# §1.4 Computing Probabilities

## I. Equally Likely Outcomes (Special case)

1. Def. If the sample space has only a finite number of outcomes and each particular outcome (elementary event) has the same probability, then it is called the equally likely outcome case

2. Example:

A fair coin is thrown twice.

$\Omega = \{HH; HT; TH; TT\}$

Question: Event A: exactly one head appears

$P(A) = ?$

Ans. Reasonably: $P(A) = \frac{2}{4} = \frac{1}{2}$

Since: Equally likely outcomes.

So $$P(A) = \frac{\text{number of ways A can occur}}{\text{total number of outcomes}}$$

3. Conclusion: Suppose the sample space has $n$ elements, $\{e_1, e_2, \cdots, e_n\}$ say, and suppose that each elementary event $\{e_i\}$ has the same probability $\frac{1}{n}$.

then the probability of any event is the number of ways this event can occur over the total number $n$. i.e. for any event $A$,

$$P(A) = \frac{\text{number of ways } A \text{ can occur}}{\text{total number of outcomes}}.$$

4. Note:

① "Equally likely outcomes" is a special case. For example, for unfair coin, the above formula is not true.

② We need the method to calculate the number of outcomes.

# II. Permutations and Combinations. (4)

## 1. Problems:

① Suppose that from 5 children, 3 are to be chosen and lined up. How many different lines are possible?

② Suppose that from 5 children, 3 are to be chosen for a team. In how many ways can this be done?

## 2. Idea:

① Difference between Problems ① and ②?

Problem ①: ordered !!

Problem ②: unordered !!

② For Problem ①: the first position: 5 different ways; the second position: 4 ways; the third position: 3 ways;

Altogether: $5 \times 4 \times 3$ different ways.

③ For Problem ②, we consider as follows:

First; assume: ordered (line up!!)

then $5\times4\times3$ different way!!

Second; but actually "ordered" is no use.

Then we fix three children, then it is only

one team (!). However, this team can

line up for $3\times2\times1$ different ways.

(Think why here!!)

In other words,

(The number of teams) $\times$ $(3\times2\times1)$

= The number of ways to line up

So, the number of teams

$$= \frac{5\times4\times3}{3\times2\times1} = \frac{5\times4\times3}{3!}$$

$$= \frac{5\times4\times3\times2\times1}{3!\times2\times1} = \frac{5!}{3!\,2!}$$

3. Conclusions:

Using the above idea, we can easily get the following conclusions:

① Proposition 1: For a set of size $n$ and a sample of size $r$, there are

$$n(n-1)(n-2)\cdots(n-r+1)$$

different ordered samples.

② Proposition 2: The number of unordered samples of $r$ objects from $n$ objects is

$$\frac{n(n-1)\cdots(n-r+1)}{r!}$$

The second conclusion is most often used!

The first: (special case): Permutation

The second: Combination.

Note that, the number of

$$\frac{n(n-1)\cdots(n-r+1)}{r!}$$

can be written as

$$\frac{n(n-1)\cdots(n-r+1)}{r!}=\frac{n(n-1)\cdots(n-r+1)(n-r)\cdots 1}{r!\,(n-r)\cdots 1}$$

$$=\frac{n!}{r!\,(n-r)!}$$

4. Notation and Terminology:

① We define $\binom{n}{r}$, for $r \le n$ by

$$\binom{n}{r}=\frac{n!}{(n-r)!\,r!}$$

and say that $\binom{n}{r}$ represents the number of possible combinations of n objects taken r at a time.

※ Thus $\binom{n}{r}$ represents the number of different groups of size r that could be selected from a set of n objects when the order of selection is not considered relevant. ※

② Other notation:

$$\binom{n}{r} \; ; \quad C_n^r \; ; \quad {}^nC_r$$

for example, $\binom{5}{3}$ ; $C_5^3$ ; ${}^5C_3$

③ Notes:

By convention $0!$ is defined to be 1.

Thus $\binom{n}{0} = \binom{n}{n} = 1$.

Also, in actually calculation we use the original one

i.e. $\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{1\cdot 2\cdots r}$

5. Important Application: The Binomial Theorem

① Question: $(a+b)^n = ?$ (n: positive integer

② Idea: $(1+x)^n = ?$ Must be

$$(1+x)^n = \underbrace{(1+x)(1+x)\cdots(1+x)}_{n}$$

$$= 1 + b_1 x + b_2 x^2 + \cdots + b_r x^r + \cdots + x^n$$

$b_r = ?$

Easy to see: $b_r = \binom{n}{r}$. (Think why here)

$\Rightarrow$: $(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$

Now $(a+b)^n = b^n (1+\frac{a}{b})^n$

$= b^n \cdot \sum_{r=0}^{n} \binom{n}{r} (\frac{a}{b})^r$ (let $x = \frac{a}{b}$ !!)

