

## Exercises in Statistical Mechanics

Based on course by Doron Cohen, has to be proofed  
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This exercises pool is intended for a graduate course in “statistical mechanics”. Some of the problems are original, while other were assembled from various undocumented sources. In particular some problems originate from exams that were written by B. Horovitz (BGU), S. Fishman (Technion), and D. Cohen (BGU).

### ===== [Exercise 0010]

#### Average distance between two particles in a box

In a one dimensional box with length  $L$ , two particles have random positions  $x_1, x_2$ . The particles do not know about each other. The probability function for finding a particle in a specific location in the box is uniform. Let  $r = x_1 - x_2$  be the relative distance of the particles. Find  $\langle \hat{r} \rangle$  and the dispersion  $\sigma_r$  as follows:

- (1) By using theorems for “summing” the expectation values and variances of independent variables.
- (2) By calculating the probability function  $f(r) dr = P(r < \hat{r} < r + dr)$ .

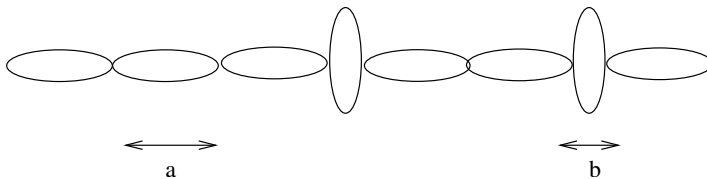
### ===== [Exercise 0020]

#### Average length of a polymer

A polymer can be described as a chain of  $N$  monomers. Each monomer has the probability  $p$  to be positioned horizontally, adding length  $a$  to the polymer, otherwise the monomer adds length  $b$ . Let  $L$  be the total length of the polymer. Define random variables  $\hat{X}_n$  such that:

$$X_n = \begin{cases} a, & \text{the monomer is horizontal} \\ b, & \text{the monomer is vertical} \end{cases}$$

- (a) Express  $\hat{L}$  using  $\hat{X}_n$ . Using theorems for adding independent random variables find the average length  $\langle L \rangle$  and the variance  $\text{Var}(L)$ .
- (b) Define  $f(L) \equiv P(L = na + (N - n)b)$ . Find it using combinatorial considerations. Calculate  $\langle \hat{L} \rangle$  and  $\text{Var}(L)$ .
- (c) Define  $\sigma_L = \sqrt{\text{Var}(L)}$ . What is the behavior of  $\sigma_L / \langle L \rangle$  as a function of  $N$ ?



### ===== [Exercise 0030]

#### Fluctuations in the number of particles

A closed box of volume  $V_0$  has  $N_0$  particles. The “system” is a subvolume  $V$ . The number of particles in  $V$  is a random variable  $N$ . Define the random variable  $\hat{X}_n$ , that indicates whether the  $n$ th particle is located inside the system:

$$X_n = \begin{cases} 1, & \text{the particle is in } V \\ 0, & \text{the particle is not in } V \end{cases}$$

- Express  $\hat{N}$  using  $\hat{X}_n$ . Using theorems on adding independent random variables find  $\langle N \rangle$  and  $\text{Var}(N)$ .
- Find the probability function  $f(N)$  using combinatorial considerations. Calculate from it  $\langle \hat{L} \rangle$  and  $\text{Var}(L)$ .
- Assume  $|(V/V_0) - \frac{1}{2}| \ll 1$ , and treat  $N$  as a continuous random variable. Approximate the probability function  $f(N)$  as a Gaussian, and verify agreement with the central limit theorem.

### ===== [Exercise 0050]

#### Changing random variables $x = \cos(\theta)$

Assume that the random phase  $\theta$  has a uniform distribution. Define a new random variable  $x = \cos(\theta)$ . What is the probability distribution of  $x$ ?

### ===== [Exercise 0060]

#### Oscillator in a microcanonical state

Assume that a harmonic oscillator with frequency  $\Omega$  and mass  $m$  is prepared in a microcanonical state with energy  $E$ .

- Write the probability distribution  $\rho(x, p)$
- Find the projected probability distribution  $\rho(x)$

### ===== [Exercise 0070]

#### The ergodic microcanonical density

Find an expression for  $\rho(x)$  of a particle which is confined by a potential  $V(x)$ , assuming that its state is microcanonical with energy  $E$ . Distinguish the special cases of  $d = 1, 2, 3$  dimensions. In particular show that in the  $d = 2$  case the density forms a step function. Contrast your results with the canonical expression  $\rho(x) \propto \exp(-\beta V(x))$ .

### ===== [Exercise 0080]

#### The spreading of a free particle

Given a free classic particle  $H = \frac{p^2}{2m}$ , that has been prepared in time  $t = 0$  in a state represented by the probability function

$$\rho_{t=0}(X, P) \propto \exp\left(-a(X - X_0)^2 - b(p - p_1)^2\right)$$

- Normalize  $\rho_{t=0}(X, P)$ .
- Calculate  $\langle X \rangle$ ,  $\langle P \rangle$ ,  $\sigma_X$ ,  $\sigma_P$ ,  $E$
- Express the random variables  $\hat{X}_t, \hat{P}_t$  with  $\hat{X}_{t=0}, \hat{P}_{t=0}$
- Express  $\rho_t(X, P)$  with  $\rho_{t=0}(X, P)$ . (Hint: 'variables replacement').
- Mention two ways to calculate the sizes appeared in paragraph b in time  $t$ . Use the simple one to express  $\sigma_x(t), \sigma_p(t)$  with  $\sigma_x(t=0), \sigma_p(t=0)$  (that you've calculated in b).

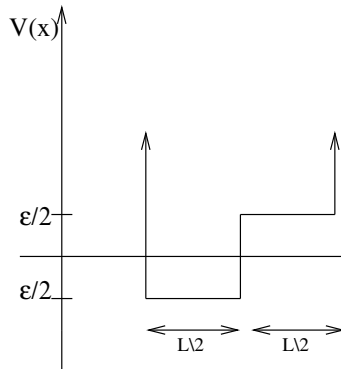
### ===== [Exercise 0100]

#### Spectral functions

### ===== [Exercise 0105]

#### Spectral functions for a particle in a double well

Consider a particle that has a mass  $m$  in a double well. The potential  $V(x)$  of the well is described in the figure.



- Describe the possible trajectories of the particle in the double well.
- Calculate  $N(E)$  and the energy levels in the semi-classical approximation.
- Calculate  $Z(\beta)$  and show that it can be written as a product of "kinetic" term and "spin" term.

### ===== [Exercise 0120]

#### Spectral functions for N particles in a box

In this question one must evaluate  $Z(\beta)$  using the next equation

$$Z(\beta) = \sum_n e^{-\beta E_n} = \int g(E) d(E) e^{-\beta E}$$

- Particle in a three dimensional space  $H = \sum_{i=1}^3 \frac{p_i^2}{2m}$   
Calculate  $g(E)$  and through that evaluate  $Z(\beta)$   
Guideline: for calculating  $\mathcal{N}(E)$  one must evaluate some points  $(n_1 n_2 n_3)$ - each point represents a state - there's in ellipse  $E_{n_1 n_2 n_3} \leq E$
- $N$  particles with equal mass in a three dimensional space. assume that it's possible to distinguish between those particles. Prove:  $\mathcal{N}(E) = \text{const} \cdot E^{\frac{3N}{2}}$   
Find the const. use Dirichlet's integral (private case) for calculating the 'volume' of an  $N$  dimensional Hyper-ball:

$$\int \dots \int \Pi dx_i = \frac{\pi^{\frac{N}{2}}}{(\frac{N}{2})!} R^N$$

$$\sum x_i^2 \leq R^2$$

Calculate  $g(E)$  and from there evaluate  $Z(\beta)$

### ===== [Exercise 0122]

#### Spectral functions for N harmonic oscillators

Consider an ensemble of  $N$  harmonic oscillators with an energy spectrum of each oscillator being  $(n + \frac{1}{2}) \hbar \omega, n = 0, 1, 2, \dots$

- (a) Evaluate the asymptotic expression for  $\Omega(E)$ , the number of ways in which a given energy  $E$  can be distributed.
- (a) Consider these oscillators as classical and find the volume in phase space for the energy  $E$ . Compare the result to (a) and show that the phase space volume corresponding to one state is  $h^N$ .

### ===== [Exercise 0130]

#### Spectral functions for general dispersion relation

Find the states density function  $g(E)$  and the distribution function  $Z(\beta)$  for a particle that moves in a  $d$  dimensional space with volume  $V = L^d$ .

Assume the particle has dispersion relation

case a'  $E = C|P|^\nu$

case b'  $E = \sqrt{m^2 + p^2}$

Make sure that you know how to get a result also in the "quantal" and the "semiclassical" way.

### ===== [Exercise 0140]

#### Spectral functions for two dimensional box

What is two dimensional gas?

Given gas in a box with dimensions  $(L \ll L) L \times L \times L$ .

Determine what are the energies of the uniparticle states. Show that there's an energy range  $0 < E < E_{max}$  where it's possible to relate the gas as a gas in a  $2 - D$  space with a states density function

$$g(E) = A \frac{m}{2\pi} \quad 0 \leq E \quad (A \equiv L^2)$$

### ===== [Exercise 0150]

#### Spectral functions for N spins

Consider an  $N$  spin system:

$$\hat{H} = \sum_{\alpha=1}^N \frac{\varepsilon}{2} \hat{\sigma}_z^{(\alpha)}$$

Calculate  $Z_N(\beta)$  in two different ways:

- (1) The short way - Calculate  $Z_N(\beta)$  by factoring the sum.
- (2) The long way - Write the energy levels  $E_n$  of the system. Mark with  $n = 0$  the ground level, and with  $n = 1, 2, 3, \dots$  the excited levels. Find the degeneracy  $g_n$  of each level. Use these results to express  $Z_N(\beta)$ , and show that the same result is obtained.

### ===== [Exercise 0160]

#### Partition function for AB and AA molecules

A diatomic molecule  $AB$  can be regarded as two atoms that are connected by a spring of length  $r_0$ , and vibration frequency  $\omega_0$ . The total angular momentum is  $\ell = 0, 1, 2, \dots$ . The masses of the atoms are  $m_A$  and  $m_B$ , and they have spins  $S_A$  and  $S_B$ .

- Explain what are the conditions that allow to ignore all the excited vibrational levels, so you can treat the molecule as a rigid body ("rotor").
- Calculate the partition function of the diatomic molecule, assuming that it is like a classical rigid rotor. Define the condition on the temperature for this approximation to hold.
- Calculate the partition function of the diatomic molecule, if the temperature is very low, taking only the  $\ell = 0, 1$  states into account.
- How the previous answers are modified for an  $AA$  molecule that is composed of two identical spin 0 atoms?
- How the previous answers are modified for an  $AA$  molecule that is composed of two identical spin 1/2 atoms? What is the probability to find the spin configuration in a triplet state? Relate to the two limits in (b) and (c).

### ===== [Exercise 1000]

#### Canonical formalism

### ===== [Exercise 1010]

#### State equations derived from $Z(T)$

Make sure you're well aware of the basic equations of the canonical ensemble, and know how to prove those equations for the state functions.

$$(*) Z(\beta, X) \equiv \sum_r e^{-\beta E_r}$$

$$E = -\frac{\partial \ln Z}{\partial \beta}$$

$$y = \frac{1}{\beta} \frac{\partial \ln Z}{\partial X}$$

$$F(T, X) \equiv -\frac{1}{\beta} \ln Z(\beta, X)$$

$$S = -\frac{\partial F}{\partial T}$$

More definitions

(Heat capacity)  $C_x \equiv \frac{\partial E}{\partial T}|_X$

(Generalized susceptibility)  $\chi \equiv \frac{\partial y}{\partial X}$

(\*) for a classical particle

$$\sum_r \mapsto \int \frac{dx dp}{2\pi} E_r \mapsto H(XP)$$

### ===== [Exercise 1020]

#### Formula for the canonical fluctuations in energy

Prove that  $\sigma_E^2 = T^2 C_X$

Where  $\sigma_E^2 \equiv \langle H^2 \rangle - \langle H \rangle^2$

and  $C_X \equiv \frac{\partial E}{\partial T}|_X$

Guideline: Express  $\sigma_E^2$  by the distribution function and use the result we got for  $E$  in order to get the requested expression.

### ===== [Exercise 1025]

#### Fluctuations of N in the grand canonical ensemble

Show that

$$\langle \Delta N^2 \rangle = \left( \frac{1}{\beta} \frac{\partial}{\partial \mu} \right)^2 \ln Z = T \left( \frac{\partial N}{\partial \mu} \right)_{TV}$$

From that prove the equation

$$\frac{\langle \Delta N^2 \rangle}{\langle N \rangle^2} = - \frac{T}{V^2} \left( \frac{\partial V}{\partial p} \right)_{N,T}$$

The last step demands manipulation of equations in thermodynamics

Hints:

$$\left( \frac{\partial N}{\partial \mu} \right)_{T,V} = \left( \frac{\partial \mu}{\partial N} \right)_{TV}^{-1}$$

$$d\mu = \nu dp + SdT, \quad \nu = \frac{V}{N}, \quad S = \frac{S}{N}$$

$$\frac{\partial \mu}{\partial \nu}|_T = \nu \frac{\partial p}{\partial \nu}|_T$$

### ===== [Exercise 1026]

#### Fluctuations in the grand canonical ensemble

A fluid in a volume  $V$  is held (by a huge reservoir) at a temperature  $T$  and chemical potential  $\mu$ . Do not assume an ideal gas. Find the relation between  $\langle (E - \langle E \rangle)^3 \rangle$  and the heat capacity  $C_V(T, z)$  at constant fugacity  $z$ . Find the relation between  $\langle (N - \langle N \rangle)^3 \rangle$  and the isothermal compressibility  $\chi T(V, \mu) = -(\partial v / \partial \mu)|_{V,T}$  where  $v = V / \langle N \rangle$ . [Hint: Evaluate 3rd derivatives of the grand canonical partition function.] Find explicitly results in case of a classical ideal gas.

### ===== [Exercise 1027]

## Fluctuations in the grand canonical ensemble

Fluctuations in the grand canonical ensemble: A fluid in a volume  $V$  is held (by a huge reservoir) at a temperature  $T$  and chemical potential  $\mu$ . Do not assume an ideal gas.

- Find the relation between  $\langle (E - \langle E \rangle)^3 \rangle$  and the heat capacity  $C_V(T, \zeta)$  at constant fugacity  $\zeta$ .
- Find the relation between  $\langle (N - \langle N \rangle)^3 \rangle$  and the isothermal compressibility  $k_T(V, \mu) = -(\partial v / \partial \mu)_{V,T}$  where  $v = V / \langle N \rangle$   
[Hint: Evaluate 3rd derivatives of the grand canonical partition function.]
- Find (a) and (b) explicitly for a classical ideal gas.

### ===== [Exercise 1030]

#### The Helmholtz function

We define  $F \equiv -\frac{1}{\beta} \ln Z_\beta$

Prove:

$$E = F + TS$$

$$\begin{cases} S = -\frac{\partial F}{\partial T} \\ y = -\frac{\partial F}{\partial X} \end{cases}$$

It is possible to rely on the expressions that express  $X, S, y$  by the distribution function and by the  $F$  definition.

### ===== [Exercise 1032]

#### The extensive property of the grand energy

Explain why  $F, \Omega$  are extensive functions in the thermodynamic limit, so that

$$F(\beta; \lambda V, \lambda N) = \lambda F(\beta; V, N)$$

$$\Omega(\beta\mu; \lambda V) = \lambda \Omega(\beta\mu; V)$$

Guideline: Note that if you split the system, then in neglecting "surface" interaction, the functions  $Z, \Omega$  will be factorized.

$$Z \approx Z^A + Z^B, F \approx F^A + F^B$$

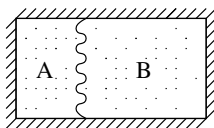
Result: therefore,

$$\Omega(\beta\mu; V) = V \Omega(\beta\mu; 1)$$

Prove that from here, we can conclude that

$$\Omega(\beta\mu; V) = -V * p(\beta\mu)$$

remark: Generalization of considerations such these were written by Euler.



### ===== [Exercise 1040]

#### Absolute temperature and entropy

In a general quasi static process we defined

$$dQ \equiv dE + dW = \frac{\partial E}{\partial \beta} d\beta + \left( \frac{\partial E}{\partial X} + y \right) dX$$

We expressed  $E$  and  $y$  by the distribution function  $Z(\beta)$

Using differential equations technic, for an integration factor to the non precise differential  $dQ$ , you learned in the course, which is solely, a function of  $\beta$ . Show that the integration factor you get is

$$T^{-1}(\beta) = \beta$$

therefore It's possible to write  $dQ = TdS$  where  $dS$  is a precise differential.  
Find the function  $S$  and show

$$S = - \frac{\partial}{\partial \left( \frac{1}{\beta} \right)} \left[ -\frac{1}{\beta} \ln Z(\beta) \right]$$

By definition,  $S$  is an entropy function, and  $T$  is called in an absolute temperature.

### ===== [Exercise 1041]

#### The Shanon definition of the entropy

Show that it's possible to write the entropy function expression as:

$$S = - \sum_r p_r \ln(p_r)$$

It is possible to account this equation as a definition when we talk about other ensembles.

### ===== [Exercise 1043]

#### Entropy, additivity

Assume that the entropy  $S$  and the number of states in phase space  $\Omega$  of a physical system are related through an arbitrary function,  $S = f(\Omega)$ . Show that the additive character of  $S$  and the multiplicative character of  $\Omega$  *necessarily* require that  $f(\Omega) \sim \ln \Omega$ .

### ===== [Exercise 1045]

#### Entropy, mixing of two gases

Consider mixing of two gases with initial different temperatures,  $T_1, T_2$ , particle numbers  $N_1, N_2$  and volumes  $V_1, V_2$ , respectively. Evaluate the mixing entropy (i.e. the change of entropy upon mixing) in two cases: (i) the gases are identical, (ii) the gases are distinct (but have equal mass). Show that the mixing entropy in case (ii) is larger and argue for the reason.

### ===== [Exercise 1047]



### Negative temperature

Consider  $N$  particles, each fixed in position and having a magnetic moment  $\mu$ , in a magnetic field  $H$ . Each particle has then two energy states,  $\pm\mu H$ . Treat the particles as distinguishable.

- Evaluate the entropy of the system  $S(n)$  where  $n$  is the number of particles in the upper energy level; assume  $n \gg 1$ . Draw a rough plot of  $S(n)$ .
- Find the most probable value of  $n$  and its mean square fluctuation.
- Relate  $n$  to the energy  $E$  of the system and find the temperature. Show that the system can have negative temperatures. Why a negative temperature is not possible for a gas in a box?
- What happens if a system of negative temperature is in contact with a heat bath of fixed temperature  $T_0$ ?

### ===== [Exercise 1060]

#### Quasi-Static processes in a mesoscopic system

Write the basic level energy of a particle with mass  $m$ , which is in a box with final volume  $V$ . (Take boundary conditions zero in the limits of the box). In temperature zero, ( $\beta^{-1} = 0$ ), calculate explicitly the pressure caused by the particle. Use the equation

$$p = \sum_r p_r \left( -\frac{\partial E_r}{\partial V} \right)$$

Compare it to the equation developed in class for general temperature

$$P = \frac{1}{V} \beta^{-1}$$

and explain why in the limit  $\beta^{-1} \rightarrow 0$  we don't get the result you calculated. (Hint - notice the title of this question).

### ===== [Exercise 1510]

#### Boltzmann approximation from the canonical ensemble

Given  $N$  particle gas with uniparticle state density function  $g(E)$ .

In the grand canonical ensemble, in Boltzmann approximation, the results we get for the state functions  $N(\beta\mu)$ ,  $E(\beta\mu)$  are

$$N(\beta\mu) = \int_0^\infty g(E) dE f(E - \mu)$$

$$E(\beta\mu) = \int g(E) dE E \cdot f(E - \mu)$$

Where  $f(E - \mu) = e^{-\beta(E - \mu)}$  is called the Boltzmann occupation function.

In this exercise you need to show that you get those equations in the framework of the approximation  $Z_N \approx \frac{1}{N!} Z_1^N$ . For that, calculate  $Z$ , that you get from this proximity for  $Z_N$  and derive the expressions for  $N(\beta\mu)$ ,  $E(\beta\mu)$ .

### ===== [Exercise 1627]

#### Equipartition theorem

This is an MCE version of A23: An equipartition type relation is obtained in the following way:

Consider  $N$  particles with coordinates  $\vec{q}_i$ , and conjugate momenta  $\vec{p}_i$  (with  $i = 1, \dots, N$ ), and subject to a Hamiltonian  $\mathcal{H}(\vec{p}_i, \vec{q}_i)$ .

- (a) Using the classical micro canonical ensemble (MCE) show that the entropy  $S$  is invariant under the rescaling  $\vec{q}_i \rightarrow \lambda \vec{q}_i$  and  $\vec{p}_i \rightarrow \vec{p}_i/\lambda$  of a pair of conjugate variables, i.e.  $S[\mathcal{H}_\lambda]$  is independent of  $\lambda$ , where  $\mathcal{H}_\lambda$  is the Hamiltonian obtained after the above rescaling.
- (b) Now assume a Hamiltonian of the form  $\mathcal{H} = \sum_i \frac{(\vec{p}_i)^2}{2m} + V(\{\vec{q}_i\})$ . Use the result that  $S[\mathcal{H}_\lambda]$  is independent of  $\lambda$  to prove the virial relation

$$\left\langle \frac{(\vec{p}_1)^2}{m} \right\rangle = \left\langle \frac{\partial V}{\partial \vec{q}_1} \cdot \vec{q}_1 \right\rangle$$

where the brackets denote MCE averages. Hint:  $S$  can also be expressed with the accumulated number of states  $\Sigma(E)$ .

- (c) Show that classical equipartition,  $\langle x_i \frac{\partial \mathcal{H}}{\partial x_j} \rangle = \delta_{ij} k_B T$ , also yields the result (b). Note that this form may fail for quantum systems.
- (d) Quantum mechanical version: Write down the expression for the entropy in the quantum case. Show that it is also invariant under the rescalings  $\vec{q}_i \rightarrow \lambda \vec{q}_i$  and  $\vec{p}_i \rightarrow \vec{p}_i/\lambda$  where  $\vec{p}_i$  and  $\vec{q}_i$  are now quantum mechanical operators. (Hint: Use Schrödinger's equation and  $\vec{p}_i = -i\hbar \partial / \partial \vec{q}_i$ .) Show that the result in (b) is valid also in the quantum case.

===== [Exercise 1800]

## Thermodynamic processes

===== [Exercise 1808]

### Adiabatic law for generalized dispersion

Consider a gas of noninteracting particles with kinetic energy of the form  $\varepsilon(\mathbf{p}) = \alpha |\mathbf{p}|^{3(\gamma-1)}$  where  $\alpha$  is a constant;  $\mathbf{p}$  is the momentum quantized in a box of size  $L^3$  by  $p_x = \hbar n_x/L, p_y = \hbar n_y/L, p_z = \hbar n_z/L$  with  $n_x, n_y, n_z$  integers. Examples are nonrelativistic particles with  $\gamma = 5/3$  and extreme relativistic particles with  $\gamma = 4/3$ .

- (a) Use the microcanonical ensemble to show that in an adiabatic process (i.e. constant  $S, N$ )  $PV^\gamma = \text{const.}$
- (b) Deduce from (a) that the energy is  $E = Nk_B T / (\gamma - 1)$  and the entropy is  $S = \frac{k_B N}{\gamma - 1} \ln(PV^\gamma) + f(N)$ . What is the most general form of the function  $f(N)$ ?
- (c) Show that  $C_p/C_v = \gamma$ .
- (d) Repeat (a) by using the canonical ensemble.

===== [Exercise 1814]

### Adiabatic versus sudden expansion of an ideal gas

$N$  atoms of mass  $m$  of an ideal classical gas are in a cylinder with insulating walls, closed at one end by a piston. The initial volume and temperature are  $V_0$  and  $T_0$ , respectively.

- (a) If the piston is moving out rapidly the atoms cannot perform work, i.e. their energy is constant. Find the condition on the velocity of the piston that justifies this result.
- (b) Find the change in temperature, pressure and entropy if the volume increases from  $V_0$  to  $V_1$  under the conditions found in (a).

- (c) Find the change in temperature, pressure and entropy if the volume increases from  $V_0$  to  $V_1$  with the piston moving very slowly, i.e. an adiabatic process.

### ===== [Exercise 1815]

#### Cooling by demagnetization

Consider a solid with  $N$  non-magnetic atoms and  $N_i$  non-interacting magnetic impurities with spin  $s$ . There is a weak spin-phonon interaction which allows energy transfer between the impurities and the non-magnetic atoms.

- A magnetic field is applied to the system at a constant temperature  $T$ . The field is strong enough to line up the spins completely. What is the change in entropy of the system due to the applied field? (neglect here the spin-phonon interaction).
- Now the magnetic field is reduced to zero adiabatically. What is the qualitative effect on the temperature of the solid? Why is the spin-phonon interaction relevant?
- Assume that the heat capacity of the solid is  $C_V = 3Nk_B$  in the relevant temperature range. What is the temperature change produced by the process (b)? (assume the process is at constant volume).

### ===== [Exercise 1816]

#### Cooling by adiabatic demagnetization

Consider a system of  $N$  spins on a lattice at temperature  $T$ , each spin has a magnetic moment  $\mu$ . In presence of an external magnetic field each spin has two energy levels,  $\mu H$ .

- Evaluate the changes in energy  $\delta E$  and in entropy  $\delta S$  as the magnetic field increases from 0 to  $H$ . Derive the magnetization  $M(H)$  and show that

$$\delta E = T\delta S - \int_0^H M(H') dH'.$$

Interpret this result.

