SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF MATHEMATICS

MA215 Probability Theory

Tutorial 09Solu
Set: Friday 11th November 2016; Hand in: Friday 18th November by 5pm.

1. If the joint probability density of X and Y is given by

$$f(x,y) = \begin{cases} x+y & 0<x<1, 0<y<1; \\ 0 & \text{otherwise.} \end{cases}$$

Find the joint (cumulative) distribution function (joint c.d.f) of these two random variables X and Y.

Solution: If either $x \le 0$ or $y \le 0$, it follows that $f(x,y)=0$, then

$$F(x,y) = \iint_{\substack{-\infty<s\le x \\ -\infty<t\le y}} f(s,t)\mathrm{d}s\mathrm{d}t = \int_{-\infty}^{x}\int_{-\infty}^{y} 0\mathrm{d}s\mathrm{d}t = 0.$$

For $0<x<1$ and $0<y<1$, we get

$$F(x,y) = \int_0^y \int_0^x (s+t)dsdt = \frac{1}{2}xy(x+y)$$

for $x>1$ and $0<y<1$, we get

$$F(x,y) = \int_0^y \int_0^1 (s+t)dsdt = \frac{1}{2}y(y+1)$$

for $0<x<1$ and $y>1$, we get

$$F(x,y) = \int_0^1 \int_0^x (s+t)dsdt = \frac{1}{2}x(x+1)$$

and for $x>1$ and $y>1$, we get

$$F(x,y) = \int_0^1 \int_0^1 (s+t)dsdt = 1$$

Since the joint distribution function is everywhere continuous, the boundaries between any two of these regions can be included in either one, and we can write

$$F(x,y) = \begin{cases} 0 & \text{for } x \le 0 \text{ or } y \le 0 \\ \frac{1}{2}xy(x+y) & \text{for } 0<x<1, 0<y<1 \\ \frac{1}{2}y(y+1) & \text{for } x \ge 1, 0<y<1 \\ \frac{1}{2}x(x+1) & \text{for } 0<x<1, y \ge 1 \\ 1 & \text{for } x \ge 1, y \ge 1 \end{cases}$$

2. Find the joint probability density of the two random variables X and Y whose joint cumulative distribution function (joint c.d.f)is given by

$$F(x,y)=\begin{cases}(1-e^{-x})(1-e^{-y}) & x>0, y>0;\\ 0 & \text{otherwise.}\end{cases}$$

Also use the joint probability density to determine $\mathrm{P}(1<\mathrm{X}<3, 1<\mathrm{Y}<2)$.

Solution: For $x>0$ and $y>0$, since F(x,y) is the binary continuous, yields

$$\begin{aligned}\frac{\partial^2}{\partial y\partial x}F(x,y)=\frac{\partial^2}{\partial x\partial y}F(x,y)&=\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}F(x,y)\right)\\ &=\frac{\partial}{\partial y}\left((-e^{-x}\cdot(-1))(1-\mathrm{e}^{-y})\right)\\ &=e^{-x}\cdot\frac{\partial}{\partial y}\left((1-\mathrm{e}^{-y})\right)=e^{-x}\cdot(-e^{-y}\cdot(-1))\\ &=e^{-(x+y)}.\end{aligned}$$

and 0 elsewhere.

Hence we find the joint probability density of X and Y is:

$$f(x,y)=\begin{cases}e^{-(x+y)} & x>0, y>0;\\ 0 & \text{otherwise.}\end{cases}$$

Thus, integration yields

$$\begin{aligned}P(1<X<3,1<Y<2)&=\int_1^3\int_1^2\mathrm{e}^{-(x+y)}\mathrm{d}y\mathrm{d}x\\ &=\int_1^3\int_1^2\mathrm{e}^{-x}\cdot\mathrm{e}^{-y}\mathrm{d}y\mathrm{d}x=\int_1^3\mathrm{e}^{-x}\mathrm{d}x\cdot\int_1^2\mathrm{e}^{-y}\mathrm{d}y\\ &=(-\mathrm{e}^{-x})\Big|_1^3\cdot(-\mathrm{e}^{-y})\Big|_1^2\\ &=(-\mathrm{e}^{-3}+\mathrm{e}^{-1})(-\mathrm{e}^{-2}+\mathrm{e}^{-1})\\ &=\mathrm{e}^{-2}-\mathrm{e}^{-3}-\mathrm{e}^{-4}+\mathrm{e}^{-5}.\end{aligned}$$

