SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF MATHEMATICS

MA215 Probability Theory

Tutorial 08Solu

Set: Friday 4th November 2016; Hand in: Friday 11th November by 5pm.

1. Suppose that the continuous random variable X has p.d.f

$$f_X(x) = \begin{cases} kx(1-x) & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Evaluate the constant k.
Find the non-zero range of Y and the p.d.f $f_Y(y)$ of Y when

(a) $Y = -3X + 3$;

(b) $Y = \frac{1}{X}$.

Solution: A p.d.f must integrate to 1, so

$$\begin{aligned} 1 = \int_{-\infty}^{+\infty} f_X(x)\mathrm{d}x &= \int_{-\infty}^{0} f_X(x)\mathrm{d}x + \int_{0}^{1} f_X(x)\mathrm{d}x + \int_{1}^{+\infty} f_X(x)\mathrm{d}x \\ &= \int_{-\infty}^{0} 0\mathrm{d}x + \int_{0}^{1} kx(1-x)\mathrm{d}x + \int_{1}^{+\infty} 0\mathrm{d}x \\ &= \int_{0}^{1} kx(1-x)\mathrm{d}x = \int_{0}^{1} k\left(x - x^2\right)\mathrm{d}x \\ &= \left[k\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\right]_0^1 = \frac{k}{6}. \end{aligned}$$

Hence, we get $k = 6$.

(a) If $Y = -3X + 3$. Then,

$$\begin{aligned} F_Y(y) = \mathrm{P}\{\mathrm{Y} \le \mathrm{y}\} &= \mathrm{P}\{-3\mathrm{X} + 3 \le \mathrm{y}\} = \mathrm{P}\{-3\mathrm{X} \le \mathrm{y} - 3\} \\ &= \mathrm{P}\{\mathrm{X} \ge \frac{\mathrm{y}-3}{-3}\} \\ &= 1 - \mathrm{P}\{\mathrm{X} < \frac{\mathrm{y}-3}{-3}\} = 1 - \mathrm{P}\{\mathrm{X} \le \frac{\mathrm{y}-3}{-3}\} \\ &= 1 - F_X(\frac{y-3}{-3}). \end{aligned}$$

Hence,

$$\begin{aligned} f_Y(y) &= \frac{\mathrm{d}F_Y(y)}{\mathrm{d}y} \\ &= \frac{\mathrm{d}}{\mathrm{d}y}\left(1 - F_X(\frac{y-3}{-3})\right) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{\mathrm{d}}{\mathrm{d}y} F_X(\frac{y-3}{-3}) \\
&= -\frac{\mathrm{d}}{\mathrm{d}u} F_X(u)\Big|_{u=\frac{y-3}{-3}} \cdot \frac{\mathrm{d}}{\mathrm{d}y}(\frac{y-3}{-3}) \\
&= -f_X(u)\Big|_{u=\frac{y-3}{-3}} \cdot \left(\frac{1}{-3}\right) \\
&= -f_X\left(\frac{y-3}{-3}\right) \cdot \left(\frac{1}{-3}\right) = \frac{1}{3} f_X\left(\frac{y-3}{-3}\right).
\end{aligned}
$$

Now since the continuous random variable X has p.d.f

$$
f_X(x) = \begin{cases} 6x(1-x) & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}
$$

Therefore, for $0 < \frac{y-3}{-3} < 1$, i.e. $0 < y < 3$,

$$
f_Y(y) = \frac{1}{3} f_X\left(\frac{y-3}{-3}\right) = \frac{1}{3} \times 6 \times (\frac{y-3}{-3})(1 - \frac{y-3}{-3}) = \frac{2}{9} y(3-y).
$$

otherwise, $f_Y(y) = 0$. i.e.,

$$
f_Y(y) = \begin{cases} \frac{2}{9} y(3-y) & 0 < y < 3, \\ 0 & \text{otherwise.} \end{cases}
$$

Hence, the non-zero range of f_Y is (0,3).

