Summary: Properties of Probability Measures

1. **Definition**: Let Ω be a sample space, and $\mathscr{F}$ denotes the set of events. Then $(\Omega, \mathscr{F})$ is called a **measurable space**. Let $(\Omega, \mathscr{F})$ be a measurable space, a set function P on $\mathscr{F}$ is called **probability measure**, if

   (i) for any $B \in \mathscr{F}$, $P(B) \geq 0$;

   (ii) $P(\emptyset) = 0$;

   (iii) $P(\Omega) = 1$;

   (iv) for each infinite sequence $\{A_i\}$ of disjoint sets that belong to $\mathscr{F}$, we have
   $$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i).$$

   Then $(\Omega, \mathscr{F}, P)$ is called a **probability space**, or probability triple.

2. **Terminology**

$(\Omega, \mathscr{F}, P)$	Probability triple
Ω	Sample space A point ω of Ω is called a sample point.
$\mathscr{F}$	the family of events, i.e. an element of $\mathscr{F}$ is called an event.

3. **Properties of Probability Measure** (including the ones in the Definition)

   Three Groups: $(\Omega, \mathscr{F}, P)$, a probability space

   ① Group A: Inequality

   $$0 \leq P(A) \leq 1 \qquad \forall A \in \mathscr{F}$$
   $$P(A) \leq P(B) \qquad \forall A \in \mathscr{F}, B \in \mathscr{F}, A \subset B$$
   $$P(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} P(A_n), \qquad \forall n, A_n \in \mathscr{F}$$
   $$P(\bigcup_{n=1}^{m} A_n) \leq \sum_{n=1}^{m} P(A_n), \qquad A_1, \cdots, A_n \in \mathscr{F}$$
   $$P(A \cup B) \leq P(A) + P(B)$$

② Group B: Equality

$$P(\emptyset) = 0 \qquad P(\Omega) = 1$$

$$P(B\backslash A) = P(B) - P(A), \qquad \forall A \in \mathscr{F}, B \in \mathscr{F}, A \subset B$$

$$P(A^C) = 1 - P(A), \qquad \forall A \in \mathscr{F}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B), \qquad \forall A \in \mathscr{F}, B \in \mathscr{F}$$

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j\leq n}\sum P(A_i \cap A_j) + \cdots + (-1)^{n-1} P(A_1 \cap \cdots \cap A_n), \qquad A_i \in \mathscr{F}$$

$$P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n), \qquad \text{for } \underline{\underline{\text{disjoint}}} \text{ sequence } \{A_n\} \text{ in } \mathscr{F}$$

$$P(\bigcup_{n=1}^{m} A_n) = \sum_{n=1}^{m} P(A_n), \qquad \text{for } \underline{\underline{\text{disjoint}}}\ \{A_n\} \text{ in } \mathscr{F}$$

③ Group C: Limiting property

$$P(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} P(A_n) \qquad \text{for } \underline{\underline{\text{increasing}}}\ \{A_n\} \in \mathscr{F}$$

$$P(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} P(A_n) \qquad \text{for } \underline{\underline{\text{decreasing}}}\ \{A_n\} \in \mathscr{F}$$