Properties of Joint cdf etc.

1. Joint pmf: $p(x, y)$ for random vector (X, Y)

(i) $\forall x, y,\quad p(x, y) \geq 0 \qquad (\because p(x, y) = \Pr\{X = x, Y = y\})$

(ii) $$\sum_y \sum_x p(x, y) = 1$$

Proof of (ii)

$$\sum_y \sum_x p(x, y) = \sum_y \sum_x \Pr\{X = x, Y = y\}$$

$$= \sum_y \Pr\left\{\left[\bigcup_x (X = x)\right] \cap (Y = y)\right\}$$

$$= \sum_y \Pr\{\Omega \cap (Y = y)\} = \sum_y \Pr\{Y = y\}$$

$$= 1$$

2. Properties of joint cdf $F_{(X,Y)}(x, y) \triangleq F(x, y)$

(i) For fixed x, $F(x, y)$ is an increasing function of y and for fixed y, $F(x, y)$ is an increasing function of x.

(ii) For fixed x, $F(x, y)$ is right continuous function of y, and for fixed y, $F(x, y)$ is right continuous function of x.

(iii) $\lim_{\substack{x \to +\infty \\ y \to +\infty}} F(x, y) \triangleq F(+\infty, +\infty) = 1$

(iv) $\lim_{x \to -\infty} F(x, y) = 0 \quad (\forall y)$ [could be written as $F(-\infty, y) = 0$]

$\lim_{y \to -\infty} F(x, y) = 0 \quad (\forall x)$ [could be written as $F(x, -\infty) = 0$]

(v) Suppose that $x_1 \leq x_2$, $y_1 \leq y_2$, then

$$\Pr\{x_1 < X \leq x_2, \ y_1 < Y \leq y_2\}$$

$$= F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1)$$

$$\geq 0 \qquad (*)$$

"Proof" (i) and (ii) are easy and, also, the proof is very similar as a single random variable

For example, let's show that for fixed x, if $y_1 < y_2$ then $F(x, y_1) \leq F(x, y_2)$

Indeed, $F(x, y_1) = \Pr\{X \le x,\ Y \le y_1\}$

$$= \Pr\{(X(w) \le x) \cap (Y(w) \le y_1)\}$$

If $y_1 < y_2$, then $\{w;\ Y(w) \le y_1\} \subset \{w;\ Y(w) \le y_2\}$

and thus $(X(w) \le x) \cap (Y(w) \le y_1) \subset (X(w) \le x) \cap (Y(w) \le y_2)$

$$\Rightarrow \Pr\{X \le x,\ Y \le y_1\} \le P\{X \le x,\ Y \le y_2\}$$

(iii) $\displaystyle\lim_{\substack{x \to +\infty \\ y \to +\infty}} F(x, y) = \lim_{\substack{x \to +\infty \\ y \to +\infty}} \Pr\{X \le x,\ Y \le y\}$

$$= \Pr\{X \le +\infty,\ Y \le +\infty\} = P(\Omega) = 1$$

(iv) $\displaystyle\lim_{x \to -\infty} F(x, y) = \lim_{x \to -\infty} \Pr\{X \le x,\ Y \le y\}$

$$= \Pr\{X \le -\infty,\ Y \le y\}$$

But $(X \le -\infty) \cap (Y \le y) \subset (X \le -\infty)$

$\therefore \Pr\{X \le -\infty,\ Y \le y\} \le \Pr\{X \le -\infty\} = P(\phi) = 0$

Similarly, $\displaystyle\lim_{y \to -\infty} F(x, y) = 0$

Finally we show that if $x_1 < x_2$, $y_1 < y_2$, then

$$\Pr\{x_1 < X \le x_2,\ y_1 < Y \le y_2\}$$

$$= F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1) \quad (1)$$

which is non-negative (Of course since it is a probability)

Proof. Note that since $y_1 < y_2$,

$$\{w;\ Y(w) \le y_2\} = \{w;\ Y(w) \le y_1\} \cup \{w;\ y_1 < Y(w) \le y_2\} \quad (2)$$

and the right-hand side is a union of two $\boxed{\text{disjoint}}$ sets

Hence

$$\{X(w) \le x_2\} \cap \{Y(w) \le y_2\}$$

again disjoint !!