(b) $Y = \frac{1}{X}$. Then,

$$
F_Y(y) = \mathrm{P}\{\mathrm{Y} \le \mathrm{y}\} = \mathrm{P}\{\frac{1}{\mathrm{X}} \le \mathrm{y}\}
$$

If $y = 0$, then $\{\frac{1}{X} \le 0\} = \{X < 0\}$, so

$$
\begin{aligned}
F_Y(y) = \mathrm{P}\{\mathrm{Y} \le \mathrm{y}\} &= \mathrm{P}\{\frac{1}{\mathrm{X}} \le \mathrm{y}\} = \mathrm{P}\{\frac{1}{\mathrm{X}} \le 0\} \\
&= \mathrm{P}\{\mathrm{X} < 0\} = \mathrm{P}\{\mathrm{X} \le 0\} \\
&= F_X(0) = \int_{-\infty}^{0} f_X(x)\mathrm{d}x = \int_{-\infty}^{0} 0\mathrm{d}x = 0.
\end{aligned}
$$

Since the probability density function in a point modify the value does not affect the distribution function, for $y = 0$, let $f_Y(y) = f_Y(0) = 0$.

If $y \ne 0$, then $\{\frac{y(\frac{1}{y} - X)}{X} \le 0\} = \{yX(\frac{1}{y} - X) \le 0, X \ne 0\}$, yields

$$
F_Y(y) = \mathrm{P}\{\frac{1}{\mathrm{X}} \le \mathrm{y}\} = \mathrm{P}\{\frac{1-\mathrm{yX}}{\mathrm{X}} \le 0\} = \mathrm{P}\{\frac{\mathrm{y}(\frac{1}{\mathrm{y}} - \mathrm{X})}{\mathrm{X}} \le 0\} = \mathrm{P}\{\mathrm{yX}(\frac{1}{\mathrm{y}} - \mathrm{X}) \le 0, \mathrm{X} \ne 0\}.
$$

- if $y > 0$,

$$
F_Y(y) = \mathrm{P}\{\mathrm{yX}(\frac{1}{\mathrm{y}} - \mathrm{X}) \le 0, \mathrm{X} \ne 0\} = \mathrm{P}\{\mathrm{X}(\frac{1}{\mathrm{y}} - \mathrm{X}) \le 0, \mathrm{X} \ne 0\}
$$

$$= \mathrm{P}\{\mathrm{X} < 0 \text{ or } \mathrm{X} \geq \frac{1}{\mathrm{y}}\} = \mathrm{P}\{\mathrm{X} < 0\} + \mathrm{P}\{\mathrm{X} \geq \frac{1}{\mathrm{y}}\}$$

$$= \mathrm{P}\{\mathrm{X} < 0\} + 1 - \mathrm{P}\{\mathrm{X} < \frac{1}{\mathrm{y}}\} = \mathrm{P}\{\mathrm{X} \leq 0\} + 1 - \mathrm{P}\{\mathrm{X} \leq \frac{1}{\mathrm{y}}\}$$

$$= F_X(0) + 1 - F_X(\frac{1}{y}).$$

Therefore,

$$\begin{aligned} f_Y(y) &= \frac{\mathrm{d}}{\mathrm{d}y}\left(F_Y(y)\right) = \frac{\mathrm{d}}{\mathrm{d}y}\left(F_X(0) + 1 - F_X(\frac{1}{y})\right) \\ &= -\frac{\mathrm{d}F_X(u)}{\mathrm{d}u}\Big|_{u=\frac{1}{y}} \cdot \frac{\mathrm{d}(\frac{1}{y})}{\mathrm{d}y} \\ &= -f_X(u)\Big|_{u=\frac{1}{y}} \cdot (-\frac{1}{y^2}) \\ &= -f_X(\frac{1}{y}) \cdot (-\frac{1}{y^2}) = \frac{1}{y^2} f_X(\frac{1}{y}). \end{aligned}$$

