Chapter 0

Preliminary

(Baby Set Theory)

I. Basic Concepts:

1. "Def": A set is any collection of objects.

"object" $\Longrightarrow$ "element"

2. Notations and Representations:

① "Listing": $\{a, b, c\}$, say, denoted by $A = \{a, b, c\}$. Also $\{0, 1, 2, 3, \dots\}$ etc.

② "Function form" : $\{x;\ x^2=1\}$ or $\{x \mid x^2=1\}$.

③ "Venn diagram" : 

3. Notation "$\in$" :

"1 is an element of set B" usually denoted by

$1 \in B$

"$\notin$" means "does not belong to"

Hence if $A = \{a, b, c\}$

then $a \in A$, $b \in A$, $c \in A$ but $d \notin A$

Note that we view $\{1, 2, 1, 3\}$ and $\{1, 2, 3\}$,

for example, as the same set.

4. Some Special Sets:

① Empty set : ϕ **No element !**

② Singleton : $\{1\}$ **Only one element.**

③ "Universal" set : Ω : The totality of objects under consideration.

5. Set of Sets: Recall, a set is any collection of objects and so an element of set can be a set itself.

For example, $A = \{1, -1\}$, $B = \{a, b, c\}$

then $D = \{A, B, \phi, 5, \text{Cat}, \text{Dog}\}$

is a set.

So, $\{a, \{a, b\}\}$ is a set of two elements.

(Not 3)

Also, $E = \{a, b, c, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

is a set of 7 elements.

Note also that ϕ and $\{\phi\}$ have different meaning:

the former: empty set. No element.

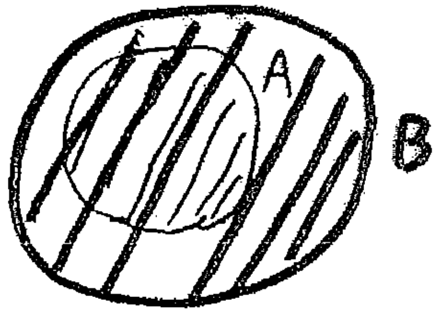
the latter: singleton set. With only one element, this element is the empty set.

II. Subsets of a Set:

1. Def.: If each element of a set A is also an element of a set B, then we say th. A is a **subset** of B.

(Or: the set A is contained in the set B or: the set B contains the set A)

Denoted by $A \subset B$ (or $B \supset A$)



In other words, $A \subset B$ means

$$\forall x \in A \Longrightarrow x \in B \quad (1.1.1)$$

Here "$\forall$" means "**for every**"

Also "$\exists$" means "**there exists**"

2. Equality of Sets:

If $A \subset B$ and $B \subset A$ we say A and B are equal and denote it by $A = B$, i.e. "$A = B$" means

$$\text{"} \quad \forall x \in A \implies x \in B \quad \text{and} \quad \forall x \in B \implies x \in A \quad \text{"} \qquad (1.1.2)$$

Note that, essentially, (1.1.2) is the only way to prove two sets are equal.

3. Simple facts:

① $A \subset A \quad \forall$ set A

② $\phi \subset A \quad \forall$ set A

③ $A \subset \Omega \quad \forall$ set A

(under the consideration)

4. Proper Subset:

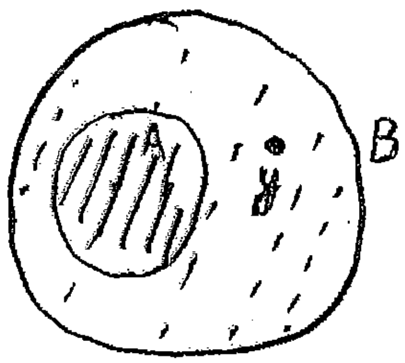
If $A \subset B$ and $A \neq B$ we say that A is a proper subset of B.

