Notes

Assignments

- 5.2, 5.5, 5.15, 5.21(a-f, h)

Tutorial problems

- 5.1, 5.3, 5.4, 5.41

Mid-term examination

- **Time: Nov. 10 (Saturday) 7:00-9:00 pm**
- **Venue: 第一教学楼 107 110 108**
- **Range: Chapters 1-4**
- **Allow: one (A4) page note**
- **Problem language: English**
- **Final examination: 40% for Chapters 1-4**

The CT Fourier Transform Pair

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

— *FT*

Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$

— Inverse *FT*

Inverse Fourier Transform

With CTFT, now the frequency response of an LTI system makes complete sense

$$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow y(t)=\sum_{k=-\infty}^{\infty} H(jk\omega_0)a_k e^{jk\omega_0 t}$$

$$a_k \longrightarrow \underbrace{H(jk\omega_0)}_{\text{"gain"}}a_k$$

$$\boxed{H(j\omega)=\int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt}$$

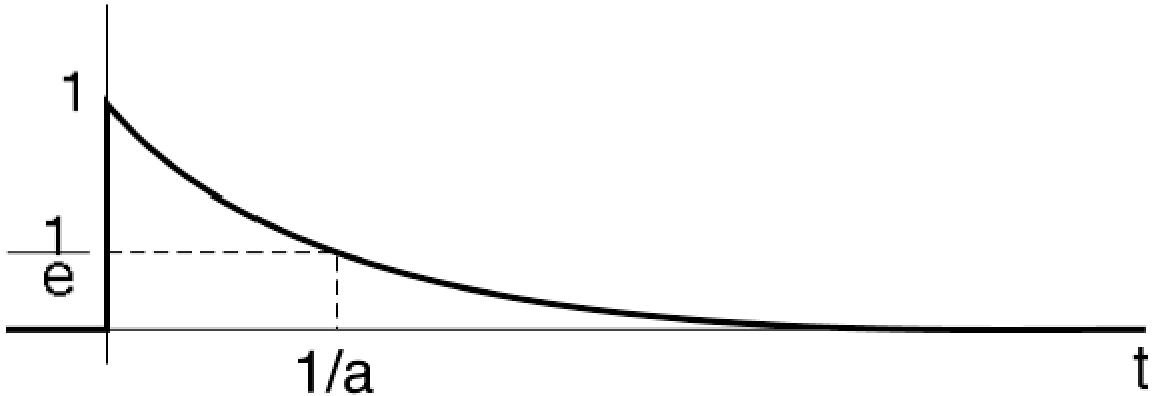
Impulse response $\overset{\mathcal{F}}{\longleftrightarrow}$ Frequency response

CTFT Properties
8) Convolution Property

$$y(t) = h(t) * x(t) \longleftrightarrow Y(j\omega) = H(j\omega)X(j\omega)$$

$$\text{where} \quad h(t) \longleftrightarrow H(j\omega) \quad x(t) \longleftrightarrow X(j\omega)$$

Review from the last lecture, right-sided exponential



$$x(t) = e^{-at}u(t)\ , \quad a > 0$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt = \int_{0}^{+\infty} \underbrace{e^{-at}e^{-j\omega t}}_{e^{-(a+j\omega)t}}dt$$

$$= -\left(\frac{1}{a+j\omega}\right)e^{-(a+j\omega)t}\Big|_0^{\infty} = \boxed{\frac{1}{a+j\omega}}$$

Example 4.19

$$h(t) = e^{-t}u(t) \ , \ x(t) = e^{-2t}u(t)$$

$$y(t) = h(t) * x(t) = ?$$

$$\Downarrow$$

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{1}{(1+j\omega)} \cdot \frac{1}{(2+j\omega)}$$

$$\Downarrow \boxed{\text{Partial fraction expansion}}$$

$$Y(j\omega) = \frac{1}{1+j\omega}\ a=1 - \frac{1}{2+j\omega}\ a=2$$

$$\Downarrow \text{inverse } FT$$

$$y(t) = [e^{-t} - e^{-2t}]u(t)$$

CTFT Properties
9) Multiplication Property

Since *FT* is highly symmetric,

$$x(t) = \mathfrak{F}^{-1}\{X(j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega, \quad X(j\omega) = \mathfrak{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

thus if $\quad x(t) * y(t) \longleftrightarrow X(j\omega)\cdot Y(j\omega)$

then the other way around is also true

$$\boxed{\begin{aligned} x(t)\cdot y(t) &\longleftrightarrow \frac{1}{2\pi} X(j\omega) * Y(j\omega) \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta) Y(j(\omega-\theta))\, d\theta \end{aligned}}$$

