Properties of Joint pdf

1. For two (absolutely) continuous random variables $X(\omega)$ and $Y(\omega)$, the joint cdf $F_{(X,Y)}(x, y)$, or simply $F(x, y)$ is defied as

$$F(x, y) = P_r\{ X(\omega) \le x,\ Y(\omega) \le y\}. \tag{1.1}$$

The joint cdf for two continuous random variables X and Y has the following properties:

(i) For any fixed x, $F(x, y)$ is an increasing function of y and for any fixed y, $F(x, y)$ is an increasing function of x.

(ii) $$\lim_{\substack{x \to +\infty \\ y \to +\infty}} F(x, y) \triangleq F(+\infty, +\infty) = 1 \tag{1.2}$$

(iii) For any fixed y, $$\lim_{x \to -\infty} F(x, y) = 0 \tag{1.3}$$

For any fixed x, $$\lim_{y \to -\infty} F(x, y) = 0 \tag{1.4}$$

「Could be denoted as

$\forall y \quad F(-\infty, y) = 0$ and $\forall x \quad F(x, -\infty) = 0$ 」

(iv) Suppose that $x_1 < x_2$, $y_1 < y_2$, then

$$\Pr\{x_1 < X(w) \leq x_2,\quad y_1 < Y(w) \leq y_2\}$$

$$= F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1) \quad (*)$$

$$\geq 0 \qquad (1.5)$$

Furthermore, we have ($\because$ X and Y are both continuous r.vs)

$$\Pr\{x_1 < X(w) \leq x_2,\quad y_1 < Y(w) \leq y_2\}$$

$$= \Pr\{x_1 < X(w) < x_2,\quad y_1 < Y(w) < y_2\}$$

$$= \Pr\{x_1 \leq X(w) \leq x_2,\quad y_1 \leq Y(w) \leq y_2\} \qquad (1.6)$$

$$= \cdots$$

$$= \cdots \qquad (**)$$

(v) $\dfrac{\partial^2 F(x, y)}{\partial x \partial y}$ exists and, in particular,

For any fixed x, $F(x, y)$ is differentiable (and thus continuous) function of y and for any fixed y, $F(x, y)$ is differentiable (and thus continuous) function of x.

2. Marginal cdfs: (Two continuous r.v.s $X(\omega)$ and $Y(\omega)$)

$$F_X(x) = \Pr\{X \le x\} \quad \text{and} \quad F_Y(y) = \Pr\{Y \le y\}$$

How to get marginal ones from joint one?

$$F_X(x) = \lim_{y \to +\infty} F(x, y) \tag{2.1}$$

and
$$F_Y(y) = \lim_{x \to +\infty} F(x, y) \tag{2.2}$$

"Proof"
$$\lim_{y \to \infty} F(x, y) = \lim_{y \to \infty} \Pr\{X \le x, \ Y \le y\}$$

$$(?) = \Pr\left\{\lim_{y \to \infty} \left[(X \le x) \cap (Y \le y)\right]\right\} \quad \text{(conti. property of Pr.)}$$

$$= \Pr\{X \le x, \ Y < \infty\}$$

$$= \Pr\{X \le x\} \qquad (\because \ (\omega;\ Y(\omega) < \infty) = \Omega \ !!)$$

$$= F_X(x)$$ This proves (2.1).

(2.2) can be similarly proven.

3. Joint pdf of two continuous random variables X and Y

(i) Joint cdf $F(x,y)$; joint pdf $f(x,y)$

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y} \equiv \frac{\partial^2 F(x,y)}{\partial y \partial x} \qquad (3.1)$$

Hence
$$F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(u,v)\,du\,dv \qquad (3.2)$$

(ii) Properties of joint pdf $f(x,y)$

(a) $$f(x,y) \geq 0 \qquad (\forall x \in \mathbb{R},\ \forall y \in \mathbb{R}) \qquad (3.3)$$

(b) $$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y)\,dx\,dy = 1 \qquad (3.4)$$

"proof" (a) follows from (1.5), i.e.

$$\boxed{\begin{array}{l} \forall x_1 < x_2, \quad y_1 < y_2, \\ F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1) \geq 0 \end{array}}$$