3. Consider a circle of radius R and suppose that a point within the circle is randomly chosen in such a manner that all regions within the circle of equal area are equally likely to contain the point. (In other words, the point is uniformly distributed within the circle.) If we let the center of the circle denote the origin and define X and Y to be the coordinates of the point chosen, it follows, since (X,Y) is equally likely to be near each point in the circle, that the joint density function of X and Y is given by

$$f(x,y)=\begin{cases}c & \text{if } x^2+y^2\le R^2,\\ 0 & \text{if } x^2+y^2>R^2.\end{cases}$$

for some value of c.

(a) Determine the constant c.

(b) Find the marginal density functions of X and Y.

(c) Compute the probability that the distance from the origin of the point selected is not greater that a. ($0 \le a \le R$.)

(d) Are X and Y independent? Specify your reasons clearly.

Proof:

(a) Because

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx = 1$$

it follows that

$$c \iint\limits_{x^2+y^2\le R^2} dydx = 1$$

We can evaluate $\iint\limits_{x^2+y^2\le R^2} dydx$ either by using coordinates, or more simply, by noting that it represents the area of the circle and is thus equal to πR^2. Hence

$$c = \frac{1}{\pi R^2}$$

(b) Notice that for fixed $x_0 \in \mathbb{R}$, $x_0^2 + y^2 \le R^2 \Leftrightarrow y^2 \le R^2 - x_0^2$
If $x_0^2 > R^2$, i.e. $|x_0| > R$, then $f(x_0, y) = 0$, so

$$f_X(x_0) = \int_{-\infty}^{+\infty} f(x_0,y)\mathrm{d}y = \int_{-\infty}^{+\infty} 0\mathrm{d}y = 0.$$

If $x_0^2 \le R^2$, i.e. $|x_0| \le R$, then

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{+\infty} f(x,y)\mathrm{d}y \\
&= \int_{-\infty}^{-\sqrt{R^2-x_0^2}} f(x,y)\mathrm{d}y + \int_{-\sqrt{R^2-x_0^2}}^{\sqrt{R^2-x_0^2}} f(x,y)\mathrm{d}y + \int_{\sqrt{R^2-x_0^2}}^{+\infty} f(x,y)\mathrm{d}y \\
&= \int_{-\infty}^{-\sqrt{R^2-x_0^2}} 0\mathrm{d}y + \int_{-\sqrt{R^2-x_0^2}}^{\sqrt{R^2-x_0^2}} (\frac{1}{\pi R^2})\mathrm{d}y + \int_{\sqrt{R^2-x_0^2}}^{+\infty} 0\mathrm{d}y \\
&= \frac{2\sqrt{R^2-x^2}}{\pi R^2}.
\end{aligned}$$

Hence the marginal density of X is:

$$f_X(x) = \begin{cases} \dfrac{2\sqrt{R^2-x^2}}{\pi R^2} & x^2 \le R^2, \\ 0 & x^2 > R^2. \end{cases}$$

By symmetry the marginal density of Y is:

$$f_Y(x) = \begin{cases} \dfrac{2\sqrt{R^2 - y^2}}{\pi R^2} & y^2 \leq R^2, \\ 0 & y^2 > R^2. \end{cases}$$

(c) Let $Z = \sqrt{X^2 + Y^2}$, then for $0 \leq a \leq R$, we have

$$\begin{aligned} \mathrm{P}\{\sqrt{\mathrm{X}^2 + \mathrm{Y}^2} \leq \mathrm{a}\} &= \mathrm{P}(\mathrm{Z} \leq \mathrm{a}) \xlongequal{\text{since } Z \geq 0} \mathrm{P}(|\mathrm{Z}| \leq \mathrm{a}) \xlongequal{\text{since } a \geq 0} \mathrm{P}(|\mathrm{Z}| \leq |\mathrm{a}|) \\ &= \mathrm{P}\{\mathrm{Z}^2 \leq \mathrm{a}^2\} = \mathrm{P}\{\mathrm{X}^2 + \mathrm{Y}^2 \leq \mathrm{a}^2\} \\ &= \iint\limits_{x^2+y^2 \leq a^2} f(x,y)\mathrm{d}y\mathrm{d}x \\ &= \iint\limits_{x^2+y^2 \leq a^2} \frac{1}{\pi R^2}\mathrm{d}y\mathrm{d}x = \frac{1}{\pi R^2} \iint\limits_{x^2+y^2 \leq a^2} \mathrm{d}y\mathrm{d}x \\ &= \frac{\pi a^2}{\pi R^2} = \frac{a^2}{R^2}. \end{aligned}$$