In other words. A is a proper subset of B means:

" $\forall x \in A \Rightarrow x \in B$

and

$\exists y \in B$ such that $y \notin A$ "

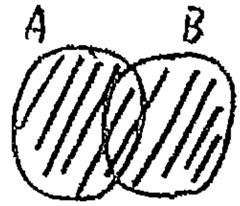


III Operations of Sets:

1. Union: ($\cup$) the elements belong to either A or B.

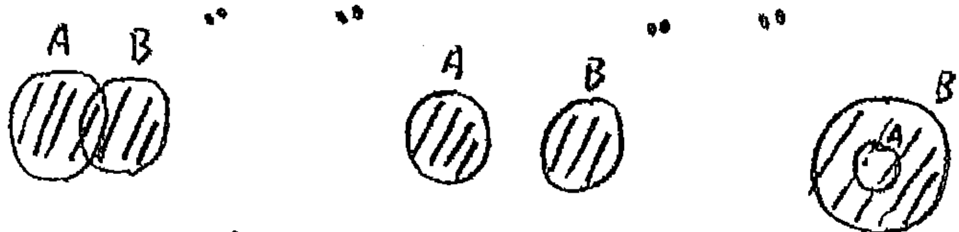
① Def., $A \cup B = \{x;\ x \in A \text{ or } x \in B\}$

② Diagram:



$\Leftarrow$ $A \cup B$ is the shaded region.

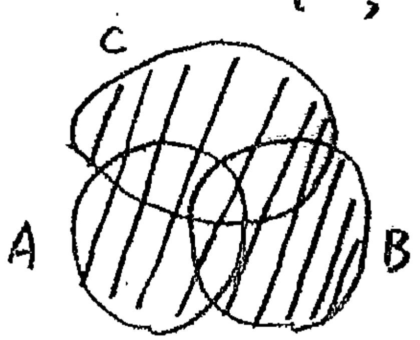
③ Three possible cases:



In the last case $A \subset B$ and so $A \cup B = B$

④ Finite union: Similarly, we may define the union of three, for example, sets.

$$A \cup B \cup C = \{x;\ x \in A \text{ or } x \in B \text{ or } x \in C\}$$



⑤ Laws:

$$A \cup B = B \cup A$$ (Commutative law)

$$(A \cup B) \cup C = A \cup (B \cup C)$$ (Associative law)

$$A \cup A = A$$ (Absorbing law)

$$A \cup \phi = A \quad (\because \phi \subset A)$$

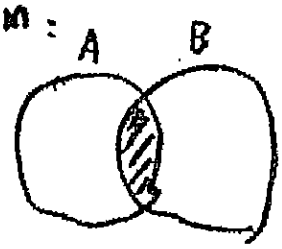
$$A \cup \Omega = \Omega \quad (\because A \subset \Omega)$$

7. Intersection: ($\cap$)

① Def.: $A \cap B = \{x;\ x \in A \text{ and } x \in B\}$

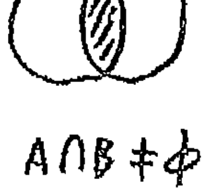
i.e. the elements that belong to both A and B

② Diagram:

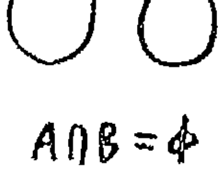


$\Leftarrow$ $A \cap B$ is the shaded region

③ Three possible cases:



$A \cap B \neq \phi$



$A \cap B = \phi$

$A \cap B = A$

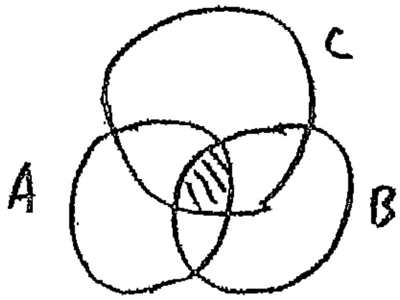
In the second case $A \cap B = \phi$.

In the third case $A \subset B$ and thus $A \cap B = A$.

④ Finite intersection:

For example,

$$A \cap B \cap C = \{x;\ x \in A \text{ and } x \in B \text{ and } x \in C\}$$



⇐ $A \cap B \cap C$ is the shaded region.

Question: what are $A \cap B$, $B \cap C$ and $A \cap C$?

⑤ Laws:

$A \cap B = B \cap A$ (Commutative law)

$(A \cap B) \cap C = A \cap (B \cap C)$ (Associative law)

$A \cap A = A$ (Absorbing law)

$A \cap \phi = \phi$ ($\because \phi \subset A$)

$A \cap \Omega = A$ ($\because A \subset \Omega$)

⑥ Distributive Laws:

For the operations of "union" and "intersection" we have:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad (1.1.3)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad (1.1.4)$$

Try to prove (1.1.3) and (1.1.4) yourself!!!

