Homework #3

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April 3, 2018

1. Find a closed form expression for each of the following generating functions:

a)

$$\begin{aligned}\sum_{n=0}^{\infty}(n+2)^2x^n &= \sum_{n=0}^{\infty}(n+1)(n+2)x^n+\sum_{n=0}^{\infty}(n+2)x^n\\ &= \sum_{n=0}^{\infty}(n+1)(n+2)x^n+\sum_{n=0}^{\infty}(n+1)x^n+\sum_{n=0}^{\infty}x^n\\ &= \sum_{n=0}^{\infty}(x^{n+2})''+\sum_{n=0}^{\infty}(x^{n+1})'+\sum_{n=0}^{\infty}x^n\\ &= \left(\sum_{n=0}^{\infty}x^n\cdot x^2\right)''+\left(\sum_{n=0}^{\infty}x^n\cdot x\right)'+\sum_{n=0}^{\infty}x^n\\ &= \left(\frac{x^2}{1-x}\right)''+\left(\frac{x}{1-x}\right)'+\frac{1}{1-x}\\ &= \left(\frac{2x(1-x)-x^2(-1)}{(1-x)^2}\right)'+\frac{1-x-x(-1)}{(1-x)^2}+\frac{1}{1-x}\\ &= \left(\frac{2x-x^2}{(1-x)^2}\right)'+\frac{2-x}{(1-x)^2}\\ &= \frac{3x^2-7x+4}{(1-x)^3}\end{aligned}$$

b)

$$\begin{aligned}\sum_{n=0}^{\infty}(n+2)^2\frac{x^n}{n!} &= \sum_{n=0}^{\infty}(n+1)(n+2)\frac{x^n}{n!}+\sum_{n=0}^{\infty}(n+2)\frac{x^n}{n!}\\ &= \sum_{n=0}^{\infty}(n+1)(n+2)\frac{x^n}{n!}+\sum_{n=0}^{\infty}(n+1)\frac{x^n}{n!}+\sum_{n=0}^{\infty}\frac{x^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\frac{x^{n+2}}{n!}\right)''+\sum_{n=0}^{\infty}\left(\frac{x^{n+1}}{n!}\right)'+\sum_{n=0}^{\infty}\frac{x^n}{n!}\\ &= \left(\sum_{n=0}^{\infty}\frac{x^{n+2}}{n!}\right)''+\left(\sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}\right)'+\sum_{n=0}^{\infty}\frac{x^n}{n!}\\ &= (x^2e^x)''+(xe^x)'+e^x\\ &= x^2e^x+5xe^x+4e^x\end{aligned}$$

c)

$$\begin{aligned}\sum_{n=0}^{\infty}(n+2)^2\binom{2n}{n}x^n &= \sum_{n=0}^{\infty}(n+1)(n+2)\binom{2n}{n}x^n+\sum_{n=0}^{\infty}(n+1)\binom{2n}{n}x^n+\sum_{n=0}^{\infty}\binom{2n}{n}x^n\\ &= \sum_{n=0}^{\infty}\left(\binom{2n}{n}x^{n+2}\right)''+\sum_{n=0}^{\infty}\left(\binom{2n}{n}x^{n+1}\right)'+\sum_{n=0}^{\infty}\binom{2n}{n}x^n\\ &= \left(\sum_{n=0}^{\infty}\binom{2n}{n}x^n\cdot x^2\right)''+\left(\sum_{n=0}^{\infty}\binom{2n}{n}x^n\cdot x\right)'+\sum_{n=0}^{\infty}\binom{2n}{n}x^n\\ &= (x^2\sqrt{1-4x})''+(x\sqrt{1-4x})'+\sqrt{1-4x}\\ &= \frac{(3-24x)\sqrt{1-4x}}{1-4x}+\frac{2(2x-10x^2)}{(1-4x)\sqrt{1-4x}}+\frac{2-10x}{\sqrt{1-4x}}\end{aligned}$$

2. Problem 14K. Consider walks in the $X-Y$ plane where each step is $R\colon(x,y)\rightarrow(x+1,y)$ or $U\colon(x,y)\rightarrow(x+1,y)$. We start at (0,0) and ask in how many ways we can reach $(2n,2n)$ without passing through one of the points $(2i-1,2i-1), i=1,\ldots,n$. Prove that this number is the Catalan number u_{2n+1}.

