Problem 4A

1. Let G be a simple graph with 10 vertices and 26 edges. Show that G has at least 5 triangles. Can equality occur?

Proof. Let $N(n,e)$ denote the minimum number of triangles that a simple graph of order n & size e can have. We will show that $N(2k, k^2+1) = k$ and $N(2k+1, k^2+k+1) = k$.

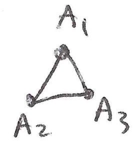
By Mantel's theorem, we know that

$$N(2k, k^2+1) \geq 1 \quad \text{and} \quad N(2k+1, k^2+k+1) \geq 1.$$

Next we observe that

$$N(n,e) \geq e - \left\lfloor \frac{n^2}{4} \right\rfloor. \quad \text{(Clear from Mantel's theorem.)}$$

Let $n = 2k$, $e = k^2+1$. Let $V(G) = A \cup B$, where

$A = \{A_1, A_2, A_3\}$, $B = \{B_1, B_2, \cdots, B_{2k-3}\}$, s.t. 

is a triangle in G. Let e_{AB} = the # of edges between A and B,

e_B = the # of edges between vertices in B.

Then $e_{AB} + e_B = e - 3 = k^2+1-3 = k^2-2$.

Let d_i, $1 \leq i \leq 2k-3$, be the # of edges in G that is incident with B_i and some vertex in A. Then $d_i = 0, 1, 2,$ or 3, and

$$d_1 + d_2 + \cdots + d_{2k-3} = e_{AB}.$$

The d_i edges incident with B_i will produce $\begin{cases} 1 \text{ triangle if } d_i = 2, \\ 3 \text{ triangles if } d_i = 3, \\ 0 \text{ triangle if } d_i = 0 \text{ or } 1. \end{cases}$

Each of these triangles has 2 vertices in A and one vertex in B. So the e_{AB} edges between A and B will produce at least $e_{AB} - (2k-3)$ triangles (with 2 vertices in A & 1 vertex in B), and by the inequality $N(n,e) \geq e - \lfloor \frac{n^2}{4} \rfloor$, the e_B edges within B will produce at least $e_B - (k-1)(k-2)$ triangles in B. Therefore we have at least

$$1 + e_{AB} - (2k-3) + e_B - (k-1)(k-2) = 1 + (k^2 - 2) - (2k-3)$$
$$= k \text{ triangles.}$$

$\Rightarrow N(2k, k^2+1) \geq k$.

The bipartite graph $K_{k,k}$ with one more edge added contains exactly k triangles. Hence $N(2k, k^2+1) = k$.

Similarly $N(2k+1, k^2+k+1) = k$. In particular, $N(10, 26) = 5$. $\square$

prob 4H. show that a graph on n vertices that does not contain a circuit on four vertices has at most $\frac{n}{4}\left(1+\sqrt{4n-3}\right)$ edges.

proof: Let G be a graph on n vertices without 4-cycles. Let $d(u)$ be the degree of u. Now we count the following set S in two ways: S is the set of pairs $(u, \{v,w\})$, where u is adjacent to both v & w, $v \neq w$.

Summing over u, we find $|S| = \sum_{u\in V} \binom{d(u)}{2}$. On the other hand, every pair $\{v,w\}$ has at most one common neighbor (by the no 4-cycle condition), Hence $|S| \leq \binom{n}{2}$. So

$$\sum_{u\in V} \binom{d(u)}{2} \leq \binom{n}{2}$$

$$\Rightarrow \quad \sum_{u\in V} d(u)^2 \leq n(n-1) + \sum_{u\in V} d(u)$$

By Cauchy-Schwartz's inequality,

$$\left(\sum_{u\in V} d(u)\right)^2 \leq n \sum_{u\in V} d(u)^2.$$

Hence

$$\frac{1}{n}\left(\sum_{u\in V} d(u)\right)^2 \leq n(n-1) + \sum_{u\in V} d(u).$$

$$4|E|^2 \leq n^2(n-1) + 2n|E|. \Rightarrow \quad |E|^2 - \frac{n}{2}|E| - \frac{n^2(n-1)}{4} \leq 0$$

Solving this quad. equation, we get $|E| \leq \frac{n}{4}\left(1+\sqrt{4n-3}\right)$.

3. Let $H_3(r)$ denote the number of 3×3 matrices with nonnegative integer entries such that every row and every column have sum r. Show that $H_3(r) = \binom{r+5}{5} - \binom{r+2}{5}$.

