THE SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA
DEPARTMENT OF MATHEMATICS

MA215 Introduction to Probability Theory

Exercise Sheet 12

Set: Friday 9th December 2016; Hand in: Friday 16th December 2016 by 5pm.

1. The covariance between X and Y, denoted by $Cov(X,Y)$, is defined by
$$Cov(X,Y) = E[(X - E(X))(Y - E(Y))].$$
Show that
$$Cov(X,Y) = E[XY] - E[X]E[Y].$$

2. Let X be a discrete random variable with pmf as
$$P\{X = 0\} = P\{X = 1\} = P\{X = -1\} = \frac{1}{3}.$$
Define
$$Y = \begin{cases} 0 & if \quad X \neq 0 \\ 1 & if \quad X = 0 \end{cases}$$
   (i) Show that $Cov(X,Y) = 0$.
   (ii) Write down the joint pmf of X and Y, and show that X and Y are not independent.

3. Show that the following conclusions are true:
   (i) $Cov(X,Y) = Cov(Y,X)$;
   (ii) $Cov(X,X) = Var(X)$;
   (iii) $Cov(aX,Y) = aCov(X,Y)$, where a is a constant;
   (iv) $Cov(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} Cov(X_i, Y_j)$;
   (v) If X is a random variable and C is a constant, then $Cov(X,C) = 0$.

4. Show that the following statements are true
$$Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i) + \sum\sum_{i\neq j} Cov(X_i, X_j),$$
or, equivalently,
$$Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i) + 2\sum\sum_{i<j} Cov(X_i, X_j).$$
Further show that,if $X_1, \ldots, X_n$ are pairwise independent, in that X_i and X_j are independent for $i \neq j$, then we have
$$Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i).$$

5. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables having common expected value μ and common variance σ^2. Let $\bar{X}$ and S^2 be defined as follows.
$$\bar{X} = \sum_{i=1}^{n} X_i/n.$$
$$S^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2.$$
The two random variables $\bar{X}$ and $S^2/(n-1)$ are called the sample mean and sample variance, respectively. Find

(i) $E[(\bar{X})]$;

(ii) $Var(\bar{X})$;

(iii) $E[S^2/(n-1)]$.

6. Let I_A and I_B be the indicator variables for the events A and B. That is,

$$I_A = \begin{cases} 1 & if \quad A \quad occurs \\ 0 & otherwise \end{cases}$$

$$I_B = \begin{cases} 1 & if \quad B \quad occurs \\ 0 & otherwise \end{cases}$$

Show that

(i)

$$E[I_A] = P(A)$$
$$E[I_B] = P(B)$$
$$E[I_A I_B] = P(AB)$$

(ii)

$$\begin{aligned} Cov(I_A, I_B) &= P(AB) - P(A)P(B) \\ &= P(B)[P(A|B) - P(A)]. \end{aligned}$$

7. Let$X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables having common variance σ^2. Show that

$$Cov(X_i - \bar{X}, \bar{X}) = 0,$$

where $\bar{X}$ is the sample mean as given in the above Question 5. (i.e.$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$.)