SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF MATHEMATICS

MA215 Probability Theory

Tutorial 01Solu

Set: Wednesday 7th September 2016; Hand in: Friday, 16th September 2016.

Note: Hand in your solutions no later than 4pm of Friday, 16th September.

1. Provide a strict proof for the following set relations.

(i) $B \setminus A = B \cap A^c$;

(ii) $(A \setminus B) \cap C = (A \cap C) \setminus (B \cap C)$;

(iii) $(\bigcup_{k=1}^{\infty} A_k)^c = \bigcap_{k=1}^{\infty} A_k^c$;

(iv) $(\bigcap_{k=1}^{\infty} A_k)^c = \bigcup_{k=1}^{\infty} A_k^c$;

(v) $A \cup (\bigcap_{k=1}^{\infty} B_k) = \bigcap_{k=1}^{\infty} (A \cup B_k)$;

(vi) $A \cap (\bigcup_{k=1}^{\infty} B_k) = \bigcup_{k=1}^{\infty} (A \cap B_k)$.

As the generalizations of (iii) to (vi) we have the following general **De Morgan's Laws** and **Distributive laws:** For any index set I, we have

(vii) $(\bigcup_{i\in I} A_i)^c = \bigcap_{i\in I} (A_i)^c$;

(viii) $(\bigcap_{i\in I} A_i)^c = \bigcup_{i\in I} (A_i)^c$;

(ix) $A \cup (\bigcap_{i\in I} B_i) = \bigcap_{i\in I} (A \cup B_i)$;

(x) $A \cap (\bigcup_{i\in I} B_i) = \bigcup_{i\in I} (A \cap B_i)$.

Proof:

(i)

$$\begin{aligned} x \in B \setminus A &\Leftrightarrow x \in B \text{ and } x \notin A \\ &\Leftrightarrow x \in B \text{ and } x \in A^c \\ &\Leftrightarrow x \in B \cap A^c. \end{aligned}$$

i.e. $B \setminus A = B \cap A^c$.

(ii)

$$\begin{aligned} x \in (A \setminus B) \cap C &\Leftrightarrow x \in A \setminus B \text{ and } x \in C \\ &\Leftrightarrow x \in A \text{ and } x \notin B \text{ and } x \in C \\ &\Leftrightarrow x \in A \text{ and } x \in C \text{ and } x \notin B \\ &\Leftrightarrow x \in A \cap C \text{ and } x \notin B \cap C \\ &\Leftrightarrow x \in (A \cap C) \setminus (B \cap C). \end{aligned}$$

i.e. $(A \setminus B) \cap C = (A \cap C) \setminus (B \cap C)$.

(iii)

$$
\begin{aligned}
x \in (\bigcup_{k=1}^{\infty} A_k)^c &\Leftrightarrow x \notin \bigcup_{k=1}^{\infty} A_k \\
&\Leftrightarrow x \notin A_k, \forall k \geq 1 \\
&\Leftrightarrow \forall k \geq 1, x \notin A_k \\
&\Leftrightarrow x \in A_k^c, \forall k \geq 1 \\
&\Leftrightarrow \forall k \geq 1, x \notin A_k \\
&\Leftrightarrow x \in \bigcap_{k=1}^{\infty} A_k^c.
\end{aligned}
$$

i.e. $(\bigcup_{k=1}^{\infty} A_k)^c = \bigcap_{k=1}^{\infty} A_k^c$.

(iv)

$$
\begin{aligned}
x \in (\bigcap_{k=1}^{\infty} A_k)^c &\Leftrightarrow x \notin \bigcap_{k=1}^{\infty} A_k \\
&\Leftrightarrow \exists k_0 \geq 1, s.t.\ x \notin A_{k_0} \\
&\Leftrightarrow \exists k_0 \geq 1, s.t.\ x \in A_{k_0}^c \\
&\Leftrightarrow x \in \bigcup_{k=1}^{\infty} A_k^c.
\end{aligned}
$$

i.e. $(\bigcap_{k=1}^{\infty} A_k)^c = \bigcup_{k=1}^{\infty} A_k^c$.

