§8 Turan's Theorem & Extremal Graphs

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How many edges must a simple graph on n vertices have to gurantee the graph to have a triangle? or what's the maximum # of edges of a simple graph on n vertices s.t. there is no Δ?

Theorem: Let G be a simple graph on n vertices. If G has no Δ, then

$$e \leqslant \lfloor \frac{n^2}{4} \rfloor$$

In otherwords. if $e(G) > \lfloor \frac{n^2}{4} \rfloor + 1$, G must have a Δ.

Proof.

Method from book:

We given vertex i a weight $z_i \geqslant 0$, number from 1 to n, such that $\sum z_i = 1$, and we wish maximize $S := \sum z_i z_j$, where the sum is taken over all edges $\{i, j\}$. Suppose that vertex k and vertex l are not joined. Let the neighbors of k have total weight x, and those of l total weight y, where $x > y$. Since $(z_k + \varepsilon)x + (z_l - \varepsilon)y \geqslant z_k x + z_l y$, we do not decrease the value of S if we shift some of the weight of vertex l to the vertex k. It follows that S is maximal if all of the weight is concentrated on some complete subgraph of G, i.e. on *one edge*! Therefore $S \leqslant \frac{1}{4}$. On the other hand, takig all z_i equal to n^{-1} would yield a value of $n^{-2}|E|$ for S. Therefore $|E| \leqslant \frac{1}{4} n^2$

(1) Any $xy \in E(G)$,

$\sum$ □

Theorem: The indpendence number of a simple graph G= The size of a largest coclique(or indep set) in G.

Proof. Let α = the independent number of G, and let A be a coclique of size α, $\beta = V(G) \backslash A$

Claim: $\forall x \in V(G)$, $d(x < \alpha)$

Claim: $\forall e \in E(G)$, at least one end of e in β

$$\text{count} \left\{ (e, x) \middle| \begin{array}{l} e \in E(G) \\ e \in \beta \\ x \,\text{is incident with}\, e \end{array} \right\}$$

$e(G) = |E(G)| \leqslant \sum_{e \in E(G)} (1 \,\text{or}\, 2) = \sum_{x \in \beta} d(x) \leqslant \alpha|\beta| \leqslant \left(\frac{\alpha + |\beta|}{2}\right)^2 = \left(\frac{n}{2}\right)^2 = \frac{n^2}{4}$ □

Notes: The difference of Maximal and Maximum. Let $(X, \leqslant)$ be a partially ordered set, then

Maximal	An element $m \in X$ is maximal, if $m \leq x$ for any $x \in X$ then $x = m$.
Maximum	An element $M \in X$ is a maximum, if $x \leq M$ for every $x \in X$.

[2]

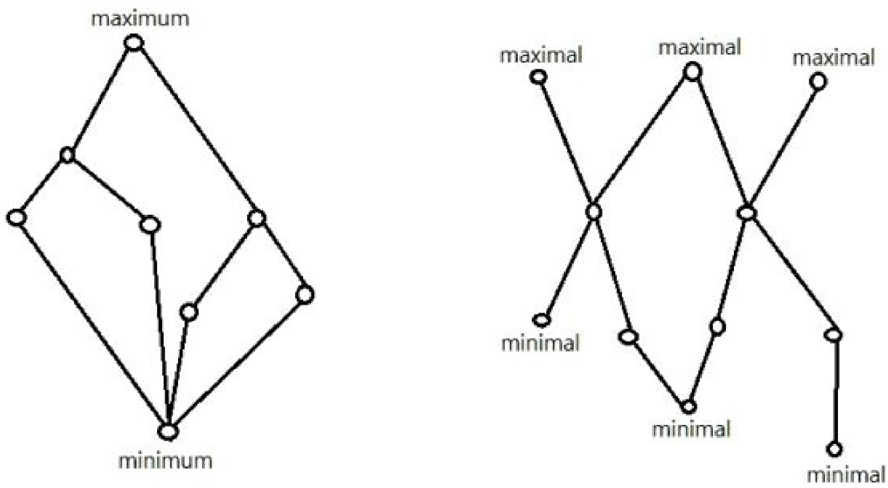

Figure 1. The difference between maximum and maximal[3]

Extremal graphs: $\mathrm{ex}(H,n)$ = the largest number of edges in a simple graph on n vertrices which doesn't contain H as a subgraph.

In general, an extremal graph is the largest graph of order n which does not contain a given graph G as a subgraph. Turan studied extremal graphs that do not contain a complete graphs K_p as a subgraph.[1]

exclude

Turan's theorem: If a simpel G on n vertics contains no copy of K_{v+1}, then it has at most $\left(1-\frac{1}{r}\right)\frac{n^2}{2}$ edges.

$$e(G)=e(A)+e(B)+e(A,B)$$

Proof. Use strong induction on $n=r$.

Assumet that the thm is true for graph on <n vetieces

Let G be a graph on n vercties without K_{r+1} with maximum # of edges,

Claim: G has a copy(subset) of K_r(otherwise we could add edges to G so that G has a K_r and still has no K_{r+1}.

