TABLE 2. Estimates of TEAM insolvency risk over multiyear horizon.

$\mu = \sigma = .15, \quad c = .05, \quad C_0 = S_0 = 0.5$

Investment Horizon (n)	5	10	15	20	25
S_0'	.7318	1.178	1.898	3.058	4.929
$E(T_n')$	1.721	3.457	7.716	18.11	43.13
$E(B_n)$	1.644	2.837	5.108	9.510	18.15
$\sqrt{\mathrm{Var}(T_n')} = \sqrt{\mathrm{Var}(B_n)}$	.2984	.8671	2.183	5.184	11.92
α	1.366	.8486	.5269	.3270	.2029
Insolvency Risk:					
Lower Bound	4.01×10^{-9}	3.45×10^{-5}	.00085	.0066	.0258
Monte Carlo	0	6.67×10^{-5}	.00333	.0220	.0696
Upper Bound	4.50×10^{-9}	1.85×10^{-4}	.00738	.0523	.1692

uous, although we can imagine a high-stakes player attempting to implement such a strategy.

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The Friendship Theorem

Craig Huneke

The 'friendship theorem' can be stated as follows [**1**, p. 183]:

> *Suppose in a group of at least three people we have the situation that any pair of persons have precisely one common friend. Then there is always a person who is everybody's friend.*

The first published proof of this theorem of which I am aware was due to Paul Erdős, Alfred Rényi, and Vera Sós [**3**]. Translating the theorem into graph theory yields the following theorem:

Theorem. *If G is a graph in which any two distinct vertices have exactly one common neighbor, then G has a vertex joined to all others.*

As a consequence, such graphs are completely determined; they consist of edge-disjoint triangles around a common vertex. The best known and simplest proof is based on computing the eigenvalues (and their multiplicities) of the square of the adjacency matrix of the graph. Such a proof is given in [**1**]. In [**6**], a similar idea is used, but phrased in terms of showing the graph has the structure of a 'projective plane'; see also [**2**]. In his recent book [**5**, p. 466], West writes, "It is startling that such a combinatorial-sounding result seems to have no short combinatorial proof. There do exist proofs avoiding eigenvalues (see [**4**]), but they require complicated numerical arguments to eliminate regular graphs." The goal of this note is to provide one proof which is more combinatorial, and another proof which does not explicitly use eigenvalues (though it does use traces), and in some sense combines the combinatorics with the linear algebra.

I first heard of this problem from William Lang as a graduate student in 1975; he challenged me to solve it. I used the idea of counting walks of length p, where p is a carefully chosen prime number, to construct a proof that same year. I recently showed this proof to my colleague Fred Galvin, who then significantly simplified it; this resulted in our first proof following. The first three paragraphs of the proof are standard; they reduce the problem to a regular graph. However, we make this reduction in a slightly different way than in the references to this paper.

Proof of Theorem. We first claim that if $x, y \in G$ and are not adjacent, then they have the same degree (i.e., the same number of adjacent vertices). Let $N(x)$ denote the vertices adjacent to x (the 'neighborhood' of x). Define a map $\alpha : N(x) \to N(y)$ by sending a vertex $z \in N(x)$ to the common neighbor of z and y. This common neighbor cannot be x, as x and y are not adjacent. The map is one-to-one since $z \in N(x)$ is uniquely determined as the common neighbor of x and $\alpha(z)$. By symmetry the map is onto.

Suppose there is a vertex of degree $k > 1$. We claim that all vertices have degree k, unless there is a universal friend. Let A be the set of all vertices of degree k, let B be the set of all vertices of degree different from k, and assume for a contradiction that B is nonempty. By the first claim, every vertex in A is adjacent to every vertex in B. If A or B is a singleton, then that singleton is a universal friend; otherwise, there are two different vertices in A, and they have two common neighbors in B, contradicting the hypothesis. It follows that G is k-regular, i.e., the degree of every vertex is k.

