Calculations of Two Absolutely Continuous rvs

1. An important conclusion (Theorem, essentially, definition!!)

Suppose the random vector (X, Y) has pdf $f(x, y)$. Then for any Borel set G in the plane, we have

$$Pr\{(X, Y) \in G\} = \iint_G f(x, y)\,dx\,dy \qquad (*)$$

2. Remarks:

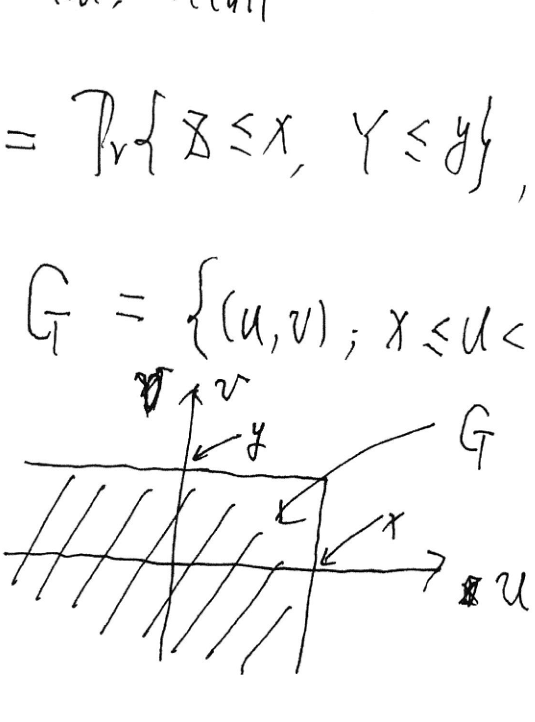
① The "area" of G could be either finite or infinite.

② The definition of joint cdf could be viewed as a special case of (*). Indeed, recall

$$F(x, y) = Pr\{X \le x, Y \le y\},$$

then let $G = \{(u, v);\ x \le u < +\infty,\ y \le v < +\infty\}$

then



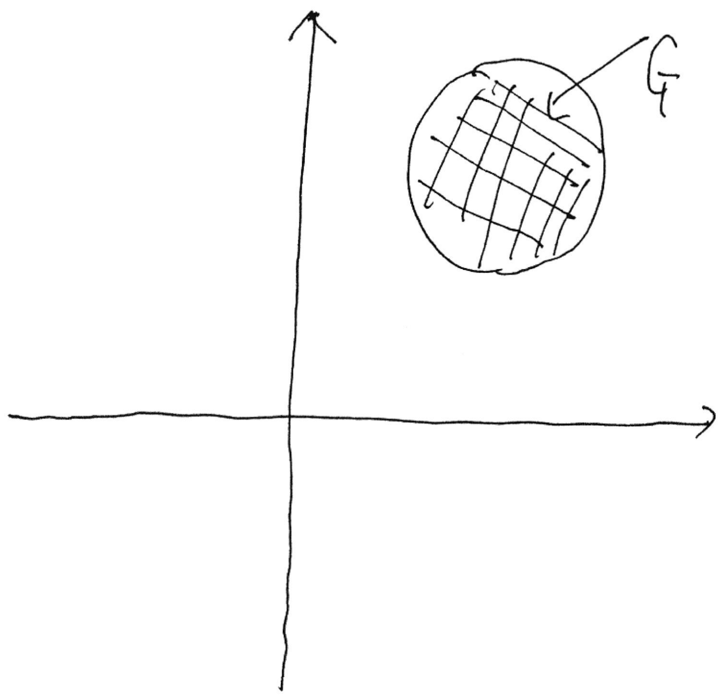
Then

$$F(x,y) = \Pr\{X \le x, Y \le y\} = \Pr\{(X,Y) \in G\}$$

$$= \iint_G f(x,y)\,dx\,dy = \iint_G f(u,v)\,du\,dv$$

$$= \int_{-\infty}^{y}\int_{-\infty}^{x} f(u,v)\,du\,dv$$

③ Geometric "meaning" (Recall single rv case!)



3. Examples

① Example 1. Suppose the random vector (X, Y) has joint pdf

$$f(x,y) = \begin{cases} cxy^2 & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the constant c.

(b) Find the two marginal pdfs. $f_X(x)$ and $f_Y(y)$.

(c) Find the ~~proper~~ probability $P\{X \le \frac{1}{2}, Y \le \frac{1}{2}\}$

(d) Find $P_r\{X < Y\}$

(e) Are X and Y independent?

