SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF MATHEMATICS

MA215 Probability Theory

Tutorial 03Solu

Set: Wednesday 21st September 2016; Hand in: Friday, 30th September 2016.
Note: Hand in your solutions no later than 4pm of Friday, 30th September.

1. Show that if the conditional probabilities exist, then

$$\mathrm{P}(\mathrm{A}_1 \cap \mathrm{A}_2 \cap \cdots \cap \mathrm{A}_n) = \mathrm{P}(\mathrm{A}_1)\mathrm{P}(\mathrm{A}_2|\mathrm{A}_1)\mathrm{P}(\mathrm{A}_3|\mathrm{A}_1 \cap \mathrm{A}_2)\cdots \mathrm{P}(\mathrm{A}_n|\mathrm{A}_1 \cap \mathrm{A}_2 \cap \cdots \cap \mathrm{A}_{n-1}).$$

Proof: Assume for $n = 2$, we have

$$\mathrm{P}(\mathrm{A}_1 \cap \mathrm{A}_2) = \cancel{\mathrm{P}(\mathrm{A}_1)}\frac{\mathrm{P}(\mathrm{A}_1 \cap \mathrm{A}_2)}{\cancel{\mathrm{P}(\mathrm{A}_1)}} = \mathrm{P}(\mathrm{A}_1)\frac{\mathrm{P}(\mathrm{A}_2 \cap \mathrm{A}_1)}{\mathrm{P}(\mathrm{A}_1)} = \mathrm{P}(\mathrm{A}_1)\mathrm{P}(\mathrm{A}_2|\mathrm{A}_1).$$

Assume, for $n = k$, we have

$$\mathrm{P}(\mathrm{A}_1 \cap \mathrm{A}_2 \cap \cdots \cap \mathrm{A}_k) = \mathrm{P}(\mathrm{A}_1)\mathrm{P}(\mathrm{A}_2|\mathrm{A}_1)\mathrm{P}(\mathrm{A}_3|\mathrm{A}_1 \cap \mathrm{A}_2)\cdots \mathrm{P}(\mathrm{A}_k|\mathrm{A}_1 \cap \mathrm{A}_2 \cap \cdots \cap \mathrm{A}_{k-1}).$$

Then for $n = k + 1$, we obtain

$$\begin{aligned}
&\mathrm{P}(\mathrm{A}_1 \cap \mathrm{A}_2 \cap \cdots \cap \mathrm{A}_{k+1}) \\
&= \mathrm{P}\left((\mathrm{A}_1 \cap \mathrm{A}_2 \cap \cdots \mathrm{A}_k) \cap \mathrm{A}_{k+1}\right) \\
&= \mathrm{P}\left(\mathrm{A}_1 \cap \mathrm{A}_2 \cap \cdots \mathrm{A}_k\right)\mathrm{P}\left(\mathrm{A}_{k+1}|\mathrm{A}_1 \cap \mathrm{A}_2 \cap \cdots \mathrm{A}_k\right) \\
&= \mathrm{P}(\mathrm{A}_1)\mathrm{P}(\mathrm{A}_2|\mathrm{A}_1)\mathrm{P}(\mathrm{A}_3|\mathrm{A}_1 \cap \mathrm{A}_2)\cdots \mathrm{P}(\mathrm{A}_k|\mathrm{A}_1 \cap \mathrm{A}_2 \cap \cdots \cap \mathrm{A}_{k-1})\mathrm{P}\left(\mathrm{A}_{k+1}|\mathrm{A}_1 \cap \mathrm{A}_2 \cap \cdots \mathrm{A}_k\right) \\
&= \mathrm{P}(\mathrm{A}_1)\mathrm{P}(\mathrm{A}_2|\mathrm{A}_1)\mathrm{P}(\mathrm{A}_3|\mathrm{A}_1 \cap \mathrm{A}_2)\cdots \mathrm{P}\left(\mathrm{A}_{k+1}|\mathrm{A}_1 \cap \mathrm{A}_2 \cap \cdots \mathrm{A}_k\right).
\end{aligned}$$

Remark:

$$\begin{aligned}
&\mathrm{P}(\mathrm{A}_1 \cap \mathrm{A}_2 \cap \cdots \cap \mathrm{A}_n) \\
&= \mathrm{P}(\mathrm{A}_1 \cap \mathrm{A}_2 \cap \cdots \cap \mathrm{A}_{n-1})\mathrm{P}(\mathrm{A}_n|\mathrm{A}_1 \cap \mathrm{A}_2 \cap \cdots \cap \mathrm{A}_{n-1}) \\
&= \mathrm{P}(\mathrm{A}_1 \cap \mathrm{A}_2 \cap \cdots \cap \mathrm{A}_{n-2})\mathrm{P}(\mathrm{A}_{n-1}|\mathrm{A}_1 \cap \mathrm{A}_2 \cap \cdots \cap \mathrm{A}_{n-2})\mathrm{P}(\mathrm{A}_n|\mathrm{A}_1 \cap \mathrm{A}_2 \cap \cdots \cap \mathrm{A}_{n-1}) \\
&= \cdots \\
&= \mathrm{P}(\mathrm{A}_1 \cap \mathrm{A}_2)\mathrm{P}(\mathrm{A}_3|\mathrm{A}_1 \cap \mathrm{A}_2)\cdots \mathrm{P}(\mathrm{A}_n|\mathrm{A}_1 \cap \mathrm{A}_2 \cap \cdots \cap \mathrm{A}_{n-1}) \\
&= \mathrm{P}(\mathrm{A}_1)\mathrm{P}(\mathrm{A}_2|\mathrm{A}_1)\mathrm{P}(\mathrm{A}_3|\mathrm{A}_1 \cap \mathrm{A}_2)\cdots \mathrm{P}(\mathrm{A}_n|\mathrm{A}_1 \cap \mathrm{A}_2 \cap \cdots \cap \mathrm{A}_{n-1}).
\end{aligned}$$

2. Urn A has three red balls and two white balls, and urn B has two red balls and five white balls. A fair coin is tossed; if it hands heads up, a ball is drawn from urn A and otherwise a ball is drawn from urn B.

   (a) What is the probability that a red ball is drawn ?

(b) If a red ball is drawn, what is the probability that the coin landed heads up?

Proof: Let H $\triangleq$\{a fair coin is tossed, if it hand head up\},

T $\triangleq$\{a fair coin is tossed, if it hand tail up\};

E $\triangleq$\{a red ball is drawn\}.

(a) Then by the law of total probability, we have

$$\begin{aligned}
\mathrm{P}(\text{a red ball is drawn}) &= \mathrm{P(E)} \\
&= \mathrm{P(E|H)P(H) + P(E|T)P(T)} \\
&= \frac{3}{3+2} \times \frac{1}{2} + \frac{2}{2+5} \times \frac{1}{2} \\
&= \frac{3}{5} \times \frac{1}{2} + \frac{2}{7} \times \frac{1}{2} = \frac{31}{70}.
\end{aligned}$$

(b) By Bayes Theorem,

$$\mathrm{P(H|E)} = \frac{\mathrm{P(E|H)P(H)}}{\mathrm{P(E)}} = \frac{\frac{3}{5} \times \frac{1}{2}}{\frac{31}{70}} = \frac{21}{31}.$$

