Statistical Mechanics II

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Lecture 6 Degenerated Quantum Gas

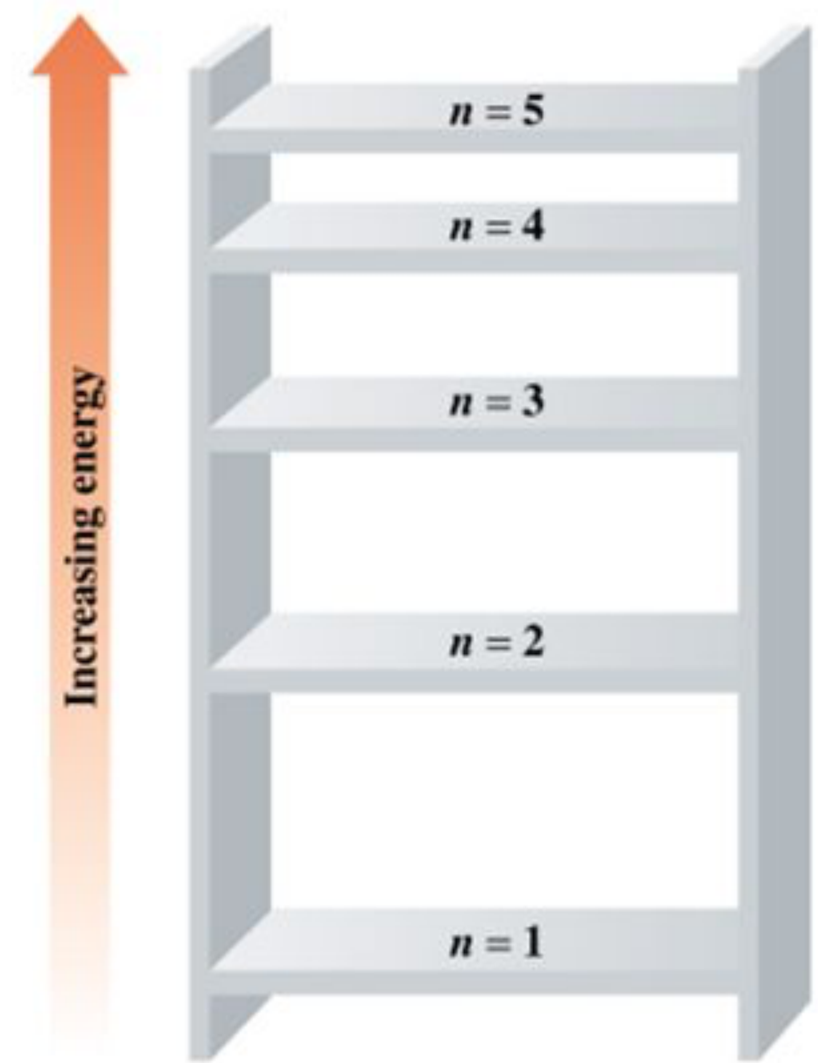
Outline

- General results for quantum gas
- High temperature expansion
- Degenerated Bose gas and superfluid
- Degenerated Fermi gas and conduction properties

The most probable distribution for quantum particles

energy level	$\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_\lambda, \cdots$
degeneracy	$g_1, g_2, \cdots, g_\lambda, \cdots$
occupation number	$n_1, n_2, \cdots, n_\lambda, \cdots$

$$\begin{cases} \sum_\lambda n_\lambda = N \\ \sum_\lambda \varepsilon_\lambda n_\lambda = E \end{cases}$$



What is the most probable distribution?

The most probable distribution for Bosons

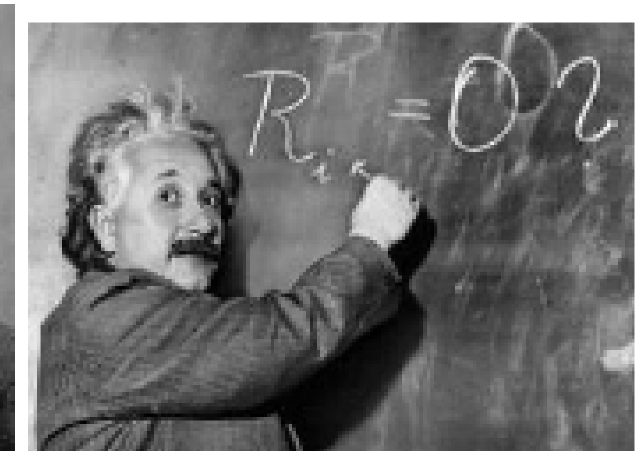
Total # of microstate for distribution $\{n_\lambda\}$ is

$$W_B(\{n_\lambda\}) = \prod_\lambda \begin{pmatrix} n_\lambda + g_\lambda - 1 \\ n_\lambda \end{pmatrix} = \prod_\lambda \frac{(n_\lambda + g_\lambda - 1)!}{n_\lambda!\,(g_\lambda - 1)!}$$

$$\begin{aligned}
\mathcal{L} &= \ln W_B - \alpha\left(\sum_\lambda n_\lambda - N\right) - \beta\left(\sum_\lambda \varepsilon_\lambda n_\lambda - E\right) \\
&= \sum_\lambda \ln \frac{(n_\lambda + g_\lambda - 1)!}{n_\lambda!\,(g_\lambda - 1)!} - \alpha\left(\sum_\lambda n_\lambda - N\right) - \beta\left(\sum_\lambda \varepsilon_\lambda n_\lambda - E\right) \\
&= (n_\lambda + g_\lambda - 1)\ln(n_\lambda + g_\lambda - 1) - n_\lambda \ln n_\lambda - (g_\lambda - 1)\ln(g_\lambda - 1) \\
&\quad - \alpha\left(\sum_\lambda n_\lambda - N\right) - \beta\left(\sum_\lambda \varepsilon_\lambda n_\lambda - E\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial n_\lambda} &= \ln(n_\lambda + g_\lambda - 1) + 1 - \ln n_\lambda - 1 - \alpha - \beta\varepsilon_\lambda \\
&= \ln\left(1 + \frac{g_\lambda - 1}{n_\lambda}\right) - \alpha - \beta\varepsilon_\lambda = 0
\end{aligned}$$

$$n_\lambda = \frac{g_\lambda - 1}{e^{\alpha + \beta\varepsilon_\lambda} - 1} \overset{g_\lambda \gg 1}{\approx} \frac{g_\lambda}{e^{\alpha + \beta\varepsilon_\lambda} - 1}$$



Bose-Einstein distribution

The most probable distribution for Fermions

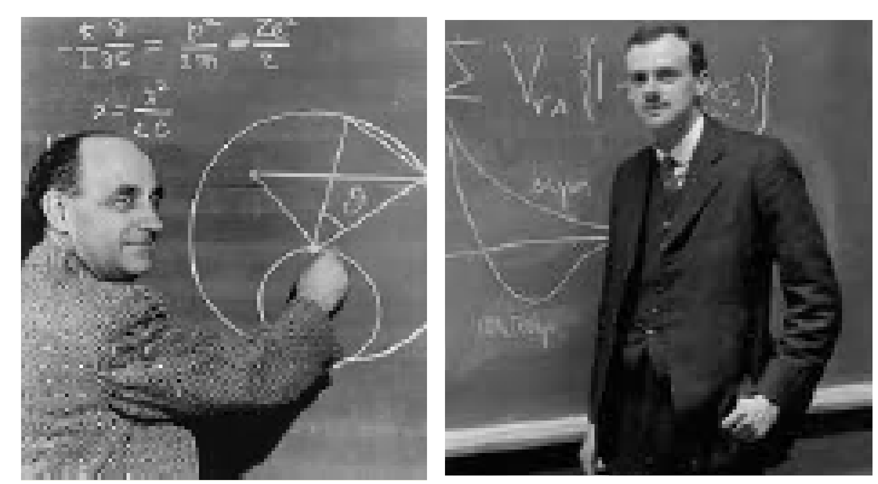
Total # of microstate for distribution $\{n_\lambda\}$ is

$$W_F(\{n_\lambda\}) = \prod_\lambda \begin{pmatrix} g_\lambda \\ n_\lambda \end{pmatrix} = \prod_\lambda \frac{g_\lambda!}{n_\lambda!\,(g_\lambda - n_\lambda)!}$$

$$\begin{aligned}
\mathcal{L} &= \ln W_F - \alpha\left(\sum_\lambda n_\lambda - N\right) - \beta\left(\sum_\lambda \varepsilon_\lambda n_\lambda - E\right) \\
&= \sum_\lambda \ln \frac{g_\lambda!}{n_\lambda!\,(g_\lambda - n_\lambda)!} - \alpha\left(\sum_\lambda n_\lambda - N\right) - \beta\left(\sum_\lambda \varepsilon_\lambda n_\lambda - E\right) \\
&= \sum_\lambda g_\lambda \ln g_\lambda - \sum_\lambda n_\lambda \ln n_\lambda - \sum_\lambda (g_\lambda - n_\lambda)\ln(g_\lambda - n_\lambda) \\
&\quad - \alpha\left(\sum_\lambda n_\lambda - N\right) - \beta\left(\sum_\lambda \varepsilon_\lambda n_\lambda - E\right)
\end{aligned}$$

$$0 = \frac{\partial \mathcal{L}}{\partial n_\lambda} = -\ln n_\lambda + 1 + \ln(g_\lambda - n_\lambda) + 1 - \alpha - \beta\,\varepsilon_\lambda$$

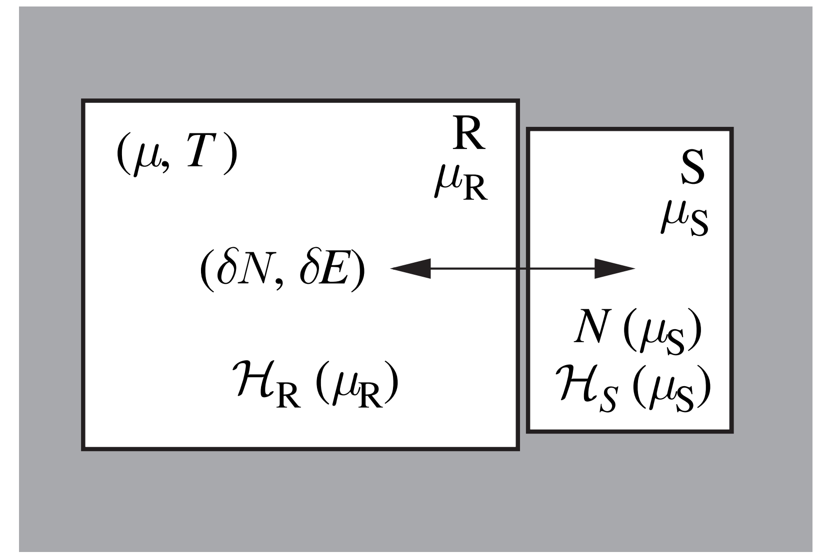
$$n_\lambda = \frac{g_\lambda}{e^{\alpha + \beta\,\varepsilon_\lambda} + 1}$$



