Chapter 3 Joint Distributions

§3.1. Introduction

I. Main Problem:

Consider two random variables, X and Y, say.

New question: the ralation between X and Y.

In general, n random variables $X_1, X_2, \cdots, X_n$.

II. Applications:

In many cases, we have to consider several random variables together since there are relations among them.

An example: In ecological studies, several species have to be considered together: prey and predators

III. Reading of the Book:

Mainly 3.2 3.3 and 3.4.

(3.5; 3.6 and 3.7: Omitted;) 3.1:

§3.2 Discrete Random Variables

I. An Example:

1. Example: Random experiment: A fair coin is tossed three times.

Sample space:

$\Omega = \{HHH; HHT; HTH; HTT; THH; THT; TTH; TTT\}$

Let X denote the number of heads on the first toss

and Y the total number of heads.

Then we can see that

(1) Both X and Y are random variables:

(depends upon the outcome. for example.

let $w_1 = \{HHH\}$. then $X(w_1) = 1$ $Y(w_1) = 3$

$w_5 = \{THH\}$. then $X(w_5) = 0$ $Y(w_5) = 2$

— — — — — — .

(2) Both X and Y are discrete random variables:

all possible values of X: $\{0, 1\}$.

all possible values of Y: $\{0, 1, 2, 3\}$.

(3) X and Y are defined on the same sample space and there exists a relation between them.

2. "Joint probability mass function" (Joint p.m.f.)

Now consider the events A and B:

A: "the number of heads on the 1st toss is zero"

B: "the total number of heads is two"

then

$$A = \{\omega \in \Omega;\ X(\omega) = 0\} = (X = 0) = \{THH; THT; TTH; TTT\}.$$

$$B = \{\omega \in \Omega;\ Y(\omega) = 2\} = (Y = 2) = \{HHT; HTH; THH\}$$

Easy to see, the intersection of A and B is the event $A \cap B = \{THH\}$

Hence $P(A \cap B) = \frac{1}{8}$ (Equally likely!!)

We write it as

$$P(A\cap B) = P\{(X=0)\cap(Y=2)\} = \frac{1}{8}$$

More simply, just write it as

$$\boxed{P(X=0,\ Y=2) = \frac{1}{8}.}$$

Similarly, A_1: "number of heads on 1st toss is one"

B_2 "number of total heads is two".

$\Rightarrow$: $A_1 = (X=1) = \{HHH;\ HHT;\ HTH;\ HTT\}$

$B_2 = (Y=2) = \{HHT;\ HTH;\ THH\}$

$\Rightarrow$: $A_1\cap B_2 = \{HHT;\ HTH\}$

$\Rightarrow$ $P(A_1\cap B_2) = \frac{2}{8}$ (Equally likely !!)

i.e. $$\boxed{P(X=1,\ Y=2) = \frac{2}{8}}$$

Similarly. (check these!)

$P(X=0, Y=0) = \frac{1}{8}$

$P(X=0, Y=1) = \frac{2}{8}$

$P(X=0, Y=2) = \frac{1}{8}$

$P(X=0, Y=3) = 0$

$P(X=1, Y=0) = 0$

$P(X=1, Y=1) = \frac{1}{8}$

$P(X=1, Y=2) = \frac{2}{8}$

$P(X=1, Y=3) = \frac{1}{8}$

The above are the all possibilities.

Table:

x \ y	0	1	2	3
0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0
1	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$

(3, 2, 1)

Now, we denote $P(X=0, Y=0) = P(0,0)$

then $P(0,0) = \frac{1}{8}$

Also $P(X=0, Y=2) = P(0,2)$

then $P(0,2) = \frac{1}{8}$

In general, let

$$p(x, y) = P(X = x, Y = y)$$

we get a function depending upon two real variables.

This function is called the joint probability mass function of X and Y.

This function gives the full information about the two random variables X and Y.

