I. Normal Density Functions

1. Standard Normal

pdf: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad (-\infty < x < +\infty)$

① $f(x) \geq 0$ for all $x \in R$: easy.

② Check $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx = 1$:

Let $I = \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx = \int_{-\infty}^{+\infty} e^{-\frac{y^2}{2}} dy$

$$\Rightarrow I^2 = \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx \cdot \int_{-\infty}^{+\infty} e^{-\frac{y^2}{2}} dy = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{x^2+y^2}{2}} dx dy$$

Changing of variables to polar coordinates yields

$x = r\cos\theta$

$y = r\sin\theta$

then $$J = \frac{D(x,y)}{D(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & r(-\sin\theta) \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r \cdot 1 = r$$

Also $x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

Hence

$$I^2 = \int_0^{\infty}\int_0^{2\pi} e^{-\frac{r^2}{2}} \cdot r\,dr\,d\theta = \int_0^{\infty} re^{-\frac{r^2}{2}}\left[\int_0^{2\pi} d\theta\right] \cdot dr$$

$$= 2\pi \int_0^{\infty} re^{-\frac{r^2}{2}}\,dr$$

$$= \left(-2\pi e^{-\frac{r^2}{2}}\right)\Big|_0^{+\infty}$$

$$= 0 - (-2\pi)$$

$$= 2\pi$$

$$\Rightarrow \quad I = \sqrt{2\pi} = \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\,dx$$

$$\Rightarrow \quad \boxed{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\,dx = 1}$$

3. General Normal

pdf: $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (-\infty < x < +\infty)$

$\sigma > 0 \quad -\infty < \mu < +\infty$

① $f(x) \geq 0$ ✓

② check: $\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$

let $y = \frac{x-\mu}{\sigma}$ then $dy = \frac{1}{\sigma} dx \Rightarrow dx = \sigma dy$

Also: $x \to -\infty \iff y \to -\infty \quad (\because \sigma > 0)$

$x \to +\infty \iff y \to +\infty \quad (\because \sigma > 0)$

Hence $\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$

$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}y^2} \cdot \sigma dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{y^2}{2}} dy$

$= 1$

II. Transformation of Normal Random Variables

1. Basic Question:

If $X \sim N(\mu, \sigma^2)$, $Y = aX + b$ $(a \neq 0)$

$Y \sim ?$

2. Solution: Let $F_X(\cdot)$ and $F_Y(\cdot)$ be the cdfs of X and Y, respectively.

Then $F_Y(y) = Pr\{Y \leq y\} = Pr\{aX + b \leq y\}$

$= Pr\{aX \leq y - b\}$ (reason?!)

Consider two cases: $a > 0$; $a < 0$

① If $a > 0$, then $F_Y(y) = Pr\{aX \leq y-b\} = Pr\{X \leq \frac{y-b}{a}\}$

$\therefore F_Y(y) = F_X(\frac{y-b}{a}) \Rightarrow \frac{dF_Y(y)}{dy} = \frac{d}{dy} F_X(\frac{y-b}{a})$

$\Rightarrow F_Y'(y) = F_X'(\frac{y-b}{a}) \times \frac{1}{a}$

Let $f_Y(y)$ and $f_X(x)$ be the pdfs of Y and X, respectively.

Then $f_Y(y) = f_X(\frac{y-b}{a}) \cdot \frac{1}{a}$

Now $X \sim N(\mu, \sigma^2)$ i.e. $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$\Rightarrow f_X(\frac{y-b}{a}) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(\frac{y-b}{a}-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(\frac{y-b-a\mu}{a})^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(y-b-a\mu)^2}{2\sigma^2 a^2}}$

Therefore $f_Y(y) = f_Z\left(\frac{y-b}{a}\right) \cdot \frac{1}{a} = \frac{1}{\sqrt{2\pi}\, a\sigma} e^{-\frac{(y-(a\mu+b))^2}{2\sigma^2 a^2}}$

Let $\tilde{\sigma} = a\sigma \quad \tilde{\mu} = a\mu + b$

Then $f_Y(y) = \frac{1}{\sqrt{2\pi}\,\tilde{\sigma}} e^{-\frac{(y-\tilde{\mu})^2}{2\tilde{\sigma}^2}}$

Hence $Y \sim N(\tilde{\mu}, \tilde{\sigma}^2)$ i.e., $Y \sim N(a\mu+b, a^2\sigma^2)$

(2) If $a<0$, then $F_Y(y) = \Pr\{aZ \le y-b\}$

$= \Pr\{Z \ge \frac{y-b}{a}\}$ ($\because a<0$!!!)

$= 1 - \Pr\{Z < \frac{y-b}{a}\} = 1 - \Pr\{Z \le \frac{y-b}{a}\}$ ($\because$ conti. r.v)

$\Rightarrow \frac{dF_Y(y)}{dy} = (-1) \cdot F_Z'\left(\frac{y-b}{a}\right) \cdot \frac{1}{a}$

$\Rightarrow f_Y(y) = (-1) \cdot \frac{1}{a} f_Z\left(\frac{y-b}{a}\right)$

$\Rightarrow f_Y(y) = (-1) \frac{1}{a} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{[y-(a\mu+b)]^2}{2\sigma^2 a^2}} = \frac{1}{\sqrt{2\pi}(-a\sigma)} e^{-\frac{[y-(a\mu+b)]^2}{2a^2\sigma^2}}$

Let $\tilde{\sigma} = (-a\sigma)$, then $\tilde{\sigma} > 0$ ($\because a<0, \sigma>0$) $\tilde{\mu} = a\mu + b$

then $f_Y(y) = \frac{1}{\sqrt{2\pi}\,\tilde{\sigma}} e^{-\frac{(y-\tilde{\mu})^2}{2\tilde{\sigma}^2}}$ ($\because (-a\sigma)^2 = a^2\sigma^2$!!)

$\Rightarrow Y \sim N(a\mu+b, (-a\sigma)^2) = N(a\mu+b, a^2\sigma^2)$.

3. Corollary: If $X \sim N(\mu, \sigma^2)$, let $Y = \frac{X-\mu}{\sigma}$,

then $Y \sim N(0,1)$

Proof: Recall if $X \sim N(\mu, \sigma^2)$, $Y = aX + b$ $(a \neq 0)$

then $Y \sim N(a\mu + b, a^2\sigma^2)$

Now, $Y = \frac{X-\mu}{\sigma}$, hence $a = \frac{1}{\sigma}$, $b = -\frac{\mu}{\sigma}$

and hence $a\mu + b = \frac{1}{\sigma} \cdot \mu - \frac{\mu}{\sigma} = 0$

$a^2\sigma^2 = (\frac{1}{\sigma})^2 \cdot \sigma^2 = 1$

Therefore $Y \sim N(0, 1)$.