(v)

$$
\begin{aligned}
x \in A \cup (\bigcap_{k=1}^{\infty} B_k) &\Leftrightarrow x \in A \text{ or } x \in \bigcap_{k=1}^{\infty} B_k \\
&\Leftrightarrow x \in A \text{ or } s.t.\ (\forall k \geq 1, x \in B_k) \\
&\Leftrightarrow \forall k \geq 1, (x \in A \text{ or } x \in B_k) \\
&\Leftrightarrow \forall k \geq 1, x \in A \cup B_k \\
&\Leftrightarrow x \in \bigcap_{k=1}^{\infty} (A \cup B_k).
\end{aligned}
$$

i.e. $A \cup (\bigcap_{k=1}^{\infty} B_k) = \bigcap_{k=1}^{\infty} (A \cup B_k)$.

(vi)

$$
\begin{aligned}
x \in A \cap (\bigcup_{k=1}^{\infty} B_k) &\Leftrightarrow x \in A \text{ and } x \in \bigcup_{k=1}^{\infty} B_k \\
&\Leftrightarrow x \in A \text{ and } \exists k_0 \geq 1, s.t.\ x \in B_{k_0} \\
&\Leftrightarrow \exists k_0 \geq 1, s.t.\ x \in A \text{ and } x \in B_{k_0} \\
&\Leftrightarrow \exists k_0 \geq 1, s.t.\ x \in A \cap B_{k_0} \\
&\Leftrightarrow x \in \bigcup_{k=1}^{\infty} (A \cap B_k).
\end{aligned}
$$

i.e. $A \cap (\bigcup_{k=1}^{\infty} B_k) = \bigcup_{k=1}^{\infty} (A \cap B_k)$.

(vii)

$$
\begin{aligned}
x \in (\bigcup_{i\in I} A_i)^c &\Leftrightarrow x \notin \bigcup_{i\in I} A_i \\
&\Leftrightarrow x \notin A_i, \forall i \in I \\
&\Leftrightarrow \forall i \in I, x \notin A_i \\
&\Leftrightarrow x \in A_i^c, \forall i \in I \\
&\Leftrightarrow \forall i \in I, x \notin A_i \\
&\Leftrightarrow x \in \bigcap_{i\in I} A_i^c.
\end{aligned}
$$

i.e. $(\bigcup_{i\in I} A_i)^c = \bigcap_{i\in I} (A_i)^c$.

(viii)

$$
\begin{aligned}
x \in (\bigcap_{i\in I} A_i)^c &\Leftrightarrow x \notin \bigcap_{i\in I} A_i \\
&\Leftrightarrow \exists i_0 \in I, s.t.\ x \notin A_{i_0} \\
&\Leftrightarrow \exists i_0 \in I, s.t.\ x \in A_{i_0}^c \\
&\Leftrightarrow x \in \bigcup_{i\in I} A_i^c.
\end{aligned}
$$

i.e. $(\bigcap_{i\in I} A_i)^c = \bigcup_{i\in I} (A_i)^c$.

(ix)

$$
\begin{aligned}
x \in A \cup (\bigcap_{i\in I} B_i) &\Leftrightarrow x \in A \text{ or } x \in \bigcap_{i\in I} B_i \\
&\Leftrightarrow x \in A \text{ or } (\forall i \in I, x \in B_i) \\
&\Leftrightarrow x \in A \text{ or } (x \in B_i, \forall i \in I) \\
&\Leftrightarrow \forall i \in I,\ s.t.\ (x \in A \text{ or } x \in B_i) \\
&\Leftrightarrow (x \in A \text{ or } x \in B_i), \forall i \in I, \\
&\Leftrightarrow \forall i \in I,\ s.t.\ x \in A \cup B_i \\
&\Leftrightarrow\ s.t.\ x \in A \cup B_i, \forall i \in I \\
&\Leftrightarrow x \in \bigcap_{i\in I} (A \cup B_i).
\end{aligned}
$$

i.e. $A \cup (\bigcap_{i\in I} B_i) = \bigcap_{i\in I} (A \cup B_i)$.

(x)

$$
\begin{aligned}
x \in A \cap (\bigcup_{i\in I} B_i) &\Leftrightarrow x \in A \text{ and } x \in \bigcup_{i\in I} B_i \\
&\Leftrightarrow x \in A \text{ and } \exists i_0 \in I, s.t.\ x \in B_{i_0} \\
&\Leftrightarrow \exists i_0 \in I, s.t.\ x \in A \text{ and } x \in B_{i_0} \\
&\Leftrightarrow \exists i_0 \in I, s.t.\ x \in A \cap B_{i_0} \\
&\Leftrightarrow x \in \bigcup_{i\in I} (A \cap B_i).
\end{aligned}
$$

i.e. $A \cap (\bigcup_{i\in I} B_i) = \bigcup_{i\in I} (A \cap B_i)$.