II. General Definition:

1. Joint probability mass function:

Suppose that X and Y are two discrete random variables defined on the same sample space and they take on values $x_1, x_2 \cdots x_n, \cdots$ (for X)
and $y_1, y_2, \cdots y_m, \cdots$ (for Y)
respectively. Then let

$$P(x_i, y_j) = P(X = x_i, Y = y_j) \quad (3.2.1)$$

(for all i and j)

we get a function of two variables. This function is called the joint probability mass function of the random variables X and Y

In short: $\boxed{\text{Joint p.m.f.}}$

2. Marginal probability mass function:

① In considering two random variables X and Y, the random variable X itself has its own distribution, which is called the marginal p.m.f of X.

Similarly, the marginal p.m.f of Y

Hence, for random variables X and Y we have two marginal p.m.f's, so we use different notations:

$p_X(\cdot)$ and $p_Y(\cdot)$.

② Def:

Suppose X and Y are two discrete random variables defined on the same sample space and taking all possible values:

(for X) $x_1, x_2, \cdots, x_n, \cdots$

(for Y) $y_1, y_2, \cdots, y_m, \cdots$ respectively.

Then the marginal probability mass function of X is a function defined by

$$P_X(x_i) = P(X = x_i) \quad (i=1, 2, \cdots) \tag{3.2.3}$$

Similarly, the marginal p.m.f. of Y:

$$P_Y(y_j) = P(Y = y_j) \quad (j=1, 2, \cdots) \tag{3.2.4}$$

3. Relation (between joint p.m.f and marginal p.m.f)

① Joint p.m.f $\xRightarrow{\text{determines}}$ Marginal p.m.f's

② (but usually) ~~even~~

all marginal p.m.f's can not determine the joint p.m.f.

③ For some special cases. all marginal p.m.f's can decide the joint p.m.f.

(No wander! Since joint p.m.f: full information of X and Y, including the relation between X and Y.

But marginal p.m.f's can only give the information about the X and Y, individually.

However, if the relation is known, then ...

4. How to get Marginal p.m.f's from joint p.m.f

Random variables X and Y

Possible values: X: $x_1, x_2, \dots x_n \dots$

Y: $y_1, y_2, \dots y_m \dots$

Joint p.m.f: $P(x_i, y_j)$

$P_X(x_i) = ?$ $\quad P_Y(y_j) = ?$

Conclusion:

$$P_X(x_i) = \sum_j P(x_i, y_j)$$

$$P_Y(y_j) = \sum_i P(x_i, y_j)$$

Think why? (Law of Total Probability!)

5. Example: (See the example in P3.2)

$P_X(\cdot)$? $\quad P_Y(\cdot)$?

Recall : Example in P 3.2

Possible values: X : 0, 1, 2, 3

X : 0, 1

Joint p.m.f . [See Table (3.2.1)]

$x \backslash y$	0	1	2	3
0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0
1	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$

$$P_X(0) = P(X=0) =$$

$$= P(X=0, Y=0) + P(X=0, Y=1) + P(X=0, Y=2) + P(X=0, Y=3)$$

$$= P(0,0) + P(0,1) + P(0,2) + P(0,3)$$

$$= \frac{1}{8} + \frac{2}{8} + \frac{1}{8} + 0 = \frac{4}{8} = \frac{1}{2}$$

$$p_X(1) = P(X=1)$$

$$= P(X=1, Y=0) + P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=$$

$$= p(1,0) + p(1,1) + p(1,2) + p(1,3)$$

$$= 0 + \frac{1}{8} + \frac{2}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

$\Rightarrow$ Marginal p.m.f of X:

$$p_X(0) = \frac{1}{2} \qquad p_X(1) = \frac{1}{2}$$

(We see: the row sum!!)

Similarly, Marginal p.m.f of Y

$$p_Y(0) = \frac{1}{8} \quad p_Y(1) = \frac{3}{8} \quad p_Y(2) = \frac{3}{8} \quad p_Y(3) = \frac{1}{8}$$

(Column sum!!)