- Show that the entropy  $S(E, N)$  can be written as  $S(M, N)$ . Deduce the temperature change when  $H$  is reduced to zero in an adiabatic process. Explain how can this operate as a cooling machine to reach  $T \approx 10^{-4}K$ . (Note: below  $10^{-4}K$  in realistic systems spin-electron or spin-spin interactions reduce  $S(T, H=0) \rightarrow 0$  as  $T \rightarrow 0$ . This method is known as cooling by adiabatic demagnetization.

### ===== [Exercise 1817]

#### Adiabatic cooling of spins

Consider an ideal gas whose  $N$  atoms have mass  $m$ , spin  $1/2$  and a magnetic moment  $\gamma$ . The kinetic energy of a particle is  $p^2/(2m)$  and the interaction with the magnetic field  $B$  is  $\pm\gamma B$  for up/down spins.

- Calculate the entropy as  $S(T, B) = S_{kinetic} + S_{spin}$ .
- Consider an adiabatic process in which the magnetic field is varied from  $B$  to zero. Show that the initial and final temperatures  $T_i$  and  $T_f$  are related by the equation:

$$\ln \frac{T_f}{T_i} = \frac{2}{3N} [S_{spin}(T_i, B) - S_{spin}(T_f, 0)]$$

- (c) Find the solution for  $\frac{T_f}{T_i}$  in the large  $B$  limit.
- (d) Extend (c) to the case of space dimensionality  $d$  and general spin  $S$ .

### ===== [Exercise 2000]

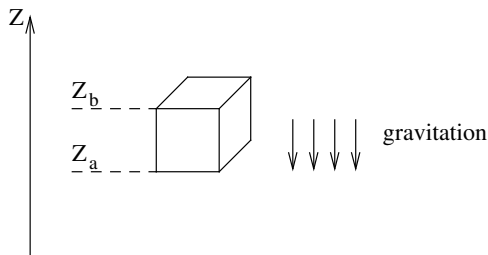
## Canonical formalism, applications

### ===== [Exercise 2040]

#### Pressure of gas in a box with gravitation

Consider an ideal gas in a 3D box of volume  $V = L^2 \times (Z_b - Z_a)$ . The box is placed in an external gravitational field that points along  $-\hat{z}$ .

- Find the one-particle partition function  $Z_1(\beta, Z_a, Z_b)$ .
- What is the  $N$ -particle partition function  $Z_N(\beta, Z_a, Z_b)$ .
- What are the forces  $F_a$  and  $F_b$  acting on the floor and on the ceiling of the box?
- What is the difference between these forces? explain your result.



### ===== [Exercise 2041]

#### Gas in gravitation confined between adhesive plates

A classical ideal gas that consists of  $N$  mass  $m$  particles is confined between two horizontal plates that have each area  $A$ , while the vertical distance between them is  $L$ . The gravitational force is  $f$  oriented towards the lower plate. In the calculation below fix the center of the box as the reference point of the potential.

The particles can be adsorbed by the plates. The adsorption energy is  $-\epsilon$ . The adsorbed particles can move along the plates freely forming a two dimensional classical gas. The system is in thermal equilibrium, the temperature is  $T$ .

- Calculate the one particle partition function  $Z(\beta, A, L, f)$  of the whole system.  
Tip: express the answer using sinh and cosh functions.
- Find the ratio  $N_A/N_V$ , where  $N_A$  and  $N_V$  are the number of adsorbed and non-adsorbed particles.
- What is the value of this ratio at high temperatures.  
Express the result using the thermal wavelength  $\lambda_T$ .
- Find an expression for  $F_V$  in the formula  $dW = (N_V F_V + N_A F_A)dL$ .  
Tip: the expression is quite simple (a single term).
- Find a high temperature approximation for  $F_V$ .  
Tip: it is possible to guess the result without any computation.

6. Find a zero temperature approximation for  $dW$ .  
 Tip: it is possible to guess the result without any computation.

### ===== [Exercise 2042]

#### Pressure of an ideal gas in the atmosphere

An ideal classical gas of  $N$  particles of mass  $m$  is in a container of height  $L$  which is in a gravitational field of a constant acceleration  $g$ . The gas is in uniform temperature  $T$ .

- (a) Find the dependence  $P(h)$  of the pressure on the height  $h$ .
- (b) Find the partition function and the internal energy. Examine the limits  $mgL \ll T$  and  $mgL \gg T$ .
- (c) Find  $P(h)$  for an adiabatic atmosphere, i.e. the atmosphere has been formed by a constant entropy process in which  $T, \mu$ , are not equilibrated, but  $Pn^{-\gamma} = \text{const}$ . The equilibrium is maintained within each atmospheric layer. Find  $T(h)$  and  $n(h)$  at height  $h$  in terms of the density  $n_0$  and the temperature  $T_0$  at  $h = 0$ .

### ===== [Exercise 2044]

#### Boltzmann gas confined in capacitor

An ideal gas is formed of  $N$  spinless particles of mass  $m$  that are inserted between two parallel plates (Z direction). The horizontal confinement is due to a two dimensional harmonic potential (XY direction). Accordingly,

$$V(x, y, z) = \begin{cases} \frac{1}{2}m\omega^2(x^2 + y^2) & z_1 < z < z_2 \\ \infty & \text{else} \end{cases}$$

The distance between the plates is  $L = z_2 - z_1$ . In the first set of questions (a) note that the partition function  $Z$  can be factorized. In the second set of questions (b) an electric field  $\mathcal{E}$  is added in the Z direction. Assume that the particles have charge  $e$ . Express your answers using  $N, m, L, \omega, e, \mathcal{E}, T$ .

- (a1) Calculate the classical partition function  $Z_1(\beta; L)$  via a phase space integral. Find the heat capacity  $C(T)$  of the gas.
- (a2) Calculate the quantum partition function for large  $L$ . Define what is large  $L$  such that the Z motion can be regarded as classical.
- (a3) Find the heat capacity  $C(T)$  of the gas using the partition function of item (a2). Define what temperature is required to get the classical limit.
- (a4) Calculate the forces  $F_1$  and  $F_2$  that the particles apply on the upper and lower plates.
- (b1) Write the one-particle Hamiltonian and calculate the classical partition function  $Z_1(\beta; z_1, z_2, \mathcal{E})$
- (b2) Calculate the forces  $F_1$  and  $F_2$  that are acting on the upper and lower plates. What is the total force on the system? What is the prefactor in  $(F_1 - F_2) = \alpha NT/L$ .
- (b3) Find the polarization  $\tilde{P}$  of the electron gas as a function of the electric field. Recall that the polarization is defined via the formula  $dW = \tilde{P}dE$ .
- (b4) Find the susceptibility by expanding  $\mathcal{P}(\mathcal{E}) = (1/L)\tilde{P} = \chi\mathcal{E} + O(\mathcal{E}^2)$ . Determine what is a weak field  $\mathcal{E}$  such that the linear approximation is justified.

### ===== [Exercise 2046]

#### Gas in a centrifuge

A cylinder of radius  $R$  rotates about its axis with a constant angular velocity  $\Omega$ . It contains an ideal classical gas of  $N$  particles at temperature  $T$ . Find the density distribution as a function of the radial distance from the axis. Write what is the pressure on the walls.

Note that the Hamiltonian in the rotating frame is  $H'(r, p; \Omega) = H(r, p) - \Omega L(r, p)$  where  $L(r, p)$  is the angular momentum.

It is conceptually useful to realize that formally the Hamiltonian is the same as that of a charged particle in a magnetic field ("Coriolis force") plus centrifugal potential  $V(r)$ . Explain how this formal equivalence can be used in order to make a shortcut in the above calculation.

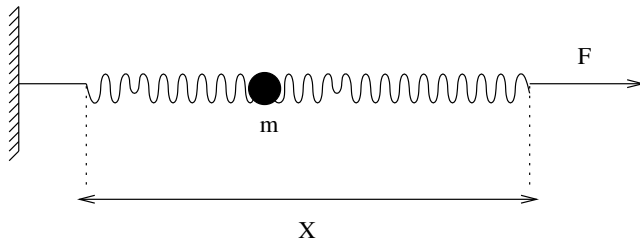
### ===== [Exercise 2050]

#### Pressure by a particle in a spring-box system

A spring that has an elastic constant  $K$  and natural length  $L$  is connected between a wall at  $x = 0$  and a piston at  $x = X$ . Consequently the force that acts on the piston is  $F_0 = -K(X - L)$ . A classical particle of mass  $m$  is attached to the middle point of the spring. The system is at equilibrium, the temperature is  $T$ .

- (1) Write the Hamiltonian (be careful).
- (2) Write an expression for the partition function  $Z(\beta, X)$ . The answer is an expression that may contain a definite integral.
- (3) Write an expression for the force  $F$  on the piston. The answer is an expression that may contain a definite integral.
- (4) Find a leading order (non-zero) expression for  $F - F_0$  in the limit of high temperature.
- (5) Find a leading order (non-zero) expression for  $F - F_0$  in the limit of low temperature.

Your answers should not involve exotic functions, and should be expressed using  $(X, L, K, m, T)$ .



### ===== [Exercise 2051]

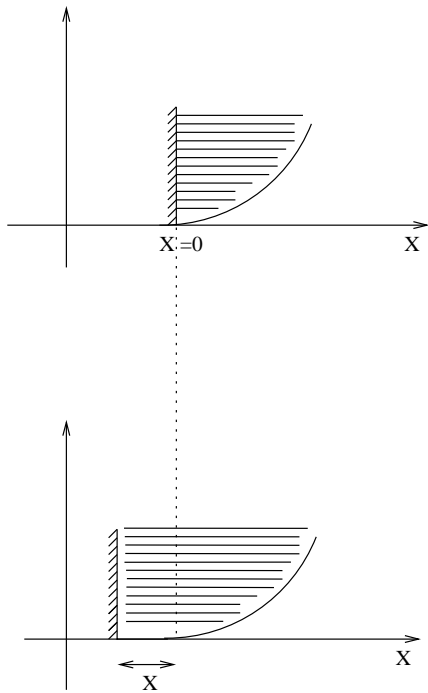
#### Gas in a box with parabolic potential wall

Consider  $N$  classical particles in a potential

$$V(x, y, z) = \begin{cases} \frac{1}{2}ax^2 & 0 < x, \quad 0 < y < L, \quad 0 < z < L \\ \infty & \text{else} \end{cases}$$

Calculate the partition function and derive from it an expression for the pressure on the wall at  $x = 0$ . Note that for this purpose you have to re-define the potential, such that it would depend on a parameter  $X$  that describes the position of the wall.

Show that the result for the pressure can be optionally obtained by assuming that the pressure is the same as that of an ideal gas. For this purpose evaluate the density of the particles in the vicinity of the wall.



### ===== [Exercise 2052]

#### Pressure in a box with $V(x)$ potential

A particle is confined by hard walls to move inside a box  $[0, L]$ . There is an added external potential  $U(x)$ . Find the force ("Pressure") on the wall at  $x = L$ .

- (1) The short way - evaluate the density of the particles in the vicinity of the wall, and assume that the pressure there is the same as that of an ideal gas.
- (2) The long way - using the Virial theorem relate the force at  $x = L$  to the expectation function of  $xU'(x)$ .
- (3) Explain why the Virial based derivation gives the force on the  $x = L$  and not on the  $x = 0$  wall.

### ===== [Exercise 2065]

#### Classical gas with general dispersion relation

Consider a gas of  $N$  non-interacting particles in a  $d$  dimensional box. The kinetic energy of a particle is  $\epsilon_p = c|p|^s$ .

- (a) Find the partition function of the gas for a given temperature is  $T$ .
- (b) Define  $\gamma = 1 + (s/d)$  and using (a) show that the energy is  $E = \frac{NT}{\gamma-1}$ .
- (c) Show that the entropy is  $S = \frac{N}{\gamma-1} \ln(PV^\gamma) + f(N)$ .
- (d) Deduce that in an adiabatic process  $PV^\gamma = \text{const}$ .
- (e) Show that the heat capacity ratio is  $C_P/C_V = \gamma$ .

===== [Exercise 2100]

Systems subjected to electric or magnetic fields

===== [Exercise 2160]

**Particle on a ring with electric field**

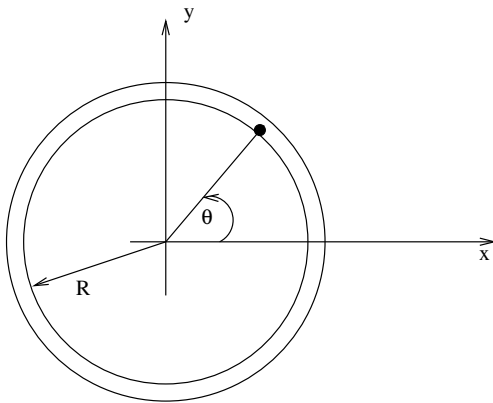
A particle of mass  $m$  and charge  $e$  is free to move on a ring of radius  $R$ . The ring is located in the  $(x, y)$  plan. The position of the particle on the ring is  $x = R \cos(\theta)$  and  $y = R \sin(\theta)$ . There is an electric field  $\mathcal{E}$  in the  $x$  direction. The temperature is  $T$ .

- (1) Write the Hamiltonian  $H(\theta, p)$  of the particle.
- (2) Calculate the partition function  $Z(\beta, \mathcal{E})$ .
- (3) Write an expression for the probability distribution  $\rho(\theta)$ .
- (4) Calculate the mean position  $\langle x \rangle$  and  $\langle y \rangle$ .
- (5) Write an expression for the probability distribution  $\rho(x)$ . Attach a schematic plot.
- (6) Write an expression for the polarization. Expand it up to first order in  $\mathcal{E}$ , and determine the susceptibility.

$$\frac{1}{2\pi} \int_0^{2\pi} \exp(z \cos(\theta)) d\theta = I_0(z)$$

$$I'_0(z) = I_1(z)$$

$$I_0(z) = 1 + \left(\frac{1}{4}\right) z^2 + \left(\frac{1}{64}\right) z^4 + \dots$$



===== [Exercise 2170]

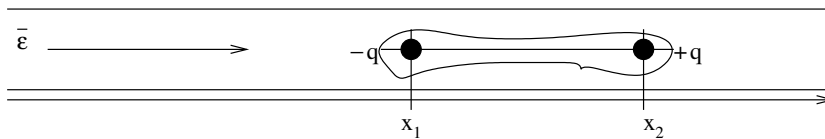
**Polarization of two-spheres system inside a tube**



Consider two spheres in a very long hollow tube of length  $L$ . The mass of each ball is  $m$ , the charge of one ball is  $-q$ , and the charge of the other one is  $+q$ . The ball radius is negligible, and the electrostatic attraction between the spheres is also negligible. The spheres are rigid and cannot pass through each other. The spheres are attached by a drop of water. Due to the surface tension there is an attraction force  $\gamma$  that does not depend on the distance. Additionally there is an applied external electric field  $f$ . The temperature is  $T$ .

- Write the hamiltonian  $H(p_1, p_2, x_1, x_2)$  of the system. Rewrite it also in terms of center-of-mass and distance  $r = |x_2 - x_1|$  coordinates.
- Calculate the partition function  $Z(\beta, f)$  assuming that the drop is not teared out. What is the condition for that?
- Find the probability density function of  $\rho(r)$ , and calculate the average distance  $\langle r \rangle$ .
- Find the polarization  $\tilde{P}$  as a function of  $f$ .
- Expand the polarization up to first order in the field, namely  $\tilde{P}(f) = \tilde{P}(0) + \chi f + \mathcal{O}(f^2)$ .

Express your answers with  $L, m, q, \gamma, T, f$ .



### ===== [Exercise 2173]

#### Polarization of classical polar molecules

Find the polarization  $\tilde{P}(\xi)$  and the electric susceptibility  $\chi$  for gas of  $N$  classical molecules with dipole moment  $\mu$ . The system's temperature is  $T$ .

### ===== [Exercise 2180]

#### Magnetization of spin 1/2 system

Find the state functions  $E(T, B)$ ,  $M(T, B)$ ,  $S(T, B)$  for  $N$  spins system:

$$H = -\gamma B \sum_{a=1}^N \sigma_t^a$$

Write the results for a weak magnetic field  $\gamma B \ll T$ . Especially find the magnetic susceptibility  $\chi$  and  $S(B \rightarrow 0)$

### ===== [Exercise 2190]

#### Electron gas in a magnetic field, Landau levels

Calculate the partition function for electrons in a 3D box subject to a homogeneous magnetic field in the  $z$  direction. Use the known results for the Landau levels and their degeneracy. Assume the Boltzmann approximation.

Find the magnetization for arbitrary field, and the susceptibility at zero field. Distinguish the orbital (Landau) and spin (Pauli) contributions.

Disregarding the spin, explain why there is no magnetism in the classical limit.

Note: The zero temperature case is treated in a different exercise, and requires to take the Pauli exclusion into account.

===== [Exercise 2200]

## Harmonic Oscillators

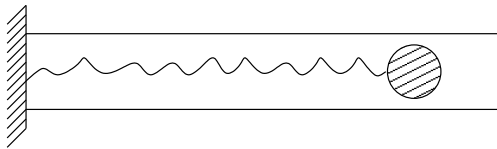
===== [Exercise 2210]

### Harmonic oscillator, Heat capacity

Find the energy and the heat capacity of the next system:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}mw^2\hat{X}^2$$

This system can be a model to a particle adsorbed to the surface of a solid. Compare to the result you get in a classical treatment.



===== [Exercise 2215]

### Heat capacity of solids

Consider a piece of solid whose low laying excitations are bosonic modes that have spectral density  $g(\omega) = C\omega^{\alpha-1}$  up to a cutoff frequency  $\omega_c$ , as in the well-known Debye model (items 1-5). Similar description applies for magnetic materials (item 6). In items 7-8 assume that the solid is a "glass", whose low laying excitations are like two level entities that have a spectral density  $g(\omega)$ .

- (1) Write a general expression for the energy  $E(T)$  of the system. This expression may involve a numerical prefactor that is defined by an  $\alpha$  dependent definite integral.
- (2) Write a general expression for the heat capacity  $C(T)$ .
- (3) Write a general expression for the variance  $\text{Var}(r)$  of an atom that reside inside the solid.
- (4) Determine what are  $\alpha$  and  $C$  and  $\omega_c$  for a piece of solid that consists of  $N$  atoms that occupy a volume  $L^d$  in  $d = 1, 2, 3$  dimensions, assuming a dispersion relation  $\omega = c|k|$ , as for "phonons".
- (5) Write explicitly what are  $C(T)$  and  $\text{Var}(r)$  for  $d = 1, 2, 3$ . Be careful with the evaluation of  $\text{Var}(r)$ . In all cases consider both low temperatures ( $T \ll \omega_c$ ), and high temperatures ( $T \gg \omega_c$ ).
- (6) Point out what would be  $\alpha$  if the low laying excitations had a dispersion relation  $\omega = a|k|^2$  as for "magnons".
- (7) What is the heat capacity of a "glass" whose two level entities have excitation energies  $\omega = \Delta$ , where  $\Delta$  has a uniform distribution with density  $C$ .
- (8) What is the heat capacity of a "glass" whose two level entities have excitation energies  $\omega = \omega_c \exp(-\Delta)$ , where the barrier  $\Delta > 0$  has a uniform distribution with density  $D$ .

===== [Exercise 2230]

## Harmonic oscillators, Photons

Find the state equations of photon gas in 1D/2D/3D cavity within the framework of the canonical formalism, regarding the electromagnetic modes as a collection of harmonic oscillators. The volume of the cavity is  $L^d$  with  $d = 1, 2, 3$ . The temperature is  $T$ .

- (1) Write the partition function for a single mode  $\omega$ .
- (2) Find the mode average occupation  $f(\omega)$ .
- (3) Find the spectral density of modes  $g(\omega)$ .
- (4) Find the energy  $E(T)$  of the photon gas.
- (5) Find the free energy  $F(T)$  of the photon gas.
- (6) Find an expression for the pressure  $P(T)$  of the photon gas.

Note: additional exercises on photon gas and blackbody radiation can be found in the context of quantum gases. Formally, photon gas is like Bose gas with chemical potential  $\mu = 0$ . Note that the same type of calculation appears in Debye model ("acoustic" phonons instead of "transverse" photons).

===== [Exercise 2300]

## Misc mechanical constructs

===== [Exercise 2311]

## Imperfect lattice with defects

A perfect lattice is composed of  $N$  atoms on  $N$  sites. If  $n$  of these atoms are shifted to interstitial sites (i.e. between regular positions) we have an imperfect lattice with  $n$  defects. The number of available interstitial sites is  $M$  and is of order  $N$ . Every atom can be shifted from lattice to any defect site. The energy needed to create a defect is  $\omega$ . The temperature is  $T$ . Define  $x \equiv e^{-\omega/T}$ .

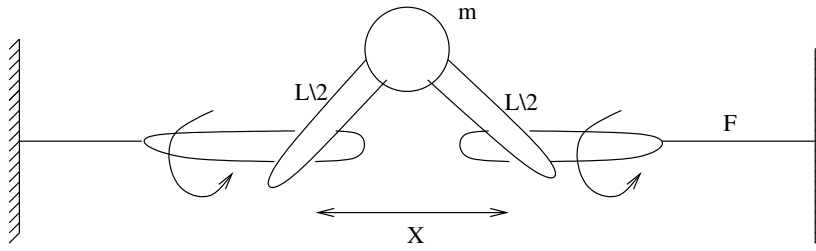
- (a) Write the expression for the partition function  $Z(x)$  as a sum over  $n$ .
- (b) Using Stirling approximation (see note) determine what is the most probable  $n$ , and write for it the simplest approximation assuming  $x \ll 1$ .
- (c) Explain why your result for  $\bar{n}$  merely reproduces the law of mass action.
- (d) Evaluate  $Z(x)$  using a Gaussian integral.
- (e) Derive the expressions for the entropy and for the specific heat.
- (f) What would be the result if instead of Gaussian integration one were taking only the largest term in the sum?

Note: Regarding  $n$  as a continuous variable the derivative of  $\ln(n!)$  is approximately  $\ln(n)$ .

===== [Exercise 2320]

## Tension of rotating device

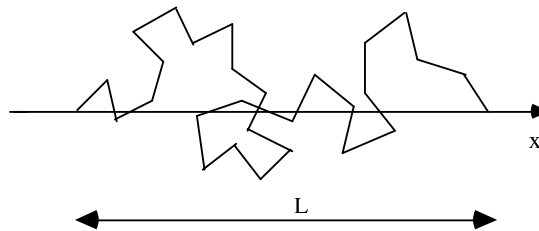
The system in the drawing is in balance (Temperature  $T$ ). Find Tension  $F$  in the axis.



### ===== [Exercise 2340]

#### Tension of a chain molecule

A chain molecule consists of  $N$  units, each having a length  $a$ , see figure. The units are joined so as to permit free rotation about the joints. At a given temperature  $T$ , derive the relation between the tension  $f$  acting between both ends of the three-dimensional chain molecule and the distance  $L$  between the ends.



### ===== [Exercise 2351]

#### Tension of a rubber band

The elasticity of a rubber band can be described by a one dimensional model of a polymer. The polymer consists of  $N$  monomers that are arranged along a straight line, hence forming a chain. Each unit can be either in a state of length  $a$  with energy  $E_a$ , or in a state of length  $b$  with energy  $E_b$ . We define  $f$  as the tension, i.e. the force that is applied while holding the polymer in equilibrium.

- (1) Write expressions for the partition function  $Z_G(\beta, f)$ .
- (2) For very high temperatures  $F_G(T, f) \approx F_G^{(\infty)}(T, f)$ , where  $F_G^{(\infty)}(T, f)$  is a linear function of  $T$ . Write the explicit expression for  $F_G^{(\infty)}(T, f)$ .
- (3) Write the expression for  $F_G(T, f) - F_G^{(\infty)}(T, f)$ . Hint: this expression is quite simple - within this expression  $f$  should appear only once in a linear combination with other parameters.
- (4) Derive an expression for the length  $L$  of the polymer at thermal equilibrium, given the tension  $f$ . Write two separate expressions: one for the infinite temperature result  $L(\infty, f)$  and one for the difference  $L(T, f) - L(\infty, f)$ .
- (5) Assuming  $E_a = E_b$ , write a linear approximation for the function  $L(T, f)$  in the limit of weak tension.
- (6) Treating  $L$  as a continuous variable, find the probability distribution  $P(L)$ , assuming  $E_a = E_b$  and  $f = 0$ .
- (7) Write an expression that relates the function  $f(L)$  to the probability distribution  $P(L)$ . Write also the result that you get from this expression.
- (8) Find what would be the results for  $Z_G(\beta, f)$  if the monomer could have any length  $\in [a, b]$ . Assume that the

energy of the monomer is independent of its length.

(9) Find what would be the results for  $L(T, f)$  in the latter case.

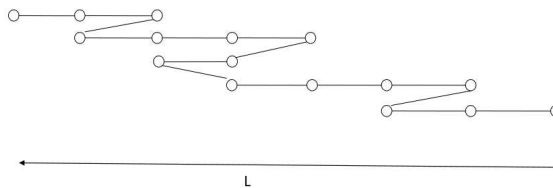
*Note:* Above a "linear function" means  $y = Ax + B$ .

Please express all results using  $(N, a, b, E_a, E_b, f, T, L)$ .

### ===== [Exercise 2353]

#### Tension of a stretched chain

A rubber band is modeled as a single chain of  $N \gg 1$  massless non-interacting links, each of fixed length  $a$ . Consider a one-dimensional model where the links are restricted to point parallel or anti-parallel to a given axis, while the endpoints are constraint to have a distance  $X = (2n - N)a$ , where  $n$  is an integer. Later you are requested to use approximations that allow to regard  $X$  as a continuous variable. Note that the body of the chain may extend beyond the length  $X$ , only its endpoints are fixed. In items (c,d) a spring is pushed between the two endpoints, such that the additional potential energy  $-KX^2$  favors large  $X$ , and the system is released (i.e.  $X$  is free to fluctuate).

