§4.8. Some Useful Facts

I. Indepence:

Easy to see (intuitively)

if X and Y are independent r.vs

$g(\cdot)$ and $h(\cdot)$ are two functions,

then the random variables $U = g(X)$ and $V = h(Y)$ are also independent

In particular,

if X and Y are independent

then $aX+b$ and $c \cdot Y+d$ are also indepent where a, b, c, d are constants $(a \neq 0, c \neq 0)$.

More generally, if $\{X_1, X_2, \cdots X_m, X_{m+1}, \cdots X_n\}$ are mutuarial independent rv's and

$g(x_1, \cdots, x_m)$ and $h(y_1, \cdots, y_{n-m})$ are two ordinary functions, then the rv's

$g(X_1, \cdots, X_m)$ and $h(X_{m+1}, \cdots, X_n)$

are also indepent.

For example, if $\{X_1, X_2, X_3, X_4, X_5\}$ are independent, then

$$2X_2 + e^{X_3} - X_1 X_2^2 X_3 \quad \text{and}$$

$$X_5 X_4 + \frac{5X_4}{e^{X_5}}$$, say are also ~~indep~~ independent.

The essential thing here is that there is no common r.v in <u>both</u> functions

For example, if $\{X_1, X_2, \cdots X_n\}$ are independent random variables, (even if not in the same type) then, for exph

$X_1 + X_2 + \cdots + X_{n-1}$, ~~may~~ and X_n are independent. But you can not say

$X_1 + X_2 + \cdots + X_n$ and X_n are independent

II. Mixing case.

Suppose X is a discrete rv but Y is a continuous r.v.
Then joint cdf can be similarly defined as $F(x, y) = P\{X \le x, Y \le y\}$ for all $(-\infty < x < \infty, -\infty < y < \infty)$.
Then, again, they are independent if joint cdf is the product of two marginal cdfs. i.e.

$$F(x, y) = F_X(x) \cdot F_Y(y)$$

We can also define the so-called "joint pdf-pmf" function

$$f(x, y) \quad \left[\begin{cases} x = \{x_1, x_2, \cdots\} \\ y : (-\infty, +\infty) \end{cases} \right]$$

such that for any real numbers, a, b we have

$$F(a, b) = \sum_{x \le a} \int_{-\infty}^{b} f(x, y)\, dy .$$

(Similar for the mixing case for more than 2 random variables)

It can be proved that X and Y are independent if and only if "joint pdf-pmf" = (Marginal pdf)·(Marginal pmf)

III Is a constant random variable?

For example, is the constant 10 a random variable?

Usually not, "we emphasise a r.v X depends upon the change!!" (<u>This difference is important!!</u>)

However, for convenience, some times we may <u>view</u> a constant as a "random variable".

Questions. If the constant, 10, say is <u>viewed</u> as a random variable X, (i.e. $X(\omega) \equiv 10$), then

① discrete type or continuous type?

(<u>Discrete type</u>, since only taking <u>one value</u>)

② "pmf"? "cdf"?

pmf: simply, $P\{X = 10\} = 1$.

cdf: For any $x \in (-\infty, +\infty)$

$$F_X(x) = Pr\{X \leq x\}$$

Hence, if $x < 10$

Then $F_X(x) = P\{X \le x\} = 0$

$(\because X \equiv 10, \Rightarrow \{X \le x\} = \{10 \le x\} = \phi)$ → Impossible event.

If $x \geq 10$,

then $F_X(x) = P\{X \le x\} = 1$

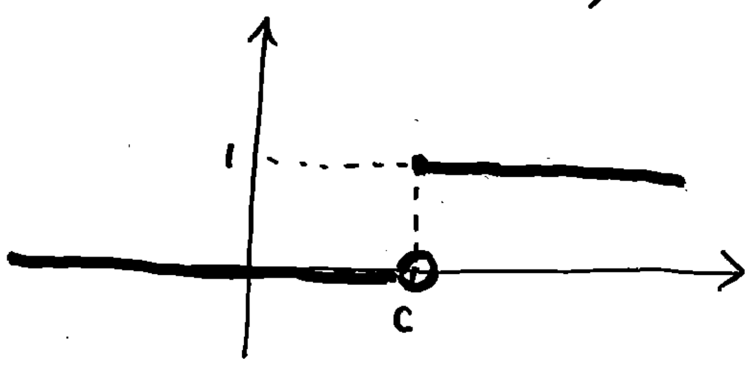
$(\because X \equiv 10 \Rightarrow \{X \le x\} = \{10 \le x\}$ must be true and thus certain event!)