Solutions.

(a) $\because \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy = 1$ and thus let $G = \{(x,y), 0 \le x \le 1, 0 \le y \le 1\}$

$$\therefore 1 = \int_0^1\int_0^1 f(x,y)\,dx\,dy + \iint_{R^2 \setminus G} f(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^1 cxy^2\,dx\,dy + \iint_{R^2 \setminus G} 0\,dx\,dy$$

$$= c\cdot\int_0^1 y^2\left[\int_0^1 x\,dx\right]dy = c\frac{1}{2}\int_0^1 y^2\,dy = c\cdot\frac{1}{2}\left[\frac{y^3}{3}\right]\Big|_0^1 = \frac{c}{6}$$

$$\Longrightarrow \boxed{c = 6}$$

(b) $f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy$

(i) If $x<0$ or $x>1$, then $f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy = \int_{-\infty}^{+\infty} 0\,dy = 0$

(ii) If $0 \le x \le 1$, then $f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy$

i.e. $f_X(x) = \int_{-\infty}^{0} f(x,y)\,dy + \int_{0}^{1} f(x,y)\,dy + \int_{1}^{+\infty} f(x,y)\,dy$

$$= \underbrace{\int_{-\infty}^{0} 0\,dy}_{=0} + \int_{0}^{1} 6xy^2\,dy + \underbrace{\int_{1}^{+\infty} 0\,dy}_{=0}$$

$$= 6x \int_0^1 y^2\,dy = 6x \left[\frac{y^3}{3}\right]_0^1 = 2x$$

$$\therefore\ f_X(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Similarly, (iii), If $y<0$ or $y>1$, then $f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx = 0$

(iv) If $0 \le y \le 1$, then $f_Y(y) = \int_0^1 6xy^2\,dx = 6y^2 \left[\frac{x^2}{2}\right]_0^1 = 3y^2$

$$\therefore\ f_Y(y) = \begin{cases} 3y^2 & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(e) Now we can see X and Y are independent.

Indeed, let $G = \{(x, y);\ 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$

Then for any $(x, y) \notin G$, $f(x, y) = 0$ and $f_X(x) = f_Y(y) = 0$

while if $(x, y) \in G$, then

$$f(x, y) = 6xy^2$$

and $f_X(x) = 2x$ and $f_Y(y) = 3y^2$

Hence $f(x, y) = f_X(x) \cdot f_Y(y)$ $\boxed{\forall\ (x, y) \in R^2}$

(c) $P\{X \leq \frac{1}{2},\ Y \leq \frac{1}{2}\}$

Let $G = \{(x, y);\ -\infty < x \leq \frac{1}{2},\ -\infty < y \leq \frac{1}{2}\}$

$\therefore P\{X \leq \frac{1}{2},\ Y \leq \frac{1}{2}\} = P\{(X, Y) \in G\}$

$$= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} 6xy^2\,dx\,dy = 6 \int_0^{\frac{1}{2}} y^2 \left[\int_0^{\frac{1}{2}} x\,dx\right] dy = 6 \int_0^{\frac{1}{2}} y^2 \left[\frac{x^2}{2}\right]\Big|_0^{\frac{1}{2}} dy$$

$$= 6 \cdot \int_0^{\frac{1}{2}} y^2 \cdot \frac{1}{8}\,dy = \frac{3}{4} \int_0^{\frac{1}{2}} y^2\,dy = \frac{3}{4} \cdot \left[\frac{y^3}{3}\right]\Big|_0^{\frac{1}{2}} = \frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}$$

of course, you could do as follows (totally the same)

$$P\{X \le \tfrac{1}{2},\ Y \le \tfrac{1}{2}\} = F(\tfrac{1}{2}, \tfrac{1}{2}) = \int_{-\infty}^{\frac{1}{2}} \int_{-\infty}^{\frac{1}{2}} f(x, y)\, dx\, dy$$

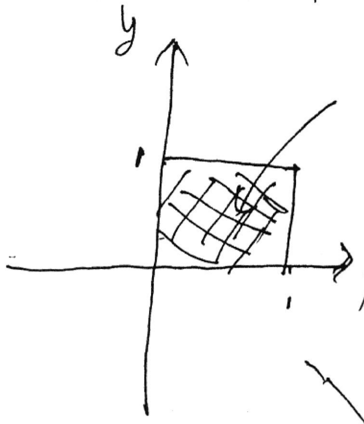
$$= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} 6xy^2\, dx\, dy = \frac{1}{32}$$

