Functions of a Random Variable (conti-)

1. We now give a very useful general result.

① Theorem: Let X be a continous random variable having pdf $f_X(x)$. Suppose that $g(x)$ is a strictly monotone (increasing or decreasing) differentiable (and thus continuous) function of x. Then the random variable Y defined by $Y = g(X)$ has a pdf given by

$$f_Y(y) = \begin{cases} f_X[g^{-1}(y)] \cdot \left| \dfrac{d}{dy} g^{-1}(y) \right| & \text{if } y = g(x) \text{ for some } x \\ 0 & \text{if } y \neq g(x) \text{ for all } x \end{cases}$$

where $g^{-1}(y)$ is defined to equal that value of x such that $g(x) = y$.

② Conditions: pdf of $X(w)$: $f_X(x)$

function $g(x)$ either $\uparrow\uparrow$ or $\downarrow\downarrow$

($\uparrow\uparrow$ means if $x_1 < x_2$ then $g(x_1) < g(x_2)$
$\downarrow\downarrow$ means if $x_1 < x_2$ then $g(x_1) > g(x_2)$)

Note that strictly monotone property guarantees that $g^{-1}(x)$ is uniquely determined.

Also, $g'(x)$ exists (and thus $g(x)$ is continuous).

③ Conclusion: (Assume that $y = g(x)$ for some x otherwise just 0)

(i) First find $g^{-1}(y)$.

(ii) Obtain $\frac{d}{dy} g^{-1}(y)$ [$g'(x)$ existing guarantees this fact]

(iii) obtain $\left|\frac{d}{dy} g^{-1}(y)\right|$ [In order to guarantee $f_Y(\cdot) \geq 0$]

(iv) Find $f_X(g^{-1}(y))$ [$f_X(\cdot)$ and $g^{-1}(\cdot)$ are given]

(v) Finally $f_Y(y)$ is the product of $f_X(g^{-1}(y))$ and $\left|\frac{d}{dy} g^{-1}(y)\right|$

If $y \neq g(x)$ for all x, then $f_Y(y) = 0$

④ Proof:

Let $F_X(x)$ and $F_Y(y)$ be the cdfs of X and Y, respectively, then

$$F_Y(y) = \Pr\{Y \le y\} = \Pr\{g(X) \le y\}$$

Now consider the w-set

$$\{w \in \Omega;\ g(X) \le y\} \qquad (*)$$

(i) If $g(\cdot)\uparrow\uparrow$, then $(*)$ happens if and only if

$$\{w \in \Omega;\ X \le g^{-1}(y)\} \text{ happens} \qquad (1)$$

while

(ii) if $g(\cdot)\downarrow\downarrow$ then $(*)$ happens if and only if

$$\{w \in \Omega;\ X \ge g^{-1}(y)\} \text{ happens} \qquad (2)$$

Therefore, if $g(\cdot) \uparrow\uparrow$, then

$$
\begin{aligned}
F_Y(y) &= \Pr\{ g(X) \le y \} \\
&= \Pr\{ X \le g^{-1}(y) \} \\
&= F_X(g^{-1}(y))
\end{aligned}
$$

$$\Rightarrow \frac{d F_Y(y)}{dy} = \frac{d}{dy} F_X(g^{-1}(y)) = F_X'(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$

(Chain rule !!)

$$\Rightarrow f_Y(y) = f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y) \qquad (3)$$

while if $g(\cdot) \downarrow\downarrow$, then

$$F_Y(y) = \Pr\{g(X) \le y\}$$

$$= \Pr\{X \ge g^{-1}(y)\} \quad (*\!*\!*!!)$$

$$= 1 - \Pr\{X < g^{-1}(y)\} \quad (\text{Think why here!})$$

$$= 1 - \Pr\{X \le g^{-1}(y)\} \quad (\because X \text{ is conti- r.v})$$

$$\Rightarrow \quad = 1 - F_X(g^{-1}(y))$$

$$\Rightarrow \frac{dF_Y(y)}{dy} = (-1)\cdot F_X'(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$

$$= (-1)\cdot f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)$$

i.e.

$$f_Y(y) = f_X(g^{-1}(y)) \cdot (-1) \cdot \frac{d}{dy} g^{-1}(y) \tag{4}$$

We now write (3) and (4) in a uniformed form as

$$\boxed{f_Y(y) = f_Z(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|} \qquad (5)$$

The proof is completed. □

⑤ Remarks. By the proof we see that

(i) If $g(\cdot) \uparrow\uparrow$, then

$$f_Y(y) = f_Z(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y).$$

Note that in this case, since $g(\cdot) \uparrow\uparrow$, and thus $g^{-1}(y) \uparrow\uparrow$

$\Rightarrow \frac{d}{dy} g^{-1}(y) > 0$ which implies $f_Y(y) \geq 0$

(ii) If $g(\cdot) \downarrow\downarrow$ then

$$f_Y(y) = f_Z(g^{-1}(y)) \cdot (-1) \cdot \frac{d}{dy} g^{-1}(y)$$

Note that since $g(\cdot) \downarrow\downarrow$ and thus $g^{-1}(y) \downarrow\downarrow$

$\Rightarrow \frac{d}{dy} g^{-1}(y) < 0$ which implies $f_Y(y) = (-1) \frac{d}{dy} g^{-1}(y) \cdot f_Z(g^{-1}(y)) \geq 0$

⑥ Notes.

