

# Independence (probability theory)

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In probability theory, two events are **independent**, **statistically independent**, or **stochastically independent**<sup>[1]</sup> if the occurrence of one does not affect the probability of the other. Similarly, two random variables are independent if the realization of one does not affect the probability distribution of the other.

The concept of independence extends to dealing with collections of more than two events or random variables, in which case the events are pairwise independent if each pair are independent of each other, and the events are mutually independent if each event is independent of each other combination of events.

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## Definition

### For events

#### Two events

Two events  $A$  and  $B$  are **independent** (often written as  $A \perp B$  or  $A \perp\!\!\!\perp B$ ) if their joint probability equals the product of their probabilities:

$$P(A \cap B) = P(A)P(B).$$

Why this defines independence is made clear by rewriting with conditional probabilities:

$$P(A \cap B) = P(A)P(B) \Leftrightarrow P(A) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(B)} = P(A | B).$$

and similarly

$$P(A \cap B) = P(A)P(B) \Leftrightarrow P(B) = P(B | A).$$

Thus, the occurrence of  $B$  does not affect the probability of  $A$ , and vice versa. Although the derived expressions may seem more intuitive, they are not the preferred definition, as the conditional probabilities may be undefined if  $P(A)$  or  $P(B)$  are 0. Furthermore, the preferred definition makes clear by symmetry that when  $A$  is independent of  $B$ ,  $B$  is also independent of  $A$ .

### More than two events

A finite set of events  $\{A_i\}$  is **pairwise independent** if and only if every pair of events is independent<sup>[2]</sup>—that is, if and only if for all distinct pairs of indices  $m, k$ ,

$$P(A_m \cap A_k) = P(A_m)P(A_k)$$

A finite set of events is **mutually independent** if and only if every event is independent of any intersection of the other events<sup>[2]</sup>—that is, if and only if for every  $n$ -element subset  $\{A_i\}$ ,

$$P\left(\bigcap_{i=1}^n A_i\right) = \prod_{i=1}^n P(A_i)$$

This is called the *multiplication rule* for independent events. Note that it is not a single condition involving only the product of all the probabilities of all single events (see below for a counterexample); it must hold true for all subset of events.

For more than two events, a mutually independent set of events is (by definition) pairwise independent; but the converse is not necessarily true (see below for a counterexample).

### For random variables

#### Two random variables

Two random variables  $X$  and  $Y$  are **independent** if and only if (iff) the elements of the  $\pi$ -system generated by them are independent; that is to say, for every  $a$  and  $b$ , the events  $\{X \leq a\}$  and  $\{Y \leq b\}$  are independent events (as defined above). That is,  $X$  and  $Y$  with cumulative distribution functions  $F_X(x)$  and  $F_Y(y)$ , and probability densities  $f_X(x)$  and  $f_Y(y)$ , are independent iff the combined random variable  $(X, Y)$  has a joint cumulative distribution function

$$F_{X,Y}(x, y) = F_X(x)F_Y(y).$$

or equivalently, if the joint density exists,

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

### More than two random variables

A set of random variables is **pairwise independent** if and only if every pair of random variables is independent.

A set of random variables is **mutually independent** if and only if for any finite subset  $X_1, \dots, X_n$  and any finite sequence of numbers  $a_1, \dots, a_n$ , the events  $\{X_1 \leq a_1\}, \dots, \{X_n \leq a_n\}$  are mutually independent events (as defined above).

The measure-theoretically inclined may prefer to substitute events  $\{X \in A\}$  for events  $\{X \leq a\}$  in the above definition, where  $A$  is any Borel set. That definition is exactly equivalent to the one above when the values of the random variables are real numbers. It has the advantage of working also for complex-valued random variables or for random variables taking values in any measurable space (which includes topological spaces endowed by appropriate  $\sigma$ -algebras).

### Conditional independence

Intuitively, two random variables  $X$  and  $Y$  are conditionally independent given  $Z$  if, once  $Z$  is known, the value of  $Y$  does not add any additional information about  $X$ . For instance, two measurements  $X$  and  $Y$  of the same underlying quantity  $Z$  are not independent, but they are **conditionally independent given  $Z$**  (unless the errors in the two measurements are somehow connected).