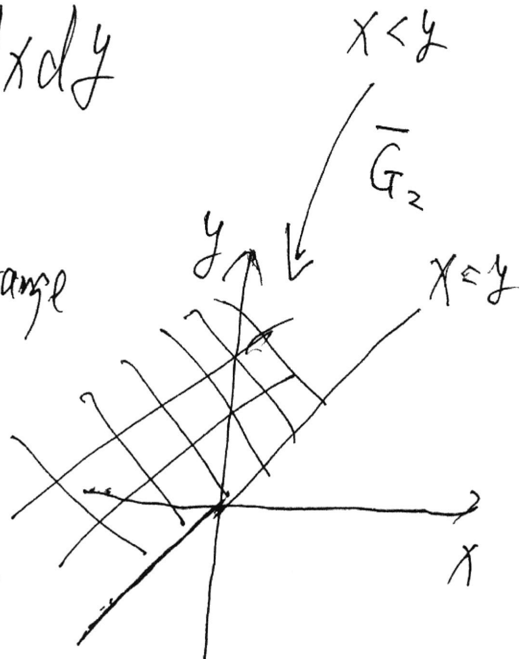
(d). How about $P\{X < Y\}$

Let $G = \{(x, y) \in R^2;\ 0 \le x \le 1,\ 0 \le y \le 1,\ x < y\}$

Then $P\{X < Y\} = \iint_G f(x, y)\, dx\, dy$

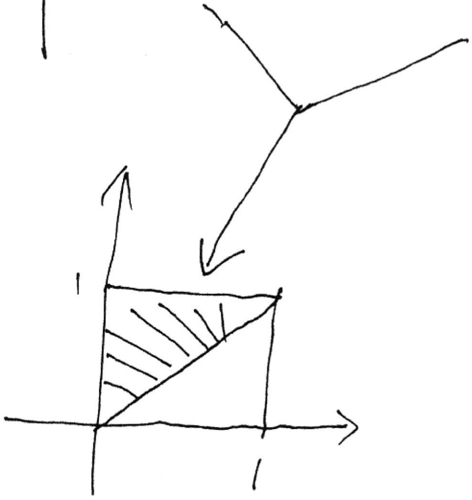
$\bar{G}_1$:





$\Rightarrow$ ~~$P\{X < Y\}$~~

G :



$$\therefore P\{X < Y\} = \iint_G f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_0^y 6xy^3\,dx\,dy = 6\int_0^1 y^3 \left[\int_0^y x\,dx\right] dy$$

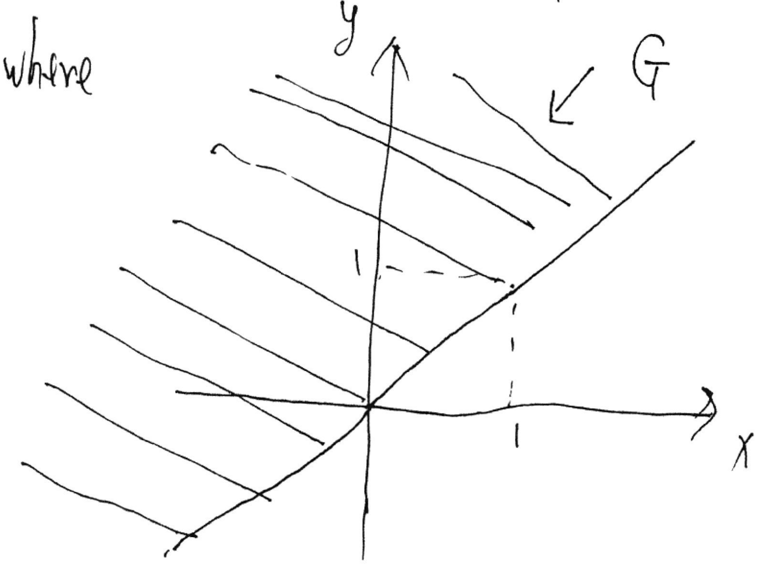
$$= 6\int_0^1 y^2 \left[\frac{x^2}{2}\right]\Big|_0^y dy = 6\int_0^1 y^2 \cdot \frac{1}{2} y^2\,dy$$

$$= 3\int_0^1 y^4\,dy = 3\left[\frac{y^5}{5}\right]\Big|_0^1 = \frac{3}{5}$$

② Note that for any random vector (X, Y) having pdf $f(x,y)$ we have

$$P\{X < Y\} = \iint_G f(x,y)\,dx\,dy = \iint_{\substack{-\infty < x < +\infty \\ -\infty < y < +\infty \\ x < y}} f(x,y)\,dx\,dy$$

where



But for our example 1, $f(x,y) = 0$ except $0 \le x \le 1,\ 0 \le y \le 1$.