$x \backslash y$	0	1	2	3	$p_X(\cdot)$
0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0	$\frac{1}{2}$
1	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{1}{2}$
$p_Y(\cdot)$	$\frac{1}{8}$	3/8	3/8	1/8	

III. Independent Random Variables:

1. Recall: Two events A and B are called independent if $P(A \cap B) = P(A) \cdot P(B)$

2. Definition: Let X and Y be two discrete random variables. Suppose all the possible values of X and Y are:

X: $x_1, x_2, x_3, \cdots, x_i, \cdots$

Y: $y_1, y_2, y_3, \cdots, y_j, \cdots$

We call X and Y are independent random variables if for all x_i and y_j, we have

$$P(X = x_i, Y = y_j) = P(X = x_i) \cdot P(Y = y_j) \quad (3.2.5)$$

i.e.

$$p(x, y) = p_X(x) \cdot p_Y(y) \quad (3.2.6)$$

where $p(x, y)$ is the joint probability mass function and $p_X(x)$ and $p_Y(y)$ are marginal probability mass functions.

3. Meaning:

① For fixed x_i and y_j, (3.2.5) means, two events $(X = x_i)$ and $(Y = y_j)$ are independent events. Since (3.2.5) holds true for all x_i and y_j and so there are many pairs of events independent.

In short, "two random variables X and Y are independent" means "many pairs of events are independent where in each pair, one event involving X and the other involving Y."

② (3.2.5) and (3.2.6) are totally the same.

③ (3.2.6) tells us, if X and Y are independent then all the marginal p.m.f can determine the joint p.m.f.

4. Remark: (Important)

① Joint p.m.f $\Longrightarrow$ all marginal p.m.f

② Joint p.m.f $\dashleftarrow$ all marginal p.m.f

(not enough, in general!)

③ For independent r.v's X and Y

Joint p.m.f $\Longleftarrow$ all marginal p.m.f.

IV. Joint Cumulative Probability Distribution Function (Joint c.d.f)

(Similar to a single random variable ...)

1. Definition. Suppose X and Y are two random variables. The function $F(x, y)$ defined by

$$F(x, y) = P(X \leq x,\ Y \leq y) \qquad (3.2.7)$$

where $-\infty < x < +\infty$, $-\infty < y < +\infty$ is called the joint c.d.f (cumulative distribution function) of the random variables X and Y.

[More exactly, the joint c.d.f of X and Y should be denoted by $F_{(X,Y)}(x, y)$!!]

2. Marginal Cumulative Probability Distribution Function (Marginal c.d.f)

① Suppose the random variables X and Y have the joint c.d.f. $F(x,y)$. Then the c.d.f of X, i.e.

$$F_X(x) = P(X \leq x) \qquad (3.2.8)$$

is called the marginal c.d.f of X.

Similarly, the c.d.f of Y, i.e

$$F_Y(y) = P(Y \leq y) \qquad (3.2.9)$$

is called the marginal c.d.f of Y.

② Relation with joint c.d.f:

Suppose the random variables X and Y have the joint c.d.f. $F(x,y)$, then the marginal c.d.f of X, $F_X(x)$ can be obtained by

$$F_X(x) = \lim_{y \to \infty} F(x, y) \qquad (3.2.10).$$

Similarly, the marginal c.d.f of Y, $F_Y(y)$

$$F_Y(y) = \lim_{x \to +\infty} F(x, y) \qquad (3.2.11)$$

"Proof": $\lim_{y \to +\infty} F(x, y) = \lim_{y \to +\infty} P(X \le x, Y \le y)$

$= P(X \le x, Y < +\infty)$

$= P(X \le x)$ $[\because (Y < +\infty) = \Omega \;!!]$

$= F_X(x)$ (Definition!)

Similarly, (3.2.11)

However, in general all marginal c.d.f's can not determine the joint c.d.f.