Remark: Assume we have a probability space $(\Omega_1, \mathscr{F}_1, P_1)$, where

$$\Omega_1 = \{\mathrm{H}, \mathrm{T}\},$$

$$\mathscr{F}_1 (= \mathscr{P}(\Omega_1) = 2^{\Omega_1}),$$

$$\mathrm{P}_1(\{\mathrm{H}\}) = \mathrm{P}_1(\{\mathrm{T}\}) = \frac{1}{2}.$$

In addition, we have

$$\Omega_A = \{\underbrace{r_1^A, r_2^A, r_3^A}_{A_r}, \overbrace{w_1^A, w_2^A}^{A_w}\},$$

$$\mathscr{F}_A (= \mathscr{P}(A) = 2^A),$$

$$\mathrm{P_A}(\{\mathrm{r_1^A}\}) = \mathrm{P_A}(\{\mathrm{r_2^A}\}) = \mathrm{P_A}(\{\mathrm{r_3^A}\}) = \mathrm{P_A}(\{\mathrm{w_4^A}\}) = \mathrm{P_A}(\{\mathrm{w_5^A}\}) = \frac{1}{5};$$

$$\Omega_B = \{\underbrace{r_1^B, r_2^B}_{B_r}, \overbrace{w_1^B, w_2^B, w_3^B, w_4^B, w_5^B}^{B_w}\},$$

$$\mathscr{F}_B (= \mathscr{P}(B) = 2^B),$$

$$\mathrm{P_B}(\{\mathrm{r_1^B}\}) = \mathrm{P_B}(\{\mathrm{r_2^B}\}) = \mathrm{P_B}(\{\mathrm{w_3^B}\}) = \mathrm{P_B}(\{\mathrm{w_4^B}\}) = \mathrm{P_B}(\{\mathrm{w_5^B}\}) = \mathrm{P_B}(\{\mathrm{w_6^B}\}) = \mathrm{P_B}(\{\mathrm{w_7^B}\}) = \frac{1}{7}.$$

Let $\Omega_2 = \Omega_A \dot\cup \Omega_B$,

$$\Omega = \Omega_1 \times \Omega_2,$$

$$\mathscr{F} (= \mathscr{P}(\Omega) = 2^\Omega),$$

P				
$\{H\}\times\Omega_A$			$\{T\}\times\Omega_B$	
(H,r_1^A)	$\frac{1}{2}\times\frac{1}{5}=\frac{1}{10}$		(T,r_1^B)	$\frac{1}{2}\times\frac{1}{7}=\frac{1}{14}$
(H,r_2^A)	$\frac{1}{2}\times\frac{1}{5}=\frac{1}{10}$		(T,r_2^B)	$\frac{1}{2}\times\frac{1}{7}=\frac{1}{14}$
(H,r_3^A)	$\frac{1}{2}\times\frac{1}{5}=\frac{1}{10}$		(T,w_3^B)	$\frac{1}{2}\times\frac{1}{7}=\frac{1}{14}$
(H,w_4^A)	$\frac{1}{2}\times\frac{1}{5}=\frac{1}{10}$		(T,w_4^B)	$\frac{1}{2}\times\frac{1}{7}=\frac{1}{14}$
(H,w_5^A)	$\frac{1}{2}\times\frac{1}{5}=\frac{1}{10}$		(T,w_5^B)	$\frac{1}{2}\times\frac{1}{7}=\frac{1}{14}$
			(T,w_6^B)	$\frac{1}{2}\times\frac{1}{7}=\frac{1}{14}$
			(T,w_7^B)	$\frac{1}{2}\times\frac{1}{7}=\frac{1}{14}$
otherwise				
0				

E={a red ball is drawn}=$\{H\}\times A_r+\{T\}\times B_r$*,

$$\begin{aligned}\mathrm{P(a\ red\ ball\ is\ drawn)} &= \mathrm{P(E)}\\ &= \mathrm{P(\{H\}\times A_r+\{T\}\times B_r)}\\ &= \mathrm{P(\{H\}\times A_r)+P(\{T\}\times B_r)}\\ &= \mathrm{P}\left(\{\mathrm{H}\}\times\{\mathrm{r_1^A}\}+\{\mathrm{H}\}\times\{\mathrm{r_1^A}\}+\{\mathrm{H}\}\times\{\mathrm{r_1^A}\}\right)\\ &\quad+\mathrm{P_1}\left(\{\mathrm{T}\}\times\{\mathrm{r_1^B}\}+\{\mathrm{T}\}\times\{\mathrm{r_1^B}\}\right)\\ &= \mathrm{P(\{H\}\times\{r_1^A\})+P(\{H\}\times\{r_1^A\})+P(\{H\}\times\{r_1^A\})}\\ &\quad+\mathrm{P_1(\{T\}\times\{r_1^B\})+P(\{T\}\times\{r_1^B\})}\\ &= \frac{1}{2}\times\frac{1}{5}+\frac{1}{2}\times\frac{1}{5}+\frac{1}{2}\times\frac{1}{5}\\ &\quad+\frac{1}{2}\times\frac{1}{7}+\frac{1}{2}\times\frac{1}{7}=\frac{31}{70}.\end{aligned}$$

Or,

$$\begin{aligned}\mathrm{P(a\ red\ ball\ is\ drawn)} &= \mathrm{P(E)}\\ &= \mathrm{P(\{H\}\times A_r+\{T\}\times B_r)}\\ &= \mathrm{P(\{H\}\times A_r)+P(\{T\}\times B_r)}\\ &= \mathrm{P_1(\{H\})P_A(A_r)+P_1(\{T\})P_B(B_r)}\\ &= \frac{1}{2}\times\frac{3}{5}+\frac{1}{2}\times\frac{2}{7}=\frac{31}{70}.\end{aligned}$$

Let $F\triangleq$\{the coin landed heads up\}=$\{H\}\times\Omega_2$, then

$$F\cap E=\{H\}\times\Omega_2\cap(\{H\}\times A_r\cap\{T\}\times B_r)=\{H\}\times A_r.$$

and

$$\mathrm{P(F|E)}=\frac{\mathrm{P(F\cap E)}}{\mathrm{P(E)}}=\frac{\mathrm{P(\{H\}\times A_r)}}{\mathrm{P(E)}}=\frac{\frac{1}{2}\times\frac{3}{5}}{\frac{31}{70}}=\frac{21}{31}.$$

*In order to facilitate writing, the sum of events $\overset{n}{\underset{i=1}{\cup}} A_i$ and $\overset{\infty}{\underset{i=1}{\cup}} A_i$ are symbolized by $\sum_{i=1}^{n} A_i$ and $\sum_{i=1}^{\infty} A_i$,respectively

In fact, we can consider a more "smaller" probability space $(S, \mathscr{F}_S, \mathrm{P_S})$, where $S = \{H\} \times \Omega_A + \{T\} \times \Omega_B = \Omega \cap S$, $\mathscr{F}_S = \mathscr{F} \cap S$, $\mathrm{P_S}(\cdot) = \mathrm{P}(\cdot \cap \mathrm{S})$.

Thus, sometimes written $(S, \mathscr{F}_S, \mathrm{P_S}) \triangleq (\Omega, \mathscr{F}, \mathrm{P})|_\mathrm{S} \triangleq (\Omega|_\mathrm{S}, \mathscr{F}|_\mathrm{S}, \mathrm{P}|_\mathrm{S})$.

The sample space S is actually happening in the actual model and the extra sample points in the sample space are theoretically needed (for the probability are zero).

For example, assume we have two probability space $(\Omega_1, \mathscr{F}_1, P_1)$, where

$$\Omega_1 = \{\mathrm{H}, \mathrm{T}\},$$
$$\mathscr{F}_1 (= \mathscr{P}(\Omega_1) = 2^{\Omega_1}),$$
$$\mathrm{P}_1(\{\mathrm{H}\}) = \mathrm{P}_1(\{\mathrm{T}\}) = \frac{1}{2}.$$

and $(\Omega_2, \mathscr{F}_2, P_2)$, where

$$\Omega_2 = \{\mathrm{H}, \mathrm{T}, \omega\},$$
$$\mathscr{F}_2 (= \mathscr{P}(\Omega_2) = 2^{\Omega_2}),$$
$$\mathrm{P}_2(\{\mathrm{H}\}) = \mathrm{P}_2(\{\mathrm{T}\}) = \frac{1}{2}, \mathrm{P}_2(\{\omega\}) = 0.$$

These two sample spaces are "equivalent". The sample space Ω_1 is actually happening in the actual model and the sample space Ω_2 is in theoretically needed (which sample point ω of the probability is zero).

3. Urn A has four red, three blue and two green balls. Urns B has two red, three blue and four green balls. A ball is drawn from urn A and put into urn B and then a ball is drawn from urn B.

   (a) What is the probability that a red ball is drawn from urn B?

   (b) If a red ball is drawn from urn B, what is the probability that a red ball was drawn from urn A?

