THE SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF MATHEMATICS

MA215 Probability Theory

Exercise Sheet 13

Set: Tuesday 13th December, 2016; Hand in: Tuesday 20th December 2016 by 5pm.

1. Suppose X is an absolutely continuous random variable with probability density function (pdf) $f(x)$. For any real value $-\infty < t < +\infty$, define a real-valued function, denoted by $M_X(t)$, as $M_X(t) = E(e^{tX})$. Further assume that $M_X(t)$ is well-defined for any t where $-\infty < t < +\infty$.
   (a) Write down the integration form of $M_X(t)$.
   (b) If X is a non-negative (absolutely continuous) random variable, show that $M_X(t)$ is a non-decreasing function of t.
   (c) If X is a non-negative (absolutely continuous) random variable, show that if $t < 0$ then $0 \le M_X(t) \le 1$ and $M_X(0) = 1$.
   (d) If $Y = aX + b$ where X is a random variable, and a and b are two constants. Show that $M_Y(t) = e^{bt}M_X(at)$.
   (e) Suppose X and Y are two independent absolutely continuous random variables. Let $Z = X+Y$. Show that $M_Z(t) = M_X(t)M_Y(t)$.
   (f) Suppose X is a discrete random variable taking values of non-negative integers (for example, Poisson random variable) with probability mass function p_k. Write down the form of $M_X(t)$ and prove, again, the above properties for $M_X(t)$.

2. Find the MGF of
   (a) the Uniform $[0, 1]$ distribution;
   (b) the discrete random variable X with $P(X = 4) = 1$;
   (c) the continuous random variable Y with probability density function $f(y) = 2y$ for $0 \le y \le 1$, density zero elsewhere.

3. Find the MGF of the Binomial random variable $Bin(n, p)$.

4. Find the MGF of a Geometric random variable with parameter p, and then applying the properties of MGFs to find the MGF of the Negative Binomial random variable with parameter p and r where r is a positive integer.

5. Suppose the random variable X obeys the uniform distribution over interval $[a, b]$. Find $M_X(t)$.

6. Suppose X is a Poisson random variable with parameter λ. Find $M_X(t)$.

7. Suppose X is a normally distributed random variable with parameters μ and σ^2. Find $M_X(t)$. (Hint: First consider the standard normal and then apply the properties of MGFs.)

8. Find the MGF of the general $\Gamma(\lambda, \alpha)$ distribution, where $\alpha > 0$ may NOT be a positive integer.

9. Suppose that the MGF of a random variable X is given by $M_X(t) = e^{3(e^t-1)}$. What is $P(x = 0)$? Also, find $E(X)$ and $Var(X)$. (Hint: You do not need to do any detailed calculations. Just find what the random variable X is, and then use the known results to answer this question.)