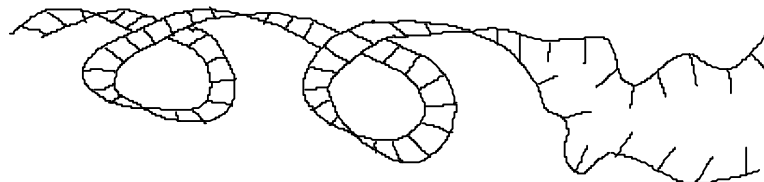


- Calculate the partition function  $Z(X)$ . Write the exact combinatorial expression. Explain how and why it is related trivially to the entropy  $S(X)$ .
- Calculate the force  $f(X)$  that the chain applies on the endpoints. Use the Stirling approximation for the derivatives of the factorials.
- Determine the temperature  $T_c$  below which the  $X = 0$  equilibrium state becomes unstable.
- For  $T < T_c$  write an equation for the stable equilibrium distance  $X(T)$ . Find an explicit solution by expanding  $f(X)$  in leading order.

### ===== [Exercise 2360]

#### The zipper model for DNA molecule

The DNA molecule forms a double stranded helix with hydrogen bonds stabilizing the double helix. Under certain conditions the two strands get separated resulting in a sharp "phase transition" in the thermodynamic limit. As a model for this unwinding, use the "zipper model" where the DNA is modeled as a polymer with  $N$  parallel links that can be opened from one end (see figure).



The energy cost of an open link is  $\varepsilon$ . A possible state of the DNA is having links  $1, 2, 3, \dots, p$  open, and the rest are closed. The last link cannot be opened. Each open link can have  $g$  orientations, corresponding to the rotational freedom about the bond. Assume a large number of links  $N$ .

- Define  $x = ge^{-\varepsilon/T}$  and find the canonical partition function  $Z(\beta, x)$ .
- Find the average number of open links  $\langle p \rangle$  as a function of  $x$ .

- (3) Find the linear approximation for  $\langle p \rangle$ .
- (4) Approximate  $\frac{\langle p \rangle}{N}$  for large  $x$ .
- (5) Describe the dependence of  $\frac{\langle p \rangle}{N}$  on  $x$ .
- (6) Find expressions for the entropy  $S(x)$  and the heat capacity  $C(x)$  at  $x = 1$ .
- (7) What is the order of the phase transition?

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### [Exercise 3000]

## Quantum gases

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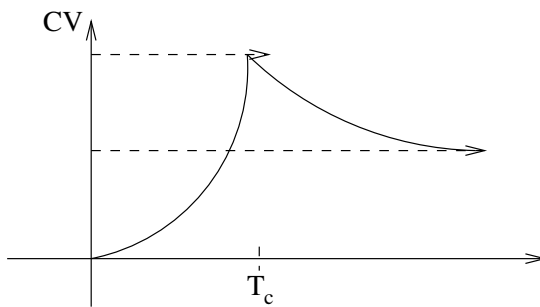
### [Exercise 3009]

#### Entropy and heat capacity of quantum ideal gases

Consider an  $N$  particle ideal gas confined in volume  $V$ . Find (a) the entropy  $S$  and (b) the heat capacity  $C$ , highlighting its dependence on the temperature  $T$ .

- (1) Consider classical gas.
- (2) Consider Fermi gas at low temperatures, using leading order Sommerfeld expansion.
- (3) Consider Bose gas below the condensation temperature.
- (4) Consider Bose gas above the condensation temperature.
- (5) What is  $C_{Bose}/C_{classical}$  at the condensation temperature?
- (6) For temperatures that are above but very close to the condensation temperature, find an approximation for  $C_V$  in terms of elementary functions.

Hints: In (4) use the Grand-Canonical formalism to express  $N$  and  $E$  as a function of the temperature  $T$  and the fugacity  $z$ . Use the equation for  $N$  in order to deduce an expression for  $(\partial z / \partial T)_N$ . Note that the derivative of the polylogarithmic function  $L_\alpha(z)$  is  $(1/z)L_{\alpha-1}(z)$ . Final results should be expressed in terms of  $(N, V, T)$ , but it is allowed to define and use the notations  $\lambda_T$  and  $\epsilon_F$  and  $T_c$ . In item (4) the final result can include ratios of polylogarithmic functions, with the fugacity  $z$  as an implicit variable. Note that such ratios are all of order unity throughout the whole temperature range provided  $\alpha > 1$ , while functions with  $\alpha < 1$  are singular at  $z = 1$ .




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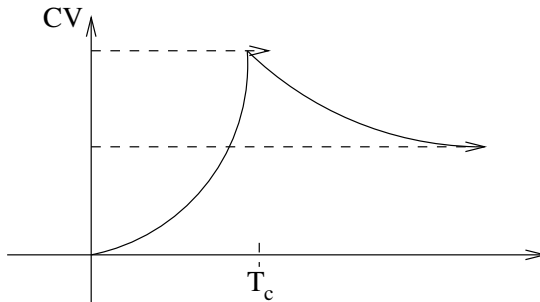
### [Exercise 3010]

#### Heat capacity of an ideal Bose gas

Consider a volume  $V$  that contains  $N$  mass  $m$  bosons. The gas is in a thermal equilibrium at temperature  $T$ .

1. Write an explicit expression for the condensation temperature  $T_c$ .

2. Calculate the chemical potential, the energy and the pressure in the Boltzmann approximation  $T \gg T_c$ .
3. Calculate the chemical potential, the energy and the pressure in the regime  $T < T_c$ .
4. Calculate  $C_v$  for  $T < T_c$
5. Calculate  $C_v$  for  $T = T_c$
6. Calculate  $C_v$  for  $T \gg T_c$
7. Express the ratio  $C_p/C_v$  using the polylogarithmic functions. Explain why  $C_p \rightarrow \infty$  in the condensed phase?
8. Find the  $\gamma$  in the adiabatic equation of state. Note that in general it does not equal  $C_p/C_v$ .



### ===== [Exercise 3021]

#### Bosons with Spin in magnetic field

$N$  Bosons that have mass  $m$  and spin 1 are placed in a box that has volume  $V$ . A magnetic field  $B$  is applied, such that the interaction is  $-\gamma B S_z$ , where  $S_z = 1, 0, -1$ , and  $\gamma$  is the gyromagnetic ratio. In items (c-f) assume the Boltzmann approximation for the occupation of the  $S_z \neq 1$  states.

- (a) Find an equation for the condensation temperature  $T_c$ .
- (b) Find the condensation temperature  $T_c(B)$  for  $B = 0$  and for  $B \rightarrow \infty$ .
- (c) Find the critical  $B$  for condensation if  $T$  is set in the range of temperatures that has been defined in item(b).
- (d) Describe how  $T_c(B)$  depends of  $B$  in a qualitatively manner. Find approximate expressions for moderate and large fields.
- (e) Find the condensate fraction as a function of  $T$  and  $B$ .
- (f) Find the heat capacity of the gas assuming large but finite field.

### ===== [Exercise 3022]

#### Bosons with spin in harmonic trap

$N$  Bosons that have spin 1 are placed in a 3D harmonic trap. The harmonic trap frequency is  $\Omega$ . A magnetic field  $B$  is applied, such that the interaction is  $-\gamma B S_z$ , where  $S_z = 1, 0, -1$ , and  $\gamma$  is the gyromagnetic ratio.

- (1) Write an expression for the density of one-particle states  $g(\epsilon)$ .
- (2) Write an expression for the  $B = \infty$  condensation temperature  $T_c$ .
- (3) Write an equation for  $T_c(B)$ . It should be expressed in terms of the appropriate polylogarithmic function.
- (4) Find the leading correction in  $T_c(B)/T_c \approx 1 + \dots$  assuming that  $B$  is very large. It should be expressed in terms

of an elementary function.

(5) Find what is  $T_c(B)/T_c$  for  $B = 0$ , and what is the first-order correction term if  $B$  is very small.

(6) Sketch a schematic plot of  $T_c(B)/T_c$  versus  $B$ . Indicate by solid line the exact dependence, and by dashed and dotted lines the approximations. It should be clear from the figure whether the approximation under-estimates or over-estimates the true result, and what is the  $B$  dependence of the slope.

**Tips:** The prefactors are important in this question. Do not use numerical substitutions.

Use the notation  $L_\alpha(z)$  for the polylogarithmic function, and recall that  $L_\alpha(1) = \zeta(\alpha)$ .

Note also that  $L'_\alpha(z) = (1/z)L_{\alpha-1}(z)$ , and that  $\Gamma(n) = (n-1)!$  for integer  $n$ .

### ===== [Exercise 3030]

#### Charged Bose gas in a divided box

Consider  $N$  bosons with mass  $m$ , positive charge  $e$  and spin 0. The particles are in a tank in thermic equilibrium, and temperature  $T$ . The tank has two zones  $A$  and  $B$ . The volume of each zone is  $L^3$ . A battery creates potential difference  $V$  between the zones. The potential in every zone is homogenous.

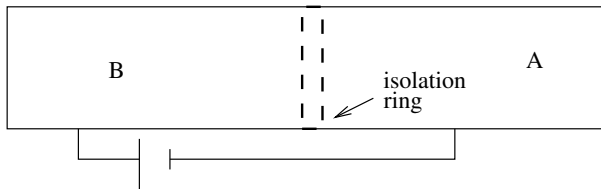
Find the condition on  $N$ , so if  $V = 0$  then there's no condensation, but if  $V = \infty$  then there's condensation.

answer:  $L^3 \zeta\left(\frac{3}{2}\right) \cdot \left(\frac{mT}{2\pi}\right)^{\frac{3}{2}} < N < 2L^3 \zeta\left(\frac{3}{2}\right) \cdot \left(\frac{mT}{2\pi}\right)^{\frac{3}{2}}$

Below, assume that the particles in zone  $A$  are in a condensation state and the particles in zone  $B$  can be described in the Boltzman proximity frame.

- What is the number of the particles in zone  $B$ . What is the condition for  $V$ , so that Boltzman proximity will be valid
- What is the number of the particles in zone  $A$ . How many of them are in condensation state?
- Show that the condensation in zone  $A$  as long as  $V_c < V$ . Find an explicit expression for  $V_c$ .

remark: This problem is formally identical to the 'bosons' problem with spin  $\frac{1}{2}$  in magnetic field. [zone  $B$ ] down, [zone  $A$ ] up, and potential difference  $eV\gamma B$ .



### ===== [Exercise 3040]

#### Quantum Bose Gas with an oscillating piston

A cylinder of length  $L$  and cross section  $A$  is divided into two compartments by a piston. The piston has mass  $M$  and it is free to move without friction. Its distance from the left basis of the cylinder is denoted by  $x$ . In the left side of the piston there is an ideal Bose gas of  $N_a$  particles with mass  $m_a$ . In the right side of the piston there is an ideal Bose gas of  $N_b$  particles with mass  $m_b$ . The temperature of the system is  $T$ .

(\*) Assume that the left gas can be treated within the framework of the Boltzmann approximation.

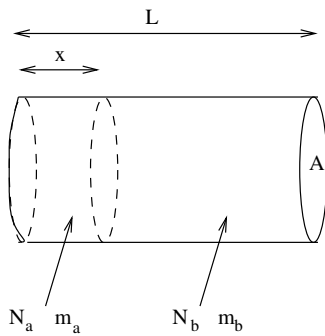
(\*\*) Assume that the right gas is in condensation.

- Find the equilibrium position of the piston.
- What is the condition for (\*) to be valid?
- Below which temperature (\*\*) holds?



(d) What is the frequency of small oscillations of the piston.

Express your answers using  $L, A, N_a, N_b, m_a, m_b, T, M$ .



### ===== [Exercise 3042]

#### Oscillations of a piston in a cylinder filled with gas

Consider a vertically aligned cylinder whose basis has an area  $A$ . A piston that has mass  $M$  is pushed from above. The piston is held by a spring that has an elastic constant  $K$ . If the cylinder is empty the piston is down at zero height ( $x = 0$ ). The cylinder is filled with  $N$  gas particles. Each particle has mass  $m$  and the temperature is  $T$ . Consequently the piston goes up a distance  $x$ , such that the gas occupies a volume  $Ax$ . Consider the following 3 cases:

- The temperature is high, such that Boltzmann approximation can be applied.
- The particles are condensed Bosons,  $T$  is lower than the condensation temperature.
- The particles are spinless Fermions, and the temperature is zero.

Answer the following questions, relating to each case separately.

- What is the equilibrium position  $x_{eq}$  of the piston?
- What is the frequency  $\omega$  of small oscillations?
- Plot schematic drawing of  $\omega$  versus  $T$ .

Express answers using  $A, M, K, N, T$ . The schematic drawing is required to be clearly displayed.

### ===== [Exercise 3230]

#### Heat Capacity of He4 system, energy gap

The specific heat of  $He^4$  at low temperatures has the form

$$C_v = A(T) + B(T)e^{-\Delta/T}$$

This is explained by the having a dispersion relation that give rise to long wavelength phonons  $\omega = c|\mathbf{k}|$  and short wavelength rotons  $\omega(k) = \Delta + b(|\mathbf{k}| - k_0)^2$ , where  $k_0 = 1/a$  is comparable to the mean interparticle separation.

- Find explicit expressions for the coefficients  $A(T)$  and  $B(T)$
- What would be the power in the  $T$  dependence of the coefficients if the system were two dimensional?

### ===== [Exercise 3240]

#### Bose gas in a uniform gravitational field

Consider an ideal Bose gas of particles of mass  $m$  in a uniform gravitational field of acceleration  $g$ .

- (1) Show that the phenomenon of Bose-Einstein condensation in this gas sets in at a temperature  $T_c$  given by

$$T_c \approx T_c^0 \left[ 1 + \frac{8}{9} \frac{1}{\zeta(3/2)} \sqrt{\frac{\pi m g L}{k T_c^0}} \right]$$

where  $L$  is the height of the tank and  $m g L \ll k T_c^0$ , where  $T_c^0 \equiv T_c^0(g = 0)$ .

- (2) Show that the condensation is accompanied by a discontinuity in the specific heat of the gas:

$$(\Delta C_V)_{T=T_c} \approx -\frac{9}{8\pi} \zeta(3/2) N k \sqrt{\frac{\pi m g L}{k T_c^0}}$$

**Hint:** note the following expansion of the polylogarithmic function:

$$L_\nu(e^{-\alpha}) = \frac{\Gamma(1-\nu)}{\alpha^{1-\nu}} + \sum_{i=0}^{\infty} \frac{(-1)^i}{i} \zeta(\nu-i) \alpha^i$$

### ===== [Exercise 3336]

#### Condensation for general dispersion

An ideal Bose gas consists of particles that have the dispersion relation  $\epsilon = c|p|^s$  with  $s > 0$ . The gas is contained in a box that has volume  $V$  in  $d$  dimensions. The gas is maintained in a uniform temperature  $T$ .

- (1) Calculate the single particle density of states.
- (2) Find a condition involving  $s$  and  $d$  for the existence of Bose-Einstein condensation. In particular relate to relativistic ( $s = 1$ ) and nonrelativistic ( $s = 2$ ) particles in two dimensions.
- (3) Find the dependence of the number of particles  $N$  on the chemical potential  $\mu$ .
- (4) Find the dependence of the total energy  $E$  on the chemical potential, and show how the pressure  $P$  is obtained from this result.
- (5) Find an expression for the heat capacity  $C_v$ . Show how this result can be expressed using  $N$  in the limit of infinite temperature.
- (6) Repeat item 1 for relativistic gas whose particles have finite mass such that their dispersion relation is  $\epsilon = \sqrt{m^2 c^4 + c^2 p^2}$ .
- (7) Consider a relativistic gas in  $2D$ . Find expressions for  $N$  and  $E$  and  $P$ . Should one expect Bose-Einstein condensation?

### ===== [Exercise 3341]

#### Bose in 2D harmonic trap

Consider a two dimensional Bose gas in a harmonic potential with energy eigenvalues  $(1 + n_1 + n_2)\omega$ , where  $n_1, n_2$  are integers. This reflects a conventional setup in actual experiments. Assume that the temperature  $T$  is below the Bose-Einstein condensation temperature  $T_c$ .

- (a) Find the average number  $N_e(T)$  of particles in the excited states. Assume  $T \gg \omega$  so that summations can be replaced by integrals.
- (b) Given that the total number of particles is  $N$  what is the Bose-Einstein condensation temperature  $T_c$ .
- (c) Deduce that the number of condensed particles is  $n_0 = N[1 - (T/T_c)^2]$

$$\int_0^\infty \frac{x}{e^x - 1} dx = \frac{\pi^2}{6}$$

### ===== [Exercise 3342]

#### Black body radiation in the universe

The universe is pervaded by a black body radiation corresponding to a temperature of  $3K$ . In a simple view, this radiation was produced from the adiabatic expansion of a much hotter photon cloud which was produced during the big bang.

- (a) Why is the recent expansion adiabatic rather than, for example, isothermal? It is also known that the expansion velocity is sufficiently small. Smallness compared with what is needed? explain.
- (b) If in the next  $10^{10}$  years the volume of the universe increases by a factor of two, what then will be the temperature of the black body radiation?
- (c) By what factor does the energy change in the process (b)? Explain the process by which the energy changes and show that this specific process indeed reproduces the change in energy.

### ===== [Exercise 3344]

#### BEC in harmonic potential

The current experimental realizations of Bose Einstein condensation rely on trapping cold atoms in a potential. Close to its minimum, the potential can be expanded to second order, and has the form

$$U(\vec{r}) = \frac{1}{2}m \sum_{\alpha} \omega_{\alpha}^2 x_{\alpha}^2$$

where  $\alpha = 1, \dots, d$ ,  $d$  is the space dimensionality and the trapping potential may have different frequencies  $\omega_{\alpha}$  in different directions.

- (a) We are interested in the limit of wide traps such that  $\hbar\omega_{\alpha} \ll k_B T$ , and the discreteness of the allowed energies can be largely ignored. Show that in this limit, the number of states  $N(E)$  with energy less than or equal to  $E$ , and the density of states  $\rho(E) = dN(E)/dE$  are given by

$$N(E) = \frac{1}{d!} \prod_{\alpha=1}^d \left( \frac{E}{\hbar\omega_{\alpha}} \right) \quad \Rightarrow \quad \rho(E) = \frac{1}{(d-1)!} \frac{E^{d-1}}{\prod_{\alpha=1}^d \hbar\omega_{\alpha}}$$

[Hint: The volume of the hyper-pyramid defined by  $\sum_{i=1}^d x_i \leq R$  and  $x_i \geq 0$ , in  $d$  dimensions is  $R^d/d!$  .]

- (b) Show that in a grand canonical ensemble, the number of particles in the trap is

$$\langle N \rangle = g_d(\zeta) \prod_{\alpha=1}^d \left( \frac{k_B T}{\hbar\omega_{\alpha}} \right)$$

where  $g_n(\zeta)$  is the usual Bose function.

- (c) Find the chemical potential in the high temperature limit.
- (d) Find the temperature  $T_c$  for BE condensation (no need to evaluate the  $g_d$  integrals). At which dimensions there is no solution with finite  $T_c$ ?

[Note that the condensate is confined by the trap to a finite size so that the system does not have a proper thermodynamic ( $N \rightarrow \infty$ ) limit. Nonetheless, there is a reasonable sharp crossover temperature  $T_c$ , at which a macroscopic fraction of particles condenses to the ground state.]

### ===== [Exercise 3500]

#### Fermi systems

### ===== [Exercise 3510]

#### State equations for ideal Fermi gas

$N$  fermions with  $\frac{1}{2}$  spin and mass  $m$  are in a tank with volume  $V$ . The gas is in thermic equilibrium in temperature  $T$ .

Assume it's possible to relate to the temperature as a low one, and find explicit expressions, up to second order in temperature, for the state equations

$$\mu = \mu\left(T; \frac{N}{V}\right)$$

$$E = E(T, V; N)$$

$$P = P(T, V; N)$$

Define what is a low temperature. Use only  $N, m, V, T$ . Write expressions also for the heat capacity  $C_v$  and the compressibility  $K_T$ .

$$K_T \equiv \frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T$$

Guideline: Write an expression for  $N = N(\beta\mu)$  and find  $\mu(\beta, N/V)$  while keeping terms up to  $O(T^2)$

Similar to the calculation of  $N(\beta\mu)$  it is possible to calculate  $E(\beta\mu)$  up to second order in temperature.

Now there's to place the expression for  $\mu(T; \frac{N}{V})$  you found earlier, and write the result as a development of  $T$  while keeping terms up to second order only! This is the "trickiest" phase..., You'll have to use the development

$$(1 + \chi)^\alpha = 1 + \alpha\chi + \frac{\alpha(\alpha-1)}{2}\chi^2 + \theta(\chi^3)$$

several times and to make sure not to losing the first and the second order terms during the algebra process.

### ===== [Exercise 3515]

#### Ideal Fermi gas in 1D space

Consider  $N$  electrons that are kept between the plates of a capacitor.

$$V(x, y, z) = \begin{cases} \frac{1}{2}m\omega^2(x^2 + y^2) & 0 \leq z \leq L \\ \infty & \text{else} \end{cases}$$

The system is in thermal equilibrium at zero temperature. Find the force that the gas exerts of the plates assuming that it can be treated as one-dimensional.

Write the condition on  $N$  for having this assumption valid.

Tip: Find first the one particle states, and illustrate them using a schematic drawing. Express your results using  $N, L, m, \omega$  only.

### ===== [Exercise 3520]

#### Ideal Fermi gas in 2D space

Consider  $N$  mass  $m$  spin  $1/2$  Fermions, that are held in a two dimensional box that has an area  $A$ . Show that:

$$N(\beta, \mu) = A \frac{m}{\pi} T \ln \left( 1 + e^{\frac{\mu}{T}} \right)$$

Tip: Define  $X = e^{\beta(E-\mu)}$  and use the integral  $\int_1^\infty \frac{dx}{x(x+1)} = \ln \left( \frac{1+X_1}{X_1} \right)$ .

Write and explain what is the  $T = 0$  result.

Find the chemical potential  $\mu(T, N)$ .

Find the Fermi energy  $E_F \equiv \mu(T \rightarrow 0, N)$ .

Show that at low temperatures

$$\mu(T) \approx E_F - T e^{-\frac{E_F}{T}}$$

Show that at high temperatures the result is consistent with the Boltzmann approximation.

Find  $E(\beta, \mu)$  and  $P(\beta, \mu)$  at zero temperature.

Derive the following results:

$$E = A \frac{\pi}{m} \frac{1}{2} \left( \frac{N}{A} \right)^2, \quad P = \frac{\pi}{m} \frac{1}{2} \left( \frac{N}{A} \right)^2$$

Clarify why at zero temperature  $P \propto 1/A^2$ , while at high temperatures  $P \propto 1/A$ .

### ===== [Exercise 3530]

#### Ideal Fermi gas in 2D box

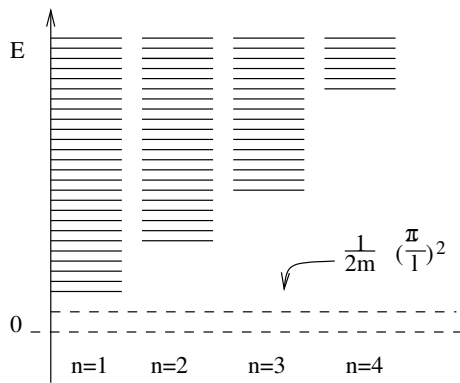
$N$  fermions with mass  $m$  and spin  $\frac{1}{2}$  are in a box, its dimensions are  $L \times L \times \gamma$ , ( $\gamma \ll L$ ). The system is kept in low temperature  $T$ . Find the pressure on the box walls and the heat capacity of the gas in conditions where it's possible to consider as 2 dimensional. The one particle states are  $|np_x p_y\rangle$

$$E_{np_x p_y} = \frac{1}{2m} \left( \frac{\pi}{\gamma} R \right)^2 + \frac{p_x^2}{2m} + \frac{p_y^2}{2m}$$

It's possible to consider the gas as a 2 dimensional as long as the occupation of the levels  $1 < n$  is negligible.

$$g(E) = 2 \cdot A \frac{m}{2\pi}, \quad \frac{1}{2m} \left( \frac{\pi}{\gamma} \right)^2 < E < 4 \cdot \frac{1}{2m} \left( \frac{\pi}{\gamma} \right)^2$$

It's convenient to take in to calculation the  $E = \frac{1}{2m} \left( \frac{\pi}{\gamma} \right)^2$  like an attribute level to the uniparticles state energy, but there's a need to be careful when you calculate the pressure on the upper and lower walls.



### ===== [Exercise 3540]

#### Ideal Fermi gas in semiconductor

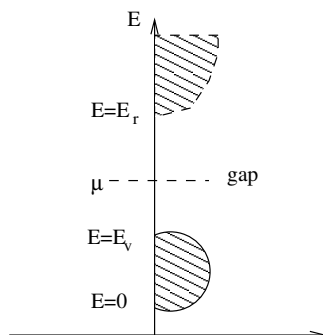
Consider a gas of electrons in a semiconductor, the temperature is  $T$ , and the chemical potential is  $\mu$ . The single particle density of states  $g(E) = g_v(E) + g_c(E)$  consists of valence and conduction bands, separated by a gap  $E_g = E_c - E_v$ . In the vicinity of the energy gap, one can use the following approximation:

$$g_c(E) \approx 2 \frac{V}{(2\pi)^2} \cdot (2m_c)^{\frac{3}{2}} (E - E_c)^{\frac{1}{2}}$$

$$g_v(E) \approx 2 \frac{V}{(2\pi)^2} \cdot (2m_v)^{\frac{3}{2}} (E_v - E)^{\frac{1}{2}}$$

The electron has Fermi occupation  $f(E - \mu)$ , optionally it is customary to define an occupations function  $\tilde{f}(E - \mu) = 1 - f$  for the holes.

