Assignments (Week 3)

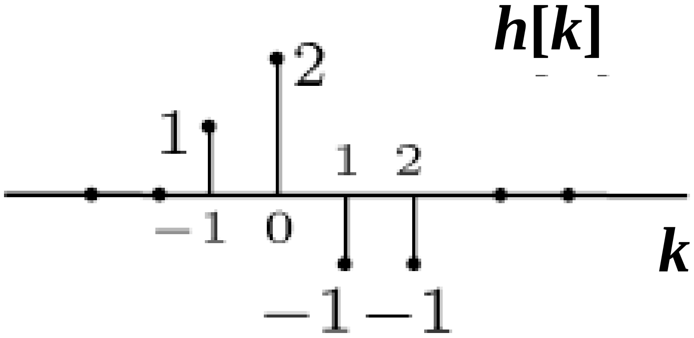
- 2.4
- 2.6
- 2.19
- 2.21 (c) (d)

Tutorial Problems (Week 4)

- Basic Problems with Answers 2.3, 2.7, 2.13
- Basic Problems 2.24, 2.26

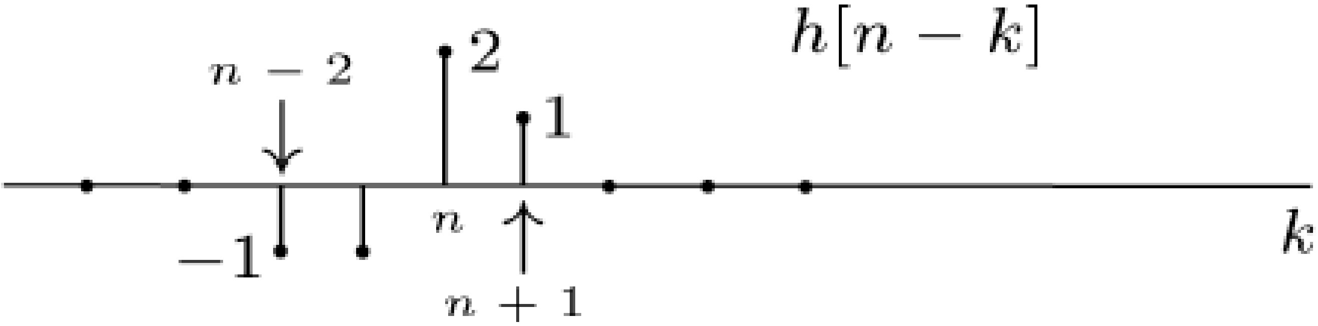
Ch 1 Review (1)

- **Time-shift and flip**



What is the plot for *h*[*n*-*k*]??
***n* is a constant**

$$h[k] \xrightarrow{\text{Flip}} h[-k] \xrightarrow{\text{Slide}} h[n-k]$$

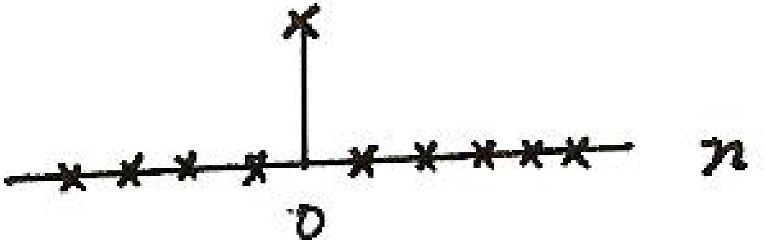


Ch 1 Review (2)

- **Unit impulse function (unit sample function)**

Discrete-time

$$\delta[n]=\begin{cases}0, & n\neq 0\\ 1, & n=0\end{cases}$$



- We can use unit impulse function to represent any other different signals, or it is a building function (or basic signal).

Ch 1 Review (3)

System properties:

1. **With memory or memoryless**

   y(n)=f (x(n))

2. **Invertible**

   for a system x→y, if $x_1 \neq x_2$, then $y_1 \neq y_2$

3. **Causal**

   … up to that time n …

4. **Stable**

   either prove the system is table, or find a specific counterexample

Ch 1 Review (4)

5. Time-invariant

If $x[n] \rightarrow y[n]$

then $x[n - n_0] \rightarrow y[n - n_0]$.

6. Linear

A (CT) system is linear if it has the $\boxed{\text{superposition property:}}$

If $x_1(t) \rightarrow y_1(t)$ and $x_2(t) \rightarrow y_2(t)$

then $ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$

Chapter 2

Linear Time-invariant (LTI) Systems

Exploiting Superposition and Time-Invariance

If we have $x_k[n] \rightarrow y_k[n]$, then

$$x[n] = \sum_k a_k x_k[n] \xrightarrow{\text{Linear System}} y[n] = \sum_k a_k y_k[n]$$

Question: Are there sets of "basic" signals $x_k[n]$ such that

a) We can represent *any* signals as linear combinations of these building block signals.

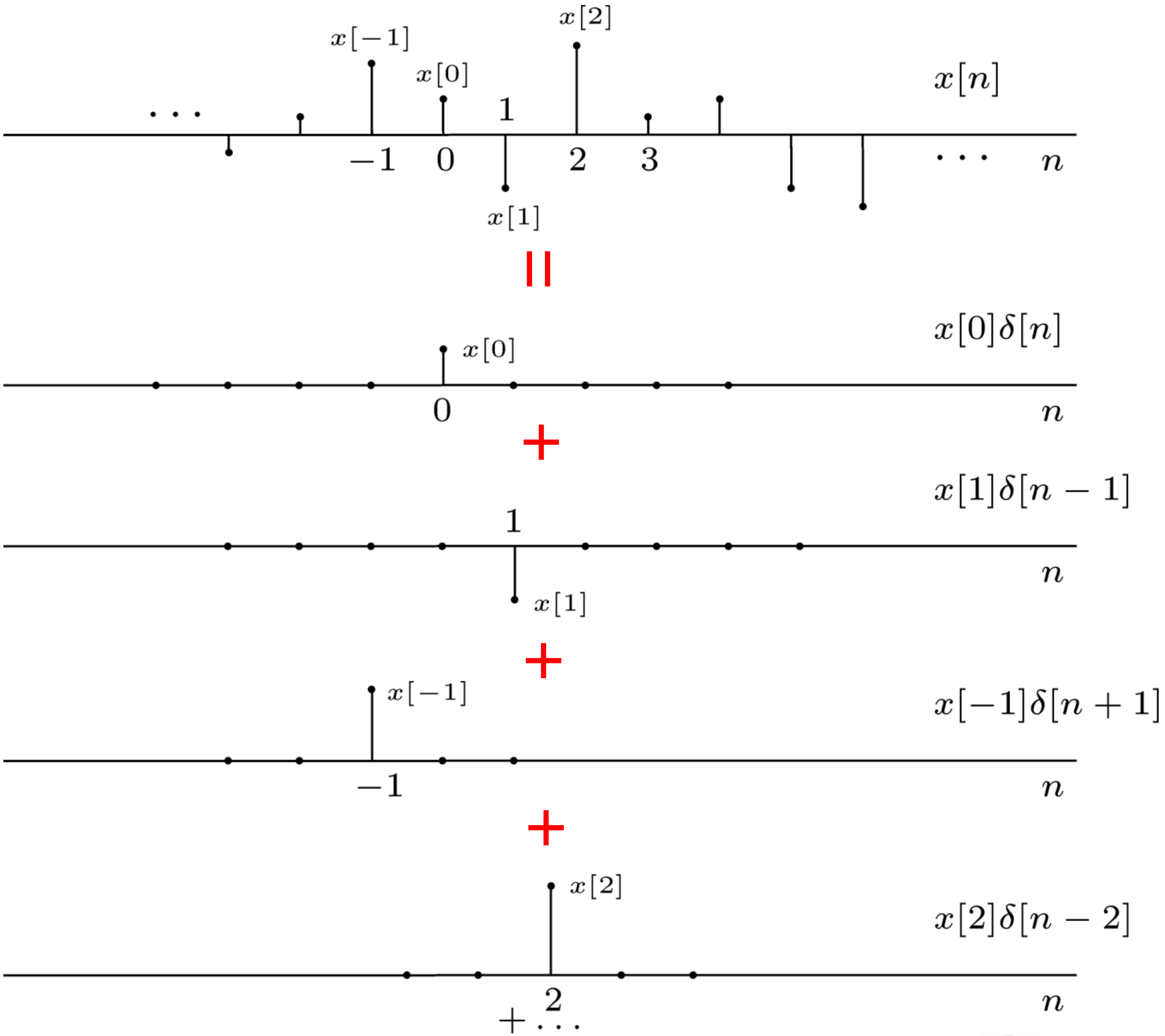
b) The response of LTI Systems to these basic signals are both *simple* and *insightful*.

