Some Other Continuous Random Variables

1. The Gamma Distribution

① Def. A r.v. X is said to follow a gamma distribution with parameter (α, λ) where $\alpha > 0$, $\lambda > 0$ if its pdf is given by

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad (1.1)$$

where $\Gamma(\alpha)$, called the gamma function, is defined as

$$\Gamma(\alpha) = \int_0^\infty e^{-y} \cdot y^{\alpha-1}\, dy \qquad (1.2)$$

Hence, X can only take non-negative values.

Easy to see: $\displaystyle\int_0^\infty f(x)dx = \frac{1}{\Gamma(\alpha)} \int_0^\infty \lambda e^{-\lambda x}(\lambda x)^{\alpha-1} dx = 1$

Indeed, letting $\lambda x = y$ (and noting $\lambda > 0$) yields

$$\int_0^\infty \lambda e^{-\lambda x}(\lambda x)^{\alpha-1} dx = \int_0^\infty e^{-y} \cdot y^{\alpha-1} dy = \Gamma(\alpha).$$

② Some simple facts about gamma function (1.2).

$$\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1} dy \quad (\alpha > 0) \qquad (1.2)$$

⌈ The integration by parts of $\Gamma(\alpha)$ yields that for $\alpha > 1$

$$\Gamma(\alpha) = -\int_0^\infty y^{\alpha-1} de^{-y} = -y^{\alpha-1}e^{-y}\Big|_0^\infty + \int_0^\infty e^{-y} dy^{\alpha-1}$$

$$= \int_0^\infty e^{-y} dy^{\alpha-1} = \int_0^\infty e^{-y}\cdot(\alpha-1)\, y^{\alpha-1-1} dy$$

$$= (\alpha-1)\int_0^\infty e^{-y} y^{\alpha-1-1} dy$$

$$= (\alpha-1)\,\Gamma(\alpha-1) \quad \text{(see (1.2)!!)} \qquad (1.3)$$

If $\alpha > 2$ we could continue

$$\Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1) = (\alpha-1)(\alpha-2)\,\Gamma(\alpha-2)$$

If $\alpha > 3$, we could get

$$\Gamma(\alpha) = (\alpha-1)(\alpha-2)(\alpha-3)\,\Gamma(\alpha-3)$$

etc. By repeated application of the recursion formula (1.3) can reduce the form of $\Gamma(\alpha)$ until $\Gamma(x)$ for $0 < x \le 1$.

In particular, if $\alpha = n$ (a positive integer)

then (1.3) yields

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = \cdots$$

$$= (n-1)(n-2)(n-3)\cdots 2\times 1\times \Gamma(1)$$

Now $\Gamma(1) = \int_0^\infty e^{-y} y^{1-1} dy$ $\quad \left(\because \Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1} dy \text{ !!}\right)$

$$= \int_0^\infty e^{-y} dy = (-e^{-y})\Big|_0^{+\infty} = 1$$

$\Rightarrow$ if $\alpha = n$ (a positive integer !)

then $\Gamma(n) = (n-1)!$ $\quad$ (n: positive integer) $\quad$ (1.4)

If $\alpha = n + \frac{1}{2}$ (n: positive integer)

then $\Gamma(n+\frac{1}{2}) = \Gamma(n+\frac{1}{2}-1)(n+\frac{1}{2}-1)$

$$= \frac{2n-1}{2}\Gamma\left(\frac{2n-1}{2}\right)$$

$$= \frac{2n-1}{2}\cdot\frac{2n-3}{2}\Gamma\left(\frac{2n-3}{2}\right) = \cdots$$

$$= \frac{(2n-1)}{2}\cdot\frac{2n-3}{2}\cdots\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)$$

In short for a positive integer n

$$\Gamma(n+\tfrac{1}{2}) = \frac{(2n-1)!!}{2^n}\,\Gamma(\tfrac{1}{2}) \qquad (1.5)$$

what is $\Gamma(\frac{1}{2})$? By definition (see 1.2 again)

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty x^{\frac{1}{2}-1} e^{-x}\,dx = \int_0^\infty x^{-\frac{1}{2}} e^{-x}\,dx = \int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$$

Let $x = +\frac{y^2}{2}$ (why so? Recall standard normal $N(0,1)$!!!)