3. Diffence: (\ ) (or just "−")

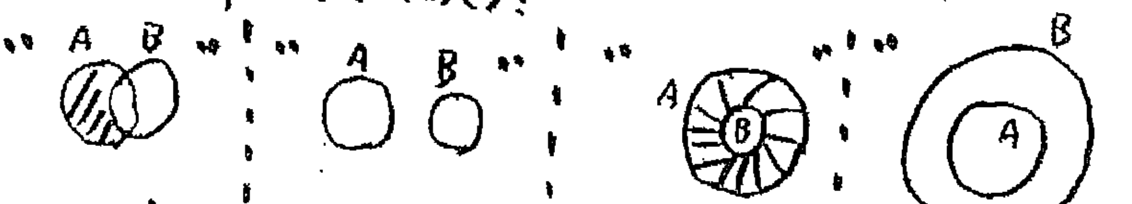
① Def.: $A \setminus B = \{x;\ x \in A \text{ and } x \notin B\}$

i.e. the set of elements that belong to A but do not belong to B.

② Diagram:

⇐ $A \setminus B$ is the shaded region

③ Four possible cases:



$A \setminus B \neq \phi$ | $A \setminus B = A$ | $A \setminus B = ?$ | $A \setminus B = \phi$

4. Complement:

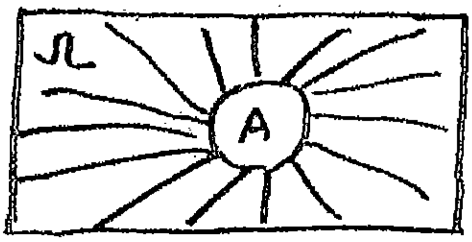
① Def.: The difference of the universary set Ω and A is called the complement of A and denoted by A^c.

i.e. $A^c = \{x;\ x \notin A\}$

「 $A^c = \Omega \setminus A = \{x;\ x \in \Omega \text{ and } x \notin A\}$

$= \{x;\ x \notin A\}$ since $x \in \Omega$ is always true.」

② Diagram:



⇐ A^c is the shaded region.

③ Laws:

$(A^c)^c = A$ $\qquad (A^c)^c = \{x;\ x \notin A^c\} = \{x;\ x \in A\}$

$\phi^c = \Omega$ $\qquad \phi^c = \{x;\ x \notin \phi\} = \Omega$

$\Omega^c = \phi$ $\qquad \Omega^c = \{x.\ x \notin \Omega\} = \phi$

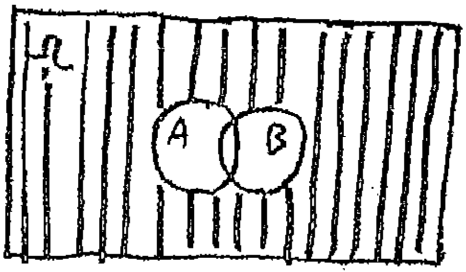
④ De Morgan's Laws. (Important !!!)

$$(A \cup B)^c = A^c \cap B^c \qquad (1.1.5)$$

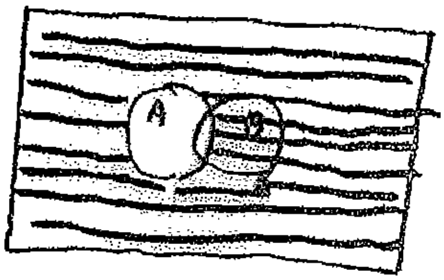
$$(A \cap B)^c = A^c \cup B^c \qquad (1.1.6)$$

See the following diagrams:

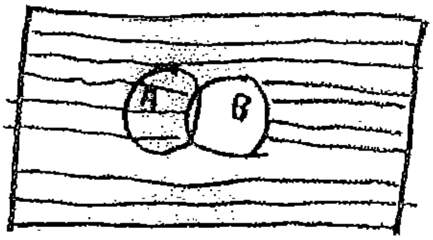
$(A \cup B)^c$: In Black



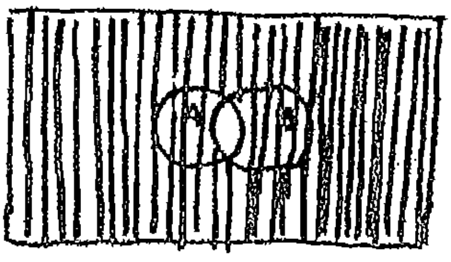
A^c : In Red

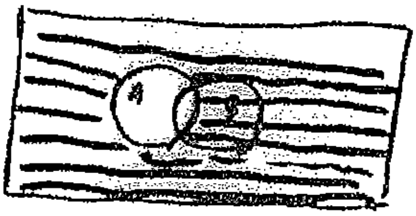


B^c : In Blue

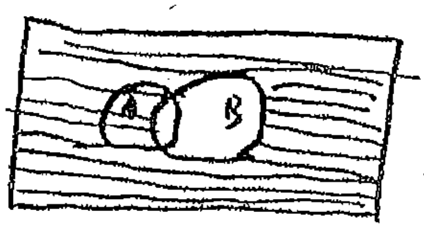


$(A \cap B)^c$: In Black





A^c : In Red



B^c : In Blue

Try to prove (1.1.5) and (1.1.6) yourself.

5. Operations of a family of sets:

① Union: For finitely many of sets, we usually write it as $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$.

Suppose we have a sequence of sets

$A_1, A_2, A_3, A_4, \cdots$

then the union of this sequence of sets is defined as the elements that belong to at least one A_k, $k=1,2,3,\cdots$ and denoted as $\bigcup_{k=1}^{\infty} A_k$,

i.e. $\bigcup_{k=1}^{\infty} A_k = A_1 \cup A_2 \cup A_3 \cup \cdots = \{x;\ \exists k \text{ such that } x \in A_k\}$

Similarly we may define the union of any family of sets as $\bigcup_{i \in I} A_i = \{x;\ \exists i \in I \text{ such that } x \in A_i\}$

② Intersection :

Similarly, suppose we have a sequence of sets

$A_1, A_2, A_3, \cdots,$

then the intersection of this sequence of sets is defined as the elements that belong to every A_k, $k=1,2,\cdots$ and denoted as

$$\bigcap_{k=1}^{\infty} A_k = A_1 \cap A_2 \cap A_3 \cap \cdots$$
$$= \{x ;\ \forall k,\ x \in A_k\}$$

③ De Morgan's Laws :

We still have

$$\left(\bigcap_{k=1}^{\infty} A_k\right)^c = \bigcup_{k=1}^{\infty} A_k^c \qquad (1.1.7)$$

$$\left(\bigcup_{k=1}^{\infty} A_k\right)^c = \bigcap_{k=1}^{\infty} A_k^c \qquad (1.1.8)$$

IV. Cartesian Product:

1. Ordered Pair:

A pair is called ordered if $(a, b) = (c, d)$ implies $a = c$, $b = d$.

In other words, usually $(a, b) \neq (b, a)$

2. Cartesian Product

Suppose A and B are two sets, then the Cartesian product of A and B, denoted by $A \times B$, is defined to be the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$,

i.e. $A \times B = \{(a, b) ;\ a \in A,\ b \in B\}$

Note that, usually $A \times B \neq B \times A$

3. Example:

$A = \{1, 2\} \qquad B = \{2, 3, 4\}$

$A \times B = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4)\}$

$B \times A = \{(2, 1), (2, 2), (3, 1)\ (3\ 2)\ (4\ 1)\ (4\ 2)\}$

4. General case:

Ordered triple (a, b, c);

Ordered n-tuple $(a_1, a_2, \cdots, a_n)$

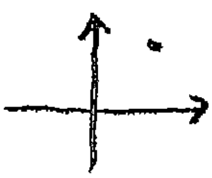
Suppose $A_1, A_2, \cdots, A_n$ are sets, then the Cartesian product of $A_1, A_2, \cdots, A_n$, denoted by $A_1 \times A_2 \times A_3 \times \cdots \times A_n$, is the set of all ordered n-tuple, $(a_1, a_2, \cdots a_n)$ where $a_1 \in A_1$, $a_2 \in A_2$, $\cdots$ $a_n \in A_n$,

i.e $A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \cdots, a_n);\ a_1 \in A_1, \cdots, a_n \in A_n$

If all the A_i are the same, then we write is as A^n. i.e. $A^n = A \times A \times \cdots \times A$.

