SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF MATHEMATICS

MA215 Probability Theory

Tutorial 02Solu

Set: Monday 19th September 2016; Hand in: Monday, 26th September 2016.
Note: Hand in your solutions no later than 4pm of Monday, 26th September.

1. Two six-sided dice are thrown sequentially, and the face values that come up are recorded.

   (a) List the sample space S.

   (b) List the elements that make up the following events:

   (1) $A =$ the sum of the two values is at least 5;

   (2) $B =$ the value for the first die is higher than the value of the second;

   (3) $C =$ the first value is 4.

   (c) List the elements of the following events:

   (1) $A \cap C$;

   (2) $B \cup C$;

   (3) $A \cap (B \cup C)$.

Proof:

(a)

$$S = \left\{ \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right\}.$$

(b)

$$A = \left\{ \begin{array}{cccccc} & & & (1,4) & (1,5) & (1,6) \\ & & (2,3) & (2,4) & (2,5) & (2,6) \\ & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right\}.$$

$$B = \left\{ \begin{array}{lllll} (2,1) & & & & \\ (3,1) & (3,2) & & & \\ (4,1) & (4,2) & (4,3) & & \\ (5,1) & (5,2) & (5,3) & (5,4) & \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) \end{array} \right\}.$$

$$C = \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\}.$$

(c)

$$A \cap C = \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\}.$$

$$B \cup C = \left\{ \begin{array}{llllll} (2,1) & & & & & \\ (3,1) & (3,2) & & & & \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & & \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & \end{array} \right\}.$$

$$A \cap (B \cup C) = \left\{ \begin{array}{llllll} & (3,2) & & & & \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & & \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & \end{array} \right\}.$$

2. Let A and B be arbitrary events. Let C be the event that either A occurs or B occurs, but not both. Express C in terms of A and B using any of the basic operations of union, intersection, and complement.

   Proof: $C = (A \cup B) \setminus (A \cap B) = (A \cup B) \cap (A \cap B)^c = (A \cup B) \cap (A^c \cup B^c)$.

3. Suppose A and B are two events such that $A \subset B$. show that

$$P(B \setminus A) = P(B) - P(A).$$

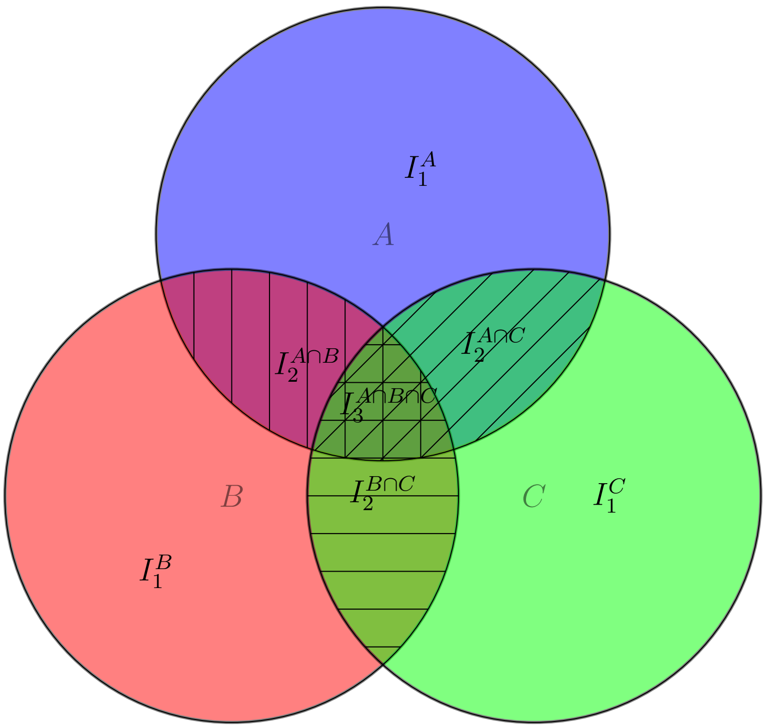
   Proof: Suppose A and B are two events such that $A \subset B$. Then we can obtain $B = A \uplus (B \setminus A)$*, so $P(B) = P(A) + P(B \setminus A), \Longrightarrow$

$$P(B \setminus A) = P(B) - P(A).$$

*Symbolic $\uplus$ represent disjoint union

4. Verify the following extension of the addition rule (a) by an appropriate Venn diagram and (b) by a formal argument using the axioms of probability and the propositions in the first chapter.

$$\begin{aligned} P(A \cup B \cup C) =& P(A) + P(B) + P(C) \\ & - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ & + P(A \cap B \cap C) \end{aligned}$$