$\frac{1}{2\pi}$ — A consequence of *Duality*

Examples of the Multiplication Property: Modulation Property

Frequency shift

$$e^{j\omega_0 t}x(t) \xleftrightarrow{F} X\left(j(\omega-\omega_0)\right)$$

$$e^{j\omega_0 t}\cdot x(t) \leftrightarrow \frac{1}{2\pi}[2\pi\delta(\omega-\omega_0) * X(j\omega)]$$
$$= X(j(\omega-\omega_0))$$

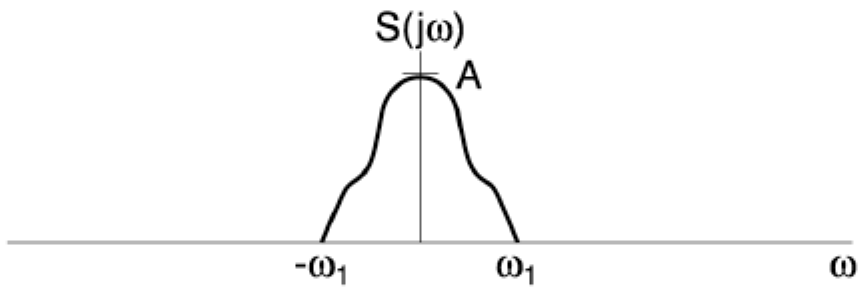
Example 4.21

$$r(t) = s(t)\cdot p(t) \longleftrightarrow R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j\omega)]$$

For $p(t) = \cos\omega_0 t \longleftrightarrow P(j\omega) = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$

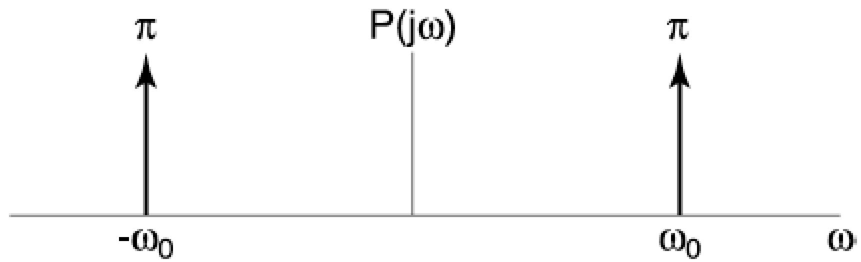
$$R(j\omega) = \frac{1}{2}S(j(\omega-\omega_0)) + \frac{1}{2}S(j(\omega+\omega_0))$$

(cont.)

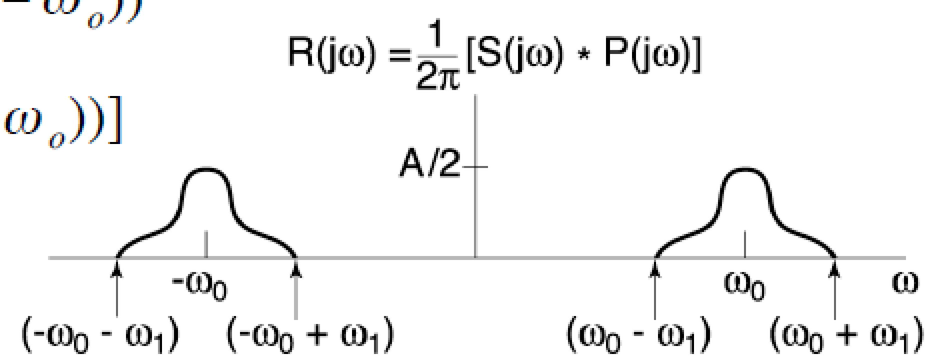


ω_1 : bandwidth

$r(t) = s(t)\cdot\cos(\omega_o t)$
Amplitude modulation (*AM*)



$$R(j\omega) = \frac{1}{2}[S(j(\omega-\omega_o)) + S(j(\omega+\omega_o))]$$



Drawn assume

$\omega_o - \omega_1 > 0$

i.e. $\omega_o > \omega_1$

Example #8: LTI Systems Described by LCCDE's

(Linear-constant-coefficient differential equations)