In general, for any constant c,

(if viewed as a random variable X) then

pmf: $P\{X = c\} = 1$ (discrete type)

cdf: $$F_X(x) = P\{X \le x\} = \begin{cases} 1 & \text{if } x \geq c \\ 0 & \text{if } x < c \end{cases}$$



③ Independent with other random variable?

If $X = 10$ (constant) is viewed as a r.v.

and Y is another r.v. (either type?)

" X and Y are independent? "

Conclusion: $\boxed{\text{Yes!}}$

Proof. $F(x, y) \overset{?}{=} F_X(x) F_Y(y)$ (for all x, and y)

If $x < 10$, then $F(x, y) = P\{X \le x, Y \le y\} = P\{10 \le x, Y \le y\}$

$= P(\phi) = 0$

but $F_X(x) F_Y(y) = P\{X \le x\} P\{Y \le y\} = P\{10 \le x\} P\{Y \le y\}$

$= P(\phi) \cdot P\{Y \le y\} = 0 \times P\{Y \le y\} = 0$

If $x \ge 10$, then $F(x, y) = P\{X \le x, Y \le y\} = P\{10 \le x, Y \le y\}$

$= P\{Y \le y\}$ (Think why here!!)

but $F_X(x) F_Y(y) = P\{X \le x\} P\{Y \le y\} = P\{10 \le x\} P\{Y \le y\}$

$= P(\Omega) P\{Y \le y\} = P\{Y \le y\}$

Hence always true. (Similar for any constant c)

II. Normal distribution:

1. If $X \sim N(\mu, \sigma^2)$

then $E(X) = \mu$, $Var(X) = \sigma^2$.

2. If $X \sim N(\mu, \sigma^2)$

then $a \cdot X + b \sim N(a\mu + b, a^2\sigma^2)$

i.e. $E(aX+b) = a\mu + b$

$Var(a \cdot X + b) = a^2\sigma^2$

Indeed, $E(aX+b) = a\,E(X) + b = a\mu + b$

$Var(a \cdot X + b) = Var(aX) = a^2 \cdot Var(X)$

$= a^2 \cdot \sigma^2$

3. If $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$ and

X and Y are independent,

then $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

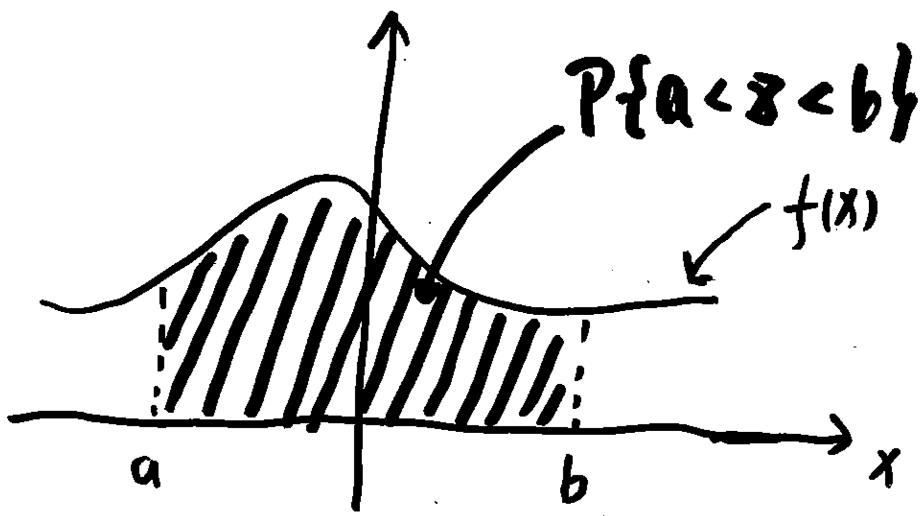
III. Other Continuous Random Variables:

1. Geometric meaning: (True for any continuous r.v)

If X is a continuous r.v. with c.d.f. $F(x)$ and p.d.f. $f(x)$, then

$$\forall a<b, \quad P\{a<X<b\} = F(b)-F(a) = \int_a^b f(x)dx$$

the geometric meaning of $\int_a^b f(x)dx$ is the area between a and b under the curve $f(x)$ (and so is the probability that $\{a<X<b\}$.



(a, b can be infinities !!)

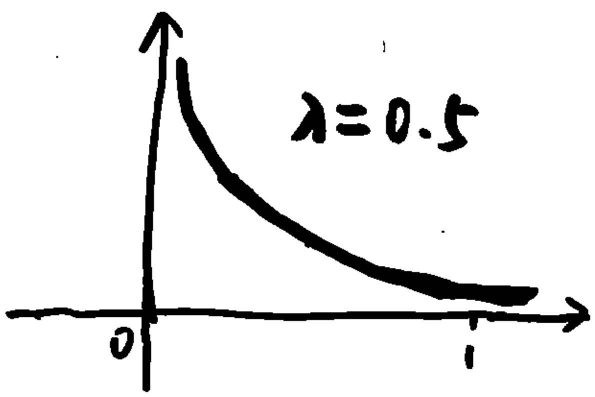
2. Exponential Distribution:

p.d.f.: $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

c.d.f.: $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$

$$\boxed{E(X) = \frac{1}{\lambda} \qquad Var(X) = \frac{1}{\lambda^2}}$$

Graph of $f(x)$ (p.d.f)



So. X can not take negative values.

§2. Distributions Derived from the Normal Distribution

The following three distributions play extremely important role in statistics. (See Sec 6.2 of the Reference book)

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3. χ^2 Distribution:

① Def: If $X_1, X_2 \cdots X_n$ are i.i.d. (independent, identically distributed) with common standard normal distribution, i.e. $X_i \sim N(0,1)$ $(\forall i)$ and $\{X_1, X_2 \cdots X_n\}$ are mutually independent.

then $Y = X_1^2 + X_2^2 + \cdots + X_n^2$

is called a χ^2 random variable with degree freedom of n.

② Notation: $Y \sim \chi^2(n)$

③ p.d.f.:

$$f(x) = \begin{cases} \dfrac{1}{2^{\frac{n}{2}} \cdot \Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} \cdot e^{-\frac{x}{2}}, & x \geq 0 \\ 0 & x < 0 \end{cases}$$

(The following three continuous distributions are useful in Statistics and will be discussed in another course)

3. χ^2 Distribution:

① Def: If $X_1, X_2 \cdots X_n$ are i.i.d. (independent, identically distributed) with common standard normal distribution, i.e. $X_i \sim N(0,1)$ $(\forall i)$ and $\{X_1, X_2 \cdots X_n\}$ are mutually independent.

then $$Y = X_1^2 + X_2^2 + \cdots + X_n^2$$

is called a χ^2 random variable with degree freedom of n.