(Recall the similar conclusions to p.m.f)

③ Independence: ((Can prove that)

Two random variables X and Y are independent if and only if that for any x and y $[x \in (-\infty, +\infty),\ y \in (-\infty, +\infty)]$

$$P(X \le x,\ Y \le y) = P(X \le x) \cdot P(Y \le y)$$

i.e. (3.2.12)

$$F(x, y) = F_X(x) \cdot F_Y(y) \qquad (3.2.13)$$

where $F(x, y)$ is the joint c.d.f of X and Y, and $F_X(x)$ and $F_Y(y)$ are the marginal c.d.f's.

(Intuitive meaning. - - -

(Actually, this is the definition of independence.)

Again, (3.2.13) tells us that for independent random variables (and only for independent), the joint c.d.f can be determined by the marginal c.d.f's.

§3.3. Continuous Random Variables

I. Joint Cumulative Probability Distribution Function (Joint c.d.f) of two continuous random variables

(For joint c.d.f. totally similar to the discrete case)

1. Definition: Suppose X and Y are two continuous random variables. Then the function

$$F_{(X,Y)}(x, y) = P(X \le x, Y \le y) \qquad (3.3.1)$$

or more simply,

$$F(x, y) = P(X \le x, Y \le y) \qquad (3.3.2)$$

is called the joint c.d.f of X and Y.

2. Marginal c.d.f:

Again,

$$F_X(x) = P(X \le x) \qquad (3.3.3)$$

$$F_Y(y) = P(Y \le y) \qquad (3.3.4)$$

are called marginal c.d.f

3. Relation:

$$F_X(x) = \lim_{y \to +\infty} F(x, y) \qquad (3.3.5)$$

$$F_Y(y) = \lim_{x \to +\infty} F(x, y) \qquad (3.3.6)$$

where $F(x, y)$ is the joint c.d.f of X and Y and $F_X(x)$ and $F_Y(y)$ are marginal c.d.f's.

4. Independence:

Two continuous random variables X and Y are called independent if for all x and y. $[x \in (-\infty, +\infty); \ y \in (-\infty, +\infty)]$.

$$P(X \le x, Y \le y) = P(X \le x) \cdot P(Y \le y) \qquad (3.3.7)$$

i.e.

$$F_{(X,Y)}(x, y) = F_X(x) \cdot F_Y(y) \qquad (3.3.8)$$

Hence, again, for independent (and only for independent) continuous random variables, the joint c.d.f can be determined by the marginal c.d.f's

II. Joint Probability Density Function (Joint p.d.f):

Recall the definition of joint c.d.f:

$$F(x, y) = P(X \le x,\ Y \le y)$$

and similar to the single continuous random variabl

1. Definition: Suppose the joint c.d.f of the continuous random variables X and Y is

$$F(x, y) = P(X \le x,\ Y \le y).$$

Then the joint probability density function (Joint p.d.f) $f(x,y)$ is defined by

$$f(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y} \qquad (3.3.9)$$

and thus by the basic formula in Calculus

$$F(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(u, v)\,du\,dv \qquad (3.3.10)$$

(Double integral !!)

2. Properties of joint p.d.f:

Let $f(x,y)$ be the joint p.d.f of X and Y, then

$$f(x,y) \geq 0 \quad (\forall x, \ \forall y) \qquad (3.3.11)$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy = 1 \qquad (3.3.12)$$

⌈ Recall for single continuous r.v. X, the p.d.f $f(x)$ satisfies

$$f(x) \geq 0 \quad (\forall x)$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1$$

⌋

Note also that for discrete random variables X and Y, the joint p.m.f has the similar properties:

(i) $p(x,y) \geq 0$

(ii) $\sum_{y}\sum_{x} p(x,y) = 1$

III. Marginal probability density function.
(Marginal p.d.f)

1. Definition: Suppose X and Y are two continuous random variables with joint p.d.f. $f(x, y)$. The p.d.f of the random variable X, $f_X(x)$ is called the marginal p.d.f of X.
(Similarly, $f_Y(y)$, the marginal p.d.f of Y)

2. Relation with joint p.d.f
Let $f(x, y)$ be the joint p.d.f of X and Y.
Then the marginal probability density functions $f_X(x)$ and $f_Y(y)$ can be obtained by

$$f_X(x) = \int_{-\infty}^{+\infty} f(x, y)\, dy \qquad (3.3.13)$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x, y)\, dx \qquad (3.3.14)$$

(Compare with the discrete case)

IV. Independence.

1. Conclusion: Two random variables X and Y are independent if and only if

$$f(x, y) = f_X(x) \cdot f_Y(y) \qquad (3.3.15)$$

where $f(x, y)$ is the joint p.d.f of X and Y and $f_X(x)$ and $f_Y(y)$ are the marginal ones.

