Solutions to Problem 1.

$$\sum_{n\geq 0}(n+2)^2x^n = \sum_{n=1}^{\infty} n^2x^n + 4\sum_{n=1}^{\infty} n\,x^n + 4\sum_{n=0}^{\infty} x^n$$

$$= \frac{x}{(1-x)^2} + \frac{2x^2}{(1-x)^3} + \frac{4x}{(1-x)^2} + \frac{4}{(1-x)}$$

$$= \frac{x^2-3x+4}{(1-x)^3}$$

$$\sum_{n\geq 0}(n+2)^2\frac{x^n}{n!} = 4\sum_{n=0}^{\infty}\frac{x^n}{n!} + 4\sum_{n=0}^{\infty} n\frac{x^n}{n!} + \sum_{n=0}^{\infty} n^2\frac{x^n}{n!}$$

$$= 4e^x + 4xe^x + (xe^x + x^2e^x)$$

$$= (x^2+5x+4)e^x$$

$$\sum_{n\geq 0}(n+2)^2\binom{2n}{n}x^n = \frac{4-22x+36x^2}{(1-4x)^{3/2}}$$

2. problem 14K, page 149

Call the number of walks A_{4n} and define

$A(z) = 1 + \sum_{n=1}^{\infty} A_{4n} z^{4n}$. If B_{2n} is the number of walks from $(0,0)$ to (n,n) that avoid points (i,i), then we know from (14.12) that $B(z) = \sum_{n=1}^{\infty} B_{2n} z^{2n} = 1 - \sqrt{1-4z^2}$.

By classifying the paths (not passing through $(2i-1, 2i-1)$) according to their FIRST intersection with the diagonal, we get

$$A_{4n} = B_4 A_{4n-4} + B_8 A_{4n-8} + \cdots + B_{4n} A_0,$$

with initial value $A_0 = 1$. In terms of generating functions, we have

$$A(z)\left(B_4 z^4 + B_8 z^8 + \cdots\right) = A(z) - 1$$

So

$$A(z) = \frac{1}{1-(B_4 z^4 + B_8 z^8 + \cdots)} = \frac{1}{1 - \frac{B(z)+B(zi)}{2}}$$

$$= \frac{\sqrt{1+4z^2} - \sqrt{1-4z^2}}{4z^2}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{1}{2k+1}\binom{4k}{2k} z^{4k}$$

QED

3. Let n be a positive integer, q a prime power, and let $f_2(n)$ = the number of co-prime pairs of monic polys of degree n over $\mathbb{F}_q$. Find a simple formula for $f_2(n)$.

Consider a pair $(f(x), g(x))$ of monic polys of degree n over $\mathbb{F}_q$. We can write

$$(f(x), g(x)) = d(x) \left(\frac{f(x)}{d(x)}, \frac{g(x)}{d(x)} \right),$$

where $d(x) = \gcd(f(x), g(x))$. From this factorization, we obtain

$$q^{2n} = \sum_{m=0}^{n} q^m f_2(n-m).$$

In terms of generating functions, we have

$$\sum_{n=0}^{\infty} q^{2n} x^n = \frac{1}{(1-qx)} \left[\sum_{n=0}^{\infty} f_2(n) x^n \right].$$

So $$\sum_{n=0}^{\infty} f_2(n) x^n = \frac{1-qx}{1-q^2x} = \frac{1}{1-q^2x} - \frac{qx}{1-q^2x}$$