Proof:

(a) See the above figure. we have

$$A \cup B \cup C = \left[I_1^A \uplus I_1^B \uplus I_1^C\right] \uplus \left[I_2^{A\cap B} \uplus I_2^{A\cap C} \uplus I_2^{B\cap C}\right] \uplus I_3^{A\cap B\cap C}.$$

Hence

$$\begin{aligned} P(A \cup B \cup C) =& \boxed{P(I_1^A) + P(I_1^B) + P(I_1^C)} \\ & + \boxed{P(I_2^{A\cap B}) + P(I_2^{A\cap C}) + P(I_2^{B\cap C})} \\ & + P(I_3^{A\cap B\cap C}) \\ =& \boxed{P\left(A \setminus \left(I_2^{A\cap B} \uplus I_2^{A\cap C} \uplus I_3^{A\cap B\cap C}\right)\right) + P\left(B \setminus \left(I_2^{A\cap B} \uplus I_2^{B\cap C} \uplus I_3^{A\cap B\cap C}\right)\right) + P\left(C \setminus \left(I_2^{A\cap C} \uplus I_2^{B\cap C} \uplus I_3^{A\cap B\cap C}\right)\right)} \\ & + \boxed{P(I_2^{A\cap B}) + P(I_2^{A\cap C}) + P(I_2^{B\cap C})} \\ & + P(I_3^{A\cap B\cap C}) \\ =& \boxed{P(A) - P\left(I_2^{A\cap B} \uplus I_2^{A\cap C} \uplus I_3^{A\cap B\cap C}\right) + P(B) - P\left(I_2^{A\cap B} \uplus I_2^{B\cap C} \uplus I_3^{A\cap B\cap C}\right) + P(C) - P\left(I_2^{A\cap C} \uplus I_2^{B\cap C} \uplus I_3^{A\cap B\cap C}\right)} \end{aligned}$$

$$
\begin{aligned}
&+ \boxed{P(I_2^{A\cap B}) + P(I_2^{A\cap C}) + P(I_2^{B\cap C})} \\
&+ P(I_3^{A\cap B\cap C}) \\
= &\boxed{P(A) - [P(I_2^{A\cap B}) + \cancel{P(I_2^{A\cap C})} + P(I_3^{A\cap B\cap C})] + P(B) - [\cancel{P(I_2^{A\cap B})} + P(I_2^{B\cap C}) + P(I_3^{A\cap B\cap C})] + P(C) - [P(I_2^{A\cap C}) + \cancel{P(I_2^{B\cap C})} + P(I_3^{A\cap B\cap C})]} \\
&+ \boxed{\cancel{P(I_2^{A\cap B})} + \cancel{P(I_2^{A\cap C})} + \cancel{P(I_2^{B\cap C})}} \\
&+ P(I_3^{A\cap B\cap C}) \\
= &P(A) + P(B) + P(C) \\
&- \boxed{[P(I_2^{A\cap B}) + P(I_3^{A\cap B\cap C})] - [P(I_2^{B\cap C}) + P(I_3^{A\cap B\cap C})] - [P(I_2^{A\cap C}) + P(I_3^{A\cap B\cap C})]} \\
&+ P(I_3^{A\cap B\cap C}) \\
= &P(A) + P(B) + P(C) \\
&- P(A\cap B) - P(A\cap C) - P(B\cap C) \\
&+ P(A\cap B\cap C)
\end{aligned}
$$

.

(b) Suppose E and F are any two events, Note that

$$E \cup F = (E \setminus F) \uplus (F \setminus E) \uplus (E \cap F).$$

Then, we can obtain

$$
\begin{aligned}
P(E \cup F) &= P\left((E \setminus F) \uplus (F \setminus E) \uplus (E \cap F)\right) \\
&= P(E \setminus F) + P(F \setminus E) + P(E \cap F) \\
&= [P(E \setminus F) + P(E \cap F)] + [P(F \setminus E) + P(E \cap F)] - P(E \cap F) \\
&= P(E) + P(F) - P(E \cap F).
\end{aligned}
$$

$$
\begin{aligned}
&P(A \cup B \cup C) \\
&= P\left((A \cup B) \cup C\right) \\
&= P(A \cup B) + P(C) - P\left((A \cup B) \cap C\right) \\
&= P(A \cup B) + P(C) - P\left((A \cap C) \cup (B \cap C)\right) \\
&= [P(A) + P(B) - P(A \cap B)] + P(C) - P\left((A \cap C) \cup (B \cap C)\right) \\
&= [P(A) + P(B) - P(A \cap B)] + P(C) - [P(A \cap C) + P(B \cap C) - P\left((A \cap C) \cap (B \cap C)\right)] \\
&= [P(A) + P(B) - P(A \cap B)] + P(C) - [P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)] \\
&= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)
\end{aligned}
$$