$= \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$

③ Conclusion: The Binomial Theorem:

For any positive integer $n$, we have

$$\boxed{(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}} \equiv \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

$$= a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + b^n$$

Thus, the values $\binom{n}{r}$ are often referred to as binomial coefficients.

④ Simple properties: (Easy proved by definition)

(i) $\binom{n}{0} = \binom{n}{n} = 1$

(ii) $\binom{n}{r} = \binom{n}{n-r}$

(iii) $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ $\quad 1 \leq r \leq n$

⑤ Keep in mind:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ & & & & \cdots & & & & \end{array}$$

The $(n+1)$st row is just

$\binom{n}{0}$ $\binom{n}{1}$ $\binom{n}{2}$ $\cdots$ $\binom{n}{n}$

# §1.5 Conditional Probability

(See P17)

## I. Definition:

1. Motivation: An example (See the ref.)

Total: 135 patients

high blood concentration (Positive test)

low blood concentration (Negative test)

toxicity (disease present)

no toxicity (disease absent)

| | Disease Present | Disease Absent | Total |
|---|---|---|---|
| Positive test | 25 | 14 | 39 |
| Negative test | 18 | 78 | 96 |
| Total | 43 | 92 | 135 |

Now, choose a patient "at random" (means: <u>equally likely</u> !!) from the <u>135</u> patients

Event A: disease present.

$P(A) = ?$

Easy $P(A) = \frac{\#(\text{Disease Present})}{\#(\text{Patients})} = \frac{43}{135} \doteqdot 0.318$

Now if a doctor knows that the test for the chosen person was positive (Event B).

what is the probability of disease present given this knowledge?

$\therefore = \frac{\#(\text{Disease Present and Positive})}{\#(\text{Positive})} = \frac{25}{39} \doteqdot 0.641$

of course $0.318 \neq 0.641$

Reason: for the second one, B has occured.

so the second probability is called the probability of event A under the condition that B has occured.

Or simply called:

"the conditional probability of A given B"

denoted by $P(A|B)$.

So, usually $P(A) \neq P(A|B)$.

But we can see

$$P(A|B) = \frac{25}{39} = \frac{\text{Number of Disease Present and Positive}}{\text{Number of Positive}}$$

$$= \frac{\frac{25}{135}}{\frac{39}{135}} = \frac{\frac{\text{Number of Disease Present and Positive}}{\text{Total Number}}}{\frac{\text{Number of Positive}}{\text{Total Number}}}$$

$$= \frac{P(A \cap B)}{P(B)}$$

Here, we need, $P(B) > 0$

otherwise undefined.

2. Definition: Let A and B be two events with $P(B) \neq 0$. The conditional probability of A given B is defined to be

$$P(A/B) = \frac{P(A \cap B)}{P(B)} . \qquad (1.5.1)$$

II. Multiplication Law: (※)

1. Conclusion: Let A and B be two events with $P(B) \neq 0$. Then

$$P(A \cap B) = P(B) \cdot P(A/B) . \qquad (1.5.2)$$

2. Proof: By (1.5.1) directly. [Also $P(A) \cdot P(B/A)$]

3. Application: Usually, $P(A \cap B) = ?$ but $P(B)$ and $P(A/B)$ are easy.

Example: An urn contains 3 red balls and one blue ball. Two balls are selected without replacement. What is the probability that they are both red.

Method 1. (Without using the conditional prob)

Total number of outcomes: $4 \times 3$

Total number of "Two reds": $3 \times 2$

$\Rightarrow$: Prob. $= \dfrac{3 \times 2}{4 \times 3} = \dfrac{1}{2}$ ($\because$ Equally likely!)

Method 2. (Using the conditional prob.)