(i) This theorem requires the condition that

$\forall x \in \mathbb{R}$, $g(x)$ is strictly monotone (either $\uparrow\uparrow$ or $\downarrow\downarrow$)

However, it is easy to see that it is still true that

$g(x)$ is piecewise strictly monotone.

(Only in different intervals, the form is different, but can use the uniformed form (5). i.e. $f_X(g^{-1}(y)) \cdot \left|\frac{d}{dy} g^{-1}(y)\right|$)

(ii) In practice, you do not need to remember the formula, just use the "cdf" method.

(iii) Why if $y \neq g(x)$ for all x we have $f_Y(y) = 0$?

Recall $Y = g(X(w))$.

If for all x, $y \neq g(x)$ that means there does not exist an $w \in \Omega$ such that $Y(w) = g(X(w))$ i.e. $Y(w)$ does not taking any value. Hence $f_Y(y) = 0$.

2. Examples

① Example 1. $y = g(x)$ where $g(x) = x^2$

($g(x)$ is piecewise strictly monotone!

Indeed, $g(x) \uparrow\uparrow$ on $[0, \infty)$ and $g(x) \downarrow\downarrow$ on $(-\infty, 0]$.)

$$y = x^2 \implies x = \pm\sqrt{y} \quad \text{(depending on } (-\infty, 0] \text{ or } [0, \infty)\text{)}$$

i.e. $g^{-1}(y) = \pm\sqrt{y}$

$$\frac{d}{dy}\{g^{-1}(y)\} = \frac{d}{dy}(\pm\sqrt{y}) = \pm\frac{1}{2\sqrt{y}}$$

$$\implies \left|\frac{d}{dy}\{g^{-1}(y)\}\right| = \frac{1}{2\sqrt{y}}$$

Recall $f_Y(y) = f_X(g^{-1}(y)) \cdot \left|\frac{d}{dy}\{g^{-1}(y)\}\right| = \frac{1}{2\sqrt{y}} f_X(g^{-1}(y))$

Now, if X is a non-negative r.v., then $f_X(x) = \begin{cases} f_X(x) & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

then $f_X(g^{-1}(y)) = f_X(\sqrt{y}) \implies f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y})$

while if X is non-positive r.v., then

$$f_X(g^{-1}(y)) = f_X(-\sqrt{y}) \implies f_Y(y) = \frac{1}{2\sqrt{y}} f_X(-\sqrt{y})$$

If X takes value of $(-\infty, +\infty)$, then we can write the pdf of $Y = X^2$ as

$$\frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right] \qquad (2.1)$$

「Conclusion (2.1) is in agreement with the result we have obtained before!」

Now, suppose $X \sim Exp(\lambda)$, i.e. the pdf

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0, \end{cases}$$

then $f_X(-\sqrt{y}) = 0$ $\quad f_X(\sqrt{y}) = \lambda e^{-\lambda\sqrt{y}}$

Therefore, for $Y = X^2$, the pdf is

$$\begin{cases} f_Y(y) = \frac{1}{2\sqrt{y}}\lambda \cdot e^{-\lambda\sqrt{y}} & \text{if } y \geq 0 \\ f_Y(y) = 0 & \text{if } y < 0 \end{cases}$$

On the other hand, if $X \sim N(0,1)$

then $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad (\forall -\infty < x < +\infty)$

$$\Rightarrow f_X(\sqrt{y}) + f_X(-\sqrt{y}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{y})^2}{2}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{(-\sqrt{y})^2}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} = \frac{2}{\sqrt{2\pi}} e^{-\frac{y}{2}} \quad (\text{for } y \geq 0)$$

Therefore, if $X \sim N(0,1)$, then the pdf of $Y = X^2$ is

$$f_Y(y) = \begin{cases} \sqrt{\frac{2}{\pi}} e^{-\frac{y}{2}} & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$$

② Example 2: Suppose X is a non-negative r.v. with pdf $f_X(\cdot)$.

Let $Y = X^n$ (where $n \geq 1$ is a positive integer)

Find the pdf of Y: $f_Y(\cdot)$.