5. More examples: $R = (-\infty, +\infty)$

$$R^2 = R \times R = \{(a, b);\ a \in R, b \in R\}$$



Also, R^3, R^n.

V. Cardinal Number of Sets:

1. Basic Concept:

① Problem:

To discuss the "size" or "number" of sets.
Try to answer the questions such as
For two sets A and B

"Do A and B have the same 'size'?"

"Does A have more elements than B?"

In particular, for <u>infinite sets</u>.

For example, $N = \{1, 2, 3, 4, 5, 6, \cdots\}$

$E = \{2, 4, 6, 8, \cdots\}$

More elements in N? (Since E is a <u>proper subset</u> of N.)

The problem is: how to compare??

② Idea: From finite case to infinite case.
<u>One-to-One correspondence between the</u> (element

③ Definition:

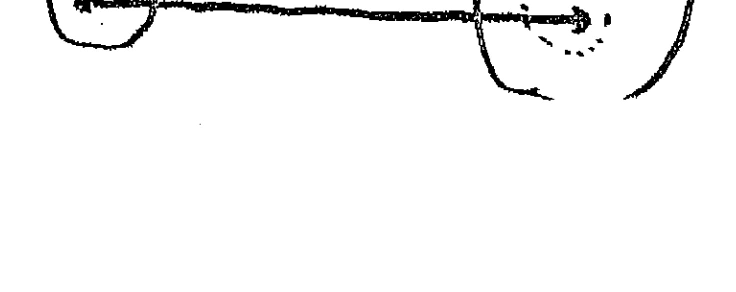
Two sets A and B are said to have the same cardinal number if and only if there exists a one-to-one correspondence between A and B.

The cardinal number of A is denoted by

$$\mathrm{Card}(A).$$

So, $\mathrm{Card}(A) = \mathrm{Card}(B)$ iff $\exists$ 1-1 correspondence

We shall say $\mathrm{Card}(A) \leq \mathrm{Card}(B)$ if and only if there exists a one-to-one correspondence between A and <u>a subset of B</u>.



Furthermore define

$$\text{Card}(A) < \text{Card}(B) \text{ iff } \text{Card}(A) \le \text{Card}(B) \text{ and } \text{Card}(A) \ne \text{Card}(B).$$

④ More Questions:

(i) Does any set have a cardinal number?

(ii) If **Yes**, then comparable for any two sets?

i.e. the following statement holds true?

"For any two sets A and B, either

$\text{Card}(A) \le \text{Card}(B)$ or $\text{Card}(B) \le \text{Card}(A)$"

(iii) If again **Yes** (Actually equivalent to "Axiom of Choice"),

"Smallest infinity"?

If Yes, which one? $\aleph_0$ say.

(iv) Is there a set A such that $\text{Card}(A) > \aleph_0$?

If Yes, which one?

(v) "Largest infinity"?

If **No**, "continue"? In particular, "the second smallest"?

For Questions (i) and (ii), we shall say "Yes", but ....

2. Countable sets:

① Def.: A set B is called countable if Card(B) is the same as the cardinal number of the natural numbers $N = \{1, 2, 3, \cdots\}$.

In other words, B is countable if and only if there exists a 1-1 correspondence between the elements of B and $N = \{1, 2, 3, \cdots\}$.

The cardinal number of a countable set is denoted by $\aleph_0$.

② Properties:

Theorem 1. A set B is countable if and only if all the elements of B can be written as a sequence, i.e., $A = \{x_1, x_2, x_3, \cdots\}$.

Proof. By definition. ($\because$ $\exists$ 1-1 correspondence

Theorem 2.

If both A and B are countable, then so is $A \cup B$.

If $A_1, A_2, \cdots, A_n$ are all countable then so is $\bigcup_{i=1}^{n} A_i$.

If $A_1, A_2, A_3, \cdots$ are all countable, then so is $\bigcup_{k=1}^{\infty} A_k$.

Proof: Just prove the last statement.

