§4 Turan's Thm & Extremal Graphs

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April 26, 2018

How many edges must a simple graph on n vertices have to gurantee the graph to have a triangle? or what's the maximum # of edges of a simple graph on n vertices s.t. there is no Δ?

Thm: Let G be a simple graph on n vertices. If G has no Δ, then

$$e \leqslant \lfloor \frac{n^2}{4} \rfloor$$

In otherwords. if $e(G) > \lfloor \frac{n^2}{4} \rfloor + 1$, G must have a Δ.

Proof. (1) Any $xy \in E(G)$,

$\sum$ □

THe indpendence number of a simple graph G= The size of a largest coclique(or indep set) in G.

Proof. Let α = the indep number of G, and let A be a coclique of size α, $\beta = V(G) \langle \backslash \rangle A$

Claim: $\forall x \in V(G)$,$d(x < \alpha)$

Claim: $\forall e \in E(G)$, at least one end of e in β

count $\left\{ (e, x) \left| \begin{array}{l} e \in E(G) \\ e \in \beta \\ x \,\text{is incident with}\, e \end{array} \right. \right\}$

$e(G) = |E(G)| \leqslant \sum_{e \in E(G)} (1 \,\text{or}\, 2) = \sum_{x \in \beta} d(x) \leqslant \alpha |\beta| \leqslant \left(\frac{\alpha + |\beta|}{2} \right)^2 = \left(\frac{n}{2} \right)^2 = \frac{n^2}{4}$ □

Clear Maximal(极大) and maximum(最大)

Extremal graphs:

$\text{ex}(H, n) =$ the largest number of edges in a simple graph on n vertrices which doesn$'t$ contauin H as a subgraph.

exclude

Turan's thm

If a simpel G on n vertics contains no copy of K_{v+1}, then it has at most $\left(1-\frac{1}{r}\right)\frac{n^2}{2}$ edges.

$$e(G)=e(A)+e(B)+e(A,B)$$

Proof. Use strong induction on $n=r$.

Assumet that the thm is true for graph on <n vetieces

Let G be a graph on n vercties without K_{r+1} with maximum # of edges,

claim: G has a copy(subset) of K_r(otherwise we could add eges to G so that G has a K_r and still has no K_{r+1})

Let A be the set of vertices of this K_r,let $\beta=V(G)\langle\backslash\rangle A$ □

Complete

Y= partitie graphs, $K_{n_1,n_2,\ldots,n_r}$

$n_1+n_2+\ldots+n_r=n$

$\#\text{of eges}=\sum_{1\leqslant i\leqslant j\leqslant r} n_o n_j$

$|n_i-n_j|\leqslant 1, \forall i\neq j$

The *girth* of a graph G = the size of a smallest cycle(polygon P_n) in G. (If G has no cycles then we say that girth of G is ∞, such as forest)

Girth >= 3 <=> G is simple

Girth >= 4 <=> G is simple and no Δ

G has girth >=5 <=>

Theorem 4.2. If a graph G on n vertices has more than $\frac{1}{2}n\sqrt{n-1}$ edges, then G has girth $\leqslant 4$. That is, G is not simple or contains a P_3 or a P_4(a triangle or a quadrilateral).

Proof. $\forall x\in V(G)$,

Claim 1: no two of $y_1, y_2, \ldots, y_n$ are adjacent

Claim 2: no vertex other than x can be adjacent to more than one of $y_1, y_2, \ldots, y_d$

$$(\deg(y_1)-1)+(\deg(y_2)-1)+\ldots+(\deg(y_d)-1)+d+1\leqslant n$$

Then

$$\frac{1}{n}(2|E(G)|)^2=\frac{1}{n}\left(\sum_{y\in V(G)}\deg(y)\right)^2\leqslant\sum_{y\in V(G)}\deg(y)^2=\sum_{x\in V(G)}\sum_{y\text{ adjacent to }x}\deg(y)\leqslant n(n-1)$$

$$\frac{1}{n}+4E(G)^2=n(n-1)??$$ □

i. $n = 1 + d^2, d\colon a\,\text{positive interger}$

ii. The grith is regular(The equality in c-s inqualigty holds four situation)

iii. no $\Delta(\text{girth} \geqslant 5)$

iv. $\forall x, y$, xy $\in E(G)$ $\exists$!z,xz, yz $\in E(G)$

A=the adjecncy matrix of G

$$(A^2)_{\text{xy}} = \text{the \#of walks of lenth 2 between } x \text{ and } d$$

Vx $\in V(G)$, $\sum_{y\sim x} \deg(y) \leqslant -1$

$A^2 = dI, +0.A_{+1(J-I-A)}$

AJ = dJ

eigenvalues s of A

(1) For adj martix A odf G trhe eighenvals are real $\exists\alpha$ baisis of R^n consosting eigenveacotr of A.

(2) A d-regualr gra;phn G has d as an eigenval . Infact d is the larigest eigencal. The mulitip of d as an eigenval=#connected comps of G

i. $1 + f + g = n = 1 + d^2$

ii. $d + f_r + g = 0$

iii.

iv.

* $d + f_r + g = 0$