Fermi-Dirac distribution

Grand Canonical ensemble
（for open systems）

巨正则系综适用于开放系统，就是与外界存在物质，热，功交还的系统



probability distribution of a microstate

$$p(\mu_S) = \exp\left[\boxed{\beta\mu N(\mu_S)} - \beta\mathcal{H}(\mu_S)\right]/\mathcal{Q}.$$

grand partition function 巨配分函数

热库提供热能和粒子源，
具有恒定温度和化学势

$$\mathcal{Q}(T, \mu, \mathbf{x}) = \sum_{\mu_S} e^{\beta\mu N(\mu_S) - \beta\mathcal{H}(\mu_S)}.$$

$$\mathcal{Q}(T, \mu, \mathbf{x}) = \sum_{N=0}^{\infty} e^{\beta\mu N} \sum_{(\mu_S|N)} e^{-\beta\mathcal{H}_N(\mu_S)}.$$

probability of finding system with particle number N is

$$p(N) = \frac{e^{\beta\mu N} Z(T, N, \mathbf{x})}{\mathcal{Q}(T, \mu, \mathbf{x})}.$$

系统存在粒子数为N的状态的概率

Entropy and grand potential

Entropy from information theory 熵

$$S = -k_B \sum_{\mu_s} p(\mu_s) \ln p(\mu_s) = -k_B \sum_{\mu_s} \frac{e^{\beta\mu N(\mu_s) - \beta H(\mu_s)}}{Q} \ln\left(\frac{e^{\beta\mu N(\mu_s) - \beta H(\mu_s)}}{Q}\right)$$

$$= -k_B \sum_{\mu} \frac{e^{\beta\mu N(\mu_s) - \beta H(\mu_s)}}{Q} \left[\beta\mu N(\mu_s) - \beta H(\mu_s) - \ln Q\right]$$

$$= -\sum_{\mu} \frac{p(\mu_S)\left[\mu N(\mu_s) - H(\mu_s)\right]}{T} + \sum_{\mu} p(\mu) \ln Q = \frac{\langle H \rangle - \mu \langle N \rangle}{T} + k_B \ln Q$$

$$-k_B \ln Q = \langle H \rangle - \mu \langle N \rangle - TS = E - \mu N - TS = \mathcal{G}(T, \mu, V)$$ 巨势

$$\mathcal{G}(T, \mu, \mathbf{x}) = E - TS - \mu N = -k_B T \ln \mathcal{Q}$$

$$\mathrm{d}\mathcal{G} = -S\mathrm{d}T - N\mathrm{d}\mu + \mathbf{J} \cdot \mathrm{d}\mathbf{x},$$ 热力学关系

entropy熵

$$-S = \left.\frac{\partial \mathcal{G}}{\partial T}\right|_{\mu, \mathbf{x}}$$

particle number 粒子数

$$N = -\left.\frac{\partial \mathcal{G}}{\partial \mu}\right|_{T, \mathbf{x}}$$

generalized force 广义力

$$J_i = \left.\frac{\partial \mathcal{G}}{\partial x_i}\right|_{T, \mu}$$

巨势能够与巨配分函数直接联系起来所以我们先计算了巨配分函数，再算巨势，在然后由热力学关系算其他热力学量

GCE：ideal gas in an open systems

$$\mathcal{Q}(T,\mu,V)=\sum_{N=0}^{\infty}\mathrm{e}^{\beta\mu N}\frac{1}{N!}\int\left(\prod_{i=1}^{N}\frac{\mathrm{d}^3\vec{q}_i\mathrm{d}^3\vec{p}_i}{h^3}\right)\exp\left[-\beta\sum_i\frac{p_i^2}{2m}\right]$$

$$=\sum_{N=0}^{\infty}\frac{\mathrm{e}^{\beta\mu N}}{N!}\left(\frac{V}{\lambda^3}\right)^N \qquad \left(\text{with } \lambda=\frac{h}{\sqrt{2\pi m k_B T}}\right)$$

$$=\exp\left[\mathrm{e}^{\beta\mu}\frac{V}{\lambda^3}\right],$$ 巨配分函数

$$\mathcal{G}(T,\mu,V)=-k_BT\ln\mathcal{Q}=-k_BT\mathrm{e}^{\beta\mu}\frac{V}{\lambda^3}.$$ 巨势

$$\mathrm{d}\mathcal{G}=-SdT-Nd\mu-PdV$$ 热力学关系

$$P=-\frac{\mathcal{G}}{V}=-\left.\frac{\partial\mathcal{G}}{\partial V}\right|_{\mu,T}=k_BT\frac{\mathrm{e}^{\beta\mu}}{\lambda^3}.$$ 压强，由此得状态方程

$$N=-\left.\frac{\partial\mathcal{G}}{\partial\mu}\right|_{T,V}=\frac{\mathrm{e}^{\beta\mu}V}{\lambda^3}.$$ 粒子数

Deriving one particle distribution function from ensemble theory

In canonical ensemble

$$Z_N = \mathrm{tr}\left(\mathrm{e}^{-\beta\mathcal{H}}\right) = \sum_{\{\vec{k}_\alpha\}}' \exp\left[-\beta\sum_{\alpha=1}^{N}\mathcal{E}(\vec{k}_\alpha)\right] = \sum_{\{n_{\vec{k}}\}}' \exp\left[-\beta\sum_{\vec{k}}\mathcal{E}(\vec{k})n(\vec{k})\right]$$

$$\textstyle\sum_{\vec{k}} n_{\vec{k}} = N$$

In grand canonical ensemble

$$\mathcal{Q}_\eta(T,\mu) = \sum_{N=0}^{\infty} \mathrm{e}^{\beta\mu N} \sum_{\{n_{\vec{k}}\}}' \exp\left[-\beta\sum_{\vec{k}}\mathcal{E}(\vec{k})n_{\vec{k}}\right] = \sum_{\{n_{\vec{k}}\}}^{\eta} \prod_{\vec{k}} \exp\left[-\beta\big(\mathcal{E}(\vec{k})-\mu\big)n_{\vec{k}}\right]$$

Deriving one particle distribution function from ensemble theory

For *fermions*, $n_{\vec{k}} = 0$ or 1, and

$$\mathcal{Q}_{-} = \prod_{\vec{k}} \left[1 + \exp\left(\beta\mu - \beta\mathcal{E}(\vec{k})\right) \right]$$

For *bosons*, $n_{\vec{k}} = 0, 1, 2, \cdots$

$$\mathcal{Q}_{+} = \prod_{\vec{k}} \left[1 - \exp\left(\beta\mu - \beta\mathcal{E}(\vec{k})\right) \right]^{-1}$$

$$\ln \mathcal{Q}_{\eta} = -\eta \sum_{\vec{k}} \ln \left[1 - \eta \exp\left(\beta\mu - \beta\mathcal{E}(\vec{k})\right) \right]$$

$$\langle n_{\vec{k}} \rangle_{\eta} = -\frac{\partial \ln \mathcal{Q}_{\eta}}{\partial\left(\beta\mathcal{E}(\vec{k})\right)} = \frac{1}{z^{-1} e^{\beta\mathcal{E}(\vec{k})} - \eta}$$

Occupation number

$$z \equiv e^{\beta\mu}$$

$$n_{\vec{k}} = \frac{1}{e^{\beta\left(\varepsilon\left(\vec{k}\right)-\mu\right)} \mp 1} \qquad \varepsilon\left(\vec{k}\right) = \frac{\hbar^2 k^2}{2m}$$

Occupation number

$$N = \sum_{\vec{k}} n_{\vec{k}} = \frac{V}{(2\pi)^3}\int n_{\vec{k}} d^3\vec{k} = \frac{V}{(2\pi)^3}\int d^3\vec{k}\frac{1}{e^{\beta\left(\varepsilon\left(\vec{k}\right)-\mu\right)} \mp 1}$$

$$n = \frac{N}{V} = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{e^{\beta(\varepsilon_k-\mu)\mp 1}}$$

$$E = \sum_{\vec{k}} \varepsilon_{\vec{k}} n_{\vec{k}}$$

$$\varepsilon = \frac{E}{V} = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{\varepsilon\left(\vec{k}\right)}{e^{\beta(\varepsilon_k - \mu)} \mp 1} = \frac{V}{(2\pi)^3} \int d^3\vec{k} \frac{\varepsilon\left(\vec{k}\right)}{e^{\beta\left(\varepsilon\left(\vec{k}\right) - \mu\right)} \mp 1}$$

$$P = \frac{F}{A} = \frac{\overline{\Delta(mv_x)/\Delta t}}{A} = \frac{\overline{2mv_x}}{2LA/v_x} = \frac{\overline{mv_x^2}}{V} = \frac{1}{3}\overline{mv^2}\frac{1}{V} = \frac{2}{3}\overline{\frac{1}{2}mv^2}\frac{1}{V}$$

$$= \frac{1}{2}\int 2mv_x^2 f(\vec{q}, p, t)\, d^3p = \frac{2}{3}\int \frac{1}{2}mv^2 f(\vec{q}, p, t)\, d^3p$$

$$= \frac{2}{3}\int \frac{d^3\vec{k}}{(2\pi)^3} \varepsilon_{\vec{k}} f(\vec{p}, t)\, \frac{1}{V} = \frac{2}{3}\frac{E}{V} = \frac{2}{3}\varepsilon$$

$$n = \frac{N}{V} = g \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{z \exp\left(\frac{\beta \hbar^2 k^2}{2m}\right) \mp 1}$$

$$\varepsilon = \frac{E}{V} = g \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{z^{-1} \exp\left(\frac{\beta \hbar^2 k^2}{2m}\right) \mp 1}$$

$$P = \frac{2}{3}\varepsilon$$

$$x \equiv \frac{\beta\hbar^2 k^2}{2m} \Rightarrow \frac{\hbar^2 k^2}{2m} = \frac{x}{\beta} \Rightarrow k = \sqrt{\frac{2mx}{\beta\hbar^2}} \Rightarrow dk = \frac{1}{2}\sqrt{\frac{2m}{\beta\hbar^2}}\frac{1}{\sqrt{x}}$$

$$\lambda = \frac{h}{\sqrt{2\pi m k_B T}} \qquad d^3k = 4\pi k^2 dk$$

$$f^{\pm}_{m}(z) = \frac{1}{(m-1)!}\int_0^\infty \frac{x^{m-1}dx}{z^{-1}e^x \mp 1}$$

$$\begin{cases} n = \frac{g}{\lambda^3}\frac{2}{\sqrt{\pi}}\int_0^\infty \frac{x^{\frac{1}{2}}dx}{z^{-1}e^x \mp 1} \\ \beta\varepsilon = \frac{g}{\lambda^3}\frac{2}{\sqrt{\pi}}\int_0^\infty \frac{x^{\frac{3}{2}}dx}{z^{-1}e^x \mp 1} \end{cases} \longrightarrow \begin{cases} n = \frac{g}{\lambda^3} f^{\pm}_{\frac{3}{2}}(z) \\ \beta\varepsilon = \frac{3g}{2\lambda^3} f^{\pm}_{\frac{5}{2}}(z) \\ \beta P = \frac{2}{3}\varepsilon = \frac{g}{\lambda^3} f^{\pm}_{\frac{5}{2}}(z) \end{cases}$$