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Using the Differentiation Property

$$\frac{d^k x(t)}{dt^k} \longleftrightarrow (j\omega)^k X(j\omega)$$

$\Downarrow$ Transform both sides of the equation

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

$$\Downarrow$$

$$Y(j\omega) = \underbrace{\left[\frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}\right]}_{H(j\omega)} X(j\omega)$$

$$\boxed{H(j\omega) = \left[\frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}\right]}$$

Problem 4.34

- A causal and stable LTI system S has the frequency response

$$H(j\omega)=\frac{j\omega+4}{6-\omega^2+5j\omega}$$

  - Determine a differential equation relating the input $x(t)$ and output $y(t)$ of S
  - Determine the impulse response $h(t)$ of S
  - What is the output of S when the input is $x(t)=e^{-4t}u(t)-te^{-4t}u(t)$

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①

$$\frac{Y(j\omega)}{X(j\omega)}=H(j\omega)=\frac{j\omega+4}{6-\omega^2+5j\omega}$$

$$Y(j\omega)\cdot(6-\omega^2+5j\omega)=X(j\omega)\cdot(j\omega+4)$$

$$\Updownarrow$$

$$\frac{d^2y(t)}{dt^2}+5\frac{dy(t)}{dt}+6y(t)=\frac{dx(t)}{dt}+4x(t)$$

②

$$H(j\omega)=\frac{j\omega+4}{(2+j\omega)(3+j\omega)}=\frac{A}{2+j\omega}-\frac{B}{3+j\omega}=\frac{2}{2+j\omega}-\frac{1}{3+j\omega}$$

$$\text{Q}\ e^{-at}u(t)\longleftrightarrow\frac{1}{a+j\omega}\qquad\therefore h(t)=2e^{-2t}u(t)-e^{-3t}u(t)$$