Fact: For LTI Systems (CT or DT) there are two natural choices for these building blocks

Focus for now: DT Shifted unit samples $\delta[n-n_o]$

Next time: CT Shifted unit impulses $\delta(t-t_o)$

Representation of DT Signals Using Unit Samples



That is ...

$$x[n] = \ldots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \ldots$$

$$\Downarrow$$

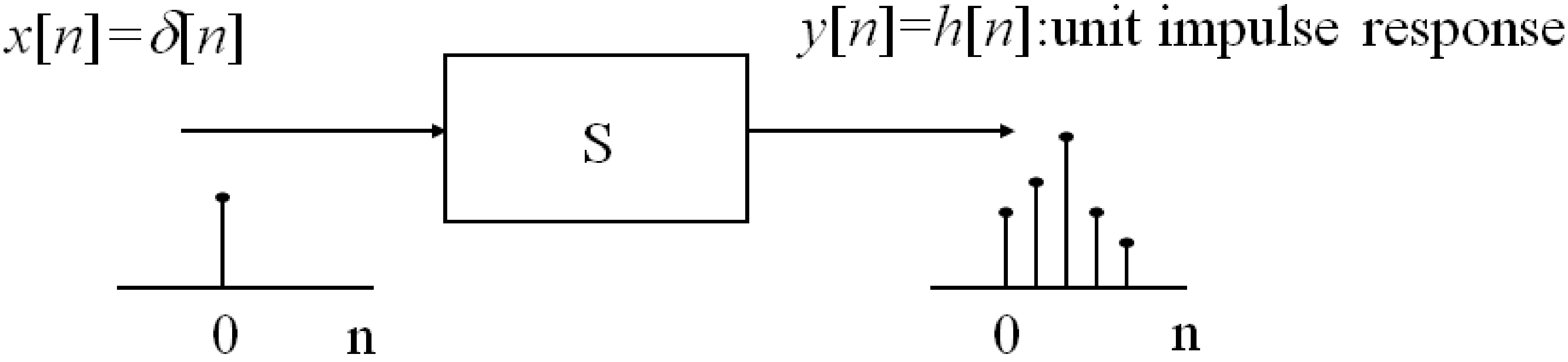
Important to note the "-" sign

$$x[n] = \sum_{k=-\infty}^{+\infty} \underbrace{x[k]}_{\text{Coefficients}}\underbrace{\delta[n-k]}_{\text{Basic Signals}}$$

- **Sifting property** of the unit impulse: looked at the index *k*, $\delta[n-k]$ is nonzero only at $k = n$, which "sifts" the value $x[n]$ out of the function $x[k]$.

Unit Impulse Response (Unit Sample Response)

- Define the output for an unit impulse input as the unit impulse response



Example: $y[n]=x[n]+2x[n-1]+4x[n-2]$

What is unit impulse response?

Response of DT LTI Systems

$$x[n] \longrightarrow \boxed{\text{DT System}} \longrightarrow y[n]$$

- Now suppose the system is **LTI**, and define the *unit impulse response* $h[n]$:

$$\delta[n] \longrightarrow h[n]$$

$$\Downarrow$$

From Time-Invariance:

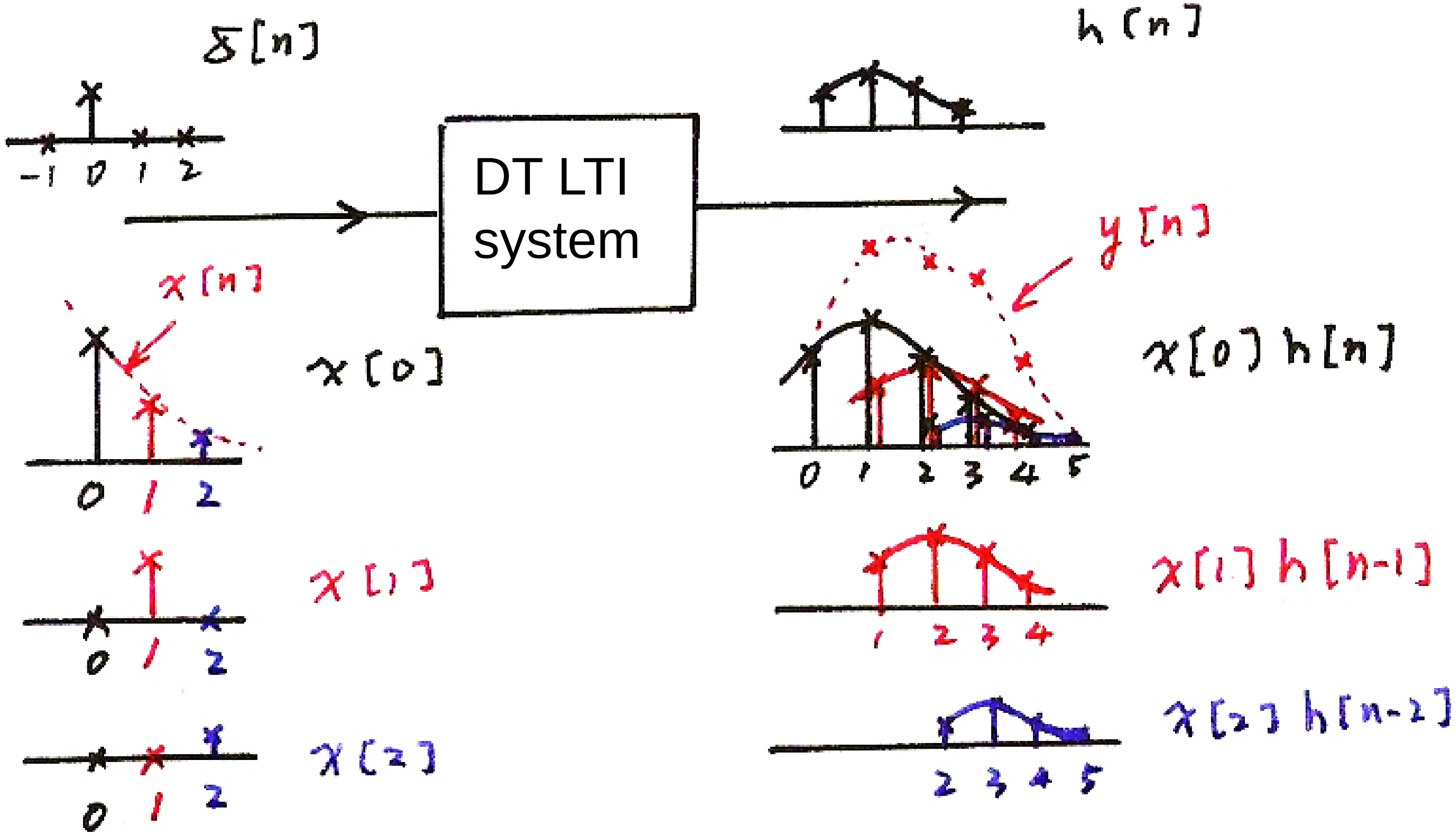
$$\delta[n-k] \longrightarrow h[n-k]$$

From Linearity:

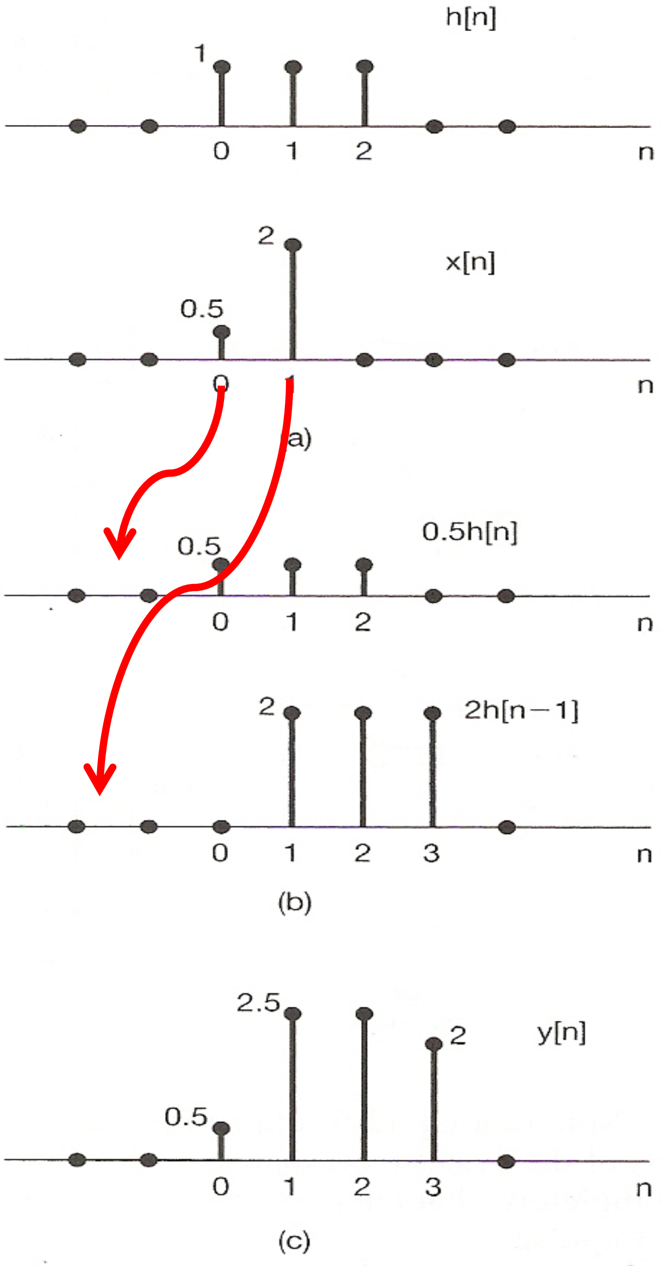
$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k] \longrightarrow y[n] = \underbrace{\sum_{k=-\infty}^{+\infty} x[k]\,h[n-k]}_{\text{convolution sum}} = x[n] * h[n]$$

The output for an arbitrary input signal is the superposition of a series of “shifted, scaled unit impulse response”

Input/Output Relation


DT LTI system

Example



$$y[n]=0.5h[n]+2h[n-1]$$

Figure 2.3 (a) The impulse response $h[n]$ of an LTI system and an input $x[n]$ to the system; (b) the responses or "echoes," $0.5h[n]$ and $2h[n-1]$, to the nonzero values of the input, namely, $x[0] = 0.5$ and $x[1] = 2$; (c) the overall response $y[n]$, which is the sum of the echos in (b).

Hence a Very Important Property of LTI Systems:

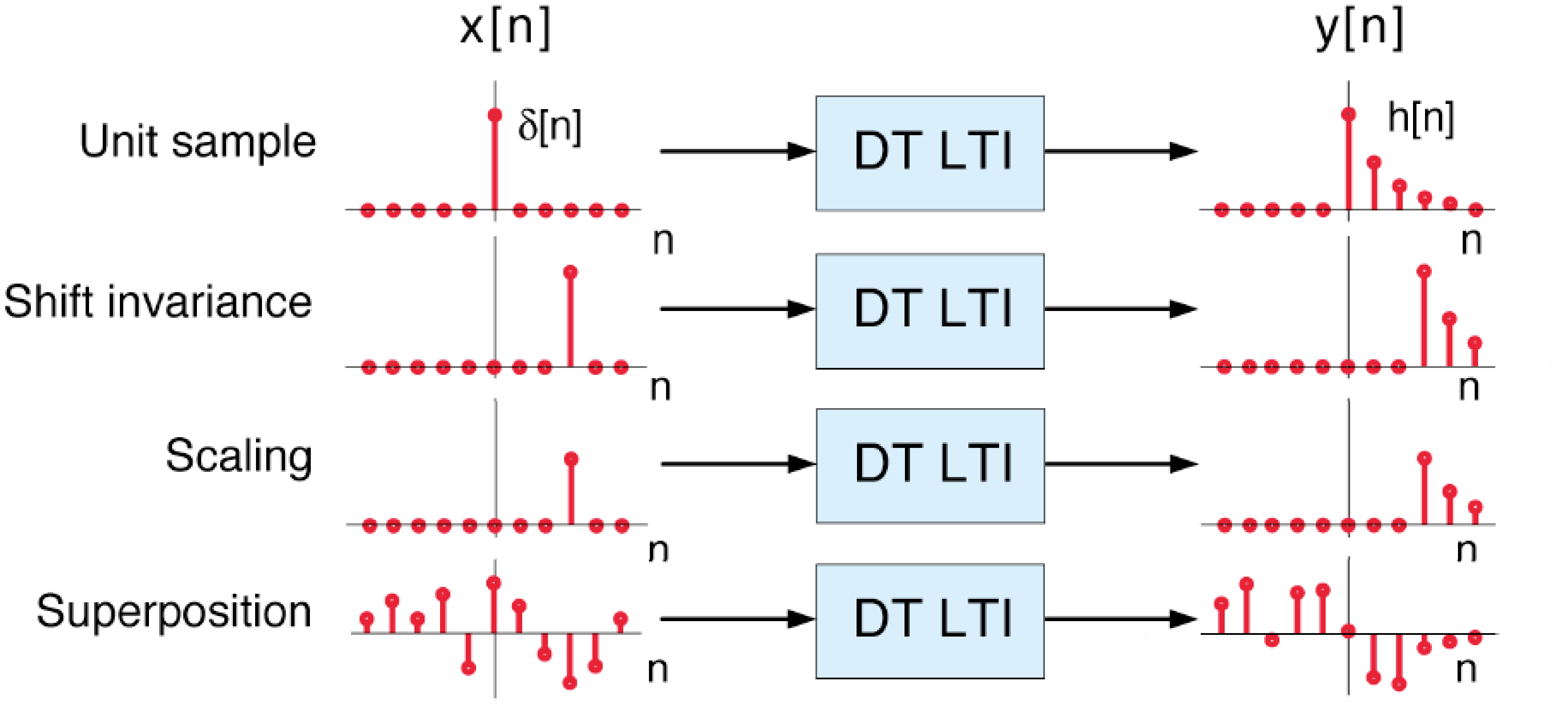
The output of *any* DT LTI System is a convolution of the input signal with the unit-sample response, *i.e.*

$$\text{Any DT LTI} \longleftrightarrow y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

As a result, any DT LTI Systems are *completely characterized* by its unit sample response

Graphic View of the Convolution Sum
Response of DT LTI systems



$$x[n]=\sum_{k=-\infty}^{+\infty}x[k]\,\delta[n-k]\xrightarrow{\text{DT LTI}}y[n]=\underbrace{\sum_{k=-\infty}^{+\infty}x[k]\,h[n-k]}_{convolution\ sum}$$

- A different way to visualize the convolution sum
  - looked at on the index *k*

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]$$

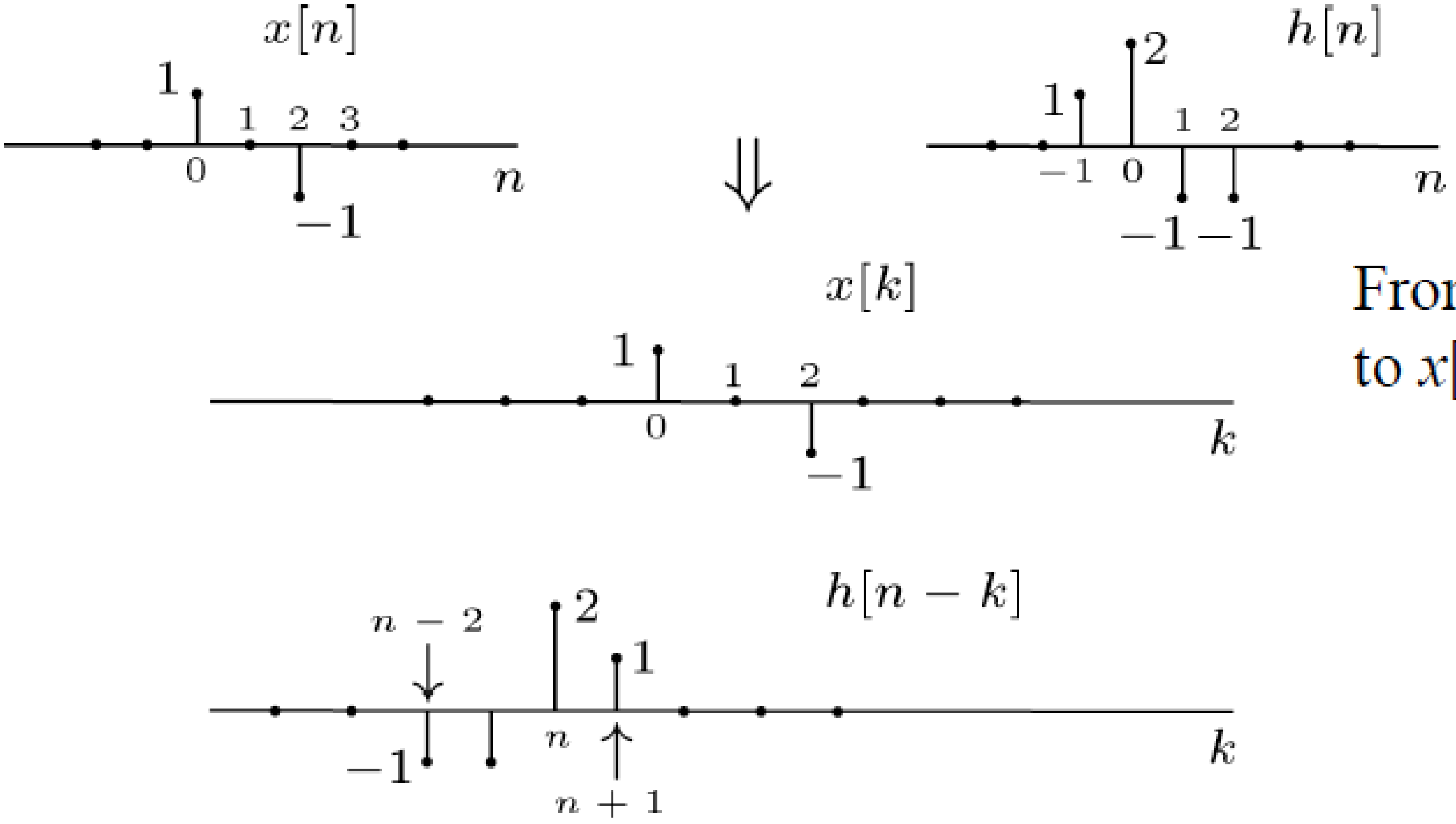
input signal

Contribution to the
output signal at time *n*

flipped version of *h*[*k*] located at *k* = *n*

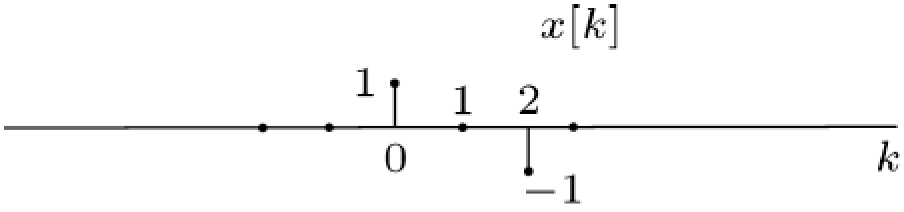
  - on the dummy index *k*, *h*[*k*] is flipped over and shifted to *k*=*n*, weighted by *x*[*k*], and summed to produce an output sample *y*[*n*] at time *n*