then $dx = +y\,dy$

$$\Rightarrow \Gamma(\tfrac{1}{2}) = \int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx = \int_0^\infty \frac{e^{-\frac{y^2}{2}}}{\frac{y}{\sqrt{2}}}\,y\,dy = \sqrt{2}\int_0^\infty e^{-\frac{y^2}{2}}\,dy$$

But easy to see $\displaystyle\int_{-\infty}^{0} e^{-\frac{y^2}{2}}\,dy = \int_{+\infty}^{0} e^{-\frac{u^2}{2}}\,d(-u) = \int_0^\infty e^{-\frac{y^2}{2}}\,dy$

Hence $\displaystyle\int_0^\infty e^{-\frac{y^2}{2}}\,dy = \frac{1}{2}\int_{-\infty}^{+\infty} e^{-\frac{y^2}{2}}\,dy$

$$\Rightarrow \Gamma(\tfrac{1}{2}) = \frac{\sqrt{2}}{2}\int_{-\infty}^{+\infty} e^{-\frac{y^2}{2}}\,dy = \frac{\sqrt{\pi}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{y^2}{2}}\,dy$$

$= \sqrt{\pi}$ (Recall ~~stand~~ $N(0,1)$!!)

Hence for $n+\frac{1}{2}$ with n: positive integer we get that

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(1+\tfrac{1}{2}\right) = \frac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma\left(2+\tfrac{1}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{3!!}{2^2}\sqrt{\pi}$$

In general

$$\Gamma\left(n+\tfrac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$$

③ Now, return to gamma distribution, if $\alpha = 1$, then

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{1-1}}{\Gamma(1)} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

i.e., $$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Exponentially distributed with parameter $\lambda > 0$.

If $\alpha = n$: positive integer

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x} (\lambda x)^{n-1}}{(n-1)!} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Very useful in Queueing theory.

2. The Weibull Distribution

see our tutorial question

Useful in engineering problem and life assurance!

3. Lognormal Distribution.

see before. Very useful in FM.

4. χ^2-distribution, t, F distributions

Very useful in Statistics.

5. The Cauchy Distribution

① Def. A r.v. Y is called to follow a Cauchy distribution with parameter μ $(-\infty < \mu < \infty)$, if its pdf $f_Y(y)$ is given by

$$f_Y(y) = \frac{1}{\pi} \cdot \frac{1}{[1+(y-\mu)^2]} \qquad (-\infty < y < \infty)$$

If the parameter $\mu = 0$, we usually called it standard Cauchy distribution.

It is easy to see that the above $f_Y(y)$ is indeed a pdf.

Indeed, $\forall -\infty < y < \infty$, $f_Y(y) > 0$ is clear.

Also, $$\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{dy}{1+(y-\mu)^2} = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{dz}{1+z^2} = \frac{1}{\pi} \cdot \pi = 1.$$

② The standard Cauchy distribution (i.e. $\mu = 0$) has an interesting explanation and also, has a close link with the uniform distribution $U[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Let X be the uniformly distributed on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ (we use ~~of~~ open interval here)

Hence the pdf is

$$f_X(x) = \begin{cases} \frac{1}{\pi} & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

Let $$Y = \tan(X)$$

⌜ i.e. the function form is $y = \tan(x)$ ⌟

i.e. $g(x) = \tan(x)$ $\therefore$ $g(x) \uparrow\uparrow$ $(-\frac{\pi}{2} < x < \frac{\pi}{2})$

Now $g^{-1}(y) = \tan^{-1}(y) \equiv \arctan(y)$

$$\Rightarrow \frac{d}{dy} g^{-1}(y) = \frac{d}{dy} \tan^{-1}(y) = \frac{1}{1+y^2}$$

Now by our general formula, we get that the pdf of Y

is
$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$= f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y) \quad (\because \text{in our case } g(\cdot) \uparrow\uparrow)$$

$$= f_X(g^{-1}(y)) \cdot \frac{1}{1+y^2}$$

Note $g^{-1}(y) = \tan^{-1}(y)$ taking values $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

But

$$f_X(x) = \begin{cases} \frac{1}{\pi} & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

Hence
$$f_X(g^{-1}(y)) = \frac{1}{\pi}$$

Therefore the pdf of Y is

$$f_Y(y) = \frac{1}{\pi} \cdot \frac{1}{1+y^2} \quad (-\infty < y < +\infty).$$

(thus standard Cauchy distribution).