③

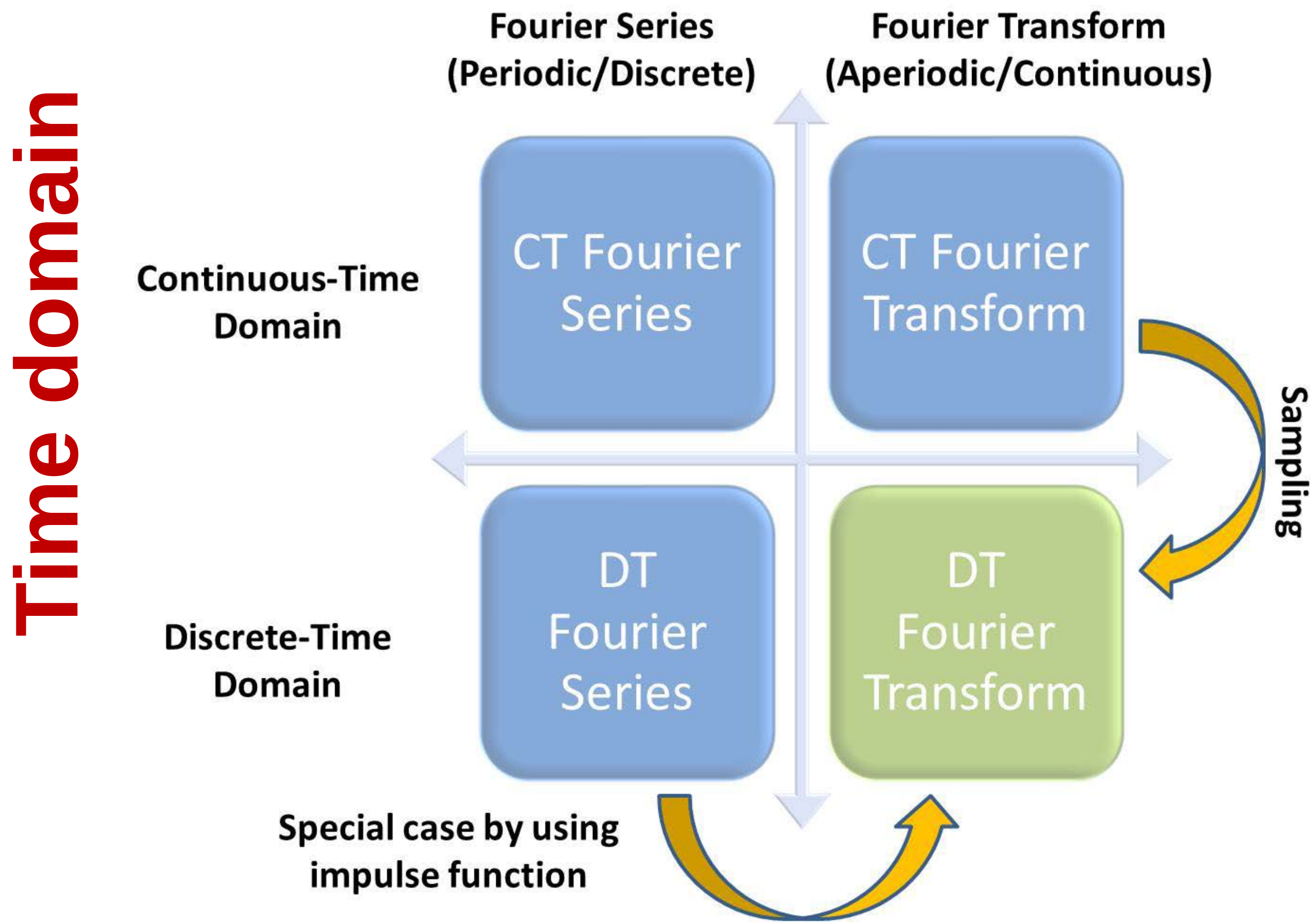
$$\text{Q}\ te^{-at}u(t)\longleftrightarrow\frac{1}{(a+j\omega)^2}$$

$$\therefore X(j\omega)=\frac{1}{4+j\omega}-\frac{1}{(4+j\omega)^2}$$

$$\therefore Y(j\omega)=H(j\omega)\cdot X(j\omega)=\frac{1}{(4+j\omega)(2+j\omega)}=\frac{A}{4+j\omega}-\frac{B}{2+j\omega}$$

Chapter 5
The Discrete-Time Fourier Transform

Frequency domain



Thinking…

- **How to move from CT Fourier series to CT Fourier transform?**

Discrete-Time Fourier Transform (DTFT)