Remark: Note that

$$\begin{aligned} A = B &\Longleftrightarrow A \subset B, A \supset B \\ &\Longleftrightarrow x \in A \Longrightarrow x \in B, x \in B \Longrightarrow x \in A \end{aligned}$$

$$\begin{aligned} x \in \bigcap_{k=1}^{\infty} A_k &\Longleftrightarrow x \in A_k, \forall k \geq 1 \Longleftrightarrow x \in A_k, \text{ 对任意的 } k(k \geq 1) \text{ 都成立} \\ &\Longleftrightarrow x \in A_1 \text{ and } x \in A_2 \text{ and } x \in A_3 \text{ and } \cdots \Longleftrightarrow x \in A_m, \forall m \geq 1 \\ &\Longleftrightarrow \forall k \geq 1, x \in A_k \Longleftrightarrow \text{ 对任意的} k(k \geq 1), \text{ 都有 } x \in A_k, \text{ 成立} \\ &\text{for any fixed } k(k \geq 1), x \in A_k \\ &\Longleftrightarrow \text{ 对任意固定的} k(k \geq 1), \text{ 都有 } x \in A_k \text{ 成立(但是此时 } x = x(k) \text{ 与 } k \text{ 有关)} \end{aligned}$$

2. A sequence of sets $\{A_1, A_2, \dots, A_n, \dots\}$ is called **increasing** if

$$A_1 \subset A_2 \subset A_3 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots.$$

Similarly, a sequence of sets $\{A_1, A_2, \dots, A_n, \dots\}$ is called **decreasing** if

$$A_1 \supset A_2 \supset A_3 \supset \cdots \supset A_n \supset A_{n+1} \supset \cdots.$$

Show that

(i) If $\{A_n; n \geq 1\}$ is an increasing set sequence, then for any $n \geq 1$, $\bigcup\limits_{k=1}^{n} A_k = A_n$ and $\lim\limits_{n\to\infty} A_n = \bigcup\limits_{k=1}^{\infty} A_k = \bigcup\limits_{n=1}^{\infty} A_n$.

(ii) If $\{A_n; n \geq 1\}$ is a decreasing set sequence, then for any $n \geq 1$, $\bigcap\limits_{k=1}^{n} A_k = A_n$ and $\lim\limits_{n\to\infty} A_n = \bigcap\limits_{k=1}^{\infty} A_k = \bigcap\limits_{n=1}^{\infty} A_n$.

Proof:

(i) 1. It is easy to see that for any $n \geq 1$, $x \in A_n \Rightarrow x \in \bigcup\limits_{k=1}^{n} A_k$, i.e.

$$A_n \subset \bigcup_{k=1}^{n} A_k.$$

However, on the other hand, $x \in \bigcup\limits_{k=1}^{n} A_k \Rightarrow \exists k_0, 1 \leq k_0 \leq n$, s.t.

$$x \in A_{k_0}.$$

Since $\{A_n; n \geq 1\}$ is an increasing set sequence, hence for $1 \leq k_0 \leq n$, we have $A_{k_0} \subset A_n$, so $x \in A_n$, i.e.

$$\bigcup_{k=1}^{n} A_k \subset A_n.$$

Altogether, we get

$$\bigcup_{k=1}^{n} A_k = A_n.$$

2. **Formal proof:** Since we have

$$A_n = \bigcup_{k=1}^{n} A_k.$$

Let $n \to \infty$, we get

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} \bigcup_{k=1}^{n} A_k = \bigcup_{k=1}^{\lim_{n\to\infty} n} A_k = \bigcup_{k=1}^{\infty} A_k.$$

Rigorous Proof: Firstly, we should show two facts: for any $k \in \mathbb{N}_+$,

$$\bigcap_{n=k}^{\infty} A_n = A_k, \quad \bigcup_{n=k}^{\infty} A_n = \bigcup_{n=1}^{\infty} A_n.$$