The formal definition of conditional independence is based on the idea of conditional distributions. If  $X$ ,  $Y$ , and  $Z$  are discrete random variables, then we define  $X$  and  $Y$  to be *conditionally independent given  $Z$*  if

$$P(X \leq x, Y \leq y \mid Z = z) = P(X \leq x \mid Z = z) \cdot P(Y \leq y \mid Z = z)$$

for all  $x, y$  and  $z$  such that  $P(Z = z) > 0$ . On the other hand, if the random variables are continuous and have a joint probability density function  $p$ , then  $X$  and  $Y$  are conditionally independent given  $Z$  if

$$p_{XY|Z}(x, y|z) = p_{X|Z}(x|z) \cdot p_{Y|Z}(y|z),$$

for all real numbers  $x, y$  and  $z$  such that  $p_Z(z) > 0$ .

If  $X$  and  $Y$  are conditionally independent given  $Z$ , then

$$P(X = x|Y = y, Z = z) = P(X = x|Z = z)$$

for any  $x, y$  and  $z$  with  $P(Z = z) > 0$ . That is, the conditional distribution for  $X$  given  $Y$  and  $Z$  is the same as that given  $Z$  alone. A similar equation holds for the conditional probability density functions in the continuous case.

Independence can be seen as a special kind of conditional independence, since probability can be seen as a kind of conditional probability given no events.

## Independent $\sigma$ -algebras

The definitions above are both generalized by the following definition of independence for  $\sigma$ -algebras. Let  $(\Omega, \Sigma, \text{Pr})$  be a probability space and let  $\mathbf{A}$  and  $\mathbf{B}$  be two sub- $\sigma$ -algebras of  $\Sigma$ .  $\mathbf{A}$  and  $\mathbf{B}$  are said to be **independent** if, whenever  $A \in \mathbf{A}$  and  $B \in \mathbf{B}$ ,

$$\text{P}(A \cap B) = \text{P}(A)\text{P}(B).$$

Likewise, a finite family of  $\sigma$ -algebras  $(\tau_i)_{i \in I}$  is said to be independent if and only if

$$\forall (A_i)_{i \in I} \in \prod_{i \in I} \tau_i : \text{P} \left( \bigcap_{i \in I} A_i \right) = \prod_{i \in I} \text{P}(A_i)$$

and an infinite family of  $\sigma$ -algebras is said to be independent if all its finite subfamilies are independent.

The new definition relates to the previous ones very directly:

- Two events are independent (in the old sense) if and only if the  $\sigma$ -algebras that they generate are independent (in the new sense). The  $\sigma$ -algebra generated by an event  $E \in \Sigma$  is, by definition,

$$\sigma(\{E\}) = \{\emptyset, E, \Omega \setminus E, \Omega\}.$$

- Two random variables  $X$  and  $Y$  defined over  $\Omega$  are independent (in the old sense) if and only if the  $\sigma$ -algebras that they generate are independent (in the new sense). The  $\sigma$ -algebra generated by a random variable  $X$  taking values in some measurable space  $S$  consists, by definition, of all subsets of  $\Omega$  of the form  $X^{-1}(U)$ , where  $U$  is any measurable subset of  $S$ .

Using this definition, it is easy to show that if  $X$  and  $Y$  are random variables and  $Y$  is constant, then  $X$  and  $Y$  are independent, since the  $\sigma$ -algebra generated by a constant random variable is the trivial  $\sigma$ -algebra  $\{\emptyset, \Omega\}$ . Probability zero events cannot affect independence so independence also holds if  $Y$  is only  $\text{Pr}$ -almost surely constant.

## Properties

### Self-independence

Note that an event is independent of itself if and only if

$$\text{P}(A) = \text{P}(A \cap A) = \text{P}(A) \cdot \text{P}(A) \Leftrightarrow \text{P}(A) = 0 \text{ or } 1.$$

Thus an event is independent of itself if and only if it almost surely occurs or its complement almost surely occurs.

### Expectation and covariance

If  $X$  and  $Y$  are independent, then the expectation operator  $E$  has the property

$$E[XY] = E[X]E[Y].$$

and the covariance  $\text{cov}(X, Y)$  is zero, since we have

$$\text{cov}[X, Y] = E[XY] - E[X]E[Y]$$

(The converse of these, i.e. the proposition that if two random variables have a covariance of 0 they must be independent, is not true. See uncorrelated.)

## Characteristic function

Two random variables  $X$  and  $Y$  are independent if and only if the characteristic function of the random vector  $(X, Y)$  satisfies

$$\varphi_{(X,Y)}(t, s) = \varphi_X(t) \cdot \varphi_Y(s).$$

In particular the characteristic function of their sum is the product of their marginal characteristic functions:

$$\varphi_{X+Y}(t) = \varphi_X(t) \cdot \varphi_Y(t)$$

though the reverse implication is not true. Random variables that satisfy the latter condition are called subindependent.