- What are the occupation functions of the electrons in the conduction band, and of the holes in the valence band, in the Boltzmann approximation.
- What is the condition for the validity of this approximation? Assume that this condition is satisfied in the following items.
- Derive expressions for the number of electrons  $N_c(\beta, \mu)$  and for the number of holes  $N_v(\beta, \mu)$  in the conduction and valence band respectively. Explain how the product  $N_c N_v$  could be optionally deduced from the law of mass action.
- Consider a *closed* system, such that at  $T = 0$  the valence band is fully occupied, while the conduction band is empty. The temperature is raised to  $T$ . Find the chemical potential and evaluate  $N_c(T)$  and  $N_v(T)$ .



### ===== [Exercise 3550]

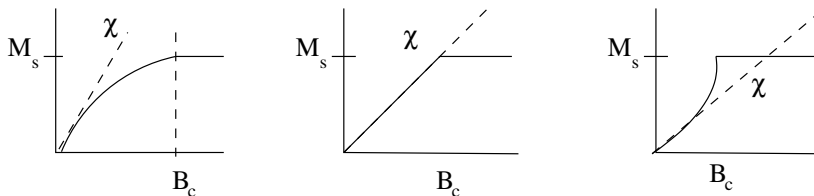
#### Fermions in magnetic field - Pauli

$N$  electrons with mass  $m$  and spin  $\frac{1}{2}$  are placed in a box at zero temperature. A magnetic field  $B$  is applied, such that the interaction is  $-\gamma B \sigma_z$  where  $\gamma$  is the gyromagnetic ratio. Consider the following cases:

- (a) one-dimensional box with length  $L$ .
- (b) two-dimensional box with area  $A$ .
- (c) three dimensional box with volume  $V$ .

Answer the following questions. Express your results using  $\gamma$ ,  $m$ ,  $N$ ,  $L$ ,  $A$ ,  $V$ .

- (1) What is the single particle density of states. Distinguish between a spin up and spin down particles.
- (2) Which is the graph that describes the magnetization  $M(B)$  of each case (a),(b),(c). Complete the missing details: what are  $M_s$ ,  $B_c$ ,  $\chi$ .



### ===== [Exercise 3555]

#### Fermions in magnetic field - Landau

Consider  $N$  spinless electrons that have mass  $m$  and charge  $e$  in a 2D box that has an area  $A$  at zero temperature. A perpendicular magnetic field  $B$  is applied. The purpose of this question is to find the magnetization of the system.

- (1) What are the threshold value  $B_n$  for which there are exactly  $n$  filled Landau levels.
- (2) Find the energy  $E(B)$  and the magnetization  $M(B)$  for strong field  $B > B_1$ . Give an optional semiclassical derivation to the result assuming that each electron is doing a cyclotron motion with minimal one-particle energy.
- (3) Find the energy  $E(B)$  and the magnetization  $M(B)$  for  $B_{n+1} < B < B_n$ . In particular find  $E(B_n)$  and explain why they are all equal to  $E(0)$ . Write what is the value of the magnetization at the endpoints of each interval.
- (4) Give a semiclassical derivation to the drops of  $M(B)$  at the threshold values  $B_n$ , using the Hall formula for the current along the Edge.

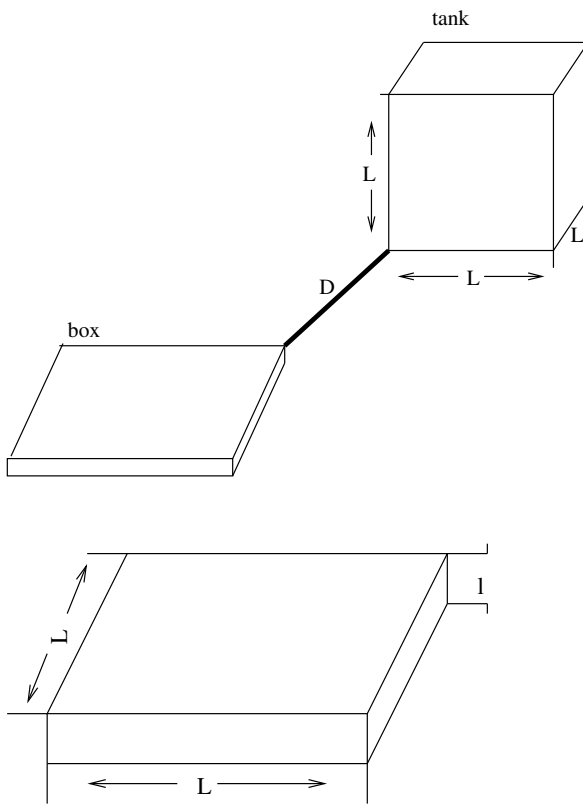
Keywords: Landau levels; Landau magnetism; The de Haas van Alphen (dHvA) oscillations; The quantum Hall effect.

### ===== [Exercise 3570]

#### Fermi gas in 2D+3D connected boxes with gravitation

Consider a mesoscopic box that has dimensions  $L \times L \times \ell$ , such that  $\ell \ll L$ . In the box there are  $N$  spin  $1/2$  electrons. The mass of an electron is  $m$ . In items (a-d) assume that the temperature is  $T = 0$ . In items (d-e) the box is attached to a tank that has dimensions  $L \times L \times L$ , that is placed at height  $D$  relative to the box, and you have to take into account the gravitational field  $g$ . Express your answers using  $m, L, \ell, g, T$ .

- Describe the single particle density of states. Specify the energy range over which it is the same as for a two dimensional box.
- Find the fermi energy  $E_F$  assuming that it is in the range defined above. What is the maximum number  $N_{max}$  of electrons that can be accommodated without violating the 2D description?
- Assuming  $N < N_{max}$  find the pressure  $P$  on the side walls of the box, and the force  $F$  on the horizontal walls.
- Assume  $N = N_{max}$ . What is the minimum height  $D_{min}$  to place the tank such that all of the electrons stay in the box?
- Assume  $N = N_{max}$  and  $D > D_{min}$ . The temperature of the system is raised a little bit. As a result some of the particles that were in the 2D box are transferred to the tank. Estimate their number  $N'$ . You are allowed to use any reasonable approximation.



### ===== [Exercise 3710]

#### Fermions in gravitation field of a star

Consider an artificial model of neutron star where the gas of  $N$  neutrons is held together by a gravitational potential  $U = -GMm/r$  generated by the solid core  $M$  of the star, where  $G$  is the gravitation constant, and  $m$  is mass of a neutron. The core has a radius  $r \sim 0$ .

- Assume the neutron gas as Fermi gas. Find the density  $n(r)$  for  $T = 0$ , and determine the Fermi energy  $\epsilon_f = -W(N)$ .



- (b) What is the radius  $R$  that is occupied by neutrons?
- (c) Find the flux of particles that escape the gravitational field assuming a low temperature  $T$ . Use the analogy to thermionic emission.
- (d) Write a differential equation for the number  $N(t)$  of remaining particles, assuming that the temperature is not changing.
- (e) The calculation of the flux in item (c) involves a Boltzmann approximation. Write the condition on  $T$  for the validity of this assumption. Additionally, point out what is the systematic error that follows from the constant  $T$  assumption.

Note: In item (a) your answer depends on a numerical constant  $C$  that you have to define in terms of an elementary definite integral.

### ===== [Exercise 3711]

#### Fermions in gravitation field of a star

Consider a neutron star as non-relativistic gas of non-interacting neutrons of mass  $m$  in a spherical symmetric equilibrium configuration. The neutrons are held together by a gravitational potential  $-mMG/r$  of a heavy object of mass  $M$  and radius  $r_0$  at the center of the star ( $G$  is the gravity constant and  $r$  is the distance from the center).

- (a) Assume that the neutrons are classical particles at temperature  $T$  and find their density  $n(r)$  at  $r > r_0$  for a given  $n(r_0)$ . Is the potential confining, i.e. is there a solution with  $n(r) \rightarrow 0$  at  $r \rightarrow \infty$ ?
- (b) Consider the neutrons as fermions at  $T = 0$  and find  $n(r)$ . Is the potential confining? [Hint: classify solutions according to their chemical potential  $\mu$ .]
- (c) Is the potential confining for fermions at  $T \neq 0$ ? when is the result (a) valid?

### ===== [Exercise 3745]

#### Fermions in a uniform gravitational field

Consider fermions of mass  $m$  and spin  $1/2$  in a gravitational field with constant acceleration  $g$  and at uniform temperature  $T$ . The density of the Fermions at zero height is  $n(0) = n_0$ . In item (3) assume that at zero height the fermions form a degenerate gas with Fermi energy  $\epsilon_F^0$  that is much larger compared with  $T$ .

1. Assume that the fermions behave as classical particles and find their density  $n(h)$  as function of the height.
2. Assume  $T = 0$ . Find the local Fermi momentum  $p_F(h)$  and the density  $n(h)$  as function of the height.
3. Assume low temperatures. Estimate the height  $h_c$  such that for  $h \gg h_c$  the fermions are non-degenerate.
4. In the latter case find  $n(h)$  for  $h \gg h_c$ , given as before  $n_0$  at zero height.

### ===== [Exercise 4000]

#### Chemical equilibrium

### ===== [Exercise 4001]

#### Two level system with $N$ particles

Consider  $N$  particles in a two level system. The one-particle energies of the two levels are  $E_1$  and  $E_2$ . Consider separately the two following cases: (i) The particles can be distinguished; (ii) The particles are identical Bosons. Find the expectation values  $n_1$  and  $n_2$  of the occupation numbers. Discuss the special limits  $N = 1$  and  $N$  infinity. Explain the connection with Fermi/Bose occupation statistics.

### ===== [Exercise 4012]

#### Classical gas in volume-surface phases equilibrium

An ideal gas composed of point particles with mass  $m$ , moves between parallel boards of a capacitor. The surface of each one of them is  $A$  and the distance between them is  $L$ , as described in the figure.

Force  $\vec{f}$  operates on the particles, in vertical direction to the boards, which pushes the particles to the lower board. particles can be adsorbed to the boards. the adsorbed particles move over them freely, and adsorbed potential  $-E$  operates on them (when  $E > 0$ ) in addition to force  $\vec{f}$ .

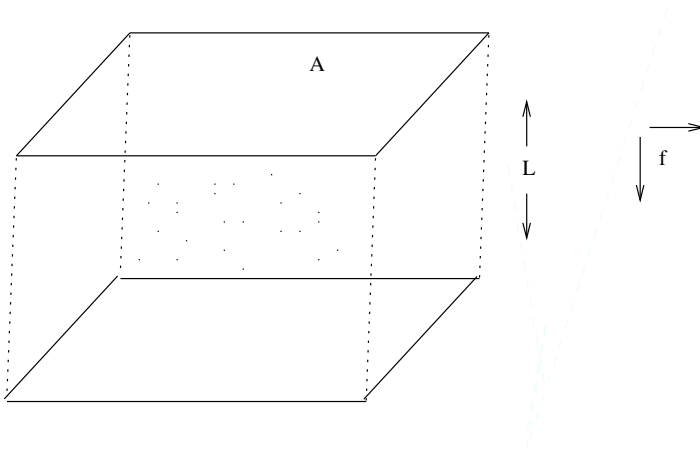
The system is in balance, in temperature  $T$ . Moreover, It's given that the average number of the particles that move between the boards and are not adsorbed over them is  $N$ , and their average density is  $\bar{n}$ .

Assume that the gas particles maintain Maxwell-Boltzman statistics and therefore it's possible to carry out the calculations in the classical statistical mechanics frame.

Express all of your answers with  $E, L, \bar{n}, T, f = |\vec{f}|, m$  and through physical and mathematical constants only.

- Calculate  $n(x, y, z)$ , The density of the particles per volume unit in some point between the boards. Define the coordinate system you use.
- Calculate the ratio  $\frac{\Phi_+}{\Phi_-}$  between the flow that hits the upper board and the flow that hits the lower board.
- Calculate  $\bar{n}_+$  and  $\bar{n}_-$  which are the densities of the particles adsorbed over the upper board and the lower board respectively. Moreover, calculate the ratio  $\frac{\bar{n}_+}{\bar{n}_-}$ .

Guideline: It's possible to make the calculation through the chemical potentials of the gas between the boards and over them.



### ===== [Exercise 4014]

#### Chemical equilibrium volume-surface

Consider a tank with water volume  $V$ , and over it oil is floating. The surface contact between the water and the oil is  $S$ . In the water and over the contact surface between the water and the oil, large molecules with mass  $m$  are moving. Assume that the potential energy of each molecule is  $E_1$  when it's in the water, and  $E_2$  when it's on the boundary between the water and the oil ( $E_2 > 0, E_1 > 0$ )  $E_2 - E_1 = E_0 > 0$ .

Assume that the large molecules are classical ideal gas (which means there's no interaction between the large molecules). What is the system's temperature  $T$ ?

- Calculate the chemical potential  $\mu_l$  of the large molecules in the water.
- Calculate the chemical potential  $\mu_s$  of the large molecules on the boundary between the water and the oil.
- What is the ratio between the large molecules density in the water, and their density on the boundary between the water and the oil in equilibrium?
- What is the total energy of the large molecules?

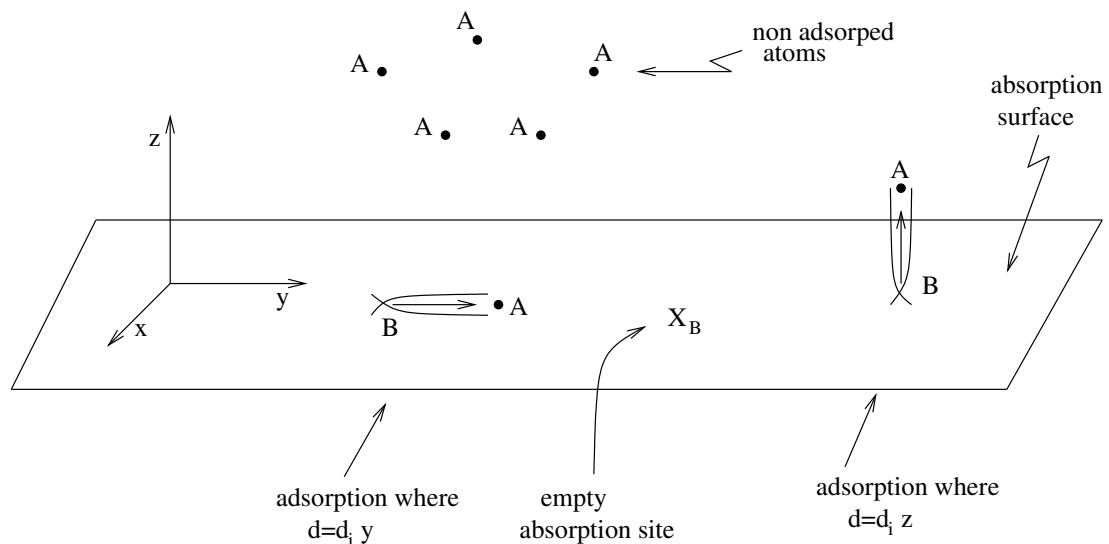
### ===== [Exercise 4015]

#### Adsorption of polar molecules to a surface

A large number  $n$  of identical mass  $m$  atoms are bounded within a surface that has  $M$  adsorption centers. Each adsorption center can connect one atom, such that a polar molecule  $AB$  is created. The dipole moment of each molecule is  $d$ , and it can be oriented either vertically (1 possible orientation) or horizontally (4 possible orientations). The binding energy is  $\epsilon_0$ . Additionally a vertical electric field  $\mathcal{E}$  is applied. The interaction energy between the field and the dipole is  $-\vec{\mathcal{E}} \cdot \vec{d}$ . The polarization of the system is defined via the expression for the work,  $dW = -Dd\mathcal{E}$ .

- Find the canonical partition function  $Z_n(\beta)$  of the system.
- Derive an expression for the chemical potential  $\mu(T; n)$ .
- Given  $\mu$ , deduce what is the coverage  $\langle n \rangle$ .
- Re-derive the expression for  $\langle n \rangle$  using the grand canonical partition function  $\mathcal{Z}(\beta, \mu)$ .
- Calculate the polarization  $D(\mathcal{E})$  of the system.

*Remarks:* In items (1-2) it is assumed the the system is closed with a given number  $n$  of adsorbed atoms. Hence it is treated within the framework of the canonical ensemble. In items (3-4) the system is in equilibrium with a gas of atoms: the chemical potential  $\mu$  is given, and the average  $\langle n \rangle$  should be calculated using the grand-canonical formalism. In item (5) it is requested to verify that the same result is obtained in the canonical and in the grand-canonical treatments.

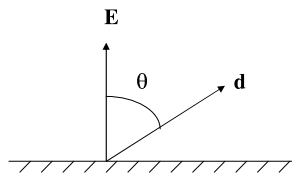


### [Exercise 4016]

#### Adsorption of polar molecules to a surface

Consider a 2D adsorbing surface in equilibrium with a 3D gas of atoms that have a temperature  $T$  and a chemical potential  $\mu$ . On the surface there are  $M$  sites. Each site can absorb at most one atom. At the adsorption site an atom forms an electric dipole  $d$  that can be oriented at any direction away from the surface (see figure). In the presence of a perpendicular electric field  $\mathcal{E}$  the dipole has energy is  $-\mathcal{E}d\cos(\theta)$ , where  $|\theta| < \pi/2$  is the angle between  $d$  and  $\mathcal{E}$ .

- Calculate the grand partition function  $\mathcal{Z}(\beta, \mu, \mathcal{E})$
- Derive the average number  $N$  of absorbed atoms.
- Use the formal approach to define the average polarization  $D$  as the expectation value of a system observable. Derive the state equation for  $D$ .
- What are the results in the limit  $\mathcal{E} \rightarrow 0$ , and in particular what is the ratio  $D/N$ . Explain how this result can be obtained without going through the formal derivation.



### [Exercise 4017]

#### Adsorption and fractal dimension

Surfactant Adsorption: A dilute solution of surfactants can be regarded as an ideal three dimensional gas. As surfactant molecules can reduce their energy by contact with air, a fraction of them migrate to the surface where they can be treated as a two dimensional ideal gas. Surfactants are similarly adsorbed by other porous media such as polymers and gels with an affinity for them.

- Consider an ideal gas of classical particles of mass  $m$  in  $d$  dimensions, moving in a uniform potential of strength  $\epsilon_d$ . Show that the chemical potential at a temperature  $T$  and particle density  $n_d$ , is given by

$$\mu_d = \epsilon_d + k_B T \ln[n_d \lambda^d] \quad \text{where} \quad \lambda = \frac{h}{\sqrt{2\pi m k_B T}}$$

- If a surfactant lowers its energy by  $\epsilon_0$  in moving from the solution to the surface, calculate the concentration of coating surfactants as a function of the solution concentration  $n$  (at  $d = 3$ ).
- Gels are formed by cross-linking linear polymers. It has been suggested that the porous gel should be regarded as fractal, and the surfactants adsorbed on its surface treated as a gas in  $d_f$  dimensional space, with a non-integer  $d_f$ . Can this assertion be tested by comparing the relative adsorption of surfactants to a gel, and to the individual polymers (assuming it is one dimensional) before cross-linking, as a function of temperature?

### [Exercise 4019]

#### Chemical equilibrium volume-polymer

Consider a polymer composed with  $M$  monomers. The polymer is in a gas with temperature  $\beta$  and chemical potential  $\mu$ . The gas molecules can absorb the polymer's monomers. The connection energy of the gas molecule to the monomer is  $\epsilon$ . The natural length of a monomer is  $a$ , when a gas molecule is absorbed to it, it's length is  $b$ .

(a) Calculate  $Z_N$  for the polymer, and from that, calculate  $Z$ .

(b) Calculate  $Z$  by the factorization.

Guideline: in paragraph b' write the polymer's states in this form  $|n_r (r = 1 \dots M) \rangle$  when  $n_r = 0, 1$ . Accordingly, if there is or there is no absorption. Write  $N_{(n_r)} E_{(r)}$ , and show the sum you need to calculate for  $Z$  is factorized.

(c) Calculate the average length  $L$  of the polymer.

Guideline: Express  $\hat{L}$  through  $\hat{N}$ . Calculate  $N \equiv \langle \hat{N} \rangle$  in two ways:

Way I - to derive from  $Z$  (page

Way II - Express  $\hat{N}$  through  $\hat{n}_r$  and then use the probability theory and the result for  $\langle \hat{n}_r \rangle$ .

### ===== [Exercise 4200]

#### The law of mass action

### ===== [Exercise 4211]

#### The law of mass action for $C \rightleftharpoons A + B$

Consider ideal gases of atoms  $A$ , atoms  $B$  and atoms  $C$  undergoing the reaction  $\nu C \leftrightarrow A + B$ , where  $\nu$  is an integer.  $n_A$ , and  $n_B$  and  $n_C$  are the respective densities of the atoms. The law of mass action states that

$$n_A^a n_B^b n_C^c = K(T)$$

(1) Determine what are the exponents  $a, b$  and  $c$ , and the equilibrium constant  $K(T)$ .

(2) Write explicit expression for  $K(T)$  for the reaction  $H_2 + D_2 \leftrightarrow 2HD$ , given the masses  $m_H, m_D$ , and the vibrational frequency  $\omega_0$  of the  $HD$  bond. Assume the temperature is high enough to allow classical approximation of the rotational motion.

(3) What is  $K(\infty)$ ?

### ===== [Exercise 4213]

#### Chemical equilibrium for $A \rightleftharpoons A + e$

$N_0$  atoms of type  $A$  are placed in an empty box of volume  $V$ , such that their initial density is  $n_0 = N_0/V$ . The ionization energy of the atoms is  $\varepsilon_0$ . The box is held in temperature  $T$ , and eventually a chemical equilibrium  $A \rightleftharpoons A^+ + e^-$  is reached. The fraction of ionized atoms is  $x = N^+/N_0$ . The masses of the particles are  $m_e$  for the electron, and  $m_{A^+} \approx m_A$  for the atoms and the ions.

(1) Define temperature  $T_0$  such that  $T \gg T_0$  is a sufficient condition for treating the gas of atoms in the Boltzmann approximation.

(2) Assuming the Boltzmann approximation for both the atoms and the electrons, write an equation for  $x$ . Write its *approximate* solution assuming  $x \ll 1$ . Write the condition for the validity of the latter assumption.

(3) Assuming that  $x \ll 1$ , write a condition on the density  $n_0$ , that above  $T_0$  it was legitimate to treat the electrons in the Boltzmann approximation. Note: the condition is a simple inequality and should be expressed using  $(m_e, m_A, \varepsilon_0)$ .

Assume that the condition in (3) breaks down. It follows that there is a regimes  $T_0 \ll T \ll T_1$  where the atoms can be treated in the Boltzmann approximation, while the electrons can be treated as a low temperature quantum gas.

(4) Write an equation for  $x$  assuming that the electrons can be treated approximately as a zero temperature Fermi

gas. Exotic functions should not appear. You are not expected to solve this transcendental equation.

(5) What would be the equation for  $x$  if the electrons were Bosons instead of Fermions.

*Note:* Express all the final answers using  $(m_e, m_A, n_0, \varepsilon_0, T)$ , and *elementary* functions. Exotic functions should not appear. It is allowed to use the notation  $\lambda_e(T) = (2\pi/m_e T)^{1/2}$ .

===== [Exercise 4215]

### Equilibrium of condensed Bosons and atoms B==2A

$N$  Boson molecules of type B are inserted into a box with volume  $V$ . The system temperature is  $T$ . Each molecule is composed from two atoms of type A. The mass of each atom is  $m$ , and the binding energy of the molecules is  $\varepsilon$ . Assume that there are molecules in condensation, and that the atoms can be treated within the framework of the Boltzmann's approximation.

1. With regard to the atoms - what is the condition for the Boltzmann approximation.
2. How many free atoms occupy the the box?
3. How many molecules occupy excited states?
4. What is the minimal  $N$  that is required to have condensation as assumed?
5. What is the pressure on the walls?
6. Who dominates the pressure - molecules or atoms?

===== [Exercise 4220]

### Chemical equilibrium: H<sub>2</sub>[3D]==2H[2D]

An  $H_2$  molecule (mass  $2m_H$ ) decomposes into  $H$  atoms when it is absorbed upon a certain metallic surface with an energy gain  $\varepsilon$  per  $H$  atom due to binding on the surface. This binding is not to particular sites on the surface: the  $H$  atoms are free to move on the surface. Consider the  $H_2$  as an ideal gas, and express the surface density of the  $H$  atoms as a function of the  $H_2$  pressure.

===== [Exercise 4441]

### Chemical equilibrium for gamma==e+e-

Consider the reaction

$$\gamma + \gamma \leftrightarrow e^+ + e^-$$

where the net charge of the system is fixed by the density difference  $n_0 = n_+ - n_-$ ;  $\gamma$  is a photon and  $e^\pm$  are the positron and electron, respectively.

- (a) Derive equations from which the densities  $n_+$  and  $n_-$  can be determined in terms of  $n_0$ , temperature  $T$ , and the mass  $m$  of either  $e^+$  or  $e^-$ .
- (b) Find the Fermi momentum  $p_F$  at  $T = 0$  for non-relativistic  $e^+$ ,  $e^-$  and the condition on  $n_0$  that allows a non-relativistic limit.
- (c) Solve (a) for  $p_F^2/2m \ll k_B T \ll mc^2$ . (Hint: Find first an expression for the product  $n_+ n_-$ ).

