MAT8010 Homework #1

Due Date: March 6, 2018

1. (5 points) Find as simple a solution as possible for each of the following problems. (I will not give partial credits for any of the subproblems of this problem.)

(a). How many subsets of the set $[10] = \{1, 2, \ldots, 10\}$ contain at least one odd integer? Answer: $2^{10} - 2^5 = 992$.

(b). Ten people split up into five groups of two each. In how many ways can this be done? Answer: $10!/(5!(2!)^5) = 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9$.

(c). In how many different ways can the letters of the word MISSISSIPPI be arranged if the four S's cannot appear consecutively? Answer: $11!/(4!4!2!) - 8!/(4!2!) = 33810$.

(d). How many sequences $(a_1, a_2, \ldots, a_{12})$ are there consisting of 4 0's and 8 1's, if no two consecutive terms are both 0's? Answer: $\binom{9}{4} = 126$

(e). How many functions $f : [5] \to [5]$ are at most 2-to-1 (i.e., $|f^{-1}(n)| \le 2$ for every $n \in [5]$)? Answer: $5! + \binom{5}{2}(5)_4 + \frac{1}{2}\binom{5}{1}\binom{4}{2}(5)_3 = 2220$.

2. (4 points) Prove the following identities (by a combinatorial argument or a generating function argument):

(a). $\sum\limits_{i=0}^{k} \binom{n+i}{i} = \binom{n+k+1}{k}$

Proof 1. Let $X = \{1, 2, 3, \ldots, n,\ a_1, a_2, \ldots, a_{k+1}\}$, where $a_1, a_2, \ldots, a_{k+1}$ are $(k+1)$ letters. The total number of k–subsets of X is $\binom{n+k+1}{k}$. We may classify the k–subsets of X as follows:

(1) The k–subsets of X which don't contain a_1; $\# = \binom{n+k}{k}$

(2) The k–subsets of X which contain a_1, but don't contain a_2; $\# = \binom{n+k-1}{k-1}$

(3) The k–subsets of X which contain a_1, a_2, but don't contain a_3; $\# = \binom{n+k-2}{k-2}$

$\vdots$

$(k+1)$ The k–subsets of X which contain $a_1, a_2, \ldots, a_k$; but don't contain a_{k+1}; $\# = \binom{n}{0}$.

So $\sum\limits_{i=0}^{k} \binom{n+i}{i} = \binom{n+k+1}{k}$.

Proof 2. For $i = 0, 1, \ldots, k$, we have $\binom{n+i}{i} = \binom{n+1+i-1}{i} = \#$ solutions to $x_1+x_2+\cdots+x_{n+1} = i$, where each x_j is a nonnegative integer. So $\sum\limits_{i=0}^{k} \binom{n+i}{i} = \#$ solutions to

$$x_1 + x_2 + \cdots + x_{n+1} \leq k,$$

where each x_j is a nonnegative integer. Given a set of solutions $x_1, x_2, \ldots, x_{n+1}$ to the inequality $x_1 + x_2 + \cdots + x_{n+1} \leq k$, we let $x_{n+2} = k - (x_1 + \cdots + x_{n+1})$, then we get a set of solutions for the equation

$$x_1 + x_2 + \cdots + x_{n+2} = k\,, \quad x_j \geq 0\,.$$

This process can be reversed. So the # of solutions to the inequality $x_1+x_2+\cdots+x_{n+1} \leq k$, x_j nonnegative integers, is equal to the # of solutions of $x_1 + x_2 + \cdots + x_{n+2} = k$, $x_j \geq 0$; which is $\binom{n+2+k-1}{k} = \binom{n+k+1}{k}$. Hence $\sum\limits_{i=0}^{k} \binom{n+i}{i} = \binom{n+k+1}{k}$.

(b). $\sum\limits_{k=0}^{n} (-1)^k \binom{n}{k}^2 = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ (-1)^m \binom{2m}{m}, & \text{if } n = 2m \end{cases}$

Proof: $(1+x)^n(1-x)^n = (1-x^2)^n$. Comparing the coefficients of x^n on both sides, we get

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \cdot \binom{n}{n-k} = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^m \binom{2m}{m}, & \text{if } n = 2m \end{cases}$$

3. (2 points) Fix $1 \leq k \leq n$. How many integer sequences $1 \leq a_1 < a_2 < \cdots < a_k \leq n$ satisfy $a_i \equiv i \pmod 2$ for all i?

Solution. Gievna sequence $1 \leq a_1 < a_2 < \cdots < a_k \leq n$ satisfying $a_i \equiv i \pmod 2$ for all i, we let $b_i = a_i - i + 1$ for all i. Then

$$1 \leq b_1 \leq b_2 \leq \cdots \leq b_k \leq n - k + 1,$$

and $b_i \equiv 1 \pmod 2$ for all i. Note that given the b_i's satisfying these conditions, we can set $a_i = b_i + i - 1$, for all i, and these a_i's will satisfy the conditions stated in the problem. Hence the problem becomes counting the sequences $1 \leq b_1 \leq b_2 \leq \cdots \leq b_k \leq n - k + 1$, where each b_i is odd. Setting $m = \lceil (n-k+1)/2 \rceil$ (which is the number of odd integers among $1, 2, 3, \ldots, n - k + 1$), we see that the answer to the problem is $\binom{m+k-1}{k}$.

4. (2 points) Let $S(n,k)$ denote a Stirling number of the 2nd kind. Prove that

$$\sum_n S(n,k)x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)} \tag{1}$$

Deduce from (1) that

$$S(n,k) = \sum 1^{a_1-1}2^{a_2-1}\cdots k^{a_k-1}, \tag{2}$$

where the sum is over all compositions $(a_1, a_2, \ldots, a_k)$ of n. (If you are ambitious, try to give a combinatorial proof for (2).)

Proof. Set $F_k(x) = \sum_n S(n,k)x^n$. Using the recurrence relation on $S(n,k)$ that we proved in class, we obtain

$$F_k(x) = kxF_k(x) + xF_{k-1}(x).$$

So

$$F_k(x) = \frac{x}{1-kx}F_{k-1}(x) = \frac{x}{(1-kx)}\frac{x}{(1-(k-1)x)}F_{k-2}(x) = \cdots = \frac{x^k}{(1-kx)(1-(k-1)x)\cdots(1-x)}$$

From the above we see that

$$\sum_n S(n,k)x^n = x^k(1+x+x^2+\cdots)(1+2x+(2x)^2+\cdots)\cdots(1+kx+(kx)^2+\cdots)$$

Comparing the coefficients of x^n on both sides, we get

$$S(n,k) = \sum 1^{a_1-1}2^{a_2-1}\cdots k^{a_k-1},$$

where the sum is over all compositions $(a_1, a_2, \ldots, a_k)$ of n.