DT Fourier Series Pair $\left(\omega_o = \frac{2\pi}{N}\right)$

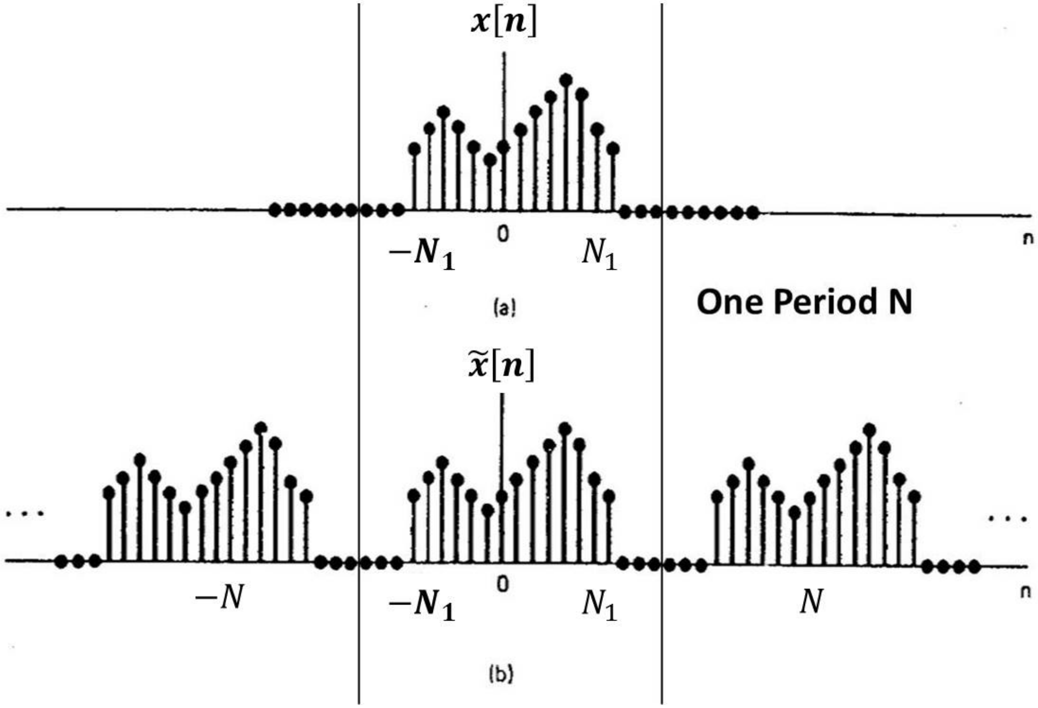
$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \qquad \text{(Synthesis equation)}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} \qquad \text{(Analysis equation)}$$

Aperiodic signals can be treated as periodic signals with period $N \to \infty$

- $x[n]$ must be like $\sum_k b_k e^{jk(2\pi/N)n}$ or $\int_\omega b(\omega) e^{j\omega n} d\omega$
- b_k or $b(\omega)$ can be calculated from $x[n]$

DTFT Derivation (1/3)



Original signal: $x[n]$

Define new periodic signal with period N: $\tilde{x}[n]$, such that

$$\tilde{x}[n] = x[n], \;\; n = -N/2, \dots, N/2 - 1$$

Notice: when $N \to \infty$, $\tilde{x}[n]$ becomes $x[n]$

DTFT Derivation (2/3)

- Look at the Fourier series of $\widetilde{x}[n]$:

$$a_k = \frac{1}{N} \sum_{n=-N/2}^{N/2+1} \widetilde{x}[n] e^{-jk(2\pi/N)n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk(2\pi/N)n}$$

DTFT Derivation (3/3)

- Therefore, we get the discrete-time Fourier transform pair

Discrete-Time Fourier Transform

$$\textit{Synthesis Equation:} \quad x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$\textit{Analysis Equation:} \quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Periodicity Properties of DT Complex Exponentials

- $x[n]$ - periodic with fundamental period N, fundamental frequency

$$x[n+N] = x[n] \qquad and \qquad \omega_o = \frac{2\pi}{N}$$

$$n = \dots, -1, 0, 1, 2, 3, \dots$$

- For DT complex exponentials, signal are periodic only when
$\omega_0 N = k \cdot 2\pi, \qquad k = 0, \pm 1, \pm 2, \cdots$

$$\boxed{e^{j\omega_0 n} = e^{j\omega_0 (n+N)} \rightarrow e^{j\omega_0 N} = 1 \rightarrow \omega_0 N = k \cdot 2\pi}$$

- For DT complex exponentials, signals with frequencies ω_0 and $\omega_0 + k \bullet 2\pi$ are identical. $\boxed{e^{j(\omega_0 + k \cdot 2\pi)n} = e^{j\omega_0 n} \cdot e^{jk \cdot 2\pi n} = e^{j\omega_0 n}}$
    - We need only consider a frequency interval of length 2π, and on most cases, we use the interval: $0 \le \omega_0 < 2\pi$, or $-\pi \le \omega_0 < \pi$

Cont.