High temperature expansion (classical limit)

$$f_m^{\pm}(z) = \frac{1}{(m-1)!}\int_0^\infty \frac{x^{m-1}dx}{z^{-1}e^x \mp 1}$$

$$= \frac{1}{(m-1)!}\int_0^\infty \frac{x^{m-1}ze^{-x}dx}{1 \mp ze^{-x}}$$

$$= \frac{1}{(m-1)!}\int_0^\infty dx x^{m-1}ze^{-x}\sum_{n=0}^{\infty}\left[\pm ze^{-x}\right]^n$$

$$= \frac{1}{(m-1)!}\sum_{n=0}^{\infty}\int_0^\infty dx x^{m-1}(\pm 1)^n z^{(1+n)}e^{-(1+n)x}$$

$$= \sum_{n=0}^{\infty}(\pm 1)^n z^{(1+n)}\frac{1}{(m-1)!}\int_0^\infty dx x^{m-1}e^{-(1+n)x}$$

$$= \sum_{n=0}^{\infty}(\pm 1)^n \frac{z^{(1+n)}}{(1+n)^m}\left[\frac{1}{(m-1)!}\int_0^\infty dx x^{m-1}e^{-x}\right]$$

$$= \sum_{n=0}^{\infty}(\pm 1)^n \frac{z^{(1+n)}}{(1+n)^m}$$

$$= z \pm \frac{z^2}{2^m} + \frac{z^3}{3^m} \pm \frac{z^4}{4^m} + \ldots$$

$$\frac{n\lambda^3}{g} = f^{\pm}_{\frac{3}{2}}(z) \approx z \pm \frac{z^2}{2^{(3/2)}} + \frac{z^3}{3^{(3/2)}} + \ldots$$

$$\frac{\beta P\lambda^3}{g} = f^{\pm}_{\frac{5}{2}}(z) \approx z \pm \frac{z^2}{2^{(5/2)}} + \frac{z^3}{3^{(5/2)}} + \ldots$$

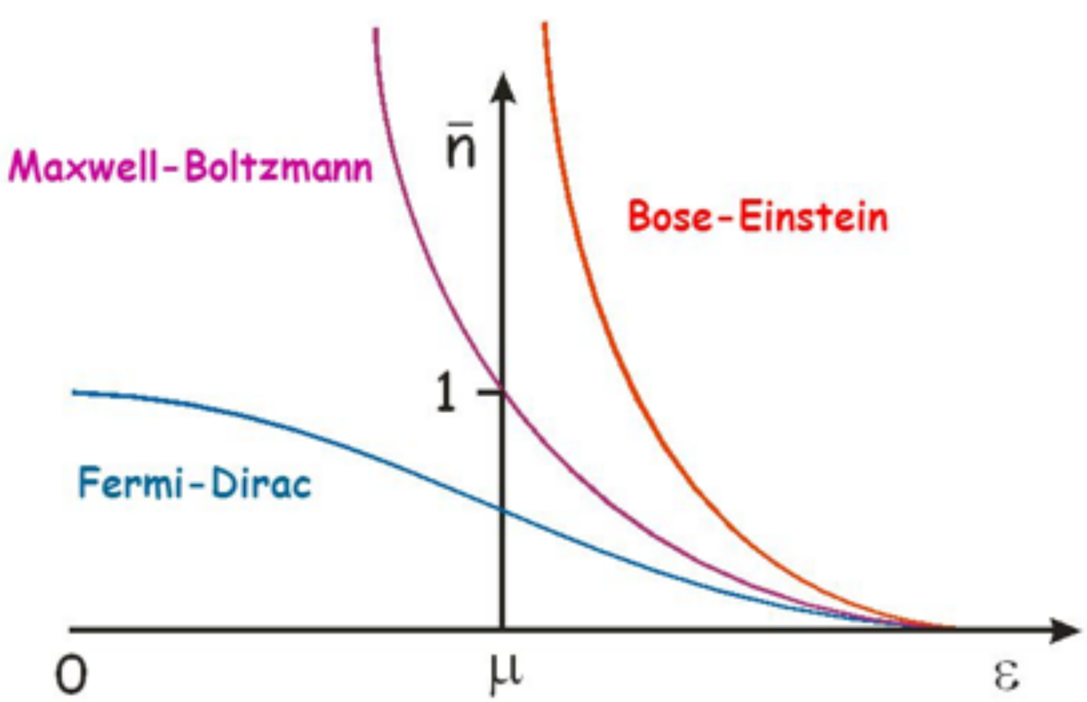
$$z \approx \frac{n\lambda^3}{g} \mp \frac{z^2}{2^{(3/2)}} \approx \frac{n\lambda^3}{g} \mp \frac{1}{2^{(3/2)}}\left(\frac{n\lambda^3}{g}\right)^2 + \ldots$$

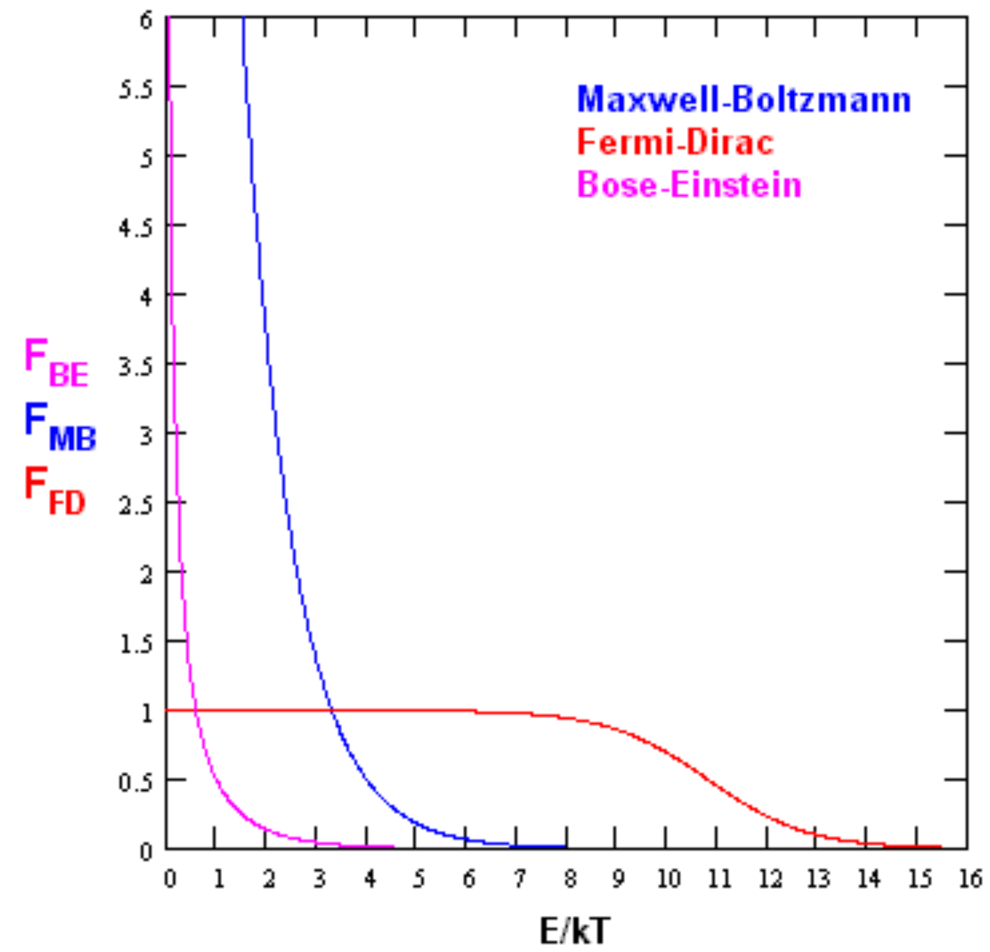
$$\frac{\beta P\lambda^3}{g} = f^{\pm}_{\frac{5}{2}}(z) \approx z \pm \frac{z^2}{2^{(5/2)}} + \ldots$$

$$\approx \left(\frac{n\lambda^3}{g} \mp \frac{1}{2^{(3/2)}}\left(\frac{n\lambda^3}{g}\right)^2\right) \pm \frac{1}{2^{(5/2)}}\left(\frac{n\lambda^3}{g} \mp \frac{1}{2^{(3/2)}}\left(\frac{n\lambda^3}{g}\right)^2\right)^2 + \ldots$$

$$\approx \frac{n\lambda^3}{g} \mp \frac{1}{2^{(5/2)}}\left(\frac{n\lambda^3}{g}\right)^2 + \ldots$$

$$P \approx nk_BT\left[1 \mp \frac{1}{2^{(5/2)}}\left(\frac{n\lambda^3}{g}\right) + \ldots\right]$$





Degenerated quantum gas

$$P \approx nk_B T \left[1 \mp \frac{1}{2^{(5/2)}}\left(\frac{n\lambda^3}{g}\right) + ...\right]$$

Work for $\left(\frac{n\lambda^3}{g}\right) \ll 1$ i.e. $e^{\alpha} \gg 1$

Work for 1) high temperature 2) low density

The opposite is called “degenerated” quantum gas

At low temperature, $\mathcal{Z} \to 1$ for Bosonic system

$\mathcal{Z} \to \infty$ for Fermionic system

Bosons and fermions behave differently.

Properties of $f^{+}_{m}(z)$

$$f^{+}_{m}(z) = \frac{1}{(m-1)!}\int\limits_0^\infty \frac{x^{m-1}dx}{z^{-1}e^{x}-1}$$

$$\frac{d}{dz} f\left(z^{-1}e^{x}\right) = f'\,\frac{d}{dz}\left(z^{-1}e^{x}\right) = f'\,e^{x}\left(-\frac{1}{z^{2}}\right) = -\frac{1}{z}\frac{d}{dx} f\left(z^{-1}e^{x}\right)$$

$$\frac{d}{dz} f^{-}_{m}(z) = \int\limits_0^\infty \frac{dx}{(m-1)!}\frac{d}{dz}\frac{x^{m-1}}{z^{-1}e^{x}-1}$$

$$= -\frac{1}{z}\int\limits_0^\infty dx\,\frac{x^{m-1}}{(m-1)!}\frac{d}{dx}\left(\frac{1}{z^{-1}e^{x}-1}\right)$$

$$= -\frac{1}{z}\left[\frac{x^{m-1}}{(m-1)!}\frac{1}{z^{-1}e^{x}-1}\Bigg|_0^\infty - \int\limits_0^\infty \frac{1}{z^{-1}e^{x}-1}\frac{d}{dx}\left(\frac{x^{m-1}}{(m-1)!}\right)\right]$$

$$= \frac{1}{z}\int\limits_0^\infty dx\,\frac{1}{z^{-1}e^{x}-1}\frac{x^{m-2}}{(m-2)!} = \frac{1}{z} f^{+}_{m-1}(z)$$

The occupation number is always finite, $$n_{\vec{k}} = \frac{1}{e^{\beta(\varepsilon(\vec{k})-\mu)} \mp 1}$$

$\varepsilon_{\mathbf{k}} - \mu$ is always positive

the ground state energy is zero, so $\mu < 0$. $0 \leqslant e^{\beta\mu} \leqslant 1$,

$$0 \leqslant \mathcal{Z} \leqslant 1$$

$$\xi_m \equiv f_m^+(1) = \frac{1}{(m-1)!} \int_0^\infty \frac{x^{m-1}}{e^x - 1} dx$$

It has a pole at $x = 0$, when $x \to 0$

$$\int dx \frac{x^{m-1}}{e^x - 1} \simeq \int dx \frac{x^{m-1}}{x} \sim \int x^{m-2} dx \sim \frac{1}{x^{m-1}}$$