Let $A$: the first one be Red

$B$: the second one be Red

then both Red: $A \cap B$

Easy to see. $P(A) = \dfrac{3}{4}$ ($\because$ total 4; Red 3)

$P(B|A) = ?$

"A has occured" $\Leftrightarrow$ "the first one Red"

$\Rightarrow$ "3 left with 2 Red"

$\Rightarrow$: $P(B|A) = \dfrac{2}{3}$

Now $P(A \cap B) = P(A) \cdot P(B|A) = \dfrac{3}{4} \cdot \dfrac{2}{3} = \dfrac{1}{2}$

We got the same results

4. More interesting example: Pólya's urn scheme (1)

(Pólya: 1887—1985; Book: "How to Solve It")

Suppose, originally we have $m$ blue balls and $n$ red balls. We draw a ball and note its color, then we replace it and add one more ball of the same color.

(Model for the spread of contagious disease)

What is the probability that the first and the second balls are both red?

Ans: $A$: "first red"; $B$: "Second red"

$P(A\cap B) = ?$ Not easy!!

But: $P(A) = \dfrac{n}{m+n}$ (very easy)

$P(B|A) = ?$ Also easy since

$$P(B|A) = \frac{n+1}{m+n+1}$$

$\Rightarrow$: $P(A\cap B) = P(A)\cdot P(B|A) = \dfrac{n}{m+n}\cdot\dfrac{n+1}{m+n+1}$.

Notes:

1. $P(A\cap B) = P(B)P(A|B)$ (if $P(B)\neq 0$

$= P(A)P(B|A)$ (if $P(A)\neq 0$

But $P(A\cap B) \neq P(A)P(A|B)$

2. If $P(B)=0$, one can not use $P(A\cap B) = P(B)P(A|B)$, then $P(A\cap B) = ?$

Ans: $P(A\cap B) = 0$

Reason: $\because \quad A\cap B \subset B$

$\therefore \quad 0 \leq P(A\cap B) \leq P(B)$

$\Rightarrow: \quad 0 \leq P(A\cap B) \leq 0$

$\Rightarrow: \quad P(A\cap B) = 0.$

3. How to choose A and B ?

According to your convenience !!!

4. $P(A \cap B \cap C) = ?$

let $A \cap B = D$, then

$$P(A \cap B \cap C) = P(D)P(C/D)$$

$$= P(A \cap B)P(C/A \cap B)$$

$$= P(A)P(B/A)P(C/A \cap B)$$

In general,

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1 \cap A_2 \cap \cdots \cap A_n)$$

$$= P(A_1)P(A_2/A_1)P(A_3/A_1 \cap A_2) \times \cdots$$

$$\times P(A_n/A_1 \cap A_2 \cap \cdots \cap A_{n-1})$$

For example, $P(A_1 \cap A_2 \cap A_3 \cap A_4)$

$$= P(A_1)P(A_2/A_1)P(A_3/A_1 \cap A_2)P(A_4/A_1 \cap A_2 \cap A_3)$$

# III. Law of Total Probability: (P20. Th. 1)

1. Idea: An example:

A school boy has 5 blue and 4 white marbles in his left pocket and 4 blue and 5 white marbles in his right pocket. If he transfers one marble at random from his left to his right pocket, what is the probability of his then drawing a blue from his right pocket?

Original

| | Blue | White |
|---|---|---|
| Left | 5 | 4 |
| Right | 4 | 5 |

One left $\longrightarrow$ right at random

Let A: drawing blue from right after transferring

$P(A) = ?$ Complicated?

But how about if we know the result of transfering? Easy, isn't it?

Indeed,

$B_1$ : the result of transfering (Blue)

then $P(A|B_1) = \frac{4+1}{9+1} = \frac{5}{10}$

$B_2$ : the result of transfering (white)

then $P(A|B_2) = \frac{4}{9+1} = \frac{4}{10}$

However, relation between $P(A)$ and $P(A|B_i)$ etc?

Note that $B_1 \cup B_2 = \Omega$ $B_1 \cap B_2 = \phi$

$\Rightarrow$: $A = A \cap \Omega = A \cap (B_1 \cup B_2)$

$= (A \cap B_1) \cup (A \cap B_2)$

Easy to see $A \cap B_1$ and $A \cap B_2$ are disjoint ($\because$ $B_1$ and $B_2$ are !!)

$\Rightarrow$: $P(A) = P(A \cap B_1) + P(A \cap B_2)$ (Think why!)

$= P(B_1) \cdot P(A|B_1) + P(B_2) \cdot P(A|B_2)$

How about $P(B_1)$ and $P(B_2)$?

Easy! $P(B_1) = \frac{5}{9}$ $P(B_2) = \frac{4}{9}$

($\because$ In left: 5 Blue + 4 white!!)

