

Extremal Graphs Without 4-Cycles

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Abstract

We prove an upper bound for the number of edges a C_4 -free graph on $q^2 + q$ vertices can contain for q even. This upper bound is achieved whenever there is an orthogonal polarity graph of a plane of even order q .

Let n be a positive integer and G a graph. We define $ex(n, G)$ to be the largest number of edges possible in a graph on n vertices that does not contain G as a subgraph; we call a graph on n vertices *extremal* if it has $ex(n, G)$ edges and does not contain G as a subgraph. $EX(n, G)$ is the set of all extremal G -free graphs on n vertices.

The problem of determining $ex(n, G)$ (and $EX(n, G)$) for general n and G belongs to an area of graph theory called *extremal graph theory*. Extremal graph theory officially began with Turán's theorem that solves $EX(n, K_m)$ for all n and m , a result that is striking in its precision. In general, however, exact results for $ex(n, G)$ (and especially $EX(n, G)$) are very rare; most results are upper or lower bounds and asymptotic results. For many bipartite G there is a large gap between upper and lower bounds.

The question of $ex(n, C_4)$ (where C_4 is a cycle of length 4) has an interesting history; Erdős originally posed the problem in 1938, and the bipartite version of this problem was solved by Reiman using a construction derived from the projective plane (see [3] and the references therein for a more detailed history). Reiman also determined the upper bound $ex(n, C_4) \leq \frac{n}{4}(1 + \sqrt{4n - 3})$ for general graphs, but this is known not to be sharp [11]. Erdős, Rényi, and Sós later showed that this is asymptotically correct using a construction known as the *Erdős-Rényi graph* derived from the orthogonal polarity graph of the classical projective plane [5] [6]. This is part of a more general family of graphs which we define below.

Let π be a finite projective plane with point set P and line set L . A polarity ϕ of π is an involutory permutation of $P \cup L$ which maps points to lines and lines to points and reverses containment. We call points absolute when they are contained in their own polar image. A polarity is called orthogonal if there are exactly $q + 1$ absolute points. We define the polarity graph of π to be the graph with vertex set P , with two distinct vertices x, y adjacent whenever $x \in \phi(y)$. The graph is called an orthogonal polarity graph if the polarity is orthogonal. This graph is C_4 -free, has q^2 vertices of degree $q + 1$, and $q + 1$ of degree q , for a total of $\frac{1}{2}q(q + 1)^2$ edges.

Füredi determined the first exact result that encompasses infinitely many n , namely that for $q > 13$ we have $ex(q^2 + q + 1, C_4) \leq \frac{1}{2}q(q + 1)^2$ [7] [8], with equality if and only if the graph is an orthogonal polarity graph of a plane of order q . In particular, this shows $ex(q^2 + q + 1, C_4) = \frac{1}{2}q(q + 1)^2$ for all prime powers q .

The question of finding $ex(n, C_4)$ exactly for general n appears to be a difficult problem. Computer searches by Clapham et al. [4] and Yuansheng and Rowlinson [12] determined $EX(n, C_4)$ for all $n \leq 31$. More general lower bounds are given in [1] by deleting carefully chosen vertices from the Erdős-Rényi graph. It is not known if any of these bounds are sharp in general. In particular it is not even known whether deleting a single vertex of degree q from an orthogonal polarity graph yields a graph which is still extremal, a question posed by Lazebnik in 2003 [10]. More generally, is $ex(q^2 + q, C_4) \leq \frac{1}{2}q(q + 1)^2 - q$? In this paper we will prove the following theorem:

Theorem 1. *For q even, $ex(q^2 + q, C_4) \leq \frac{1}{2}q(q + 1)^2 - q$.*

It follows that equality holds for all q which are powers of 2.

The question of determining $EX(q^2 + q, C_4)$ in this case is subtler; the searches referred to above showed that there are multiple constructions that achieve the bound for $q = 2, 3$, but for $q = 4, 5$ there is only one. In a subsequent paper, we will prove the following:

Theorem 2. *For all but finitely many even q , any C_4 -free graph with $ex(q^2 + q, C_4)$ edges is derived from an orthogonal polarity graph by removing a vertex of minimum degree.*

The proof of this result is much more lengthy and complicated than that of the inequality in Theorem 1, and requires q to be sufficiently large. The purpose of this paper is to give a simpler proof of the inequality and show it holds for all even q . We start with some notation.