Now since the continuous random variable X has p.d.f

$$f_X(x) = \begin{cases} 6x(1-x) & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

- for $0 < \frac{1}{y} < 1$, i.e. $y > 1$,

$$f_Y(y) = \frac{1}{y^2} f_X(\frac{1}{y}) = \frac{1}{y^2} \cdot 6\frac{1}{y}(1 - \frac{1}{y}) = \frac{6(y-1)}{y^4}.$$

- for $\frac{1}{y} \geq 1$, i.e. $0 < y \leq 1$,

$$f_Y(y) = \frac{1}{y^2} f_X(\frac{1}{y}) = \frac{1}{y^2} \cdot 0 = 0.$$

▶ if $y < 0$,

$$\begin{aligned} F_Y(y) &= \mathrm{P}\{\mathrm{yX}(\frac{1}{\mathrm{y}} - \mathrm{X}) \leq 0, \mathrm{X} \neq 0\} = \mathrm{P}\{\mathrm{X}(\frac{1}{\mathrm{y}} - \mathrm{X}) \geq 0, \mathrm{X} \neq 0\} \\ &= \mathrm{P}\{\mathrm{X}(\mathrm{X} - \frac{1}{\mathrm{y}}) \leq 0, \mathrm{X} \neq 0\} \\ &= \mathrm{P}\{\frac{1}{\mathrm{y}} \leq \mathrm{X} \leq 0, \mathrm{X} \neq 0\} = \mathrm{P}\{\frac{1}{\mathrm{y}} \leq \mathrm{X} < 0\} \\ &= F_X(0) - F_X(\frac{1}{y}). \end{aligned}$$

Therefore,

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y}\left(F_Y(y)\right) = \frac{\mathrm{d}}{\mathrm{d}y}\left(F_X(0) - F_X(\frac{1}{y})\right)$$

$$= -\frac{\mathrm{d}F_X(u)}{\mathrm{d}u}\Big|_{u=\frac{1}{y}} \cdot \frac{\mathrm{d}(\frac{1}{y})}{\mathrm{d}y}$$

$$= -f_X(u)\Big|_{u=\frac{1}{y}} \cdot (-\frac{1}{y^2})$$

$$= -f_X(\frac{1}{y}) \cdot (-\frac{1}{y^2}) = \frac{1}{y^2} f_X(\frac{1}{y}).$$

Now since the continuous random variable X has p.d.f

$$f_X(x) = \begin{cases} 6x(1-x) & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, for $y < 0$, i.e. $\frac{1}{y} < 0$,

$$f_Y(y) = \frac{1}{y^2} f_X(\frac{1}{y}) = \frac{1}{y^2} \cdot 0 = 0.$$

To sum up, we get

$$f_Y(y) = \begin{cases} \dfrac{6(y-1)}{y^4} & y > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the non-zero range of f_Y is $(1, +\infty)$.

Remark: (Reference Version)

(a) Easy to see the non-zero range of X is $0 < x < 1$ and, hence, the non-zero range of Y is $0 < y < 3$. Since $X = \frac{y-3}{-3}$ and thus easy to get that for $0 < y < 3$,

$$f_Y(y) = f_X\left(\frac{y-3}{-3}\right) \times \frac{1}{3}$$

$$= 6\left(\frac{y-3}{-3}\right)\left(1 - \frac{y-3}{-3}\right) \times \frac{1}{3}$$

$$= \frac{6}{27}(y-3)(-y)$$

$$= \frac{2}{9}y(3-y). \qquad 0 < y < 3$$

(b) NON-ZERO Range: $1 < Y < \infty$. Easy to get, if $y > 1$

$$f_Y(y) = f_X\left(\frac{1}{y}\right) \cdot \frac{1}{y^2}$$

$$= 6\left(\frac{1}{y}\right)\left(1 - \frac{1}{y}\right) \cdot \frac{1}{y^2}$$

$$= 6\frac{(y-1)}{y^4}. \qquad y > 1$$

2. Suppose that the random variable X has (cumulative) distribution function

$$F_X(x) = \begin{cases} 0 & x < 0, \\ \dfrac{1-\cos(x)}{2} & 0 \le x \le \pi, \\ 1 & x > \pi. \end{cases}$$

and that $Y = \sqrt{X}$.