By Th 1. all the elements of $A_1, A_2, A_3, \cdots$ can be written as sequences:

$$A_1: \{ a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, \cdots \}$$
$$A_2: \{ a_{21}, a_{22}, a_{23}, a_{24}, a_{25}, \cdots \}$$
$$A_3: \{ a_{31}, a_{32}, a_{33}, a_{34}, a_{35}, \cdots \}$$

— — — — — — —

But then all the elements of $\bigcup_{k=1}^{\infty} A_k$ can be written as a sequence as well. For example.

$$\{ a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31} \cdots \}$$

Theorem 3. (The "smallest Property).

Any infinite set contains a countable subset.

Proof: Suppose A is an infinite set, then $A \neq \phi$

choose $a_1 \in A$ then $A \setminus \{a_1\} \neq \phi$

we can then choose $a_2 \in A \setminus \{a_1\}$.

In general, after choosing $a_1, a_2, \cdots, a_n$,

then $A \setminus \{a_1, a_2, \cdots, a_n\} \neq \phi$ (otherwise A is finite)

we can therefore choose $a_{n+1} \in A \setminus \{a_1, a_2, \cdots, a_n\}$

So we can extract a sequence from A. ※

Meaning of Theorem 3:

For any infinite set A, A has a subset which is countable and thus there exists a 1-1 correspondence between the countable set and a subset of A.

$\Rightarrow$: $\text{Card}(A) \geq \aleph_0$

i.e. $\aleph_0$ is the smallest cardinal number among the infinite sets

Theorem 4.

If $A_1, A_2, \cdots, A_n$ are all countable sets, then so is the Cartesian product $A_1 \times A_2 \times \cdots \times A_n$.

In particular,

$$A \text{ countable} \Longrightarrow A^n \text{ countable}.$$

Proof. Similar to Th. 2.

③ Examples of countable sets:

$$N = \{1, 2, 3, 4, 5, \cdots\} \quad ✓$$

$$E = \{2, 4, 6, 8, \cdots\} \quad ✓$$

Direct proof: $n \Longleftrightarrow 2n$.

Same "size" !!! **Astonishing** ?

More "sparse" examples? Sure!

$$F = \{10, 100, 1000, 10000, \cdots\}$$

Even

$$G = \{10, 10^{10}, 10^{10^{10}}, 10^{10^{10^{10}}}, \cdots\}$$

Can you image how sparse the G is ??

On the other hand, more "dense" examples ?

Conclusion: The set of all rational numbers is countable.

Proof. Just consider the non-negative ones. (why?)
Note that each rational number, r say, can be written as (r is also non-negative)

$$r = \frac{m}{n}$$

where both m and n are positive integers.
(except, of course, $r=0$ but this is trivial).
Hence, all the rational numbers are as follows

$$\frac{1}{1},\ \frac{2}{1},\ \frac{3}{1},\ \frac{4}{1},\ \frac{5}{1},\ \cdots$$

$$\frac{1}{2},\ \frac{2}{2},\ \frac{3}{2},\ \frac{4}{2},\ \frac{5}{2},\ \cdots$$

$$\frac{1}{3},\ \frac{2}{3},\ \frac{3}{3},\ \frac{4}{3},\ \frac{5}{3},\ \cdots$$

$$- - - - - -$$

So, the set of rational numbers is countable. 證完.

· How about the set of all irrational numbers ?
Interesting question ! See later.
However, if it were, then all real numbers $R = (-\infty, +\infty)$ would be also countable.
(See theorem 2)

Here, we first give another more "dense" example.

Conclusion: The set of all algebraic numbers is countable

An algebraic number is a real number that is the root of some polynomial with integer cofficients.
rational number must be algebraic number
($\because$ $r = \frac{m}{n}$ is the root of $nx - m = 0$)

Many irrational numbers are also algebraic number.
For example, $\sqrt{2}$ is the root of $x^2 - 2 = 0$
Now, is there a non-algebraic number ??

3. Cardinal Number c:

$\exists A$ such that $\text{Card}(A) > \aleph_0$??

① Definition: The cardinal number of set $[0,1]$ is denoted by c.

$[0,1] = \{x;\ 0 \leq x \leq 1\}$ is, of course, infinite and thus $c \geq \aleph_0$.

The question is whether $c = \aleph_0$??

② Conclusion (Th. 5): The set $[0,1]$ is not countable.

(hence $c \neq \aleph_0 \Rightarrow c > \aleph_0$!!)

