MAT8010 Homework #5

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1. Let G be a simple graph with 10 vertices and 26 edges. Show that G has at least 5 triangles. Can equality occur?

   Solution:

   (i) Color K10 in such a way that red corresponds

   to an edge of G, blue to a nonedge. There are triangles with 3 red

   edges, 2 red and 1 blue, 2 blue and 1 red, and finally with 3 blue

   edges. Let ai (i = 1, 2, 3, 4) be the numbers of these. Set up a

   system of equations and inequalities for these numbers, expressed

   in the degrees of the vertices of G. These should show that there

   are at least four triangles in G and equality can then be excluded,

   again by looking at the equations.

   (ii) A second solution is as follows. Show there is a triangle.

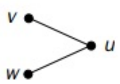
   Then consider a triangle, the ary 7-set, and edges in between. This

   gives a number of new triangles of two types; estimate it.

2. Show that a graph on n vertices that does not contain a circuit on four vertices has at most $\frac{n}{4}(1+\sqrt{4n-3})$ edges

   Proof. (Refer form Reiman 1958):

   Let G be a graph statified above requirement, and $d_1, d_2, \ldots, d_n$ be the degrees of its vertices. We now count in two ways the number of elements in the following set S. The set S consists of all (ordered) pairs $(u, \{v, w\})$ such that $v \neq w$ and u is adjacent to both v and w in G. That is, we count all occurrences of "cherries"

   in G. For each vertex u, we have $\binom{d_u}{2}$ possibilities to choose a 2-element subset of its d_u neighbors. Thus, summing over u, we find $|S| = \sum_{u=1}^{n} \binom{d_u}{2}$. On the other hand, because G are C_4-free graph, which implies that no pair of vertics $v \neq w$ can have more than one common neighbor. Thus, summing over all pairs we obtain that $|S| \leq \binom{n}{2}$. Altogether this gives

   $$\sum_{i=1}^{n} \binom{d_i}{2} \leq \binom{n}{2}$$

   or

   $$\sum_{i=1}^{n} d_i^2 \leq n(n-1) + \sum_{i=1}^{n} d_i \tag{1}$$

Now, we use the Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$$

with $x_i = d_i$ and $y_i = 1$, and obtain

$$\left(\sum_{i=1}^{n} d_i\right)^2 \leq d_i \sum_{i=1}^{n} d_i^2$$

and hence by (1)

$$\left(\sum_{i=1}^{n} d_i\right)^2 \leq n^2(n-1) + n\sum_{i=1}^{n} d_i$$

Euler's theorem gives $\sum_{i=1}^{n} d_i = 2|E|$. In voking this fact, we obtain

$$4|E|^2 \leq n^2(n-1) + 2n|E|$$

or

$$|E|^2 - \frac{n}{2}|E| \leq \frac{n^2(n-1)}{4} \leq 0$$

Solving the corresponding quadratic equation yields the desired upper bound on $|E|$.

3. Let $H_3(r)$ denote the number of 3×3 matrices with nonegative integer entries such that each row and each column sum to r. Show that

$$H_3(r) = \left(\begin{array}{c} r+5 \\ 5 \end{array}\right) - \left(\begin{array}{c} r+2 \\ 5 \end{array}\right)$$

Do this problem by using Theorem 5.5. Do not use the PLE.

Proof:

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4. For $n \geq 6$, let $\mathcal{H} \leq 2^{[n]}$ be such that for any two distinct $A, B \in \mathcal{H}$, $|A \cap B|$ is even. Prove that $\mathcal{H} \geq 2^{n/2}$ if n is even, and $\mathcal{H} \leq 1 + 2^{(n-1)/2}$ if n is odd. (Here note that we do not require $|A|$ to be even for $A \in \mathcal{H}$)

Proof:

5. Suppose that in a town of n residents, every club has an even number of members, and any two distinct clubs have an odd number of members in common. Then the maximum of clubs is n if n is odd, and *n-1* if n is even.

Proof:

6. Let $\mathcal{A} = \{A_1, ..., A_m\}$be a collection of m distinct subsets of $N := \{1, 2, ..., n\}$ such that if $i \neq j$ then $A_i \not\subseteq A_j, A_i \cap A_j \neq \phi, A_i \cup A_j \neq N$. Prove that

$$m \leqslant \left(\begin{array}{c} n-1 \\ \left[\frac{n}{2}\right] - 1 \end{array}\right)$$

Proof:

Show that large sets can be replaced by their complements, and apply Theorem 6.5.