5. Suppose $\{A_i; 1 \leq i \leq n\}$ are events.

(i) Show that **inclusion-exclusion formula:**

$$P(\overset{n}{\underset{i=1}{\cup}} A_i) = \sum_{i=1}^{n} P(A_i) - \sum\sum_{1\leq i<j\leq n} P(A_i \cap A_j) + \sum\sum\sum_{1\leq i<j<k\leq n} P(A_i \cap A_j \cap A_k)$$
$$- \ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n).$$

(ii) Write this formula for cases of $n = 2, n = 3, n = 4$ and $n = 5$ clearly.

Proof:

(i) (**Use the Mathematical induction**) For $n = 2$, we have

$$\begin{aligned} P(A_1 \cup A_2) &= P\left((A_1 \setminus A_2) \dot\cup (A_2 \setminus A_1) \dot\cup (A_1 \cap A_2)\right) \\ &= P(A_1 \setminus A_2) + P(A_2 \setminus A_1) + P(A_1 \cap A_2) \\ &= [P(A_1 \setminus A_2) + P(A_1 \cap A_2)] + [P(A_2 \setminus A_1) + P(A_1 \cap A_2)] - P(A_1 \cap A_2) \\ &= P(A_1) + P(A_2) - P(A_1 \cap A_2). \end{aligned}$$

Assume, for $n = m$, we have

$$P(\overset{m}{\underset{i=1}{\cup}} A_i) = \sum_{i=1}^{m} P(A_i) - \sum\sum_{1\leq i<j\leq m} P(A_i \cap A_j) + \sum\sum\sum_{1\leq i<j<k\leq m} P(A_i \cap A_j \cap A_k)$$
$$- \ldots + (-1)^{m-1} P(A_1 \cap A_2 \cap \ldots \cap A_m).$$

Then, for $n = m + 1$,

$$\begin{aligned} P(\overset{m+1}{\underset{i=1}{\cup}} A_i) &= P\left((\overset{m}{\underset{i=1}{\cup}} A_i) \cup A_{m+1}\right) \\ &= \boxed{P(\overset{m}{\underset{i=1}{\cup}} A_i)} + P(A_{m+1}) - P\left((\overset{m}{\underset{i=1}{\cup}} A_i) \cap A_{m+1}\right) \\ &= \boxed{\sum_{i=1}^{m} P(A_i) - \sum\sum_{1\leq i<j\leq m} P(A_i \cap A_j) + \ldots + (-1)^{m-1} P(A_1 \cap A_2 \cap \ldots \cap A_m)} \\ &\quad + P(A_{m+1}) - \boxed{P\left((\overset{m}{\underset{i=1}{\cup}} A_i) \cap A_{m+1}\right)} \\ &= \sum_{i=1}^{m} P(A_i) - \sum\sum_{1\leq i<j\leq m} P(A_i \cap A_j) + \ldots + (-1)^{m-1} P(A_1 \cap A_2 \cap \ldots \cap A_m) \\ &\quad + P(A_{m+1}) - \boxed{P\left(\overset{m}{\underset{i=1}{\cup}} (A_i \cap A_{m+1})\right)} \\ &= \sum_{i=1}^{m} P(A_i) - \sum\sum_{1\leq i<j\leq m} P(A_i \cap A_j) + \ldots + (-1)^{m-1} P(A_1 \cap A_2 \cap \ldots \cap A_m) \end{aligned}$$

