MAT8010 Homework #2

Due Date: March 20, 2018

1. (3 points) Let $H_3(r)$ denote the number of 3×3 matrices with nonnegative integer entries such that each row and each column sum to r. Show that

$$H_3(r) = \binom{r+5}{5} - \binom{r+2}{5}.$$

Solution. Let

$X = \{(a_1, a_2, a_3, b_1, b_2, b_3) \mid a_1+a_2+a_3 = r, b_1+b_2+b_3 = r, a_i, b_i \text{ are nonnegative integers for all } i\}.$

For $i = 1, 2, 3$, define

$$X_i = \{(a_1, a_2, a_3, b_1, b_2, b_3) \in X \mid a_i + b_i \geq r + 1\}.$$

Then $H_3(r) = |\bar{X}_1 \cap \bar{X}_2 \cap \bar{X}_3| = |X| - (|X_1| + |X_2| + |X_3|) + (|X_1 \cap X_2| + \cdots) - |X_1 \cap X_2 \cap X_3|$. Clearly $|X_1 \cap X_2| = |X_1 \cap X_3| = |X_2 \cap X_3| = 0$, and $|X_1 \cap X_2 \cap X_3| = 0$. Now we compute $|X_1|$. Define

$$Y_1 = \{(A, a_2, a_3, b_2, b_3) \mid A \geq r + 1, A + a_2 + a_3 + b_2 + b_3 = 2r, a_i \geq 0, b_i \geq 0 \text{ for all } i\}.$$

There is a bijection θ from X_1 to Y_1 as shown below: $\theta : X_1 \to Y_1$,

$$\theta(a_1, a_2, a_3, b_1, b_2, b_3) = (a_1 + b_1, a_2, a_3, b_2, b_3).$$

Given an element $(A, a_2, a_3, b_2, b_3) \in Y_1$, we must have $a_2 + a_3 \leq r$; otherwise $A + a_2 + a_3 \geq 2r+2$, impossible. Let $a_1 = r-(a_2+a_3)$ and $b_1 = A-a_1$. Then we see that $a_1 \geq 0$ and $b_1 \geq 0$. And we get a tuple $(a_1, a_2, a_3, b_1, b_2, b_3)$ satisfying $a_1 + a_2 + a_3 = r$ and $b_1 + b_2 + b_3 = r$, and $a_i \geq 0$, $b_i \geq 0$, for all i. This shows that θ is surjective. Also the map θ is clearly injective. Hence we have shown that θ is indeed a bijection as claimed. It follows that $|X_1| = |Y_1| = \binom{5+(r-1)-1}{r-1} = \binom{r+3}{4}$. Hence

$$H_3(r) = \binom{r+2}{2}^2 - 3\binom{r+3}{4} = \binom{r+5}{5} - \binom{r+2}{5}.$$

2. (4 points) Let n and r be positive integers. A function $f : [n] \to [r]$ is called *monotone* if $x \leq y$ implies $f(x) \leq f(y)$, for all $x, y \in [n]$.

- Prove that the number of monotone surjections from $[n]$ to $[r]$ is $\binom{n-1}{r-1}$.

  A function $f : [n] \to [r]$ is uniquely determined by $f(1), f(2), \ldots, f(n)$. f is monotone if and only if $1 \le f(1) \le f(2) \le \cdots \le f(n) \le r$. For each i, $1 \le i \le r$, let $x_i =$ the number of i's in the sequence $f(1), f(2), \ldots, f(n)$. The number of monotone surjections is equal to the number of solutions to $x_1 + x_2 + \cdots + x_r = n$ $(\forall i, x_i \ge 1)$. (Here we require $x_i \ge 1$ for all i since we want to count surjections.) From class we know that the number of such solutions is $\binom{n-1}{r-1}$.

- Count monotone surjections from $[n]$ to $[r]$ by the PIE to prove that

$$\binom{n-1}{r-1} = \sum_{k=0}^{r-1} (-1)^k \binom{r}{k} \binom{r+n-k-1}{n}.$$

Define $S = \{(x_1, x_2, \ldots, x_j, \ldots, x_r) \mid \sum_{j=1}^{r} x_j = n, \forall j, x_j \ge 0\}$. For each i, $1 \le i \le r$, let $E_i = \{(x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_r) \mid \sum_{j=1}^{r} x_j = n, \forall j, x_j \ge 0\}$. Then the number of monotone surjections is equal to $|\bar{E}_1 \cap \bar{E}_2 \cap \cdots \cap \bar{E}_r|$, which, by the Principle of Inclusion and Exclusion, is equal to

$$|S| - \sum_{i=1}^{r} |E_i| + \sum_{i<j} |E_i \cap E_j| - \cdots$$

We can easily compute each term in the above alternating sum. So

$$|\bar{E}_1 \cap \bar{E}_2 \cap \cdots \cap \bar{E}_r| = \binom{r+n-1}{n} - \binom{r}{1}\binom{r+(n-1)-1}{n} + \binom{r}{2}\binom{r+(n-2)-1}{n} - \cdots.$$

Hence $\binom{n-1}{r-1} = \sum_{k=0}^{r-1} (-1)^k \binom{r}{k} \binom{r+n-k-1}{n}$.