$$= \left[\{X(w) \le x_2\} \cap \{Y(w) \le y_1\}\right] \cup \left[\{X(w) \le x_2\} \cap \{w;\ y_1 < Y(w) \le y_2\}\right]$$

$$\Rightarrow \Pr\{X \le x_2,\ Y \le y_2\}$$

$$= P\{X \le x_2,\ Y \le y_1\} + P\{X \le x_2,\ y_1 < Y(w) \le y_2\} \quad (3)$$

Similarly, since $x_1 < x_2$ we have

disjoint !!

$$\{\omega \in \Omega;\ X \le x_2\} = \{\omega \in \Omega;\ X \le x_1\} \cup \{\omega;\ x_1 < X \le x_2\}$$

and hence

$$\{X \le x_2\} \cap \{y_1 < Y \le y_2\}$$

disjoint !!

$$= \big[\{X \le x_1\} \cap \{y_1 < Y \le y_2\}\big] \cup \big[\{x_1 < X \le x_2\} \cap \{y_1 < Y \le y_2\}\big]$$

and therefore

$$P\{X \le x_2,\ y_1 < Y \le y_2\}$$

$$= P\{X \le x_1,\ y_1 < Y \le y_2\} + P\{x_1 < X \le x_2,\ y_1 < Y \le y_2\}$$

Substituting the above into (3) yields [the above is just the 2nd term of the right-hand side of (3)]

$$P_r\{X \le x_2,\ Y \le y_2\}$$

$$= P\{X \le x_2,\ Y \le y_1\} + P\{X \le x_1,\ y_1 < Y \le y_2\}$$

$$+ P\{x_1 < X \le x_2,\ y_1 < Y \le y_2\} \qquad (4)$$

On the other hand, since $y_1 < y_2$ we also have

disjoint

$$\{Y \leq y_2\} = \{Y \leq y_1\} \cup \{y_1 < Y \leq y_2\}$$

and thus

disjoint

$$\{X \leq x_1\} \cap \{Y \leq y_2\} = \left[\{X \leq x_1\} \cap \{Y \leq y_1\}\right] \cup \left[\{X \leq x_1\} \cap \{y_1 < Y \leq y_2\}\right]$$

and therefore

$$P\{X \leq x_1, Y \leq y_2\} = P\{X \leq x_1, Y \leq y_1\} + P\{X \leq x_1, y_1 < Y \leq y_2\}$$

i.e. (after moving the 1st term of the right-hand side to the other side)

$$P\{X \leq x_1, y_1 < Y \leq y_2\} = P\{X \leq x_1, Y \leq y_2\} - P\{X \leq x_1, Y \leq y_1\} \quad (5)$$

Substituting the above (5) into (4) (and noting (5) is just the 2nd term in the right-hand side of (4)) yields

$$P\{X \leq x_2, Y \leq y_2\} = P_r\{X \leq x_2, Y \leq y_1\} + P\{X \leq x_1, Y \leq y_2\}$$
$$- P\{X \leq x_1, Y \leq y_1\} + P\{x_1 < X \leq x_2, y_1 < Y \leq y_2\}$$

Hence

$$P\{x_1 < X \leq x_2, y_1 < Y \leq y_2\} = F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1)$$

This completes the proof. #

3. Remarks on Properties of joint cdf:

① The properties stated in Section 2 of this note is true for any two rvs (no matter discrete, continuous or even mixed ones)

