Topic 1. Subsets, Partitions, Permuations

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A subject dealing with ways of arranging and distributing objects, and which involves ideas from geometryl, algebra and analysis.

The basic problem of enmerative combinatorics is that of counting the number of elements of a finite set. Usually we are given an infinte collection of finite sets S_i where i ranges over some index set I(such as the nonegative integers $\mathbb{N}$), and we wish to count the number $f(i)$ of elements in each S_i "stimultaneously."(why must be) Immediate philophical difficulties arise. What does it mean to "counte" the number of elements of S_i? There is no definitive answer to the question. Only through experience does one develop an idea of what is meant by a "determination" of a counting function $f(i)$. The counting function $f(i)$ can be gaven in several standard ways.

1. The most satisfactory form of $f(i)$ is a completely explict closed formula involving only well-known functions, and free from summation symbols. Only in rare cases will such formula exists. As formulas for $f(i)$ become more complicated, our willingness to accept them as "determinations" of $f(i)$ decrease. Consider the following examples.

1.1.1 Example. For each $n \in \mathbb{N}$, let $f(n)$ be the number of subsets of the set $[n] = \{1, 2, \ldots, n\}$. Then $f(n) = 2^n$, and no one will quarrel about being a satisfactory formula for $f(n)$

1.1.2 Example. Suppose n mean give their n hats to a hat-check person. Let $f(n)$ be the number of ways that the hats can be given back to the men, each man revceiving one hat, so that no man revceives his own hat. For instance, $f(1) = 0$, $f(2) = 1$, $f(3) = 2$. We will see in Chapter 2(Example 2.2.1) that

$$f(n) = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}.$$

This formula for $f(n)$ is not as elegant as the formula in Example 1.1.1, but for lack of a simpler answer we are willing to accept (1.1) as a satisfactory formula. It certainly has the virtue of making it easy (in a sense that can be made precise) to compute the values $f(n)$. Moreover, once the derivatino of (1.1) is understood(using the Principle of Inclusion-Exclusion), every term of (1.1) has an easily understood combinatorial meaning. This enables us to "understand" (1.1) intuitively, so our willingness to accept it is enhanced. We also remark that it follows easily from (1.1) that $f(n)$ is the nearest integer to $n!/e$. This is certainly a simple explict formula, but is has the disadvantage of being "non-combinatiorial"; that is. dividing by e and rounding off to the nearest integer has no direct combinatorial significance.

1.1.3 Example. There are actually formulas in the literature("nameless here for evermore") for certain counting functions $f(n)$ whose evalution requires listing all (or almost all) of the $f(n)$ objects being counted! Such a "formula" is completely worthless.

2. A recurrence for $f(i)$ may be given in terms of prevously calculated $f(j)$'s, thereby giving a simple procedure for calculating $f(i)$ for any desired $i \in I$. For instance, let $f(n)$ be the number of subsets of $[n]$ that do not contain two consecutive integers. For example, for $n=4$ we have the subsets $\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,3\}, \{1,4\}, \{2,4\}, \mathrm{sof}\ f(4)=8$. It is easily seen that $f(n)=f(n-1)$ for $n \geqslant 2$. This makes it trivial, for example, to compute $f(20)=17711$. On the other hand, it can be shown(see Setion 4.1 for the underlying theory) that

$$f(n)=\frac{1}{\sqrt{5}}(\tau^{n+2}-\bar{\tau}^{n+2}),$$

where $\tau=\frac{1}{2}(1+\sqrt{5})$, $\bar{\tau}=\frac{1}{2}(1-\sqrt{5})$. This is an explict answer, but because it involves irratinal numbers it is a matter of opinion(which may depend on the context) whether it is a better answer than the recurrence $f(n)=f(n-1)+f(n-2)$.

3. An algorithm may be given for computing $f(i)$. This method of dtermining f subsumes the previous two, as well as method 5 below. Any counting function likely to arise in practice can be computed from an algorithm, so the acceptability of this method will depend on the elegance and performance of the algorithm. In general, we would like the time that it takes the algorithm to compute $f(i)$ to be "substantially less" than $f(i)$ itself. Otherwise we are accomplishing little more that a brute force listing of the objects counted by $f(i)$. It would take us too far afield to discuss the profound contributions that computer science has made to the problem of analyzing, constructing, and evaluating algorithms. We will be concered almost exclusively with enumerative problems that admit solutions that are more concrete than an algorithm.