$y = x^n \quad \because X(\omega) \geq 0 \Rightarrow \forall x \geq 0, \ g(x) = x^n \uparrow\uparrow$

$\because y = x^n \Rightarrow x = y^{\frac{1}{n}} \quad (y \geq 0 \because x \geq 0)$

i.e. $g^{-1}(y) = y^{\frac{1}{n}}$

$$\frac{d}{dy}\{g^{-1}(y)\} = \frac{d}{dy} y^{\frac{1}{n}} = \frac{1}{n} y^{\frac{1}{n}-1} \geq 0. \quad (\text{for } y \geq 0)$$

$$\Rightarrow f_Y(y) = f_X(g^{-1}(y)) \left|\frac{d}{dy} g^{-1}(y)\right| = f_X(y^{\frac{1}{n}}) \cdot \frac{1}{n} y^{\frac{1}{n}-1} \quad (y \geq 0)$$

$\Rightarrow$ For nonnegative r.v. X, the pdf of $Y = X^n$ is

「Note. when $X(w) \geq 0 \Rightarrow Y(w) = X^n(w) \geq 0$」

$$f_Y(y) = \begin{cases} \frac{1}{n} y^{\frac{1}{n}-1} \cdot f_X(y^{\frac{1}{n}}) & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$$

(3) Example 3. Suppose $X \sim N(\mu, \sigma^2)$.

Let $Y = e^X$. Find pdf of Y: $f_Y(y)$

「Y is usually called Lognormal r.v. which has important applications in FM」

$y = e^x$, i.e. $g(x) = e^x$ Hence $g(x) \uparrow\uparrow$

Note also that $Y = e^X$ can only take positive values.

$\because$ $y = e^x$ $\therefore$ $x = \ln y$ $(y > 0)$

i.e. $g(x) = e^x$, $g^{-1}(y) = \ln y$

$$\Rightarrow \frac{d}{dy} g^{-1}(y) = \frac{d}{dy} \ln y = \frac{1}{y} \quad (\text{for } y > 0)$$

$\Rightarrow$ The pdf of Y:
$$f_Y(y) = f_X[g^{-1}(y)] \cdot \left| \frac{d}{dy} g^{-1}(y) \right|$$
$$= f_X(\ln y) \cdot \frac{1}{y} \quad (\text{for } y > 0)$$

But $X \sim N(\mu, \sigma^2)$

$$\therefore \quad f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (-\infty < x < \infty)$$

Therefore, for $y > 0$,

$$f_Y(y) = f_X(\ln y) \cdot \frac{1}{y} = \frac{1}{y} \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(\ln y - \mu)^2}{2\sigma^2}}$$

Finally,

$$f_Y(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\,\sigma y} e^{-\frac{(\ln y - \mu)^2}{2\sigma^2}} & \text{if } y > 0 \\ 0 & \text{if } y \leq 0 \end{cases}$$

Hence, we have

$$\frac{1}{\sqrt{2\pi}\,\sigma} \int_0^{+\infty} \frac{e^{-\frac{(\ln y - \mu)^2}{2\sigma^2}}}{y} \, dy = 1$$

3. Use cdf method to reobtain the pdf of Lognormal.

Suppose $X \sim N(\mu, \sigma^2)$, and $Y = e^X$. Find $f_Y(\cdot)$.

Let $F_Y(\cdot)$ and $F_X(\cdot)$ be the cdfs of Y and X respectively.

Then $F_Y(y) = \Pr\{Y \le y\} = \Pr\{e^X \le y\}$

If $y \le 0$, then $\{w;\ e^X \le y\} = \phi \Rightarrow F_Y(y) = 0$.

If $y > 0$, then $\{w;\ e^X \le y\} = \{w;\ X \le \ln y\}$.

Hence, if $y > 0$, $F_Y(y) = \Pr\{e^X \le y\} = P\{X \le \ln y\}$

$$\Rightarrow \quad F_Y(y) = F_X(\ln y)$$

$$\Rightarrow \quad \frac{d}{dy} F_Y(y) = F_X'(\ln y) \cdot \frac{d}{dy} \ln y = \frac{1}{y} F_X'(\ln y)$$

$$\Rightarrow \quad f_Y(y) = \frac{1}{y} f_X(\ln y) \qquad (y > 0)$$

But $X \sim N(\mu, \sigma^2) \Rightarrow f_X(\ln y) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(\ln y - \mu)^2}{2\sigma^2}}$

Finally, the pdf of $Y = e^X$ (for $X \sim N(\mu, \sigma^2)$) is

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}\,\sigma y} e^{-\frac{(\ln y - \mu)^2}{2\sigma^2}} & \text{if } y > 0 \\ 0 & \text{if } y \le 0. \end{cases}$$