3. (4 points) 10F (Van Lint/Wilson, page 96) The second part "Can you prove this identity directly?" is mandatory.

Solution. First we note that if i $(1 \le i \le 2n)$ is colored red, then all j $(j \le i)$ must be colored red. Therefore there are $2n+1$ ways of coloring the integers, namely

1	2	3	4	·	·	·	$2n-1$	$2n$
B	B	B	B	·	·	·	B	B
R	R	R	R	·	·	·	R	R
R	R	R	R	·	·	·	R	B
R	R	R	R	·	·	·	B	B
		·	·	·	·			
R	R	B	B	·	·	·	B	B
R	B	B	B	·	·	·	B	B

Table I

Next we count the number of ways of coloring by the PIE. For $i = 2, 3, \ldots, 2n$, we define E_i = the set of colorings in which i is colored red and $(i-1)$ is colored blue. Note that $|E_i \cap E_{i-1}| = 0$ for all i, $3 \le i \le 2n$. The number of colorings we are seeking in this question is $|\bar{E}_2 \cap \bar{E}_3 \cap \cdots \cap \bar{E}_{2n}|$, which, by the PIE, is equal to

$$2^{2n} - \sum_{i=2}^{2n} |E_i| + \sum_{i<j} |E_i \cap E_j| - \cdots .$$

Note that the number of ways to select k integers (no two consecutive) from $\{2, 3, \ldots, 2n\}$ is $\binom{(2n-1)-k+1}{k}$. We obtain

$$|\bar{E}_2 \cap \bar{E}_3 \cap \cdots \cap \bar{E}_{2n}| = \sum_{k=0}^{n} (-1)^k \binom{2n-k}{k} 2^{2n-2k}.$$

Hence $\sum_{k=0}^{n} (-1)^k \binom{2n-k}{k} 2^{2n-2k} = 2n+1$.

We now prove the above identity directly. One can prove the identity by using generating functions or by using recursions. I will use the latter method. Define

$$f_n = \sum_{k=0}^{n} (-1)^k \binom{2n-k+1}{k} 2^{2n-2k}, \quad g_n = \sum_{k=0}^{n} (-1)^k \binom{2n-k}{k} 2^{2n-2k}$$

Then $f_n = \sum_{k=0}^{n} (-1)^k \left(\binom{2n-k}{k} + \binom{2n-k}{k-1} \right) 2^{2n-2k} = g_n + \sum_{k=1}^{n} (-1)^k \binom{2n-k}{k-1} 2^{2n-2k}$; in the second sum, set $k-1 = i$, we find that $f_n = g_n - f_{n-1}$.

Similarly, we have $g_n = \sum_{k=0}^{n} (-1)^k \left(\binom{2n-k-1}{k} + \binom{2n-k-1}{k-1} \right) 2^{2n-2k} = 4f_{n-1} - g_{n-1}$.

So $f_n + f_{n-1} = g_n = 4f_{n-1} - (f_{n-1} + f_{n-2}) = 3f_{n-1} - f_{n-2}$. Hence $f_n = 2f_{n-1} - f_{n-2}$, with initial values $f_0 = 1$ and $f_1 = 2$. Solving this recursion we get $f_n = n+1$. It follows that $g_n = f_n + f_{n-1} = 2n+1$. This gives a direct proof of the identity in this question.

4. (2 points) 10G (Van Lint/Wilson, page 96)

Solution. Define S to be the set of all permutations of $1, 2, \ldots, 2n$. So $|S| = (2n)!$. For $i = 1, 2, \ldots, 2n-1$, define $E_i = \{x_1 x_2 \ldots x_{2n} \mid x_i + x_{i+1} = 2n+1\}$. Note that $|E_i \cap E_{i+1}| = 0$ for all i, $1 \leq i \leq 2n-2$. The number we are seeking in this question is equal to $|\bar{E}_1 \cap \bar{E}_2 \cap \cdots \cap \bar{E}_{2n-1}|$, which, by the Principle of Inclusion and Exclusion, is

$$(2n)! + \sum_{k=1}^{n} (-1)^k \binom{2n-k}{k} 2n \cdot (2n-2) \cdot (2n-4) \cdots (2n-2k+2) \cdot (2n-2k)!$$

5. (2 points) Let k, n be fixed positive integers. Show that

$$\sum c_1 c_2 \cdots c_k = \binom{n+k-1}{2k-1},$$

where the sum ranges over all compositions $c_1 + c_2 + \cdots + c_k$ of n into k parts.

Solution. It is clear that the sum in this question is equal to the coefficient of x^n in

$$f(x) := (x + 2x^2 + 3x^3 + \cdots + ix^i + \cdots)^k$$

Note that $x + 2x^2 + 3x^3 + \cdots + ix^i + \cdots = x \cdot (\frac{1}{1-x})' = \frac{x}{(1-x)^2}$. Hence the coefficient of x^n in $f(x)$ is the coefficient of x^{n-k} in $(1-x)^{-2k}$, which is

$$\binom{2k-1+n-k}{n-k} = \binom{n+k-1}{2k-1}.$$