   Proof: Let $E \triangleq$ {a red ball is drawn from urn B};
   $E_1 \triangleq$ {a red ball is drawn from urn A};
   $E_2 \triangleq$ {a blue ball is drawn from urn A};
   $E_3 \triangleq$ {a green ball is drawn from urn A};

   (a) Then, use the law of total probability, we have

$$\begin{aligned}
&\mathrm{P}(\text{a red ball is drawn from urn } B) = \mathrm{P(E)} \\
&= \mathrm{P(E|E_1)P(E_1) + P(E|E_2)P(E_2) + P(E|E_3)P(E_3)} \\
&= \frac{3}{2+1+3+4} \times \frac{4}{4+3+2} + \frac{2}{2+3+1+4} \times \frac{3}{4+3+2} + \frac{2}{2+3+4+1} \times \frac{2}{4+3+2} \\
&= \frac{3}{10} \times \frac{4}{9} + \frac{2}{10} \times \frac{3}{9} + \frac{2}{10} \times \frac{2}{9} \\
&= \frac{12+6+4}{90} = \frac{22}{90} = \frac{11}{45}.
\end{aligned}$$

(b) By Bayes Theorem,

$$\mathrm{P}(\mathrm{E}_1|\mathrm{E}) = \frac{\mathrm{P}(\mathrm{E}|\mathrm{E}_1)\mathrm{P}(\mathrm{E}_1)}{\mathrm{P}(\mathrm{E})} = \frac{\frac{3}{10}\times\frac{4}{9}}{\frac{11}{45}} = \frac{12}{22} = \frac{6}{11}.$$

4. There are three cabinets A, B, C, each of which has two drawers. Each drawer contains one coin; A has two gold coins, B has two silver coins and C has one gold and one silver coin. Take a experiment as a cabinet is chosen at random, one drawer is opened and a silver coin has found. What is the probability that the other drawer in that cabinet contains a silver coin?

   Modification: There are three cabinets A, B, C, each of which has two drawers. Each drawer contains one coin; A has two gold coins, B has two silver coins and C has one gold and one silver coin. If/Aussme/Suppose (the event) a cabinet is chosen at random, one drawer is opened and a silver coin has found (are given), what is the probability that the other drawer in that cabinet contains a silver coin?

   Proof: Let $E_A \triangleq$\{cabinet A is chosen at random\};

   $E_B \triangleq$\{cabinet B is chosen at random\};

   $E_C \triangleq$\{cabinet C is chosen at random\};

   $E \triangleq$\{one drawer is opened and a silver coin has found\};

   $D \triangleq$\{the other drawer in that cabinet contains a silver coin\}.

   Notice that only the cabinet B has two silver coins, so

   $D =$\{the other drawer in that cabinet contains a silver coin\}

   $=$\{chose the cabinet B, one drawer is opened at random and a silver coin has found\}.
   Then, use the law of total probability, we have

$$\begin{aligned}
&\mathrm{P}(A \text{ cabinet is chosen at random, one drawer is opened and a silver coin has found})\\
&= \mathrm{P}(\mathrm{E}) = \mathrm{P}(\mathrm{E}|\mathrm{E_A})\mathrm{P}(\mathrm{E_A}) + \mathrm{P}(\mathrm{E}|\mathrm{E_B})\mathrm{P}(\mathrm{E_B}) + \mathrm{P}(\mathrm{E}|\mathrm{E_C})\mathrm{P}(\mathrm{E_C})\\
&= 0\times\frac{1}{3} + 1\times\frac{1}{3} + \frac{1}{2}\times\frac{1}{3}\\
&= \frac{1}{2}.
\end{aligned}$$

   By Bayes Theorem,

$$\mathrm{P}(\mathrm{E_B}|\mathrm{E}) = \frac{\mathrm{P}(\mathrm{E}|\mathrm{E_B})\mathrm{P}(\mathrm{E_B})}{\mathrm{P}(\mathrm{E})} = \frac{1\times\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}.$$

   Remark: Assume we have a probability space $(\Omega_1, \mathscr{F}_1, P_1)$, where

$$\Omega_1 = \{A, B, C\},$$

$$\mathscr{F}_1 (= \mathscr{P}(\Omega_1) = 2^{\Omega_1}),$$

$$\mathrm{P_1(A) = P_1(B) = P_1(C) = \frac{1}{3}}.$$

In addition, we have

$$\Omega_A = \{g_1^A, g_2^A\},$$
$$\mathscr{F}_A (= \mathscr{P}(A) = 2^A),$$
$$\mathrm{P_A(\{g_1^A\}) = P_A(\{g_2^A\}) = \frac{1}{2}};$$
$$\Omega_B = \{s_1^B, s_2^B\},$$
$$\mathscr{F}_B (= \mathscr{P}(B) = 2^B),$$
$$\mathrm{P_B(\{s_1^B\}) = P_B(\{s_2^B\}) = \frac{1}{2}}.$$
$$\Omega_C = \{g_1^C, s_2^C\},$$
$$\mathscr{F}_B (= \mathscr{P}(B) = 2^B),$$
$$\mathrm{P_B(\{g_1^C\}) = P_B(\{s_2^C\}) = \frac{1}{2}}.$$

Let $\Omega_2 = \Omega_A \cupdot \Omega_B \cupdot \Omega_C$,

$$\Omega = \Omega_1 \times \Omega_2,$$
$$\mathscr{F} (= \mathscr{P}(\Omega) = 2^\Omega),$$

Table 1: Probability distribution table

P								
cabinet	$drawer^{first}$	$drawer^{second}$	*cabinet*	$drawer^{first}$	$drawer^{second}$	*cabinet*	$drawer^{first}$	$drawer^{second}$
A	(A, g_1^A)	(A, g_2^A)	B	(B, g_1^A)	(B, g_2^A)	C	(C, g_1^A)	(C, g_2^A)
	$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$	$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$		0	0		0	0
	(A, s_1^B)	(A, s_2^B)		(B, s_1^B)	(B, s_2^B)		(C, s_1^B)	(C, s_2^B)
	0	0		$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$	$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$		0	0
	(A, g_1^C)	(A, s_2^C)		(B, g_1^C)	(B, s_2^C)		(C, g_1^C)	(C, s_2^C)
	0	0		0	0		$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$	$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$
otherwise								
0								

E={one drawer is opened and a silver coin has found}
$=B \times \{s_1^B\} + B \times \{s_2^B\} + C \times \{s_2^C\}$,

$$\begin{aligned}
&\mathrm{P(one\ drawer\ is\ opened\ and\ a\ silver\ coin\ has\ found)}\\
&= \mathrm{P(E) = P\left(B \times \{s_1^B\} + B \times \{s_2^B\} + C \times \{s_2^C\}\right)}\\
&= \mathrm{P(B \times \{s_1^B\}) + P(B \times \{s_2^B\}) + P(C \times \{s_2^C\})}\\
&= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}.
\end{aligned}$$

Or,

$$\mathrm{P(one\ drawer\ is\ opened\ and\ a\ silver\ coin\ has\ found)}$$

$$
\begin{aligned}
&= \mathrm{P(E)} \\
&= \mathrm{P(B \times \Omega_B + C \times \{s_2^C\})} \\
&= \mathrm{P(B \times \Omega_B) + P(C \times \{s_2^C\})} \\
&= \mathrm{P_1(B)P_B(\Omega_B) + P_1(C)P_C(\{s_2^C\})} \\
&= \frac{1}{3} \times 1 + \frac{1}{3} \times \frac{1}{2} = \frac{1}{3} \times (\frac{1}{2} + \frac{1}{2}) + \frac{1}{3} \times \frac{1}{2} \\
&= \frac{1}{2}.
\end{aligned}
$$

Note that $D = B \times \Omega_B = B \times \{s_1^B\} + B \times \{s_2^B\}$, then

$$D \cap E = D.$$

and

$$\mathrm{P(D|E)} = \frac{\mathrm{P(D \cap E)}}{\mathrm{P(E)}} = \frac{\mathrm{P(D)}}{\mathrm{P(E)}} = \frac{\mathrm{P(B \times \{s_1^B\} + B \times \{s_2^B\})}}{\mathrm{P(E)}} = \frac{\frac{1}{6} + \frac{1}{6}}{\frac{1}{2}} = \frac{2}{3}.$$

In fact, we can consider a more "smaller" probability space $(S, \mathscr{F}_S, \mathrm{P_S})$, where $S = A \times \Omega_A + B \times \Omega_B + C \times \Omega_C = \Omega \cap S$, $\mathscr{F}_S = \mathscr{F} \cap S$, $\mathrm{P_S}(\cdot) = \mathrm{P}(\cdot \cap \mathrm{S})$. Thus, sometimes written $(S, \mathscr{F}_S, \mathrm{P_S}) \triangleq (\Omega, \mathscr{F}, \mathrm{P})|_\mathrm{S} \triangleq (\Omega|_\mathrm{S}, \mathscr{F}|_\mathrm{S}, \mathrm{P}|_\mathrm{S})$.