It is finite when $m > 1$, and infinite when $m \leqslant 1$, so if $m > 1$

$$\frac{d}{dZ} f_m^+(Z) = \frac{1}{Z} f_{m-1}^+(Z) \geqslant 0$$

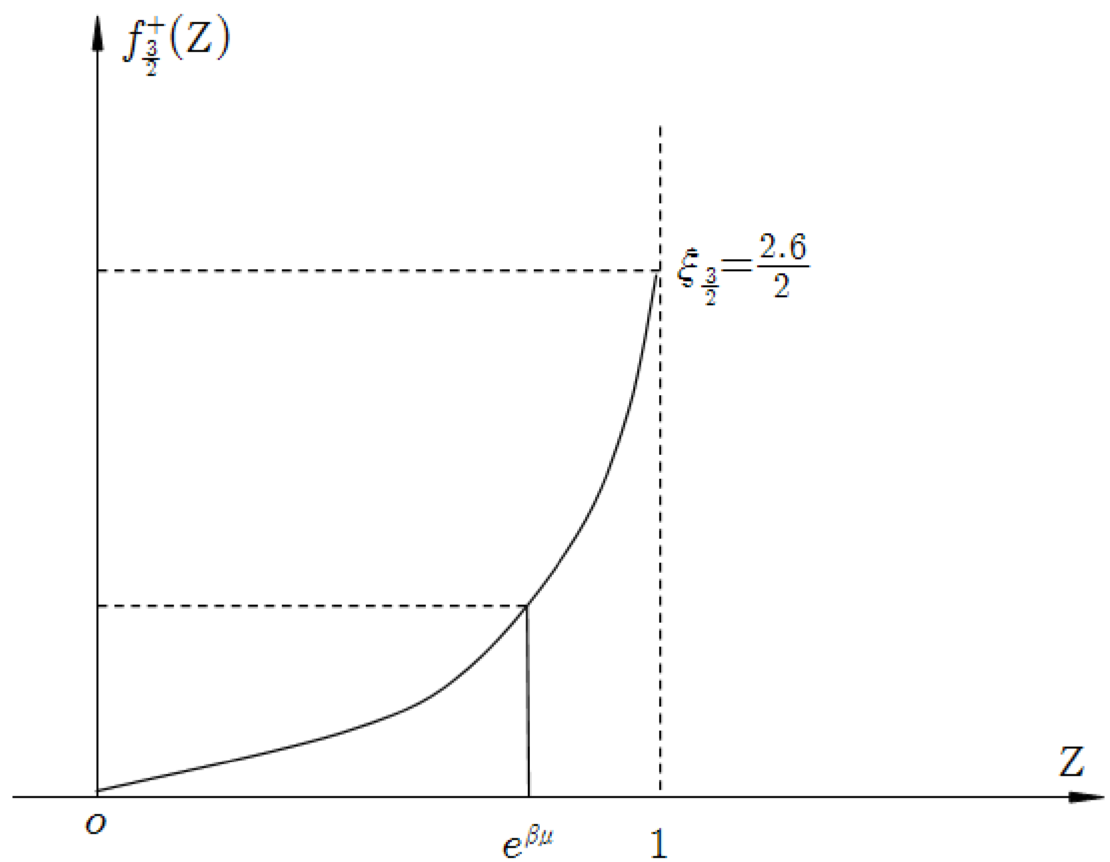
$$\xi_m \equiv f_m^+(1) = \text{finite value} \qquad m > 1.$$

$$\frac{d}{dZ} f_{\frac{3}{2}}^+ \sim \frac{1}{Z} f_{\frac{1}{2}}^+ \to \infty$$

$$\xi_{\frac{3}{2}} = \frac{2.6}{2}$$

$$n = \frac{g}{\lambda^3} f_{\frac{3}{2}}^+(Z)$$

$$\frac{n\lambda^3}{g} = f_{\frac{3}{2}}^+(Z) = f_{\frac{3}{2}}^+\left(e^{\beta\mu}\right) \leqslant \xi_{\frac{3}{2}}$$

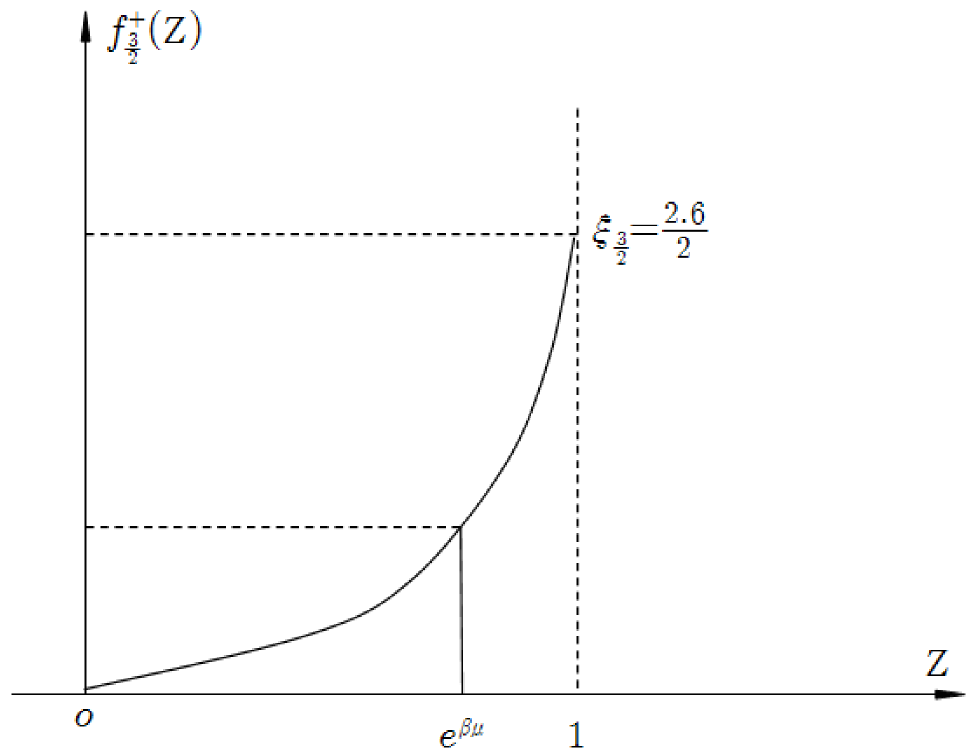


Bose-Einstein condensation

$$\frac{n\lambda^3}{g} = f^+_{\frac{3}{2}}(z) = f^+_{\frac{3}{2}}\left(e^{\beta\mu}\right) \leqslant \xi_{\frac{3}{2}}$$

n and μ are determining each other.

$$\boxed{\frac{n\lambda^3}{g} = \frac{n}{g}\left(\frac{h}{\sqrt{2\pi m k_B T}}\right)^3 \leqslant \xi_{\frac{3}{2}}}$$



The particle number has a maximum value?
Why!

What is missing?

$$\sum_{\vec{k}} \longrightarrow \int \frac{d^3\vec{k}}{(2\pi)^3} = \int \frac{k^2}{2\pi^2} dk$$

the ground state $k = 0$ is missing when we replace the summation with integral.

$$\langle n_{\vec{k}=0} \rangle = \frac{1}{e^{\beta(\varepsilon_{\vec{k}=0}-\mu)} - 1} = \frac{1}{\mathcal{Z}^{-1} e^0 - 1} = \frac{1}{\mathcal{Z}^{-1} - 1}$$

$$\lim_{T \to 0} \frac{1}{\mathcal{Z}^{-1} - 1} \longrightarrow \infty$$

here we have $\mu \to 0$ and $\mathcal{Z} \to 1$ when $T \to 0$.

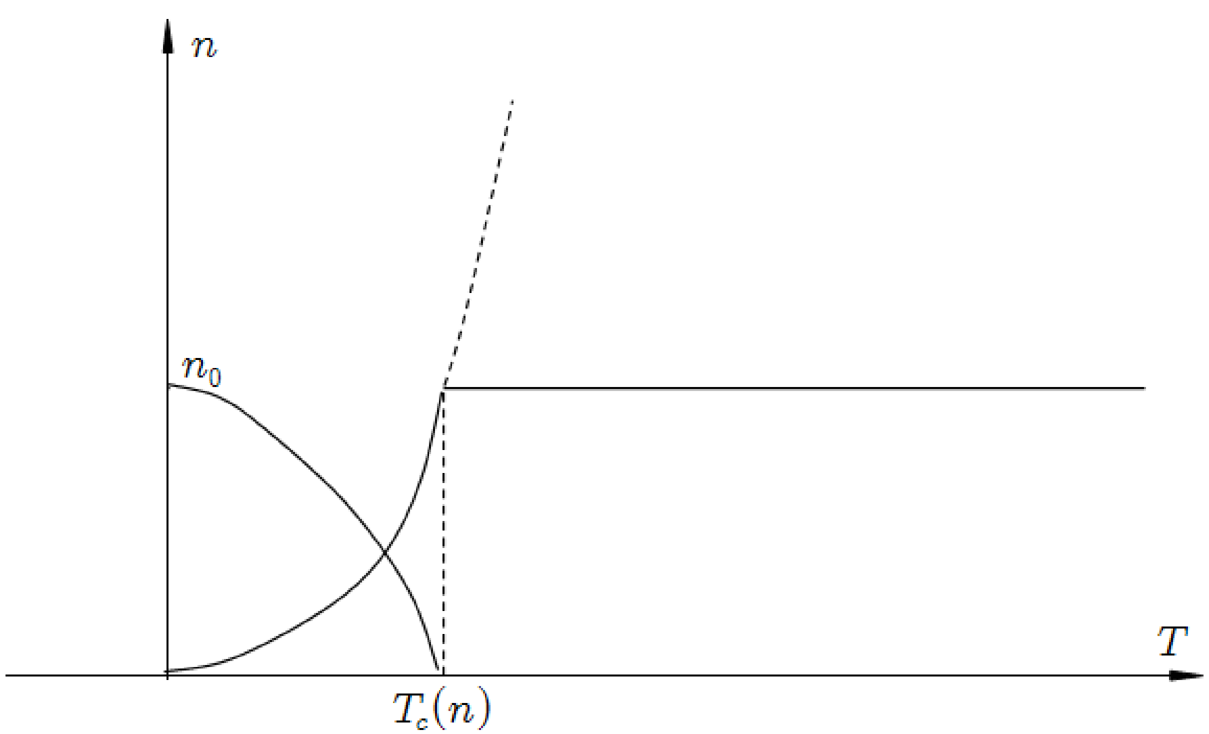
$$n = \frac{1}{\mathcal{Z}^{-1} - 1} + n^* = n_0 + n^*$$

$$n^* = \frac{g}{\lambda^3} \xi_{\frac{3}{2}} = g \left(\frac{\sqrt{2\pi m k_B T}}{h} \right)^3 \xi_{\frac{3}{2}}$$

How BEC occur?

$$n = \frac{1}{z^{-1}-1} + n^* = n_0 + n^*$$

$$n^* = \frac{g}{\lambda^3}\xi_{\frac{3}{2}} = g\left(\frac{\sqrt{2\pi m k_B T}}{h}\right)^3 \xi_{\frac{3}{2}}$$



1) high density for given T

n^* is the upper limit of the occupation number of the excited states

2) low temperature for given density

$$T_c(n) \equiv \frac{h}{2\pi m k_B}\left(\frac{n}{\xi_{\frac{3}{2}} g}\right)^{2/3}$$

Properties of BEC

When $T < T_c(n)$, $\mu = 0$, $z = 1$, most calculation can be simplified.