Proof. Let A be a 3×3 matrix with nonnegative integer entries and every row and every column of A sum to r. By Thm 5.5, A is a sum of r permutation matrices. That is,

$$(*) \qquad A = \sum_{\pi \in S_3} \alpha_\pi P_\pi, \quad \text{where } \alpha_\pi \geq 0,\ P_\pi \text{ is a permutation matrix}$$

and S_3 = the symmetric group on 3 letters

Since every row and every column sum to r, we have

$$(**) \qquad \sum_{\pi \in S_3} \alpha_\pi = r, \quad \alpha_\pi \geq 0.$$

The number of solutions to $(**)$ is $\binom{r+6-1}{6-1} = \binom{r+5}{5}$. However, two different solutions to $(**)$ may give rise to the same matrix A in $(*)$. So we need to see where the duplicates come from. Let

$$P_{123} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_{132} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_{213} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_{231} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$P_{312} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_{321} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

If $\alpha_{123} P_{123} + \alpha_{132} P_{132} + \alpha_{213} P_{213} + \alpha_{231} P_{231} + \alpha_{312} P_{312} + \alpha_{321} P_{321} = 0$,

then

$$\begin{pmatrix} \alpha_{123} + \alpha_{132} & \alpha_{213} + \alpha_{231} & \alpha_{312} + \alpha_{321} \\ \alpha_{213} + \alpha_{312} & \alpha_{321} + \alpha_{123} & \alpha_{132} + \alpha_{231} \\ \alpha_{321} + \alpha_{231} & \alpha_{312} + \alpha_{132} & \alpha_{213} + \alpha_{123} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence $\alpha_{123} = \alpha_{312} = \alpha_{231} = -\alpha_{132} = -\alpha_{213} = -\alpha_{321}$.

Therefore we see that if

$$A = \beta_{123} P_{123} + \beta_{132} P_{132} + \beta_{213} P_{213} + \beta_{231} P_{231} + \beta_{312} P_{312} + \beta_{321} P_{321}$$

with $\beta_{132} \geq 1$, $\beta_{213} \geq 1$, $\beta_{321} \geq 1$, then we can also write A as

$$A = (\beta_{123}+1) P_{123} + (\beta_{132}-1) P_{132} + (\beta_{213}-1) P_{213} + (\beta_{231}+1) P_{231}$$

$$+ (\beta_{312}+1) P_{312} + (\beta_{321}-1) P_{321}$$

While $(\beta_{123}, \beta_{132}, \beta_{213}, \beta_{231}, \beta_{312}, \beta_{321})$ and $(\beta_{123}+1, \beta_{132}-1, \beta_{213}-1, \beta_{231}+1, \beta_{312}+1, \beta_{321}-1)$

are two distinct solutions to $(**)$.

This tells us that if $(\alpha_{123}, \alpha_{132}, \alpha_{213}, \alpha_{231}, \alpha_{312}, \alpha_{321})$ is a solution to $(**)$

with $\alpha_{132} \geq 1, \alpha_{213} \geq 1, \alpha_{321} \geq 1$, with $\min\{\alpha_{132}, \alpha_{213}, \alpha_{321}\} = \alpha_{132} := a$, then

the matrix $A = \sum_{\pi \in S_3} \alpha_\pi P_\pi$ given by the solution $(\alpha_\pi : \pi \in S_3)$

was already given by $(\alpha_{123}+a, 0, \alpha_{213}-a, \alpha_{231}+a, \alpha_{312}+a, \alpha_{321}-a)$.

Therefore # of A's

$$= \binom{r+5}{5} - \Big(\text{the \# of solutions to } \sum_{\pi \in S_3} \alpha_\pi = r,$$

$$\alpha_{132} \geq 1, \alpha_{213} \geq 1, \alpha_{321} \geq 1\Big)$$

$$= \binom{r+5}{5} - \binom{r+2}{5}$$

4.

For $n \geq 6$, the maximum number of subsets of $[n]$ s.t. any two distinct of them intersect in an even number of elements is $2^{n/2}$ if n is even & $2^{(n-1)/2}+1$ if n is odd.

proof. Let $\mathcal{A} = \{A_1, A_2, \cdots, A_m\}$, $A_i \subseteq [n]$, $\forall i$

& $|A_i \cap A_j| =$ even $\forall\, i \neq j$.