5. If B is an event with $\mathrm{P(B)} > 0$, show that the set function $\mathrm{Q(A)} = \mathrm{P(A|B)}$ is a probability measure. Thus, we can use the following formulas in lectures

$$
\begin{aligned}
&\mathrm{P(A \cup C|B) = P(A|B) + P(C|B) - P(A \cap C|B)} \\
&\text{and } \mathrm{P(A^c|B) = 1 - P(A|B)}.
\end{aligned}
$$

Proof:

1° $\mathrm{Q}(\emptyset) = \mathrm{P}(\emptyset|\mathrm{B}) = \frac{\mathrm{P}(\emptyset \cap \mathrm{B})}{\mathrm{P(B)}} = \frac{\mathrm{P}(\emptyset)}{\mathrm{P(B)}} = 0$;

2° $\mathrm{Q}(\Omega) = \mathrm{P}(\Omega|\mathrm{B}) = \frac{\mathrm{P}(\Omega \cap \mathrm{B})}{\mathrm{P(B)}} = \frac{\mathrm{P(B)}}{\mathrm{P(B)}} = 1$;

3° Assume $\{A_i\}_{i=1}^{\infty}$ are mutually exclusive events, then

$$
\begin{aligned}
&\mathrm{Q}(\bigcup_{\mathrm{i}=1}^{\infty} \mathrm{A_i}) = \mathrm{P}\left(\bigcup_{\mathrm{i}=1}^{\infty} \mathrm{A_i}|\mathrm{B}\right) \\
&= \frac{\mathrm{P}((\bigcup_{\mathrm{i}=1}^{\infty} \mathrm{A_i}) \cap \mathrm{B})}{\mathrm{P(B)}} \\
&= \frac{\mathrm{P}\left(\bigcup_{\mathrm{i}=1}^{\infty} (\mathrm{A_i} \cap \mathrm{B})\right)}{\mathrm{P(B)}} \\
&= \frac{\sum_{i=1}^{\infty} \mathrm{P(A_i \cap B)}}{\mathrm{P(B)}} \qquad \text{Notice that } \{A_i \cap B\}_{i=1}^{\infty} \text{ are mutually exclusive events}
\end{aligned}
$$

$$= \sum_{i=1}^{\infty} \frac{\mathrm{P}(\mathrm{A_i} \cap \mathrm{B})}{\mathrm{P}(\mathrm{B})} = \sum_{i=1}^{\infty} \mathrm{P}(\mathrm{A_i}|\mathrm{B})$$
$$= \sum_{i=1}^{\infty} \mathrm{Q}(\mathrm{A_i}).$$

Therefore, we have shown that the set function $\mathrm{Q}(\mathrm{A}) = \mathrm{P}(\mathrm{A}|\mathrm{B})$ is a probability measure. So, according to the property of the probability measure, we get

$$\mathrm{Q}(\mathrm{A} \cup \mathrm{C}) = \mathrm{Q}(\mathrm{A}) + \mathrm{Q}(\mathrm{C}) - \mathrm{Q}(\mathrm{A} \cap \mathrm{C}), \quad \mathrm{Q}(\mathrm{A^c}) = 1 - \mathrm{Q}(\mathrm{A}).$$

i.e.,

$$\begin{aligned} &\mathrm{P}(\mathrm{A} \cup \mathrm{C}|\mathrm{B}) = \mathrm{P}(\mathrm{A}|\mathrm{B}) + \mathrm{P}(\mathrm{C}|\mathrm{B}) - \mathrm{P}(\mathrm{A} \cap \mathrm{C}|\mathrm{B}) \\ &\text{and } \mathrm{P}(\mathrm{A^c}|\mathrm{B}) = 1 - \mathrm{P}(\mathrm{A}|\mathrm{B}). \end{aligned}$$

Thus, we can use the above formulas in lectures.

6. Show that if A, B, C are mutually independent, then $A \cap B$ and C are independent and $A \cup B$ and C are independent.

   Proof: Since

$$\begin{aligned} \mathrm{P}\left((\mathrm{A} \cap \mathrm{B}) \cap \mathrm{C}\right) &= \mathrm{P}(\mathrm{A} \cap \mathrm{B} \cap \mathrm{C}) = \mathrm{P}(\mathrm{ABC}) \\ &= \mathrm{P}(\mathrm{A})\mathrm{P}(\mathrm{B})\mathrm{P}(\mathrm{C}) = (\mathrm{P}(\mathrm{A})\mathrm{P}(\mathrm{B}))\mathrm{P}(\mathrm{C}) \\ &= \mathrm{P}(\mathrm{AB})\mathrm{P}(\mathrm{C}) = \mathrm{P}(\mathrm{A} \cap \mathrm{B})\mathrm{P}(\mathrm{C}). \end{aligned}$$

$$\begin{aligned} \mathrm{P}\left((\mathrm{A} \cup \mathrm{B}) \cap \mathrm{C}\right) &= \mathrm{P}\left((\mathrm{A} \cap \mathrm{C}) \cup (\mathrm{B} \cap \mathrm{C})\right) = \mathrm{P}((\mathrm{AC}) \cup (\mathrm{BC})) \\ &= \mathrm{P}(\mathrm{AC}) + \mathrm{P}(\mathrm{BC}) - \mathrm{P}\left((\mathrm{AC}) \cap (\mathrm{BC})\right) = \mathrm{P}(\mathrm{AC}) + \mathrm{P}(\mathrm{BC}) - \mathrm{P}(\mathrm{ABC}) \\ &= \mathrm{P}(\mathrm{A})\mathrm{P}(\mathrm{C}) + \mathrm{P}(\mathrm{B})\mathrm{P}(\mathrm{C}) - \mathrm{P}(\mathrm{A})\mathrm{P}(\mathrm{B})\mathrm{P}(\mathrm{C}) \\ &= [\mathrm{P}(\mathrm{A}) + \mathrm{P}(\mathrm{B}) - \mathrm{P}(\mathrm{A})\mathrm{P}(\mathrm{B})]\mathrm{P}(\mathrm{C}) \\ &= \mathrm{P}(\mathrm{A} \cup \mathrm{B})\mathrm{P}(\mathrm{C}). \end{aligned}$$

   Hence, $A \cap B$ and C are independent and $A \cup B$ and C are independent.

   Recall: A, B are independent:

$$\mathrm{P}(\mathrm{A} \cap \mathrm{B}) = \mathrm{P}(\mathrm{A})\mathrm{P}(\mathrm{B});$$

   A, B, C are pairwise independent:

$$P(AB) = P(A)P(B), P(AC) = P(A)P(C), P(BC) = P(B)P(C);$$

   A, B, C are (mutually) independent:

$$\mathrm{P}(\mathrm{AB}) = \mathrm{P}(\mathrm{A})\mathrm{P}(\mathrm{B}), \mathrm{P}(\mathrm{AC}) = \mathrm{P}(\mathrm{A})\mathrm{P}(\mathrm{C}), \mathrm{P}(\mathrm{BC}) = \mathrm{P}(\mathrm{B})\mathrm{P}(\mathrm{C});$$

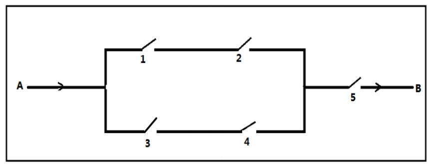
$$\text{and} \quad \mathrm{P}(\mathrm{ABC}) = \mathrm{P}(\mathrm{A})\mathrm{P}(\mathrm{B})\mathrm{P}(\mathrm{C}).$$

   More general, $\{A_i\}_{i \in I}$ are (mutually) independent:

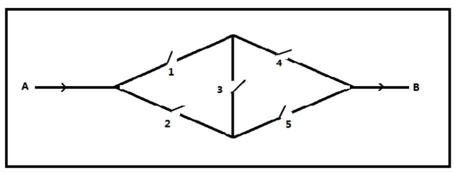
For every finite set F of index set I, one has

$$\mathrm{P}(\bigcap_{\mathrm{i}\in\mathrm{F}} \mathrm{A_i}) = \prod_{\mathrm{i}\in\mathrm{F}} \mathrm{P}(\mathrm{A_i}).$$

7. The probability of the closing of the ith relay in the circuits shown is given by $p_i, i = 1, 2, 3, 4, 5$. If all relays function independently, what is the probability that a current flows between A and B for the respective circuits?