\- $e^{j\omega_0 n}$ does ***not*** have a continually increasing rate of oscillation as ω_0 is increased in magnitude.

low-frequency (slowly varying): ω_0 near 0, 2π, …, or 2k • π

high-frequency (rapid variation): ω_0 near ± π, …, or (2k+1) • π

$$e^{j(2k+1)\pi n} = e^{j\pi n} = (e^{j\pi})^n = (-1)^n$$

Discrete-Time Fourier Transform

$$\textit{Synthesis Equation:} \quad x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

$$\textit{Analysis Equation:} \quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

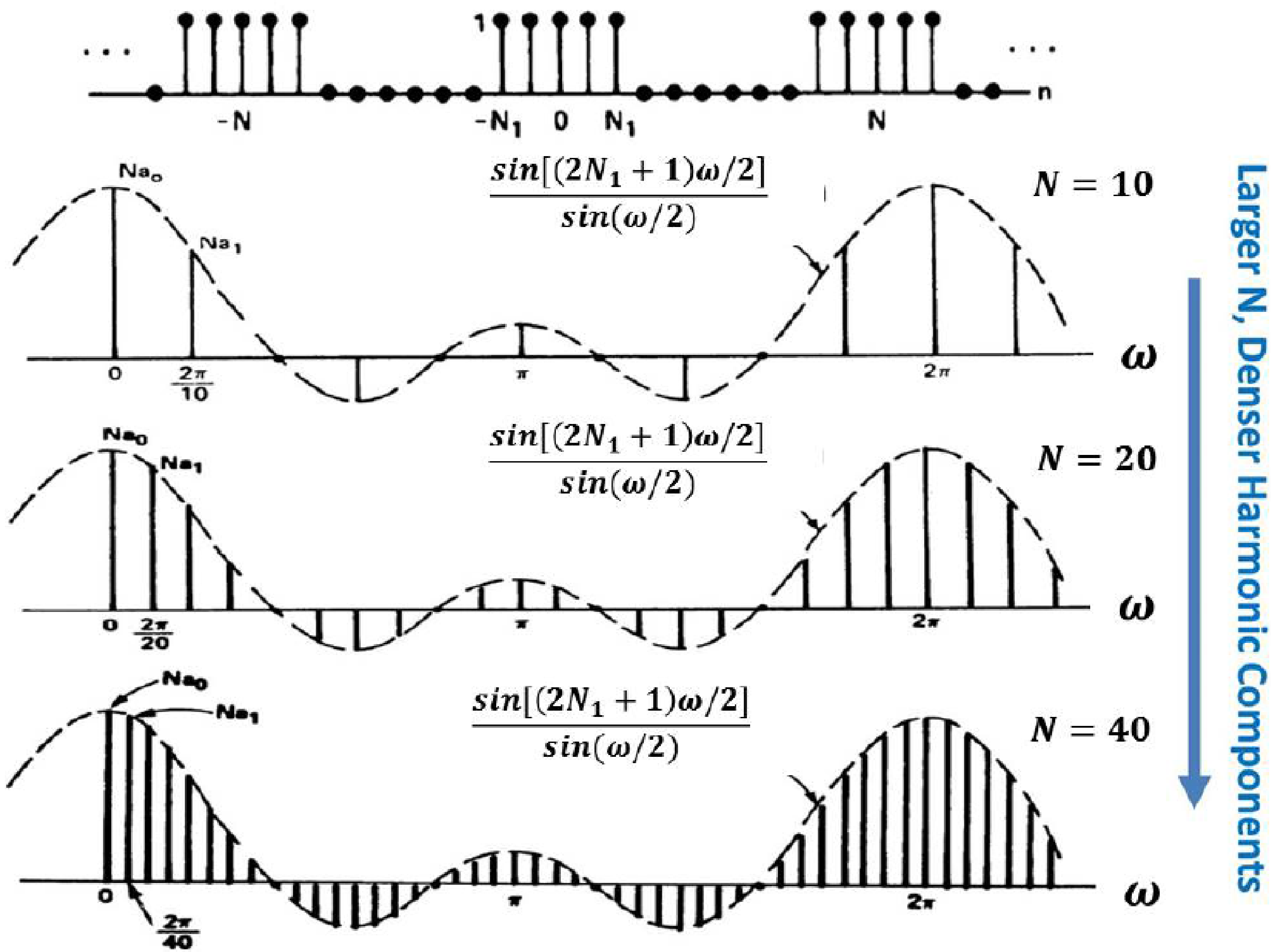
DT Fourier Series Pair $\left(\omega_o = \frac{2\pi}{N}\right)$