4. An estimate may be given for $f(i)$. If $I=\mathbb{N}$, this estimate frequently takes the form of an *asymptotic formula* $f(n)\sim g(n)$, where $g(n)$ is a familiar function." The notation $f(n)\sim g(n)$ means that $\lim_{n\longrightarrow\infty} f(n)/g(n)=1$. For instance, let $f(n)$ be the function of Example 1.1.3. It can be shown that

$$f(n)\sim e^{-2}36^{-n}(3n)!$$

For many purposes this estimate is superior to the "explicit" formulat(1.2).

5. The most useful but most difficult to understand method for evaluating $f(i)$ is to give its *generating function*. We will not develop in this chapter a rigorous abstract theory of generating functions, but will instead content ourselves with an informal discussion and some example. Informally, a generating fucntion is an "object" that represents a counting function $f(i)$. Usually this object is a *formal power series*. The two most common types of *generating functions* are *oridinary*

generating functions and *exponential* generating functions. If $I = \mathbb{N}$, then the ordinary generating funcion of $f(n)$ is the formal power series

$$\sum_{n \geqslant 0} f(n) x^n,$$

while the exponential generating function of $f(n)$ is the formal power series

$$\sum_{n \geqslant 0} f(n) \frac{x^n}{n!}.$$

(If $I = \mathbb{P}$, the positive integers, then these sums begin at $n = 1$.) These power series are called "formal" because we are not concerned with letting x take on particular values, and we ignore questions of convergence and divergence. The term x^n or $x^n/n!$ merely marks the place where $f(n)$ is written.

If $F(x) = \sum_{n \geqslant 0} a_n x^n$, then we call a_n the coefficient of x^n in $F(x)$ and write

$$a_n = [x^n] F(x).$$

Similarly, if $F(x) = \sum_{n \geqslant 0} a_n x^n / n!$, then we write

$$a_n = n! [x^n] F(x).$$

In the same way we can deal with generaing functions of several variables, such as

$$\sum_{l \geqslant 0} \sum_{m \geqslant 0} \sum_{n \geqslant 0} f(l, m, n) \frac{x^l y^m z^n}{n!}$$

(which may be considered as "oridinary" in the indices l, m and "exponential" in n), or even of infinitely many variables. In this latter case every term should involve only finitely many of the variables. A simple generating function infinitely many vaiables in $x_1 + x_2 + x_3 + \cdots$.

Useally we are give an infintec

What is a good answer for a counting qusetion?

Some are better than others, but different answers can have different advantanges.

Example (Ardila 1.1)

tiling

Let $a_n = \#$ of tiling of a 2xn recrtangle by dominoes, what is a_n

Answer1. Recurrence:

$a_n = a_{n-1} + a_{n-2} \quad \text{for} \quad n \geqslant 2$

$a_0 = a_1 = 1 \quad \text{for}\, n \leqslant 1$

Compare with Fibonacci Recurrence:

$F_n = F_{n-1} + F_{n-2}\,\text{for}\, n \geqslant 2$

$F_0 = 0, F_1 = 1$

n	a_n	F_n
0	1	0
1	1	1
2	2	1
3	3	2
4	5	3
5	8	5
6	13	8

Table 1.

It will take a while to compute a_{1000} this way, and we don't have too much sense of its order of magnitude.