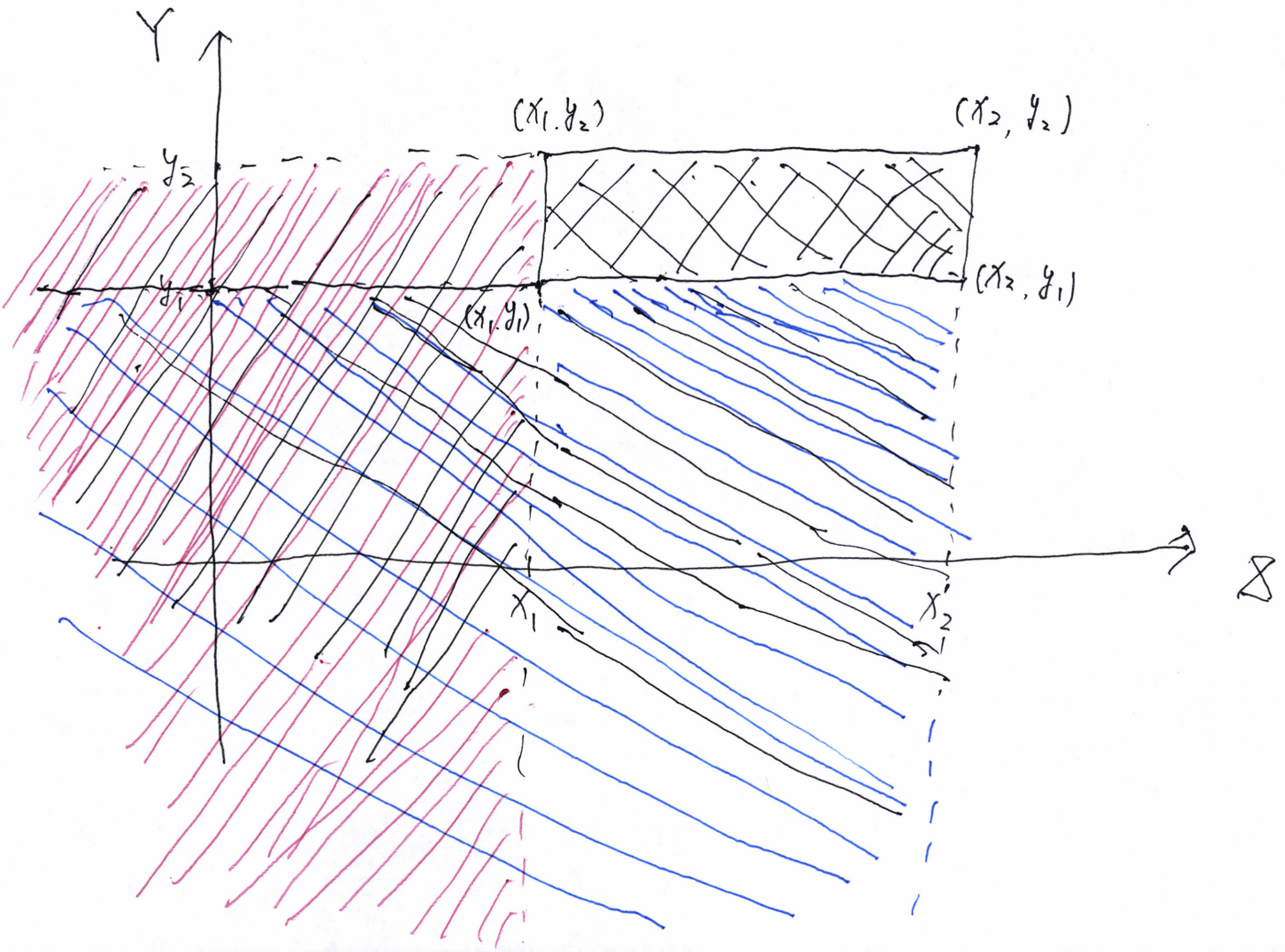
Of course, for conti- random vector, conclusion (ii) (i.e. right continuous property) will be much stronger.

② Conclusion (v) has clear geometric interpretation

i.e. $\forall\ x_1 < x_2,\ y_1 < y_2$

$$\Pr\{x_1 < X \le x_2,\ y_1 < Y \le y_2\} = F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1)$$

see below: Red $F(x_1, y_2)$; Blue: $F(x_2, y_1)$



4. Relationship between joint pmf and joint cdf for discrete random vector

Easy to see

$$F_{(X,Y)}(x, y) = \Pr\{X \le x, Y \le y\}$$

$$= \sum_{y_j \le y} \sum_{x_i \le x} \Pr\{X = x_i, Y = y_j\}$$

$$= \sum_{y_j \le y} \sum_{x_i \le x} P(x_i, y_j)$$

Hence, essentially speaking, all the calculations regarding the probabilities of the discrete random vector (X, Y) can be done in terms of joint pmf.

For example,

$$P\{X \ge x, Y \le y\}$$

$$= \sum_{y_j \le y} \sum_{x_i \ge x} P(x_i, y_j)$$