Example: Flip h[n]

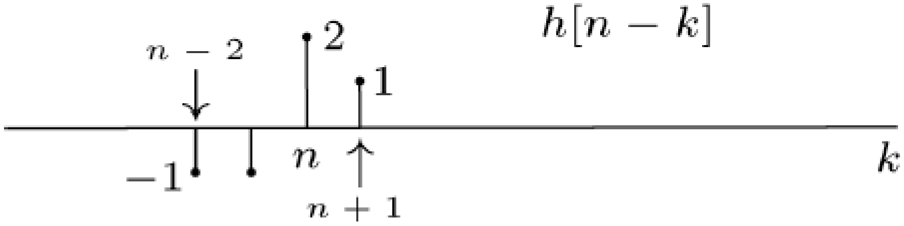


From $x[n]$ and $h[n]$ to $x[k]$ and $h[n\text{-}k]$

Calculating Successive Values: Shift, Multiply, Sum



$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$



$y[n] = 0 \quad$ for $n <$

$y[-1] =$

$y[0] =$

$y[1] =$

$y[2] =$

$y[3] =$

$y[4] =$

$y[n] = 0 \quad$ for $n >$

Convolution operation procedure:

$$h[k] \xrightarrow{\text{Flip}} h[-k] \xrightarrow{\text{Slide}} h[n-k] \xrightarrow{\text{Multiply}} x[k]h[n-k]$$

$$\xrightarrow{\text{Sum}} \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

F S M S

$$\Big\Downarrow$$

$$y[n]$$

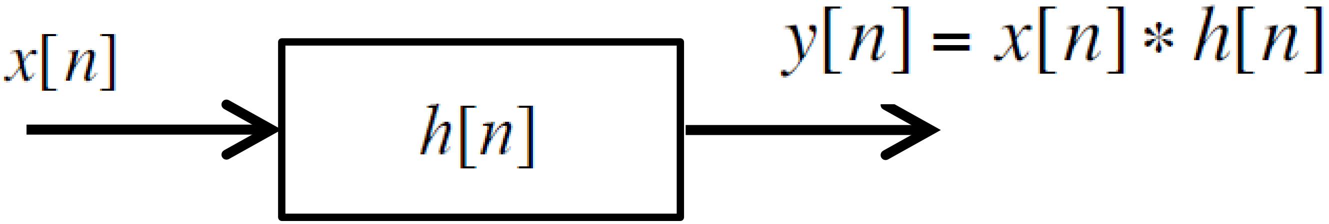
$$\text{Any DT LTI} \longleftrightarrow y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

$x[n]$: input

$y[n]$: output

$h[n]$: impulse response of the system



$x[n]$

$h[n]$

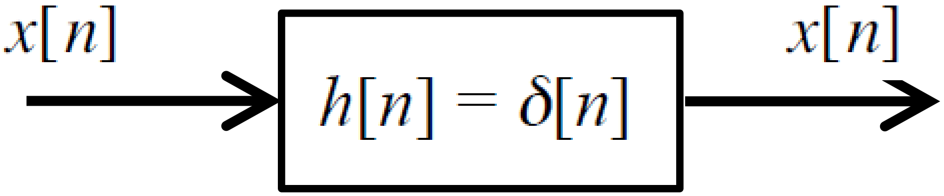
$y[n] = x[n] * h[n]$

Examples of Convolution and DT LTI Systems

Ex. #1: $h[n] = \delta[n]$

$$y[n] = x[n] * \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$= x[n] \quad \text{— An Identity system}$$

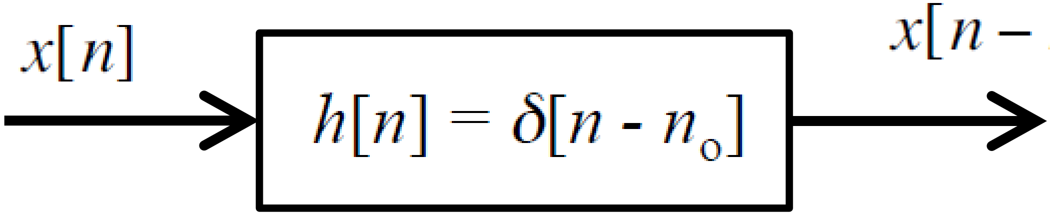


– sifting property, i.e., convolution sum (or integral) with an unit impulse function gives the original signal

Ex. #2: $h[n] = \delta[n - n_o]$

$$y[n] = x[n] * \delta[n - n_o] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - n_o - k]$$

$$= x[n - n_o] \quad — \quad \text{A Shift}$$



$x[n]$

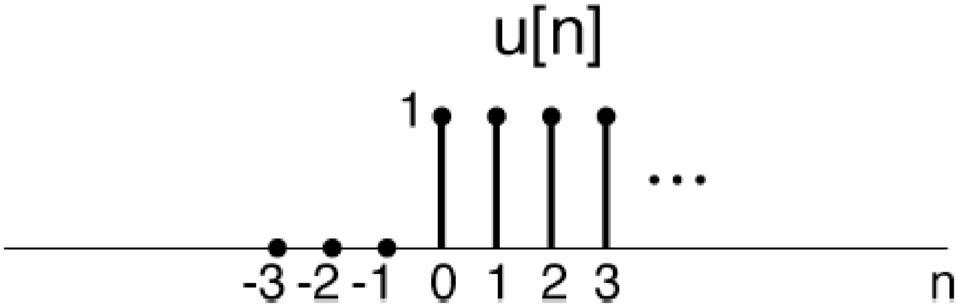
$h[n] = \delta[n - n_o]$

$x[n - n_o]$

Ex. #3 $y[n]=\sum_{k=-\infty}^{n} x[k]$ – An accumulator

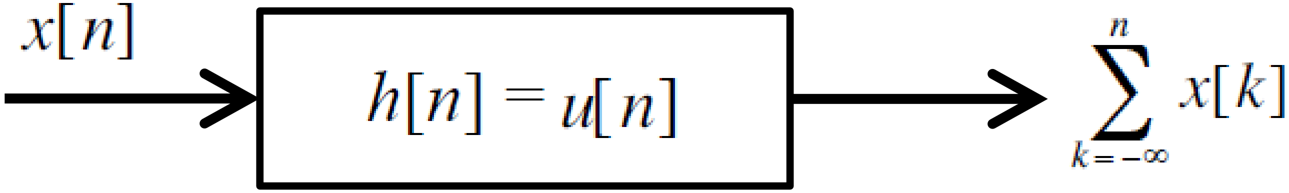
Unit Sample response

$$h[n]=\sum_{k=-\infty}^{n}\delta[k]=u[n]$$



$$\Downarrow$$

$$x[n]*u[n]=\sum_{k=-\infty}^{n} x[k]$$


$x[n]$

$h[n]=u[n]$

$\sum_{k=-\infty}^{n} x[k]$

Ex. #4 (Example 2.3)