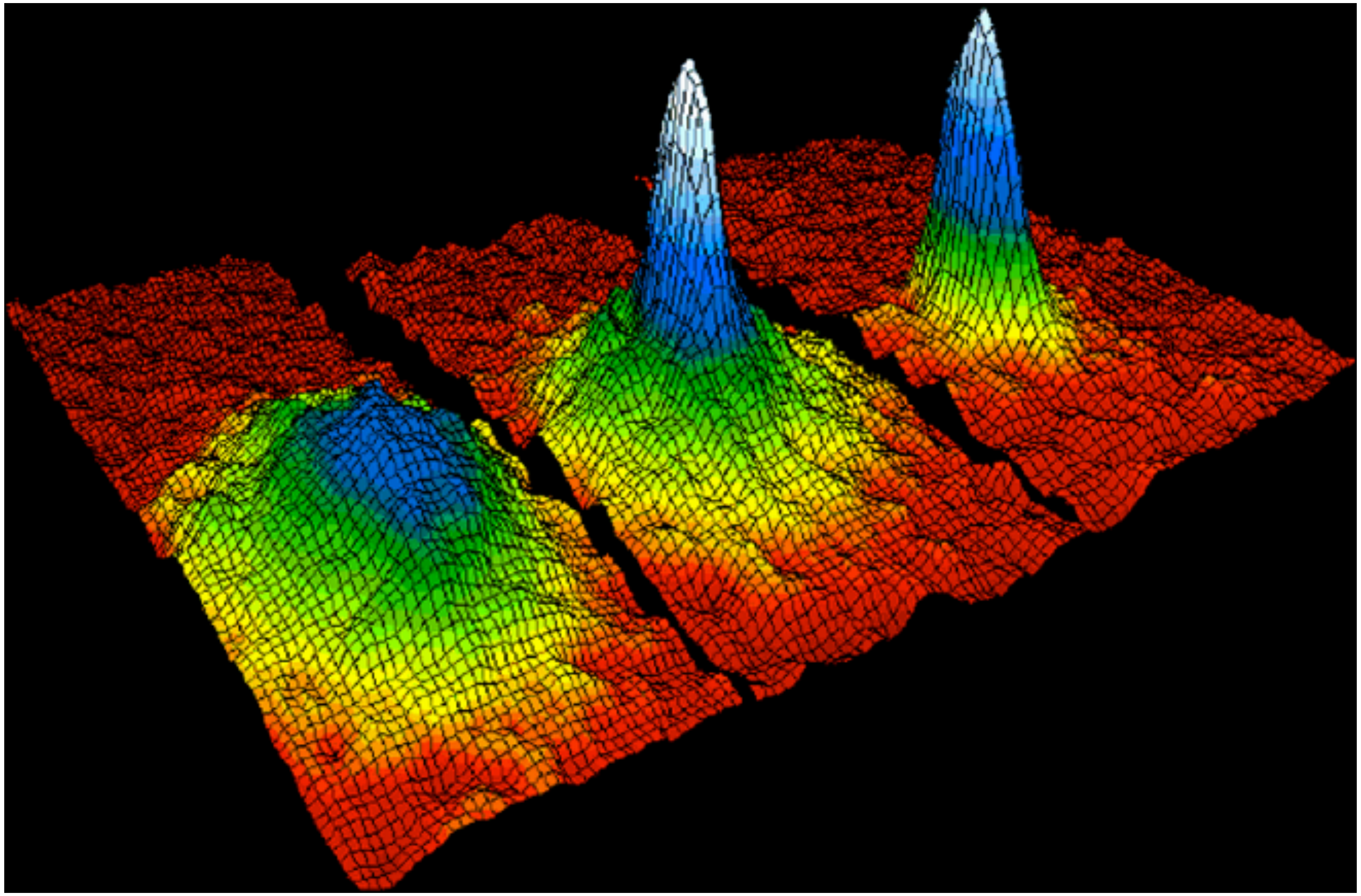
$$\begin{aligned}\beta P &= \frac{g}{\lambda^3} f^{+}_{\frac{5}{2}}(1) = \frac{g}{\lambda^3}\xi_{\frac{5}{2}} \approx 1.341\frac{g}{\lambda^3} \\ &= 1.341 g\left(\frac{\sqrt{2\pi m k_B T}}{h}\right)^3 \sim T^{\frac{3}{2}}\end{aligned}$$

$P \propto T^{\frac{3}{2}} T = T^{\frac{5}{2}}$ which is independent of n (not like $P = n k_B T$ in classical gas).

$$\varepsilon = \frac{2}{3}\frac{g}{\lambda^3} f^{+}_{\frac{5}{2}}(1) = \frac{2}{3}\frac{g}{\lambda^3}\xi_{\frac{5}{2}} \sim T^{\frac{3}{2}}$$

energy and pressure are independent of density!
Why?

Properties of BEC



https://en.wikipedia.org/wiki/Bose–Einstein_condensate

The Nobel Prize in Physics 1997

Steven Chu

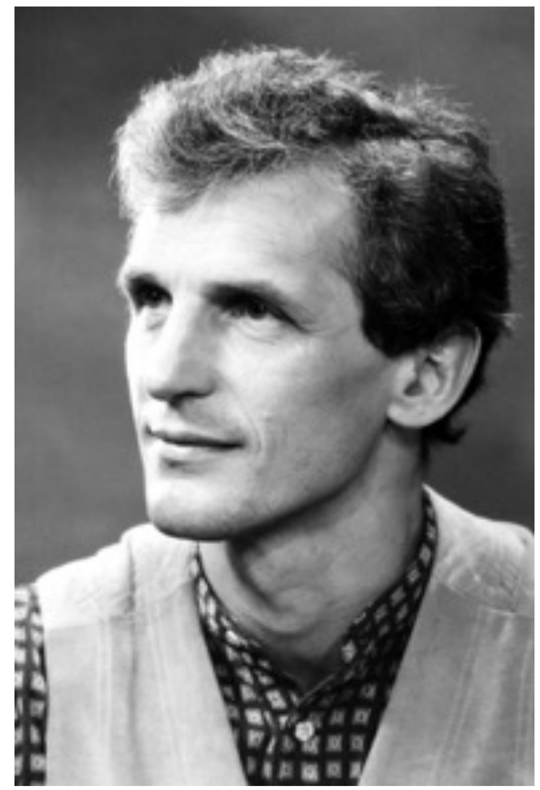
Claude
Cohen-Tannoudji

William
D. Phillips

“for development of methods to cool and trap atoms with laser light.”

The Nobel Prize in Physics 2001

Eric A. Cornell



Wolfgang Ketterle

Carl E. Wieman

"for the achievement of Bose-Einstein condensation in dilute gases of alkali atoms, and for early fundamental studies of the properties of the condensates."

Properties of degenerated Bose gas

When the temperature is above the BEC temperature $T_c(n)$, $\mathcal{Z} \neq 1$, we have to determine μ from the particle number N. In this case

$$P = \frac{2}{3}\varepsilon = \frac{2}{3}\frac{E}{V}$$

$$E = \frac{3}{2}PV = \frac{3}{2}V\frac{g}{\lambda^3}k_B T f^+_{\frac{5}{2}} \propto T^{\frac{5}{2}} f^+_{\frac{5}{2}}(\mathcal{Z}), \quad \mathcal{Z} = e^{\beta\mu}$$

$$\begin{aligned} C_{V,N} &= \left.\frac{dE}{dT}\right|_{V,N} = \frac{3}{2}V\frac{g}{\lambda^3}\left[\frac{\frac{5}{2}k_B T}{T}f^+_{\frac{5}{2}}(\mathcal{Z}) + k_B T\frac{d}{d\mathcal{Z}}f^+_{\frac{5}{2}}(\mathcal{Z})\left.\frac{d\mathcal{Z}}{dT}\right|_{V,N}\right] \\ &= \frac{3}{2}V\frac{g k_B T}{\lambda^3}\left[\frac{5}{2T}f^+_{\frac{5}{2}}(\mathcal{Z}) + \frac{1}{\mathcal{Z}}f^+_{\frac{3}{2}}(\mathcal{Z})\left.\frac{d\mathcal{Z}}{dT}\right|_{V,N}\right] \end{aligned}$$

Properties of degenerated Bose gas

$\left.\frac{dz}{dT}\right|_{V,N}$ can be obtain by particle number conservation.

$$\left.\frac{dN}{dT}\right|_{V} = 0 = \frac{d}{dT}\left[\frac{1}{V}\frac{g}{\lambda^3} f^{+}_{\frac{3}{2}}(z)\right]$$

$$= \frac{1}{V}\frac{g}{\lambda^3}\left[\frac{3}{2T} f^{+}_{\frac{3}{2}}(z) + \frac{1}{z} f^{+}_{\frac{1}{2}}(z)\frac{dz}{dT}\right]$$

$$\frac{dz}{dT}\frac{T}{z} = -\frac{3}{2}\frac{f^{+}_{\frac{3}{2}}(z)}{f^{+}_{\frac{1}{2}}(z)}$$

$$C_{V,N} = \frac{3}{2} V \frac{g k_B}{\lambda^3}\left[\frac{5}{2} f^{+}_{\frac{5}{2}}(z) - \frac{3}{2}\frac{f^{+2}_{\frac{3}{2}}(z)}{f^{+}_{\frac{1}{2}}(z)}\right]$$