Indeed, It is easy to see that $\bigcap_{n=k}^{\infty} A_n \subset A_k$. On the other hand, $\forall x \in A_k$, note that$\{A_n; n \geq 1\}$ is an increasing set sequence, hence $n \geq k$, we have $A_k \subset A_n$, so $x \in A_n, n \geq k$, i.e. $x \in \bigcap_{n=k}^{\infty} A_n$. Then, $\bigcap_{n=k}^{\infty} A_n \supset A_k$. Altogether, we get

$$\bigcap_{n=k}^{\infty} A_n = A_k.$$

Note that $\bigcup_{n=k}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} A_n$. To show the converse case, suppose $x \in \bigcup_{n=1}^{\infty} A_n$, thus $\exists n_0 \geq 1$, $s.t.$ $x \in A_{n_0}$

- if $n_0 < k$, then

$$x \in A_{n_0} \subset A_k \subset \bigcup_{n=k}^{\infty} A_n.$$

- if $n_0 \geq k$, then

$$x \in A_{n_0} \subset \bigcup_{n=k}^{\infty} A_n.$$

Hence, $x \in \bigcup_{n=k}^{\infty} A_n$, i.e. $\bigcup_{n=k}^{\infty} A_n \supset \bigcup_{n=1}^{\infty} A_n$. So,

$$\bigcup_{n=k}^{\infty} A_n = \bigcup_{n=1}^{\infty} A_n$$

Therefore, we can get

$$\begin{aligned}
\overline{\lim_{n\to\infty}} A_n &= \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} A_n = \bigcap_{k=1}^{\infty}\bigcup_{n=1}^{\infty} A_n \\
&= \bigcup_{n=1}^{\infty} A_n (x \in \bigcup_{n=1}^{\infty} A_n \Longleftrightarrow x \in A_1 \text{ or } x \in A_2 \text{ or } x \in A_3 \text{ or } \cdots) \\
&= \bigcup_{k=1}^{\infty} A_k \\
&= \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} A_n = \underline{\lim}_{n\to\infty} A_n.
\end{aligned}$$

Hence, $\lim\limits_{n\to\infty} A_n = \bigcup\limits_{k=1}^{\infty} A_k = \bigcup\limits_{n=1}^{\infty} A_n$.

(ii) 1. It is easy to see that $\bigcap\limits_{k=1}^{n} A_k \subset A_n$. However, on the other hand, if $\forall x \in A_n$. Since $\{A_n; n \geq 1\}$ is a decreasing set sequence, hence for any $k, 1 \leq k \leq n$, we have $A_n \subset A_k$, so $x \in A_k, 1 \leq k \leq n$, i.e. $A_n \subset \bigcap\limits_{k=1}^{n} A_k$.

Altogether, we get

$$\bigcap_{k=1}^{n} A_k = A_n.$$

2.**Formal proof:** Since we have

$$A_n = \bigcap_{k=1}^{n} A_k.$$

Let $n \to \infty$, we get

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} \bigcap_{k=1}^{n} A_k = \bigcap_{k=1}^{\lim_{n\to\infty} n} A_k = \bigcap_{k=1}^{\infty} A_k.$$

Rigorous Proof: Firstly, we should show two facts: for any $k \in \mathbb{N}_+$,

$$\bigcup_{n=k}^{\infty} A_n = A_k, \quad \bigcap_{n=k}^{\infty} A_n = \bigcap_{n=1}^{\infty} A_n.$$

Indeed, It is easy to see that

$$\bigcup_{n=k}^{\infty} A_n \supset A_k.$$

On the other hand, $\forall x \in \bigcup\limits_{n=k}^{\infty} A_n, \Rightarrow \exists n_0, n_0 \geq k$, *s.t.* $x \in A_{n_0}$, note that $\{A_n; n \geq 1\}$ is a decreasing set sequence, hence for $n_0 \geq k$, we have $A_{n_0} \subset A_k$, so $x \in A_k$, i.e. $\bigcup\limits_{n=k}^{\infty} A_n \subset A_k$.

