Topic 4 Recusion and Generating Functions

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Given a sequence $\{a_n\}_{n=0}^{\infty}$ of complex number, we define

$$f(x) := \sum_{n=0}^{\infty} a_n x^n,$$

the ordering generating function of $\{a_n\}_{n=0}^{\infty}$,

$$g(x) := \sum_{n=0}^{\infty} a_n \frac{x^n}{n!},$$

the exponential generating function. We can using the

- Generating function can be used to prove and/or discovery identities.
- Generating function can be used to solve recursion
- Generating function can be used to prove recursion

Example 14.4. A formal bookkeeping device to list all possible configurations is to let the named one correspond to the term $x_1^{r_1} x_2^{r_2} x_3^{r_3} \cdots x_k^{r_k}$ in the product

$$(1 + x_1 + x_1^2 + \cdots)(1 + x_2 + x_2^2 + \cdots) \cdots (1 + x_k + x_k^2 + \cdots)$$

We can collect all the terms involving exactly n balls by taking $x_i = x$ for all i, and considering the terms equal to x^n. Therefore we find that the number of ways to divide n balls over k distinguishable boxes is the coefficient of x^n in the expansion of $(1 - x)^{-k}$.

What is field?

Z/pZ= a field = $\{\}$

Field: let α be the a root of $x^2 + x + 1$ in . What is th size of fied

$d \geqslant 1$, N_d= the # of irreducible mnic polys of dogree in z/pz[x].

$x^d + a_{n-1} x^{d-1} + \cdots + a_1 x + a_0$

UFD

N_2

$p^2 - \sum z$

List all monic irred polys of deg>=1 over Z/pZ as follows:

$f_1(x), \ldots\ldots,$

$\deg, d_1, d_2 \ldots\ldots$

Given any sequence i_1, i_2, i_3 all but finitely we can define

$f(x) = f_1(x)^{x_1} + f_2()$

Conversley V deg n, monic play can be obtain uniquetly in this way (UFD)

We just established an 1-to-1 resserp between

the set of monic polys of deg n $\longleftrightarrow$ the set of tuples $(i_1,)$

$\frac{1}{1 - p x}$=(1+x)

Example 14.6

Square free

$M_n = \#$ of monic sequare free polys of deg n in $Z/\text{pZ}[x]$

$f(x) = f_1(x)^{c1} + f_2(x)^{c2} + \cdots$

$i_1 \leqslant 1, i_2 \leqslant 1$

$\sum M_n x^n = (1 + x^{d_1})(1 + x^{d_2}) + (1 + x^{d_3}) \cdots$

$= \Pi_{d \geqslant 1} \left(\frac{1 - x^{2d}}{1 - x^d} \right)^{N_d} = \frac{1 - p x^2}{1 - p x}$

$= \frac{1}{1 - p x} - \frac{p x^2}{1 - p x}$

$= 1 + p x + (p x)^2 + \cdots + (p x)^n + \cdots$

$M_n = p^n - p^{n+1} \qquad V n \geqslant 2$

Let $n \geqslant 0$, let $f_2(n) =$ the $\#$ of pairs of monic deg npolys in $Z/\text{pZ}[x]$ where relative

$= \#\{(f(x), g(x))\colon f(x), g(x)$ monic deg n in Z/pZ, gcd $(f, g) = 1\}$

$M_n = \#$ of monic sequare free polys of deg n in $Z/\text{pZ}[x]$

$f(x) = f_1(x)^{c1} + f_2(x)^{c2} + \cdots$

$i_1 \leqslant 1, i_2 \leqslant 1$

$\sum M_n x^n = (1 + x^{d_1})(1 + x^{d_2}) + (1 + x^{d_3}) \cdots$

$= \Pi_{d \geqslant 1} \left(\frac{1 - x^{2d}}{1 - x^d} \right)^{N_d} = \frac{1 - p x^2}{1 - p x}$

$= \frac{1}{1 - p x} - \frac{p x^2}{1 - p x}$

$= 1 + p x + (p x)^2 + \cdots + (p x)^n + \cdots$

$M_n = p^n - p^{n+1} \qquad V n \geqslant 2$

$$\frac{1}{1-qx^2}=\Pi_{d\geqslant 1}\left(\frac{1}{1-x^{2d}}\right)^{N_d}$$

$\frac{1-9x^6}{(1-9x^2)(1-9x^3)}=\sum_{m=0}^{\infty}9^m x^{2m+1}+\sum_{m\geqslant 0}^{\infty}q^m x^{2m+1}$

$n=\begin{cases}\text{even}\\ \text{odd}\end{cases}$

$P(n)=q^{\frac{n}{2}-1}-q^{\frac{n-1}{3}}$

$$u_n=\frac{1}{n}\binom{2n-2}{n-1}$$

are called *Catalan numbers*.