(a) circuit 1



(b) circuit 2

Figure 1: circuits

Proof: Let $R_i^a \triangleq$ {the closing of the ith relay in the circuits a}(i=1,2,3,4,5);
$R_i^b \triangleq$ {the closing of the ith relay in the circuits b}(i=1,2,3,4,5).

(a) **Loops are allowed:**

$$\begin{aligned}
\mathrm{P}(\mathrm{A} \to \mathrm{B}) &= \mathrm{P}(\mathrm{R}_5^a + \square) = \mathrm{P}(\mathrm{R}_5^a)\mathrm{P}(\square) \\
&= \mathrm{P}(\mathrm{R}_5^a)\mathrm{P}(\square_{\text{upper}}\square_{\text{lower}}^c + \square_{\text{upper}}^c\square_{\text{lower}} + \square_{\text{upper}}\square_{\text{lower}}) \\
&= \mathrm{P}(\mathrm{R}_5^a)\Big\{\mathrm{P}(\square_{\text{upper}}\square_{\text{lower}}^c) + \mathrm{P}(\square_{\text{upper}}^c\square_{\text{lower}}) + \mathrm{P}(\square_{\text{upper}}\square_{\text{lower}})\Big\} \\
&= \mathrm{P}(\mathrm{R}_5^a)\Big\{\mathrm{P}(\square_{\text{upper}})\mathrm{P}(\square_{\text{lower}}^c) + \mathrm{P}(\square_{\text{upper}}^c)\mathrm{P}(\square_{\text{lower}}) + \mathrm{P}(\square_{\text{upper}})\mathrm{P}(\square_{\text{lower}})\Big\} \\
&= \mathrm{P}(\mathrm{R}_5^a)\Big\{\mathrm{P}(\square_{\text{upper}})[1-\mathrm{P}(\square_{\text{lower}})] + [1-\mathrm{P}(\square_{\text{upper}})]\mathrm{P}(\square_{\text{lower}}) + \mathrm{P}(\square_{\text{upper}})\mathrm{P}(\square_{\text{lower}})\Big\} \\
&= \mathrm{P}(\mathrm{R}_5^a)\Big\{\mathrm{P}(\mathrm{R}_1^a\mathrm{R}_2^a)[1-\mathrm{P}(\mathrm{R}_3^a\mathrm{R}_4^a)] + [1-\mathrm{P}(\mathrm{R}_1^a\mathrm{R}_2^a)]\mathrm{P}(\mathrm{R}_3^a\mathrm{R}_4^a) + \mathrm{P}(\mathrm{R}_1^a\mathrm{R}_2^a)\mathrm{P}(\mathrm{R}_3^a\mathrm{R}_4^a)\Big\} \\
&= \mathrm{P}(\mathrm{R}_5^a)\Big\{\mathrm{P}(\mathrm{R}_1^a)\mathrm{P}(\mathrm{R}_2^a)[1-\mathrm{P}(\mathrm{R}_3^a)\mathrm{P}(\mathrm{R}_4^a)] + [1-\mathrm{P}(\mathrm{R}_1^a)\mathrm{P}(\mathrm{R}_2^a)]\mathrm{P}(\mathrm{R}_3^a)\mathrm{P}(\mathrm{R}_4^a) + \mathrm{P}(\mathrm{R}_1^a)\mathrm{P}(\mathrm{R}_2^a)\mathrm{P}(\mathrm{R}_3^a)\mathrm{P}(\mathrm{R}_4^a)\Big\} \\
&= p_5[p_1p_2(1-p_3p_4) + (1-p_1p_2)p_3p_4 + p_1p_2p_3p_4] \\
&= p_1p_2p_5 + p_3p_4p_5 - p_1p_2p_3p_4p_5.
\end{aligned}$$

No loops are allowed:

$$\begin{aligned}
\mathrm{P}(\mathrm{A} \to \mathrm{B}) &= \mathrm{P}(\mathrm{R}_5^a + \square) = \mathrm{P}(\mathrm{R}_5^a)\mathrm{P}(\square) \\
&= \mathrm{P}(\mathrm{R}_5^a)\mathrm{P}(\square_{\text{upper}}\square_{\text{lower}}^c + \square_{\text{upper}}^c\square_{\text{lower}}) \\
&= \mathrm{P}(\mathrm{R}_5^a)\Big\{\mathrm{P}(\square_{\text{upper}}\square_{\text{lower}}^c) + \mathrm{P}(\square_{\text{upper}}^c\square_{\text{lower}})\Big\} \\
&= \mathrm{P}(\mathrm{R}_5^a)\Big\{\mathrm{P}(\square_{\text{upper}})\mathrm{P}(\square_{\text{lower}}^c) + \mathrm{P}(\square_{\text{upper}}^c)\mathrm{P}(\square_{\text{lower}})\Big\} \\
&= \mathrm{P}(\mathrm{R}_5^a)\Big\{\mathrm{P}(\square_{\text{upper}})[1-\mathrm{P}(\square_{\text{lower}})] + [1-\mathrm{P}(\square_{\text{upper}})]\mathrm{P}(\square_{\text{lower}})\Big\} \\
&= \mathrm{P}(\mathrm{R}_5^a)\Big\{\mathrm{P}(\mathrm{R}_1^a\mathrm{R}_2^a)[1-\mathrm{P}(\mathrm{R}_3^a\mathrm{R}_4^a)] + [1-\mathrm{P}(\mathrm{R}_1^a\mathrm{R}_2^a)]\mathrm{P}(\mathrm{R}_3^a\mathrm{R}_4^a)\Big\} \\
&= \mathrm{P}(\mathrm{R}_5^a)\Big\{\mathrm{P}(\mathrm{R}_1^a)\mathrm{P}(\mathrm{R}_2^a)[1-\mathrm{P}(\mathrm{R}_3^a)\mathrm{P}(\mathrm{R}_4^a)] + [1-\mathrm{P}(\mathrm{R}_1^a)\mathrm{P}(\mathrm{R}_2^a)]\mathrm{P}(\mathrm{R}_3^a)\mathrm{P}(\mathrm{R}_4^a)\Big\} \\
&= p_5[p_1p_2(1-p_3p_4) + (1-p_1p_2)p_3p_4] \\
&= p_1p_2p_5 + p_3p_4p_5 - 2p_1p_2p_3p_4p_5.
\end{aligned}$$