② Notation: $Y \sim \chi^2(n)$

③ P.d.f:

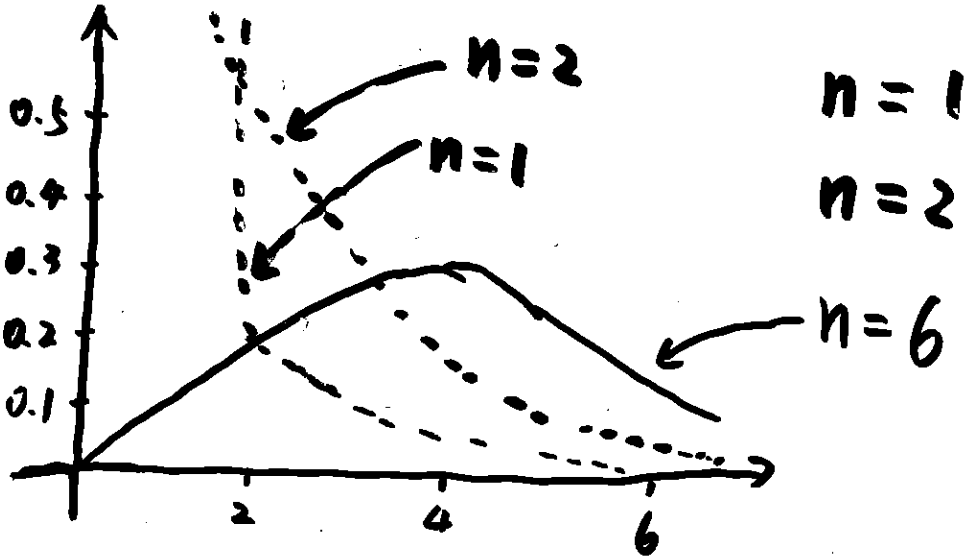
$$f(x) = \begin{cases} \dfrac{1}{2^{\frac{n}{2}} \cdot \Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} \cdot e^{-\frac{x}{2}}, & x \geq 0 \\ 0 & x < 0 \end{cases}$$

④ Expectation and Variance:

If $Y \sim \chi^2(n)$ ($\Rightarrow$ moment generating function $M(t) = (1-2t)^{-\frac{n}{2}}$)

then $E(Y) = n$, $Var(Y) = 2n$

⑤ Graph of p.d.f. (depend upon the parameter n where n is a positive integer)



⑥ $\Gamma(\frac{n}{2})$:

if n is even then $\Gamma(\frac{n}{2}) = [(\frac{n}{2})-1]!$

for example, $n=4$. $\Gamma(\frac{4}{2}) = [(\frac{4}{2})-1]! = 1! = 1$

$n=10$ $\Gamma(\frac{10}{2}) = [(\frac{10}{2})-1]! = 4! = 24$

if n is odd then $\Gamma(\frac{n}{2}) = \frac{n-2}{2} \cdot \frac{n-4}{2} \cdots \cdot \frac{1}{2} \cdot \sqrt{\pi}$

ex. $\Gamma(\frac{5}{2}) = \frac{5-2}{2} \cdot \frac{5-4}{2} \sqrt{\pi} = \frac{3}{4}\sqrt{\pi}$

4. t-distribution (student-distribution)

① Def: If $X \sim N(0,1)$, $Y \sim \chi^2(n)$

X and Y are independent

then $T = \dfrac{X}{\sqrt{\frac{Y}{n}}}$

is called to obey the t-distribution with n degree of freedom. (Also a parameter)

② Notation: $T \sim t(n)$

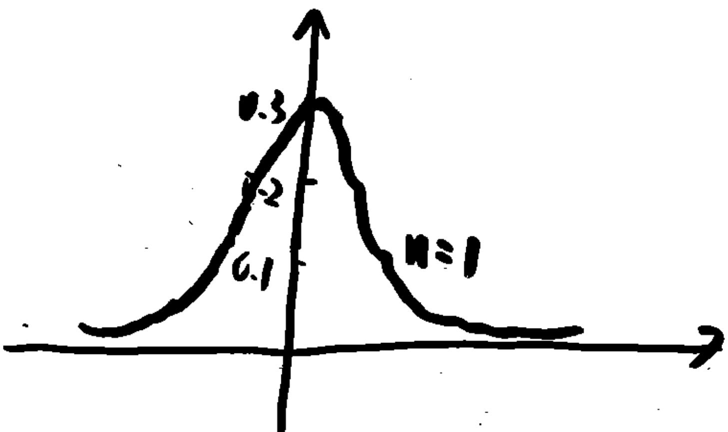
③ P.d.f.:

$$f(x) = \begin{cases} \dfrac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})\sqrt{n\pi}} (1+x^2/n)^{-\frac{n+1}{2}} \\ \forall -\infty < x < +\infty \end{cases}$$