We let X_k be the set of vertices of degree k , $X_{\leq k}$ be the set of vertices of degree at most k , E_0 be $\frac{1}{2}q(q+1)^2 - q$, and n be the number of vertices ($q^2 + q$). We will use $\Gamma(x)$ to represent the vertices in the neighborhood of x . For our various lemmas, we will specify in each case whether q is an even number or simply a positive integer; however, in all cases we consider $q \geq 6$. (We know from [4] and [12] that the inequality in Theorem 1 is true for $q \leq 5$.)

In general, we proceed indirectly. We will show that no C_4 -free graph with $E_0 + 1$ edges can exist, from which we conclude that a graph cannot have more than that number of edges (as it would contain an impossible subgraph). We will use and generalize the techniques found in [7], [8], and [9].

Lemma 1. *Let q be a natural number greater than 2 and let G be a C_4 -free graph on $q^2 + q$ vertices with at least E_0 edges. Then the maximum degree of a vertex in G is at most $q + 2$.*

PROOF. Let u be a vertex of G of maximum degree d . Let e be the number of edges of G , $e \geq \frac{1}{2}q(q+1)^2 - q$. We proceed by bounding the number of 2-paths in G which have no endpoints in $\Gamma(u)$. This gives us:

$$\binom{n-d}{2} \geq \sum_{v \neq u} \binom{d(v)-1}{2}$$

Using Jensen's inequality for the function $f(x) = \binom{x}{2}$ we have:

$$\sum_{v \neq u} \binom{d(v)-1}{2} \geq (n-1) \binom{(2e - (n-1) - d)/(n-1)}{2}$$

Multiplying by $2(n-1)(q+1)$ and simplifying yields:

$$(q+1)(n-1)(n-d)(n-d-1) \geq (q+1)(2e - n - d + 1)(2e - 2n - d + 2) \quad (1)$$

However, we also have:

$$\begin{aligned} (q+1)(2e - 2n - d + 2) - (n-1)(n-d-1) &\geq (q^2 - 2)d - q^3 - 2q^2 + q + 1 \\ &\geq (q^2 - 2)(q+3) - q^3 - 3q^2 + 1 \geq q^2 - q - 5 \end{aligned}$$

with $q^2 - q - 5 > 0$ for $q > 2$,

which gives us:

$$(q+1)(2e - 2n - d + 2) > (n-1)(n-d). \quad (2)$$

We also have the inequality:

$$\begin{aligned} (2e - n - d + 1) - (q+1)(n-d) &\geq -q^2 - 3q + 1 + qd \\ &\geq -q^2 - 3q + 1 + q(q+3) = 1 > 0, \end{aligned}$$

which demonstrates that:

$$(2e - n - d + 1) > (q + 1)(n - d - 1). \quad (3)$$

Therefore the product of (2) and (3) contradict (1), and the theorem follows.

Since we now have an upper bound on the maximum degree of an extremal graph, we focus on the lower bound.

Theorem 3. *Let q be an even number and let G be a C_4 -free graph on $q^2 + q$ vertices, with maximum degree $q + 1$ or less. Then, if e denotes the number of edges of G , we have $e \leq E_0$. Furthermore, if equality holds, then the degree sequence of G must be one of the following (where z is a parameter):*

$ X_{q+1} $	$ X_q $	$ X_{q-1} $	$ X_{q-2} $
$q^2 - q + z$	$2q - 2z$	z	0
$q^2 - q + z + 1$	$2q - 2z - 1$	$z - 1$	1

PROOF. If the maximum degree of G is q or less, we have $e \leq \frac{1}{2}q(q^2 + q) \leq E_0$ and the theorem is immediate. Therefore, we take G to have maximum degree $q + 1$. It is clear that:

$$|X_{q+1}| + |X_q| + |X_{\leq q-1}| = q^2 + q$$

Noting that the degree sum of all the vertices of a graph is equal to $2e$, we have:

$$2e \leq (q + 1)|X_{q+1}| + q|X_q| + (q - 1)|X_{\leq q-1}| = q(q^2 + q) + |X_{q+1}| - |X_{\leq q-1}| \quad (4)$$

Let v be a vertex of degree $q + 1$. We wish to bound the average degree of $\Gamma(v)$. We will do this by noting that the collection $C(v)$ of vertices distance 2 or less from v must naturally be less than $q^2 + q$. We then show that $|\bigcup_{w \in \Gamma(v)} \Gamma(w)| + 1 \leq |C(v)|$.

