , i = 1, . . . , n. Prove that this number is the Catalan number u_{2n+1}.

3. Let n be a positive integer, q a prime power, and let $f_2(n, q)$=the number of co-prime pairs of monic polynomials of degree n over GF(q). Find a simple formula for $f_2(n, q)$.

4. An L-tile is a 2 × 2 square with the upper right 1 × 1 square removed; no rotations are allowed. Let an be the number of tilings of a 4 × n rectangle using tiles that are either 1 × 1 squares or L-tiles. Find a closed form for the generating function $1+a_1x+a_2x^2+a_3x^3+\cdots$ (i.e., write the generating function as a rational function).

5. For positive integers m and n, let $f(m,n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that

$$|x_1| + |x_2| + \cdots + |x_n| \leqslant m.$$

a) Find a recurrence formula for $f(m,n)$.

b) Use the recurrence formula to prove that $f(m,n) = f(n,m)$.

c) Do part (b) by using the generating function method.

d) Find a closed formula for $\sum_{n \geqslant 0} f(n,n) x^n$