Altogether, we get

$$\bigcup_{n=k}^{\infty} A_n = A_k.$$

Note that $\bigcap\limits_{n=k}^{\infty} A_n \supset \bigcap\limits_{n=1}^{\infty} A_n$. To show the converse case, suppose $x \in \bigcap\limits_{n=k}^{\infty} A_n$, thus $\forall n \geq k$, *s.t.* $x \in A_n$. then for $1 \leq n < k$, since $\{A_n; n \geq 1\}$ is a decreasing set sequence,

$$\Rightarrow x \in A_k \subset A_{k-1} \subset A_n$$

Hence, $x \in A_n, \forall n \geq 1$, i.e. $x \in \bigcap\limits_{n=1}^{\infty} A_n$. Therefore,

$$\bigcap_{n=k}^{\infty} A_n \subset \bigcap_{n=1}^{\infty} A_n.$$

So, we obtain

$$\bigcap_{n=k}^{\infty} A_n = \bigcap_{n=1}^{\infty} A_n$$

Therefore, we can get

$$\begin{aligned}
\overline{\lim_{n\to\infty}} A_n &= \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n = \bigcap_{k=1}^{\infty} A_k \\
&= \bigcap_{n=1}^{\infty} A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} A_n \\
&= \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n = \underline{\lim_{n\to\infty}} A_n.
\end{aligned}$$

Hence, $\lim\limits_{n\to\infty} A_n = \bigcap\limits_{k=1}^{\infty} A_k = \bigcap\limits_{n=1}^{\infty} A_n$.

Method 2: dual principle Since $\{A_n; n \geq 1\}$ is a decreasing set sequence, so $\{A_n^c; n \geq 1\}$ is an increasing set sequence, use the above conclusion of (i), we have

$$\underline{\lim_{n\to\infty}} A_n^c = \overline{\lim_{n\to\infty}} A_n^c = \lim_{n\to\infty} A_n^c = \bigcup_{k=1}^{\infty} A_k^c = \bigcup_{n=1}^{\infty} A_n^c.$$

Therefore,

$$\begin{aligned}
\overline{\lim_{n\to\infty}} A_n &= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} (A_k^c)^c \\
&= \left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k^c \right)^c = \left(\underline{\lim_{n\to\infty}} A_n^c \right)^c \\
&= \left(\lim_{n\to\infty} A_n^c \right)^c = \left(\bigcup_{n=1}^{\infty} A_n^c \right)^c = \bigcap_{n=1}^{\infty} (A_n^c)^c \\
&= \bigcap_{n=1}^{\infty} A_n \left(x \in \bigcap_{n=1}^{\infty} A_n \Longleftrightarrow x \in A_1 \text{ and } x \in A_2 \text{ and } x \in A_3 \text{ and } \cdots \right) \\
&= \bigcap_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} (A_k^c)^c \\
&= \left(\overline{\lim_{n\to\infty}} A_n^c \right)^c = \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k^c \right)^c \\
&= \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} (A_k^c)^c = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \\
&= \underline{\lim_{n\to\infty}} A_n.
\end{aligned}$$

Hence, $\lim\limits_{n\to\infty} A_n = \bigcap\limits_{k=1}^{\infty} A_k = \bigcap\limits_{n=1}^{\infty} A_n$.

Remark:

$$\begin{aligned}\limsup_{k\to\infty} A_k = \overline{\lim_{k\to\infty}} A_k &= \{\text{ consists of elements which belong to } A_k \text{ for infinitely many } k\} \\ &= \{x\colon \forall n\in\mathbb{N}, \exists k\geq n,\ s.t.\ x\in A_k\}\end{aligned}$$

$$\begin{aligned}\liminf_{k\to\infty} A_k = \underline{\lim_{k\to\infty}} A_k &= \{\text{ consists of elements which belong to } A_k \text{ for all but finitely many } k\} \\ &= \{\text{ consists of elements which not belong to } A_k \text{ for finitely many } k\} \\ &= \{x\colon \exists n_0\in\mathbb{N}, \forall k\geq n_0,\ s.t.\ x\in A_k\}\end{aligned}$$

The $\lim_{n\to\infty} A_n$ exists if and only if $\underline{\lim_{n\to\infty}} A_n = \overline{\lim_{n\to\infty}} A_n$, in which case

$$\lim_{n\to\infty} A_n = \overline{\lim_{n\to\infty}} A_n = \underline{\lim_{n\to\infty}} A_n.$$

From the above definition, we can get

$$\bigcap_{k=1}^{\infty} A_k \subset \underline{\lim_{k\to\infty}} A_k \subset \overline{\lim_{k\to\infty}} A_k \subset \bigcup_{k=1}^{\infty} A_k.$$

Prop: Assume $\{A_n; n\geq 1\}$ is an set sequence, then

(a) $\overline{\lim_{k\to\infty}} A_k = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k$;

(b) $\underline{\lim_{k\to\infty}} A_k = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k$.