What is the non-zero range of Y? Find the (cumulative) distribution function $F_Y(y)$ of Y, and hence find the p.d.f of Y.

Solution: Let $f_X(x)$ and $f_Y(y)$ be the p.d.fs of the random variable X and Y,respectively. Then $F_Y(y) = \mathrm{P}\{\mathrm{Y} \le \mathrm{y}\} = \mathrm{P}\{\sqrt{\mathrm{X}} \le \mathrm{y}\}$.

Now,

1° if $y < 0$, then the event $\{w \in \Omega : \sqrt{X}(\omega) \le y\} = \{\sqrt{X} \le y\} = \emptyset$ and hence

$$F_Y(y) = \mathrm{P}\{\mathrm{Y} \le \mathrm{y}\} = \mathrm{P}\{\sqrt{\mathrm{X}} \le \mathrm{y}\} = \mathrm{P}(\emptyset) = 0.$$

So, $f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y}(0) = 0$.

2° if $y = 0$, then the event $\{\sqrt{X} \le y\} = \{\sqrt{X} \le 0\} = \{\sqrt{X} = 0\} = \{X = 0\}$ and hence

$$F_Y(y) = \mathrm{P}\{\mathrm{Y} \le \mathrm{y}\} = \mathrm{P}\{\sqrt{\mathrm{X}} \le \mathrm{y}\} = \mathrm{P}(\mathrm{X} = 0) = 0.$$

Since the probability density function in a point modify the value does not affect the distribution function, for $y = 0$, let $f_Y(y) = f_Y(0) = 0$.

3° if $y > 0$, then

$$\begin{aligned} F_Y(y) &= \mathrm{P}(\sqrt{\mathrm{X}} \le \mathrm{y}) = \mathrm{P}\{0 \le \mathrm{X} \le \mathrm{y}^2\} \\ &= F_X(y^2) - F_X(0). \end{aligned}$$

Notice that the random variable X has (cumulative) distribution function

$$F_X(x) = \begin{cases} 0 & x < 0, \\ \dfrac{1-\cos(x)}{2} & 0 \le x \le \pi, \\ 1 & x > \pi. \end{cases}$$

It follows that, if $0 \le y^2 \le \pi$, i.e. $|y| \le \sqrt{\pi}$, $\Rightarrow 0 < y \le \sqrt{\pi}$, then

$$F_Y(y) = F_X(y^2) - F_X(0) = \frac{1-\cos(y^2)}{2} - 0 = \frac{1-\cos(y^2)}{2}.$$

About the variable y differentiating, for $0 < y < \sqrt{\pi}$, we have

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1-\cos(y^2)}{2}\right) = -\frac{1}{2} \cdot \left(-\sin(y^2)\right) \cdot (2y) = y\sin(y^2).$$

Since the probability density function in a point modify the value does not affect the distribution function, for $y = \sqrt{\pi}$, let $f_Y(y) = f_Y(\sqrt{\pi}) = 0$.

if $y^2 > \pi$, i.e. $|y| > \sqrt{\pi}$, $\Rightarrow y > \sqrt{\pi}$, then

$$F_Y(y) = F_X(y^2) - F_X(0) = 1 - 0 = 1.$$

Differentiating, we have

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y}(1) = 0.$$

In sum up, we get that the (cumulative) distribution function of Y is

$$F_Y(y) = \begin{cases} 0 & y \leq 0, \\ \dfrac{1-\cos(y^2)}{2} & 0 < y < \sqrt{\pi}, \\ 1 & y \geq \sqrt{\pi}. \end{cases}$$

and the probability density function(p.d.f) of Y is

$$f_Y(y) = \begin{cases} y \sin(y^2) & 0 < y < \sqrt{\pi}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the non-zero range of f_Y is $(1, \sqrt{\pi}) \setminus \{\sqrt{k\pi}\}_{k=1}^{\infty}$.