Insert the equation of z from high temperature expansion,

Properties of degenerated Bose gas

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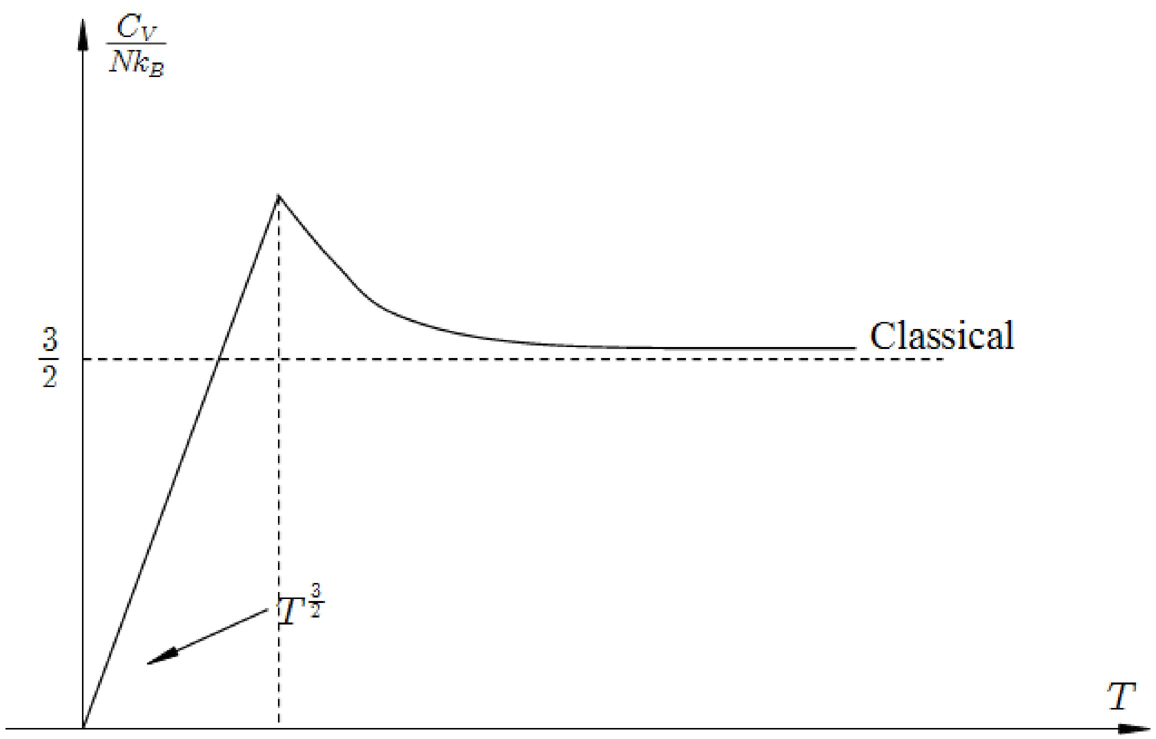


$$C_V/Nk_B = 3/2[1 + n\lambda^3/2^{7/2} + \cdots]$$

At low temperatures, $z = 1$

$$\frac{C_V}{Nk_B} = \frac{15}{4}\frac{g}{n\lambda^3}\zeta_{5/2} = \frac{15}{4}\frac{\zeta_{5/2}}{\zeta_{3/2}}\left(\frac{T}{T_c}\right)^{3/2}$$

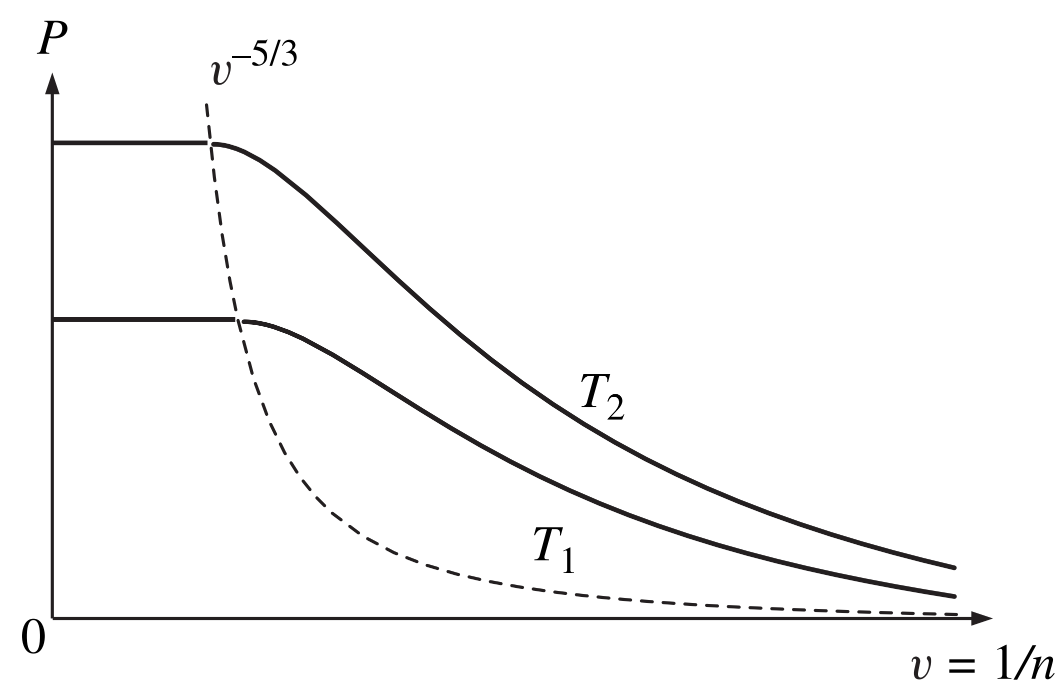
How to understand the behavior of $C_{V,N} \sim T^{\frac{3}{2}}$ as $T \to 0$?

Properties of degenerated Bose gas

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathsf{f}\left(z^{-1}e^{x}\right) = \mathsf{f}'\frac{\mathrm{d}}{\mathrm{d}z}\left(z^{-1}e^{x}\right) = \mathsf{f}'e^{x}\left(-\frac{1}{z^{2}}\right) = -\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{f}\left(z^{-1}e^{x}\right)$$

$$\begin{cases} \dfrac{\mathrm{d}P}{\mathrm{d}z} = \dfrac{gk_BT}{\lambda^3}\dfrac{1}{z}f^{+}_{3/2}(z) \\ \dfrac{\mathrm{d}n}{\mathrm{d}z} = \dfrac{g}{\lambda^3}\dfrac{1}{z}f^{+}_{1/2}(z) \end{cases}.$$

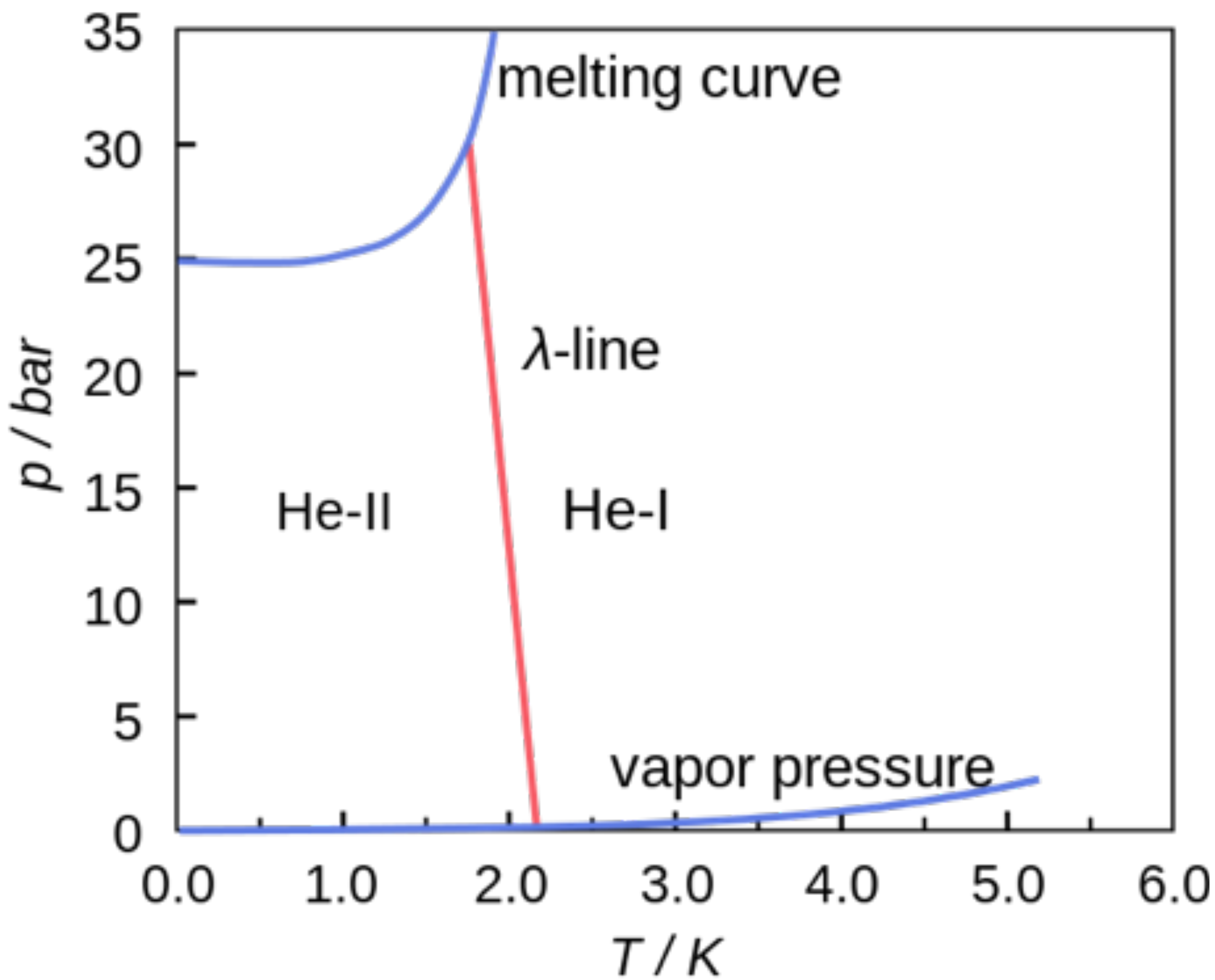
$$\kappa_T = \partial n/\partial P|_T/n = \frac{f^{+}_{1/2}(z)}{nk_BTf^{+}_{3/2}(z)}$$



which diverges at the transition since $\lim_{z\to 1} f^{+}_{1/2}(z) \to \infty$, that is, the isotherms approach the flat coexistence portion tangentially.