③ Example 2. Suppose the joint pdf of (X, Y) is given by

$$f(x,y) = \begin{cases} x e^{-(x+xy)} & \text{if } x \geq 0, \text{ and } y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(Easy to check that $f(x,y) \geq 0$ and $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy = \int_0^{+\infty}\int_0^{+\infty} x e^{-(x+xy)}\,dx\,dy = 1$)

(a). Find the two marginal pdfs and see whether X and Y are independent

(b) Find the two marginal cdfs $F_X(x)$ and $F_Y(y)$.

Solutions:

(a) $f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy$ and $f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx$

First consider $f_X(x)$.

If $x \leq 0$, then $f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy = \int_{-\infty}^{+\infty} 0\,dy = 0$.

If $x > 0$, then $f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy = \int_{-\infty}^{0} f(x,y)\,dy + \int_0^{+\infty} f(x,y)\,dy$

$$= \int_{-\infty}^{0} 0\,dy + \int_0^{+\infty} x e^{-x(1+y)}\,dy = x\int_0^{+\infty} e^{-x} \cdot e^{-xy}\,dy$$

$$= x e^{-x} \cdot \int_0^{+\infty} e^{-xy}\,dy = x e^{-x}\left[-\frac{e^{-xy}}{x}\right]\Bigg|_0^{+\infty} = x e^{-x} \cdot \frac{1}{x} = e^{-x}$$

Hence
$$f_X(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$
(Exponentially distributed)

For $f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx$, we see that

if $y \leq 0$, we still have $f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)dx = \int_{-\infty}^{+\infty} 0\,dx = 0$

while if $y > 0$, then

$$f_Y(y) = \int_{-\infty}^{0} f(x,y)dx + \int_{0}^{+\infty} f(x,y)dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{+\infty} xe^{-(1+y)x}\,dx$$

$$= \int_0^{+\infty} x e^{-x(1+y)}\,dx = \int_0^{+\infty} x \frac{(-1)(1+y)}{(-1)\,1+y} \cdot e^{-x(1+y)}\,dx$$

$$= -\frac{1}{1+y}\int_0^{+\infty} x\, d e^{-x(1+y)}$$

$$= -\frac{1}{1+y}\left\{ xe^{-x(1+y)}\Big|_{x=0}^{x=+\infty} - \int_0^{+\infty} e^{-x(1+y)}\,dx \right\}$$ (Integration by parts!!)

$$= -\frac{1}{1+y}\left\{0 - 0 - \int_0^{+\infty} e^{-x(1+y)}dx\right\} = \frac{1}{1+y}\int_0^{+\infty} e^{-(1+y)x}\,dx$$

$$= \frac{1}{(1+y)^2}\left[e^{(1+y)x}\right]\Big|_{x=0}^{x=+\infty} = \left[-0+1\right]\cdot\frac{1}{(1+y)^2} = \frac{1}{(1+y)^2}$$

Hence
$$f_Y(y) = \begin{cases} 0 & \text{if } y \le 0 \\ \dfrac{1}{(1+y)^2} & \text{if } y > 0 \end{cases}$$

or
$$f_Y(y) = \begin{cases} \dfrac{1}{(1+y)^2} & \text{if } y \ge 0 \\ 0 & \text{if } y < 0 \end{cases}$$

(b) We have two ways to find the two cdfs.

Method 1. Use the two pdfs. For example, $F_Z(x) = \int_0^x f_Z(u)\,du$

Method 2. For $F_Z(x)$, use our general formula. For $a \ge 0$

$$F_Z(a) = \Pr\{Z \le a\} = \Pr\{0 \le Z \le a\} \quad (\because \Pr\{Z \le 0\} = 0)$$

$$= \Pr\{0 \le Z \le a,\ -\infty \le Y < +\infty\}$$

$$= \Pr\{0 \le Z \le a,\ 0 \le Y < +\infty\}$$

$$= \int_0^a \int_0^{+\infty} f(x,y)\,dy\,dx = \int_0^a \int_0^{+\infty} x e^{-(x+xy)}\,dy\,dx$$

$$= \int_0^a x e^{-x}\left[\int_0^{+\infty} e^{-xy}\,dy\right]dx = \int_0^a x e^{-x} \cdot \frac{1}{x}\,dx = \int_0^a e^{-x}\,dx = 1 - e^{-a}$$

while if $a \le 0$, then $F_Z(a) = 0$

Hence

$$F_X(x) = \begin{cases} 1-e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Similarly, we could get

$$F_Y(y) = \begin{cases} 1-\frac{1}{1+y} & \text{if } y \geq 0 \\ 0 & \text{otherwise (i.e. } y < 0) \end{cases}$$

④ Example 3

The joint pdf of X and Y is given by

$$f(x, y) = \begin{cases} e^{-(x+y)} & 0 < x < \infty, \ 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find the pdf of the random variable $\frac{X}{Y}$.