2. Remark (Important):

For two continuous random variables X and Y,

① Joint p.d.f $\Longrightarrow$ Marginal ones

See (3.3.13) and (3.3.14).

② In general,

Joint p.d.f $\xleftarrow{\text{Not enough}}$ Marginal ones

※ ③ However, if X and Y are independent then Joint p.d.f $\Longleftarrow$ Marginal ones

(See (3.3.15))

V. Example: Bivariate Normal p.d.f.:

1. Expression: Two random variables X and Y are called bivariate normally distributed if the joint p.d.f.: $f(x, y)$ is given by

$$f(x,y) = \frac{1}{2\pi \sigma_X \cdot \sigma_Y \cdot \sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X \cdot \sigma_Y} \right] \right\}$$

(3.3.16)

where μ_X, μ_Y, σ_X, σ_Y and ρ are constants satisfying

$$-\infty < \mu_X < +\infty, \quad -\infty < \mu_Y < +\infty$$

$$\sigma_X > 0, \quad \sigma_Y > 0,$$

and

$$-1 < \rho < 1 \qquad (3.3.17)$$

[No need to remember (3.3.16) !!]

Form (3.3.16) is called the bivariate normal density and the five constants μ_X, μ_Y, σ_X, σ_Y and ρ are called parameters (and so, five parameters.)

[Better let $b_1 = \mu_X$, $b_2 = \mu_Y$, $\sigma_1 = \sigma_X$, $\sigma_2 = \sigma_Y$, then

$$f(x,y) = \frac{e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-b_1)^2}{\sigma_1^2} + \frac{(y-b_2)^2}{\sigma_2^2} - \frac{2\rho(x-b_1)(y-b_2)}{\sigma_1\sigma_2}\right]}}{2\pi\cdot\sigma_1\cdot\sigma_2\cdot\sqrt{1-\rho^2}} \quad (3.3.17)$$

]

2. Marginal p.d.f's:

It can be proved [by using (3.3.13) and (3.3.14) !!] that, the two marginal p.d.f's are given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{(x-b_1)^2}{2\sigma_1^2}} \quad (3.3.18)$$

and
$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{(y-b_2)^2}{2\sigma_2^2}} \quad (3.3.19)$$

i.e. $X \sim N(b_1, \sigma_1^2)$, $Y \sim N(b_2, \sigma_2^2)$

3. Independence.

Easy to see that the bivariate normally distributed random variables X and Y (with the joint p.d.f (3.3.16)) are independent *if and only if*

$$\rho = 0 \qquad (3.3.20)$$

§3.4 General Case

(We could also consider n random variables

$X_1, X_2, \cdots, X_n$ say)

I. Joint c.d.f:

The joint c.d.f of $X_1, X_2, \cdots X_n$ is defined by

$$F(x_1, x_2, \cdots, x_n) = P(X_1 \le x_1, X_2 \le x_2, \cdots, X_n \le x_n)$$

For example, four random variables X_1, X_2, X_3, X_4

then $F(x_1, x_2, x_3, x_4) = P(X_1 \le x_1, X_2 \le x_2, X_3 \le x_3, X_4 \le x_4)$