Now:

$$P(A) = \frac{5}{9} \times \frac{5}{10} + \frac{4}{9} \times \frac{4}{10} = \frac{25+16}{90} = \frac{41}{90}.$$

2. Law of Total Probability: Conclusion:

① Let $B_1, B_2, \cdots, B_n$ be such that

$$\bigcup_{i=1}^{n} B_i = \Omega \quad \text{and} \quad B_i \cap B_j = \phi \text{ for } i \neq j.$$

with $P(B_i) > 0$ for all $i$.

Then for any event $A$, we have

$$P(A) = \sum_{i=1}^{n} P(B_i) \cdot P(A/B_i) \quad (1.5.3)$$

② Proof: $\because A = A \cap \Omega = A \cap \left(\bigcup_{i=1}^{n} B_i\right) = \bigcup_{i=1}^{n} (A \cap B_i)$

$$\therefore P(A) = P\left(\bigcup_{i=1}^{n} (A \cap B_i)\right) = \sum_{i=1}^{n} P(A \cap B_i)$$

$$= \sum_{i=1}^{n} P(B_i) \cdot P(A/B_i)$$

$\because (A \cap B_i)$ are disjoint!

3. Notes:

① $\{B_i ; i=1,2,\cdots n\}$ is called a partition of $\Omega$

if $\bigcup_{i=1}^{n} B_i = \Omega$ and for any $i \neq j$, $B_i \cap B_j = \phi$

② The law is still true if the partition is "a sequence of events".

③ In application, the most important thing is to find a suitable partition. This very important method is called "Conditioning"

4. Example again:

In a certain population 5% of the females and 8% of the males are left-handed; 48% of the population are males. What is the probability that a randomly chosen member of the population is left-handed?

Analysis: Let $A$ be the event

"the chosen member is left-handed".

$P(A) = ?$

Conditioning on what ?

Certainly "gender"! (Since if we know the gender

then the conditional prob. is easy!)

Solution: Let $A$ : event "left-handed".

$B_1$ : event "male"

$B_2$ : event "Female"

then $P(B_1) = 0.48 \quad P(B_2) = 1 - 0.48 = 0.52$

$P(A|B_1) = 0.08 \quad P(A|B_2) = 0.05$

Now, by the law of total probability

$$P(A) = P(A \cap B_1) + P(A \cap B_2) = P(B_1) P(A|B_1) + P(B_2) \cdot P(A|B_2)$$

$$= 0.48 \times 0.08 + 0.52 \times 0.05 = 0.0644$$

(Check: $B_1 \cup B_2 = \Omega$ ; $B_1 \cap B_2 = \phi$ !!)

# IV. Bayes' Rule (Δ)

1. Example: Now return to the "left-handed" problem. We want to ask:

Suppose a member has been chosen and found left-handed. What's probability that the person is male?

Anal: A has occured, we want $P(B_1|A)$

What can we do? (When in doubt about a conditional prob., try the definition.)

$$P(B_1|A) = \frac{P(B_1 \cap A)}{P(A)}$$ (Does this help?

Sure!

Try to find
- denominator $P(A)$: ✓
- numerator: $P(B_1 \cap A) = P(B_1)P(A|B_1)$ ✓

Solution: Let $A$ be the event "left-handed", $B_1$ the event "male"; $B_2$ "female".

then $P(B_1) = 0.48 \quad P(B_2) = 0.52$

$P(A|B_1) = 0.08 \quad P(A|B_2) = 0.05$

Now

$$P(B_1|A) = \frac{P(A \cap B_1)}{P(A)}$$

$$= \frac{P(B_1) \cdot P(A|B_1)}{P(B_1) \cdot P(A|B_1) + P(B_2) \cdot P(A|B_2)}$$

$$= \frac{0.48 \times 0.08}{0.48 \times 0.08 + 0.52 \times 0.05} = \cdots$$

Similarly, we can get $P(B_2|A)$.

2. Generalization:

$\{B_k\}$ is a partition of $\Omega$;

$A$: another event.

$P(A|B_k)$ etc. easy to get.

