

MAT8010 Homework #5

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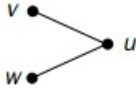
1. Let G be a simple graph with 10 vertices and 26 edges. Show that G has at least 5 triangles. Can equality occur?

Solution:

2. Show that a graph on n vertices that does not contain a circuit on four vertices has at most $\frac{n}{4}(1 + \sqrt{4n - 3})$ edges

Proof. (Refer from Reiman 1958):

Let G be a graph statified above requirement, and d_1, d_2, \dots, d_n be the degrees of its vertices. We now count in two ways the number of elements in the following set S . The set S consists of all (ordered) pairs $(u, \{v, w\})$ such that $v \neq w$ and u is adjacent to both v and w in G . That is, we count all occurrences of “cherries”



in G . For each vertex u , we have $\binom{d_u}{2}$ possibilities to choose a 2-element subset of its d_u neighbors. Thus, summing over u , we find $|S| = \sum_{u=1}^n \binom{d_u}{2}$. On the other hand, because G are C_4 -free graph, which implies that no pair of vertices $v \neq w$ can have more than one common neighbor. Thus, summing over all pairs we obtain that $|S| \leq \binom{n}{2}$. Altogether this gives

$$\sum_{i=1}^n \binom{d_i}{2} \leq \binom{n}{2}$$

or

$$\sum_{i=1}^n d_i^2 \leq n(n-1) + \sum_{i=1}^n d_i \tag{1}$$

Now, we use the Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^n x_i y_i \right)^2 \leq \left(\sum_{i=1}^n x_i^2 \right) \left(\sum_{i=1}^n y_i^2 \right)$$

with $x_i = d_i$ and $y_i = 1$, and obtain

$$\left(\sum_{i=1}^n d_i \right)^2 \leq \sum_{i=1}^n d_i^2 \cdot n$$

and hence by (1)

$$\left(\sum_{i=1}^n d_i \right)^2 \leq n^2(n-1) + n \sum_{i=1}^n d_i$$

Euler's theorem gives $\sum_{i=1}^n d_i = 2|E|$. In voking this fact, we obtain

$$4|E|^2 \leq n^2(n-1) + 2n|E|$$

or

$$|E|^2 - \frac{n}{2}|E| \leq \frac{n^2(n-1)}{4} \leq 0$$

Solving the corresponding quadratic equation yields the desired upper bound on $|E|$.

3. Let $H_3(r)$ denote the number of 3×3 matrices with nonnegative integer entries such that each row and each column sum to r . Show that

$$H_3(r) = \binom{r+5}{5} - \binom{r+2}{5}$$

Do this problem by using Theorem 5.5. Do not use the PLE.

4. For $n \geq 6$, let
5. Suppose that in a town of n residents, every club has an even number of members, and any two distinct clubs have an odd number of members in common. Then the maximum of clubs is n if n is odd, and $n-1$ if n is even.
6. Let $\mathcal{A} = \{A_1, \dots, A_m\}$ be a collection of m distinct subsets of $N := \{1, 2, \dots, n\}$ such that if $i \neq j$ then $A_i \not\subseteq A_j$, $A_i \cap A_j \neq \emptyset$, $A_i \cup A_j \neq N$. Prove that

$$m \leq \binom{n-1}{\left\lfloor \frac{n}{2} \right\rfloor - 1}$$