Answer2. Note that $a_n = \#\{\text{sequence of } 1's \text{ and } 2's \text{ totalling to } n\}$

$n = 4 \Rightarrow$ 1+1+1+1, 1+1+2, 1+2+1,2+1+1,2+2

So $a_n = \sum_{k=0}^{[n/2]} \#\{\text{sequence of } k\, 2's \,\text{and}\, n - 2k\, 1's\} = \sum_{k=0}^{[n/2]} \begin{pmatrix} (n-2k)+k \\ k \end{pmatrix} = \sum_{k=0}^{[n/2]} \begin{pmatrix} n-k \\ k \end{pmatrix}$

eg: $a_4 = \binom{4}{0} + \binom{3}{1} + \binom{2}{2} = 1 + 3 + 1 = 5$

Answer3. 2^{th} explicat formula

We'll derive soo that

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)$$
$$\varphi = \left(\frac{1+\sqrt{5}}{2}\right) \approx 1.618$$
$$p = \left(\frac{1-\sqrt{5}}{2}\right) \approx 0.618$$

which is very explicat but still not so good for computing a_{1000} on the nose.

one has from above that

$$a_n \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$$

(and in fact, a_n is the near nearest integer to $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$)

This tell us a lot about its growth

e.g. its number of base 10 digits is $\log_{10}(a_n) = (n+1)\log_{10}\left(\frac{1+\sqrt{5}}{2}\right) + \log_{10}\left(\frac{1}{\sqrt{5}}\right)$

$\log_{10}\left(\frac{1+\sqrt{5}}{2}\right) \approx 0.20899$

Answer5. Generating function for $(a_0, a_1, a_2, \ldots)$

$A(x) := a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$= 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots$

$= \sum_{n \geqslant 0} a_n x^n$

as an element of $\mathbb{C}[|x|]$the ringing of formal power series in x with C coefficiednts.

Perhaps not clear yet they why we would even consider $A(x)$, but let's find a simple formula for it rnow and derive evertything else from it!

"Slow way"

Then recurrence $a_n = a_{n-1} + a_{n-2}$ for $n \geqslant 2$, $a_0 = a_1 = 1$

given $\sum_{n \geqslant 2} a_n x^n = \sum_{n \geqslant 2} a_{n-1} x^n + \sum_{n \geqslant 2} a_{n-2} x^n = x \sum_{n \geqslant 2} a_{n-1} x^{n-1} + x^2 \sum_{n \geqslant 2} a_{n-2} x^{n-2} = x \sum_{m \geqslant 1} a_m x^m + x^2 \sum_{m \geqslant 0} a_m x^m$

$A(x) - a_0 x^0 - a_1 x = x(A(x) - a_0 x^0) + x^2 A(x) = x(A(x) - 1) + x^2 A(x)$

$\Longrightarrow A(x) = \frac{1}{1 - x - x^2}$

What is the good is this? Pelenty! It depends on how we try to extract or estimate coefficients

(a) $A(x) = \frac{1}{1 - (x + x^2)} = 1 + (x + x^2) + (x + x^2)^2 + (x + x^2)^3 + \cdots$

$\sum_{n \geqslant 0} a_n x^n = \sum_{d \geqslant 0} (x + x^2)^d = \sum_{d \geqslant 0} \left(\sum_{k=0}^{d} \binom{d}{k}\right)(x^2)^k x^{d-k} =_{n=d+k, d=n-k} \sum_{n \geqslant 0} x^n \left(\sum_{k=0}^{[n/2]} \binom{n-k}{k}\right)$

(b)

$$A(x) = \frac{1}{1 - x - x^2} = \frac{\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)}{1 - \frac{1+\sqrt{5}}{2}x} + \frac{-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)}{1 - \frac{1-\sqrt{5}}{2}x} = \frac{1}{\sqrt{5}} \sum_{n \geqslant 0} \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} x^n - \frac{1}{\sqrt{5}} \sum_{n \geqslant 0} \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} x^n$$

Thus $a_n = \cdots = .$

(c) The a symptatic $a_n \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$ was controled by the reciprocal γ of the pde γ_1 of $A(x) = \frac{1}{1-x-x^2}$ nearest the origism in $\mathbb{C}$ (Walf \$2.4)

The fast way is via Polyga's "picture-writing"

$\frac{1}{1-(\text{Vertical}(A)+\text{Horitial}(B))}$ = 1 + (Vertical + Horital)^2 + (Vertical + Horital)^2 + (Vertical + Horital)$^3 + \cdots =$