$$
\begin{aligned}
&+ P(A_{m+1}) \\
&- \boxed{\sum_{i=1}^{m} P(A_i \cap A_{m+1}) - \sum\sum_{1 \leq i < j \leq m} P\left((A_i \cap A_{m+1}) \cap (A_j \cap A_{m+1})\right) + \ldots + (-1)^{m-1} P\left((A_1 \cap A_{m+1}) \cap \ldots \cap (A_m \cap A_{m+1})\right)} \\
= &\sum_{i=1}^{m} P(A_i) - \sum\sum_{1 \leq i < j \leq m} P(A_i \cap A_j) + \ldots + (-1)^{m-1} P(A_1 \cap A_2 \cap \ldots \cap A_m) \\
&+ P(A_{m+1}) \\
&- \boxed{\sum_{i=1}^{m} P(A_i \cap A_{m+1}) - \sum\sum_{1 \leq i < j \leq m} P\left(A_i \cap A_j \cap A_{m+1}\right) + \ldots + (-1)^{m-1} P\left(A_1 \cap \cap \ldots \cap A_m \cap A_{m+1}\right)} \\
= &\sum_{i=1}^{m+1} P(A_i) \\
&- \boxed{\sum\sum_{1 \leq i < j \leq m} P(A_i \cap A_j) + \sum_{i=1}^{m} P(A_i \cap A_{m+1})} \\
&+ \boxed{\sum\sum\sum_{1 \leq i < j < k \leq m} P(A_i \cap A_j \cap A_k) + \sum\sum_{1 \leq i < j \leq m} P\left(A_i \cap A_j \cap A_{m+1}\right)} \\
&+ \ldots + (-1)(-1)^{m-1} P\left(A_1 \cap \cap \ldots \cap A_m \cap A_{m+1}\right) \\
= &\sum_{i=1}^{m+1} P(A_i) \\
&- \boxed{\sum\sum_{1 \leq i < j \leq m+1} P(A_i \cap A_j)} \\
&+ \boxed{\sum\sum\sum_{1 \leq i < j < k \leq m+1} P(A_i \cap A_j \cap A_k)} \\
&+ \ldots + (-1)^{(m+1)-1} P\left(A_1 \cap \cap \ldots \cap A_m \cap A_{m+1}\right)
\end{aligned}
$$

Altogether, we get the inclusion-exclusion formula

$$
\begin{aligned}
P(\bigcup_{i=1}^{n} A_i) = &\sum_{i=1}^{n} P(A_i) - \sum\sum_{1 \leq i < j \leq n} P(A_i \cap A_j) + \sum\sum\sum_{1 \leq i < j < k \leq n} P(A_i \cap A_j \cap A_k) \\
&- \ldots + (-1)^{n-1} P(A_1 \cap A_2 \ldots A_n).
\end{aligned}
$$

(ii) - if $n = 2$, then

$$
P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2).
$$

- if $n = 3$, then

$$
\begin{aligned}
P(A_1 \cup A_2 \cup A_3) = &P(A_1) + P(A_2) + P(A_3) \\
&- P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) \\
&+ P(A_1 \cap A_2 \cap A_3).
\end{aligned}
$$

- if $n = 4$, then

$$P(\overset{4}{\underset{i=1}{\cup}} A_i) = P(A_1) + P(A_2) + P(A_3) + P(A_4)$$

$$-P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_1 \cap A_4) - P(A_2 \cap A_3) - P(A_2 \cap A_4) - P(A_3 \cap A_4)$$

$$+P(A_1 \cap A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_4) + P(A_1 \cap A_3 \cap A_4) + P(A_2 \cap A_3 \cap A_4)$$

$$- P(A_1 \cap A_2 \cap A_3 \cap A_4).$$

- if $n = 5$, then

$$P(\overset{5}{\underset{i=1}{\cup}} A_i) = P(A_1) + P(A_2) + P(A_3) + P(A_4) + P(A_5)$$

$$-P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_1 \cap A_4) - P(A_1 \cap A_5) - P(A_2 \cap A_3) - P(A_2 \cap A_4) - P(A_2 \cap A_5) - P(A_3 \cap A_4) - P(A_3 \cap A_5) - P(A_4 \cap A_5)$$

$$+P(A_1 \cap A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_4) + P(A_1 \cap A_2 \cap A_5) + P(A_1 \cap A_3 \cap A_4) + P(A_1 \cap A_3 \cap A_5) + P(A_1 \cap A_4 \cap A_5) + P(A_2 \cap A_3 \cap A_4) + P(A_2 \cap A_3 \cap A_5) + P(A_2 \cap A_4 \cap A_5) + P(A_3 \cap A_4 \cap A_5)$$

$$+P(A_1 \cap A_2 \cap A_3 \cap A_4) + +P(A_1 \cap A_2 \cap A_3 \cap A_5) + P(A_1 \cap A_2 \cap A_4 \cap A_5) + P(A_1 \cap A_3 \cap A_4 \cap A_5) + P(A_2 \cap A_3 \cap A_4 \cap A_5)$$

$$- P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5).$$

6. (i) If $\{A_n; n \geq 1\}$ is an increasing sequence of events, i.e. for all $n \geq 1, A_n \subset A_{n+1}$, then $\lim_{n\to\infty} P(A_n) = P(\overset{\infty}{\underset{n=1}{\cup}} A_n)$.

   (ii) If $\{A_n; n \geq 1\}$ is a decreasing sequence of events, i.e. for all $n \geq 1, A_n \supset A_{n+1}$, then $\lim_{n\to\infty} P(A_n) = P(\overset{\infty}{\underset{n=1}{\cap}} A_n)$.