Then how to get $P(B_k|A)$?

$$P(B_k|A) = \frac{P(A \cap B_k)}{P(A)} = \frac{P(B_k)\cdot P(A|B_k)}{\sum_n P(B_n)\cdot P(A|B_n)}$$

3. Bayes formula:

① Suppose the events $B_1, B_2, \cdots, B_n$ form a partition of $\Omega$ and if $P(B_i) > 0$ for each $B_i$, then for any other event $A$ and any $B_i$ in the partition

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{P(B_1)\cdot P(A|B_1) + P(B_2)\cdot P(A|B_2) + \cdots + P(B_n)P(A|}$$

② Proof:

$$P(B_i|A) = \frac{P(A\cap B_i)}{P(A)} \quad \text{(Definition !!)}$$

$$= \frac{P(B_i)\cdot P(A|B_i)}{P(A)} \quad \text{(Multiplication rule)}$$

$$= \frac{P(B_i)\cdot P(A|B_i)}{\sum_{k=1}^{n} P(B_k)\cdot P(A|B_k)}$$

4. Notes:

① Bayes rule also holds true if the partition of $\Omega$, $\{B_k\}$ is a sequence of events. We usually write it as

$$P(B_i|A) = \frac{P(B_i)\cdot P(A|B_i)}{\sum_{k} P(B_k)\cdot P(A|B_k)} \quad (1.5.4)$$

(The denominator in (1.5.4) is either a sum of finite terms or a series.)

② Memory: Using the "proof". !!

Desired prob.: $P(B_i|A)$

Numerator: the "reverse" conditional probability $P(A|B_i)$ <u>times</u> the prob. of the condition

Denominator: the sum of <u>all possible</u> terms <u>like the numerator.</u>

5. Further examples:

See the Problems in the book.

# §1.6 Independent Events (See P27)

## I. Independence of Two Events: (※)

### 1. Motivation and Idea:

"Independence" is a very important concept in Probability Theory and Statistics.

Suppose A and B are two events, we know usually $P(A) \neq P(A|B)$.

However, in some cases, they might be the same. In these cases, "given B occured" does not affect the prob. of event A. We then say "A and B are independent".

For some reasons, we give another equivalent definition.

Note that if $P(A) = P(A|B) \equiv \dfrac{P(A \cap B)}{P(B)}$

then $P(A \cap B) = P(A) \cdot P(B)$

2. Definition: Two events A and B are called independent events if

$$P(A \cap B) = P(A) \cdot P(B). \quad (1.6.1)$$

3. Notes:

① If A and B are independent, then so is B and A (See (1.6.1)).

Hence Def. (1.6.1) is "symmetric".

Also, condition $P(B) \neq 0$ is not needed.

② Condition (1.6.1) is convenient for checking the independence.

4. Example:

Experiment: A card is selected randomly from a deck.

Event A: "It is an ace"

Event B: "It is a diamond"

$\Rightarrow$: $P(A) = \frac{4}{52} = \frac{1}{13}$ $\quad P(B) = \frac{13}{52} = \frac{1}{4}$

$A \cap B$: "It is a diamond ace"

$P(A \cap B) = \frac{1}{52} = \frac{1}{13} \cdot \frac{1}{4}$ $\Rightarrow$ A B are independent

5. Property:

Theorem: If $A$ and $B$ are two independent events, then the following pairs of events are also independent.

(i) $A$ and $B^c$ ;

(ii) $A^c$ and $B$ ;

(iii) $A^c$ and $B^c$ .

Proof. Easy and thus omitted.

II. Independence of More than Two events:

1. Independence of Three Events:

① Definition: Three events $A$, $B$, and $C$, are called (mutually) independent if

(i) $P(A \cap B) = P(A) \cdot P(B)$ , $P(A \cap C) = P(A) \cdot P(C)$

$P(B \cap C) = P(B) \cdot P(C)$ and

(ii) $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$ .

⊙ Note:

Only (i) holds true can not imply A, B, and C are independent.

((i) only is usually called pair-wise indep...)

Also, only (ii) is not enough for "independence".

2. Independence of n events:

Def.: n events $A_1, A_2, \cdots, A_n$ are called (mutually) independent if the following hold:

(i) for all pairs $A_i$ and $A_j$ $(i \neq j)$

$$P(A_i \cap A_j) = P(A_i) \cdot P(A_j)$$

(ii) for all triples $A_i, A_j, A_k$ ($i, j, k$ all different)

$$P(A_i \cap A_j \cap A_k) = P(A_i) P(A_j) \cdot P(A_k)$$

(iii) for all quadreples $A_i, A_j, A_k, A_l$, ($i, j, k, l$ are all different)

$$P(A_i \cap A_j \cap A_k \cap A_l) = P(A_i)P(A_j)P(A_k)P(A_l);$$

- - - -

(until finally)

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2)\cdots P(A_n).$$

3. Independence of infinitely many events.

We define an infinite set of events to be independent if every finite subset of these events is independent.