(Vertical + Horital)^2(Vertical + Horital) $= AA + \text{BA} + AB + BB$

$$A(x) = \sum_{n \geqslant 0} (x+x^2)^n = 1 + (x+x^2) + (x+x^2)^2 + \cdots = \frac{1}{1-(x+x^2)}$$

The generating function can often be refined to keep track of more statistics, e.g, what if we want to compute $a_{m,n} = \#\{\text{tilings of } 2 \times n \text{ rectangle be dominas with } m \text{ vertial domines}\}$

we'll see how to write down $\sum_{n,m>0} a_{m,n} x^n V^m = \frac{1}{1-Vx-x^2}$

this let us find out he of the experted number of verical domaino in a large , which should be $\sum_m a_{m,m} m$, or the fraction of n tilles that are vertical.

$$\frac{\sum_m a_{m,n} m}{n a_n}$$

$$\sum_{x \geqslant 0}\left(\sum_{m \geqslant 0} a_{mn} m\right) x^n = \begin{aligned} &\left[\frac{\partial}{\partial V} \sum_{m,n \geqslant 0} a_{mn} x^n v^m\right]_{v=1} \\ &= \left[\frac{\partial}{\partial V} \frac{1}{1-Vx-x^2}\right]_{v=1} \\ &= \left[\frac{x}{(1-Vx-x^2)}\right]_{v=1} \\ &= \frac{x}{(1-x-x^2)^2} \end{aligned}$$

using partial functions on this, one can show$\sum_{m \geqslant 0} a_{mn} m \approx \frac{n}{5}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} \approx \frac{1}{\sqrt{5}} n a_n$

Thus the expectation $\approx \frac{\sqrt{n}}{5}$. So out of the n tiles, expect rougly $\frac{1}{\sqrt{5}}$,

Picture writing

In fact $A(x,\, v) := \sum_{n,m\geqslant 0} a_{mn} x^n v^m = \left[\frac{1}{1-(A+B)}\right]$, where $A = vx \quad (\because)$, $B = x^2(:)$

$$=1+(A+B)+(A+B)^2+(A+B)^3+\cdots+$$

The ringe of formal pwoer series $R[[x]]$, where $R = \mathbb{C}\,\text{or}\,R\,\text{or}\,\mathbb{Q}\,\text{or}\,\mathbb{C}[v]\,\text{or}\,\mathbb{F}_q\,\text{or any commultative}\quad\text{ring with}\,1.$

Def $\mathbb{R}[[x]] = \{\} = \sum_{m\geqslant 0} a_n x^n$ is a and mulitpliction x via convolution

$$C(x) = A(x)B(x) = \sum_{n=0}^{\infty} C_n \lambda^n$$

with

$$c_n = \sum_{i=0}^{n} a_i b_{n-i}$$

$C(x) = a_0 b_0 + (a_1 b_0 + a_0 b_1)x + (a_0 b_2)$

Prop

$$A(x) = \sum_{n=0}^{\infty} x^n \in R[[x]]\,\text{is}\,a\,\text{unit}$$

i.e $\exists B(x)\,\text{with}\,1 = A(x)B(x)$

$\Leftrightarrow a_0$ is a unit of R, i.e $\exists b_0 \in R$ with $1 = a_0 b_0$

$$A(x)(x-x-x^2) = 1 \quad \Rightarrow A(x) = \frac{1}{1-x-x^2}\,\text{exists in}\,\mathbb{C}[[x]]$$

Proof. $1 = Ax\,B(x) = a_0 b_0 + (a_1 b0 + a_0 b_1)x^1 + (a_2 b_0 + a_1 b_1 + a_{10} b_2)x^2 + \cdots +$ $\square$

$\Leftrightarrow a_0 b_0 = 1, i.e. b_0 = a_n^{-1}\,\text{in}\,R$

$\text{and}\,a_0 b_1 + a_1 b_1 = 0,\text{means}\,b_1 = -\frac{a_1 b_0}{a_0}$

$(a_2 b_0 + a_1 b_1 + a_{10} b_2 = 0),\text{means}\quad b_2 = -\frac{a_2 b_1 + a_2 b_0}{a_n}$