Each vertex in $\Gamma(v)$ is connected to at most one other vertex in $\Gamma(v)$, as otherwise it would imply a C_4 is in G . Also, since $q + 1$ is odd, there must be a vertex in $\Gamma(v)$ which is connected to no other vertex in $\Gamma(v)$. Rephrasing, this means that there is at least one vertex in $C(v)$ that is not in $\bigcup_{w \in \Gamma(v)} \Gamma(w)$. We also know that for $w, u \in \Gamma(v)$ we have $\Gamma(w) \cup \Gamma(u) = \{v\}$, otherwise it again implies G has a C_4 . We then have:

$$\left| \bigcup_{w \in \Gamma(v)} \Gamma(w) \right| + 1 = \left(\sum_{w \in \Gamma(v)} d(w) \right) - q + 1 \leq |C(v)| \leq q^2 + q$$

Then we have:

$$\frac{\sum_{w \in \Gamma(v)} d(w)}{q + 1} \leq \frac{q^2 + 2q - 1}{q + 1} = q + 1 - \frac{2}{q + 1}$$

Therefore, we can conclude that if a vertex of degree $q + 1$ is connected to no vertex of degree $q - 1$, then it must be connected to at least two vertices of degree q . Let A be the set of vertices of degree $q + 1$ which are connected to at least two vertices of X_q but no vertex of $X_{\leq q-1}$, and B be the set of vertices of X_{q+1} connected to at least one vertex of $X_{\leq q-1}$. Let $a = |A|$ and $b = |B|$. Naturally $a + b = X_{q+1}$.

We consider the number of edges e' with one endpoint in A and the other in X_q . As each vertex of A is connected to at least two vertices of X_q , and each vertex of X_q is connected to at most q vertices of X_{q+1} , we have:

$$2a \leq e' \leq q|X_q| \quad (5)$$

Let e'' be the number of edges with one endpoint in B and the other in $X_{\leq q-1}$. As each vertex of B is connected to at least one vertex of $X_{\leq q-1}$ and each vertex in $X_{\leq q-1}$ is connected to at most $q - 1$ vertices in B , we have:

$$b \leq e'' \leq (q-1)|X_{\leq q-1}| \quad (6)$$

Adding twice (6) to (5) we get:

$$2|X_{q+1}| \leq q|X_q| + (2q-2)|X_{\leq q-1}|$$

Adding $q|X_{q+1}|$ to both sides we have:

$$(q+2)|X_{q+1}| \leq q(|X_{q+1}| + |X_q| + |X_{\leq q-1}|) + (q-2)|X_{\leq q-1}| = q^3 + q^2 + (q-2)|X_{\leq q-1}|$$

Dividing both sides by $(q+2)$ and expanding, we obtain:

$$|X_{q+1}| \leq q^2 - q + |X_{\leq q-1}| - \frac{4|X_{\leq q-1}|}{q+2} - \frac{4}{q+2} + 2$$

This implies

$$|X_{q+1}| - |X_{\leq q-1}| \leq q^2 - q + 1$$

Using this with (4) we have:

$$2e \leq q^3 + q^2 + q^2 - q + 1$$

Since q and $2e$ are even, we must have:

$$e \leq \frac{1}{2}q(q+1)^2 - q$$

If equality holds, then we have $q^2 - q \leq |X_{q+1}| - |X_{\leq q-1}| \leq q^2 - q + 1$ and $2e \leq (q+1)|X_{q+1}| + q|X_q| + (q-1)|X_{\leq q-1}| \leq 2e + 1$. This implies that at most one of the vertices in $X_{\leq q-1}$ has degree $q-2$, and the rest have degree $q-1$. The remainder of the theorem follows.

It is now clear that any graph with more than E_0 edges must have maximum degree equal to $q+2$. The rest of the paper is devoted to showing that no such graph exists. This is done by utilizing a connection between vertices of degree $q+2$ and vertices of relatively small degree.