We want to prove that if $n \geq 6$, then $m \leq 2^{n/2}$ when n is even

& $m \leq 2^{\frac{n-1}{2}}+1$ when n is odd.

Here we don't require $|A_i|$ to be even for all i. This problem is called the Strong Eventown theorem.

Let $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$, $\mathcal{A}_0$ consists of even sets & $\mathcal{A}_1$ consists of odd sets, such that any two distinct elements of $\mathcal{A}$ intersect in an even number of elems. Suppose that $|\mathcal{A}_0| = k$ and $|\mathcal{A}_1| = l$. We claim that $k \leq 2^{\lfloor \frac{n-l}{2} \rfloor}$, from this inequality we can deduce the claim in the problem easily.

Let $\mathcal{A}_1 = \{A_1, A_2, \cdots, A_l\}$. For $1 \leq i \leq l$, define $A_i^* = A_i \cup \{n+i\}$.
By construction, $\mathcal{A}_1^* = \{A_1^*, A_2^*, \cdots, A_l^*\}$ consists of even sets & so the family $\mathcal{A}^* = \mathcal{A}_0 \cup \mathcal{A}_1^*$ of even subsets of $[n+l]$ is such that any two of its sets intersect in an even number of elements.

Let's identify $\mathcal{A}^* = \mathcal{A}_0 \cup \mathcal{A}_1^*$ with subset S of the vector space $\mathbb{F}_2^{n+l}$, by letting each set $A \in \mathcal{A}^*$ correspond to its characteristic vector. $S = S_0 \cup S_1^*$ is a self-orthogonal set.

Let U, U_0 & U_1 be spanned by S, S_0 & S_1^*, respectively, so that $U = U_0 + U_1$.

Since A_i^* is the only member of $\mathcal{A}_0 \cup \mathcal{A}_1^*$ that contain $n+i$, S_1^* is an indep set and $U_0 \cap U_1 = \{0\}$. So $U = U_0 \oplus U_1$. $\dim(U_1) = |S_1^*| = \ell$.

$$\dim U = \dim U_0 + \dim U_1 = \dim U_0 + \ell \leq \left\lfloor \frac{n+\ell}{2} \right\rfloor .$$

$$\Rightarrow \dim U_0 \leq \left\lfloor \frac{n-\ell}{2} \right\rfloor \Rightarrow k \leq 2^{\left\lfloor \frac{n-\ell}{2} \right\rfloor} .$$

For this inequality, we can easily show that for $n \geq 6$,

$k+\ell \leq 2^{n/2}$ if n is even

& $k+\ell \leq 2^{\frac{n-1}{2}}+1$ if n is odd. □

5. Suppose that in a town with n residents, every club has an <u>even</u> number of members and any two clubs have an odd number of members in common. Then the maximum number of clubs is n if n is odd, and $n-1$ if n is even.

Proof: <u>Case 1</u>. n is odd.

$\mathcal{A} = \{A_1, \cdots, A_m\}$, $\quad A_i \subseteq [n] \ \forall i$

$|A_i| = \text{even} \ \forall i$, $|A_i \cap A_j| = \text{odd}$, $\forall i \neq j$.

Let $\mathcal{A}' = \{A_1', A_2', \cdots, A_m'\}$, where $A_i' = [n] \setminus A_i$, $\forall i$.

Then $|A_i'| = n - |A_i| = \text{odd}$, $\forall i$.

$$|A_i' \cap A_j'| = \underbrace{n}_{\text{odd}} - \underbrace{|A_i|}_{\text{even}} - \underbrace{|A_j|}_{\text{even}} + \underbrace{|A_i \cap A_j|}_{\text{odd}} = \text{even}, \ \forall i \neq j .$$

Therefore $\mathcal{A}'$ satisfies Oddtown rules. It follows that $|\mathcal{A}'| \leq n \Rightarrow |\mathcal{A}| \leq n$.

Case 2. n is even.

Let $\mathcal{A} = \{A_1, A_2, \cdots, A_\ell\}$, be such that $|A_i| = \text{even}$
$A_i \subseteq [n]$, $\forall i$ & $|A_i \cap A_j| = \text{odd}$ $\forall i \neq j$.

Define $A_i^* = A_i \cup \{n+1\}$. Then

$\mathcal{A}^* = \{A_1^*, \cdots, A_\ell^*\}$ consists of subsets of $[n+1]$ s.t. $|A_i^*| = \text{odd}$,
$A_i^* \subseteq [n+1]$ $\forall i$
& $|A_i^* \cap A_j^*| = \text{even}$, $\forall i \neq j$.