$0<\alpha<1$

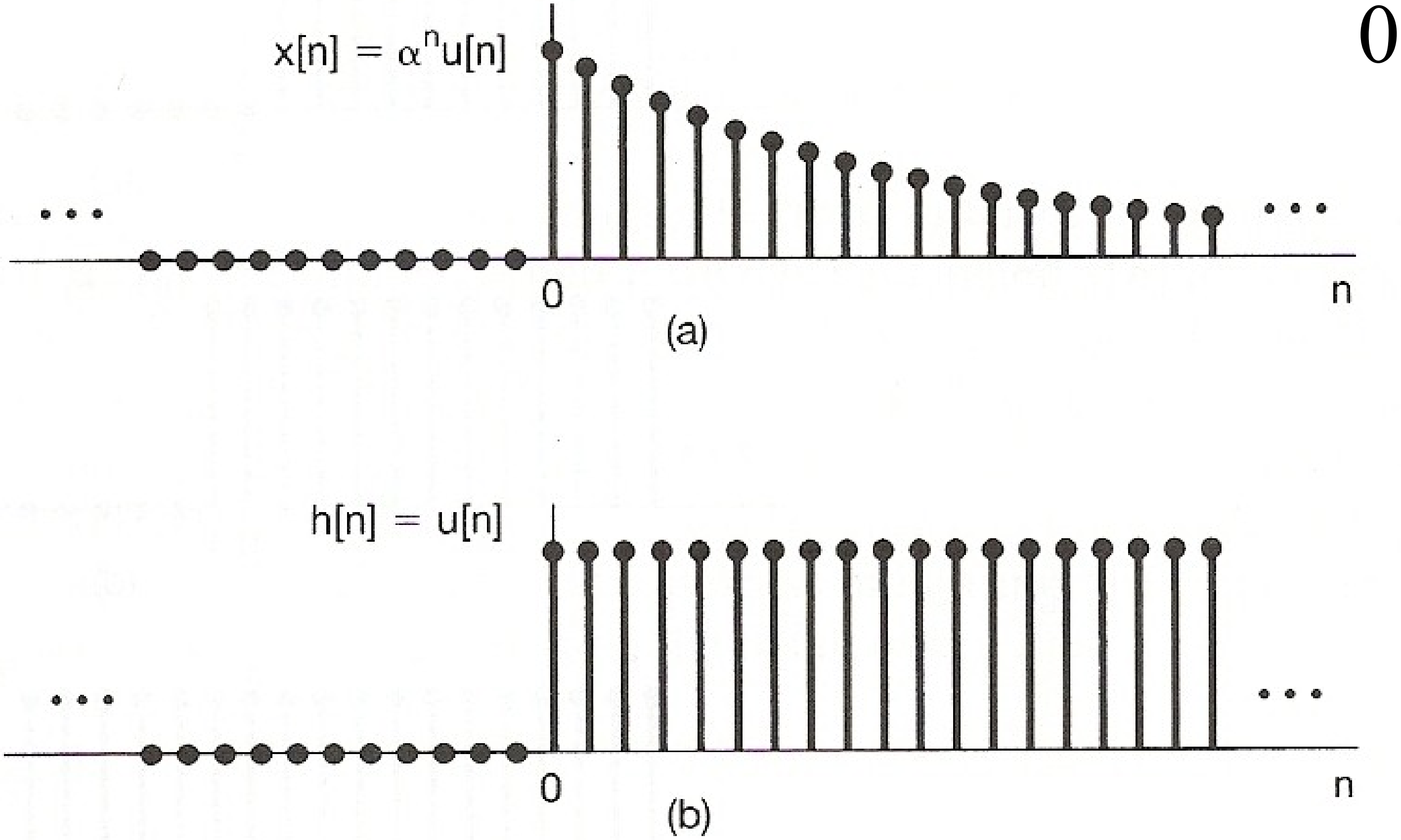
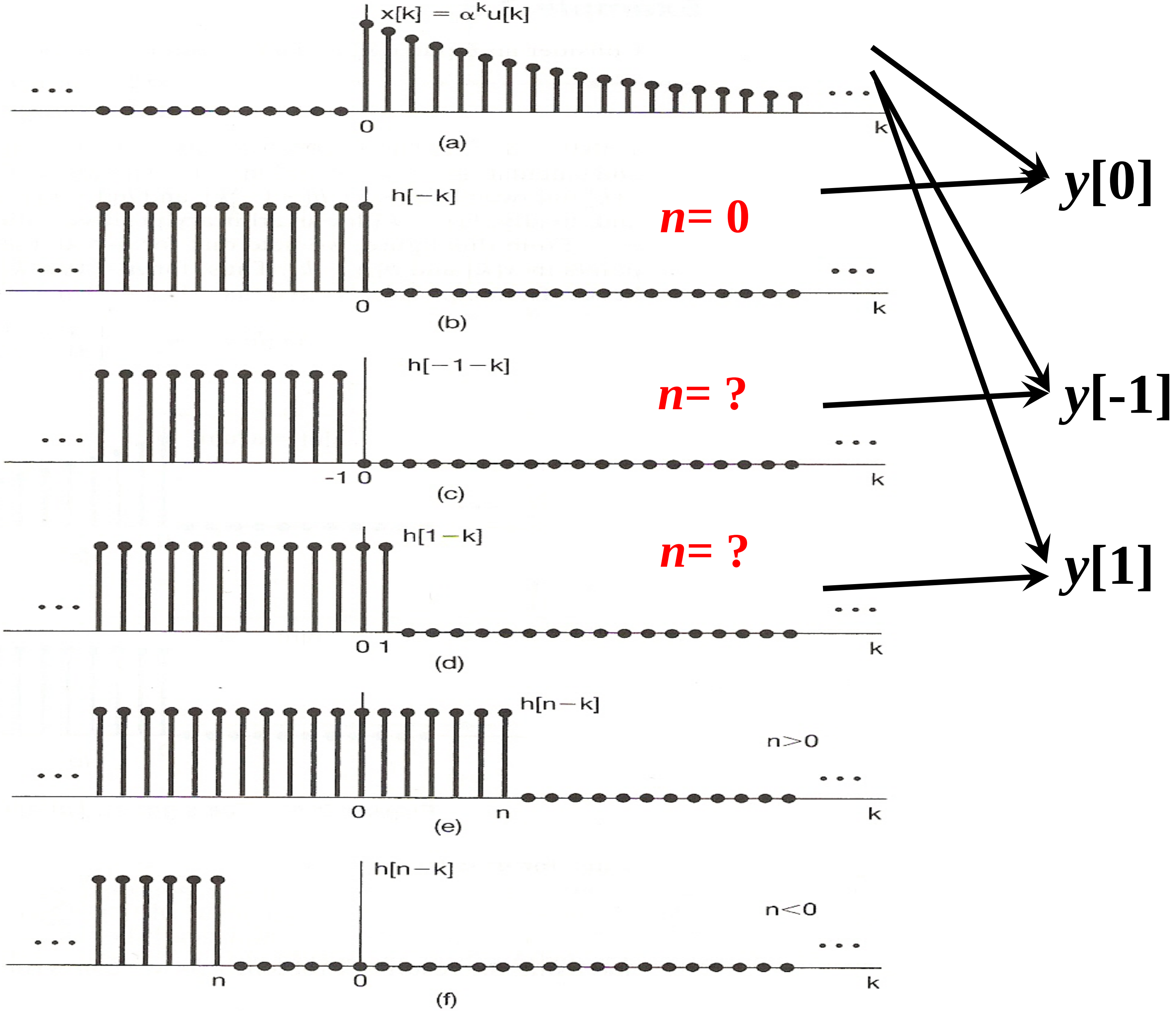


Figure 2.5 The signals $x[n]$ and $h[n]$ in Example 2.3.