Proof:

(i) Let $A_0 = \emptyset, B_n = A_n \setminus A_{n-1}, n \geq 1$. Then (from the Tutorial 01.7) we know

$\{B_n, n \geq 1\}$ are (pairwise) adjoint and $\bigcup\limits_{n=1}^{\infty} A_n = \bigcup\limits_{n=1}^{\infty} B_n$. Therefore

$$\begin{aligned}
P(\bigcup_{n=1}^{\infty} A_n) = P(\bigcup_{n=1}^{\infty} B_n) &= \sum_{n=1}^{\infty} P(B_n) \\
&= \sum_{n=1}^{\infty} P(A_n \setminus A_{n-1}) \\
&= \sum_{n=1}^{\infty} [P(A_n) - P(A_{n-1})] \text{ since } A_{n-1} \subset A_n, n \geq 1 \\
&= \lim_{k\to\infty} \sum_{n=1}^{k} [P(A_n) - P(A_{n-1})] \\
&= \lim_{k\to\infty} (P(A_k) - P(A_0)) \\
&= \lim_{k\to\infty} (P(A_k) - P(\emptyset)) = \lim_{k\to\infty} (P(A_k) - 0) \\
&= \lim_{k\to\infty} P(A_k) = \lim_{n\to\infty} P(A_n).
\end{aligned}$$

i.e. $\lim\limits_{n\to\infty} P(A_n) = P(\bigcup\limits_{n=1}^{\infty} A_n)$.

(ii) Let $B_n = A_1 \setminus A_n, n \geq 1$, then $B_n \uparrow$.[†] Using the above result of (i), we obtain

$$\lim_{n\to\infty} P(B_n) = P(\bigcup_{n=1}^{\infty} B_n).$$

Then, we can get

$$\begin{aligned}
P(A_1) - P(\bigcap_{n=1}^{\infty} A_n) &= P\left(A_1 \setminus (\bigcap_{n=1}^{\infty} A_n)\right) = P\left(A_1 \cap (\bigcap_{n=1}^{\infty} A_n)^c\right) \\
&= P\left(A_1 \cap (\bigcup_{n=1}^{\infty} A_n^c)\right) = P\left(\bigcup_{n=1}^{\infty} (A_1 \cap A_n^c)\right) \\
&= P\left(\bigcup_{n=1}^{\infty} (A_1 \setminus A_n)\right) \\
&= \boxed{P(\bigcup_{n=1}^{\infty} B_n) = \lim_{n\to\infty} P(B_n)} \\
&= \lim_{n\to\infty} P(A_1 \setminus A_n) \\
&= \lim_{n\to\infty} [P(A_1) - P(A_n)] \text{ note that } A_n \subset A_1, n \geq 1 \\
&= P(A_1) - \lim_{n\to\infty} P(A_n).
\end{aligned}$$

$\Longrightarrow \lim\limits_{n\to\infty} P(A_n) = P(\bigcap\limits_{n=1}^{\infty} A_n)$.

<u>Method 2:</u> Assume $(\Omega, \mathscr{F}, P)$ is a probability space,let $B_n = A_n^c = \Omega \setminus A_n, n \geq 1$, then $B_n \uparrow$. Using the above result of (i), we obtain

$$\lim_{n\to\infty} P(B_n) = P(\bigcup_{n=1}^{\infty} B_n).$$

[†] i.e. $\{B_n; n \geq 1\}$ is an increasing sequence of events

Then, we can get

$$
\begin{aligned}
1 - P(\bigcap_{n=1}^{\infty} A_n) &= P\left((\bigcap_{n=1}^{\infty} A_n)^c \right) \\
&= P\left(\bigcup_{n=1}^{\infty} A_n^c \right) \\
&= \boxed{P(\bigcup_{n=1}^{\infty} B_n) = \lim_{n\to\infty} P(B_n)} \\
&= \lim_{n\to\infty} P(A_n^c) \\
&= \lim_{n\to\infty} [1 - P(A_n)] \text{ note that } P(A_n^c) = 1 - P(A_n), n \geq 1 \\
&= 1 - \lim_{n\to\infty} P(A_n).
\end{aligned}
$$

$$\Longrightarrow \lim_{n\to\infty} P(A_n) = P(\bigcap_{n=1}^{\infty} A_n).$$

7. Suppose that $\{A_n; n \geq 1\}$ is a sequence of events which may not be disjoint. Show that the **sub-additive property:**

$$P(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} P(A_n).$$

Also for any $k \geq 2$, we have

$$P(\bigcup_{n=1}^{k} A_n) \leq \sum_{n=1}^{k} P(A_n).$$

In particular, for any two events A and B, we have $P(A \cup B) \leq P(A) + P(B)$.