4. Remark:

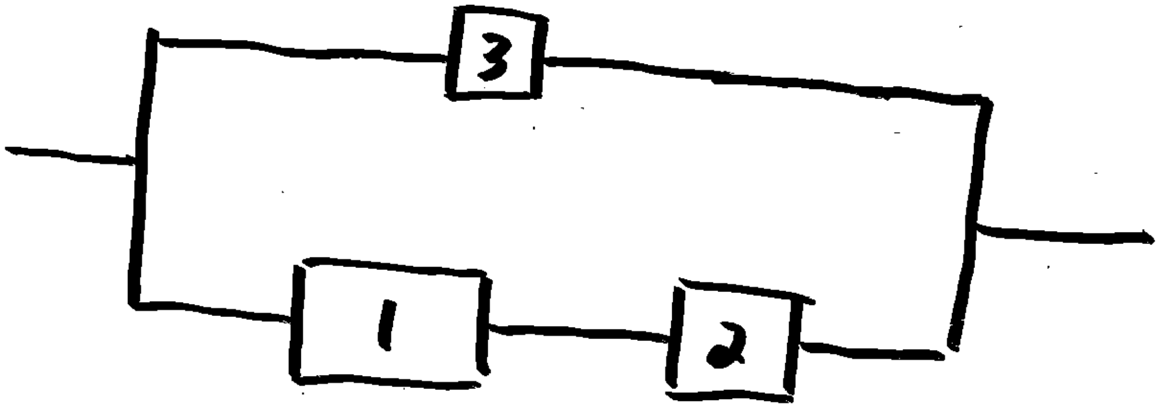
If $A_1, A_2, \cdots A_n$ are independent then

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2)\cdots P(A_n)$$

i.e.
$$P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P(A_i) \qquad (1.6.2)$$

5. Example: (Δ)

Consider a circuit with three relays:




Assume that three relays are mutually independent and the working probability of each relay is $p$.

What is the prob. that current flows through the circuit?

Analysis: Let $A_i$ denote the event that the $i$th relay works ($i=1,2,3$).

Let $F$ denote the event that current flows through the circuit.

Then $F = A_3 \cup (A_1 \cap A_2)$ (Think why here!)

So, $P(F) = P(A_3) + P(A_1 \cap A_2) - P(A_1 \cap A_2 \cap A_3)$

$= p + p^2 - p^3$

# Summary of Chapter 1 ※



## I. Basic Concepts:

1. Sample space: $\Omega$
2. Events: (Impossible event $\phi$; Certain event $\Omega$; Elementary event; General event...)
3. Probability: (Set function: event $\to R$)
4. Independence:
5. Conditional probability:
6. Disjoint Events:
7. Partition of $\Omega$:

## II. Operations of Events:

1. Union: $A \cup B = \{$Either A or B occurs$\}$
2. Intersection: $A \cap B = \{$Both A and B occur$\}$
3. Complement: $A^c = \{$A does not occur$\}$
4. $\bigcup_{k=1}^{n} A_k$ and $\bigcup_{k=1}^{\infty} A_k$; $\bigcap_{k=1}^{n} A_k$ and $\bigcap_{k=1}^{\infty} A_k$.

## III. Properties of Probability:

1. $0 \le P(A) \le 1, \quad \forall A.$
2. $P(\phi) = 0; \quad P(\Omega) = 1.$
3. $A \subset B \implies P(A) \le P(B)$
4. $\{B_k\}$ disjoint $\implies P(\bigcup_k B_k) = \sum_k P(B_k)$
5. $\{B_k\}$ independent $\implies P(\bigcap_{k=1}^{n} B_k) = \prod_{k=1}^{n} P(B_k)$

## IV. Important Formulae:

1. $P(A^c) = 1 - P(A)$
2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
3. $P(A \cap B) = P(A) \cdot P(B|A) = P(B) \cdot P(A|B)$
4. $P(A \cup B) = P(A) + P(B)$, if $A, B$ disjoint,
5. $P(A \cap B) = P(A) \cdot P(B)$, if $A, B$ independent
6. If $\{B_k\}$ is a partition of $\Omega$, then

$$\forall A, \quad P(A) = \sum_k P(B_k) \cdot P(A|B_k)$$

and
$$P(B_n|A) = \frac{P(B_n) \cdot P(A|B_n)}{\sum_k P(B_k) \cdot P(A|B_k)}.$$