Remark: (Reference Version) Since the non-zero range for X is $0 \leq x \leq \pi$ and $Y = \sqrt{X}$, and so the non-zero range for Y is $0 \leq y \leq \sqrt{\pi}$. For $0 \leq y \leq \sqrt{\pi}$,

$$\begin{aligned} F_Y(y) &= \Pr(Y \leq y) = \Pr\left(\sqrt{X} \leq y\right) = \Pr\left(X \leq y^2\right) \\ &= F_X\left(y^2\right) = \frac{1-\cos(y^2)}{2} \end{aligned}$$

In full,

$$F_Y(y) = \begin{cases} 0 & y \leq 0, \\ \dfrac{1-\cos(y^2)}{2} & 0 < y < \sqrt{\pi}, \\ 1 & y \geq \sqrt{\pi}. \end{cases}$$

Differentiating, pdf is

$$f_Y(y) = \begin{cases} y \sin(y^2) & 0 < y < \sqrt{\pi}, \\ 0 & \text{otherwise.} \end{cases}$$

3. Suppose that the two random variables X and Y have joint probability cumulative function $F(x,y)$. Show that $F(x,y)$ possesses the following properties:

   (a) For any fixed x, $F(x,y)$ in a non-decreasing function of y and, similarly, for any fixed y, $F(x,y)$ in a non-decreasing function of x.

   (b) $F(x,y) \to 1$ when both $x \to +\infty$ and $y \to +\infty$.

   (c) $F(x,y) \to 0$ when either $x \to -\infty$ or $y \to -\infty$.

   (d) If $x_1 < x_2$ and $y_1 < y_2$, then

$$\Pr(x_1 < X \le x_2, y_1 < Y \le y_2) = F(x_2,y_2) - F(x_2,y_1) - F(x_1,y_2) + F(x_1,y_1)$$

   .

   Proof: Easy. For detailed proofs, see the lecture notes.

4. Suppose that the two discrete random variables X and Y have joint probability mass function given by

$X \backslash Y$	$Y=1$	$Y=2$	$Y=3$	$Y=4$
$X=1$	2/32	3/32	4/32	5/32
$X=2$	3/32	4/32	5/32	6/32

Obtain the marginal probability mass function(p.m.f.) of X.

Solution: Notice that

$$\{X=1\} = \{X=1,Y=1\} \uplus \{X=1,Y=2\} \uplus \{X=1,Y=3\} \uplus \{X=1,Y=4\}.$$

then, the marginal probability mass function of X is:

$$\begin{aligned}
\mathrm{P(X=1)} &= \mathrm{P}\left(\{\mathrm{X=1,Y=1}\} \uplus \{\mathrm{X=1,Y=2}\} \uplus \{\mathrm{X=1,Y=3}\} \uplus \{\mathrm{X=1,Y=4}\}\right) \\
&= \mathrm{P}\{\mathrm{X=1,Y=1}\} + \mathrm{P}\{\mathrm{X=1,Y=2}\} + \mathrm{P}\{\mathrm{X=1,Y=3}\} + \mathrm{P}\{\mathrm{X=1,Y=4}\} \\
&= \frac{2}{32} + \frac{3}{32} + \frac{4}{32} + \frac{5}{32} \\
&= \frac{14}{32} = \frac{7}{16}.
\end{aligned}$$