Prove:

3. Let n be a positive integer, q a prime power, and let $f_2(n, q)$=the number of co-prime pairs of monic polynomials of degree n over GF(q). Find a simple formula for $f_2(n, q)$.

4. An L-tile is a 2×2 square with the upper right 1×1 square removed; no rotations are allowed. Let an be the number of tilings of a $4 \times$ n rectangle using tiles that are either 1×1 squares or L-tiles. Find a closed form for the generating function $1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ (i.e., write the generating function as a rational function).

5. For positive integers m and n, let $f(m, n)$ denote the number of n-tuples $(x_1, x_2, ..., x_n)$ of integers such that

$$|x_1| + |x_2| + \cdots + |x_n| \leqslant m.$$

a) Find a recurrence formula for $f(m, n)$.

$f(m, 0) = f(0, n) = 1$

$f(m, n) = f(m-1, n) + f(m-1, n-1) + f(m-1, n-1)$

b) Use the recurrence formula to prove that $f(m, n) = f(n, m)$.

Because the recurrence formula of $f(m, n)$ is symmetric, and the meric boundary is sysmmetric, f is sysmmetric.

c) Do part (b) by using the generating function method.

Firstly, it is convenient to allow $f(m, 0) = f(0, n) = 1$. Then

$$\begin{aligned}
G(x, y) &= \sum_{m \geqslant 0} \sum_{n \geqslant 0} f(m, n) x^m y^n \\
&= \sum_{m \geqslant 0} \sum_{n \geqslant 0} x^m y^n \sum_{k_1, ..., k_n \in Z, |k_1| + |k_2| + ... + |k_n| < m} 1 \\
&= \sum_{m \geqslant 0} y^n \sum_{k_1, ..., k_n \in Z} \sum_{m \geqslant |k_1| + ... + |k_n|} x^m \\
&= \sum_{n \geqslant 0} y^n \sum_{k_1, ..., k_n \in Z} \frac{x^{|k_1 + ... + k_n|}}{1-x} \\
&= \frac{1}{1-x} \sum_{n \geqslant 0} y^n (\sum_{k \in Z} x^{|k|})^n \\
&= \frac{1}{1-x} \sum_{n \geqslant 0} y^n \left(\frac{1+x}{1-x}\right)^n \\
&= \frac{1}{1-x} \frac{1}{1 - y(1+x)/(1-x)} \\
&= \frac{1}{1-x-y-xy}
\end{aligned}$$

Because the $G(x, y) = G(y, x)$, $f(m, n) = f(n, m)$.

(Refer from http://www.mathcs.emory.edu/~mic/math/files/kedlaya/)

d) Find a closed formula for $\sum_{n \geqslant 0} f(n, n) x^n$

$$\begin{aligned}
\sum_{n \geqslant 0} f(n, n) x^n &= \sum_{n \geqslant 0} x^n \sum_{k_1, ..., k_2 \in Z, |k_1| + ... + |k_n| < n} 1 \\
&= \sum_{k_1, ..., k_n \in Z} \sum_{n \geqslant |k_1| + ... + |k_n|} x^n \\
&= \sum_{k_1, ..., k_2 \in Z} \frac{x^{|k_1| + ... + |x_n|}}{1-x} \\
&= \frac{1}{1-x} \sum_{k \in Z} \left(\frac{1+x}{1-x}\right)^n \\
&= \frac{1}{1-x} \frac{1}{1 - (1+x)/(1-x)} \\
&= \frac{1}{1-2x}
\end{aligned}$$