For (b), recall $\lim_{\substack{x \to +\infty \\ y \to +\infty}} F(x,y) = 1$

and thus

$$1 = \lim_{\substack{x \to +\infty \\ y \to +\infty}} F(x, y) = \lim_{\substack{x \to +\infty \\ y \to +\infty}} \Pr\{X \leq x, Y \leq y\}$$

$$= \lim_{\substack{x \to +\infty \\ y \to +\infty}} \int_{-\infty}^{y} \int_{-\infty}^{x} f(u, v)\, du\, dv \qquad \text{(see (3.21))}$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(u, v)\, du\, dv$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)\, dx\, dy.$$

(3.4) is thus proven.

4. Joint pdf and Marginal pdfs:

$$f_X(x) = \int_{-\infty}^{+\infty} f(x, y)\, dy \tag{4.1}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x, y)\, dx \tag{4.2}$$

"Proof"

$$F_X(x) = P_r\{X \le x\}$$

$$= P_r\{X \le x,\ Y < \infty\}$$

$$= \lim_{y \to \infty} P_r\{X \le x,\ Y \le y\} \quad \text{(shown before !!)}$$

$$= \lim_{y \to +\infty} F(x, y)$$

$$= \lim_{y \to +\infty} \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v)\, du\, dv \quad \text{(Again, see (3.2) !!)}$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{+\infty} f(u, v)\, du\, dv$$

$$= \int_{-\infty}^{x} \left[\int_{-\infty}^{+\infty} f(u, v)\, dv \right] du$$

$$= \int_{-\infty}^{x} \left[\int_{-\infty}^{+\infty} f(u, y)\, dy \right] du \tag{4.3}$$

By (4.3) we have

$$\frac{d F_X(x)}{dx} = \int_{-\infty}^{+\infty} f(u, y) dy = \int_{-\infty}^{+\infty} f(x, y) dy$$

Recall if $g(x) = \int_{-\infty}^{x} h(y) dy$

then $g'(x) = h(x)$

Obvious !!

This proves (4.1). (4.2) can be similarly proven.

5. Independence

(i) Two continuous random variables X and Y are independent if and only if

$$f(x, y) = f_X(x) \cdot f_Y(y) \qquad (5.1)$$

The definition tells us that X and Y are independent if

$$F(x, y) = F_X(x) \cdot F_Y(y) \qquad (5.2)$$

where $F(x, y)$, $F_X(x)$, $F_Y(y)$ are (joint/marginal) cdfs
and $f(x, y)$, $f_X(x)$, $f_Y(y)$ are (joint/marginal) pdf's

"Proof": (5.1) $\Longleftrightarrow$ (5.2)

(i) (5.1) $\Longrightarrow$ (5.2). If $f(x,y) = f_X(x)\cdot f_Y(y)$ is true, then

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,du\,dv$$

$$= \int_{-\infty}^{x}\int_{-\infty}^{y} f_X(u)\cdot f_Y(v)\,du\,dv \qquad \text{(By (5.1)!!)}$$

$$= \int_{-\infty}^{x} f_X(u)\,du \cdot \int_{-\infty}^{y} f_Y(v)\,dv$$

$$= F_X(x)\cdot F_Y(y) \qquad \text{(and hence (5.2))}$$

(ii) (5.2) $\Longrightarrow$ (5.1)

Conversely, if (5.2) is true, i.e. $F(x,y) = F_X(x)\cdot F_Y(y)$

Hence
$$\frac{\partial F(x,y)}{\partial x} = \frac{\partial}{\partial x}\left[F_X(x)\,F_Y(y)\right] = \left(\frac{\partial}{\partial x}F_X(x)\right)\cdot F_Y(y)$$

$$= f_X(x)\cdot F_Y(y)$$

Furthermore,
$$\frac{\partial^2 F(x,y)}{\partial x\,\partial y} = \frac{\partial}{\partial y}\left[f_X(x)\cdot F_Y(y)\right] = f_X(x)\left[\frac{d}{dy}F_Y(y)\right]$$

$$= f_X(x)\cdot f_Y(y).$$

The proof is complete. $\square$