superfluidity of He^4

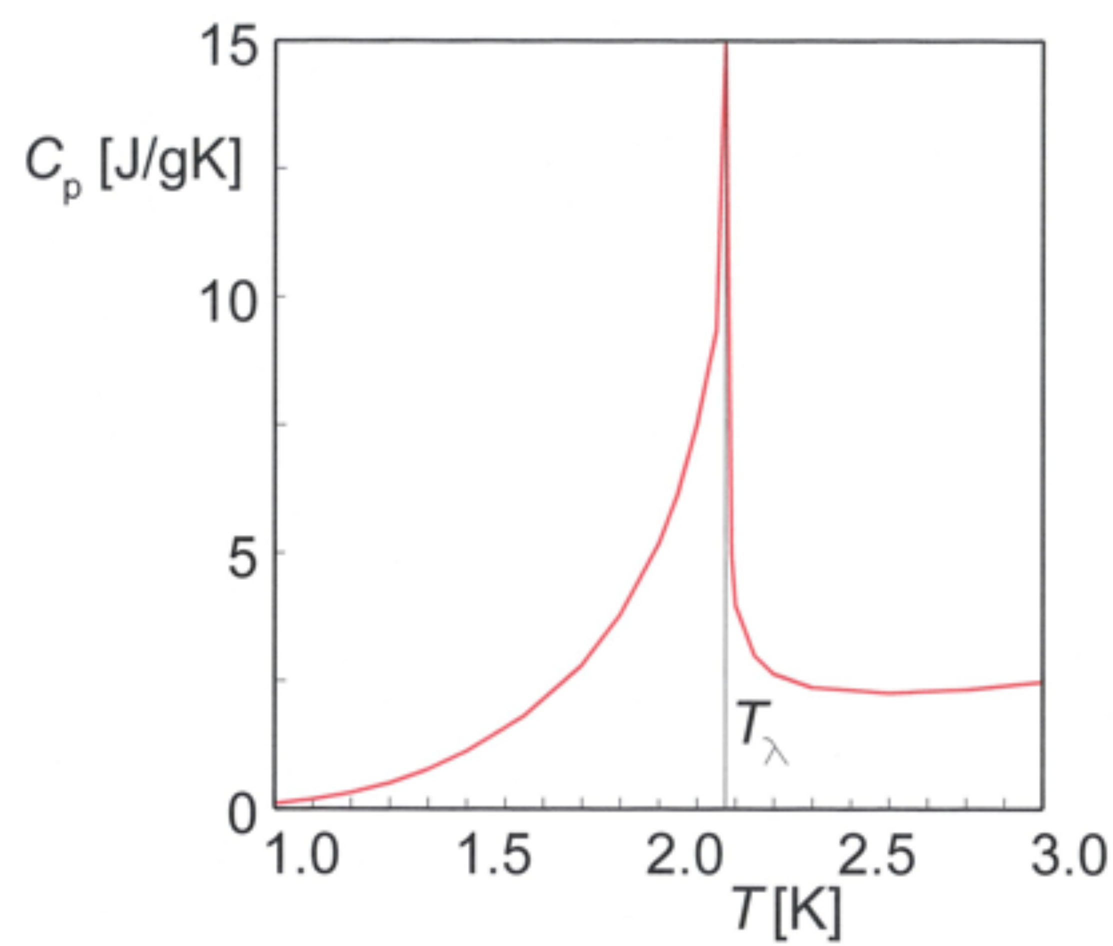
Phase diagram



1930s

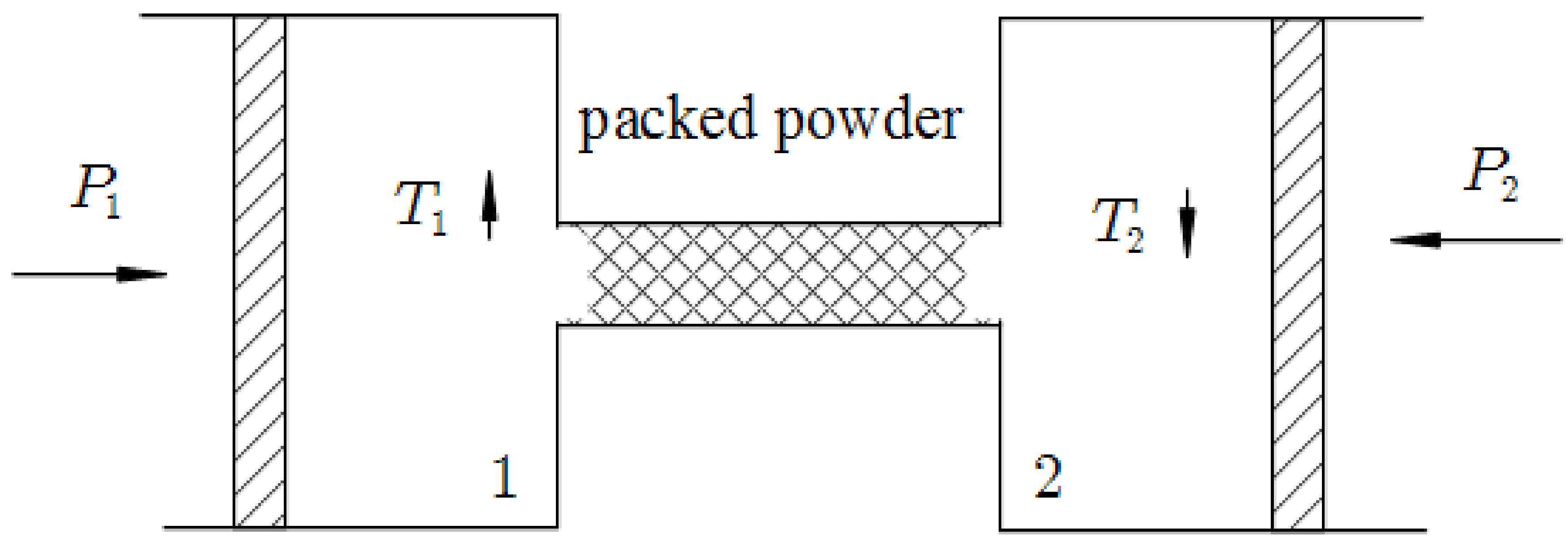
Heat capacity of liquid 4He

2nd order phase transition



Properties of superfluid

a) Zero viscosity ($T < T_c$)

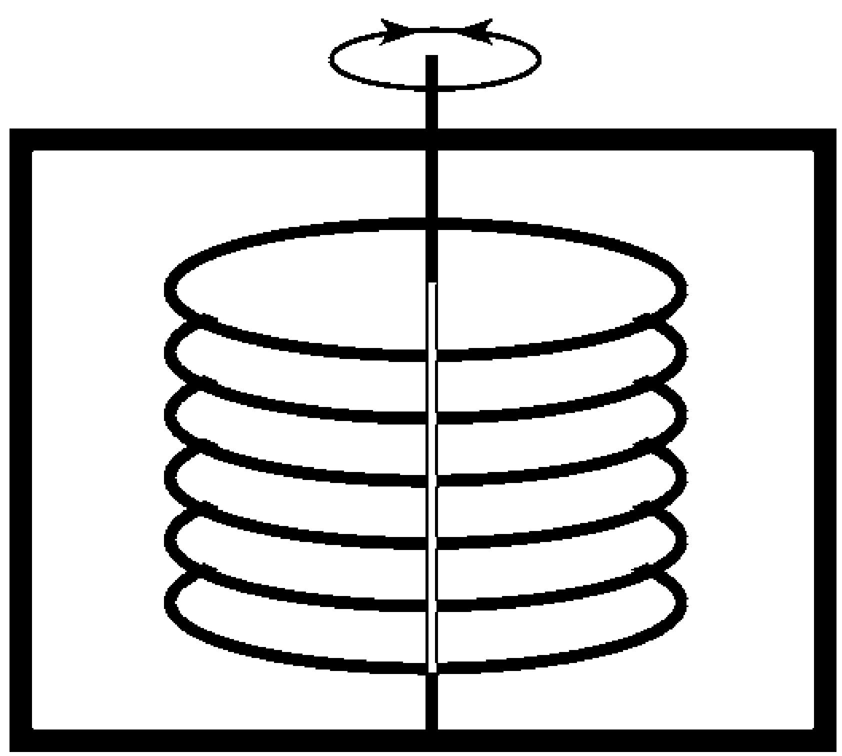


For ordinary fluid, $\Delta P = P_1 - P_2$ is need to maintain the flow from 1 to 2.

$He^4{}_{II}$ flow even in the limit of zero $\Delta P \longrightarrow$ Zero viscosity (Superfluid)

Properties of superfluid

b) Torsional oscillator:

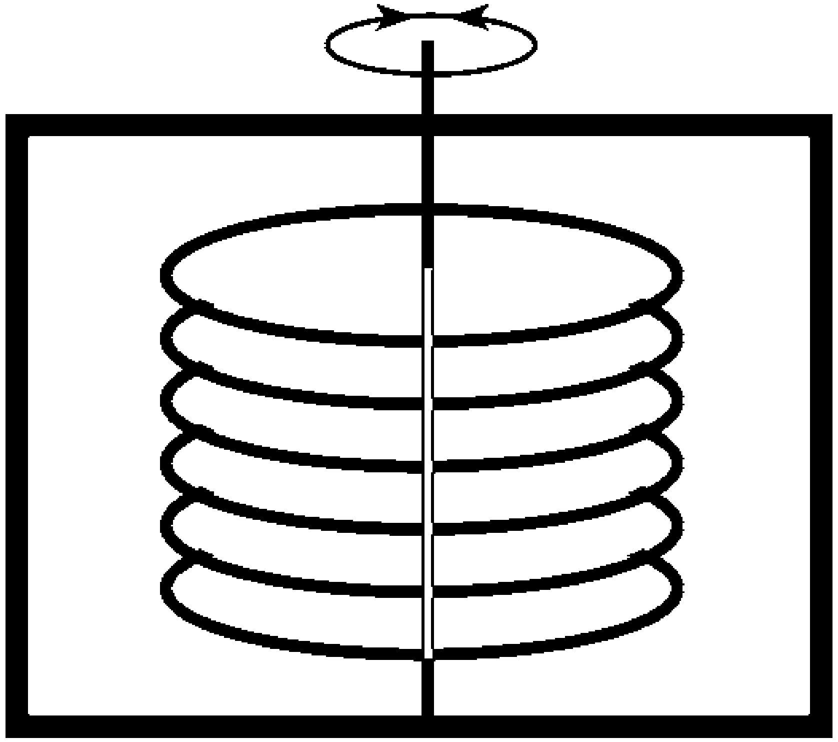


Nonzero viscosity!

Work like normal fluid even when $T < T_c$, normal density $\longrightarrow 0$ as $T \to 0$.

Properties of superfluid

b) Torsional oscillator:



Nonzero viscosity!

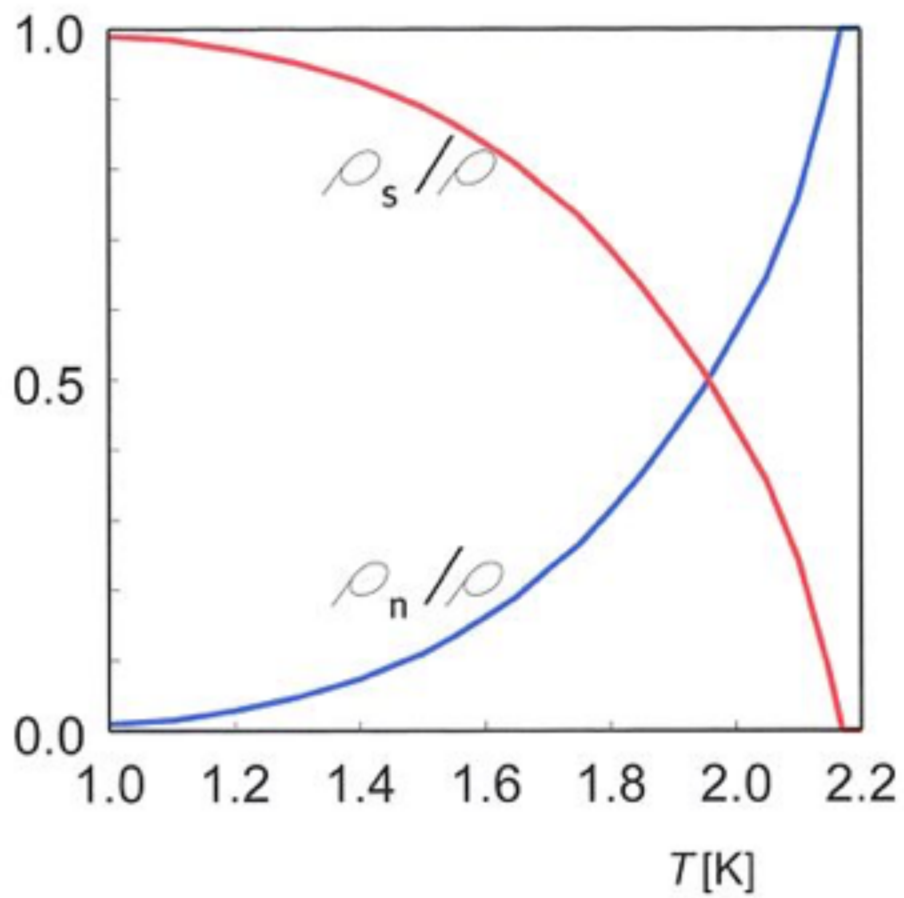
a) b) contradict to each other!