Example 14.8. Considere walk in the x-y plane, where each step is

$U\colon(x,y)\longrightarrow(x+1,y+1)\,\text{or}\,D\colon(x,y)\longrightarrow(x+1,y-1)$

From $(0,0)$, how many way can we reach $(2n,0)$ without cross with x-axis.

By reflecting the part of the path bwtteen A and the 1st meeting C of the path with $x-$ axis. Consider walks in the $X-Y$ plane, where each step if we find a path from the reflected at pt to A' to B. This establish an 1-to-1 corresp bettween $(A\longrightarrow B)$ and $(A'\longrightarrow B)$.

$\binom{n}{l2}$=# of paths bettween A and B which do not meet $x-$axis.

Q1.How many do not meet

$\frac{1}{n}\binom{2n-2}{n-1}$

Q2. How many pathways from (0,0) to (2n,0) ar there no (you can touch, but no cross)

$\binom{2n-2}{n-1}-\binom{2n-2}{n-2}$

Relection principle: We consider two point A and B in the upper half plane, and a possible path between them which meet and/or cross the $x-$axis.

(H.Wilf Generationfunctionology)

Don't try to evaluate the sum that you are looking at. Instead, find the g.f. of the whole parameartion family of them, then read off the coefficients.

a) Identify the free variable, call it n (free index), that the sum depends on. GIve a name to the sum that you are lloking at, say $f(n)$.

b) From the g.f. of $f(n)$, $F(x)=\Sigma_{n=0}^{\infty}f(n)x^n$

c) Interchange the order of two summations

d) Identify the coefficient of the g.f.

Useful techniques

$$\binom{x}{m} = \begin{cases} 0 & \text{if } m < 0 \\ 0 & \text{if } x < m \end{cases}$$

$x^{-r}(1+x)^n = \sum_k \binom{n}{r+k} x^k, \forall n \geqslant 1$

$\frac{x^k}{(1-x)^{k+1}} = \sum_{r=0}^{\infty} \binom{r}{k} x^r$

$\sum_{n \geqslant 0} \frac{1}{n+1} \binom{2n}{n} x^n = \frac{1}{2x}(1 - \sqrt{1-4x})$

Ex 1

$f(n) = \sum_{k \geqslant 0} \binom{k}{n-k}, n = 0, 1, 2....$

$F(x) = \sum_{n=0}^{\infty} f(n) x^n = \sum_{k=0}^{n} x^k (\sum_{n \geqslant k}^{\infty} \binom{k}{n-k} x^{n-k}) = \sum_{k=0}^{\infty} x^k (1+k)^k = \frac{1}{1-x-x^2} = F_0 + F_1 + F_2 + \cdots$

Ex 2

$\sum_{k \geqslant 0} ()()$

$F(x) = \sum_{n=0}^{\infty} f(n) x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \binom{n+k}{n+2k} \binom{2k}{k} \frac{(-1)^k}{k+1} \right) x^n = \sum_{k=0}^{\infty} \binom{2k+1}{k} \frac{(-1)^k}{k+1} x^{-k} \frac{x^{m+1k}}{(1-x)^{m+2k+1}} = \frac{x^m}{(1-x)^{m+1}} \sum_{k=0}^{\infty} \frac{1}{k+1} \binom{2k}{k} \left(\frac{-x}{(1-x)^2} \right)^k = \frac{x^m}{(1-x)^{m+1}} \frac{1}{2\frac{-x}{(1-x)^2}} \left(1 - \sqrt{1 + \frac{4x}{(1-x)^2}} \right) = \frac{x^m}{(1-x)^m} = \sum_{j=0}^{\infty} \binom{m+j-1}{j} x^j =$

Let $m + j = n$, then $\binom{n-1}{m-1}$

Compute can prove the WZ problem.?