Graphs with three eigenvalues

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Joint work with Ximing Cheng and it is work in progress

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Outline

Definitions

- Let $\Gamma = (V, E)$ be a graph.
- The distance $d(x, y)$ between two vertices x and y is the length of a shortest path connecting them.
- The maximum distance between two vertices in Γ is the diameter $D = D(\Gamma)$.
- The valency k_x of x is the number of vertices adjacent to it.
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- A graph is regular with valency k if each vertex has k neighbours.
- The adjacency matrix A of Γ is the matrix whose rows and columns are indexed by the vertices of Γ and the (x, y)-entry is 1 whenever x and y are adjacent and 0 otherwise.
- The eigenvalues of the graph Γ are the eigenvalues of A.

Strongly regular graphs

A strongly regular graph (SRG) with parameters (n, k, λ, μ) is a k-regular graph on n vertices such that

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- The Petersen graph is a strongly regular graph with parameters $(10, 3, 0, 1)$.
- The line graph of a complete graph on t vertices $L(K_t)$ is a SRG $(t(t-1)/2, 2(t-2), t-2, 4)$.

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- The line graph of a complete graph on t vertices $L(K_t)$ is a SRG $(t(t-1)/2, 2(t-2), t-2, 4)$.
- The line graph of a complete bipartite graph $K_{t,t}$, $L(K_{t,t})$, is a SRG $(t^2, 2(t-1), t-2, 2)$.
- There are many more examples, coming from all parts in combinatorics.

Strongly regular graphs 2

A strongly regular graph has at most diameter two, and has at most three distinct eigenvalues. We can characterize the strongly regular graphs by this property.

Theorem

A connected regular graph Γ has at most three eigenvalues if and only if it is strongly regular.

Small number of distinct eigenvalues

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Also the cone over the Petersen graph (i.e. you add a new vertex and join the new vertex with all the other vertices) is a non-regular graph with exactly three distinct eigenvalues.

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1. Introduction
   - Definitions
   - History
2. Theory
   - Basic Theory
3. Our results
   - Bound
   - Neumaier's result

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- In 1998 Muzychuk and Klin gave more examples of such graphs.
- In 1998 E. van Dam gave the basic theory for such graphs, and also give some new examples. Also he classified the graphs with exactly three distinct eigenvalues having smallest eigenvalue at least -2.

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- Then by the Perron-Frobenius Theorem θ_0 has multiplicity one.
- Let A be the adjacency matrix of Γ.
- As $B := (A - \theta_1 I)(A - \theta_2 I)$ has rank 1 and is positive semi-definite we have $B = \mathbf{x}\mathbf{x}^T$ for some eigenvector $\mathbf{x}$ of A corresponding to eigenvalue θ_0.

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- By looking at the uv entries of B, this gives $k_u = -\theta_1\theta_2 + x_u^2$ for u a vertex,
- $\lambda_{uv} = \theta_1 + \theta_2 + x_u x_v$, for $u \sim v$,
- $\mu_{xy} = x_u x_v$ for u and v non-adjacent.

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- $\theta_2 \leq -\sqrt{2}$ with equality if and only if Γ is the path of length 2.

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From now on we will assume $\theta_1 > 0$ and hence θ_0 is an integer.

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A bound on the number of vertices

Lemma

Let Γ be a non-regular connected graph with exactly three distinct eigenvalues $\theta_0 > \theta_1 > \theta_2$. Let $u \sim v$ with $k_u < k_v$. Then $k_u \geq \lambda_{uv} + 1$. This gives $x_v - 1 \leq x_u(x_v - x_u) \leq -\theta_1\theta_2 + \theta_1 + \theta_2$, and hence $x_v \leq -(\theta_1 + 1)(\theta_2 + 1)$.

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This implies:

Proposition

Let Γ be a non-regular connected graph on n vertices with three distinct eigenvalues $\theta_0 > \theta_1 > \theta_2$ with respective multiplicities $1, m_1, m_2$. Let Δ be the maximal valency in Γ and let $\ell := \min\{1 - (\theta_1 + 1)(\theta_2 + 1), -\theta_1\theta_2 + 1\}$. Then the following hold:

1. $\Delta \leq (1 - (\theta_1 + 1)(\theta_2 + 1))^2 - \theta_1\theta_2)$;
2. If $\Delta \neq n - 1$ and $\theta_1 \neq 0$, then $\Delta \leq \ell^2 - \theta_1\theta_2$;
3. $n \leq \max\{(\ell^2 - \theta_1\theta_2 - 1)^2 + 1, (1 - (\theta_1 + 1)(\theta_2 + 1))^2 - \theta_1\theta_2 + 1\}$.

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Neumaier's Theorem

Neumaier (1979) showed the following result.

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Let m be a positive integer. Let Γ be a connected and coconnected (i.e the complement is connected) strongly regular graph with minimal eigenvalue $-m$. Then either the number of vertices is bounded by a function in m, or Γ belongs to one of two infinite (one parameter) families of strongly regular graphs (and we know how to construct all of them if the number of vertices is large enough)

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How can we generalize this result to graphs with three distinct eigenvalues?

Neumaier's Theorem 2

Question 1:

Let m be a positive integer.

- Are there only finitely many non-regular connected graphs with distinct eigenvalues $\theta_0 > \theta_1 > \theta_2$ such that $0 < \theta_1 \leq m$?
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- Note that the answer for the question (i) is negative if you allow four distinct eigenvalues as the friendship graphs show.

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This was shown for cones, i.e. graphs with a vertex of valency $n-1$. We end with a challenge.

Challenge:

Construct more connected non-regular graphs with three distinct eigenvalues.

Thank you for your attention.