Why?

Work like normal fluid even when $T < T_c$, normal density $\longrightarrow 0$ as $T \to 0$.

Two-fluid model

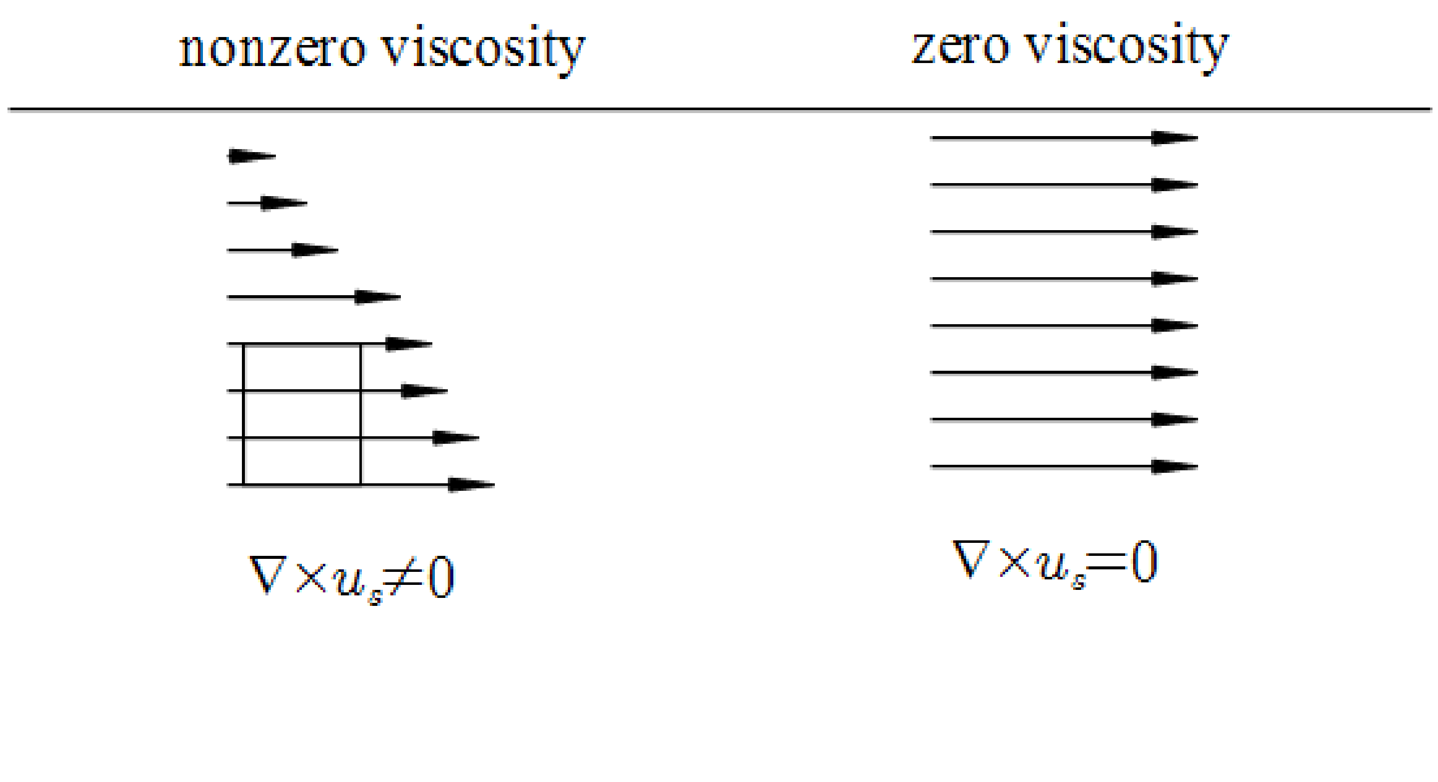
- A normal component of density ρ_n, moving with velocity $\vec{v}_n$, having a finite entropy S_n.
- A superfluid component of density ρ_s, flow without viscosity and with no vorticity ($\nabla \times \vec{u}_s = 0$), has zero entropy $S_s = 0$.



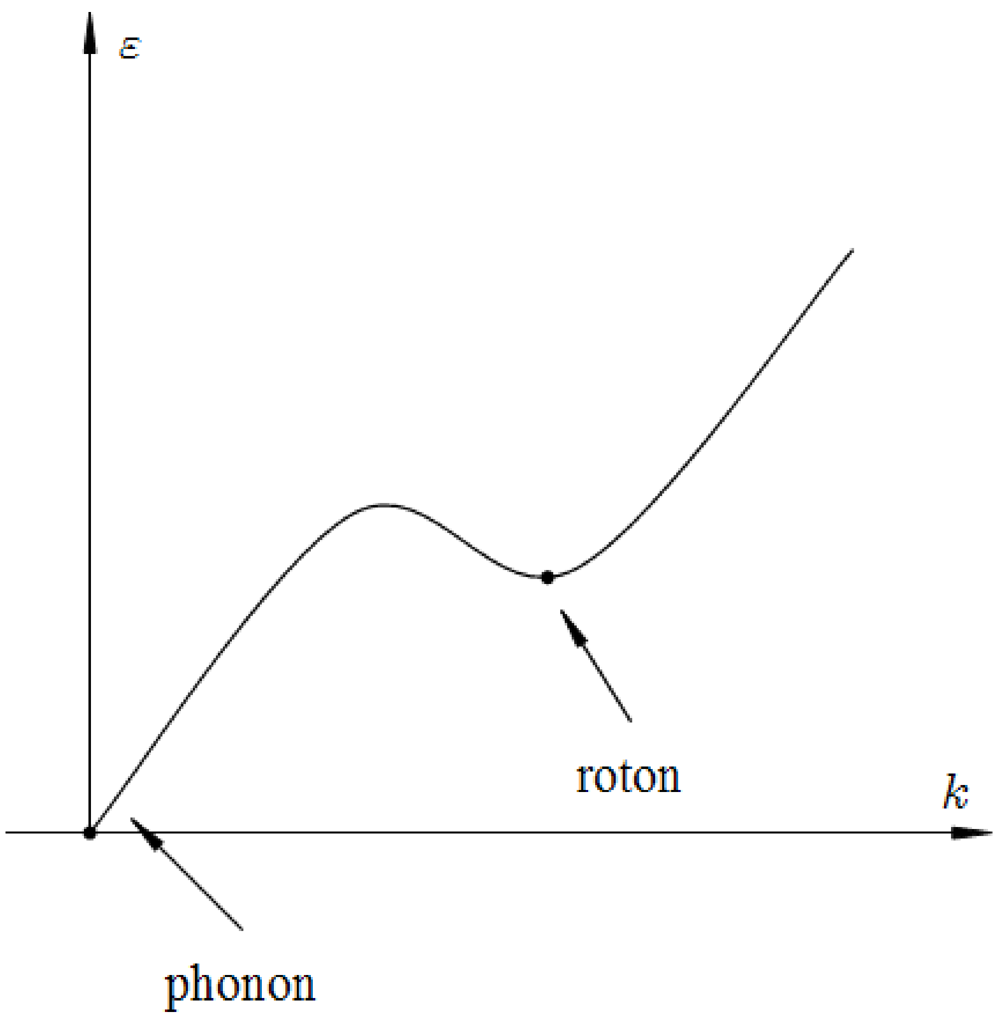
Superfluid as a quantum effect

many particles occupied the same state (B.E.C.)

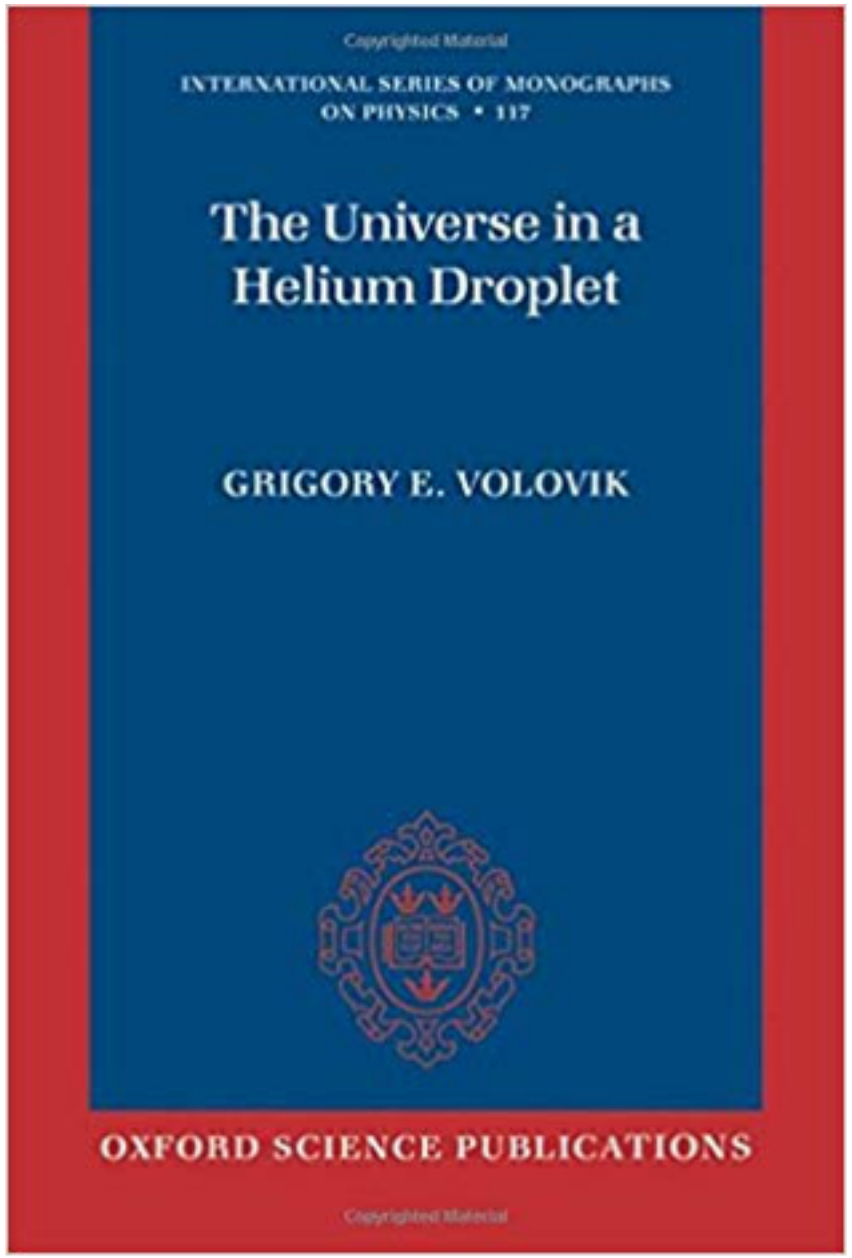
$$\nabla \times \vec{u}_s = \nabla \times (\psi \nabla \psi) = \nabla \times \left(|\psi|^2 \nabla \theta \right) = |\psi|^2 \nabla \times \nabla \theta = 0$$



Interaction effects



$$\mathcal{E}_{\text{roton}}(\vec{k}) = \Delta + \frac{\hbar^2}{2\mu}(k-k_0)^2$$



Summary

- General results for quantum gas
- High temperature expansion
- Degenerated Bose gas and superfluid