So, $f(x)$ is symmetric with $x=0$

④ Property: $t(n) \to N(0,1)$ $(n \to \infty)$

⑤ Graph



5. F-distribution:

① Def: If $X \sim \chi^2(m)$, $Y \sim \chi^2(n)$

X and Y are independent

then $F = \dfrac{X/m}{Y/n}$ is called to obey

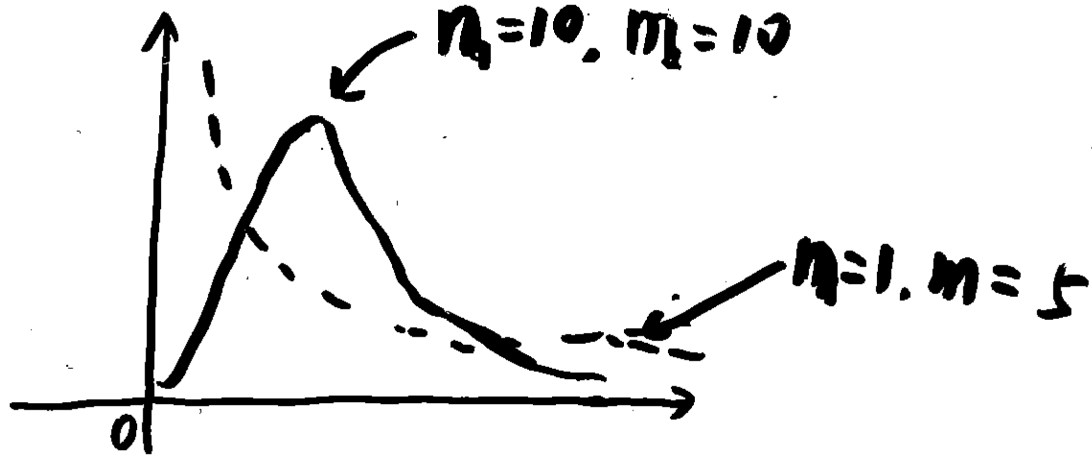
the F-distribution with (m, n) degree of freedom

② Notation: $F \sim F(m, n)$ (Two parameter)

③ p.d.f:

$$f(x) = \begin{cases} \dfrac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\cdot\Gamma\left(\frac{n}{2}\right)}\left(\frac{m}{n}\right)\left(\frac{m}{n}x\right)^{\frac{m}{2}-1}\left(1+\frac{m}{n}x\right)^{-\frac{m+n}{2}} & (\text{if } x>0) \\ 0 & \text{if } x \le 0 \end{cases}$$

④ Graph:



⑤ Property: If $X \sim F(m, n)$, then $X^{-1} \sim F(n, m)$.

If $T \sim t(n)$, then $T^2 \sim F(1, n)$

IV. Discrete Random Variables:

1. Bernoulli: p.m.f.:

x	0	1
$p(x)$	$1-p$	p

where $0 \le p \le 1$

$E(X) = p$ $\quad$ $Var(X) = pq$

2. Binomial: $X \sim B(n, p)$

p.m.f.: $p(k) = P\{X = k\} = \binom{n}{k} p^k q^{n-k}$

$(k = 0, 1, 2 \cdots n)$

where $0 \le p \le 1$ and $p + q = 1$

If $X \sim B(n, p)$

then $E(X) = np$

$Var(X) = npq$

P4.70

3. Poisson: $X \sim \text{Poisson}(\lambda)$

p.m.f.: $p(k) = P\{X=k\} = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$

$(k = 0, 1, 2, 3, \cdots)$

$\lambda > 0$ is a constant.

If $X \sim \text{Poisson}(\lambda)$

then $E(X) = \lambda$. $\quad Var(X) = \lambda$.