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \qquad \text{(Synthesis equation)}$$

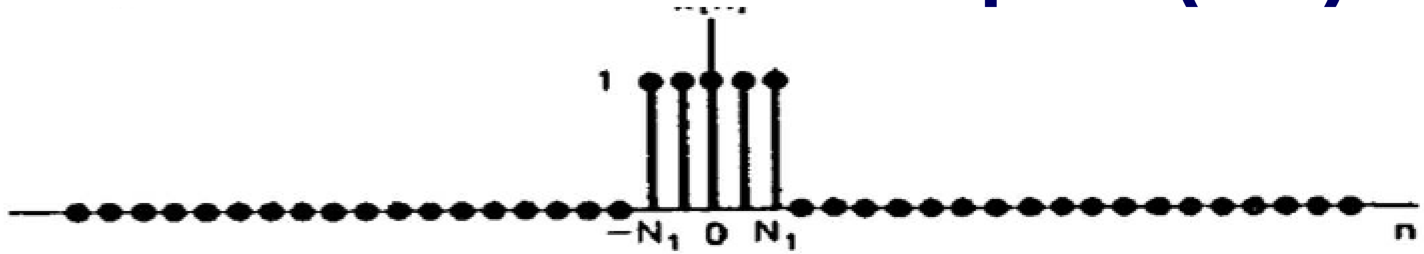
$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n} \qquad \text{(Analysis equation)}$$

$k =<N>$

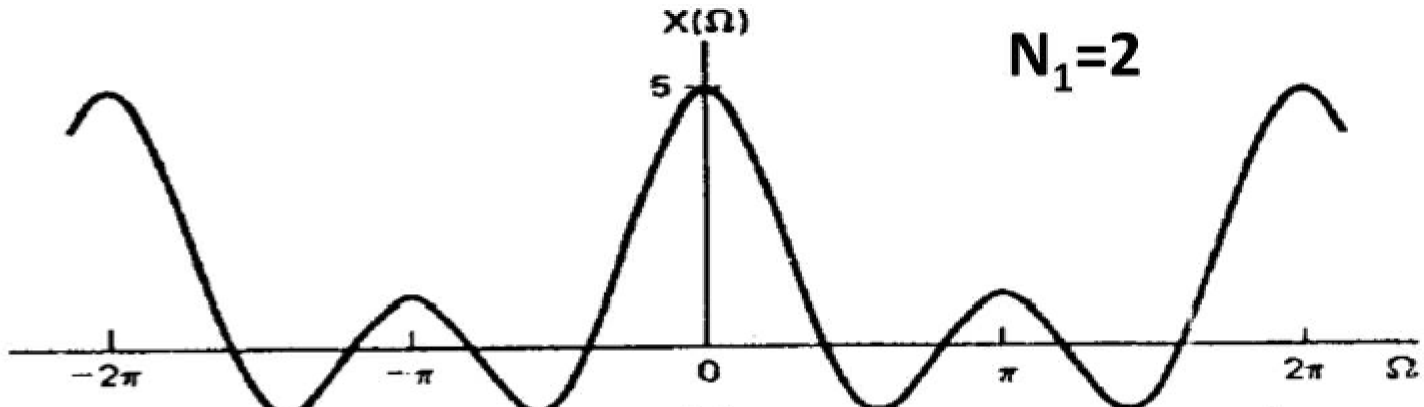
Example: From Periodic To Aperiodic



Fourier Transform Examples (1/2)



(a)