(d) X and Y are not independent, since for $x^2 + y^2 \leq R^2$,

$$f(x,y) \neq f_X(x)f_Y(y)$$

OR,$\exists x, y \in \mathbb{R}$,

$$F(x,y) \neq F_X(x)F_Y(y)$$

Reacll: For any $x \in \mathbb{R}, y \in \mathbb{R}$,

$$\mathrm{P}(\mathrm{X} \leq \mathrm{x}, \mathrm{Y} \leq \mathrm{y}) = \boxed{F(x,y) = F_X(x)F_Y(y)} = \mathrm{P}(\mathrm{X} \leq \mathrm{x})\mathrm{P}(\mathrm{Y} \leq \mathrm{y}).$$

Remark: try to write the details of the probability space and X obeys uniform distribution. blablablabla...

4. A man and a woman decide to meet at a certain location. If each person independently arrives at a time uniformly distributed between 12 noon and 1 P.M., find the probability that the first to arrive has to wait longer than 10 minutes.

Solution: If we let X and Y denote, respectively, the time past 12 that the man and the woman arrive, then X and Y are independent random variables, each of which is uniformly distributed over $(0, 60)$. The desired probability, $P\{X + 10 < Y\} + P\{Y + 10 < X\}$, which by symmetry equals $2P\{X + 10 < Y\}$, is obtained as follows:

$$\begin{aligned} 2P\{X + 10 < Y\} &= 2 \iint\limits_{x+10<y} f(x,y)dxdy \\ &= 2 \iint\limits_{x+10<y} f_X(x)f_Y(y)dxdy \end{aligned}$$

$$
\begin{aligned}
&= 2\int_{10}^{60}\int_{0}^{y-10}\left(\frac{1}{60}\right)^2 dxdy \\
&= \frac{2}{(60)^2}\int_{10}^{60}(y-10)dy \\
&= \frac{25}{36}.
\end{aligned}
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or,

$$
\begin{aligned}
\mathrm{P}(|\mathrm{X}-\mathrm{Y}|>10) &= \iint\limits_{|x-y|>10} f(x,y)\mathrm{d}x\mathrm{d}y = \iint\limits_{\substack{|x-y|>10\\-\infty<x<+\infty\\-\infty<y<+\infty}} f(x,y)\mathrm{d}x\mathrm{d}y \\
&= \iint\limits_{\substack{|x-y|>10\\-\infty<x<+\infty\\-\infty<y<+\infty}} f_X(x)f_Y(y)\mathrm{d}x\mathrm{d}y = \iint\limits_{\substack{|x-y|>10\\0\le x\le 60\\0\le y\le 60}} (\frac{1}{60}\cdot\frac{1}{60})\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{3600}\iint\limits_{\substack{|x-y|>10\\0\le x\le 60\\0\le y\le 60}} \mathrm{d}x\mathrm{d}y = \frac{1}{3600}\cdot(\frac{1}{2}\times 50\times 50\times 2) = \frac{25}{36} \\
&= \frac{1}{3600}\left[\int_0^{10}\int_{x+10}^{60}\mathrm{d}y\mathrm{d}x + \int_{10}^{50}(\int_0^{x-10}\mathrm{d}y + \int_{x+10}^{60}\mathrm{d}y)\mathrm{d}x + \int_{50}^{60}\int_0^{x-10}\mathrm{d}y\mathrm{d}x\right] \\
&= \frac{1}{3600}\left[\int_0^{10}(50-x)\mathrm{d}x + \int_{10}^{50}\left[(x-10)+(50-x)\right]\mathrm{d}x + \int_{50}^{60}(x-10)\mathrm{d}x\right] \\
&= \frac{1}{3600}\left[(50x-\frac{x^2}{2})\Big|_0^{10} + 40\times 40 + (\frac{x^2}{2}-10x)\Big|_{50}^{60}\right] \\
&= \frac{2500}{3600} = \frac{25}{36}.
\end{aligned}
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