Continous r.v. (Examples)

1. Example 1. A conti. r.v. whose pdf is given by

$$f(x) = \begin{cases} c\,(3-x) & \text{if } 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

① Find the value of c.

② what is the cdf of this r.v.?

Solution. ① $\int_{-\infty}^{+\infty} f(x)\,dx = 1$ i.e $\int_0^3 c\,(3-x)\,dx = 1$

$$\Rightarrow c\left[\int_0^3 (3-x)\,dx\right] = 1 \quad \text{or} \quad \frac{1}{c} = \int_0^3 (3-x)\,dx = \left[3x - \frac{x^2}{2}\right]\Bigg|_0^3$$

$$\text{i.e, } \frac{1}{c} = 9 - \frac{3^2}{2} - 0 = \frac{18-9}{2} = \frac{9}{2}$$

$$\therefore\ c = \frac{2}{9}$$

② cdf. $F(x) = P\{X \le x\} = \int_{-\infty}^{x} f(y)\,dy$

If $x < 0$, then $\int_{-\infty}^{x} f(y)\,dy = \int_{-\infty}^{x} 0\,dy = 0$

If $0 \le x \le 3$ then $\int_{-\infty}^{x} f(y)\,dy = \int_0^x f(y)\,dy = \int_0^x \frac{2}{9}(3-y)\,dy$

$$= \frac{2}{9}\left[3y - \frac{y^2}{2}\right]\Bigg|_0^x = \frac{2}{9}\left[3x - \frac{x^2}{2}\right] = \frac{2}{3}x - \frac{x^2}{9} \equiv 1 - \frac{1}{9}(3-x)^2.$$

[可写为 $1 - \frac{1}{9}(3-x)^2 = 1 - \frac{1}{9}(9 - 6x + x^2) = 1 - 1 + \frac{2}{3}x - \frac{1}{9}x^2$]

If $x \geq 1$. then $F(x) = 1$

Solution.
$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - \frac{1}{9}(3-x)^2 & \text{if } 0 \le x < 3 \\ 1 & \text{if } x \geq 3 \end{cases}$$

2. Example 2. Suppose the cdf of a certain continuous rv is given by

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x \ge 1. \end{cases}$$

What is the pdf of this r.v.?

⌈Remark: It is clear that $F(x)$ is differentiable everywhere except possibly at points $\{0, 1\}$. (In fact, it has a derivative at zero)

Note, Changing a density at a finite set of points does not change the cdf. (※※) even countable !!

Hence, strictly speaking, it is incorrect to speak of the density of a rv; rather, we should refer to a density.

But we shall not pay much attention to this point.⌋

Return to example, then it is easy to see. (except at points $\{0, 1\}$) we have

$$F'(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2x & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1. \end{cases}$$

Then how about the points $\{0, 1\}$? We could assign any value !!!
(but must be non-negative !!)

Suppose we assign $f(0) = f(1) = 0$, then the pdf is given by

$$f(x) = \begin{cases} 2x & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Another pdf (for the same r.v. with the unique cdf $F(x)$ above) could be

$$f_2(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

[Note: At point 1, $f(1) = 0$ but $f_2(1) = 2$.]

We could even give another density by changing the values at $x = \frac{1}{7}$, 1 and -6, say, as follows

$$f_3(x) = \begin{cases} 2x & \text{if } 0 < x < 1 \text{ but } x \ne \frac{1}{7} \\ 3 & \text{if } x = \frac{1}{7} \\ \pi & \text{if } x = 1 \\ 17.6 & \text{if } x = -6 \\ 0 & \text{otherwise} \end{cases}$$

The three densities are, of course, different, but they have the same cdf $F(x)$ defined above. $\boxed{\text{Remember: Any r.v has just one cdf.}}$

3. Example 3.

Consider the r.v. whose cdf is

$$F(x) = \begin{cases} 0 & \text{if } x<0 \\ 1-e^{-2x}-2xe^{-2x} & \text{if } x\geq 0 \end{cases}$$

Solution. Clearly, we may differentiate this function everywhere except perhaps at $x=0$

If $x<0$, then $F'(x)=0$.

If $x>0$, then $F'(x) = (-1)\cdot e^{-2x}\cdot(-2) - \left[2e^{-2x} + 2x\cdot e^{-2x}\cdot(-2)\right]$

i.e. $F'(x) = 2e^{-2x} - 2e^{-2x} + 4xe^{-2x} = 4xe^{-2x}$.

As to $x=0$, we could let it being 0.

Hence, a density is given by

$$f(x) = \begin{cases} 4xe^{-2x} & \text{if } x>0 \\ 0 & \text{otherwise.} \end{cases}$$