(b) **Loops are allowed:**

$$\begin{aligned}
&\mathrm{P}(\mathrm{A} \to \mathrm{B}) \\
&= \mathrm{P}\Big\{\mathrm{R}_1^b(\mathrm{R}_2^b)^c\big[(\mathrm{R}_3^b)^c\mathrm{R}_4^b + \mathrm{R}_3^b\left\{\mathrm{R}_4^b(\mathrm{R}_5^b)^c + \mathrm{R}_4^b\mathrm{R}_5^b + (\mathrm{R}_4^b)^c\mathrm{R}_5^b\right\}\big] \\
&\qquad + (R_1^b)^cR_2^b\big[(R_3^b)^cR_5^b + R_3^b\left\{R_4^b(R_5^b)^c + R_4^bR_5^b + (R_4^b)^cR_5^b\right\}\big] \\
&\qquad + R_1^bR_2^b\big[(R_3^b)^c\left\{R_4^b(R_5^b)^c + R_4^bR_5^b + (R_4^b)^cR_5^b\right\} + R_3^b\left\{R_4^b(R_5^b)^c + R_4^bR_5^b + (R_4^b)^cR_5^b\right\}\big]\Big\} \\
&= \mathrm{P}\Big\{\mathrm{R}_1^b(\mathrm{R}_2^b)^c\big[(\mathrm{R}_3^b)^c\mathrm{R}_4^b + \underbrace{R_3^b\left\{R_4^b(R_5^b)^c + R_4^bR_5^b + (R_4^b)^cR_5^b\right\}}_{\mathrm{R}_3^b-\mathrm{R}_3^b\mathrm{R}_4^b\mathrm{R}_5^b}\big]\Big\}
\end{aligned}$$

$$
\begin{aligned}
&+\mathrm{P}\Big\{(\mathrm{R}_1^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_2^{\mathrm{b}}\big[(\mathrm{R}_3^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_5^{\mathrm{b}}+\mathrm{R}_3^{\mathrm{b}}\left\{\mathrm{R}_4^{\mathrm{b}}(\mathrm{R}_5^{\mathrm{b}})^{\mathrm{c}}+\mathrm{R}_4^{\mathrm{b}}\mathrm{R}_5^{\mathrm{b}}+(\mathrm{R}_4^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_5^{\mathrm{b}}\right\}\big]\Big\}\\
&+\mathrm{P}\Big\{\mathrm{R}_1^{\mathrm{b}}\mathrm{R}_2^{\mathrm{b}}\big[(\mathrm{R}_3^{\mathrm{b}})^{\mathrm{c}}\left\{\mathrm{R}_4^{\mathrm{b}}(\mathrm{R}_5^{\mathrm{b}})^{\mathrm{c}}+\mathrm{R}_4^{\mathrm{b}}\mathrm{R}_5^{\mathrm{b}}+(\mathrm{R}_4^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_5^{\mathrm{b}}\right\}+\mathrm{R}_3^{\mathrm{b}}\left\{\mathrm{R}_4^{\mathrm{b}}(\mathrm{R}_5^{\mathrm{b}})^{\mathrm{c}}+\mathrm{R}_4^{\mathrm{b}}\mathrm{R}_5^{\mathrm{b}}+(\mathrm{R}_4^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_5^{\mathrm{b}}\right\}\big]\Big\}\\
=\;&\mathrm{P}\Big\{\mathrm{R}_1^{\mathrm{b}}(\mathrm{R}_2^{\mathrm{b}})^{\mathrm{c}}\big[(\mathrm{R}_3^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_4^{\mathrm{b}}+\mathrm{R}_3^{\mathrm{b}}-\mathrm{R}_3^{\mathrm{b}}\mathrm{R}_4^{\mathrm{b}}\mathrm{R}_5^{\mathrm{b}}\big]\Big\}\\
&+\mathrm{P}\Big\{(\mathrm{R}_1^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_2^{\mathrm{b}}\big[(\mathrm{R}_3^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_5^{\mathrm{b}}+\mathrm{R}_3^{\mathrm{b}}\left\{\mathrm{R}_4^{\mathrm{b}}(\mathrm{R}_5^{\mathrm{b}})^{\mathrm{c}}+\mathrm{R}_4^{\mathrm{b}}\mathrm{R}_5^{\mathrm{b}}+(\mathrm{R}_4^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_5^{\mathrm{b}}\right\}\big]\Big\}\\
&+\mathrm{P}\Big\{\mathrm{R}_1^{\mathrm{b}}\mathrm{R}_2^{\mathrm{b}}\big[\mathrm{R}_4^{\mathrm{b}}(\mathrm{R}_5^{\mathrm{b}})^{\mathrm{c}}+\mathrm{R}_4^{\mathrm{b}}\mathrm{R}_5^{\mathrm{b}}+(\mathrm{R}_4^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_5^{\mathrm{b}}\big]\Big\}\\
=\;&\mathrm{P}\Big\{\mathrm{R}_1^{\mathrm{b}}(\mathrm{R}_2^{\mathrm{b}})^{\mathrm{c}}(\mathrm{R}_3^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_4^{\mathrm{b}}+\mathrm{R}_1^{\mathrm{b}}(\mathrm{R}_2^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_3^{\mathrm{b}}-\mathrm{R}_1^{\mathrm{b}}(\mathrm{R}_2^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_3^{\mathrm{b}}\mathrm{R}_4^{\mathrm{b}}\mathrm{R}_5^{\mathrm{b}}\Big\}\\
&+\mathrm{P}\Big\{(\mathrm{R}_1^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_2^{\mathrm{b}}(\mathrm{R}_3^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_5^{\mathrm{b}}+(\mathrm{R}_1^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_2^{\mathrm{b}}\mathrm{R}_3^{\mathrm{b}}\mathrm{R}_4^{\mathrm{b}}(\mathrm{R}_5^{\mathrm{b}})^{\mathrm{c}}+(\mathrm{R}_1^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_2^{\mathrm{b}}\mathrm{R}_3^{\mathrm{b}}\mathrm{R}_4^{\mathrm{b}}\mathrm{R}_5^{\mathrm{b}}+(\mathrm{R}_1^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_2^{\mathrm{b}}\mathrm{R}_3^{\mathrm{b}}(\mathrm{R}_4^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_5^{\mathrm{b}}\Big\}\\
&+\mathrm{P}\Big\{\mathrm{R}_1^{\mathrm{b}}\mathrm{R}_2^{\mathrm{b}}\mathrm{R}_4^{\mathrm{b}}(\mathrm{R}_5^{\mathrm{b}})^{\mathrm{c}}+\mathrm{R}_1^{\mathrm{b}}\mathrm{R}_2^{\mathrm{b}}\mathrm{R}_4^{\mathrm{b}}\mathrm{R}_5^{\mathrm{b}}+\mathrm{R}_1^{\mathrm{b}}\mathrm{R}_2^{\mathrm{b}}(\mathrm{R}_4^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_5^{\mathrm{b}}\Big\}\\