II. Marginal c.d.f: (now n !!)

$$F_{X_i}(x_i) = P(X_i \le x_i) \quad (i = 1, 2, \cdots, n)$$

For example,

$$F_{X_1}(x_1) = P(X_1 \le x_1).$$

III. Joint p.d.f. (for continuous random variables)

$$f(x_1, x_2, \cdots, x_n) = \frac{\partial^n F(x_1, x_2, \cdots x_n)}{\partial x_1 \partial x_2 \cdots \partial x_n}$$

$$F(x_1, x_2, \cdots x_n) = \int_{-\infty}^{x_n} \int_{-\infty}^{x_{n-1}} \cdots \int_{-\infty}^{x_1} f(x_1, \cdots, x_n) dx_1 \cdots dx_n$$

「 For example. four r.v's X_1, X_2, X_3 and X_4,

$$f(x_1, x_2, x_3, x_4) = \frac{\partial^4 F(x_1, x_2, x_3, x_4)}{\partial x_1 \partial x_2 \partial x_3 \partial x_4}$$

$$F(x_1, x_2, x_3, x_4) = \int_{-\infty}^{x_4} \int_{-\infty}^{x_3} \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} f(x_1, x_2, x_3, x_4) dx_1 dx_2 dx_3 dx_4$$

IV. Marginal p.d.f. (now, n)

$$F_{X_i}(x_i) = P(X \leq x_i) \quad (i = 1, 2, \cdots n)$$

is the marginal ~~p.d.f~~ c.d.f of X_i.

* V. Independence:

Random variables $X_1, X_2, \cdots, X_n$ are called <u>(mutually) independent</u> if for any $x_1, x_2, \cdots x_n$

$[x_i \in (-\infty, +\infty), \quad i=1, 2, \cdots n]$

$$P(X_1 \le x_1, X_2 \le x_2, \cdots, X_n \le x_n) = P(X_1 \le x_1) \cdot P(X_2 \le x_2) \cdots \cdot P(X_n \le x_n)$$

i.e.

$$F(x_1, x_2, \cdots, x_n) = \prod_{i=1}^{n} F_{X_i}(x_i) \qquad (3.3.21)$$

If all the random variables are continuous, then they are independent if and only if

$$f(x_1, x_2, \cdots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)$$

where $f(x_1, \cdots, x_n)$: joint p.d.f

$f_{X_1}(x_1), f_{X_2}(x_2), \cdots, f_{X_n}(x_n)$: marginal p.d.f's

Summary of Chapter 3

I. Basic Concepts:

1. Joint c.d.f and Marginal c.d.f
2. Joint p.m.f and Marginal p.m.f (discrete case)
3. Joint p.d.f and Marginal p.d.f (continuous case)

※4. Independence

II. Basic Conclusions:

※ 1. Two random variables X and Y are independent if and only if

$$F(x, y) = F_X(x) \cdot F_Y(y)$$

where $F(x, y)$: joint c.d.f

$F_X(x)$ and $F_Y(y)$: marginal c.d.f

(True for both discrete and continuous ...)

For continuous random variables X and Y, independent if and only if

$$f(x, y) = f_X(x) \cdot f_Y(y)$$

Where $f(x, y)$: joint p.d.f

$f_X(x)$, $f_Y(y)$: marginal p.d.f

For discrete random variables X and Y

: independent if and only if

$$p(x, y) = p_X(x) \cdot p_Y(y)$$

where $p(x, y)$: joint p.m.f

$p_X(x)$, $p_Y(y)$: marginal p.m.f

2. For general random variables X and Y

joint c.d.f $\xrightarrow{\text{determines}}$ marginal c.d.f's

joint p.m.f $\xrightarrow{\text{determines}}$ marginal p.m.f's (discrete case)

joint p.d.f $\xrightarrow{\text{determines}}$ marginal p.d.f's (Continuous case)

3. For n random variables (※)

n random variables: $X_1, X_2, \cdots X_n$
They are (mutually) independent
if and only if
"Joint c.d.f is the product of n
marginal c.d.f's"

If all are continuous random variables
then "independent" if and only if
"Joint p.d.f is the product of
n marginal p.d.f's".

If all are discrete random variables
then "independent" if and only if
"Joint p.m.f is the product of n marginal p.m.f's