***n*= 0**

***n*= ?**

***n*= ?**

***y*[0]**

***y*[-1]**

***y*[1]**

Figure 2.6 Graphical interpretation of the calculation of the convolution sum for Example 2.3.

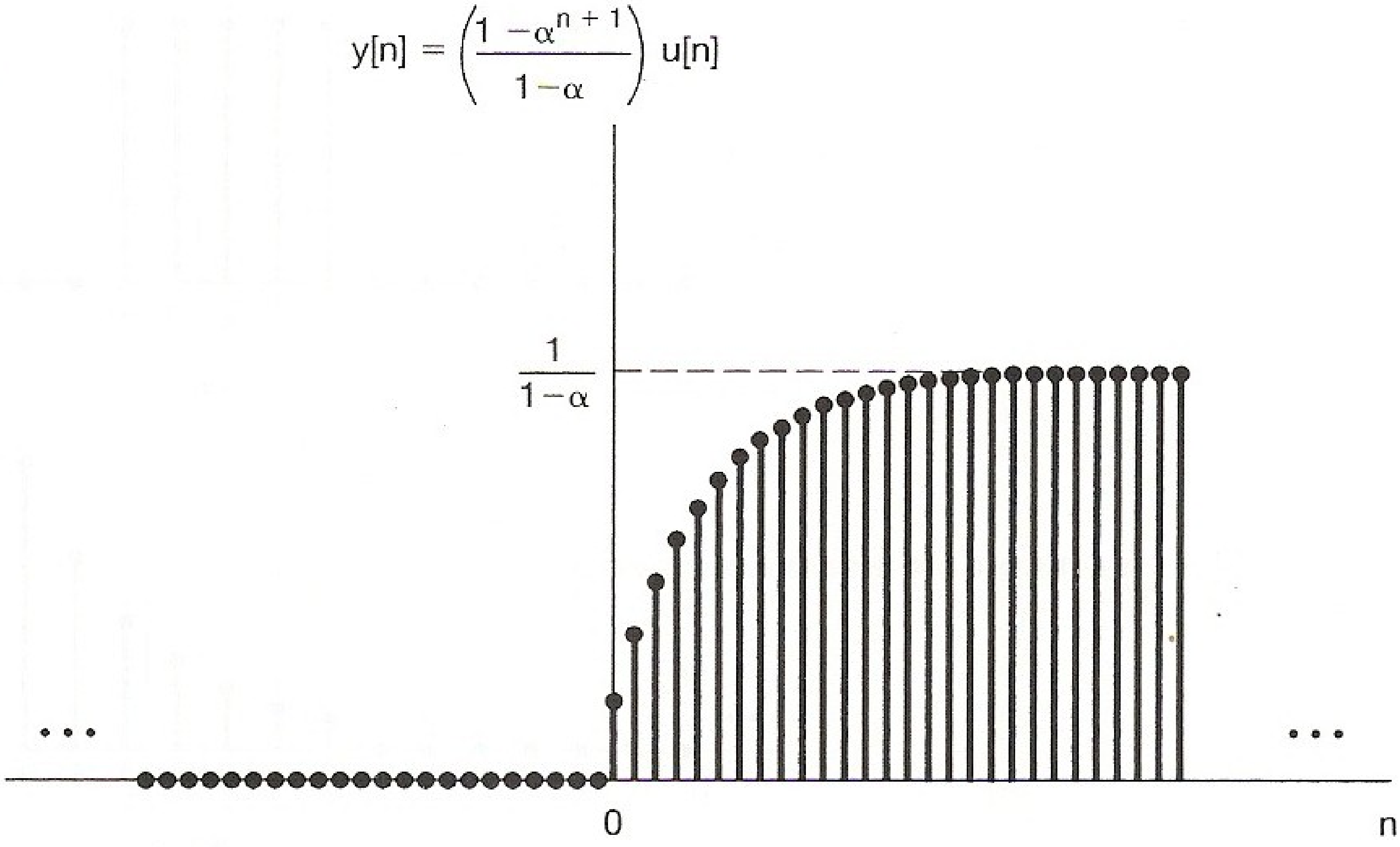


Figure 2.7 Output for Example 2.3.

- **Characteristics of an LTI system are completely determined by its impulse response.**
- ***What if the system is nonlinear?***

Consider a discrete-time system with unit impulse response

$$h[n] = \begin{cases} 1 & n = 0,1 \\ 0 & \text{otherwise} \end{cases}$$

If the system is LTI, the input/output relationship is

$$y[n] = x[n] + x[n-1].$$

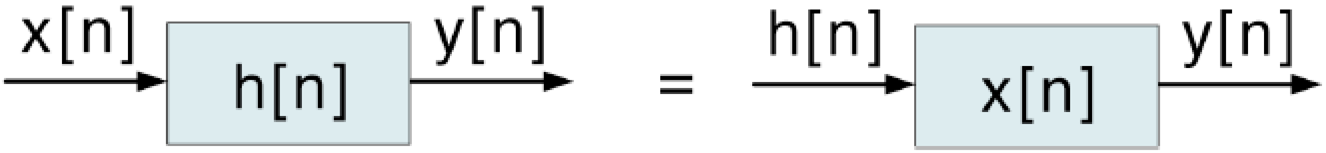
On the other hand, there are ***many*** nonlinear systems with the same response to the input $\delta[n]$.

$$y[n] = (x[n] + x[n-1])^2,$$

$$y[n] = \max(x[n], x[n-1]).$$

The Commutative Property of Convolution

$$y[n] = x[n] * h[n] \quad = \quad h[n] * x[n]$$



Example: Step response $s[n]$ of an LTI system

input: unit step function

$$s[n] = u[n] * h[n] = \underset{\substack{\uparrow \\ \text{“Input”}}}{h[n]} * \underset{\substack{\uparrow \\ \text{Unit Sample response} \\ \text{of accumulator}}}{u[n]}$$