### ===== [Exercise 4442]

#### Chemical equilibrium for $\gamma \rightleftharpoons e^+ + e^-$ and $\gamma \rightleftharpoons \pi^+ + \pi^-$

The reaction  $\gamma + \gamma \rightarrow e^+ + e^-$  occurs inside a star, where  $\gamma$  is a photon and  $e$  are the positron and electron (mass  $m$ ). In general  $e$  are relativistic. Assume overall charge neutrality, and that the system is in equilibrium at temperature  $T$ . Find an expression for the densities of  $e$ . Also find these densities in the limit  $T \ll mc^2$ . Solve the same problem for the reaction  $\gamma + \gamma \rightarrow \pi^+ + \pi^-$  where  $\pi$  are bosons with mass  $M$ . Can these bosons become Bose-condensed if the temperature is sufficiently lowered? Explain the result physically.

### ===== [Exercise 4445]

#### Adsorption and decomposition

- Evaluate the chemical potential of a classical ideal gas in two dimensions in terms of the temperature and the density per unit area.
- An  $H_2$  molecule decomposes into  $H$  atoms when it is absorbed upon a certain metallic surface with an energy gain  $\epsilon$  per  $H$  atom due to binding on the surface. (This binding is not to a particular site on the surface, i.e. the  $H$  atoms are free to move parallel to the surface). Consider  $H_2$  as an ideal gas with mass  $2m_H$  and derive the density adsorbed per unit area as function of  $\epsilon$ , temperature and the  $H_2$  pressure.  
[Hint: Chemical equilibrium is obtained by minimizing the total free energy with respect to one of the densities.]

### ===== [Exercise 4446]

#### Relative abundance of isotopes

The partition functions of a diatomic molecules  $AB$  or  $A_2$  (within an ideal gas) has the form

$$f_{AB} = g_{AB}(T)(m_A m_B)^{3/2} \quad \text{or} \quad f_{A_2} = \frac{1}{2} g_{A_2}(T) m_A^3$$

where  $m_A$ ,  $m_B$  are atomic masses and  $B$  is an isotope of  $A$ ;  $g_{AB}$  and  $g_{A_2}$  are independent of the isotope masses.

- Explain the origin of the factor  $\frac{1}{2}$ .
- In the reaction  $H_2 + Cl_2 \rightleftharpoons 2HCl$  the  $Cl$  atom has two isotopes  $Cl^{35}$  and  $Cl^{37}$ . Write the relevant four reactions and their laws of mass action.
- Show that the relative abundance of  $Cl^{35}$  and  $Cl^{37}$  in  $Cl_2$  is the same as in  $HCl$ , i.e. the various densities  $n$  satisfy

$$\frac{2n_{Cl_2^{37}} + n_{Cl^{35}Cl^{37}}}{2n_{Cl_2^{35}} + n_{Cl^{35}Cl^{37}}} = \frac{n_{HCl^{37}}}{n_{HCl^{35}}}$$

### ===== [Exercise 4450]

#### Chemical equilibrium for $n \rightleftharpoons p + e^- + \bar{\nu}$

In a certain medium, there were at the beginning  $N$  neutrons per volume unit. Some of them decomposed according to

$$n \rightleftharpoons p + e^- + \bar{\nu}$$

All of the particles are fermions with spin  $\frac{1}{2}$ . Their masses are  $m_n, m_p, m_e$ , and  $m_\nu = 0$ . Assume temperature  $T$ . Denote by  $N'$  density of the neutrons in a thermal equilibrium. Write the equation for  $N'$  in four cases:

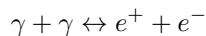
- (1) The particles are non-relativistic (except the neutrino)
- (2) The particles are hyper relativistic (negligible mass).
- (3) The temperature is zero.
- (4) The temperature is high (Boltzmann approximation).

Define the conditions for the assumptions to be valid. Write the equations using the data only. There is no need to solve the equations.

### ===== [Exercise 4546]

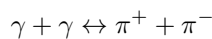
#### Chemical equilibrium for misc reactions

- (a) The following reaction occurs inside a star



where  $\gamma$  is a photon and  $e^\pm$  are the positron and electron, respectively. Assume overall charge neutrality and that the system is in equilibrium at temperature  $T$ . Find an expression for the densities of  $e^\pm$ . (In general  $e^\pm$  with mass  $m$  are relativistic). Find an explicit result for these densities in the limit  $k_B T \ll mc^2$ . (Hint: no conservation law for photons).

- (b) Repeat (a) for the reaction



where  $\pi^\pm$  are bosons with mass  $m_\pi$ . Can these bosons become Bose-condensed if the temperature is sufficiently lowered? What are the densities at  $T = 0$ ?

- (c) The photons in (b) are now eliminated from the system, but the density of the bosons is maintained. If these bosons are cooled, is there a temperature at which Bose condensation is possible? [Assume that the initial temperature satisfies  $k_B T \ll m_\pi c^2$ ]. What are the densities at  $T = 0$ ?

- (d) Consider initial equal densities  $\bar{n}$  for  $e^+$  and for  $e^-$ . The  $e^\pm$  are allowed to equilibrate with  $\pi^\pm$  via the reaction



Write equations that determine the chemical potentials at  $T \neq 0$ . Consider now  $T = 0$  and determine if Bose Einstein condensation is possible and if so what is the condition for that. Consider both  $m_e > m_\pi$  and  $m_e < m_\pi$  situations.

### ===== [Exercise 4549]

#### Consevation law for the number of nucleons

A collection of free nucleons is enclosed in a box of volume  $V$ . The energy of a single nucleon of momentum  $\mathbf{p}$  is  $\epsilon_{\mathbf{p}} = p^2/2m + mc^2$  where  $mc^2 = 1000 MeV$ .

- (a) Pretending that there is no conservation law for the number of nucleons, calculate the partition function at temperature  $T$ . (Nucleons are fermions).
- (b) Calculate the average energy density and average particle density.
- (c) In view of (a) and (b), discuss the necessity for a consevation law for the number of nucleons.



### ===== [Exercise 4550]

#### Equilibrium of Fermions and condensed Bosons $F \rightleftharpoons A+B$

$N$  fermions of type  $F$  that have spin  $1/2$  are placed in a box of volume  $V$ . Each fermion can decay into a boson of type  $B$  that have spin  $0$ , and a fermion of type  $A$  that has spin  $1/2$ . The reaction is  $F \rightarrow A + B$ , and it has an energy gain  $\epsilon_0$ . This means that  $A + B$  has a lower binding energy than  $F$ . The masses of the particles are  $m_F, m_A, m_B$  respectively.

- (1) Assuming ideal gases at temperature  $T$ , write the chemical equilibrium condition that determine the densities  $n_F, n_A, n_B$  at equilibrium.
- (2) Write the chemical equilibrium condition at  $T = 0$ . Describe the dependence of the densities on  $\epsilon_0$ . Find  $\epsilon_c$  such that for  $\epsilon_0 > \epsilon_c$  the number of  $F$  fermions vanishes.
- (3) Assume that the condensation of bosons  $B$  occurs at  $T_c$  such that  $T_c \ll p_F^2/(2m_A) < \epsilon_0$ , where  $p_F$  is the Fermi momentum of fermions  $A$ . Evaluate  $T_c$  and rewrite the condition on  $T_c$  in terms of the given parameters.

### ===== [Exercise 4551]

#### Velocity of sound of a degenerate Fermi gas

Determine the velocity of sound in a degenerate ( $T = 0$ ) Fermi gas in terms of the Fermi velocity.

### ===== [Exercise 4552]

#### Liquid He3, cooling by adiabatic pressurization

If liquid  $^3\text{He}$  is pressurized adiabatically, it becomes a solid and the temperature drops. This is a method of cooling by pressurization. Develop the theory of this process in the following steps:

- (a) Assume that the liquid state is an ideal Fermi liquid with a low temperature entropy  $S = \frac{1}{2}\pi^2 N k_B T/T_F$  where  $N$  is the number of particles and  $T_F \approx 5^\circ\text{K}$  is the Fermi temperature. Find the temperature-pressure relation in an adiabatic process for  $T \ll T_F$ .
- (b) At low temperatures the entropy of solid  $^3\text{He}$  comes almost entirely from the spins while below  $10^{-3}^\circ\text{K}$  the spins become antiferromagnetically ordered; assume that at  $T \gtrsim 10^{-3}^\circ\text{K}$  the spins are independent. Draw schematically the entropy of both solid and liquid  $^3\text{He}$  as function of temperature and draw the adiabatic trajectory for increasing pressure. Below which temperature  $T^*$  must the initial temperature be for the method to work?
- (c) Of what order is the liquid-solid transition? Evaluate the jump in the specific heat.
- (d) Use Clapeyron's relation to deduce the shape of the  $P(T)$  coexistence solid-liquid curve near  $T^*$ . Assume that the difference  $\Delta v$  of the specific volumes is temperature independent and that the solid is denser.
- (e) Consider an initial pressure that is below the  $P(T)$  coexistence line. Draw schematically the adiabatic trajectory on the  $P$ - $T$  plane, using the result (a). What is now the condition on the initial temperature for the cooling method to work, in terms of the initial  $(P, T)$  and the coexistence line  $P(T)$ ?

### ===== [Exercise 4553]

#### Beta decay $n \rightarrow p + e + \bar{\nu}$

The neutrino is a massless fermion with spin  $\frac{1}{2}$ .

- (a) Assuming neutrinos in thermal equilibrium, write the density in terms of the fugacity  $\zeta$  and the temperature  $T$  (take in account the quantum nature of the neutrinos). Evaluate the integral in the limit of high  $T$  and at  $T = 0$ . What is the condition for high  $T$ ?
- (b) Given an initial density  $n_0$  of neutrons. Some of these neutrons  $\beta$  decay to protons, electrons and antineutrinos via the reaction

$$n \rightarrow p + e^- + \bar{\nu}.$$

All the particles are fermions with spin  $\frac{1}{2}$ . The masses  $m_n, m_p, m_e$  are given and the antineutrino is massless. Assuming quantum ideal gases with  $p, n, e^-$  nonrelativistic, write the equilibrium equations for the proton density  $n_p$ . Consider only the following two cases (i) a high temperature limit, and (ii)  $T = 0$ . In both cases solve explicitly in the limit  $n_p \ll n_0$ .

### ===== [Exercise 4554]

#### Fermions in magnetic field, quantum phase transition

A  $d$  dimensional container ( $d = 1, 2, 3$ ) contains fermions of density  $n$ , temperature  $T = 0$ , mass  $m$  and spin  $\frac{1}{2}$ , having a magnetic moment  $\bar{m}$ . The container is placed in a magnetic field  $H/\bar{m}$  so that the fermion spectra is  $\epsilon_{\mathbf{p}} = \frac{\mathbf{p}^2}{2m} \pm H$  where  $\mathbf{p}$  is the momentum. (Note that orbital effects are neglected, possible e.g. at  $d=2$  with the field parallel to the layer).

- (a) Evaluate the chemical potential  $\mu(H)$ , for small  $H$ : Consider first an expansion to lowest order in  $H$  and then evaluate  $d\mu/dH$  to note the change at finite  $H$ .
- (b) Beyond which  $H_c$  does the consideration in (a) fail? Find  $\mu(H)$  at  $H > H_c$  and plot qualitatively  $\mu(H)/\mu_0$  as function of  $H/\mu_0$  (where  $\mu_0 = \mu(H = 0)$ ) for  $d = 1, 2, 3$ , indicating the values of  $\mu(H)/\mu_0$  at  $H_c$ .
- (c) Of what order is the phase transition at  $H_c$ , at either  $d = 1, 2, 3$ ? Does the phase transition survive at finite  $T$ ? (no need for finite  $T$  calculations – just note analytic properties of thermodynamic functions).
- (d) The container above, called A, with  $H \neq 0$  is now attached to an identical container B (same fermions at density  $n$ ,  $T = 0$ ), but with  $H = 0$ . In which direction will the fermions flow initially? Specify your answer for  $d = 1, 2, 3$  at relevant ranges of  $H$ .

### ===== [Exercise 5000]

#### Interacting systems, phase transition

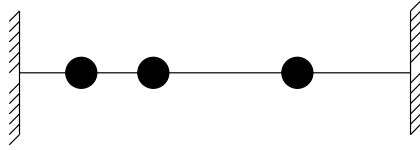
### ===== [Exercise 5010]

#### One dimensional hard sphere gas

$N$  spheres with diameter  $a$  are threaded over a wire of length  $L$ . Assume  $N \gg 1$  but  $Na \ll L$ . The system is in thermic equilibrium, temperature  $T$ . Find the force  $F$  that operates on the edges of the wire. Write the result in the shape  $F = NT/L_{eff}$ . Express  $L_{eff}$  using the data and explain it's physical meaning. Hints:

- (a) While calculating the distribution function, notice that if the beads permutation were permitted, it was causing  $Z \rightarrow N!Z$ .
- (b) Assume that a typical distance between two beads is much bigger than  $a$ .

- (c) To calculate a product  $A = \prod_{n=1}^N a_n$  look at the sum  $\ln A$ , and use reasonable approximations.



### ===== [Exercise 5012]

#### The Van der Waals equation

$N$  spheres with radius  $R$  are contained in box volume  $V$ . The temperature is  $T$ . Find the pressure using a mean-field one particle approximation. Extend the result if there is an extra potential  $u(r)$  between the particles. Show that you get the Van der Waals equation. Define the term "excluded volume" in this context, and identify the  $a$  and  $b$  coefficients.

### ===== [Exercise 5021]

#### virial/equi theorems

An equipartition type relation is obtained in the following way:

Consider  $N$  particles with coordinates  $\vec{q}_i$ , and conjugate momenta  $\vec{p}_i$  (with  $i = 1, \dots, N$ ), and subject to a Hamiltonian  $\mathcal{H}(\vec{p}_i, \vec{q}_i)$ .

- Write down the expression for the classical canonic partition function  $Z[\mathcal{H}]$  and show that it is invariant under the rescaling  $\vec{q}_i \rightarrow \lambda \vec{q}_i$  and  $\vec{p}_i \rightarrow \vec{p}_i/\lambda$  of a pair of conjugate variables, i.e.  $Z[\mathcal{H}_\lambda]$  is independent of  $\lambda$ , where  $\mathcal{H}_\lambda$  is the Hamiltonian obtained after the above rescaling.
- Now assume a Hamiltonian of the form  $\mathcal{H} = \sum_i \frac{(\vec{p}_i)^2}{2m} + V(\{\vec{q}_i\})$ . Use the result that  $Z[\mathcal{H}_\lambda]$  is independent of  $\lambda$  to prove the virial relation

$$\left\langle \frac{(\vec{p}_1)^2}{m} \right\rangle = \left\langle \frac{\partial V}{\partial \vec{q}_1} \cdot \vec{q}_1 \right\rangle$$

where the brackets denote thermal averages.

- Show that classical equipartition,  $\langle x_i \frac{\partial \mathcal{H}}{\partial x_j} \rangle = \delta_{ij} k_B T$ , also yields the result (b). Give an example of a quantum system where classical equipartition fails.
- Quantum mechanical version: Write down the expression for the quantum partition function. Show that it is also invariant under the rescalings  $\vec{q}_i \rightarrow \lambda \vec{q}_i$  and  $\vec{p}_i \rightarrow \vec{p}_i/\lambda$  where  $\vec{p}_i$  and  $\vec{q}_i$  are now quantum mechanical operators. (Hint: Use Schrödinger's equation and  $\vec{p}_i = -i\hbar \partial / \partial \vec{q}_i$ .) Show that the result in (b) is valid also in the quantum case.

### ===== [Exercise 5022]

#### Pressure via the virial theorem

A gas of  $N$  particles is confined in a box of volume  $V$  at temperature of  $T$ . The two-body interaction between the particles is  $u(r) \propto r^{-\gamma}$ . Write the virial theorem and deduce that the mean kinetic energy is

$$K = \frac{1}{\gamma + 2} (3PV + \gamma E)$$

where  $E = K + U$  is the total energy. What happens for  $\gamma = -2$  ?

### ===== [Exercise 5023]

#### Pressure of hard spheres

Consider a one-dimensional classical gas of  $N$  particles in a length  $L$  at temperature  $T$ . The particles have mass  $m$  and interact via a 2-body "hard sphere" interaction ( $x_i$  is the position of the  $i$ -th particle):

$$\begin{aligned} V(x_i - x_j) &= \infty & |x_i - x_j| < a \\ &= 0 & |x_i - x_j| > a \end{aligned}$$

- Evaluate the exact free energy  $F(T, L, N)$ .
- Find the equation of state and identify the first virial coefficient; compare with its direct definition.
- Show that the energy is  $E = Nk_B T/2$ . Why is there no effect of the interactions on  $E$  ?

### ===== [Exercise 5024]

#### Pressure of Lenard Jones gas

A gas of  $N$  particles is confined in a box of volume  $V$  at temperature of  $T$ . The two-body interaction between the particles is given by the Lenard Jones expression:

$$u(r) = \frac{a}{r^{12}} - \frac{b}{r^6}$$

Note that this interaction is characterized by a length scale  $r_0$  and an energy scale  $\epsilon_0$  that correspond to the position and the depth of the potential.

- Find an expression for the pressure via the Virial theorem, assuming that the moments  $\langle r^n \rangle_T$  are known.
- Using the Virial expansion, find an explicit expression for the pressure assuming low temperatures.
- Using the Virial expansion, find an explicit expression for the pressure assuming high temperatures.
- Comparing your answers to items (a) and (c) deduce explicit expressions for the  $n = -6$  and for the  $n = -12$  moments. Express your result in terms of  $(V, r_0, \epsilon_0, T)$ .

### ===== [Exercise 5030]

#### Virial coefficients - standard examples

Find the second virial coefficient for: Ideal Bose gas; Ideal Fermi gas; Classical hard sphere gas.

### ===== [Exercise 5040]

#### Virial coefficients - ideal Bose/Fermi

For a single quantum particle of mass  $m$ , spectra  $p^2/2m$  in a volume  $V$  the partition function is  $Z_1(m) = gV/\lambda^3$  with  $\lambda = h/\sqrt{2\pi m k_B T}$ . The particle has a spin degeneracy  $g$  ( $g = 2s + 1$  for spin  $s$ ).

- Calculate the partition function of two such particles if they are either bosons or fermions.

- (b) Calculate the corrections to the energy  $E$ , and the heat capacity  $C$ , due to Bose or Fermi statistics.
- (c) Find the second virial coefficient  $a_2$ , defined as  $PV = NkT[1 + a_2 n \lambda^3]$  to leading order in the small parameter  $n \lambda^3$ .

===== [Exercise 5400]

Ising type models, exact treatment

===== [Exercise 5420]

### Correlation function for Ising model

Consider the Ising model in one dimension with periodic boundary condition and with zero external field.

- (a) Consider an Ising spin  $\sigma_i$  ( $\sigma_i = \pm 1$ ) at site  $i$  and explain why do you expect  $\langle \sigma_i \rangle = 0$  at any temperature  $T \neq 0$ . Evaluate  $\langle \sigma_i \rangle$  by using the transfer matrix method. What is  $\langle \sigma_i \rangle$  at  $T = 0$ ?
- (b) Find the correlation function  $G(r) = \langle \sigma_1 \sigma_{r+1} \rangle$  and show that when  $N \rightarrow \infty$  ( $N$  is the number of spins)  $G(r)$  has the form  $G(r) \sim e^{-r/\xi}$ . At what temperature  $\xi$  diverges and what is its significance?

===== [Exercise 5440]

### One dimensional XY model

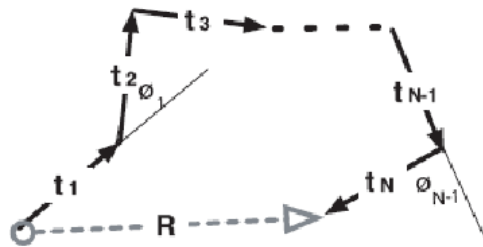
Polymer in two dimensions: Configurations of a polymer are described by a set of vectors  $\mathbf{t}_i$  of length  $a$  in two dimensions (for  $i = 1, \dots, N$ ), or alternatively by the angles  $\phi_i$  between successive vectors, as indicated in the figure below. The energy of a configuration  $\{\phi_i\}$  is

$$\mathcal{H} = -\kappa \sum_{i=1}^{N-1} \mathbf{t}_i \cdot \mathbf{t}_{i+1} = -\kappa a^2 \sum_{i=1}^{N-1} \cos \phi_i$$

- (a) Show that the correlations  $\langle \mathbf{t}_n \cdot \mathbf{t}_m \rangle$  decay exponentially with distance and obtain an expression for the "persistence length"  $a\xi$ ; you can leave the answer in terms of simple integrals.

Hint: Show  $\mathbf{t}_n \cdot \mathbf{t}_m = a^2 \text{Re} \{ e^{i \sum_{j=n}^{m-1} \phi_j} \}$ .

- (b) The end-to-end distance  $\mathbf{R}$  is defined as illustrated in the figure. Calculate  $\langle R^2 \rangle$  in the limit  $N \gg 1$ .



===== [Exercise 5641]

### Ising with long range interaction

Consider a cluster of  $N$  spins  $s_i = \pm 1$ . The interaction between *any* two spins is  $-\epsilon s_i s_j$ , with  $\epsilon > 0$ . The interaction of each spin with the external magnetic field is  $-H s_i$ . The total magnetization is defined as  $\mathbf{m} = \sum s_i$ . The inverse temperature is  $\beta$ .

- Show that the partition function can be written as  $Z(\beta, H) = \sum_{\mathbf{m}} g(\mathbf{m}) \exp(\frac{1}{2} B m^2 + h m)$ . Express  $g(\mathbf{m})$  and  $B$  and  $h$  using  $(N, \epsilon, \beta, H)$ .
- Assume that  $B = b/N$ , and define the magnetization as  $M = \mathbf{m}/N$ . Write the partition function as  $Z(b, h) = \sum_M \exp(-N \mathcal{A}(M))$ . Write the expressions for  $\mathcal{A}(M)$  and for its derivatives  $\mathcal{A}'(M)$  and  $\mathcal{A}''(M)$ .
- Determine the critical temperature  $T_c$ , and write an equation for the mean field value of  $M$ . Make a qualitative plot of  $\mathcal{A}(M)$  below and above the critical temperature.
- Write an approximation for  $\mathcal{A}(M)$  up to order  $M^4$ . On the basis of this expression determine the temperature range where mean field theory cannot be trusted. Hint: you have to estimate the variance  $\langle M^2 \rangle$  in the Gaussian approximation. What happens with this condition in the thermodynamic limit ( $N \rightarrow \infty$ )?
- Find an expression for the heat capacity in the mean field and in the Gaussian approximations.

### ===== [Exercise 5645]

#### Potts model in one dimension

A set of  $N$  atoms is arranged on a one-dimensional chain. Each atom has  $p$  possible *orientations*, labelled by  $\sigma = 1, 2, \dots, p$ . Two neighboring atoms  $\sigma_i$  and  $\sigma_j$  have a negative interaction energy  $-\epsilon$  if they are in the same orientation, and zero otherwise. It is useful to define bond variables  $s_i = \sigma_{i+1} - \sigma_i \pmod{p}$ .

- The partition function  $Z_{\text{chain}}(\beta)$  of an open chain can be written as  $Z = A q^{N-1}$ . Write what are  $A$  and  $q$ . Tip: the partition sum factorizes in the "bond" representation.
- The partition function  $Z_{\text{ring}}(\beta)$  of a closed chain, with periodic boundary conditions, can be written as  $Z = \text{trace}(T^N)$ . Write what is the matrix  $T$  for  $p = 4$ .
- Find what are the eigenvalues of the transfer matrix  $T$  for general  $p$ , and deduce an explicit expression for  $Z_{\text{ring}}(\beta)$ . Tip: The  $T$  matrix is diagonal in the "momentum" representation.
- Find the energy per atom at the  $N \rightarrow \infty$  limit. Write the result as  $E(T)/N = \epsilon f(\epsilon - \mu)$ . Provide expressions for  $\mu$  and for  $f()$  using  $p$  and the temperature  $T$ .

### ===== [Exercise 5651]

#### Ising spins with interaction that is mediated by atoms

Consider a one dimensional Ising model of spins  $\sigma_i = \pm 1$  labeled  $i = 1, 2, 3, \dots, M$ , with periodic boundary condition. Between each two spins there is a site  $n_i = 0, 1$  that can be occupied by an atom. If the atom is present the ferromagnetic coupling is decreased from  $J$  to  $(1 - \lambda)J$ .

- Evaluate the partition sum assuming that there are  $N$  atoms in the  $M$  sites. Allow all configurations of spins and of atoms. Calculate the free energy  $F$ .
- If the atoms are stationary impurities one needs to evaluate the free energy  $F$  for some random configuration of the atoms. What is the entropy difference between the results?

### ===== [Exercise 5660]

### Ising spins mediated by adsorption sites

Consider a ring along which  $M$  absorption sites are arranged. The number of particles that can be absorbed at site  $i$  is  $n_i = 0, 1$ . Between every two absorption sites a spin  $\sigma_i = \pm 1$  is located. The ring is surrounded by gas in temperature  $T$  and chemical potential  $\mu$ . The absorption energy is  $\epsilon > 0$  if the two adjacent spins are in the same direction, and  $-\epsilon$  otherwise.

1. Write an expression for the energy  $E[\sigma_i, n_i]$  of a given configuration.
2. Calculate the partition function  $\mathcal{Z}(\beta, \mu)$  using the transfer matrix method. Write what is  $T_{\sigma_i, \sigma_{i+1}}$  in this problem.
3. Find the Helmholtz function  $F(T, \mu)$  assuming  $M \gg 1$ .
4. Write an expression for the average number of adsorbed particles  $N = \sum_i \langle n_i \rangle$  as a function of  $(\beta, \mu)$ .
5. Write an expression for the correlation length  $\xi$  that characterizes arrangement of the spins in the system.