Lemma 2. *If there is a C_4 -free graph on n vertices and $E_0 + 1$ edges (with $q \in \mathbb{N}$), then any vertex of degree $\delta \leq \frac{q}{2} + 1$ connects to every vertex of degree $q+2$.*

PROOF. Assume for the sake of contradiction that this statement is not true. Then there is a vertex v of degree δ (where $\delta \leq \frac{q}{2} + 1$) and a vertex u of degree $q+2$ such that $u \not\sim v$. Then we remove all δ edges in which v is incident and add a new edge from u to v , so that our graph now has $e = E_0 + 1 - \delta$ edges, a vertex v of degree 1, and a vertex u of degree $q+3$.

Now we extend the lemma used in [7] concerning 2-paths containing no endpoints in $\Gamma(u)$, the neighborhood of u . We know that we can bound the number of such 2-paths above with $\binom{n-d(u)}{2}$, as there can be at most one 2-path between any pair of points not in $\Gamma(u)$. We also know that each vertex that is not u has at most one neighbor in $\Gamma(u)$, which means that this inequality must hold:

$$\binom{n-d(u)}{2} \geq \sum_{x \neq u} \binom{d(x)-1}{2}$$

as the right side of that inequality is a lower bound on the number of actual 2-paths in the graph.

To actually calculate this, we first note that every 2-path involving v must have an endpoint in $\Gamma(u)$, which means that we can write

$$\sum_{x \neq u} \binom{d(x)-1}{2} = \sum_{x \neq u, v} \binom{d(x)-1}{2}$$

If we consider the total sum being chosen from (i.e. $\sum_{x \neq u, v} d(x) - 1$) we get the number

$$2e - (n - 2) - (q + 3) - 1$$

(since the total degree sum is $2e$ and we subtract first the degrees of the two uncounted vertices and then 1 from the remaining $n - 2$ terms). We can thus use Jensen's inequality to obtain this expression:

$$\binom{n - (q + 3)}{2} \geq (n - 2) \binom{\frac{2e - (n - 2) - (q + 3) - 1}{n - 2}}{2}$$

Now we take

$$e = E_0 - \frac{q}{2} + 1$$

(which corresponds to $\delta = \frac{q}{2} + 1$). Since the left side is not dependent on e and the right side is, if the inequality fails for our chosen e then it will certainly fail for larger values e , which is equivalent to smaller values of δ . When we expand and simplify the above inequality, we find it is equivalent to the following:

$$-\frac{2q^3 - 2q^2 - 10q + 12}{(q^2 + q - 2)} \geq 0$$

which is not true for any relevant q .

Corollary 1. *For any q , if $|X_{q+2}| \geq 2$ in a C_4 -free graph with $E_0 + 1$ edges on n vertices, there can be only one vertex v of degree $\frac{q}{2} + 1$ or less. In that case, $|X_{q+2}| \leq d(v)$.*

PROOF. The first part follows from the prior lemma and the fact that the graph is C_4 -free; the second part follows from the lemma and the first part of the corollary.

Lemma 3. *The maximum number of 2-paths in a graph with n vertices and $E_0 + 1$ edges (with q even) is $qe - |X_{q+1}| + \frac{1}{2}|X_{q+2}|$.*

PROOF. Since there can only be one 2-path between any two vertices (if there are more, there would be a C_4 in the graph), we can bound the number of 2-paths by $\binom{n}{2}$. However, this may be improved by bounding the number of pairs of vertices which are not the endpoints of a 2-path. We consider how many other vertices cannot be reached in two steps from a given vertex, a function we will denote by $f(v)$. The exact number of 2-paths in the graph is $\binom{n}{2} - \frac{1}{2} \sum_v f(v)$, so any lower bound on $\sum_v f(v)$ will in turn yield an upper bound on the number of 2-paths in the graph.

To bound $\sum_v f(v)$, we can compute what $f(v)$ would be if v is connected only to vertices of degree $q + 1$, giving us a function we call $g(v)$. This leads us to this table:

$d(v)$	$g(v)$
$q - 2$	$3q - 1$
$q - 1$	$2q - 1$
q	$q - 1$
$q + 1$	1
$q + 2$	0

As a sample calculation, if $d(v) = q$ then $g(v) = q^2 + q - (1 + q \cdot q) = q - 1$ because it has q neighbors that each have q neighbors other than v . The 1 corresponding to degree $q + 1$ comes from the fact that a vertex of degree $q + 1$ must have one neighbor it cannot be connected to in a 2-path since (by assumption) $q + 1$ is odd.