=\;&\Big[\mathrm{P}\Big\{\mathrm{R}_1^{\mathrm{b}}(\mathrm{R}_2^{\mathrm{b}})^{\mathrm{c}}(\mathrm{R}_3^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_4^{\mathrm{b}})\Big\}+\mathrm{P}\Big\{\mathrm{R}_1^{\mathrm{b}}(\mathrm{R}_2^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_3^{\mathrm{b}}\Big\}-\mathrm{P}\Big\{\mathrm{R}_1^{\mathrm{b}}(\mathrm{R}_2^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_3^{\mathrm{b}}\mathrm{R}_4^{\mathrm{b}}\mathrm{R}_5^{\mathrm{b}}\Big\}\Big]\\
&+\Big[\mathrm{P}\Big\{(\mathrm{R}_1^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_2^{\mathrm{b}}(\mathrm{R}_3^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_5^{\mathrm{b}}\Big\}+\mathrm{P}\Big\{(\mathrm{R}_1^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_2^{\mathrm{b}}\mathrm{R}_3^{\mathrm{b}}\mathrm{R}_4^{\mathrm{b}}(\mathrm{R}_5^{\mathrm{b}})^{\mathrm{c}}\Big\}+\mathrm{P}\Big\{(\mathrm{R}_1^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_2^{\mathrm{b}}\mathrm{R}_3^{\mathrm{b}}\mathrm{R}_4^{\mathrm{b}}\mathrm{R}_5^{\mathrm{b}}\Big\}+\mathrm{P}\Big\{(\mathrm{R}_1^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_2^{\mathrm{b}}\mathrm{R}_3^{\mathrm{b}}(\mathrm{R}_4^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_5^{\mathrm{b}}\Big\}\Big]\\
&+\Big[\mathrm{P}\Big\{\mathrm{R}_1^{\mathrm{b}}\mathrm{R}_2^{\mathrm{b}}\mathrm{R}_4^{\mathrm{b}}(\mathrm{R}_5^{\mathrm{b}})^{\mathrm{c}}\Big\}+\mathrm{P}\Big\{\mathrm{R}_1^{\mathrm{b}}\mathrm{R}_2^{\mathrm{b}}\mathrm{R}_4^{\mathrm{b}}\mathrm{R}_5^{\mathrm{b}}\Big\}+\mathrm{P}\Big\{\mathrm{R}_1^{\mathrm{b}}\mathrm{R}_2^{\mathrm{b}}(\mathrm{R}_4^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_5^{\mathrm{b}}\Big\}\Big]\\
=\;&\Big[\mathrm{P}(\mathrm{R}_1^{\mathrm{b}})\mathrm{P}\left((\mathrm{R}_2^{\mathrm{b}})^{\mathrm{c}}\right)\mathrm{P}\left((\mathrm{R}_3^{\mathrm{b}})^{\mathrm{c}}\right)\mathrm{P}(\mathrm{R}_4^{\mathrm{b}})+\mathrm{P}(\mathrm{R}_1^{\mathrm{b}})\mathrm{P}\left((\mathrm{R}_2^{\mathrm{b}})^{\mathrm{c}}\right)\mathrm{P}(\mathrm{R}_3^{\mathrm{b}})-\mathrm{P}(\mathrm{R}_1^{\mathrm{b}})\mathrm{P}\left((\mathrm{R}_2^{\mathrm{b}})^{\mathrm{c}}\right)\mathrm{P}(\mathrm{R}_3^{\mathrm{b}})\mathrm{P}(\mathrm{R}_4^{\mathrm{b}})\mathrm{P}(\mathrm{R}_5^{\mathrm{b}})\Big]\\
&+\Big[\mathrm{P}\left((\mathrm{R}_1^{\mathrm{b}})^{\mathrm{c}}\right)\mathrm{P}(\mathrm{R}_2^{\mathrm{b}})\mathrm{P}\left((\mathrm{R}_3^{\mathrm{b}})^{\mathrm{c}}\right)\mathrm{P}(\mathrm{R}_5^{\mathrm{b}})+\mathrm{P}\left((\mathrm{R}_1^{\mathrm{b}})^{\mathrm{c}}\right)\mathrm{P}(\mathrm{R}_2^{\mathrm{b}})\mathrm{P}(\mathrm{R}_3^{\mathrm{b}})\mathrm{P}(\mathrm{R}_4^{\mathrm{b}})\mathrm{P}\left((\mathrm{R}_5^{\mathrm{b}})^{\mathrm{c}}\right)+\mathrm{P}\left((\mathrm{R}_1^{\mathrm{b}})^{\mathrm{c}}\right)\mathrm{P}(\mathrm{R}_2^{\mathrm{b}})\mathrm{P}(\mathrm{R}_3^{\mathrm{b}})\mathrm{P}(\mathrm{R}_4^{\mathrm{b}})\mathrm{P}(\mathrm{R}_5^{\mathrm{b}})+\mathrm{P}\left((\mathrm{R}_1^{\mathrm{b}})^{\mathrm{c}}\right)\mathrm{P}(\mathrm{R}_2^{\mathrm{b}})\mathrm{P}(\mathrm{R}_3^{\mathrm{b}})\mathrm{P}\left((\mathrm{R}_4^{\mathrm{b}})^{\mathrm{c}}\right)\mathrm{P}(\mathrm{R}_5^{\mathrm{b}})\Big]\\
&+\Big[\mathrm{P}(\mathrm{R}_1^{\mathrm{b}})\mathrm{P}(\mathrm{R}_2^{\mathrm{b}})\mathrm{P}(\mathrm{R}_4^{\mathrm{b}})\mathrm{P}\left((\mathrm{R}_5^{\mathrm{b}})^{\mathrm{c}}\right)+\mathrm{P}(\mathrm{R}_1^{\mathrm{b}})\mathrm{P}(\mathrm{R}_2^{\mathrm{b}})\mathrm{P}(\mathrm{R}_4^{\mathrm{b}})\mathrm{P}(\mathrm{R}_5^{\mathrm{b}})+\mathrm{P}(\mathrm{R}_1^{\mathrm{b}})\mathrm{P}(\mathrm{R}_2^{\mathrm{b}})\mathrm{P}\left((\mathrm{R}_4^{\mathrm{b}})^{\mathrm{c}}\right)\mathrm{P}(\mathrm{R}_5^{\mathrm{b}})\Big]\\
=\;&p_1(1-p_2)(1-p_3)p_4\times 1+p_1(1-p_2)p_3\times 1\times 1-p_1(1-p_2)p_3p_4p_5\\
&+(1-p_1)p_2(1-p_3)\times 1\times p_5+(1-p_1)p_2p_3p_4(1-p_5)+(1-p_1)p_2p_3p_4p_5+(1-p_1)p_2p_3(1-p_4)p_5\\
&+p_1p_2\times 1\times p_4(1-p_5)+p_1p_2\times 1\times p_4p_5+p_1p_2\times 1\times(1-p_4)p_5\\
=\;&(p_1p_3+p_1p_4-p_2p_5)\\
&-(p_1p_2p_3+p_1p_3p_4-p_2p_3p_4)\\
&-(p_1p_2p_4p_5+p_1p_3p_4p_5+p_2p_3p_4p_5)\\
&+2p_1p_2p_3p_4p_5.
\end{aligned}
$$