4. Geometric distribution:

p.m.f.: all possible values: $\{1, 2, \cdots\}$

$p_k = P\{X=k\} = p \cdot q^{k-1} \quad (k = 1, 2, \cdots)$

where $p > 0$, $q > 0$, $p + q = 1$

$E(X) = \frac{1}{p} \quad Var(X) = \frac{q}{p^2} = \frac{1-p}{p^2}$

Summary of Chapter 4.

I. Basic Concepts:

1. Expected value: $E(X)$ (Weighted average)
2. Variance: $Var(X) = E\left[(X - E(X))^2\right]$
3. Standard Deviation: $\sqrt{Var(X)}$
4. Covariance

$$Cov(X, Y) = E\left[(X - E(X))(Y - E(Y))\right]$$

5. Correlation

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X) \cdot Var(Y)}}$$

6. Function of random variables:

$g(X)$, $g(X, Y)$ etc.

II. Basic Properties:

1. Expectation:

① $E(c) = c$ for constant c.

※ ② $E(a \cdot X + b \cdot Y) = a \cdot E(X) + b \cdot E(Y)$

(a, b: constants) (linear property!)

※ ③ If X and Y are independent, then

$$E(X \cdot Y) = E(X) \cdot E(Y)$$

2. Variance:

① $Var(c) = 0$ for constant c.

② $Var(a \cdot X + b) = a^2 \cdot Var(X)$

(a, b: constants)

③ If X and Y are independent, then

$$Var(X + Y) = Var(X) + Var(Y).$$

3. Covariance:

① $Cov(a, b) = 0$ for constants a and b

② $Cov(a_1 X + b_1 Y + c_1,\ a_2 U + b_2 V + c_2)$

$= a_1 \cdot a_2\, Cov(X, U) + a_1 \cdot b_2\, Cov(X, V)$

$+ a_2 \cdot b_1\, Cov(Y, U) + a_2 \cdot b_2\, Cov(Y, V)$

for constants a_1, b_1, c_1, a_2, b_2, and c_2 and random variables X, Y, U, and V.

③ If X and Y are independent

then $Cov(X, Y) = 0$.

(but the converse is not true)

(for bivariate normal the converse holds true)

4. Correlation: for any two r.vs

$-1 \leq \rho(X, Y) \leq 1$

III. Calculation:

1. Function of random variables

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x)\cdot f(x)\,dx$$

$$E[g(X)] = \sum_{i} g(x_i)\cdot p(x_i)$$

$$E[g(X,Y)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y) f(x,y)\,dx\,dy$$

$$E[g(X,Y)] = \sum_{j}\sum_{i} g(x_i, y_j)\cdot f(x_i, y_j)$$

2. Variance:

$$Var(X) = E(X^2) - (E(X))^2$$

3. Covariance:

$$Cov(X,Y) = E(X\cdot Y) - E(X)\cdot E(Y)$$

IV. Some important facts:

1. Normal distribution:

① If $X \sim N(\mu, \sigma^2)$ then $E(X) = \mu$

$Var(X) = \sigma^2$

② If $X \sim N(\mu, \sigma^2)$, a, b are constants, $a \neq 0$.

then $a \cdot X + b \sim N(a\mu + b, a^2\sigma^2)$

③ If $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$

and X and Y are independent

then $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

④ If $X \sim N(\mu, \sigma^2)$.

let $Y = \dfrac{X - \mu}{\sigma}$

then $Y \sim N(0, 1)$

2. Poisson distribution

If $X \sim Poisson(\lambda)$ then $E(X) = \lambda$

$Var(X) = \lambda$

3. Binomial distribution

If $X \sim B(n, p)$ then $E(X) = np$

$Var(X) = npq$

$= np(1-p)$

4. Exponential distribution:

If $X \sim Expon(\lambda)$

then $E(X) = \frac{1}{\lambda}$ $Var(X) = \frac{1}{\lambda^2}$

P.d.f. $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

V. Other useful facts:

See §4.6