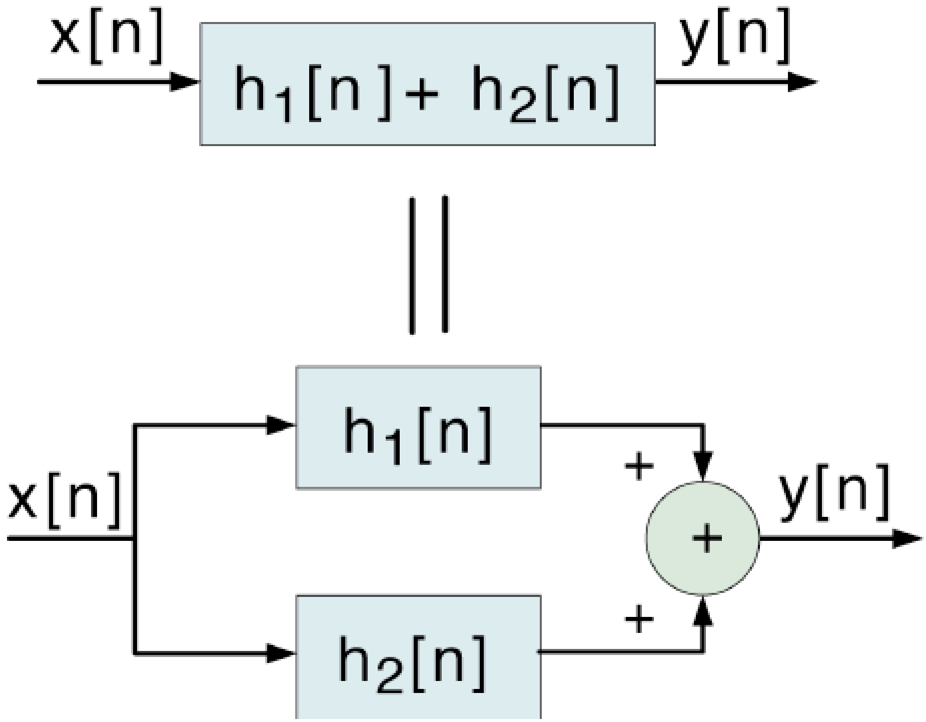
$$\Downarrow$$

$$s[n] = \sum_{k=-\infty}^{n} h[k]$$

The Distributive Property of Convolution

$$x[n]*\{h_1[n]+h_2[n]\}=x[n]*h_1[n]+x[n]*h_2[n]$$

Interpretation



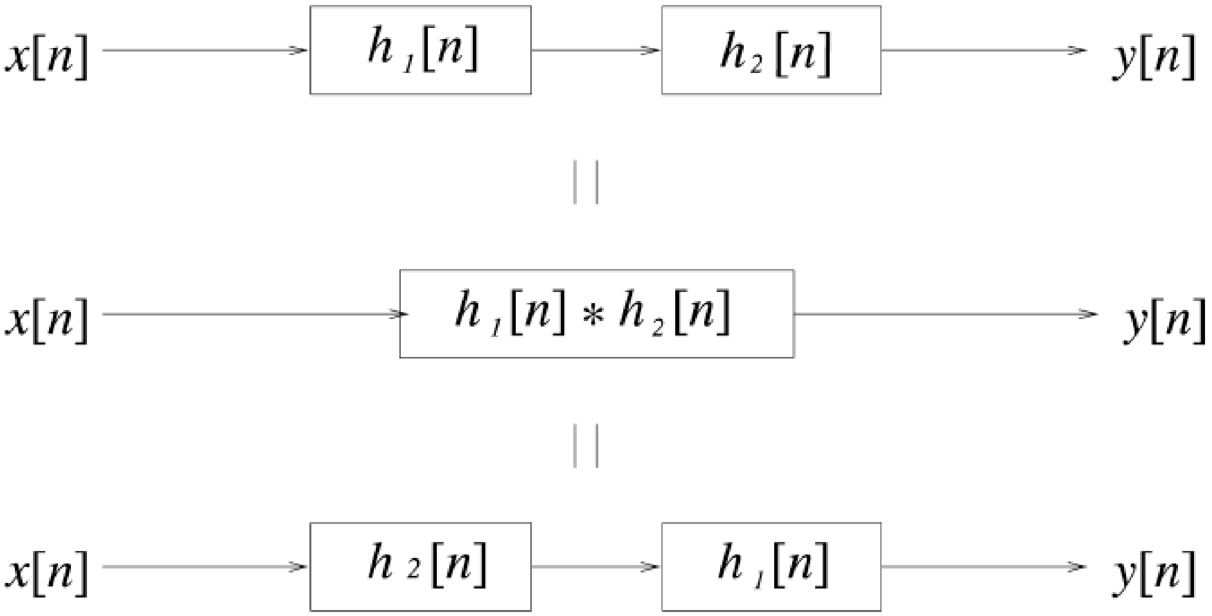
The Associative Property of Convolution

$$x[n]*(h_1[n]*h_2[n]) = (x[n]*h_1[n])*h_2[n]$$

(Commutativity) $\|$

$$x[n]*(h_2[n]*h_1[n]) = (x[n]*h_2[n])*h_1[n]$$

Implication (Very special to LTI Systems)



Properties of Convolution

Combining the Commutative property,

$$y[n] = x[n] * h[n] \quad = \quad h[n] * x[n]$$

Distributive property,

$$x[n] * \{h_1[n] + h_2[n]\} = x[n] * h_1[n] + x[n] * h_2[n]$$

and Associative property,

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$$

symbolically, we can treat “$*$” as a “×”. Easy, piece of cake!

The hard part is the actual calculation of the convolution.

Flip → Slide → Multiply → Sum.

Soon we will develop a clever way (*transformation*) to perform “×” instead of “$*$” operation.

Some Useful Properties of LTI Systems

1) Causality $\Leftrightarrow$ $h[n] = 0$ for all $n < 0$

2) Stability $\Leftrightarrow$ $\sum_{k=-\infty}^{+\infty} |h[k]| < \infty$

BIBO — Bounded Input $\Rightarrow$ Bounded Output

→ Sufficient condition: For $|x[n]| \le x_{\max} < \infty$.

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} x[k]h[n-k] \right| \le x_{\max} \left| \sum_{k=-\infty}^{\infty} h[n-k] \right| < \infty.$$

→ Necessary condition: If $\sum_{k=-\infty}^{\infty} |h[k]| = \infty$

Let $x[n] = h^*[-n]/|h[-n]|$, then $|x[n]| \equiv 1$ bounded

But $y[0] = \sum_{k=-\infty}^{\infty} x[k]h[-k] = \sum_{k=-\infty}^{\infty} h^*[-k]h[-k]/|h[-k]| = \sum_{k=-\infty}^{\infty} |h[-k]| = \infty$

- Memoryless / with Memory
  - A linear, time-invariant, causal system is memoryless only

if $h[n] = K\delta[n]$ $\quad$ $h(t) = K\delta(t)$

$y[n] = Kx[n]$ $\quad$ $y(t) = Kx(t)$

if k=1 further, they are identity systems

$$y[n] = x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = x[n] * \delta[n]$$

$$y(t) = x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t) * \delta(t)$$

Summary

Understand the following new concepts:

1. Use unit impulse function to represent any function
2. Unit impulse response h[n]
   - ◆ Given the system input/output equation, how to decide the unit impulse response?
3. Convolution, its properties, and calculation steps (FSMS)
   - ◆ Understand the meaning of index ‘k’ and index ‘n’
4. Decide LTI system property by using unit impulse response h[n]