Strictly speaking, the values of $g(v)$ are not lower bounds on $f(v)$, since there are also vertices of degree $q + 2$. In general, if v is adjacent to k vertices of degree $q + 2$, we have $f(v) \geq g(v) - k$. Then subtracting

$(q+2)|X_{q+2}|$ from $\sum_v g(v)$ gives us the bound $\sum_v f(v) \geq \sum_v g(v) - (q+2)|X_{q+2}|$. We note that, for $d(v) \leq q$, $g(v) = q(q+1-d(v)) - 1$; for $d(v) = q+1$, we must add 2 to that formula, while for $d(v) = q+2$ we must add $q+1$.

This allows us to establish an upper bound on the number of 2-paths in the graph as follows:

$$\begin{aligned}
\binom{n}{2} - \frac{1}{2} \sum_{v \in V(G)} f(v) &\leq \binom{q^2+q}{2} - \frac{1}{2} \sum_{v \in V(G)} (g(v) - |X_{q+2}|) \\
&= \frac{1}{2} [(q^2+q)(q^2+q-1) - (\sum_{v \in V(G)} (q(q+1-d(v)) - 1) + 2|X_{q+1}| - |X_{q+2}|)] \\
&= \frac{1}{2} [(q^2+q)(q^2+q-1) - |V(G)|(q^2+q-1) - 2|X_{q+1}| + |X_{q+2}| + q \sum_{v \in V(G)} d(v)] \\
&= qe - |X_{q+1}| + \frac{1}{2}|X_{q+2}|
\end{aligned}$$

and so we have our result.

Lemma 4. *For any C_4 -free graph G on n vertices with $E_0 + 1$ edges (q even), $\delta(G) > \frac{q}{2} + 1$.*

PROOF. Assume for the sake of contradiction that there is a vertex v such that $d(v) = \delta \leq \frac{q}{2} + 1$. We know from Lemma 2 and Corollary 1 that v is unique and v is connected to every vertex in X_{q+2} . We also know from the Lemma 3 that this inequality must hold:

$$qe - |X_{q+1}| + \frac{1}{2}|X_{q+2}| \geq |X_{q+2}| \binom{q+2}{2} + \sum_{x \neq v, x \notin X_{q+2}} \left(\binom{d(x)}{2} + \binom{\delta}{2} \right) \quad (7)$$

since the right hand side is the total number of 2-paths in the graph.

Take A to be the set of vertices that are neither v nor in X_{q+2} . We wish to find the average degree of A , which we will denote by c . Since the maximum degree of any vertex in A is $q+1$, $c \leq q+1$. Moreover, c will be minimized when the $|X_{q+2}| = \delta = \frac{q}{2} + 1$; in that case, we can calculate c :

$$c = \frac{2(E_0 + 1) - (\frac{q}{2} + 1) - (\frac{q}{2} + 1)(q + 2)}{n - (\frac{q}{2} + 1 + 1)}$$

which yields $c = q + 1 - \frac{4q-2}{2q^2+q-4}$. Since the subtracted term is less than 1 for all q , $q < c \leq q + 1$.

Now, clearly, for a fixed $|X_{q+2}|$ the left side of 7 is maximized when $|X_{q+1}|$ is minimized. We also wish to minimize the term $M = \sum_{x \in A} \binom{d(x)}{2}$. If there is a vertex y of degree $q-k$ (for some integer $k \geq 1$), then we know that we keep the same degree sum in A (which is fixed, since δ and $|X_{q+2}|$ are fixed) if we were to take a vertex of degree $q+1$, turn it into a vertex of degree q , and raise the degree of y by 1. Moreover, this will actually decrease M , because of the following arithmetic:

$$\begin{aligned}
\binom{q}{2} - \binom{q+1}{2} + \binom{q-k+1}{2} - \binom{q-k}{2} &= -2q + 2(q-k) \\
&= -2k < 0
\end{aligned}$$

and so if there is a vertex of degree $q-1$ or less in A then M is not minimal.

Thus we see that both M and $|X_{q+1}|$ are minimized when every vertex in A has degree q or degree $q+1$. Thus, if 7 does not hold in that case, it cannot hold in any case. To obtain values for $|X_{q+1}|$ and $|X_q|$, we solve this system of equations:

$$\begin{aligned}
|X_{q+2}| + |X_{q+1}| + |X_q| + 1 &= n \\
|X_{q+2}|(q+2) + |X_{q+1}|(q+1) + |X_q|q + \delta &= 2(E_0 + 1)
\end{aligned}$$

which makes $|X_{q+1}| = q^2 + 2 - \delta - 2|X_{q+2}|$ and $|X_q| = q - 3 + |X_{q+2}| + \delta$.