We say that a sequence A0 (x), A 1 (x), . . . of formal power series converges i.e. $A(x)\ A_f(x)$ exists, if $Vn \geqslant 0$, the coefficnt of x^0 in $A_j(x)$ stab for $j \geqslant 0$. V

$$A(x) = \frac{1}{1-x-x^2} = 1 + (x+x^2) + (x+x^2)^2 + (x+x^2)^3 + \cdots$$

Converges in $\mathbb{C}[[x]]]$

e.g. $[x^2]A(x)=[x^2]A_2(x)=[x^2]A_4(x)=\cdots=a_3=3$

e.g

$$e^{x+1}=1+\frac{x+1}{1}+\frac{(x+1)^2}{2}+\frac{(x+1)^3}{3}+\cdots+$$

does not

$$[x]A_j=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{j!}$$

$$e^x=1+\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+$$

is converges

Alternative, $\{A_j(x\}$ con

$$\min\deg(A_j(x)-A_{j+1}(x))=\infty$$

where min deg$A(x)=\sum_{n=0}^{\infty}a_nx^n=$smallest d with $a_d\neq 0$

e.g. $A_j(x)-A_{j-1}(x)=(x+x^2)^j$, having mindeg$=j\to\infty$

Cor $\sum_{j=0}^{\infty}B_j(x)-B_0(x)+B_1(x)+\cdots+$

converges in $R[|x|]\Leftrightarrow B_j(x)\longrightarrow\infty$

Cor Infinite product of the form

$$\mathbb{I}_{j=1}^{\infty}(1+B_j\,(x))\,\text{with}\min\deg B_j\geqslant 1\,\text{Vj}$$

converge in $R[|x|]\Leftrightarrow\lim_{j->\infty}\min\deg B_j(x)\longrightarrow\infty$

Proof. $A_j(x)=(1+B_1)(1+B^2)+\cdots+(1+B_j)$

has $A_j-A_{j-1}=^2B_j(1+B_1)...(1+B_{j-1})=B_j(1+\cdots)$ his mindeg $=\min\deg B_j$ □

Example 1. (Staley 1.8) Partition generating

Def A pation $\lambda=(\lambda_1,\,\lambda_2,\,\lambda_3...)$ of n is a weakly decresing sequence, $\lambda_1\geqslant\lambda_2\geqslant\lambda_3\geqslant\cdots\geqslant 0$ with , evetally 0 sequence of nonegtive intergers, such that $\lambda_1+\lambda_2+\cdots=n$. We ant $\lambda 1-n$ and $n=|\lambda|$.

e.g. $\lambda=()$ It's length $l(\lambda)=\#\{i|\lambda_i>0\}=\#$of nonzero parts λ_i. Its Ferrers diagram is a left and top justfied arrrange unit squares, with λ_i in row i from the top.

e.g . $\lambda=(5,5,3,1)\leftrightarrow$, let $p(n)=\#$ of partitions $\lambda|-n$

$p(n)$	n
7	5
5	4
3	3
2	2
1	1
1	0

$\mathbb{Y} = \text{Young's lattife poset on all partitions}$

1. $\sum_{a>0} p(n) a^n = \sum_{\text{all} \quad \text{partitions}} q^{|\lambda|}$

$\mathbb{C}[|q|] = (1 + \ . + \vdots + \vdots +)(1+ \ . +) = (1 + q + q^2 + q^3 + \cdots +)(1 + q^2 + q^{2^2} + \cdots +)(1 + q^3 + q^{3^2} + \ldots) = \frac{1}{1-q}\frac{1}{1-q^2}\frac{1}{1-q^3} = \frac{1}{(1-q)(1-q^2)(1-q^3)}$

$$A(V_1 x) = \frac{1}{1 - vx - x^2} = \sum_{n-m>0} a_{n,m} v^m x^n$$

$$\frac{x}{(1-x-x^2)^2} = \frac{1}{(x^2+x-1)^2} = \frac{x}{()}$$

x^n has coefficent

$$\sum_{n\geqslant 0} p(n) q^n = \sum_{\text{all} \quad \text{partition} \quad \lambda} q^{|\lambda|} = (1+)$$

No closed-form expression for $p(n)$ is known

ass $\quad \log(p(n)) \sim C\sqrt{n} \quad n \to \infty \ \ C = \pi\sqrt{\frac{2}{3}}$

Conjection $P_{\text{odd}}(n) = q(n)$, for any $n \geqslant 0$, Why?

