# Homework #4

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1. Problem 15C. Let $a_1$, $a_2$, ..., at be positive integers (not necessarily distinct) with greatest common divisor 1. Let $f(n)$ denote the number of solutions of

$$n = a_1x_1 + a_2x_2 + \ldots + a_tx_t$$

in nonnegative integers $x_1$, ..., $x_t$. What is the generating function $F(x)$: $= f(n)x^n$? Show that $f(n) \sim \mathrm{cn}^{t-1}$ for some constant $c$ and explicitly give $c$ as a function of $a_1, \ldots, a_t$.

2. Find a bijection proof of the fact that $u(n) = o(n)$, where $u(n)$ is the number of partitions of $n$ with unequal parts, and $o(n)$ is the number of partitions of $n$ with odd parts.

Solve:

**Odd parts → Distinct parts**

Every number can be expressed uniquely as a power of 2 multiplied by an odd number. Given a partition of $n$ into odd parts. Count the number of times each odd number $o(n)$ occurs: suppose 1 occurs $a_1$ times, similaryly 3 occurs $a_3$ times, etc. So $n = a_1 1 + a_3 3 + a_5 5 + \cdots$.

Now write each $a_i$ "in binary", i.e., as a sum of distinct powers of two. So we havee $n = (2^{b11} + 2^{b12} + \cdots)1 + (2^{b31} + 2^{b32} + \cdots)3 + \cdots$. After getting rid of the brackets, we can found all terms are distinct.

**Distinct → Odd parts**

Given a partition into distinct parts, we can write each part as a power of 2 mulitplied by an odd number, and collect the coefficients of each odd number, and wirte the odd number those many times, to get a partition into odd parts.(ref from https://math.stackexchange.com)

4. Problem 1F. The girth of a graph is the length of the smallest polygon in the graph. Let G be a graph with girth 5 for which all vertices have degree ≥ d. Show that G has at least $d^2 + 1$ vertices.

Prove: Let fix a vertex $v$ of $G$. Since each vertex of $G$ has degress ⩾ d, there are at least of vertices $v_1, v_2 \cdots$ and $v_d$ with distance 1 from $v$. Since the girth of $G$ is 5, $G$ has no 3 or 4-cycles. Using this fact that and the vertices $v_1, v_2, \cdots, v_d$, we can construct at leat $d(d-1)$ new vertices with distance two from $v$. We choose $d-1$ distance 1 vertices $v_{i1}, v_{i2}, \ldots, v_{i(d-1)}$ from each vertex $v$, (different than $v$) for $1 < i < d$. There new vertices have distance 2 from $v$. Thus $|V_G| \geqslant 1 + d + d(d-1) = d^2 + 1$.(ref http://www-users.math.umn.edu/~akhmedov/)