When we plug those values into 7 and group terms in terms of δ we obtain this expression:

$$-\frac{1}{2}\delta^2 + (q + \frac{3}{2})\delta + \frac{3}{2}|X_{q+2}| - \frac{3}{2}q - \frac{1}{2}q^2 - 2 \geq 0$$

Since, viewed as a function of δ , that is a downward-opening quadratic, we know that the inequality will only be true between the zeros of that function. Applying the quadratic formula to the expression yields this equivalent expression:

$$\frac{3}{2} + q + \frac{1}{2}\sqrt{-7 + 12|X_{q+2}|} \geq \delta \geq \frac{3}{2} + q - \frac{1}{2}\sqrt{-7 + 12|X_{q+2}|}$$

and since we know that $\delta \geq |X_{q+2}|$, we know this must be true:

$$\begin{aligned} \frac{3}{2} + q + \frac{1}{2}\sqrt{-7 + 12|X_{q+2}|} &\geq |X_{q+2}| \\ \frac{3}{2} + q + \frac{1}{2}\sqrt{-7 + 12|X_{q+2}|} - |X_{q+2}| &\geq 0 \end{aligned}$$

Solving that inequality for $|X_{q+2}|$ yields a requirement that

$$|X_{q+2}| \geq 3 + q - \sqrt{5 + 3q}$$

but that (given our constraint that q be at least 6) implies $|X_{q+2}| > \frac{q}{2} + 1$. That means that $\delta > \frac{q}{2} + 1$, which contradicts our initial assumption. Therefore $\delta(G) > \frac{q}{2} + 1$.

Lemma 5. *If G is a C_4 -free graph on n vertices with $E_0 + 1$ edges (with no restrictions on q), then any two vertices of degree $q + 2$ in G must share exactly one neighbor.*

PROOF. By way of contradiction, suppose that there exist two vertices $u, v \in G$ both of degree $q + 2$ that share no neighbors. We will expand on a technique used by Füredi in [7] to consider 2-paths without endpoints in either $\Gamma(u)$ or $\Gamma(v)$. Denote this quantity by P . Let $d = q + 2$ be the degree of u and v .

We know that $\binom{n-d(x)}{2}$ is an upper bound for the number of 2-paths without endpoints in the neighborhood of a chosen vertex x with degree $d(x)$, since every two vertices not in the neighborhood of x are endpoints of at most one 2-path. In our case, we have an upper bound on P of $\binom{n-2d}{2}$ as we are removing two disjoint neighborhoods of degree d .

Now, we know that $\sum_{w \neq u, v} \binom{d(w)}{2}$ is precisely the number of 2-paths with the central vertex not equal to u, v , but we must subtract two from each degree to account for the possibility that a given w shares a neighbor with both u and v . Thus we find that $\sum_{w \neq u, v} \binom{d(w)-2}{2}$ is the lower bound for P , as it assumes all vertices w share a neighbor with both u and v . We use Jensen's inequality to get the following result:

$$\sum_{w \neq u, v} \binom{d(w)-2}{2} \geq (n-2) \binom{(2(E_0+1) - 2(n-2) - 2d)/(n-2)}{2}.$$

Then we have an inequality that must hold for the graph G to exist:

$$\binom{n-2d}{2} - 1 \geq P \geq (n-2) \binom{(2(E_0+1) - 2(n-2) - 2d)/(n-2)}{2}.$$

After simplifying and solving for q , we get the following inequality:

$$\frac{-q^4 - 6q^3 + 17q^2 + 34q - 48}{q^2 + q - 2} \geq 0.$$

However, this inequality cannot hold for any q in the range we are concerned with. This contradiction shows that u and v must share at least one neighbor, and we know they cannot share more than one because this would create a C_4 . This result implies that every pair of vertices of degree $q + 2$ must have exactly one neighbor in common.