Proof: Let $B_1 = A_1, B_n = A_n \setminus (\bigcup_{i=1}^{n-1} A_i), n \geq 2$. Then for any $k \geq 1$,

$$\{B_n, n \geq 1\} \text{ are (pairwise) disjoint }, \bigcup_{n=1}^{k} A_n = \bigcup_{n=1}^{k} B_n \text{ and } \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n.$$

Firstly, for any $m, n \in \mathbb{N}_+$, we show that $\{B_n; n \geq 1\}$ are (pairwise) disjoint.

Indeed, without loss of generality, we can suppose $m > n$, note that $m, n \in \mathbb{N}_+$, so

$$m - 1 \geq n \geq 1.$$

then $\bigcup_{i=1}^{m-1} A_i \supset A_n$, yields $(\bigcup_{i=1}^{m-1} A_i)^c \subset A_n^c$. However,

$$B_m = A_m \setminus (\bigcup_{i=1}^{m-1} A_i) = A_m \cap (\bigcup_{i=1}^{m-1} A_i)^c \subset A_n^c \subset A_n^c \cup (\bigcup_{i=1}^{n-1} A_i) = \left(A_n \setminus (\bigcup_{i=1}^{n-1} A_i) \right)^c = B_n^c.$$

Altogether, we have for any $m > n \geq 1$,

$$B_m \subset B_n^c,$$

i.e. $B_n \cap B_m = B_m \cap B_n = \emptyset$.

Hence, $\{B_n; n \geq 1\}$ are (pairwise) disjoint.

Secondly, we show that $\overset{k}{\underset{n=1}{\cup}} A_n = \overset{k}{\underset{n=1}{\cup}} B_n$.

Indeed, for $n = 1$,

$$\overset{1}{\underset{n=1}{\cup}} A_n = A_1 = B_1 = \overset{1}{\underset{n=1}{\cup}} B_n.$$

Assume for $k = m$, we have $\overset{m}{\underset{n=1}{\cup}} A_n = \overset{m}{\underset{n=1}{\cup}} B_n$, then

$$\begin{aligned}
\overset{m+1}{\underset{n=1}{\cup}} B_n &= B_{m+1} \cup (\overset{m}{\underset{n=1}{\cup}} B_n) \overset{asd}{=\!=\!=} (B_{m+1}) \cup (\overset{m}{\underset{n=1}{\cup}} A_n) \\
&= \left(A_{m+1} \setminus (\overset{(m+1)-1}{\underset{i=1}{\cup}} A_i)\right) \cup (\overset{m}{\underset{n=1}{\cup}} A_n) \\
&= \left(A_{m+1} \setminus (\overset{m}{\underset{i=1}{\cup}} A_i)\right) \cup (\overset{m}{\underset{n=1}{\cup}} A_n) \\
&= \left(A_{m+1} \setminus (\overset{m}{\underset{n=1}{\cup}} A_n)\right) \cup (\overset{m}{\underset{n=1}{\cup}} A_n) \\
&= \left(A_{m+1} \cap (\overset{m}{\underset{n=1}{\cup}} A_n)^c\right) \cup (\overset{m}{\underset{n=1}{\cup}} A_n) \\
&= A_{m+1} \cup (\overset{m}{\underset{n=1}{\cup}} A_n) = \overset{m+1}{\underset{n=1}{\cup}} A_n.
\end{aligned}$$

Finally, we have $\overset{k}{\underset{n=1}{\cup}} A_n = \overset{k}{\underset{n=1}{\cup}} B_n$.

Or **Method 2: direct proof**

For any fixed $k \geq 2$, note that $B_n = A_n \setminus (\overset{n-1}{\underset{i=1}{\cup}} A_i) \subset A_n, n \geq 1$, then

$$\overset{k}{\underset{n=1}{\cup}} A_n \supset \overset{k}{\underset{n=1}{\cup}} B_n.$$

