MAT8010 Practice Problems

1. Let p be a prime, and let $n = \sum a_i p^i$ and $m = \sum b_i p^i$ be the p-ary expansions of the positive integers m and n, respectively. (So here a_i and b_i are the base-p digits of n and m, respectively, and $0 \leq a_i \leq p-1$ and $0 \leq b_i \leq p-1$ for all i.)

(a). Show that

$$\binom{n}{m} \equiv \binom{a_0}{b_0}\binom{a_1}{b_1}\cdots \pmod{p}$$

(b). Use (a) to determine when $\binom{n}{m}$ is odd. For what positive integers n are $\binom{n}{m}$ odd for all $0 \leq m \leq n$.

2. Show that

$$(1-4x)^{-\frac{1}{2}} = \sum_{n\geq 0}\binom{2n}{n}x^n.$$

Find a closed form for $\sum_{n\geq 0}\binom{2n-1}{n}x^n$.

3. Use generating functions to prove the following identitities.

$$(1).\quad \sum_{k=0}^{m}\binom{m+n}{n+k} = \sum_{\ell=0}^{m}\binom{\ell+n-1}{\ell}2^{m-\ell}$$

$$(2).\quad \sum_{k=0}^{n}\binom{2n+1}{2k+1}\binom{m+k}{2n} = \binom{2m}{2n}$$

4. Let $\bar{c}(m,n)$ denote the number of compositions of n with largest part at most m. Show that

$$\sum_{n\geq 0}\bar{c}(m,n)x^n = \frac{1-x}{1-2x+x^{m+1}}.$$

5. Let $f(m,n)$ be the number of $m \times n$ matrices of 0's and 1's with at least one 1 in every row and column. Use the PIE to show that

$$f(m,n) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}(2^{n-k}-1)^m.$$

Furthermore, show that

$$\sum_{m\geq 0}\sum_{n\geq 0} f(m,n)\frac{x^m y^n}{m!n!} = e^{-x-y}\sum_{i\geq 0}\sum_{j\geq 0}\frac{2^{ij}x^i y^j}{i!j!}.$$

6. Using only the combinatorial definitions of the Stirling numbers $S(n,k)$ and $c(n,k)$ (here $c(n,k)$ is the signless Stirling number of the first kind), give formulas for $S(n,1)$, $S(n,2)$, $S(n,n)$, $S(n,n-1)$, $S(n,n-2)$ and $c(n,1)$, $c(n,2)$, $c(n,n)$, $c(n,n-1)$, $c(n,n-2)$. For the case $c(n,2)$, express your answer in terms of the harmonic number $H_m = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}$ for suitable m.

7. Let $n \geq 1$, and let $f(n)$ be the number of partitions of n such that for all k, the part k occurs at most k times. Let $g(n)$ be the number of partitions of n such that no part has the form $i(i+1)$, i.e., no parts equal to $2, 6, 12, 20, \ldots$. Show that $f(n) = g(n)$.

8. Problem 10G, p.96 (Van Lint/Wilson).

9. Problem 3I, p.34 (Van Lint/Wilson). (You need the fact that the set of nonzero elements of $\mathbf{F}_{16}$ forms a cyclic group with respect to multiplication.)

10. Find the Prüfer sequences of the following trees:

$$T_1 = ([6], \{12, 13, 14, 15, 56\}).$$

$$T_2 = ([8], \{12, 13, 14, 18, 25, 26, 27\}).$$

$$T_3 = ([11], \{12, 13, 24, 25, 36, 37, 48, 49, \{5, 10\}, \{5, 11\}\}).$$

11. The n-cube is the graph Q_n with set of vertices $\{0,1\}^n$ and where two vertices $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$ are adjacent if they differ exactly in one coordinate.

(a) Draw Q_n for $1 \leq n \leq 4$.

(b) Determine the order (number of vertices), the size (number of edges) and the degree sequence of Q_n.

(c) Let ℓ be a positive integer. Prove that the number of closed walks of length ℓ in Q_n is

$$\sum_{i=0}^{n}\binom{n}{i}(n-2i)^{\ell}.$$