「 Easy to see, $f(x, y) \geq 0 \ \forall x, y$ and $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x, y)\,dxdy = \int_0^{\infty}\int_0^{\infty} f(x,y)\,dxdy = \int_0^{+\infty}\int_0^{+\infty} e^{-(x+y)}\,dxdy = 1$ 」

Let $Z = \frac{X}{Y}$, we try to find the cdf of Z.

Denote the cdf of Z is $F_Z(z)$, then

$$F_Z(z) = \Pr\{Z \leq z\} = \Pr\left\{\frac{X}{Y} \leq z\right\}$$

$$= \iint_{\frac{x}{y} \leq z} f(x, y)\,dxdy$$

Now, if $z \leq 0$, then clearly, either "$x \leq 0$ and $y > 0$"

or "$x \geq 0$ and $y < 0$"

and thus $f(x, y) = 0$ which implies that $F_Z(z) = 0$

for $z \leq 0$

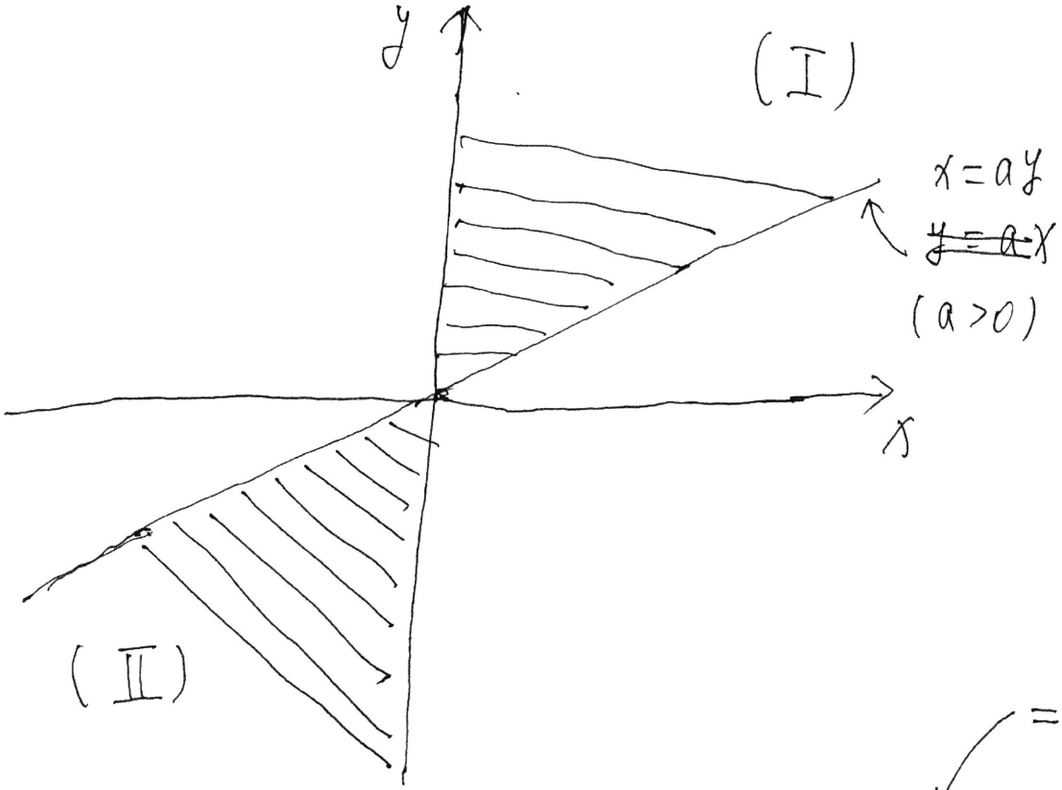
Hence only need to consider $z > 0$.