Proof. Recall each real number in $[0,1]$ can be written as the form $0.xxx\cdots$

Now, suppose $[0,1]$ is countable, then it can be written as a sequence (Th. 1) $\{x_1, x_2, x_3, \cdots\}$ say.

Suppose

$$x_1 = 0.\,a_{11}\,a_{12}\,a_{13}\,a_{14}\cdots a_{1n}\cdots$$

$$x_2 = 0.\,a_{21}\,a_{22}\,a_{23}\,a_{24}\cdots a_{2n}\cdots$$

$$x_3 = 0.\,a_{31}\,a_{32}\,a_{33}\,a_{34}\cdots a_{3n}\cdots \qquad (1.1.9)$$

— — — — — — .

$$x_n = 0.\,a_{n1}\,a_{n2}\,a_{n3}\,a_{n4}\cdots a_{nn}\cdots$$

— — — — — — .

(Remember all of the numbers in [0, 1] are listed in (1.1.9)!!)

where a_{ij} are all $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Now we define a number, x^*, say, as

$$x^* = 0.\,a_{*1}\,a_{*2}\,a_{*3}\cdots a_{*n}\cdots$$

where $a_{*1} \neq a_{11}$, $a_{*2} \neq a_{22}$, $\cdots$

$$a_{*n} \neq a_{nn}$$

and all a_{*k} take values in $\{0, 1, 2, \cdots, 9\}$.

Surely $x^* \in [0,1]$

but x^* is not in (1.1.9)

(Since it equals neither of the x_n !!)

Contradiction. !!!

③ Properties :

(i) If $Card(A) = c$, $card(B) = c$

then $Card(A \cup B) = c$

(ii) $\forall i = 1, 2, \dots, n.$ $Card(A_i) = c \Longrightarrow Card(\bigcup_{i=1}^{n} A_i) = c$

(iii) $\forall i = 1, 2, \dots$ $Card(A_i) = c \Longrightarrow Card(\bigcup_{i=1}^{\infty} A_i) = c$

(iv) $\forall i = 1, 2, \dots, n$ $Card(A_i) = c$

$\Longrightarrow Card(A_1 \times A_2 \times A_3 \times \dots \times A_n) = c.$

In particular,

$Card(A) = c \Longrightarrow Card(A^n) = c.$

④ Examples:

The following sets all have cardinal number C:

$[0,1]$; $(0,1)$; $R=(-\infty,+\infty)$,

$R^+=[0,\infty)$; R^2; R^3; R^n.

Conclusion: The cardinal number of the set of irrational numbers is C.

More astonishing, we have

Conclusion: The cardinal number of the set of non-algebraic numbers (usually called transcendental numbers) is C.

In order to understand how strong the final conclusion is, let me review a "story" ...
"Before 1874 when George Canton ... "

4. "Maximal Cardinal Number" ??

No!!

Can easily prove that: there is no maximal cardinal number.

5. Continuum Hypothesis

① Question: Is there a cardinal number K, say, such that

$$\aleph_0 < K < C \quad ??$$

This is the famous Continuum Hypothesis

(C. H. states: no such kind of K)

② Historical Notes:

George Cantor (1845 — 1918)

1874 — 1897: Cantor published many papers on set theory.

Cantor conjectured that the continuum hypothesis was true.

David Hilbert later published a proof, but incorrect.

In 1939, Gödel proved that the "C.H" could not be disproved on the basis of our axioms for set theory.

In 1963, Paul Cohen proved that the "C.H" could not be proved on the basis of our axioms for set theory.

6. Remark on the term "countable"

finite set $-$ countable set $-$ uncountable set

$\underbrace{\text{finite set} - \text{countable}}_{\text{countable (Denumerable)}}$

7. Some Remarks for thinking:

① True for the following statement? why?

"If there exists a 1-1 correspondence between the set A and a subset of B, then

$$Card(A) < Card(B),$$

even a proper setset of B"

② Meaning of the "there exists a 1-1 corres...":

Does it mean "we can find the exact form"?

③ Relationship between the so-called "well-ordered set" and "countable set":

Z^2: countable but not "well-ordered" in the usual sence!

[0,1]: "Well-ordered" but not countable!

④ Is the following set countable?

"The set of all the sequences with 0 and 1 only"

Not countable! Think why? (Binary digit --)