We can deduce immediately that $\ell \leq n+1$. But that is not enough.

We add to $\mathcal{A}^*$ two sets, namely ϕ, & $[n]$, to obtain

$$\mathcal{B} = \{A_1^*, A_2^*, \cdots, A_\ell^*, \phi, [n]\}.$$

Note that any two distinct subsets in $\mathcal{B}$ intersect in even # of pts.

By problem 4 (& its proof), we have

$$k = 2 \leq 2^{\left\lfloor \frac{(n+1)-\ell}{2} \right\rfloor} \Rightarrow n+1-\ell \geq 2 \Rightarrow \ell \leq n-1$$

$\square$

6. Problem 6C, Let $\mathcal{A} = \{A_1, A_2, \cdots, A_m\}$ be a collection of m distinct subsets of $[n] = N$ such that if $i \neq j$ then $A_i \not\subseteq A_j$, $A_i \cap A_j \neq \phi$ and $A_i \cup A_j \neq N$. Prove that $m \leq \binom{n-1}{\lfloor \frac{n}{2} \rfloor - 1}$.

proof: we will use Thm 6.6, page 57 of the textbook. First we prove the following claim: Let $\mathcal{A}$ be an antichain of subsets of N such that $A \in \mathcal{A} \Rightarrow A' = N \setminus A \in \mathcal{A}$. Then $\sum_{A \in \mathcal{A}} \frac{1}{\theta(A)} \leq 2$, where

$$\theta(A) = \min\left\{ \binom{n-1}{|A|-1}, \binom{n-1}{n-|A|-1} \right\}.$$

proof of the claim: First note that if $\mathcal{A}$ contains any set of size $\frac{n}{2}$, these sets must occur in complementary pairs. Form a subfamily $\mathcal{B}$ of $\mathcal{A}$ by putting into all those sets $A \in \mathcal{A}$, $|A| < \frac{n}{2}$, and one set from each complementary pair of sets of size $\frac{n}{2}$ in $\mathcal{A}$. Then $\mathcal{B}$ must be an intersecting family, for if $X, Y \in \mathcal{B}$ and $X \cap Y = \phi \Rightarrow X \subseteq Y'$, contradicting the antichain property. Since $\theta(A) = \binom{n-1}{|A|-1}$ for each $A \in \mathcal{B}$, we can apply Thm 6.6 of the textbook to $\mathcal{B}$ to obtain

$$\sum_{A \in \mathcal{B}} \frac{1}{\theta(A)} \leq 1.$$

The remaining members of $\mathcal{A}$ are precisely the complements of the sets in $\mathcal{B}$. Apply Thm 6.6 to their complements, we obtain

$$\sum_{A \in \mathcal{A} \setminus \mathcal{B}} \frac{1}{\binom{n-1}{n-|A|-1}} \leq 1 \Rightarrow \sum_{A \in \mathcal{A} \setminus \mathcal{B}} \frac{1}{\theta(A)} \leq 1.$$

Adding the two inequalities, we obtain $\sum_{A \in \mathcal{A}} \frac{1}{\theta(A)} \leq 2$. The claim is now proved.

We note that $\theta(A) \leq \binom{n-1}{\lfloor \frac{n}{2} \rfloor - 1}$. So we have $\frac{|\mathcal{A}|}{\binom{n-1}{\lfloor \frac{n}{2} \rfloor - 1}} \leq \sum_{A \in \mathcal{A}} \frac{1}{\theta(A)} \leq 2$.

To prove the fact stated in Problem 6C, let

$\mathcal{A}' = \{A' \mid A \in \mathcal{A}\}$. Then $\mathcal{A} \cup \mathcal{A}'$ is an antichain satisfying the conditions of the claim above. So we have $\sum_{A \in \mathcal{A} \cup \mathcal{A}'} \frac{1}{\theta(A)} \leq 2$.

Thus $$\frac{2|\mathcal{A}|}{\binom{n-1}{\lfloor \frac{n}{2} \rfloor - 1}} \leq \sum_{A \in \mathcal{A} \cup \mathcal{A}'} \frac{1}{\theta(A)} \leq 2 \Rightarrow |\mathcal{A}| \leq \binom{n-1}{\lfloor \frac{n}{2} \rfloor - 1}$$ $\square$