No loops are allowed:

$$
\begin{aligned}
&\mathrm{P}(\mathrm{A}\to\mathrm{B})\\
=\;&\mathrm{P}\Big\{\mathrm{R}_1^{\mathrm{b}}(\mathrm{R}_2^{\mathrm{b}})^{\mathrm{c}}\big[(\mathrm{R}_3^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_4^{\mathrm{b}}+\mathrm{R}_3^{\mathrm{b}}\left\{\mathrm{R}_4^{\mathrm{b}}(\mathrm{R}_5^{\mathrm{b}})^{\mathrm{c}}+(\mathrm{R}_4^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_5^{\mathrm{b}}\right\}\big]\\
&\qquad+(R_1^b)^cR_2^b\big[(R_3^b)^cR_5^b+R_3^b\left\{R_4^b(R_5^b)^c+(R_4^b)^cR_5^b\right\}\big]\\
&\qquad+R_1^bR_2^b\big[(R_3^b)^c\left\{R_4^b(R_5^b)^c+(R_4^b)^cR_5^b\right\}\big]\Big\}\\
=\;&\mathrm{P}\Big\{\mathrm{R}_1^{\mathrm{b}}(\mathrm{R}_2^{\mathrm{b}})^{\mathrm{c}}\big[(\mathrm{R}_3^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_4^{\mathrm{b}}+\mathrm{R}_3^{\mathrm{b}}\left\{\mathrm{R}_4^{\mathrm{b}}(\mathrm{R}_5^{\mathrm{b}})^{\mathrm{c}}+(\mathrm{R}_4^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_5^{\mathrm{b}}\right\}\big]\Big\}\\
&+\mathrm{P}\Big\{(\mathrm{R}_1^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_2^{\mathrm{b}}\big[(\mathrm{R}_3^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_5^{\mathrm{b}}+\mathrm{R}_3^{\mathrm{b}}\left\{\mathrm{R}_4^{\mathrm{b}}(\mathrm{R}_5^{\mathrm{b}})^{\mathrm{c}}+(\mathrm{R}_4^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_5^{\mathrm{b}}\right\}\big]\Big\}\\
&+\mathrm{P}\Big\{\mathrm{R}_1^{\mathrm{b}}\mathrm{R}_2^{\mathrm{b}}\big[(\mathrm{R}_3^{\mathrm{b}})^{\mathrm{c}}\left\{\mathrm{R}_4^{\mathrm{b}}(\mathrm{R}_5^{\mathrm{b}})^{\mathrm{c}}+(\mathrm{R}_4^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_5^{\mathrm{b}}\right\}\big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{P}\Big\{\mathrm{R}_1^{\mathrm{b}}(\mathrm{R}_2^{\mathrm{b}})^{\mathrm{c}}(\mathrm{R}_3^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_4^{\mathrm{b}} + \mathrm{R}_1^{\mathrm{b}}(\mathrm{R}_2^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_3^{\mathrm{b}}\mathrm{R}_4^{\mathrm{b}}(\mathrm{R}_5^{\mathrm{b}})^{\mathrm{c}} + \mathrm{R}_1^{\mathrm{b}}(\mathrm{R}_2^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_3^{\mathrm{b}}(\mathrm{R}_4^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_5^{\mathrm{b}}\Big\} \\
&\quad + \mathrm{P}\Big\{(\mathrm{R}_1^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_2^{\mathrm{b}}(\mathrm{R}_3^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_5^{\mathrm{b}} + (\mathrm{R}_1^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_2^{\mathrm{b}}\mathrm{R}_3^{\mathrm{b}}\mathrm{R}_4^{\mathrm{b}}(\mathrm{R}_5^{\mathrm{b}})^{\mathrm{c}} + (\mathrm{R}_1^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_2^{\mathrm{b}}\mathrm{R}_3^{\mathrm{b}}(\mathrm{R}_4^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_5^{\mathrm{b}}\Big\} \\
&\quad + \mathrm{P}\Big\{\mathrm{R}_1^{\mathrm{b}}\mathrm{R}_2^{\mathrm{b}}(\mathrm{R}_3^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_4^{\mathrm{b}}(\mathrm{R}_5^{\mathrm{b}})^{\mathrm{c}} + \mathrm{R}_1^{\mathrm{b}}\mathrm{R}_2^{\mathrm{b}}(\mathrm{R}_3^{\mathrm{b}})^{\mathrm{c}}(\mathrm{R}_4^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_5^{\mathrm{b}}\Big\} \\
&= \Big[\mathrm{P}\Big\{\mathrm{R}_1^{\mathrm{b}}(\mathrm{R}_2^{\mathrm{b}})^{\mathrm{c}}(\mathrm{R}_3^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_4^{\mathrm{b}}\Big\} + \mathrm{P}\Big\{\mathrm{R}_1^{\mathrm{b}}(\mathrm{R}_2^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_3^{\mathrm{b}}\mathrm{R}_4^{\mathrm{b}}(\mathrm{R}_5^{\mathrm{b}})^{\mathrm{c}}\Big\} + \mathrm{P}\Big\{\mathrm{R}_1^{\mathrm{b}}(\mathrm{R}_2^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_3^{\mathrm{b}}(\mathrm{R}_4^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_5^{\mathrm{b}}\Big\}\Big] \\
&\quad + \Big[\mathrm{P}\Big\{(\mathrm{R}_1^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_2^{\mathrm{b}}(\mathrm{R}_3^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_5^{\mathrm{b}}\Big\} + \mathrm{P}\Big\{(\mathrm{R}_1^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_2^{\mathrm{b}}\mathrm{R}_3^{\mathrm{b}}\mathrm{R}_4^{\mathrm{b}}(\mathrm{R}_5^{\mathrm{b}})^{\mathrm{c}}\Big\} + \mathrm{P}\Big\{(\mathrm{R}_1^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_2^{\mathrm{b}}\mathrm{R}_3^{\mathrm{b}}(\mathrm{R}_4^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_5^{\mathrm{b}}\Big\}\Big] \\
&\quad + \Big[\mathrm{P}\Big\{\mathrm{R}_1^{\mathrm{b}}\mathrm{R}_2^{\mathrm{b}}(\mathrm{R}_3^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_4^{\mathrm{b}}(\mathrm{R}_5^{\mathrm{b}})^{\mathrm{c}}\Big\} + \mathrm{P}\Big\{\mathrm{R}_1^{\mathrm{b}}\mathrm{R}_2^{\mathrm{b}}(\mathrm{R}_3^{\mathrm{b}})^{\mathrm{c}}(\mathrm{R}_4^{\mathrm{b}})^{\mathrm{c}}\mathrm{R}_5^{\mathrm{b}}\Big\}\Big] \\
&= \Big[\mathrm{P}(\mathrm{R}_1^{\mathrm{b}})\mathrm{P}\left((\mathrm{R}_2^{\mathrm{b}})^{\mathrm{c}}\right)\mathrm{P}\left((\mathrm{R}_3^{\mathrm{b}})^{\mathrm{c}}\right)\mathrm{P}(\mathrm{R}_4^{\mathrm{b}}) + \mathrm{P}(\mathrm{R}_1^{\mathrm{b}})\mathrm{P}\left((\mathrm{R}_2^{\mathrm{b}})^{\mathrm{c}}\right)\mathrm{P}(\mathrm{R}_3^{\mathrm{b}})\mathrm{P}\left((\mathrm{R}_4^{\mathrm{b}})^{\mathrm{c}}\right)\mathrm{P}(\mathrm{R}_5^{\mathrm{b}}) + \mathrm{P}(\mathrm{R}_1^{\mathrm{b}})\mathrm{P}\left((\mathrm{R}_2^{\mathrm{b}})^{\mathrm{c}}\right)\mathrm{P}(\mathrm{R}_3^{\mathrm{b}})\mathrm{P}(\mathrm{R}_4^{\mathrm{b}})\mathrm{P}\left((\mathrm{R}_5^{\mathrm{b}})^{\mathrm{c}}\right)\Big] \\
&\quad + \Big[\mathrm{P}\left((\mathrm{R}_1^{\mathrm{b}})^{\mathrm{c}}\right)\mathrm{P}(\mathrm{R}_2^{\mathrm{b}})\mathrm{P}\left((\mathrm{R}_3^{\mathrm{b}})^{\mathrm{c}}\right)\mathrm{P}(\mathrm{R}_5^{\mathrm{b}}) + \mathrm{P}\left((\mathrm{R}_1^{\mathrm{b}})^{\mathrm{c}}\right)\mathrm{P}(\mathrm{R}_2^{\mathrm{b}})\mathrm{P}(\mathrm{R}_3^{\mathrm{b}})\mathrm{P}(\mathrm{R}_4^{\mathrm{b}})\mathrm{P}\left((\mathrm{R}_5^{\mathrm{b}})^{\mathrm{c}}\right) + \mathrm{P}\left((\mathrm{R}_1^{\mathrm{b}})^{\mathrm{c}}\right)\mathrm{P}(\mathrm{R}_2^{\mathrm{b}})\mathrm{P}(\mathrm{R}_3^{\mathrm{b}})\mathrm{P}\left((\mathrm{R}_4^{\mathrm{b}})^{\mathrm{c}}\right)\mathrm{P}(\mathrm{R}_5^{\mathrm{b}})\Big] \\
&\quad + \Big[\mathrm{P}(\mathrm{R}_1^{\mathrm{b}})\mathrm{P}(\mathrm{R}_2^{\mathrm{b}})\mathrm{P}\left((\mathrm{R}_3^{\mathrm{b}})^{\mathrm{c}}\right)\mathrm{P}(\mathrm{R}_4^{\mathrm{b}})\mathrm{P}\left((\mathrm{R}_5^{\mathrm{b}})^{\mathrm{c}}\right) + \mathrm{P}(\mathrm{R}_1^{\mathrm{b}})\mathrm{P}(\mathrm{R}_2^{\mathrm{b}})\mathrm{P}\left((\mathrm{R}_3^{\mathrm{b}})^{\mathrm{c}}\right)\mathrm{P}\left((\mathrm{R}_4^{\mathrm{b}})^{\mathrm{c}}\right)\mathrm{P}(\mathrm{R}_5^{\mathrm{b}})\Big] \\
&= p_1(1-p_2)(1-p_3)p_4 \times 1 + p_1(1-p_2)p_3(1-p_4)p_5 + p_1(1-p_2)p_3p_4(1-p_5) \\
&\quad + (1-p_1)p_2(1-p_3) \times 1 \times p_5 + (1-p_1)p_2p_3p_4(1-p_5) + (1-p_1)p_2p_3(1-p_4)p_5 \\
&\quad + p_1p_2(1-p_3)p_4(1-p_5) + p_1p_2(1-p_3)(1-p_4)p_5 \\
&= (p_1p_4 + p_2p_5) \\
&\quad - (p_1p_2p_4 + p_1p_2p_5 - p_1pp_3p_5 - p_2p_3p_4) \\
&\quad - (p_1p_2p_3p_4 + p_1p_2p_3p_5 + 2p_1p_3p_4p_5 + 2p_2p_3p_4p_5) \\
&\quad + 4p_1p_2p_3p_4p_5.
\end{aligned}
$$