Hence $f_2(n) = q^{2n} - q^{2n-1} = q^{2n-1}(q-1), \ \forall n \geq 1.$

Solution to Problem 10877 of *The American Mathematical Monthly*

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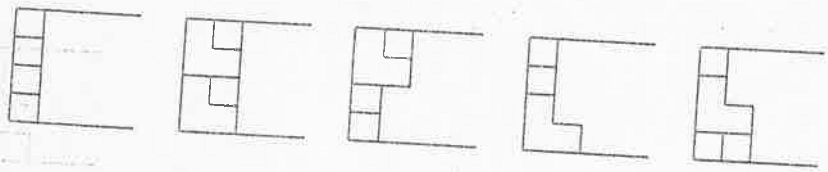
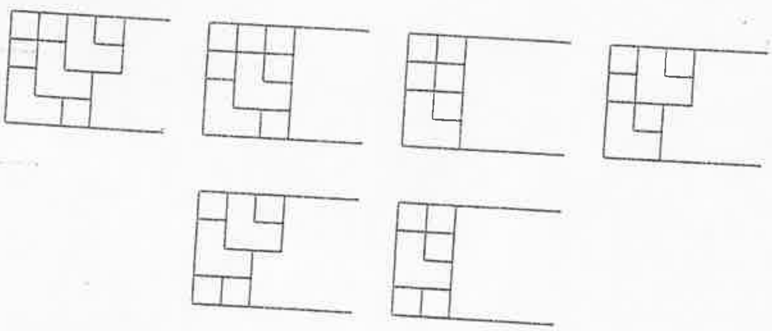
Problem. An L-tile is a 2-by-2 square with the upper right 1-by-1 subsquare removed; no rotations are allowed. Let a_n be the number of tilings of a 4-by-n rectangle using tiles that are either 1-by-1 squares or L-tiles. Find a closed form for the generating function $1 + a_1x + a_2x^2 + \ldots$.

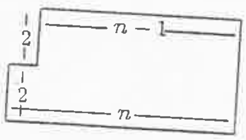
Solution. If we look at the leftmost columns of a 4-by-n rectangle, we can explicitly describe all the ways to tile these with 1-by-1 squares and L-tiles:



We can look at a few more columns to be even more precise about what must occur in the fourth and fifth examples above.



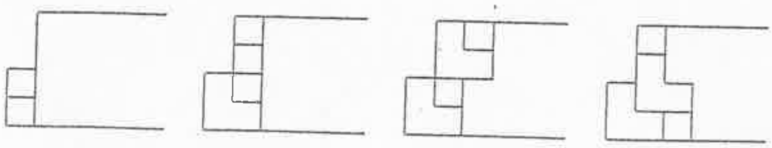
Thus any L-tiling of a 4-by-n rectangle must fall into one of these categories: the first, second, or third of the top row of diagrams, or any in the second or third rows of diagrams. If we let b_n be the number of tilings of the rectilinear region:



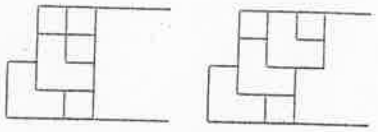
by 1-by-1 squares and L-tiles, then we have the relation:

$$a_n = a_{n-1} + 3a_{n-2} + a_{n-3} + b_{n-1} + 2b_{n-2} + b_{n-3}. \tag{1}$$

We can similarly categorize all tilings of the rectilinear figure above, by again considering how the leftmost columns must be tiled:



Notice that the fourth of these diagrams must have one of the following two forms:



Thus we also get the relation:

$$b_n = a_{n-1} + a_{n-2} + a_{n-3} + b_{n-2} + b_{n-3}. \tag{2}$$

Define $a_0 = 1$, $a_{-1} = a_{-2} = 0$, and $b_0 = b_{-1} = b_{-2} = 0$. Combinatorially these are the logical choices since not all the reductions are valid for small values of n, and these definitions allow Equations (1) and (2) to hold for all $n > 0$.

If we let $A(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum\limits_{n=0}^{\infty} b_n x^n$, we can find a closed form for $A(x)$ using standard techniques for solving recurrence relations:

Multiply Equation (1) by x^n and sum both sides for $n \geq 3$. This gives:

$$\sum_{n=3}^{\infty} a_n x^n = x\sum_{n=2}^{\infty} a_n x^n + 3x^2 \sum_{n=1}^{\infty} a_n x^n + x^3 \sum_{n=0}^{\infty} a_n x^n + x\sum_{n=2}^{\infty} b_n x^n + 2x^2 \sum_{n=1}^{\infty} b_n x^n + x^3 \sum_{n=0}^{\infty} b_n x^n. \tag{3}$$

Given the definitions of $A(x)$ and $B(x)$, we can now simplify this to:

$$\begin{aligned} A(x)(1 - x - 3x^2 - x^3) &= B(x)(x + 2x^2 + x^3) + (a_0 + a_1 x + a_2 x^2) - x(a_0 + a_1 x) \\ &\quad -3x^2(a_0) - x(b_0 + b_1 x) - 2x^2(b_0). \end{aligned}$$