3. Show that if $A_1, A_2, \ldots, A_n$ are all countable sets, then so is the n-tuple **Cartesian product**

$$A_1\times A_2\times\cdots\times A_n.$$

In particular, if A is a countable set, then so is A^n.

Proof: (Use Mathematical induction)

Assume $n=1$, then $A_1\times A_2\times\cdots\times A_n = A_1$ is a countable set.

If $n=2$, for any fixed $a_2\in A_2$, $\{(a_1,a_2)\colon a_1\in A_1\}$ is a countable set (In fact for any fixed $a_2\in A_2$,$a_1 \xleftrightarrow{\textit{one to one correspondence}} (a_1,a_2)$, so $\overline{\overline{A_1}} = \overline{\overline{A_1\times\{a_2\}}}$). Notice that

$$A_1\times A_2 = \bigcup_{a_2\in A_2}\{(a_1,a_2)\colon a_1\in A_1\},$$

then it also is a countable set.

Suppose $n=k$,$A_1\times A_2\times\cdots\times A_k$ is a countable set. Then, for $n=k+1$, only need to notice that for any fixed $a_{k+1}\in A_{k+1}$, $A_1\times A_2\times\cdots\times A_k\times\{a_{k+1}\}$ is a countable set and

$$A_1\times A_2\times\cdots\times A_k\times A_{k+1} = \bigcup_{a_{k+1}\in A_{k+1}} A_1\times A_2\times\cdots\times A_k\times\{a_{k+1}\},$$

then it also is a countable set.

In particular, if A is a countable set, then so is A^n.

4. Suppose that the three sets A, B and C have the relationship $A \subset B \subset C$ and that $\mathrm{Card}(A) = \mathrm{Card}(C)$, then

$$\mathrm{Card}(A) = \mathrm{Card}(B) = \mathrm{Card}(C),$$

where $\mathrm{Card}(A)$ denotes the cardinal number of the set A etc.

Proof: Since $A \subset B \subset C$, consider maps

$$i_A : A \to A \subset B, x \mapsto x;$$

$$i_B : B \to B \subset C, x \mapsto x.$$

Note that identity map is a special bijective map(one to one and onto), so

$$\mathrm{Card}\, A \le \mathrm{Card}\, B; \mathrm{Card}\, B \le \mathrm{Card}\, C,$$

but $\mathrm{Card}(A) = \mathrm{Card}(C)$. Thus, by using the Cantor-Bernstein's theorem, we get

$$\mathrm{Card}(A) = \mathrm{Card}(B) = \mathrm{Card}(C).$$

5. Show that the set $[0, 1]$ is not countable.

Proof: Now, suppose [0,1] is countable, then it can be written as a sequence $\{x_1, x_2, x_3, \cdots\}$ say. Suppose

$$x_1 = 0.a_{11}a_{12}a_{13}a_{14}\cdots a_{1n}\cdots$$

$$x_2 = 0.a_{21}a_{22}a_{23}a_{24}\cdots a_{2n}\cdots$$

$$x_3 = 0.a_{31}a_{32}a_{33}a_{34}\cdots a_{3n}\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_n = 0.a_{n1}a_{n2}a_{n3}a_{n4}\cdots a_{nn}\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

(Remember all of the numbers in [0,1] are be listed. *) where a_{ij} are all $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Now we define a number, x_*, say, as

$$x_* = 0.a_{*1}a_{*2}a_{*3}\cdots a_{*n}\cdots,$$

*

$$0.314 = 0.3139999\cdots \checkmark \qquad 1 = 0.99999\cdots \checkmark$$
$$= 0.3140000\cdots \qquad = 1.0000$$

where $a_{*1} \neq a_{11}, a_{*2} \neq a_{22}, \dots, a_{*n} \neq a_{nn}, \cdots$ and all $a_{*k}, k \geq 1$ take value in $\{0, 1, 2, \cdots, 9\}$. Surely $x_* \in [0, 1]$, but x_* is not be listed. Since it equals neither of the x_n, hence we have a contradiction, i.e. the set $[0, 1]$ is not countable.

6. Show that the Cardinal number of the real number $\mathbb{R}$ is equal to the cardinal number of the open unit internal $(0, 1)$.

   Proof: Let $f\colon (0,1) \to \mathbb{R}, x \mapsto f(x) = \tan(\pi x - \frac{\pi}{2})$.