Let A be the set of vertices of this K_r,let $\beta=V(G)\backslash A$.(???) □

Complete

Y= partitie graphs, $K_{n_1,n_2,\ldots,n_r}$

$n_1+n_2+\ldots+n_r=n$

$\#\text{of edges}=\sum_{1\leqslant i\leqslant j\leqslant r} n_o n_j$

$|n_i-n_j|\leqslant 1, \forall i\neq j$

The *girth* of a graph G = the size of a smallest cycle(polygon P_n) in G. (If G has no cycles then we say that girth of G is ∞, such as forest). Here are different state of girth:

- Girth $\geqslant 3 \Leftrightarrow G$ is simple
- Girth $\geqslant 4 \Leftrightarrow G$ is simple and no Δ
- Girth $\geqslant 5 \Leftrightarrow$

Theorem 4.2. If a graph G on n vertices has more than $\frac{1}{2}n\sqrt{n-1}$ edges, then G has girth $\leqslant 4$. That is, G is not simple or contains a P_3 or a P_4(a triangle or a quadrilateral).

Proof. $\forall x \in V(G)$,

Claim 1: no two of $y_1, y_2, \ldots, y_n$ are adjacent

Claim 2: no vertex other than x can be adjacent to more than one of $y_1, y_2, \ldots, y_d$

$$(\deg(y_1)-1)+(\deg(y_2)-1)+\ldots+(\deg(y_d)-1)+d+1 \leqslant n$$

Then

$$\frac{1}{n}(2|E(G)|)^2 = \frac{1}{n}\left(\sum_{y\in V(G)} \deg(y)\right)^2 \leqslant \sum_{y\in V(G)} \deg(y)^2 = \sum_{x\in V(G)} \sum_{y \text{ adjacent to } x} \deg(y) \leqslant n(n-1)$$

$$\frac{1}{n} + 4E(G)^2 = n(n-1)?? \qquad \square$$

i. $n = 1 + d^2$, d: a positive interger

ii. The grith is regular(The equality in c-s inqualigty holds four situation)

iii. no Δ(girth $\geqslant 5$)

iv. $\forall x, y$, xy $\in E(G)$ $\exists!$z,xz, yz $\in E(G)$

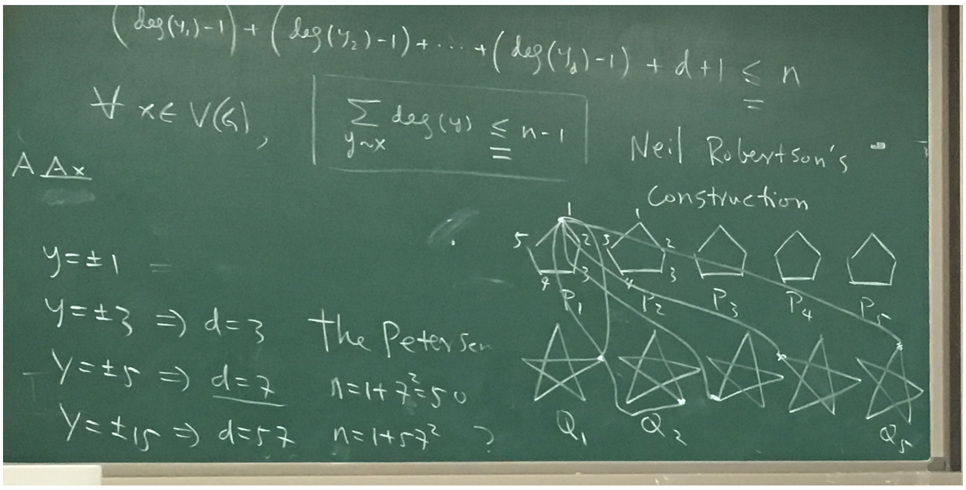

Figure 2. Note (Under understanding)

A=the adjecncy matrix of G

$$(A^2)_{\text{xy}} = \text{the \#of walks of lenth 2 between } x \text{ and } d$$

Vx $\in V(G)$, $\sum_{y\sim x} \deg(y) \leqslant -1$

$A^2 = dI, +0. A_{+1(J-I-A)}$

AJ $=$ dJ

eigenvalues s of A

(1) For adj martix A odf G trhe eighenvals are real $\exists\alpha$ baisis of R^n consosting eigenveacotr of A.

(2) A d-regualr gra;phn G has d as an eigenval . Infact d is the larigest eigencal. The mulitip of d as an eigenval=#connected comps of G

i. $1 + f + g = n = 1 + d^2$

ii. $d + f_r + g = 0$

iii.

iv.

$d + f_r + g = 0$

References

- [1]. http://mathworld.wolfram.com/ExtremalGraph.html
- [2]. http://www.math3ma.com/mathema/2015/4/20/maximal-not-maximum
- [3]. https://www.zhihu.com/question/22319675