## Examples

### Rolling a die

The event of getting a 6 the first time a die is rolled and the event of getting a 6 the second time are *independent*. By contrast, the event of getting a 6 the first time a die is rolled and the event that the sum of the numbers seen on the first and second trials is 8 are *not* independent.

### Drawing cards

If two cards are drawn *with* replacement from a deck of cards, the event of drawing a red card on the first trial and that of drawing a red card on the second trial are *independent*. By contrast, if two cards are drawn *without* replacement from a deck of cards, the event of drawing a red card on the first trial and that of drawing a red card on the second trial are again *not* independent.

### Pairwise and mutual independence

Consider the two probability spaces shown. In both cases,  $P(A) = P(B) = 1/2$  and  $P(C) = 1/4$ . The first space is pairwise independent but not mutually independent. The second space is both pairwise independent and mutually independent. To illustrate the difference, consider conditioning on two events. In the pairwise independent case, although any one event is independent of each of the other two individually, it is not independent of the intersection of the other two:

$$P(A|BC) = \frac{\frac{4}{40}}{\frac{4}{40} + \frac{1}{40}} = \frac{4}{5} \neq P(A)$$

$$P(B|AC) = \frac{\frac{4}{40}}{\frac{4}{40} + \frac{1}{40}} = \frac{4}{5} \neq P(B)$$

$$P(C|AB) = \frac{\frac{4}{40}}{\frac{4}{40} + \frac{6}{40}} = \frac{2}{5} \neq P(C)$$

In the mutually independent case however:

$$P(A|BC) = \frac{\frac{1}{16}}{\frac{1}{16} + \frac{1}{16}} = \frac{1}{2} = P(A)$$

$$P(B|AC) = \frac{\frac{1}{16}}{\frac{1}{16} + \frac{1}{16}} = \frac{1}{2} = P(B)$$

$$P(C|AB) = \frac{\frac{1}{16}}{\frac{1}{16} + \frac{3}{16}} = \frac{1}{4} = P(C)$$

## Mutual independence

It is possible to create a three-event example in which

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

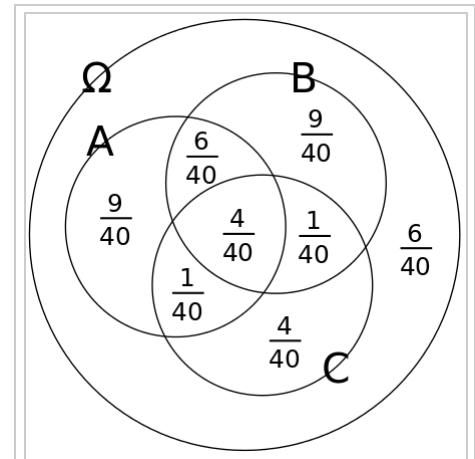
and yet no two of the three events are pairwise independent (and hence the set of events are not mutually independent).<sup>[3]</sup> This example shows that mutual independence involves requirements on the products of probabilities of all combinations of events, not just the single events as in this example.

## See also

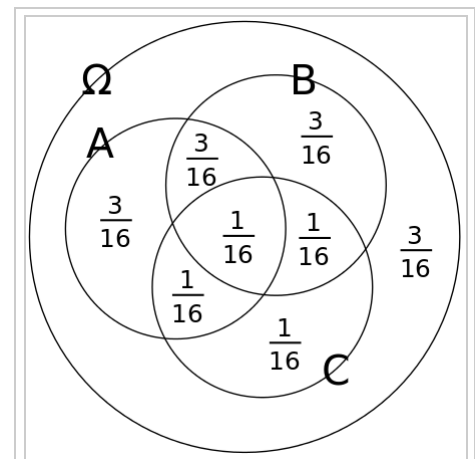
- Copula (statistics)
- Independent and identically distributed random variables
- Mutually exclusive events
- Subindependence
- Linear dependence between random variables
- Conditional independence
- Normally distributed and uncorrelated does not imply independent
- Mean dependence

## References

1. Russell, Stuart; Norvig, Peter (2002). *Artificial Intelligence: A Modern Approach*. Prentice Hall. p. 478. ISBN 0-13-790395-2.
2. Feller, W (1971). "Stochastic Independence". *An Introduction to Probability Theory and Its Applications*. Wiley.



Pairwise independent, but not mutually independent, events.



Mutually independent events.

3. George, Glyn, "Testing for the independence of three events," *Mathematical Gazette* 88, November 2004, 568. PDF (<http://www.engr.mun.ca/~ggeorge/MathGaz04.pdf>)

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