   - For any $0 < x_1 < 1, 0 < x_2 < 1$,

$$\begin{aligned} f(x_1) = f(x_2) &\Longleftrightarrow \tan(\pi x_1 - \frac{\pi}{2}) = \tan(\pi x_2 - \frac{\pi}{2}) \\ &\Longleftrightarrow \pi x_1 - \frac{\pi}{2} = \pi x_2 - \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \\ &\Longleftrightarrow x_1 = x_2 + k, k \in \mathbb{Z}. \end{aligned}$$

     Note that $0 < x_1 < 1, 0 < x_2 < 1$, then $k \equiv 0$, so $x_1 = x_2$. Therefore, f is a map and also is a injective map.

   - For any $y \in \mathbb{R}$, let $x = \frac{\arctan y}{\pi} + \frac{1}{2}$, then $0 < x < 1$ and $y = f(x) = \tan(\pi x - \frac{\pi}{2})$. Therefore f is a surjective map.

   Hence, we have show that f is a bijective map, so $\mathrm{Card}\,((0, 1)) = \mathrm{Card}\,(\mathbb{R})$.

7. Suppose $\{A_n; n = 1, 2, \dots\}$ is an increasing sequence of sets.

   Define $B_1 = A_1$, $B_2 = A_2 \setminus A_1$, and in general, $B_n = A_n \setminus A_{n-1} (n \geq 2)$. Show that

   (i) $\{B_n; n \geq 1\}$ are (pairwise) disjoint.

   (ii) For any $k \geq 1$, $\bigcup\limits_{n=1}^{k} B_n = A_k$;

   (iii) $\bigcup\limits_{n=1}^{\infty} B_n = \bigcup\limits_{n=1}^{\infty} A_n$.

   Proof:

   (i) For any $m, n \in \mathbb{N}_+$, $B_n = A_n \setminus A_{n-1} \subset A_n$(where $A_0 \triangleq \emptyset$).

   Without loss of generality, we can suppose $m > n$, note that $m, n \in \mathbb{N}_+$, so

$$m - 1 \geq n \geq 1.$$

   Since $\{A_n; n = 1, 2, \dots\}$ is an increasing sequence of sets, then $A_n \subset A_{m-1}$, yields

$$B_n \subset A_n \subset A_{m-1}.$$

   However, $B_m^c = (A_m \setminus A_{m-1})^c = (A_m \cap A_{m-1}^c)^c = A_m^c \cup (A_{m-1}^c)^c = A_m^c \cup A_{m-1} \supset A_{m-1}$. Altogether, we have for any $m > n \geq 1$,

$$B_n \subset A_n \subset A_{m-1} \subset B_m^c$$

i.e. $B_n \cap B_m = B_m \cap B_n = \emptyset$.

Hence, $\{B_n; n \geq 1\}$ are (pairwise) disjoint.

(ii) **(By Mathematical induction)** For $k = 1$, we have

$$\bigcup_{n=1}^{1} B_n = B_1 = A_1.$$

Suppose for $k = m$, we have $\bigcup\limits_{n=1}^{m} B_n = A_m$, thus

$$\begin{aligned}\bigcup_{n=1}^{m+1} B_n &= \left(\bigcup_{n=1}^{m} B_n\right) \cup B_{m+1} = A_m \cup B_{m+1} \\ &= A_m \cup (A_{m+1} \setminus A_m) = A_{m+1} \quad \text{since } A_m \subset A_{m+1}.\end{aligned}$$

According to mathematical induction, we get for any $k \geq 1$,

$$\bigcup_{n=1}^{k} B_n = A_k.$$

<u>**Method 2: Rigorous proof**</u>

If $x \in \bigcup\limits_{n=1}^{k} B_n$, then $\exists k_0, 1 \leq k_0 \leq k$, s.t. $x \in B_{k_0} = A_{k_0} \setminus A_{k_0-1} \Rightarrow x \in A_{k_0}$. Since $\{A_n; n = 1, 2, \ldots\}$ is an increasing sequence of sets, so $A_{k_0} \subset A_k$, therefore $x \in A_k$, hence $\bigcup\limits_{n=1}^{k} B_n \subset A_k$.