Lemma 6. *If G is a C_4 -free graph on n vertices with $E_0 + 1$ edges (where q can be even or odd), any two vertices of degree $q + 2$ must share a neighbor of degree $d < \frac{q}{2}$.*

PROOF. We know from the previous lemma that two vertices of degree $q + 2$ must have a neighbor in common. Consider two such vertices x and y , and let their unique common neighbor be u . By adapting the argument of the previous lemma and applying it to $B = \Gamma(x) \cup \Gamma(y)$, i.e. looking at P' , the number of 2-paths with no endpoints in B , we obtain the following inequality:

$$\binom{n - 2(q + 2) + 1}{2} \geq \sum_{v \notin \Gamma(u)} \binom{d(u) - 2}{2} + \sum_{v \in \Gamma(u), v \neq x, y} \binom{d(u) - 1}{2}$$

We see the left hand side is an upper bound on P' ; we add 1 back in to the number of vertices used in 2-paths because of the intersection between $\Gamma(x)$ and $\Gamma(y)$. The right hand side is separated into two sums. The first summation sums over all vertices v not in the neighborhood of vertex u , and relies on the fact that v can connect to at most 1 other vertex in the neighborhood of each of the $q + 2$ vertices before creating a C_4 . The second summation is over all vertices $w \in \Gamma(u)$ and comes from the fact that w can connect to at most 1 other vertex in the union of the neighborhoods of the two $q + 2$ vertices before creating a C_4 .

We use Jensen's inequality on the right hand side to obtain the following expression:

$$\sum_{v \notin \Gamma(u)} \binom{d(v) - 2}{2} + \sum_{w \in \Gamma(u)} \binom{d(w) - 1}{2} \geq (n - 2) \binom{\frac{2e - 2(q + 2) - 2(n - 2) + d(u) - 2}{n - 2}}{2}$$

(We subtract 2 because we should not count x and y for u 's total.)

Clearly, if this inequality fails for a given value of $d(u)$, it must fail for any larger value, since the right side increases with $d(u)$ and the left side is static. Thus, we plug in $\frac{q}{2}$, which yields the following expression:

$$-\frac{6q^3 - 25q^2 - 28q + 96}{8q^2 + 8 - 16} \geq 0$$

That inequality fails for all relevant q , thus the statement is proven.

Corollary 2. *If q is even, any C_4 -free graph with $E_0 + 1$ edges and n vertices has exactly one vertex of degree $q + 2$.*

PROOF. This follows from the three previous lemmas.

Having reduced the hypothetical counterexamples to a single case, we proceed with the proof of the theorem.

Theorem 4. *For q even, $ex(q^2 + q, C_4) \leq \frac{1}{2}q(q + 1)^2 - q$.*

PROOF. We know from Corollary 2 that we must consider only the case when $|X_{q+2}| = 1$. It is clear that any graph G with n vertices, $E_0 + 1$ edges, and $|X_{q+2}| = 1$ can only be created by taking a graph G' with E_0 edges and $\Delta(G) = q + 1$ and connecting a vertex of degree $q + 1$ to a vertex of strictly lower degree. We know from Lemma 3 that G' can have one of two possible degree sequences up to a parameter z . When we examine all the ways to make the necessary connection, we get these four possible degree sequences for G :

	$ X_{q+2} $	$ X_{q+1} $	$ X_q $	$ X_{q-1} $	$ X_{q-2} $
A	1	$q^2 - q + z$	$2q - 2z - 1$	z	0
B	1	$q^2 - q + z$	$2q - 2z$	$z - 2$	1
C	1	$q^2 - q + z + 1$	$2q - 2z - 2$	$z - 1$	1
D	1	$q^2 - q + z - 1$	$2q - 2z + 1$	$z - 1$	0

Since we have a specific degree sequence, we can use Lemma 3 concerning the total number of 2-paths in G to generate the following inequality:

$$qe - |X_{q+1}| + \frac{1}{2}|X_{q+2}| \geq |X_{q+2}| \binom{q+2}{2} + |X_{q+1}| \binom{q+1}{2} + |X_q| \binom{q}{2} + |X_{q-1}| \binom{q-1}{2} + |X_{q-2}| \binom{q-2}{2}$$

When we solve that inequality for z , we get the following results:

A	$z \leq -\frac{1}{4}$
B	$z \leq -\frac{3}{4}$
C	$z \leq -\frac{7}{4}$
D	$z \leq \frac{3}{4}$

Obviously, these values of z lead to impossible degree sequences, thus no such G is possible.

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