Next we claim that the number n of vertices in G is exactly $k(k-1)+1$. This follows by counting the paths of length two in G: by assumption there are $\binom{n}{2}$ such paths. For each vertex v, there are exactly $\binom{k}{2}$ paths of length two having v in the middle, giving $n \cdot \binom{k}{2}$ total paths of length two. Equating both counts permits us to conclude that $n = k(k-1)+1$. We can assume that $k \geq 3$, else $n = 3$ and the theorem is clear.

A *walk of length n* on G is an ordered sequence $v_0 v_1 \ldots v_n$ of vertices such that v_i and v_{i+1} are neighbors. We say the walk is *closed* if $v_n = v_0$. A closed walk is considered to have a starting point and an orientation, always returning to the starting vertex; thus, if $uvwu$ is a closed walk in G, then $uvwu$, $vwuv$, $vuwv$, etc. are considered distinct closed walks. It follows that, if p is a prime number, then the number of closed walks of length p is divisible by p.

For a fixed vertex v, let $f(n)$ be the number of walks from v to v of length n. If $n > 1$, then the number of closed n-walks $v_0 \ldots v_{n-2} v_{n-1} v_n$ from $v = v_0 = v_n$ with $v_{n-2} = v$ is $kf(n-2)$, and the number of such walks with $v_{n-2} \neq v$ is $k^{n-2} - f(n-2)$. The total number of walks $v_0 v_1 \ldots v_{n-2}$ from a fixed vertex $v = v_0$ is k^{n-2} as G is k-regular. Thus $f(n) = (k-1)f(n-2) + k^{n-2}$. Let p be a prime divisor of $k-1$; then $f(p) \equiv 1 \pmod{p}$. Finally, the total number of closed walks of length p is $[k(k-1)+1]f(p) \equiv 1 \pmod{p}$, contradicting the fact that the number of such walks is divisible by p. ■

Although the preceding proof requires no linear algebra, one can compare it to the proof of [**1**], which uses eigenvalues. Thinking about it gives another simple proof, provided one knows a little linear algebra in characteristic p.

Second Proof of the Friendship Theorem. We again reduce to the case in which the graph is k-regular, i.e., each vertex has exactly k adjacent vertices and the total number of vertices is $n = k(k-1)+1$, with $k \geq 3$. Let $G = \{v_1, \ldots, v_n\}$. We let A be the adjacency matrix of G, whose (i, j) entry is 1 if v_i and v_j are adjacent, and 0 otherwise. The matrix A has zeroes on the diagonal, so the trace of A is 0. We let B be the n by n matrix having a 1 in every entry. The trace of B is n.

By assumption and the fact that G is k-regular, $A^2 = (k-1)I + B$, and $AB = kB$, where I is the identity matrix of size n by n. We now pass to the field $\mathbf{Z}_p$, where p is a prime dividing $k-1$. We continue to call the matrices A and B, though we now think of them with entries in $\mathbf{Z}_p$. Observe that both n and k are now equal to 1. Hence $A^2 = B$, and furthermore $AB = kB = B$. It follows that for all $l \geq 2$, $A^l = B$. Let $\operatorname{tr} C$ denote the trace of a square matrix C. In characteristic p, $\operatorname{tr} A^p = (\operatorname{tr} A)^p$. We reach a contradiction: $1 = n = \operatorname{tr} B = \operatorname{tr} A^p = (\operatorname{tr} A)^p = 0$. ■

The relation between the first proof and the second is simply that the trace of A^p counts the closed walks of length p in the graph. The relationship between the second proof and the usual proof is clear: in characteristic 0, one computes the eigenvalues of A^2 and then proves that A could not have trace 0. The second proof takes advantage of the fact that $(a+b)^p = a^p + b^p$ in characteristic p. This allows us to avoid the actual computation of the eigenvalues, to push the calculation of the trace out to the pth power.

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A Short Proof of Lebesgue's Density Theorem

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Lebesgue's one-dimensional density theorem [1] says that almost all points of an arbitrary set $E \subseteq \mathbb{R}$ are points of density for E. We recall that a point $x \in \mathbb{R}$ is a point of density for E if one has