MAT8010 Homework #7

1. (4 points) Problem 19H, page 226 (Van Lint/Wilson).

2. (4 points) Let Γ be an srg(v, k, λ, μ) and let $-s$ be its smallest eigenvalue. If C is a coclique (independent set) of Γ, then $|C| \leq sv/(k+s)$, equality holds if and only if every vertex x of Γ not in C has exactly s neighbors in C. This is a special case of the Hoffman bound, which is stated as follows.

If C is a coclique (independent set of vertices) in a k regular graph G on v vertices, $k > 0$, then

$$|C| \leq \frac{-\theta v}{k - \theta},$$

where θ is the least eigenvalue of G.

3. (4 points) Problem 21Q. Prove the so-called Friendship Theorem: At a party with n people ($n > 3$) every two persons have exactly one mutual friend. Then there exists a unique person at the party who is a friend of all the others.

Proof. Suppose the friendship theorem is false, and let G be a friendship graph with the largest degree $\Delta < n-1$. Let us show first of all that G is regular. Consider two nonadjacent vertices x and y, where, without loss of generality, $d(x) \geq d(y)$. By assumption, x and y have exactly one common neighbour, z. For each neighbour v of x other than z, denote by $f(v)$ the common neighbour of v and y. Then f is a one-to-one mapping from $N(x) \setminus \{z\}$ to $N(y) \setminus \{z\}$. Because $|N(x)| = d(x) \geq d(y) = |N(y)|$, we conclude that f is a bijection and hence that $d(x) = d(y)$. Thus any two nonadjacent vertices of G have the same degree; equivalently, any two adjacent vertices of G have the same degree. In order to prove that G is regular, it therefore suffices to show that G is connected. But G has no singleton component, because $\delta(G) = n - 1 - \Delta(G) > 0$, and cannot have two components of order two or more, because G would then contain a 4-cycle, thus two vertices with two common neighbours. Therefore G is k-regular for some positive integer k. Moreover, by counting the number of 2-paths in G in two ways, we have $n\binom{k}{2} = \binom{n}{2}$; that is, $n = k^2 - k + 1$. Let A be the adjacency matrix of G. Then we have $A^2 = J + (k-1)I$, where J is the $n \times n$ matrix all of whose entries are 1, and I is the $n \times n$ identity matrix. Because the eigenvalues of J are 0, with multiplicity $n-1$, and n, with multiplicity 1, the eigenvalues of A^2 are $k-1$, with multiplicity $n-1$, and $n + k - 1 = k^2$, with multiplicity 1. The graph G therefore has eigenvalues $\pm\sqrt{k-1}$, with total multiplicity $n-1$, and k, with multiplicity 1. Because G is simple, the sum of its eigenvalues, the trace of A, is zero. Thus $t\sqrt{k-1} = k$ for some integer t. But this implies that $k = 2$ and $n = 3$, contradicting the assumption that $\Delta < n-1$. So there is a person who is a friend of all others (such a person is called a politician). The uniqueness is clear.