Assume $x \in \bigcup_{n=1}^{k} A_n$,

- if $x \in A_1$, then $x \in A_1 = B_1 \subset \overset{k}{\underset{n=1}{\cup}} B_n$, i.e. $x \in \overset{k}{\underset{n=1}{\cup}} B_n$;
- if $x \notin A_1$, then
  - if $x \in A_2$, then $x \in A_2 \setminus A_1 = B_2 \subset \overset{k}{\underset{n=1}{\cup}} B_n$, i.e. $x \in \overset{k}{\underset{n=1}{\cup}} B_n$;
  - if $x \notin A_2$, then

    after $k - 1$ step $\cdots$

- if $x \in A_{k-1}$, then $x \in A_{k-1} \setminus \overset{k-2}{\underset{i=1}{\cup}} A_i = B_{k-1} \subset \bigcup_{n=1}^{k} B_n$, i.e. $x \in \bigcup_{n=1}^{k} B_n$.
- if $x \notin A_{k-1}$, then
  - Notice that $x \in \bigcup_{n=1}^{k} A_n$, so $x \in A_k$, then

$$x \in A_k \setminus \overset{k-1}{\underset{i=1}{\cup}} A_i = B_k \subset \bigcup_{n=1}^{k} B_n, \text{ i.e. } x \in \bigcup_{n=1}^{k} B_n.$$

Finally, we obtain $x \in \bigcup_{n=1}^{k} B_n$, i.e. $\overset{k}{\underset{n=1}{\cup}} A_n \subset \overset{k}{\underset{n=1}{\cup}} B_n$. Hence for any fixed $k \geq 2$,

$$\overset{k}{\underset{n=1}{\cup}} A_n = \overset{k}{\underset{n=1}{\cup}} B_n.$$

Thirdly, we will show that $\overset{\infty}{\underset{n=1}{\cup}} A_n = \overset{\infty}{\underset{n=1}{\cup}} B_n$.

In fact, since $A_k \subset \overset{k}{\underset{n=1}{\cup}} A_n = \overset{k}{\underset{n=1}{\cup}} B_n \subset \overset{\infty}{\underset{n=1}{\cup}} B_n$, then $\overset{\infty}{\underset{n=1}{\cup}} A_n = \overset{\infty}{\underset{k=1}{\cup}} A_k \subset \overset{\infty}{\underset{k=1}{\cup}} \overset{\infty}{\underset{n=1}{\cup}} B_n = \overset{\infty}{\underset{n=1}{\cup}} B_n$. Note that $B_n = A_n \setminus (\overset{n-1}{\underset{i=1}{\cup}} A_i \subset A_n), n \geq 1, \Rightarrow \overset{\infty}{\underset{n=1}{\cup}} A_n \supset \overset{\infty}{\underset{n=1}{\cup}} B_n$. Hence, we obtain

$$\overset{\infty}{\underset{n=1}{\cup}} A_n = \overset{\infty}{\underset{n=1}{\cup}} B_n.$$

Or sending $n \to \infty$, then

$$\begin{aligned}
\overset{\infty}{\underset{n=1}{\cup}} A_n = \overset{\infty}{\underset{k=1}{\cup}} A_k &= \overset{\infty}{\underset{k=1}{\cup}} \overset{k}{\underset{n=1}{\cup}} A_n \\
&= \boxed{\lim_{k\to\infty} \overset{k}{\underset{n=1}{\cup}} A_n = \lim_{k\to\infty} \overset{k}{\underset{n=1}{\cup}} B_n} \\
&= \overset{\infty}{\underset{k=1}{\cup}} \overset{k}{\underset{n=1}{\cup}} B_n \\
&= \overset{\infty}{\underset{k=1}{\cup}} B_k = \overset{\infty}{\underset{n=1}{\cup}} B_n.
\end{aligned}$$

1°. Note that $B_n \subset A_n, n \geq 1$, so $P(B_n) \leq P(A_n), n \geq 1$. Therefore

$$P(\overset{\infty}{\underset{n=1}{\cup}} A_n) = P(\overset{\infty}{\underset{n=1}{\cup}} B_n) = \sum_{n=1}^{\infty} P(B_n) \leq \sum_{n=1}^{\infty} P(A_n).$$

2°. <u>**Method 1:**</u> Note that $B_n = A_n \setminus (\overset{n-1}{\underset{i=1}{\cup}} A_i) \subset A_n, 1 \leq n \leq k$, so

$$P(B_n) \leq P(A_n), 1 \leq n \leq k.$$

Therefore

$$P(\overset{k}{\underset{n=1}{\cup}} A_n) = P(\overset{k}{\underset{n=1}{\cup}} B_n) = \sum_{n=1}^{k} P(B_n) \leq \sum_{n=1}^{k} P(A_n).$$

i.e. for any $k \geq 2$, we have $P(\bigcup\limits_{n=1}^{k} A_n) \leq \sum\limits_{n=1}^{k} P(A_n)$.

