2. Linear Time-invariant(LI) Systems

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1 Discrete-Time LTI System: The Convolution Sum

Discrete-Time Signals can be represented in terms of impulses:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

Where $x(k)$ is a coeffecients, $\delta[n-k]$ is a function of n, basic signal, k is time-shift.

We define the output for an unit impluse input as the unit impulse response

$$x[n] = \delta[n] \xrightarrow{\text{System}} y[n] = h[n]\text{: unit impulse response}$$

Now suppose the system is **LTI**, and define the *unit impulse response h(n)*. By $h(n)$, we can easy to analyse the profile of system, try to find out the output.

$$\delta[n] \xrightarrow{\text{System}} h[n]$$
$$\Downarrow$$
$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \xrightarrow{\text{System}} y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = x[n] * h[n]$$

The output for an input signal is the superpositin of a series of "shifted, scaled unit impulse response"

If look the index k.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h(n-k)$$

Represent the contribution of $x[k]$ to the output at time n.

$$x[n] \xRightarrow{h(n)} y(n) = x(n) * h(n)$$

Example:

1. $h[n] = \delta[n]$:

$$y[n] = x[n]$$

2. $h[n] = \delta[n-n_0]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n-n_0]$$

3. $h[n] = u[n] = \sum_{k=-\infty}^{n} \delta[n-k]$

$$y[n] = \sum_{k=-\infty}^{n} x[n]$$

Commutative Property:

$$y[n] = x[n] * h[n] = h[n] * x[n]$$

Distributive Property:（⟶并联系统）

$$x[n] * \{h_1[n] + h_2[n]\} = x[n] * h_1[n] + x[n] * h_2[n]$$

Associative Propery: (⟶Very special to LTI 串联系统)

$$x[n] * (h_1[n] * h_2[n]) = (x[n) * h_1[n]) * h_2[n]$$

1. Causality $\iff$ $h[n] = 0$ for all $n < 0$
2. Stability $\iff \sum_{k=-\infty}^{+\infty} |h[k]| < \infty$(也就是绝对收敛)
3. Memoryless

$$h[n] = K\delta[n]$$

2 Continuous-Time: The Convolution Integral

$$\delta(t) = \begin{cases} 0 & \text{for } t \neq 0 \\ \infty & \text{for } t = 0 \end{cases}$$
$$\int_{0^-}^{0^+} \delta(t) dt = 1$$

Construction of the Unit-impulse function $\delta(t)$

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \leq t < \Delta \\ 0, & \text{otherwise} \end{cases}$$

Since, $\Delta\delta_\Delta(t)$ has unit amplitude, we have the expression

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\Delta\delta_\Delta(t - k\Delta)$$
$$\Big\Downarrow \text{limit as } \Delta \longrightarrow 0$$
$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$$
$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t - \tau) d\tau$$

Nwo suppose the sytem if LTI, and define the *unit impluse response h(t)*:

$$\delta(t) \longrightarrow h(t)$$

For Time-Invaiance and Linearity:

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau \longrightarrow y(t) = \int_{\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t-\tau)$$

The properties:

1. Commutative: $x(t) * h(t) = h(t) * x(t)$

2. Distributive:

   同一个信号：$y(t) = x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$

   同一个系统: $[x_1(t) + x_2(t)]h(t) = x_1(t)h(t) + x_2(t)h(t) y(t) = x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$

   并联系统

3. Associative: $y(t) = [x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$

   串联

   independent of the order

4. Memory / Mmemoryless

5. Invertibility

6. Causality

Bloclk diagram representation of 1-st order system

延迟一次: 一阶系统