===== [Exercise 5700]

### Mean field theory

===== [Exercise 5713]

#### Mean field approximation for a classical Heisenberg model

Apply the mean field approximation to the classical spin vector model

$$\mathcal{H} = -\epsilon \sum_{\langle i, j \rangle} \mathbf{s}_i \cdot \mathbf{s}_j - \mathbf{h} \cdot \sum_i \mathbf{s}_i$$

where  $\mathbf{s}_i$  is a unit vector and  $i, j$  are neighboring sites on a lattice with coordination number  $c$ . The lattice has  $N$  sites and each site has  $c$  neighbors.

- (a) Assume that  $\mathbf{h} = (0, 0, h)$ , define a mean field  $\mathbf{h}_{eff}$ , and evaluate the partition function  $Z$  in terms of  $\mathbf{h}_{eff}$ .
- (b) Define  $\theta_i$  as the inclination angle of  $\mathbf{s}_i$  with respect to  $\mathbf{h}$ . Assume that at equilibrium  $\mathbf{s}_i = (0, 0, M)$ , where  $M = \langle \cos \theta \rangle$ . Find the equation for  $M$ , and find the transition temperature  $T_c$ .
- (c) Write an expression for the mean field energy of the system assuming that  $M(T)$  is known.
- (d) Identify exponents  $\gamma$  and  $\beta$  that describe the susceptibility  $\chi \sim (T - T_c)^{-\gamma}$  above  $T_c$ , and the magnetization  $M \sim (T_c - T)^\beta$  below  $T_c$ .
- (e) Find the jump in the heat capacity  $C_V$  at  $T_c$ .

===== [Exercise 5716]

#### Ferromagnetism for cubic crystal

A cubic crystal which exhibits ferromagnetism at low temperature, can be described near the critical temperature  $T_c$  by an expansion of a Gibbs free energy

$$G(\mathbf{H}, T) = G_0 + \frac{1}{2} r \mathbf{M}^2 + u \mathbf{M}^4 + v \sum_{i=1}^3 M_i^4 - \mathbf{H} \cdot \mathbf{M}$$

where  $\mathbf{H} = (H_1, H_2, H_3)$  is the external field and  $\mathbf{M} = (M_1, M_2, M_3)$  is the total magnetization;  $r = a(T - T_c)$  and  $G_0, a, u$  and  $v$  are independent of  $\mathbf{H}$  and  $T, a > 0, u > 0$ . The constant  $v$  is called the cubic anisotropy and can be either positive or negative.

- (a) At  $\mathbf{H} = 0$ , find the possible solutions of  $\mathbf{M}$  which minimize  $G$  and the corresponding values of  $G(0, T)$  (these solutions are characterized by the magnitude and direction of  $\mathbf{M}$ ). Show that the region of stability of  $G$  is  $u + v > 0$  and determine the stable equilibrium phases when  $T < T_c$  for the cases (i)  $v > 0$ , (ii)  $-u < v < 0$ .
- (b) Show that there is a second order phase transition at  $T = T_c$ , and determine the critical indices  $\alpha, \beta$  and  $\gamma$  for this transition, i.e.  $C_{V, H=0} \sim |T - T_c|^{-\alpha}$  for both  $T > T_c$  and  $T < T_c$ ,  $|\mathbf{M}|_{H=0} \sim (T_c - T)^\beta$  for  $T < T_c$  and  $\chi_{ij} = \partial M_i / \partial H_j \sim \delta_{ij} |T - T_c|^{-\gamma}$  for  $T > T_c$ .

### ===== [Exercise 5721]

#### Mean field for antiferromagnetism

Consider Ising model on a 2D lattice with antiferromagnetic interaction ( $\epsilon = -\epsilon_0$ ). You can regard the lattice as composed of two sublattices A and B, such that  $M = \frac{1}{2}(M_A + M_B)$  is the averaged magnetization per spin, and  $M_s = \frac{1}{2}(M_A - M_B)$  is the staggered magnetization

- (a) Explain the claim: for zero field ( $h = 0$ ), Ising antiferromagnet is the *same* as Ising ferromagnet, where  $M_s$  is the order parameter. Write the expression for  $M_s(T)$  for  $T \sim T_c$ , based on the familiar solution of the ferromagnetic case.
- (b) Given  $h$  and  $\epsilon_0$ , find the coupled mean-field equations for  $M_A$  and  $M_B$ .
- (c) Find the critical temperature  $T_c$  for  $h = 0$ , and also for small  $h$ . Hints: for  $h = 0$  use the same procedure of expanding  $\text{arctanh}(x)$  as in the ferromagnetic case; for small  $h$  you may use the most extreme simplification that does not give a trivial solution.
- (d) Find the critical magnetic field  $h_c$  above which the system no longer acts as an antiferromagnet at zero temperature.
- (e) Find an expression for the susceptibility  $\chi(T)$ , expressed as a function of the staggered magnetization  $M_s(T)$ .
- (f) In the region of  $T \sim T_c$  give a linear approximation for  $1/\chi$  as a function of the temperature  $T$ .

### ===== [Exercise 5732]

#### Mean field for ferroelectricity

Consider electric dipoles  $\mathbf{p}$  that are situated on sites of a simple cubic lattice, which point along the crystal axes  $\pm\langle 100 \rangle$ . The interaction between dipoles is

$$U = \frac{\mathbf{p}_1 \cdot \mathbf{p}_2 - 3(\mathbf{p}_1 \cdot \mathbf{r})(\mathbf{p}_2 \cdot \mathbf{r})/r^2}{4\pi r^3}$$

where  $\mathbf{r}$  is the distance between the dipoles, and  $r = |\mathbf{r}|$ .

- (a) Assume nearest neighbour interactions and find the ground state configuration. Consider either ferroelectric (parallel dipoles) or anti-ferroelectric alignment (anti-parallel) between neighbours in various directions.
- (b) Develop a mean field theory for the ordering in (a) for the average polarization  $P$  at temperature  $T$ . Write the mean field equation for  $P(T)$ , and find the critical temperature  $T_c$ .
- (c) Within the mean field approximation find the susceptibility  $\chi = (\partial P / \partial E)_{E=0}$  for  $T > T_c$  with respect to the electric field  $E||\langle 100 \rangle$ .



### ===== [Exercise 5741]

#### Correlation function for ferromagnet - mean field

Consider a ferromagnet with magnetic moments  $m(\mathbf{r})$  on a simple cubic lattice interacting with their nearest neighbors. [The symmetry is an Ising type, i.e.  $m(\mathbf{r})$  is the moment's amplitude in a preferred direction]. The ferromagnetic coupling is  $J$  and the lattice constant is  $a$ . Extend the mean field theory to the situation that the magnetization is not uniform but is slowly varying:

- (a) Find the mean field equation in terms of  $m(\mathbf{r})$ , its gradients (to lowest order) and an external magnetic  $H(\mathbf{r})$ , which in general can be a function of  $\mathbf{r}$ .
- (b) Consider  $T > T_c$  where  $T_c$  is the critical temperature so that only lowest order in  $m(\mathbf{r})$  is needed. For a small  $H(\mathbf{r})$  find the response  $m(\mathbf{r})$  and evaluate it explicitly in two limits: (i) uniform  $H$ , i.e. find the susceptibility, and (ii)  $H(\mathbf{r}) \sim \delta^3(\mathbf{r})$ . Explain why in case (ii) the response is the correlation function and identify the correlation length.

### ===== [Exercise 5800]

#### Phase transions, misc problems

### ===== [Exercise 5811]

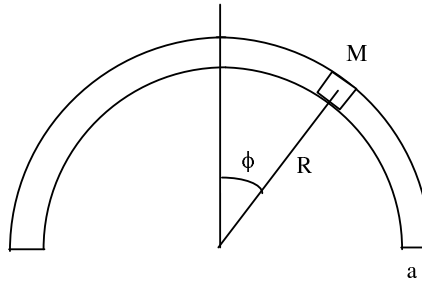
#### Mechanical model for symmetry breaking

An airtight piston of mass  $M$  is free to move inside a cylindrical tube of cross sectional area  $a$ . The tube is bent into a semicircular shape of radius  $R$ . On each side of the piston there is an ideal gas of  $N$  atoms at a temperature  $T$ . The angular position of the piston is  $\varphi$  (see figure). The gravitation field of Earth exerts a force  $Mg$  on the piston, while its effect on the gas particles can be neglected.

The partition function of the system can be written as  $d\varphi$  integral over  $\exp[-A(\varphi)]$ . The variable  $\varphi$  is regarded as the "order parameter" of the system. A small difference  $\Delta N$  in the occupation of the two sides is regarded as the conjugate field. The susceptibility is defined via the relation  $\langle \varphi \rangle \approx \chi \Delta N$ .

- (1) Write an explicit expression for  $A(\varphi)$ .
- (2) Find the coefficients in the expansion  $A(\varphi) = (a/2)\varphi^2 + (u/4)\varphi^4 - h\varphi$ .
- (3) Deduce what is the critical temperature  $T_c$ .
- (4) Using Gaussian approximation find what is  $\chi$  for  $T > T_c$ .
- (5) Using Gaussian approximation find what is  $\chi$  for  $T < T_c$ .
- (6) Sketch a plot of  $\chi$  versus  $T$  indicating by dashed lines the Gaussian approximations and by solid line the expected exact result. Write what is the range  $\Delta T$  around  $T_c$  where the Gaussian approximation fails.
- (7) What is the way to take the "thermodynamic limit" such as to have a phase transition at finite temperature?
- (8) In reality, as the temperature is lowered, droplets condense on the walls of the left (larger) chamber. What do you expect to find in the right chamber (gas? liquid? both?).

**Guidelines:** In items (4) and (5) simplify the result assuming  $T \sim T_c$  and express it in terms of  $T_c$  and  $T - T_c$ . The final answer should include one term only. Care about numerical prefactors - their correctness indicates that the algebra is done properly. In item (7) you are requested to identify the parameter that should be taken to infinity in order to get a "phase transition". Please specify what are the other parameters that should be kept constant while taking this limit.



### ===== [Exercise 5821]

#### Lattice gas

Lattice gas model: Consider  $N$  classical particles of mass  $m$  where each particle is located on a unit cell of a simple cubic lattice with a lattice constant  $a$ . Each unit cell can contain either 0 or 1 particles, providing an "excluded volume" type interaction. The number of unit cells is  $M$ , i.e. the volume is  $V = Ma^3$ . Therefore  $0 < N < M$  and the density is  $0 < n < 1/a^3$ . There is no constraint on the momentum of each particle.

- Evaluate the grand partition function and the density  $n(\mu, T)$  where  $\mu$  is the chemical potential and  $T$  is the temperature.
- Find the pressure  $P$  in terms of  $T$  and  $n$ . Identify the limit  $n \rightarrow 0$  and explain what happens in the limit  $n \rightarrow 1/a^3$ .
- This model does not show a first order transition as in a full lattice gas model. What ingredient is missing here?

### ===== [Exercise 5825]

#### Ising model 1D, domain walls

Consider the one dimensional Ising model with the Hamiltonian  $\mathcal{H} = -\sum_{n,n'} J(n-n')\sigma(n)\sigma(n')$  with  $\sigma(n) = \pm 1$  at each site  $n$ , and long range interaction  $J(n) = b/n^\gamma$  with  $b > 0$ . Find the energy of a domain wall at  $n = 0$ , i.e. all the  $n < 0$  spins are "down" and the others are "up". Show that the standard argument for the absence of spontaneous magnetization at finite temperatures fails if  $\gamma < 2$ .

### ===== [Exercise 5831]

#### Scaling form for the free energy

Given a free energy with the homogenous form

$$F = t^{2-\alpha} f(t/h^{1/\phi})$$

where  $h$  is the magnetic field and  $t = (T - T_c)/T_c$ .

- Show that  $\alpha$  is the conventional critical exponent of the specific heat.
- Express the conventional  $\beta$ ,  $\delta$  exponents in terms of  $\alpha$ ,  $\phi$  and show that  $2 - \alpha = \beta(\delta + 1)$ .

### ===== [Exercise 5841]

### Disorder averaging

Consider a system with random impurities. An experiment measures one realization of the impurity distribution and many experiments yield an average denoted by  $\langle \dots \rangle$ . Consider the free energy as being a sum over  $N$  independent subsystems, i.e. parts of the original system, with average value  $F = (1/N) \sum_{i=1}^N F_i$ ; the subsystems are identical in average, i.e.  $\langle F_i \rangle = \langle F \rangle$ .

- (a) The subsystems are independent, i.e.  $\langle F_i F_j \rangle = \langle F_i \rangle \langle F_j \rangle$  for  $i \neq j$ , although they may interact through their surface. Explain this.
- (b) Show that  $\langle (F - \langle F \rangle)^2 \rangle \sim 1/N$  so that even if the variance  $\langle (F_i - \langle F \rangle)^2 \rangle$  may not be small any measurement of  $F$  is typically near its average.
- (c) Would the conclusion (b) apply to the average of the partition function  $Z$ , i.e. replacing  $F_i$  by  $Z_i$ ?

### ===== [Exercise 5955]

#### Change of boiling point with altitude

Consider an atmosphere as an ideal gas whose average mass is 30 gr/mole, with uniform temperature  $T_A = 27^\circ C$ . The atmospheric pressure at sea level ( $h = 0$ ) equals  $P_0$ .

We take liquid whose latent heat is  $Q = 1000 \text{ cal/mole}$ , and we find that its boiling point is  $105^\circ C$  at sea level, and  $95^\circ C$  at the top of a mountain. Assume that the gas phase of this liquid is an ideal gas with density much lower than that of the liquid.

- (1) Calculate the atmospheric pressure  $P_A$  as a function of height  $h$ .
- (2) Calculate the liquid vapor pressure as a function of its temperature.
- (3) From above deduce what is the height of the mountain.

### ===== [Exercise 5963]

#### Stoner ferromagnetism

Consider Fermi gas of  $N$  spin 1/2 electrons, at temperature  $T = 0$ . Define  $N_+$  and  $N_-$  as the number of "up" and "down" electrons respectively, such that  $N = N_+ + N_-$ . Due to the antisymmetry of the total wave function the energy of the system is  $U = \alpha N_+ N_- / V$ , where  $V$  is the volume. Note that this interaction favors parallel spin states. Define the magnetization as  $M = (N_+ - N_-) / V$ .

- (a) Write the total energy  $E(M)$ , including both the kinetic energy and the interaction, and expand up to 4th order in  $M$ .
- (b) Find the critical value  $\alpha_c$ , such that for  $\alpha > \alpha_c$  the electron gas can lower its total energy by spontaneously developing magnetization. This is known as the Stoner instability.
- (c) Explain the instability qualitatively, and sketch the behavior of the spontaneous magnetization versus  $\alpha$ .
- (d) Repeat (a) at finite but low temperatures  $T$ , and find  $\alpha_c(T)$  to second order in  $T$ .

**Guidance:** In the last item explain why the energy  $E(M)$  should be replaced by the  $M$ -constrained "free energy"  $F(M)$ . Use known results [Patria] for the free energy of electrons at finite temperature.

### ===== [Exercise 5969]

## 2D Coulomb gas

$N$  ions of positive charge  $q$  and  $N$  ions of negative charge  $-q$  are constrained to move in a two dimensional square of side  $L$  and area  $A = L^2$ . The interaction energy of charge  $q_i$  at position  $r_i$  with another charge  $q_j$  at position  $r_j$  is  $-q_i q_j \ln |r_i - r_j|$ , where  $q_i, q_j = \pm q$ . The mass of the ions is  $m$ .

- By rescaling space variables to  $r_i := r_i/L$ , the partition function can be written as  $Z(L) = CL^\alpha$ , where  $C$  does not depend on  $L$ . Find  $\alpha$ . Hint:  $\sum_{ij} q_i q_j$  has a very simple dependence on  $N$ .
- Calculate the pressure, and show that for  $T < T_c$  the system is unstable. Determine what is  $T_c$ . Comment on the reason for this instability.
- Determine what is  $C$  if the interaction can be neglected. Handle the Gibbs factor correctly.
- Find the chemical potential  $\mu(T, N, L)$ , and solve for  $N(\mu, T, L)$ . In particular write what is  $N$  for a fixed  $\mu$  in the limit  $L \rightarrow \infty$  for both  $T > T_c$  and  $T < T_c$ .

## ===== [Exercise 5980]

### BEC regarded as a phase transition

Consider  $N$  bosons that each have mass  $M$  in a box of volume  $V$ . The overall density of the particles is  $\rho = N/V$ . The temperature is  $T$ . Denote by  $m$  the number of particles that occupy the ground state orbital of the box. The canonical partition function of the system can be written as

$$Z = \sum_{m=0}^N Z_{N-m} = \sum_{m=0}^N e^{-\tilde{A}(m)} = \int d\varphi e^{-N A(\varphi) + \text{const}}$$

In this question you are requested to regard the Bose-Einstein condensation as phase transition that can be handled within the framework of the canonical formalism where  $m$  is the order parameter. Whenever approximations are required assume that  $1 \ll m \ll N$  such that  $\varphi = (m/N)$  can be treated as a continuous variable. In the first part of the question assume that the gas is ideal, and that  $Z_{N-m}$  can be calculated using the Gibbs prescription. In item 5 you are requested to take into account the interactions between the particles. Due to the interactions the dispersion relation in the presence of  $m$  condensed bosons is modified as follows:

$$E_k = \sqrt{\left(\epsilon_k + 2g \frac{m}{V}\right) \epsilon_k}$$

where  $\epsilon_k$  are the one-particle energies in the absence of interaction, and  $g$  is the interaction strength. For the purpose of evaluating  $Z_{N-m}$  for large  $m$  assume that the above dispersion relation can be approximated by a linear function  $E_k \propto k$

- Write an explicit expression for the probability  $p_m$  of finding  $m$  particles in the ground state orbital. Calculation of the overall normalization factor is not required.
- Find the most probable value  $\bar{m}$ . Determine what is the condensation temperature  $T_c$  below which the result is non-zero.
- Assuming  $T < T_c$  write a Gaussian approximation for  $p_m$
- Using the Gaussian approximation determine the dispersion  $\delta m$
- Correct your answer for  $p_m$  in the large  $m$  range where the interactions dominate.
- On the basis of your answer to item 3, write an expression for  $A(\varphi; f)$  that involves a single parameter  $f$  whose definition should be provided using  $\rho, M, T$ .
- On the basis of your answer to item 5, write an expression for  $A(\varphi; a)$  that involves a single parameter  $a$  whose definition should be provided using  $\rho, M, T$  and  $g$ .

## ===== [Exercise 6000]

### Kinetics

## ===== [Exercise 6010]

### Effusion from a box with Bose gas and magnetic field

Bosons that have mass  $m$  and spin 1 with gyromagnetic ratio  $\gamma$  are placed in a box. The temperature  $T$  is below the condensation temperature. A strong magnetic field  $B$  is applied in the  $z$  direction. A hole that has small area  $\delta A$  is drilled in the box so the particles can flow out. The flux is separated into 3 beams using a Stern-Gerlach apparatus. Each beam is directed into a different container.

- Write the single particle Hamiltonian.
- Find the velocity distribution  $F_{S_z}(v)$  for  $S_z = -1, 0, 1$ .
- Define what does it mean a strong magnetic field, and explain why and how it helps for the solution of the next item.
- Find how many particles are accumulated in each container after time  $t$ .
- Find what would be the velocity distribution for horizontal filtering  $S_x = -1, 0, 1$  of the beam.

Express your answer using  $m, \gamma, B, \delta A, T, t$ . In the last item assume that  $F_{S_z}(v)$  is known, irrespective of whether the second item has been solved.

$$\int_0^\infty x^3 e^{-x^2} dx = \frac{1}{2}, \int_0^\infty \frac{x^3}{e^{x^2} - 1} dx = \frac{\pi^2}{12}$$

## ===== [Exercise 6020]

### A divided box with a hole in one side

A cylinder of length  $L$  and cross section  $A$  is divided into two compartments by a piston. The piston has mass  $M$  and it is free to move without friction. Its distance from the left basis of the cylinder is denoted by  $x$ . In the left side of the piston there is an ideal Bose gas of  $N_a$  particles with mass  $m_a$ . In the right side of the piston there is an ideal Bose gas of  $N_b$  particles with mass  $m_b$ . The temperature of the system is  $T$ . Assume that the left gas can be treated within the framework of the Boltzmann approximation. Assume that the right gas is in condensation. In items (3-5) consider separately two cases:

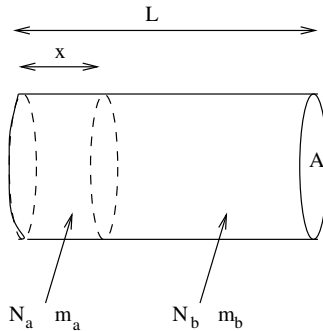
- A small hole is drilled in the left wall of the box.
- A small hole is drilled in the right wall of the box.

The area of the hole is  $\delta A$ .

- Find the equilibrium position of the piston.
- What is the frequency of small oscillations of the piston.
- What is the velocity distribution  $N(v)$  of the emitted particles?
- What is the flux (particles per unit time) of the emitted particles?
- Is the piston going to move? If yes write an expression for its velocity.

In item (3) use normalization that makes sense for the calculation in item (4). In item (5) assume that the process is quasi-static, such that at any moment the system is at equilibrium. Express your answers using  $L, A, \delta A, N_a, N_b, m_a, m_b, T, M$ .

$$\int_0^\infty \frac{x dx}{e^x - 1} = \frac{\pi^2}{6}$$

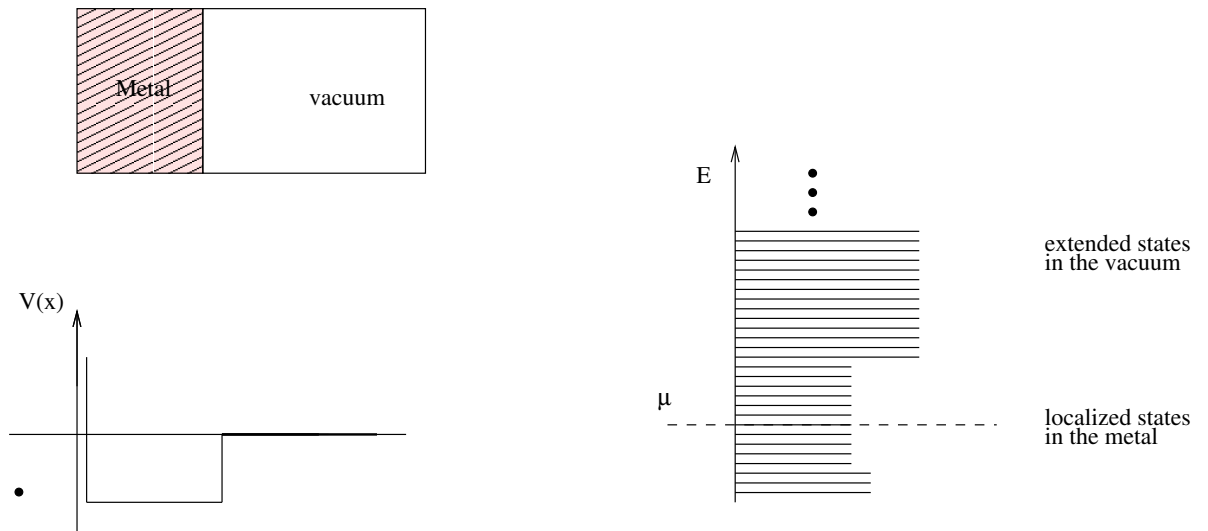


### ===== [Exercise 6030]

#### Thermionic emission of electrons from a metal

A piece of metal ("cathode") is placed inside a vacuum metal tube ("anode"). The cathode has a work function  $W$  and surface area  $A$ , while the anode has work function  $W'$ . The cathode by itself can be regarded as a potential-well: the depth of the potential floor close to the surface is zero, while deeper inside the metal it is  $V_0 (\gg W)$ . The system is held at temperature  $T$ .

- (1) Write an integral expression for the saturation current  $I_s$  that would be measured if the bias voltage is very large.
- (1a) Show that  $V_0$  does not appear in the final result: the outcome of the calculation is the same for sections that are close to the surface or deep in the metal.
- (1b) Calculate the integral in the Boltzmann approximation, and explain whether the validity condition corresponds to low temperatures ( $T \ll W$ ) or high temperatures.
- (2) Using the result of the previous item write an estimate for the current if a reverse (stopping) voltage  $V_{\text{battery}}$  is applied. Explain whether  $W$  or  $W'$  is relevant.
- (2a) Explain the relation to the analysis of the stopping voltage in the photoelectric effect.
- (3) Assume that the cathode is detached and left alone in free space. Calculate the charge  $Q(t)$  of the cathode as a function of time assuming that  $Q(0) = 0$ .
- (3a) Explain the limitations of the result that you have obtained.

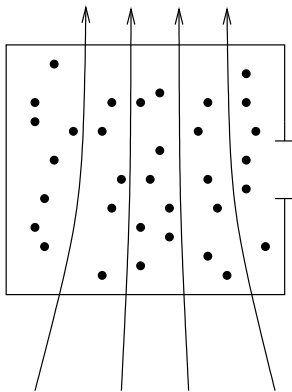


### ===== [Exercise 6040]

#### Effusion of electrons from a box in magnetic field

A box with electrons of mass  $m$  is subjected to a magnetic field  $B$ . The single particle interaction is described by  $-\gamma B \sigma_z$ . The chemical potential of the electrons inside the box is  $\mu$ . A hole through one of the walls is drilled. The electrons that are emitted from the hole with a velocity in the range  $v < v' < v + dv$  are filtered, and subsequently their spin is measured. The measured current is defined as  $I = I_{\uparrow} + I_{\downarrow}$ .