Method 2: For any $k \geq 2$, let $A_{k+1} = A_{k+2} = A_{k+3} = \cdots = \emptyset$, thus

$$\overset{\infty}{\underset{n=1}{\cup}} A_n = \overset{k}{\underset{n=1}{\cup}} A_n, \sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{k} P(A_n).$$

Indeed, it is easy to see $\overset{\infty}{\underset{n=1}{\cup}} A_n \supset \overset{k}{\underset{n=1}{\cup}} A_n$. On the other hand, if $x \in \overset{\infty}{\underset{n=1}{\cup}} A_n$, then $\exists n_0 \geq 1, s.t.\ x \in A_{n_0}$. Since $A_m = \emptyset, m \geq k+1$, so $1 \leq n_0 \leq k$, hence $x \in \overset{k}{\underset{n=1}{\cup}} A_n$, i.e. $\overset{\infty}{\underset{n=1}{\cup}} A_n \subset \overset{k}{\underset{n=1}{\cup}} A_n$. Altogether we have $\overset{\infty}{\underset{n=1}{\cup}} A_n = \overset{k}{\underset{n=1}{\cup}} A_n$.
Assume $k(k \geq 2)$ is fixed, for any $\varepsilon > 0$, let $m = k^{\ddagger}$, then

$$\left| \sum_{n=1}^{m} P(A_n) - \sum_{n=1}^{k} P(A_n) \right| = 0 < \varepsilon.$$

Therefore, $\sum\limits_{n=1}^{\infty} P(A_n) = \lim\limits_{m \to \infty} \sum\limits_{n=1}^{m} P(A_n) = \sum\limits_{n=1}^{k} P(A_n)$. Hence,

$$P(\overset{k}{\underset{n=1}{\cup}} A_n) = P(\overset{\infty}{\underset{n=1}{\cup}} A_n) \leq \sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{k} P(A_n).$$

i.e. for any $k \geq 2$, we have $P(\bigcup\limits_{n=1}^{k} A_n) \leq \sum\limits_{n=1}^{k} P(A_n)$.
In particular, let $k = 2, A_1 = A, A_2 = B$, then we have

$$\begin{aligned} P(A \cup B) &= P(A_1 \cup A_2) \\ &= \boxed{P(\overset{2}{\underset{n=1}{\cup}} A_n) \leq \sum_{n=1}^{2} P(A_n)} \\ &= P(A_1) + P(A_2) = P(A) + P(B). \end{aligned}$$

i.e. $P(A \cup B) \leq P(A) + P(B)$.

Remark: we even have a fact, assume $(\Omega, \mathscr{F}, P)$ is a probability space:

Prop: $P(\bigcup\limits_{n=1}^{\infty} A_n) \leq \sum\limits_{n=1}^{\infty} P(A_n)$ is equivalent to for any $k \geq 2$, $P(\overset{k}{\underset{n=1}{\cup}} A_n) \leq \sum\limits_{n=1}^{k} P(A_n)$.

we have proven necessity, now just prove sufficiency. Suppose, for any $k \geq 2$,

$$P(\overset{k}{\underset{n=1}{\cup}} A_n) \leq \sum_{n=1}^{k} P(A_n).$$

‡Only need to take any a number of N s.t. $N \geq k$

Firstly, we show that

$$\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{k} A_n.$$

Indeed, since $A_k \subset \bigcup_{n=1}^{k} A_n, \forall k \geq 1$, then $\bigcup_{k=1}^{\infty} A_k \subset \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{k} A_n$. On the other hand, if $x \in \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{k} A_n$, then $\exists k_0 \geq 1$, s.t. $x \in \bigcup_{n=1}^{k_0} A_n$, therefore, $\exists n_{k_0}, 1 \leq n_{k_0} \leq k_0$, s.t. $x \in A_{n_{k_0}}$, so $x \in \bigcup_{k=1}^{\infty} A_k$. Hence $\bigcup_{k=1}^{\infty} A_k \supset \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{k} A_n$. Finally, we get

$$\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{k} A_n.$$

Note that $\{\bigcup_{n=1}^{k} A_n\}_{k=1}^{\infty}$ is a increasing sequence, sending $k \to \infty$, then

$$\begin{aligned} P(\bigcup_{n=1}^{\infty} A_n) &= P(\bigcup_{k=1}^{\infty} A_k) = P(\bigcup_{k=1}^{\infty} \bigcup_{n=1}^{k} A_n) = P(\lim_{k\to\infty} \bigcup_{n=1}^{k} A_n) \\ &= \boxed{\lim_{k\to\infty} P(\bigcup_{n=1}^{k} A_n) \leq \lim_{k\to\infty} \sum_{n=1}^{k} P(A_n)} = \sum_{n=1}^{\infty} P(A_n). \end{aligned}$$