For the notational convenience, let $z = a$ and hence $a > 0$

What is the area G for which $\frac{x}{y} \leq a$? see below.

$$\left(\because \text{ if } y>0, \text{ then } \boxed{\begin{array}{l} \frac{x}{y} \leq a \iff x \leq ay \\ \frac{x}{y} \leq a \iff x \geq ay \end{array}} \text{ while if } y<0 \text{ then} \right.$$

Hence, $\{(x,y);\ \frac{x}{y} \leq a\}$ is as follows.

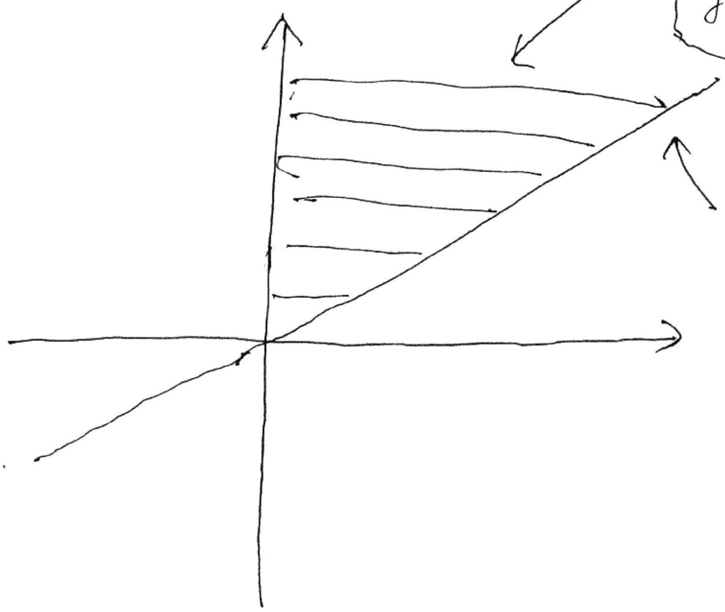


Therefore $F_Z(z) = \iint_{\frac{x}{y} \leq z} f(x,y)\,dxdy = \iint_{(I)} f(x,y)\,dxdy + \underbrace{\iint_{(II)} f(x,y)\,dxdy}_{=0}$

$$= \iint_{(I)} f(x,y)\,dxdy = \iint_{\substack{x>0,\ y>0 \\ \frac{x}{y} \leq z}} e^{-(x+y)}\,dxdy$$

Hence $F_2(z) = \iint\limits_{\substack{x>0 \\ y>0 \\ x \le yz}} e^{-(x+y)}\,dx\,dy$ (view z as a constant)

$x: 0 \to zy$
$y: 0 \to +\infty$



$$\Rightarrow F_2(z) = \iint\limits_{\substack{x>0 \\ y>0 \\ x \le yz}} e^{-(x+y)}\,dx\,dy = \int_0^{\infty}\int_0^{zy} e^{-(x+y)}\,dx\,dy$$

$$= \int_0^{\infty} e^{-y}\left[\int_0^{zy} e^{-x}\,dx\right]dy = \int_0^{\infty} e^{-y}\left[-e^{-x}\right]\Big|_{x=0}^{x=zy} dy$$

$$= \int_0^{\infty} e^{-y}\left[1-e^{-zy}\right]dy = \left[-e^{-y} + \frac{e^{-(z+1)y}}{z+1}\right]\Bigg|_{y=0}^{y=+\infty}$$

$\because z>0 \Rightarrow z+1>0$

$$= \left[-0 + 0 - \left(-e^{-0} + \frac{e^{-(z+1)\times 0}}{z+1}\right)\right]$$

$$= 1 - \frac{1}{z+1}$$

In short

$$F_Z(z) = \begin{cases} 1 - \dfrac{1}{z+1} & \text{if } z > 0 \\ 0 & \text{if } z \leq 0. \end{cases}$$

Differentiation yields $\dfrac{d}{dz} F_Z(z) = f_Z(z) \equiv f_{\frac{X}{Y}}(z)$

$$f_{\frac{X}{Y}}(z) = \begin{cases} \dfrac{1}{(z+1)^2} & \text{if } z > 0 \\ 0 & \text{if } z \leq 0 \end{cases}$$

「Note: We do have $f_{\frac{X}{Y}}(z) \geq 0 \ \forall z$ and $\int_{-\infty}^{+\infty} f_{\frac{X}{Y}}(z)\,dz = \int_0^{+\infty} \frac{dz}{(z+1)^2} = \left[-\frac{1}{z+1}\right]_0^{\infty} = 1$ 」