The gen fcsn will expliain it.

$P_{\text{odd}}(x) = (1 + q + q^2 + \cdots +)(1 + q^3 + q^6 + \cdots +)(1 + q^5 + q^{10} + \cdots +) = \frac{1}{(1-q)}\frac{1}{(1-q^2)} = \frac{1}{\mathbb{I}_{j\geqslant 0}(1-q^{2j+1})}$

In fact,

$Q(q) = (1+q^1)(1+q^2)(1+q^3)\ldots = \frac{1-q^2}{1-q}\frac{1-q^{2^2}}{1-q^2}\frac{1-q^{3^2}}{1-q^3} = \frac{1}{(1-q)(1-q^3)(1-q^5)\ldots} = P_{odd}(2)$

Let's justify it differently, let

$$\in R(q) := (1-q)(1-q^3)(1-q^5)... = \frac{1}{P_{\text{odd}(q)}} \text{in } \mathbb{C}[|q|]$$

It suffices to show $1 = Q(q)R(q)$ Since multiplication inverse are unique. $Q(q) = ((1+q)(1+q^2)(1+q^3)...)((1-q)(1-q^2)(1-q^3)...) = (1+q^1)(1-q)((1+q^2)(1+q^3)...)((1-q^3)(1-q^5)...) = (: w)$

Bijective proof: $P_{\text{odd}}(n) = q(n)$ Standly 64

Given λ partition with odd parts $2j-1$ of mulitiplicity $r_j = 2^{i_1} + 2^{i_2} + ...$ in its binary expansion. and create e.g. $\lambda = (9^{(5)}, 5^{(12)}, 3^{(2)}, 1^{(3)}) = (9^{(2^1+2^2)})$

$$\mu = (q2^0, q2^2, 5)$$

Reversible

$$\mu = (20, 10, 7, 6, 4) = (5 \cdot 2^2, 5 \cdot 2^1, 7 \cdot 2^0, 3 \cdot 2^1, 1 \cdot 2^2) \Updownarrow \lambda = (5^{(2^2+2^1)}, 7^{(2)})$$

1 Poset

A **partially ordered set** or **poset** $(P, \leq)$ is a set P together with a binary relation $\leq$, called a *partial order*, such that

- For all $p \in P$, we have $p \leq p$.
- For all $p, q \in P$, if $p \leq q$ and $q \leq p$ then $p = q$
- For all $p, q, r \in P$, if $p \leq q$ and $q \leq r$ then $p \leq r$

We say that $p < q$ if $p \leqslant q$ and $p \neq q$. We say that p and q are **comparable** if $p < q$ or $p > q$, and they are **imcomparable** otherwise. We say that q **covers** p if $q > p$ and there is no $r \in P$ such that $q > r > p$. When q covers p we write $q > p$.

Example 2. Many sets in combinatorics come with a natural partial order, and often the resulting poset structure is very useful for enumerative purpose. Some of the most imporant example are the following:

1. (Chian) The *poset* $\boldsymbol{n} = \{1, 2, ..., n\}$ with the usual total order.(n>=1)
2. (Boolean lattice) The *poset* 2^A of subsets of a set A, where $S \leq T$ if $S \subseteq T$.
3. (Divisor lattice) The *poset* D_n of divisors of n, where $c \leq d$ if c divides d.$(n \geqslant 1)$

4. (Young's lattice) The *poset* Y of integer partitions, where $\lambda \leq \mu$ if $\lambda_i \leqslant \mu_i$ for all i.