A similar procedure with Equation (2) yields:

$$B(x)(1 - x^2 - x^3) = A(x)(x + x^2 + x^3) + (b_0 + b_1 x + b_2 x^2) - x(a_0 + a_1 x) - x^2(a_0) - x^2(b_0).$$

Using the Equations (1) and (2), we find that $a_1 = 1$, $a_2 = 5$, $b_1 = 1$, and $b_2 = 2$, so we can simplify these further to obtain:

$$A(x)(1 - x - 3x^2 - x^3) = B(x)(x + 2x^2 + x^3) + 1; \tag{4}$$

$$B(x)(1 - x^2 - x^3) = A(x)(x + x^2 + x^3). \tag{5}$$

Solving for $B(x)$ in Equation (5) and substituting this into Equation (4) gives the desired generating function:

$$A(x) = \frac{1 - x^2 - x^3}{1 - x - 5x^2 - 4x^3 + x^5} = \frac{1 - x^2 - x^3}{(1 + x)(1 - 2x - 3x^2 - x^3 + x^4)}. \tag{6}$$

(a) Extend the definition of $f(m,n)$ to include $m=0$ or $n=0$ but not both. Set $f(0,n)=1$ for all $n\geq 1$, and $f(m,0)=1$ for all $m\geq 1$.

Consider the possible values of x_n, we see that if $n\geq 1$, then

$$f(m,n) = \sum_{i=-m}^{m} f(m-|i|, n-1) \qquad (\because x_n \text{ can be any } i,\ i=-m,\cdots,0,\cdots,m)$$

$$= f(m,n-1) + 2\sum_{i=1}^{m} f(m-i,n-1).$$

Applying the above to $(m-1)$ for positive m, we obtain

$$f(m-1,n) = f(m-1,n-1) + 2\sum_{i=1}^{m-1} f(m-1-i,n-1). \quad \text{So}$$

$$f(m,n) = f(m,n-1) + 2f(m-1,n-1) + 2\sum_{i=1}^{m-1} f(m-1-i,n-1)$$

$$= f(m,n-1) + f(m-1,n) + f(m-1,n-1).$$

(b) Using strong induction on $m+n$, one can easily show that $f(m,n)=f(n,m)$

(c) By the solution to a HW problem, we know that

$$\sum_{m\geq 0} f(m,n)x^m = \frac{(1+x)^n}{(1-x)^{n+1}}, \quad \forall n\geq 0.$$

It follows that $\displaystyle\sum_{n\geq 0}\sum_{m\geq 0} f(m,n)x^m y^n = \sum_{n\geq 0}\frac{(1+x)^n}{(1-x)^{n+1}}y^n$

$$= \frac{1}{1-x}\,\frac{1}{1-\frac{1+x}{1-x}y} = \frac{1}{1-x-y-xy}$$

Note that the last expression is symmetric in x and y. Thus

$$\sum_{n\geq 0}\sum_{m\geq 0} f(m,n)x^m y^n = \sum_{n\geq 0}\sum_{m\geq 0} f(m,n)y^m x^n = \sum_{n\geq 0}\sum_{m\geq 0} f(n,m)x^m y^n$$

$$\Rightarrow f(m,n)=f(n,m).$$

(d) Again using the Solution to a HW problem, we have

$$f(n,n) = \sum_{i=0}^{n} \binom{n}{i}\binom{2n-i}{n}$$

$$\sum_{n\geq 0} f(n,n)x^n = \sum_{n\geq 0}\sum_{i=0}^{n}\binom{n}{i}\binom{2n-i}{n}x^n$$

$$= \sum_{n\geq 0}\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{n}x^n$$

$$= \sum_{n\geq 0}\sum_{k=0}^{n}\frac{n!}{k!\,(n-k)!}\,\frac{(n+k)!}{n!\,k!}\,x^n$$

$$= \sum_{n\geq 0}\sum_{k=0}^{n}\binom{2k}{k}\binom{n+k}{2k}x^n$$

$$= \sum_{k=0}^{\infty}\binom{2k}{k}\sum_{n\geq k}\binom{n+k}{2k}x^n$$

$$= \sum_{k=0}^{\infty}\binom{2k}{k}\frac{x^k}{(1-x)^{2k+1}} = \frac{1}{1-x}\left(1-\frac{4x}{(1-x)^2}\right)^{-1/2}$$

$$= \frac{1}{\sqrt{1-6x+x^2}}.$$