The conclusion can also be obtained by using cdf method.

Recall X is uniformly distributed on $(-\frac{\pi}{2}, \frac{\pi}{2})$

Hence for any $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ we have

$$P\{X \leq x\} = \int_{-\frac{\pi}{2}}^{x} f_X(y)\,dy = \int_{-\frac{\pi}{2}}^{x} \frac{1}{\pi}\,dy = \frac{x-(-\frac{\pi}{2})}{\pi}$$

i.e. $P\{X \leq x\} = \frac{x}{\pi} + \frac{1}{2}$

Now, $F_Y(y) = \Pr\{Y \leq y\}$

$= \Pr\{\tan X \leq y\}$ $(\because Y = \tan X)$

$= \Pr\{X \leq \tan^{-1}(y)\}$ (should be clear here !!)

$= \frac{1}{2} + \frac{1}{\pi}\cdot\tan^{-1}(y)$ (see above)

$\Rightarrow$

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}\left[\frac{1}{2} + \frac{1}{\pi}\cdot\tan^{-1}(y)\right]$$

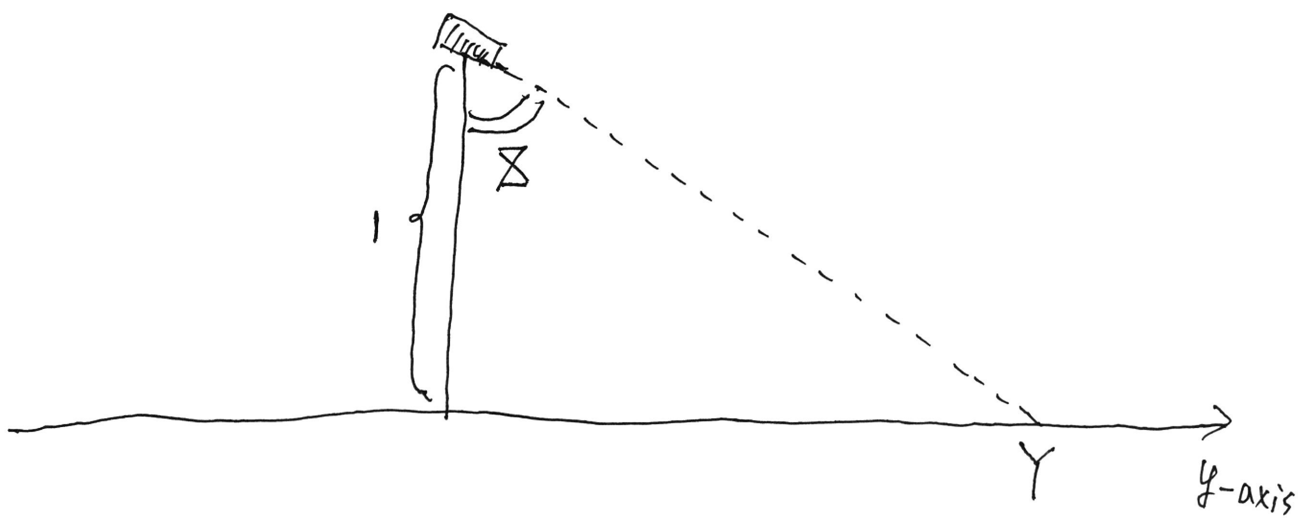
$$= \frac{1}{\pi}\frac{d}{dy}\tan^{-1}(y) = \frac{1}{\pi}\cdot\frac{1}{1+y^2}$$

We get the same conclusion.

③ " Experiment "

Suppose that a narrow beam flashlight is spun around its centre, which is located a unit distance from the y-axis. When the flashlight has stopped spinning, then the point Y (a r.v) at which the beam intersects the y-axis is just the standard Cauchy distribution.

See Figure below.



Clearly: $Y = g(X) = \tan X$

and X is uniformly distributed on $(-\frac{\pi}{2}, \frac{\pi}{2})$

「 Of course, we could use x to replace y.

i.e. X obeys the standard Cauchy dis if $f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$