- Find the ratio  $\alpha(B; \mu) = (I_{\uparrow} - I_{\downarrow})/I$ .
- Find a linear approximation for  $\alpha(B; \mu)$  regarded as a function of the magnetic field.
- What is the maximal value of  $\alpha(B; \mu)/B$ , and what is the range for which the result is valid.



### ===== [Exercise 6050]

#### Radiation from a 1D blackbody fiber

Consider an optical fiber that has a length  $L$ . Its section area is  $A$ . The fiber is in thermal equilibrium at temperature  $T$ . Assume the fiber is a one dimensional medium for the electromagnetic field. Regard the system as a 1D photon gas.

- (a) What is the electromagnetic energy density per unit length?
- (b) What is the radiation pressure on the fiber edges?
- (c) Assuming that the radiation is freely emitted from the boundary of the fiber, find the energy flow per unit time.
- (d) What is the spectral distribution  $J(\omega)$  of the emitted radiation?
- (e) What is the entropy and what is the heat capacity of the system?

You can use the following integral

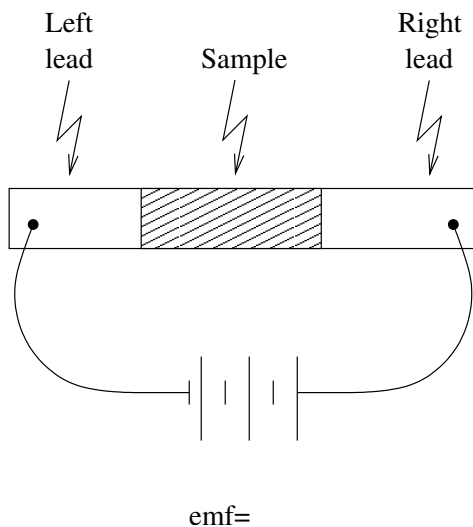
$$\int_0^\infty \frac{x}{e^x - 1} dx = \frac{\pi^2}{6}$$

### ===== [Exercise 6070]

#### Landauer formula for a 1D conductance

Consider 1D conductor that has transmission coefficient  $g$ . The conductor is connected to 1D leads that have chemical potentials  $\mu_a$  and  $\mu_b$ . Assume  $\mu_a = \mu$  and  $\mu_b = \mu + eV$ , where  $V$  is the bias.

- (1) Write the expression for the current  $I$  as an integral over the occupation function  $f(\epsilon)$ .
- (2) For small bias write the relation as  $I = GV$  and obtain an expression for  $G$ . Write explicit results for zero temperature Fermi occupation (Landauer formula) and for high temperature Boltzman occupation.
- (3) Find expressions for  $I(V)$  in the case of arbitrary (possibly large) bias, for zero temperature Fermi occupation and for high temperature Boltzmann occupation. Assume that  $g$  is independent of energy.



### ===== [Exercise 6071]

#### Generalize incident current formula for 1D and 2D boxes

Generalize the equation for  $J$  incident for the cases of two dimensional gas and one dimensional gas. in each case, note what is the 'volume', what are the units of  $J$  and especially, what is the geometric factor in the equation.

### ===== [Exercise 6080]



### Einstein relation for the conductivity of electrons

Given a metal design. We mark with  $\varphi(x)$  the electrical potential in the sample and with  $N(x)$  the spacial density of the electrons in the design. According to the kinetic theory s

$$\vec{J}(x) = -\sigma \nabla \varphi - eD \nabla N$$

$\sigma$  is the conductivity and  $D$  is the diffusion coefficient. In an equilibrium state  $\vec{J}(x) \equiv 0$ , especially in a state of equilibrium that we get in the presence of outer field  $\varphi(x) \neq \text{const}$  and therefore has to :  $\frac{\sigma}{eD} = \frac{\nabla N}{\nabla \varphi}$ .

Use the principles of the statistical mechanics to show that from here derives

$$\frac{\sigma}{D} = -e^2 \int dE g(E) f'(E - \mu)$$

low temperatures  $\frac{\sigma}{D} = e^2 g(E_F)$

High temperatures  $\frac{\sigma}{D} = N \frac{e^2}{T}$

$g(E)$  is the uniparticles states density per volume unit. Hint - notice that

$$N(x) = \int g(E - eV(x)) dE f(E - \mu)$$

===== [Exercise 6110]

### Radiometer

Radiometer

===== [Exercise 6700]

### Boltzmann Equation

===== [Exercise 6772]

### Boltzmann equation: distribution function

Consider an ideal gas in an external potential  $\phi(\mathbf{r})$ .

- (a) Let  $H = \int d^3v \int d^3r f(\mathbf{r}, \mathbf{v}, t) \ln f(\mathbf{r}, \mathbf{v}, t)$

where  $f(\mathbf{r}, \mathbf{v}, t)$  is arbitrary except for the conditions on density  $n$  and energy  $E$

$$\int d^3r \int d^3v f(\mathbf{r}, \mathbf{v}, t) = n, \quad \int d^3r \int d^3v \left[ \frac{1}{2} m v^2 + \phi(\mathbf{r}) \right] f(\mathbf{r}, \mathbf{v}, t) = E.$$

Find  $f(\mathbf{r}, \mathbf{v})$  (i.e.  $t$  independent) which maximizes  $H$ . (Note: do not assume binary collisions, i.e. the Boltzmann equation).

- (b) Use Boltzmann's equation to show that the general form of the equilibrium distribution of the ideal gas (i.e. no collision term) is  $f[\frac{1}{2} m v^2 + \phi(\mathbf{r})]$  where the local force is  $\nabla \phi$ . Determine this solution by allowing for collisions and requiring that the collision term vanishes. Find also the average density  $n(r)$ .

===== [Exercise 6773]

### Dissipation phase space volume and entropy

Consider the derivation of Liouville's theorem for the ensemble density  $\rho(p, q, t)$  in phase space  $(p, q)$  corresponding to the motion of a particle of mass  $m$  with friction  $\gamma$

$$\frac{dq}{dt} = \frac{p}{m}, \quad \frac{dp}{dt} = -\gamma p.$$

- Show that Liouville's theorem is replaced by  $d\rho/dt = \gamma\rho$ .
- Assume that the initial  $\rho(p, q, t=0)$  is uniform in a volume  $\omega_0$  in phase space and zero outside of this volume. Find  $\rho(p, q, t)$  if  $\omega_0$  is a rectangle  $-\bar{p} < p < \bar{p}$ ,  $-\bar{q} < q < \bar{q}$ . Find implicitly  $\rho(p, q, t)$  for a general  $\omega_0$ .
- what happens to the occupied volume  $\omega_0$  as time evolves? (assume a general shape of  $\omega_0$ ). Explain at what  $t$  this description breaks down due to quantization.
- Find the Boltzmann entropy as function of time for case (b). Discuss the meaning of the result.

### ===== [Exercise 6774]

#### Boltzmann equation: Conductivity

Electrons in a metal can be described by a spectrum  $\epsilon(\mathbf{k})$ , where  $\mathbf{k}$  is the crystal momentum, and a Fermi distribution  $f_0(\mathbf{k})$  at temperature  $T$ .

- Find the correction to the Fermi distribution due to a weak electric field  $\mathbf{E}$  using the Boltzmann equation and assuming that the collision term can be replaced by  $-[f(\mathbf{k}) - f_0(\mathbf{k})]/\tau$  where  $\tau$  is the relaxation time. Note that  $d\mathbf{k}/dt = e\mathbf{E}/\hbar$  and the velocity is  $\mathbf{v}_\mathbf{k} = \nabla_\mathbf{k}\epsilon(\mathbf{k})/\hbar$ , i.e. in general  $d\mathbf{v}_\mathbf{k}/dt$  is  $\mathbf{k}$  dependent.
- Find the conductivity tensor  $\sigma$ , where  $\mathbf{J} = \sigma\mathbf{E}$ . In what situation would  $\sigma$  be non-diagonal? Show that  $\sigma$  is non-diagonal if the mass tensor  $(\frac{1}{m*})_{i,j} = \frac{1}{\hbar^2} \frac{\partial^2 \epsilon(\mathbf{k})}{\partial k_i \partial k_j}$  is not diagonal.
- Find  $\sigma$  explicitly for  $\epsilon = \hbar^2 k^2 / 2m^*$  in terms of the electron density  $n$ . ( $m^*$  is an effective mass).

### ===== [Exercise 6775]

#### Coarse grained entropy

Coarse grained entropy. The usual  $\rho(p, q, t)$ , i.e. the normalized state density in the  $6N$  dimensional phase space  $(p, q)$ , satisfies Liouville's theorem  $d\rho/dt = 0$ . We wish to redefine  $\rho(p, q, t)$  so that the corresponding entropy increases with time.

Divide phase space to small sub-volumes  $\Omega_\ell$  and define a coarse grained density

$$\bar{\rho}(p, q, t) = \bar{\rho}_\ell = \frac{1}{\Omega_\ell} \int_{\Omega_\ell} \rho(p, q, t) dp dq \quad (p, q) \in \Omega_\ell$$

so that  $\bar{\rho}(p, q, t)$  is constant within each cell  $\Omega_\ell$ . Define the entropy as

$$\eta(t) = - \int \bar{\rho}(p, q, t) \ln \bar{\rho}(p, q, t) dp dq = - \sum_\ell \Omega_\ell \bar{\rho}_\ell \ln \bar{\rho}_\ell.$$

Assume that at  $t=0$   $\rho(p, q, 0)$  is uniform so that  $\rho(p, q, 0) = \bar{\rho}(p, q, 0)$ .

- Show that  $\eta(0) = - \int \rho(p, q, 0) \ln \rho(p, q, 0) dp dq$ .
- Show that  $\eta(t)$  increases with time, i.e.

$$\eta(t) - \eta(0) = - \int \rho \left[ \ln \frac{\bar{\rho}}{\rho} + 1 - \frac{\bar{\rho}}{\rho} \right] dp dq \geq 0.$$

Hint: Show that  $\ln x + 1 - x \leq 0$  for all  $x > 0$ .

### ===== [Exercise 6776]

#### Boltzmann equation: Emission

Equilibrium and kinetics of light and matter:

- (a) Consider atoms with fixed positions that can be either in their ground state  $a_0$ , or in an excited state  $a_1$ , which has a higher energy  $\epsilon$ . If  $n_0$  and  $n_1$  are the densities of atoms in the two levels, find the ratio  $n_1/n_0$  at temperature  $T$ .
- (b) Consider photons  $\gamma$  of frequency  $\omega = \epsilon/\hbar$  and momentum  $|\mathbf{p}| = \hbar\omega/c$ , which can interact with the atoms through the following processes:
- (i) *Spontaneous emission*:  $a_1 \rightarrow a_0 + \gamma$
  - (ii) *Absorption*:  $a_0 + \gamma \rightarrow a_1$
  - (iii) *Stimulated emission*:  $a_1 + \gamma \rightarrow a_0 + \gamma + \gamma$ .

Assume that spontaneous emission occurs with a probability  $\sigma_1$  (per unit time and per unit (momentum)<sup>3</sup>) and that absorption and stimulated emission have constant (angle independent) differential cross-sections of  $\sigma_2$  and  $\sigma_3/4\pi$ , respectively.

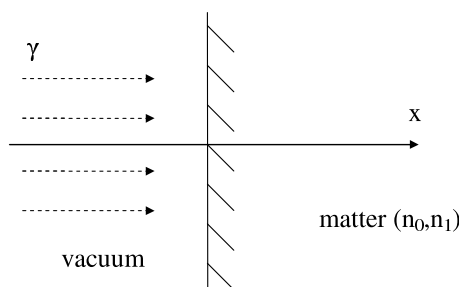
Show that the Boltzmann equation for the density  $f(\mathbf{r}, p, t)$  of the photon gas, treating the atoms as fixed scatterers of densities  $n_0$  and  $n_1$  is

$$\frac{\partial f(\mathbf{r}, p, t)}{\partial t} + \frac{\mathbf{p}c}{|\mathbf{p}|} \cdot \frac{\partial f(\mathbf{r}, p, t)}{\partial \mathbf{r}} = -\sigma_2 n_0 c f(\mathbf{r}, p, t) + \sigma_3 n_1 c f(\mathbf{r}, p, t) + \sigma_1 n_1$$

- (c) Find the equilibrium solution  $f_{eq}$ . Equate the result, using (a), to that the expected value per state  $f_{eq} = \frac{1}{h^3} \frac{1}{e^{\hbar\omega/k_B T} - 1}$  and deduce relations between the cross sections.
- (d) Consider a situation in which light shines along the  $x$  axis on a collection of atoms whose boundary is at  $x = 0$  (see figure). The incoming flux is uniform and has photons of momentum  $\mathbf{p} = \hbar\omega\hat{x}/c$  where  $\hat{x}$  is a unit vector in the  $x$  direction. Show that the solution has the form

$$Ae^{-x/a} + f_{eq}$$

and find the penetration length  $a$ .



### ===== [Exercise 6777]

#### Phase space evolution of confined particle

A thermalized gas particle at temperature  $T$  is suddenly confined to positions  $q$  in a one dimensional trap. The corresponding state is described by an initial density function  $\rho(q, p, t = 0) = \delta(q)f(p)$  where  $\delta(q)$  is Dirac's delta function and

$$f(p) = \frac{e^{-p^2/2mk_B T}}{\sqrt{2\pi mk_B T}}. \quad (1)$$

- Starting from Liouville's equation with the Hamiltonian  $\mathcal{H} = p^2/2m$  derive  $\rho(q, p, t)$ . For a given time  $t$  draw the points in the  $(p, q)$  plane where  $\rho(q, p, t)$  is finite and emphasize the segment where  $f(p)$  is large,  $p < \sqrt{mk_B T} \equiv p_0$ .
- Derive the expressions for the averages  $\langle q^2 \rangle$  and  $\langle p^2 \rangle$  at  $t > 0$ .
- Suppose that hard walls are placed at  $q = \pm Q$ . Repeat the plot of (a) and again emphasize the range  $p < p_0$ . What happens in this plot at long times  $t > 2Qm/p_0 \equiv \tau_0$ ? What is the meaning of the time  $\tau_0$ ?
- A "coarse grained" density  $\tilde{\rho}$  is obtained by ignoring variations of  $\rho$  below some small resolution in the  $(q, p)$  plane; e.g., by averaging  $\rho$  over cells of the resolution area. Find  $\tilde{\rho}(q, p)$  for the situation in part (c) at long time  $t \gg \tau_0$ , and show that it is stationary.

### ===== [Exercise 6778]

#### Boltzmann equation: particles between two plates

Consider a classical gas of particles with mass  $m$  between two plates separated by a distance  $W$ . One plate at  $y = 0$  is maintained at a temperature  $T_1$ , while the other plate at  $y = W$  is at a different temperature  $T_2$ . A zeroth order approximation to the particle density is,

$$f_0(\mathbf{p}, x, y, z) = \frac{n(y)}{[2\pi mk_B T(y)]^{3/2}} e^{-\frac{p^2}{2mk_B T(y)}}$$

- The steady state solution has a uniform pressure; it does not have a uniform chemical potential. Explain this statement and find the relation between  $n(y)$  and  $T(y)$ .
- Show that  $f_0$  does not solve Boltzmann's equation.

Consider a relaxation approximation, where the collision term of Boltzmann's equation is replaced by a term that drives a solution  $f_1$  towards  $f_0$ , i.e.

$$\left[ \frac{\partial}{\partial t} + \frac{p_y}{m} \frac{\partial}{\partial y} \right] f_0(\mathbf{p}, y) = -\frac{f_1(\mathbf{p}, y) - f_0(\mathbf{p}, y)}{\tau}$$

and solve for  $f_1$ .

- The rate of heat transfer is  $Q = n \langle p_y p^2 \rangle_1 / (2m^2)$ ;  $\langle \dots \rangle_1$  is an average with respect to  $f_1$ . Justify this form and evaluate  $Q$  using the integrals  $\langle p_y^2 p^4 \rangle_0 = 35(mk_B T)^3$  and  $\langle p_y^2 p^2 \rangle_0 = 5(mk_B T)^2$ . Identify the coefficient of thermal conductivity  $\kappa$ , where  $Q = -\kappa \frac{\partial T}{\partial y}$ .
- Find the profile  $T(y)$ .
- Show that the current is  $\langle J_y \rangle = 0$ . Explain why this result is to be expected.
- For particles with charge  $e$  add an external field  $E_y$  and extend Boltzmann's equation from (b). Evaluate, for uniform temperature,  $J_y$  and the conductivity  $\sigma$ , where  $J_y = \sigma E_y$ . Check the Wiedemann-Franz law,  $\kappa/\sigma T = \text{const.}$

### ===== [Exercise 7000]

## The FD realation

### ===== [Exercise 7001]

#### Definition of power spectrum

Prove that the Fourier components of a stationary noisy signal have a variance which is proportional to the time of the measurement.

Show that the coefficient of proportionality is just the power spectrum (defined as the Fourier transform of the correlation function).

### ===== [Exercise 7005]

#### Shot noise

The discreteness of the electron charge  $e$  implies that the current is not uniform in time and is a source of noise. Consider a vacuum tube in which electrons are emitted from the negative electrode and flow to the positive electrode; the probability of emitting any one electron is independent of when other electrons are emitted. Suppose that the current meter has a response time  $\tau$ . If  $T_e$  is the average time between the emission of two electrons, then the average current is  $\langle I \rangle = e/T_e = \frac{e}{\tau}\eta$ , where  $\eta = \tau/T_e$  is the transmission probability,  $0 \leq \eta \leq 1$ .

- Show that the fluctuations in  $I$  are  $\langle (\Delta I)^2 \rangle = \frac{e^2}{\tau^2}\eta(1-\eta)$ . Why would you expect the fluctuations to vanish at both  $\eta = 0$  and  $\eta = 1$ ? [Hint: For each  $\tau$  interval  $n_i$  is the number of electrons hitting the positive electrode. Therefore, it can be equal to  $n_i = 0$  or  $n_i = 1$  which results in an average  $\langle n_i \rangle = \tau/T_e$ ; discretize time in units of  $\tau$ .]
- Consider the meter response to be in the range  $0 < |\omega| < 2\pi/\tau$ . Show that for  $\eta \ll 1$  the fluctuations in the frequency domain are  $\langle (\Delta I)^2 \rangle = e\langle I \rangle$ . What is the condition for this noise to dominate over the Johnson-Nyquist noise in the circuit?
- Show that the 3rd order cumulant is  $\langle (I - \langle I \rangle)^3 \rangle = \frac{e^3}{\tau^3}\eta(1-\eta)(1-2\eta)$ .

### ===== [Exercise 7010]

#### Site occupation during a sweep process

Consider the occupation  $n$  of a site whose binding energy  $\varepsilon$  can be controlled, say by changing a gate voltage. The temperature of the environment is  $T$  and its chemical potential is  $\mu$ . Consider separately 3 cases:

- The occupation  $n$  can be either 0 or 1.
- The occupation  $n$  can be any natural number  $(0, 1, 2, 3, \dots)$
- The occupation  $n$  can be any real positive number  $\in [0, \infty]$

We define  $\bar{n}$  as the average occupation at equilibrium. The fluctuations of  $\delta n(t) = n(t) - \bar{n}$  are characterized by a correlation function  $C(\tau)$ . Assume that it has exponential relaxation with time constant  $\tau_0$ . Later we define  $\langle n \rangle$  as the average occupation during a sweep process, where the potential is varied with rate  $\dot{\varepsilon}$ .

- Calculate  $\bar{n}$ , express it using  $(T, \varepsilon, \mu)$ .
- Calculate  $\text{Var}(n)$ , express the result using  $\bar{n}$ .
- Write an expression for the  $\omega=0$  intensity  $\nu$  of the fluctuations.
- Write an expression for  $\langle n \rangle$  during a sweep process.

Irrespective of whether you have solved (1) and (2), in item (3) express the result using  $\text{Var}(n)$ . In item (4) use the classical version of the fluctuation-dissipation relation, and express the result using  $(T, \tau_0, \bar{n}, \varepsilon)$ , where  $\bar{n}$  had been given by your answer to item (1). Note that the time dependence is *implicit* via  $\bar{n}$ .

### ===== [Exercise 7020]

#### FDT for harmonic oscillator

A particle of mass  $m$  is described by its position  $x$  and velocity  $v$ . It is bounded by a harmonic potential of frequency  $\Omega$ , and experiences a damping with a coefficient  $\eta$ . Additionally It is subject to an external force  $f(t)$ . The system is at temperature  $T$ .

- Write the generalized susceptibility that describes the response of  $x$  to the driving by  $f(t)$ .
- Using the FD relation deduce what are the power spectra of  $x$  and of  $v$ .
- Write an integral expression for the autocorrelation function  $\langle v(t)v(0) \rangle$ . Find explicit results in various limits, e.g. for damped particle ( $\Omega \rightarrow 0$ ).
- Find  $\langle x^2 \rangle$  and  $\langle v^2 \rangle$  for  $\eta \rightarrow 0$ , both in the quantum and in the classical case. Verify consistency with the canonical results.

### ===== [Exercise 7040]

#### FDT for RL-circuit, Nyquist theory

Derive the Nyquist expression for the current-current correlation function in a closed ring, taking into account its inductance. Use the following procedure:

- Cite an expression for the inductance  $L$  of a torus shaped ring given its radius  $R$  and its cross-section radius  $r$ .
- Write the R-L circuit equation for the current  $I$ , where the flux  $\Phi(t)$  through the ring is the driving parameter.
- Identify the generalized susceptibility  $\chi(\omega)$ , and observe that it is formally the same expression as in the problem of Brownian motion.
- Calculate the current-current correlation function  $\langle I(t)I(0) \rangle$ , taking the classical / high temperature limit.
- Verify that  $\langle I^2 \rangle$  agree with the canonical result.

### ===== [Exercise 7041]

#### FDT for RLC circuit

An electrical circuit has in series components with capacitance  $C$ , inductance  $L$ , resistance  $R$  and a voltage source  $V_0 \cos \omega t$  with frequency  $\omega$ .

- Identify the responsefunction  $\alpha_Q(\omega) = \langle Q(\omega) \rangle / (\frac{1}{2} V_0)$ . Use this to write the energy dissipation rate.
- Use the fluctuation dissipation relation to identify the Fourier transform  $\Phi_Q(\omega)$  of the charge correlation function. Evaluate  $\langle Q^2(t) \rangle$  and compare with the result from equipartition.
- Evaluate the current fluctuations  $\langle I^2(t) \rangle$  and compare with the result from equipartition. Under what conditions does one get Nyquist's result  $\langle I^2 \rangle_{\omega_1 \leftrightarrow \omega_2} = \frac{2k_B T}{\pi R} (\omega_2 - \omega_1)$  ?

Hint:  $\int_{-\infty}^{\infty} \frac{d\omega/2\pi}{(\omega^2 - \omega_0^2)^2 + \gamma^2 \omega^2} = \frac{1}{2\gamma \omega_0^2}$ ,  $\int_{-\infty}^{\infty} \frac{\omega^2 d\omega/2\pi}{(\omega^2 - \omega_0^2)^2 + \gamma^2 \omega^2} = \frac{1}{2\gamma}$ .

### ===== [Exercise 7050]

#### The Drude formula

Consider a ring of length  $L$ , with a particle that has the Drude velocity-velocity correlation function with a time constant  $\tau$ . The temperature is  $T$ .

- Find the conductance of the ring using the canonical FDT.
- What is the conductance if there are  $N$  fermions at zero temperature instead of a single particle.
- What is  $\tau$ , and hence what is the conductance, if the scattering in the ring is due to a stochastic segment that has a transmission  $g$ .

### ===== [Exercise 7060]

#### The Wall formula

Consider a “piston” of area  $A$ , moving in a ideal-gas chamber. Find an expression for the friction coefficient. [See lecture notes].

### ===== [Exercise 7481]

#### FDT for velocities

Fluctuation Dissipation Theorem (FDT) for velocities: Consider an external  $F(t) = \frac{1}{2}f_0 e^{-i\omega t} + \frac{1}{2}f_0^* e^{i\omega t}$  coupled to the momentum as

$$H = \frac{p^2}{2M} + V(x; \text{env}) - \frac{1}{M} F(t)p$$

where “env” stands for the environment’s coordinates and momenta.

- Define the velocity response function by  $\langle v(\omega) \rangle = \alpha_v(\omega) F(\omega)$  and show that the average dissipation rate is

$$\frac{dE}{dt} = \frac{1}{2} \omega |f_0|^2 \text{Im} \alpha_v(\omega).$$

- Construct a Langevin’s equation with  $F(t)$  and identify  $\alpha_v(\omega)$ . [Identify also  $\alpha_{p/M}(\omega)$  and show that  $\text{Im} \alpha_v(\omega) = \text{Im} \alpha_{p/M}(\omega)$ .]

Using the known velocity correlations  $\phi_v(\omega)$  (for  $F = 0$ ) show the FDT

$$\phi_v(\omega) = \frac{2k_B T}{\omega} \text{Im} \alpha_v(\omega).$$

### ===== [Exercise 7486]

#### Linear response and Kubo

Consider a classical system of charged particles with a Hamiltonian  $H_0(p, q)$ . Turning on an external field  $\mathbf{E}(t)$  leads to the Hamiltonian  $H = H_0(p, q) - e \sum_i \mathbf{q}_i \cdot \mathbf{E}(t)$ .