<u>**Method 1:**</u>

$$\begin{aligned}
\mathrm{P(X=2)} &= \mathrm{P}\left(\{\mathrm{X=2,Y=1}\} \uplus \{\mathrm{X=2,Y=2}\} \uplus \{\mathrm{X=2,Y=3}\} \uplus \{\mathrm{X=2,Y=4}\}\right) \\
&= \mathrm{P}\{\mathrm{X=2,Y=1}\} + \mathrm{P}\{\mathrm{X=2,Y=2}\} + \mathrm{P}\{\mathrm{X=2,Y=3}\} + \mathrm{P}\{\mathrm{X=2,Y=4}\} \\
&= \frac{3}{32} + \frac{4}{32} + \frac{5}{32} + \frac{6}{32} \\
&= \frac{18}{32} = \frac{9}{16}.
\end{aligned}$$

<u>Method 2:</u> Observed that $\{X=1\}\uplus\{X=2\}=\Omega$, yield $\{X=1\}=\{X=2\}^c$. Then

$$\mathrm{P(X=2)=1-P(X=1)}=1-\frac{7}{16}=\frac{9}{16}.$$

5. Continuous random variables X and Y have joint p.d.f

$$f(x,y)=\begin{cases} x+y & 0\le x\le 1, 0\le y\le 1,\\ 0 & \text{otherwise.}\end{cases}$$

   (a) Find the marginal p.d.fs of X and Y.
   (b) Find $\mathrm{P(X>Y)}$,
   (c) Find $\mathrm{P(X\le 0.5)}$.

   Proof:

   (a) For $x<0$ or $x>1$, we have $f(x,y)=0$, thus

$$f_X(x)=\int_{-\infty}^{+\infty} f(x,y)\mathrm{d}y=\int_{-\infty}^{+\infty} 0\mathrm{d}y=0.$$

   For $0\le x\le 1$,

$$\begin{aligned} f_X(x)&=\int_{-\infty}^{+\infty} f(x,y)\mathrm{d}y\\ &=\int_{-\infty}^{0} f(x,y)\mathrm{d}y+\int_{0}^{1} f(x,y)\mathrm{d}y+\int_{1}^{+\infty} f(x,y)\mathrm{d}y\\ &=\int_{-\infty}^{0} 0\mathrm{d}y+\int_{0}^{1}(x+y)\mathrm{d}y+\int_{1}^{+\infty} 0\mathrm{d}y\\ &=\int_{0}^{1}(x+y)\mathrm{d}y=(xy+\frac{y^2}{2})\Big|_{y=0}^{y=1}=x+\frac{1}{2}.\end{aligned}$$

   Similarly, for $y<0$ or $y>1$, we have $f(x,y)=0$, thus

$$f_Y(y)=\int_{-\infty}^{+\infty} f(x,y)\mathrm{d}x=\int_{-\infty}^{+\infty} 0\mathrm{d}x=0.$$

   For $0\le y\le 1$,

$$\begin{aligned} f_Y(y)&=\int_{-\infty}^{+\infty} f(x,y)\mathrm{d}x\\ &=\int_{-\infty}^{0} f(x,y)\mathrm{d}x+\int_{0}^{1} f(x,y)\mathrm{d}x+\int_{1}^{+\infty} f(x,y)\mathrm{d}x\\ &=\int_{-\infty}^{0} 0\mathrm{d}x+\int_{0}^{1}(x+y)\mathrm{d}x+\int_{1}^{+\infty} 0\mathrm{d}x\end{aligned}$$

$$= \int_0^1 (x+y)\mathrm{d}x = (\frac{x^2}{2} + yx)\Big|_{x=0}^{x=1} = y + \frac{1}{2}.$$

In sum up, we obtain the marginal p.d.f of X:

$$f_X(x) = \begin{cases} x + \frac{1}{2} & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