5. (Partition lattice) The *poset* $\prod_n$ of set partitions of $[n]$, where $\pi \leq p$ if π refines ρ; that is, if every block of ρ is a union of blocks of $\pi.(n \geqslant 1)$

6. (Non-crossing partition lattice) The *subposet* NC_n of $\prod_n$ consisting of the non-crossing set partitions of $[n]$, where there are no elements $a < b < c < d$ such that a, c are together in one block and b, d are together in a different block.$(n \geqslant 1)$

A peek in to Posets(Stanly ch3)

Def: A poset $(P, \leqslant p)$, is a set of P with a binary relation $x \leqslant_p y$ statisfying

$$x \leqslant_p x(\text{refletion}),$$

$$x \leqslant_p y$$

Example:

([n]:={1,2,..,n})

$$Y = \text{yong}'s\,\text{latice in all partition}\,\lambda$$

Example 3:

YF:=Yong - Fibonacci lattioce(See Stanly, 3.21example #4)

For S_a a set, $(2^s, \leqslant) = \text{Boolean algebra}\{\text{all subsets of } S\}$ with S<=T, if $S \leqslant T$. When $S = [n] = \{1, 2, \ldots, n\}$ e.g. $B_1 = \{1\} - \phi$

When P is locally finite (or even locally chanins finite) i.e. all interal $[x, y]$ are chain-finite). then $\leqslant p$ is the transtive closure of the covering relation $x <_p y$.define by $x \leqslant_p y$ and $\exists\!/\ \delta$ in P with $x \leqslant_p$

The **Hasse diagram** of a finite poset P is obtained by drawing a dot for each element of P and an edge going down from p to q if p covers q.

Def: If P is finite(resp is it is locally chain-finite and has a bottom element $o^\wedge$), sat P is graded if every maximal chain has same size

Back to formal power series for a bit, we'll have use for those elements of $\mathbb{C}[|x|]$

Def $e^x = \sum_{n \geqslant 0} \frac{x^n}{n!} =$

$\log(1+x) := \sum_{n \geqslant 1} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots +$

$\forall \lambda \in \mathbb{C}, (1+x)^\lambda := \sum_{k \geqslant 0} \begin{pmatrix} l \\ k \end{pmatrix} x^k$

Thet do have all the usual propartres you expcet

Ex:

1. $(1+x)^{\lambda}(1+x)^{\mu}=(1+x)^{\mu+\lambda}$ in $\mathbb{C}\{[x]\}$

2. $e^{\log(1+\lambda)}=1+\lambda$

3. $e^{x}e^{y}=e^{x+y}$, etc...

Ex: $e^{\log(1+x)}:=1+\log(1+x)+\frac{(\log(1+x))^2}{2!}+\cdots+=1+\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\right)+\cdot$

Inf fact, Prop If $A(x)$, $B(x)$ and $b_0=0$, where $A(x)=\sum_{n\geqslant 0}a_n x^n$, $B(x)=\sum_{n\geqslant 0}b_n x^n$, then $A(B(x)):=\sum_{n\geqslant 0}a_n B(x)^n$ converges in $\mathbb{C}[|x|]$

How to justify (1)(2)(3) etc...?

(2) is laborous without a chert from caluclus(Talylor scney) and complex analysis.

Then If $f(\delta 2)=\sum_{n\geqslant 0}a_n\delta^n$ is analytif

(4) $(1+x)^n=\sum_{k\geqslant 0}\left(\begin{array}{c}n\\k\end{array}\right)n^k$, for $n\leqslant|\lambda|$ but also

$$\frac{1}{(1-x)^n}=(1+(-x))^{-n}=\sum_{k\geqslant 0}\left(\begin{array}{c}-n\\k\end{array}\right)(-x)^k=\sum_{K\geqslant 0}\left(\left(\begin{array}{c}n\\k\end{array}\right)\right)x^k$$

$$\frac{1}{1-4x}=(1+(-4x))^{-1}=\sum_{k\geqslant 0}\left(\begin{array}{c}-1\\k\end{array}\right)(-4x)^k=\sum_{k\geqslant 0}\left(\begin{array}{c}1+k-1\\k\end{array}\right)4^k x^k=\sum_{k\geqslant 0}4^k\lambda^k$$

but also

$$\frac{1}{(1-4x)}=\sum_{k\geqslant 0}\left(\begin{array}{c}2+k-1\\k\end{array}\right)4^k x^k=\sum(k+1)4^k x^k$$