Suppose $x \in A_k$,

- if $x \notin A_{k-1}$, then $x \in A_k \setminus A_{k-1} = B_k \subset \bigcup\limits_{n=1}^{k} B_n$, i.e. $x \in \bigcup\limits_{n=1}^{k} B_n$;
- if $x \in A_{k-1}$,
  - if $x \notin A_{k-2}$, then $x \in A_{k-1} \setminus A_{k-2} = B_{k-1} \subset \bigcup\limits_{n=1}^{k} B_n$, i.e. $x \in \bigcup\limits_{n=1}^{k} B_n$;
  - if $x \in A_{k-2}$,
    - if $x \notin A_{k-3}$, then $x \in A_{k-2} \setminus A_{k-3} = B_{k-2} \subset \bigcup\limits_{n=1}^{k} B_n$, i.e. $x \in \bigcup\limits_{n=1}^{k} B_n$;
    - if $x \in A_{k-3}$,

      ....
      - if $x \notin A_1$, then $x \in A_2 \setminus A_1 = B_2 \subset \bigcup\limits_{n=1}^{k} B_n$, i.e. $x \in \bigcup\limits_{n=1}^{k} B_n$;
      - if $x \in A_1 = B_1 \subset \bigcup\limits_{n=1}^{k} B_n$, i.e. $x \in \bigcup\limits_{n=1}^{k} B_n$.

(iii) Since for any $n \geq 1$, $B_n \subset A_n, \Rightarrow \bigcup\limits_{n=1}^{\infty} B_n \subset \bigcup\limits_{n=1}^{\infty} A_n$.

In the other hand, for any $x \in \bigcup\limits_{n=1}^{\infty} A_n, \Rightarrow \exists k_0 \geq 1$, *s.t.* $x \in A_{k_0}$. According to the above conclusion (ii), we have $A_{k_0} = \bigcup\limits_{n=1}^{k_0} B_n \subset \bigcup\limits_{n=1}^{\infty} B_n$, so $x \in \bigcup\limits_{n=1}^{\infty} B_n$, yields

$$\bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} B_n.$$

All in all, we can get $\bigcup\limits_{n=1}^{\infty} B_n = \bigcup\limits_{n=1}^{\infty} A_n$.

Formal proof: Since we know $\bigcup\limits_{n=1}^{k} B_n = A_k$, let $k \to \infty$, we have

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\lim\limits_{k\to\infty} k} B_n = \lim_{k\to\infty} \bigcup_{n=1}^{k} B_n = \lim_{k\to\infty} A_k = \bigcup_{n=1}^{\infty} A_n.$$

Remark:

$$A \cap B = \emptyset \Longleftrightarrow A \subset B^c \Longleftrightarrow B \subset A^c.$$

8. Let S be the set of all the sequences with elements 0 and 1 only.

   Is S countable or not? Prove your conclusion.

   Proof: Note that

$$S = \{\xi = \{x_n\}_{n=1}^{\infty} \colon x_n = 0 \text{ or } 1, n \geq 1\}.$$

We will show that S is uncountable.

In fact, suppose S is countable, then it can be written as a sequence $\{\xi_1, \xi_2, \xi_3, \cdots\}$ say. Suppose

$$\xi_1 = \{x_{1n}\}_{n=1}^{\infty} = \{x_{11}, x_{12}, x_{13}, x_{14}, \cdots, x_{1n}, \cdots\}$$

$$\xi_2 = \{x_{2n}\}_{n=1}^{\infty} = \{x_{21}, x_{22}, x_{23}, x_{24}, \cdots, x_{2n}, \cdots\}$$

$$\xi_3 = \{x_{3n}\}_{n=1}^{\infty} = \{x_{31}, x_{32}, x_{33}, x_{34}, \cdots, x_{3n}, \cdots\}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\xi_n = \{x_{nn}\}_{n=1}^{\infty} = \{x_{n1}, x_{n2}, x_{n3}, x_{n4}, \cdots x_{nn}, \cdots\}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

where $x_{ij}, i, j \geq 1$ is 0 or 1. Now we define a new sequence, ξ^*, say, as

$$\xi^* = \{x_n^*\}_{n=1}^{\infty} = \{x_1^*, x_2^*, x_3^*, \cdots, x_n^*, \cdots\}$$

where for any $n \geq 1$,

$$x_n^* = 1, \quad if \quad x_{nn} = 0$$
$$x_n^* = 0, \quad if \quad x_{nn} = 1$$

Surely $\xi^* \in S$, but ξ^* is not be listed. Since it equals neither of the ξ_n, we have a contradiction! Hence S is uncountable.