The marginal p.d.f of Y:

$$f_Y(y) = \begin{cases} y + \frac{1}{2} & 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b)

$$\begin{aligned} \mathrm{P}\{\mathrm{X} > \mathrm{Y}\} &= \iint_{x>y} f(x,y)\mathrm{d}x\mathrm{d}y = \iint_{\substack{x>y \\ -\infty<x<+\infty \\ -\infty<y<+\infty}} f(x,y)\mathrm{d}x\mathrm{d}y \\ &= \iint_{\substack{x>y \\ 0\le x\le 1 \\ 0\le y\le 1}} f(x,y)\mathrm{d}x\mathrm{d}y = \iint_{\substack{x>y \\ 0\le x\le 1 \\ 0\le y\le 1}} (x+y)\mathrm{d}x\mathrm{d}y \\ &= \int_0^1 \int_0^x (x+y)\mathrm{d}y\mathrm{d}x = \int_0^1 [xy + \frac{y^2}{2}]\Big|_0^x \mathrm{d}x = \int_0^1 (x^2 + \frac{x^2}{2})\mathrm{d}x \\ &= \frac{3}{2}\int_0^1 x^2\mathrm{d}x = \frac{3}{2}[\frac{x^3}{3}]\Big|_0^1 = \frac{1}{2}. \end{aligned}$$

(c) <u>**Method 1:**</u>

$$\begin{aligned} \mathrm{P}\{\mathrm{X} \le \frac{1}{2}\} &= \int_{-\infty}^{\frac{1}{2}} f_X(x)\mathrm{d}x = \int_{-\infty}^{0} f_X(x)\mathrm{d}x + \int_0^{\frac{1}{2}} f_X(x)\mathrm{d}x \\ &= \int_{-\infty}^{0} 0\mathrm{d}x + \int_0^{\frac{1}{2}} (x + \frac{1}{2})\mathrm{d}x \\ &= \int_0^{\frac{1}{2}} (x + \frac{1}{2})\mathrm{d}x = (\frac{x^2}{2} + \frac{x}{2})\Big|_0^{\frac{1}{2}} \\ &= \frac{1}{2}\cdot(\frac{1}{2})^2 + \frac{1}{2}\cdot\frac{1}{2} = \frac{3}{8}. \end{aligned}$$

<u>**Method 2:**</u> Note that $\{X \le \frac{1}{2}\} = \{X \le \frac{1}{2}, -\infty < Y < +\infty\}$, so

$$\begin{aligned} &\mathrm{P}\{\mathrm{X} \le \frac{1}{2}\} = \mathrm{P}\{\mathrm{X} \le \frac{1}{2}, -\infty < \mathrm{Y} < +\infty\} \\ &= \iint_{\substack{x\le\frac{1}{2} \\ -\infty<y<+\infty}} f(x,y)\mathrm{d}x\mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&= \iint\limits_{\substack{x\le\frac{1}{2}\\0\le x\le 1\\0\le y\le 1}} f(x,y)\mathrm{d}x\mathrm{d}y = \iint\limits_{\substack{0\le x\le\frac{1}{2}\\0\le y\le 1}} (x+y)\mathrm{d}x\mathrm{d}y\\
&= \int_0^{\frac{1}{2}}\int_0^1 (x+y)\mathrm{d}y\mathrm{d}x = \int_0^{\frac{1}{2}}[xy+\frac{y^2}{2}]\Big|_0^1\mathrm{d}x\\
&= \int_0^{\frac{1}{2}}(x+\frac{1}{2})\mathrm{d}x = (\frac{x^2}{2}+\frac{x}{2})\Big|_0^{\frac{1}{2}}\\
&= \frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\times\frac{1}{2} = \frac{3}{8}.
\end{aligned}
$$

Remark: Remember the following formula: for any $B \in \mathscr{B}(\mathbb{R}^2)$,

$$
\mathrm{P}\left((\mathrm{X},\mathrm{Y})\in\mathrm{B}\right) = \iint\limits_{\mathrm{B}} \mathrm{f}(\mathrm{x},\mathrm{y})\mathrm{dxdy}.
$$

Moreover, we have

blablablabla...

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{+\infty} f(x,y)\mathrm{d}y\\
f_Y(y) &= \int_{-\infty}^{+\infty} f(x,y)\mathrm{d}x
\end{aligned}
$$