- Show that the solution of Liouville’s equation to first order in  $\mathbf{E}(t)$  is

$$\rho(p, q, t) = e^{-\beta H_0(p, q)} \left[ 1 + \beta e \sum_i \int_{-\infty}^t \dot{\mathbf{q}}_i(t') \cdot \mathbf{E}(t') dt' \right].$$

- (b) In terms of the current density  $\mathbf{j}(\mathbf{r}, t) = e \sum_i \dot{\mathbf{q}}_i \delta^3(\mathbf{r} - \mathbf{q}_i)$  show that for  $\mathbf{E} = \mathbf{E}(\omega) e^{i\omega t}$  the linear response is  $\langle j^\mu(t) \rangle = \sigma^{\mu\nu}(\omega) E^\nu(\omega) e^{i\omega t}$  where  $\mu, \nu$ , are vector components and

$$\sigma^{\mu\nu}(\omega) = \beta \int_0^\infty d\tau e^{-i\omega\tau} d^3r \langle j^\mu(0, 0) j^\nu(\mathbf{r}, -\tau) \rangle_0$$

where  $\langle \dots \rangle_0$  is an average of the  $\mathbf{E} = 0$  system. This is the (classical) Kubo's formula.

- c) Rewrite (b) for  $\mathbf{j}(\mathbf{r}, t)$  in presence of a position dependent  $\mathbf{E}(\mathbf{r}, t)$ . Integrating  $\mathbf{j}(\mathbf{r}, t)$  over a cross section perpendicular to  $\mathbf{E}(\mathbf{r}, t)$  yields the current  $I(t)$ . Show that the resistance  $R(\omega)$  satisfies

$$R^{-1}(\omega) = \beta \int_0^\infty d\tau e^{-i\omega\tau} \langle I(0) I(\tau) \rangle_0$$

For a real  $R(\omega)$  (usually valid below some frequency) deduce Nyquist's theorem.

### ===== [Exercise 7487]

#### Velocity-velocity correlation and diffusion

- (a) Write the Diffusion constant  $D$  in terms of the velocity-velocity correlation function. [Assume that this correlation has a finite range in time].
- (b) Use Kubo's formula, assuming uncorrelated particles, to derive the Einstein-Nernst formula for the mobility  $\mu = eD/k_B T$ . [ $\mu = \sigma(\omega = 0)/ne$  and  $n$  is the particle density].

### ===== [Exercise 7489]

#### The Kubo formula

Particles with charge  $e$  and velocities  $\mathbf{v}_i$  couple to an external vector potential by  $V_{int} = -\frac{e}{c} \sum_i \mathbf{v}_i \cdot \mathbf{A}$  and the electric field is  $\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$ . The current density (per unit volume) is  $\mathbf{j} = e \sum_i \mathbf{v}_i$ .

- (a) Identify the response function for an  $a$  component field with a given frequency,  $E_a(\omega)$ , in terms of the conductivity  $\sigma(\omega)$  where  $\mathbf{j}_a = \sigma(\omega) \mathbf{E}_a$  (assume an isotropic system so that  $\sigma(\omega)$  is a scalar). Deduce the energy dissipation rate in terms of  $\sigma(\omega)$  and  $E_a(\omega)$ . Compare with Ohm's law. What is the symmetry of  $\text{Re}\sigma(\omega)$  when  $\omega$  changes sign?
- (b) Use the fluctuation dissipation theorem to show the (classical) Kubo formula:

$$\text{Re}\sigma(\omega) = \frac{1}{k_B T} \int_0^\infty \langle j_a(0) \cdot j_a(t) \rangle \cos(\omega t) dt$$

- (c) Write the Diffusion constant  $D$  in terms of the velocity-velocity correlation function, assuming that this correlation has a finite range in time.  
Use Kubo's formula from (b) in the DC limit of zero frequency to derive the Einstein-Nernst formula for the mobility  $\mu = \frac{\sigma}{ne} = eD/k_B T$ , where  $n$  is the particle density. (assume here uncorrelated particles).
- (d) The quantum current noise is defined as

$$S(\omega) = \int_0^\infty dt \langle j_a(t) j_a(0) + j_a(0) j_a(t) \rangle \cos(\omega t).$$

Use the quantum FDT to relate this noise to the conductivity. When is the classical result (b) valid? What is the noise at  $T = 0$ ?



### ===== [Exercise 7491]

#### Onsager

Consider a fluid in two compartments connected with a small hole. Although particles can pass easily through the hole, it is small enough so that within each compartment the fluid is in thermodynamic equilibrium. The compartments have pressure, temperature, volume and particle number  $P_1, T_1, V_1, N_1$  and  $P_2, T_2, V_2, N_2$ , respectively. There is an energy transfer rate  $dE/dt$  and particle transfer rate  $dN/dt$  through the hole.

- Identify the kinetic coefficients for  $dE/dt$  and  $dN/dt$  driven by temperature and chemical potential differences. Rewrite the equations in terms of  $\Delta T = T_1 - T_2$  and  $\Delta P = P_1 - P_2$  to first order in  $\Delta T$  and  $\Delta P$ .
- If  $\Delta T = 0$  one measures  $\epsilon_1 = (dE/dt)/(dN/dt)$ . One can also adjust the ratio  $\epsilon_2 = \Delta P/\Delta T$  so that  $dN/dt = 0$ . Show the relation

$$\epsilon_2 = \frac{1}{T} \left[ \frac{E}{V} + P - \frac{N}{V} \epsilon_1 \right]$$

( $E/V$  or  $P$  for either compartment).

- Assume that the work done during the transfer by the pressure is via reducing the effective volume to zero within the hole. Evaluate  $\epsilon_1$  and show that  $\epsilon_2 = 0$ .

### ===== [Exercise 7492]

#### Onsager

Consider the coefficients  $\gamma_{ij}$  in Onsager's relations for heat and current transport (see lecture notes pages 70-71). Consider also Boltzmann's equation as in Ex. D07.

- Show that  $\gamma_{22}$  is related to the conductivity  $\sigma = ne^2\tau/m$ .
- Show that  $\langle J_y \rangle = 0$  and identify  $\gamma_{21}$ . [Note that  $eV_i = \mu_i$  the local chemical potential.]
- Identify the thermal conductivity  $\kappa$  in terms of  $\gamma_{ij}$ . Use  $\kappa = \frac{5}{2m} k_B^2 n \tau T$  (result of D07c) to find  $\gamma_{11}$ .

### ===== [Exercise 8000]

#### Stochastic picture, Langevin

### ===== [Exercise 8001]

#### Random walk with correlations

The total displacement of a particle is a sum over steps  $X(t)$ , where  $t$  is discrete. If we define the velocity as  $v(t) = \frac{X(t) - X(0)}{t}$ , where  $\tau_0$  is the time between steps, then the random walk is described by the equation  $\left(\frac{dx}{dt}\right) = v(t)$ .

- Given the velocity-velocity correlation function  $c(t_2 - t_1) = \langle v(t_1) v(t_2) \rangle$ , write down an expression for the spreading  $S(t) = \sqrt{\langle (x(t) - x(0))^2 \rangle}$ .
- Find an expression the diffusion coefficient, assuming that  $c(\tau)$  is short range.

- (d) More generally, show that  $\frac{dS(t)}{dt}$  is equal to the  $[-t, t]$  integral of  $c(\tau)$ .
- (e) Assume that  $c(\tau)$  has zero integral and power law tails  $c(\tau) = \frac{-c_0}{\tau^\alpha}$ . Determine the sub-diffusive behavior of  $S(t)$  depending on the value of  $\alpha$ .

===== [Exercise 8020]

**Correlation functions from Langevin dynamics**

Consider the Langevin equation for a particle with mass  $M$  and velocity  $\mathbf{v}(t)$  in a medium with viscosity  $\gamma$  and a random force  $\mathbf{A}(t)$ .

- (a) Find the equilibrium value of  $\langle \mathbf{v}(t) \mathbf{A}(t) \rangle$ .
- (b) Given  $\langle \mathbf{v}(t) \mathbf{v}(0) \rangle \sim e^{-\gamma|t|}$  and  $\langle \mathbf{v} \rangle = 0$ , use  $\mathbf{v}(t) = \dot{\mathbf{x}}(t)$  to evaluate  $\langle \mathbf{x}^2(t) \rangle$  [do not use Langevin's equation] .

===== [Exercise 8025]

**Thermal flow via a Brownian particle**

A Brownian particle in one dimension that has mass  $m = 1$ , is in contact with two baths: A hot bath that has temperature  $T_2$  that induces friction with coefficient  $\gamma_2$ , and a cold bath that has temperature  $T_1$  that induces friction with coefficient  $\gamma_1$ . Accordingly the motion of the particle is described by a Langevin equation that includes two friction terms and two independent white noise terms  $f_1(t)$  and  $f_2(t)$ . The purpose of this question is to calculate the rate of heat flow  $\dot{Q}$  from the hot to the cold bath.

*Note:* Each bath exerts on the particle a force that has two components: a systematic "friction" component plus a fluctuating component. The rate of heat flow  $\dot{Q}$  equals the rate of work which is done by the force that is exerted on the particle by the hot bath. In steady state, on the average, it equals in absolute value to the rate of work which is done by the force that is exerted on the particle by the cold bath.

- (1) Write the Langevin equation for the velocity  $v(t)$ . Specify the intensity of the noise terms.
- (2) Find the steady state value of  $\langle v^2 \rangle$ .
- (3) Express the instantaneous  $\dot{Q}$  at time  $t$ , given  $v(t)$  and  $f_2(t)$ .
- (4) Find an expression for  $\langle \dot{Q} \rangle$  at steady state.

===== [Exercise 8027]

**Bosons with spin in harmonic trap**

Consider the 1D motion of two beads of mass  $m$ , attached by a very flexible bond with spring constant  $k$ . The beads are immersed in viscous liquid with friction coefficients  $\gamma_1, \gamma_2$ , and temperature  $T$ . Disregard the hydrodynamic interactions between the beads and the direct collisions of the beads.

- (1) Write down Langevin equations for the beads. Neglect accelerations.
- (2) For  $\gamma_1 = \gamma_2 \equiv \gamma$ , define  $R = \frac{x_1 + x_2}{2}$ , and  $r = x_1 - x_2$ . Find  $\langle (r(t) - r_0)^2 \rangle$  and  $\langle (R(t) - R_0)^2 \rangle$ , where  $r_0$  is half the initial distance between the "atoms" and  $R_0$  is the initial location of the "molecule".
- (3) Solve the equations for  $\gamma_1 \neq \gamma_2$ , and show that the same solution as in (2) is obtained by setting  $\gamma_1 = \gamma_2 \equiv \gamma$ .
- (4) Generalize the results to 3D.

### ===== [Exercise 8030]

#### Diffusion of Brownian particle from Langevin

Brownian motion is formally obtained as the  $\Omega \rightarrow \infty$  limit of the previous problem.

- Calculate the velocity-velocity correlation function of the Brownian particle in the limit of high temperature.
- Show that it is an exponential function, and identify the correlation time.
- Write the relation between the dispersion  $\sqrt{\langle (x(t) - x(0))^2 \rangle}$  and the velocity correlation function.
- Deduce that the particle diffuses in space and write the expression for the diffusion coefficient.
- Show that in the limit of zero temperature the velocity-velocity correlation function has a zero integral and power law tails (recall Exe.701).
- In the latter case deduce that instead of diffusive spreading one should observe slow logarithmic growth of the variance.

### ===== [Exercise 8032]

#### Sub diffusion of Brownian particle

The motion of a brownian particle in 1D is given by the Hamiltonian:

$$H_{total}(x, p; A(t)) = \frac{1}{2m}(p - A(t))^2 + H_{bath}(x)$$

Assume that the equation of motion for the average velocity is:

$$m \frac{\partial \langle v \rangle}{\partial t} = -\eta \langle v \rangle + f(t)$$

In items 5-6-7 assume a zero temperature bath, and define

$$S(t) = \langle (x(t) - x(0))^2 \rangle$$

- Relate  $f(t)$  to  $A(t)$ .
- What is the generalised susceptibility  $\chi(\omega)$  that relates  $v$  to  $A$ .
- Find the power spectrum  $\tilde{C}(\omega)$  of the velocity  $v$ .
- Find an explicit expression for the correlation function  $C(\tau)$  in the limit of high temperature.
- In the limit of zero temperature find the coefficient  $C_0$  in  $C(\tau) \sim -C_0/\tau^2$ .
- Express  $dS(t)/dt$  using the correlation function  $C(\tau)$ .
- Given  $S(t_0) = S_0$ , find what is  $S(t)$  for  $t > t_0$ .

### ===== [Exercise 8034]

#### Brownian particle on a ring

The motion of a classical Brownian particle on a 1D ring is described by the Langevin equation  $m\ddot{\theta} + \eta\dot{\theta} = f(t)$ , where  $f(t)$  is due to a noisy electromotive force that has a correlation function  $\langle f(t')f(t'') \rangle = C_f(t' - t'')$ . The power spectrum  $\tilde{C}_f(\omega)$  is defined as the Fourier transform of the correlation function. We consider two cases:

- (a) High temperature white noise  $\tilde{C}_f(\omega) = \nu$ .
- (b) Zero temperature noise  $\tilde{C}_f(\omega) = c|\omega|$ .

We define the angular velocity of the particle as  $v = \dot{\theta}$ , and its Cartesian coordinate as  $x = \sin(\theta)$ . In the absence of noise the dynamics is characterized by the damping time  $t_c = \mathbf{m}/\eta$ .

In items (3)-(5) you should assume a spreading scenario: the particle is initially ( $t = 0$ ) located at  $\theta \sim 0$ . The spreading during the transient period  $0 < t < t_c$  is assumed to be negligible. In item (6) assume that the particle had been launched in the far past ( $t = -\infty$ ): accordingly there is no preferred location on the ring.

1. Find the exact correlation function  $\langle v(t)v(0) \rangle$  in case (a).
2. Find the correlation function  $\langle v(t)v(0) \rangle$  for  $t \gg t_c$  in case (b).
3. Find the spreading  $S(t) \equiv \langle \theta(t)^2 \rangle$  for  $t \gg t_c$  in case (a).
4. Find the spreading  $S(t) \equiv \langle \theta(t)^2 \rangle$  for  $t \gg t_c$  in case (b).
5. Express  $\langle x(t)^2 \rangle$  for a spreading scenario given  $S(t)$ .
6. Express the correlation function  $\langle x(t)x(0) \rangle$  given  $S(t)$ .
7. Write the explicit long time expression for  $\langle x(t)x(0) \rangle$  in case (b), and deduce what is the critical value  $\eta_c$  above which a “phase transition” is expected in the response characteristics of the system.

Tips: For a Gaussian variable that has zero average  $\langle e^{i\varphi} \rangle = \exp[-(1/2)\langle \varphi^2 \rangle]$ .

The Fourier transform of  $|\omega|$  has zero area, with negative tails  $-1/(\pi t^2)$ .

If you fail to solve (6), assume that the answer is the same as in (5), and proceed to (7).

### ===== [Exercise 8481]

#### Mass on a spring

A balance for measuring weight consists of a sensitive spring which hangs from a fixed point. The spring constant is  $K$ . The balance is at temperature  $T$  and gravity acceleration is  $g$  in the  $x$  direction. A small mass  $\mathbf{m}$  hangs at the end of the spring. There is an option to apply an external force  $F(t)$ , to which  $x$  is conjugate or apply an external vector potential  $A(t)$ .

- (a) Find the partition function  $Z$ .
- (b) Find  $\langle x \rangle$  and  $\langle x^2 \rangle$  and  $\text{Var}(x)$ .
- (c) What is the minimal mass that can be meaningfully measured?
- (d) Write a Langevin equation for  $x(t)$ , with friction  $\gamma$ , and a random force  $f(t)$ .
- (e) Assuming  $\langle f(t)f(0) \rangle = C\delta(t)$ , find  $\text{Var}(x)$ , and deduce what is  $C$  by comparing with the canonical result.
- (f) Describe the external force  $F(t)$  by a scalar potential and demonstrate FDT.
- (g) Describe the external force  $F(t)$  by a vector potential and demonstrate FDT.

Note:  $\int \frac{d\omega}{(\omega^2 - \omega_0^2)^2 + \gamma^2 \omega^2} = \frac{\pi}{\gamma \omega_0^2}$ .

### ===== [Exercise 8483]

#### Millikan experiment

Consider a Millikan type experiment to measure the charge  $e$  of a particle with mass  $m$ . The particle is in an electric field  $E$  in the  $z$  direction, produced by a capacitor whose plates are distance  $d$  apart. The experiment is at temperature  $T$  and in a poor vacuum: the average time  $\tau_{col}$  between collisions of the air molecules and the charged particle is short. The field is opposite to the force of gravity, and the experiment attempts to find the exact field  $E^*$ , for which  $eE^* = mg$ , by monitoring the current.

1. Write a Langevin equation for the velocity  $v$  of the particle, with a friction coefficient  $\gamma$ .
2. For  $E = E^*$  find the time  $T_D$  after which the diffusion is observed.
3. For  $E \neq E^*$  the equation has a steady state solution  $\langle v_z \rangle = v_d$ . Find the drift velocity  $v_d$ .
4. Rewrite the equation in terms of  $v_d$  and find the long time limit of  $\langle z^2 \rangle$ . From the condition that the observation time is  $t \ll T_D$ , deduce a limit on the accuracy in measuring  $E^*$ .
5. If the air density is lowered such that the vacuum is improved, maintaining the same temperature  $T$ , would the accuracy be improved?

### ===== [Exercise 8484]

#### Galvanometer

A galvanometer can be regarded as a spring-held pointer that has mass  $M$ , natural oscillation frequency  $\omega_0$ , and a damping coefficient  $\gamma$ . The position  $x$  of the spring indicates the current  $I$ . It obeys the equation

$$\ddot{x} + \omega_0^2 x = -\gamma \dot{x} + A(t) + \alpha I$$

where  $A(t)$  represents an environmentally induced white noise that has a spectral intensity  $\nu$ , and  $\alpha$  is a coupling constant.

- (1) On the basis of the above Langevin equation write a  $d\omega$  integral for the variance  $\langle x^2 \rangle$  in the absence of current.
- (2) Based on canonical FDT considerations deduce what is the result of the integral that you wrote in the previous item.
- (3) For a constant  $I$ , what is the average position  $\langle x \rangle$  of the pointer?
- (4) Regarding  $I$  as a driving source, write what is the conjugate variable, what is the interaction term  $\mathcal{H}_{int}$  in the Hamiltonian, and what is the associate susceptibility  $\chi(\omega)$ .
- (5) Write an expression for the average rate of energy absorption  $\dot{W}$ , given that the current source has a frequency  $\omega$  and RMS amplitude  $I_0$ .
- (6) The expression for  $\dot{W}$  is formally the same as for a current source that is connected to a parallel RLC circuit. Write expressions for the effective values of  $R$  and  $L$  and  $C$ .

**Tip:** The equation of a parallel RLC circuit can be written as  $G(\omega)V_\omega = I_\omega$  where  $G(\omega)$  is a sum of three terms. Capacitors and inductors are described by  $I = C\dot{V}$  and by  $V = L\dot{I}$  respectively.

### ===== [Exercise 8490]

#### Stochastic rate equation

Consider  $N$  classical particles in a two site system. The two sites are subjected to a potential difference  $\varepsilon$ . The temperature of the system is  $T$ . Define  $n \in [-N, N]$  as the occupation difference. In items (3-6) assume that the thermalization process can be described by a stochastic rate equation

$$\frac{dn}{dt} = -\gamma n + A(t)$$

where  $A(t)$  is a noisy term that reflects the fluctuations of the potential difference. Assuming that it has an average value  $A_0$  and a power spectrum  $\phi(\omega)$ , it follows that  $n$  relaxes to an average value  $\langle n \rangle$ , with fluctuations that are

characterized by a power spectrum  $C(\omega)$ .

- (1) Write what is the interaction energy  $H_{\text{int}}$  of  $n$  with the field  $\varepsilon$ . Later you will have to be careful with the identification of the conjugate variables.
- (2) Using the canonical formalism find what are  $\langle n \rangle$  and  $\text{Var}(n)$ . Additionally provide approximations for small  $\varepsilon$ .
- (3) Determine what is  $A_0$  such that  $\langle n \rangle$  would be consistent with the canonical result. Assuming small  $\varepsilon$  deduce that  $A_0 \propto \varepsilon$ , and find the pre-factor.
- (4) What is the  $\chi(\omega)$  that characterizes the response of  $n$  to the applied potential in the linear-response regime? Assume that the dynamics is described by the stochastic rate equation; care to identify correctly the conjugate variables; and take into account your answer to item (3).
- (5) Deduce from the fluctuation-dissipation relation what is the power spectrum  $C(\omega)$ . Care to use the appropriate definition for  $\chi(\omega)$ , else the result will come out wrong.
- (6) Deduce what is the power spectrum  $\phi(\omega)$  that is required in order to reproduce  $C(\omega)$  from the stochastic rate equation.

**Advice:** In item (5) verify that your result is consistent with the answer to item (2). Likewise you can debug the numerical pre-factor in your answer to item (6). Care about factors of "2" in your answers. Failure to provide strictly correct pre-factors will be regarded as an essential error.

## ===== [Exercise 8492]

### Rate Equation with noise

Consider molecules A and B in a box. The molecules are subjected to weak electric field  $\mathcal{E}$  that modifies the binding energy of the B molecules such that  $\varepsilon_B = \varepsilon_B^{(0)} + \alpha\mathcal{E}$ , where  $\alpha$  is a constant. In the lack of electric field the molecules reach chemical equilibrium  $A \leftrightarrow B$ , such that their fractions are  $f_A = \langle N_A \rangle / N$  and  $f_B = \langle N_B \rangle / N$ . In item (c) it is assumed that the system can be described by the rate equation

$$\frac{dN_A}{dt} = k_B N_B - k_A N_A + A(t)$$

Where  $A(t)$  is a stochastic term with zero average and correlation function  $\langle A(t)A(t') \rangle = C\delta(t - t')$ . In item (e) assume that only  $k_B$  is affected by the weak electric field.

- (a) Express  $\langle N_A \rangle$ ,  $\langle N_B \rangle$  by  $N$ , and  $f_A$  and  $f_B$ , and  $\alpha\mathcal{E}$ .
- (b) Express  $\text{Var}(N_A)$  by  $N$ , and  $\langle N_A \rangle$  and  $\langle N_B \rangle$ .
- (c) Determine the constants  $k_A$  and  $k_B$  and  $D$  in the stochastic rate equation such that  $\langle N_A \rangle$  and  $\text{Var}(N_A)$  will agree with the canonical expectation of items (a) and (b).
- (d) Determine how  $k_B$  is modified by the weak electric field. In other words find the coefficient in  $\delta k_B \propto \mathcal{E}$ .
- (e) Find the generalized susceptibility  $\chi(\omega)$  that relates the variation  $\delta N_A$  to  $\mathcal{E}$ .
- (f) Find the power spectrum of  $\delta N_A$  in steady state.

## ===== [Exercise 9000]

### System-Bath

## ===== [Exercise 9010]

## Spin resonance

Spin Resonance: Consider a spin  $\frac{1}{2}$  particle with magnetic moment in a constant magnetic field  $B_0$  in the  $z$  direction and a perpendicular rotating magnetic field with frequency  $\omega$  and amplitude  $B_1$ ; the Hamiltonian is

$$\hat{H} = \hat{H}_0 + \frac{1}{2}\hbar\omega_1 [\sigma_x \cos(\omega t) + \sigma_y \sin(\omega t)]$$

where  $\hat{H}_0 = \frac{1}{2}\hbar\omega_0\sigma_z$ ,  $\frac{1}{2}\hbar\omega_0 = \mu B_0$ ,  $\frac{1}{2}\hbar\omega_1 = \mu B_1$  and  $\sigma_x, \sigma_y, \sigma_z$  are the Pauli matrices. The equilibrium density matrix is  $\hat{\rho}_{eq} = \exp(-\beta\hat{H}_0) / \text{Tr}[\exp(-\beta\hat{H}_0)]$ , so that the heat bath drives the system towards equilibrium with  $\hat{H}_0$  while the weak field  $B_1$  opposes this tendency. Assume that the time evolution of the density matrix  $\hat{\rho}(t)$  is determined by

$$d\hat{\rho}/dt = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] - \frac{\hat{\rho} - \hat{\rho}_{eq}}{\tau}$$

- (a) Show that this equation has a stationary solution of the form  $\delta\rho_{11} = -\delta\rho_{22} = a$ ,  $\delta\rho_{12} = \delta\rho_{21}^* = be^{-i\omega t}$  where  $\delta\hat{\rho} = \hat{\rho} - \hat{\rho}_{eq}$ .
- (b) The term  $-(\hat{\rho} - \hat{\rho}_{eq})/\tau$  represents  $(-i/\hbar) [\hat{H}_{bath}, \hat{\rho}]$  where  $\hat{H}_{bath}$  is the interaction Hamiltonian with a heat bath. Show that the power absorption is

$$\frac{d}{dt} \text{Tr}[(\hat{H} + \hat{H}_{bath})\hat{\rho}] = \text{Tr}\left[\frac{d\hat{H}}{dt}\hat{\rho}\right]$$

- (c) Determine  $b$  to first order in  $B_1$  (for which  $a = 0$  can be assumed), derive the power absorption and show that it has a maximum at  $\omega = \omega_0$ , i.e. a resonance phenomena. Show that  $(d/dt)\text{Tr}(\hat{\rho}\hat{H}) = 0$ , i.e. the absorption is dissipation into the heat bath.

## ===== [Exercise 9012]

### Equilibrium of a two level system

Consider  $N$  particles in a two level system,  $n_1$  particles in energy level  $E_1$  and  $n_2$  particles in energy level  $E_2$ . The system is in contact with a heat reservoir at temperature  $T$ . Energy can be transferred to the reservoir by a quantum emission in which  $n_2 \rightarrow n_2 - 1$ ,  $n_1 \rightarrow n_1 + 1$  and energy  $E_2 - E_1$  is released. [Note:  $n_1, n_2 \gg 1$ .]

- (a) Find the entropy change of the two level system as a result of a quantum emission.
- (b) Find the entropy change of the reservoir corresponding to (a).
- (c) Derive the ratio  $n_2/n_1$ ; do not assume a known temperature for the two level system. (Note: equilibrium is maintained by these type of energy transfers).