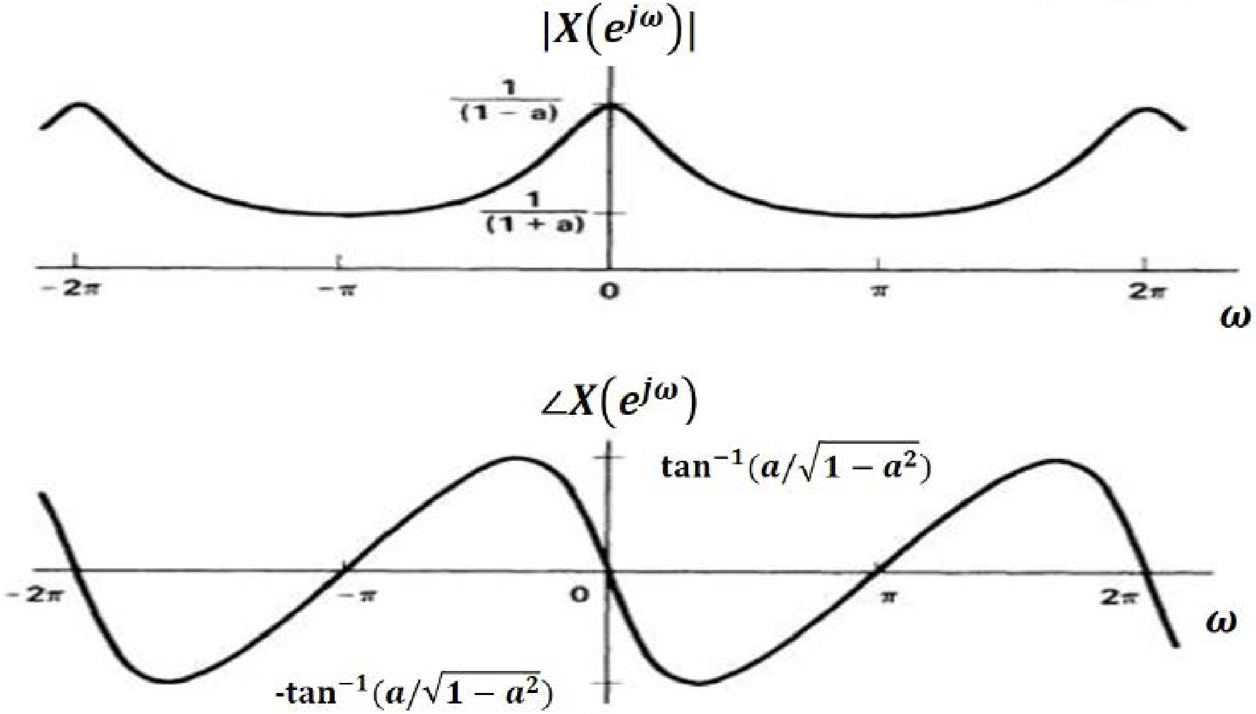


(b)

- $x[n] = 1 \ (n = -N_1, ..., 0, ..., N_1)$
- $X(e^{j\omega}) = \frac{sin\omega(N_1+1/2)}{sin(\omega/2)}$
- Width of $x[n]$: $W_t = 2N_1 + 1$; width of $X(e^{j\omega})$: $W_f = \frac{4\pi}{2N_1+1}$
- $W_t \times W_f = 4\pi$, which is a constant

Fourier Transform Examples (2/2)

$$x[n] = a^n u[n] \quad 0 < a < 1 \quad \Leftrightarrow \quad X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$



- See textbook, Example 5.1
- What's the shape of magnitude when $a \to 1$ or $a \to 0$?

Convergence Issue of Analysis Equation

Sufficient Condition of Convergence

The analysis equation $X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$ *will converge either if* $x[n]$ *is absolutely summable or if the sequence has finite energy, thus,*

$$\sum_{-\infty}^{\infty} |x[n]| < \infty \text{ or } \sum_{-\infty}^{\infty} |x[n]|^2 < \infty$$

Cont.

- Do the following signals have Fourier transform:
  - $a^n u[n]$ $(0 < a < 1)$
  - $\delta[n]$
  - $u[n]$
  - $e^{j\frac{2}{5}\pi n}$, $cos(\frac{2}{5}\pi n)$
  - $a^n u[n]$ $(a > 1)$

Can Periodic Signals Have DTFT?

- Definition of DTFT:

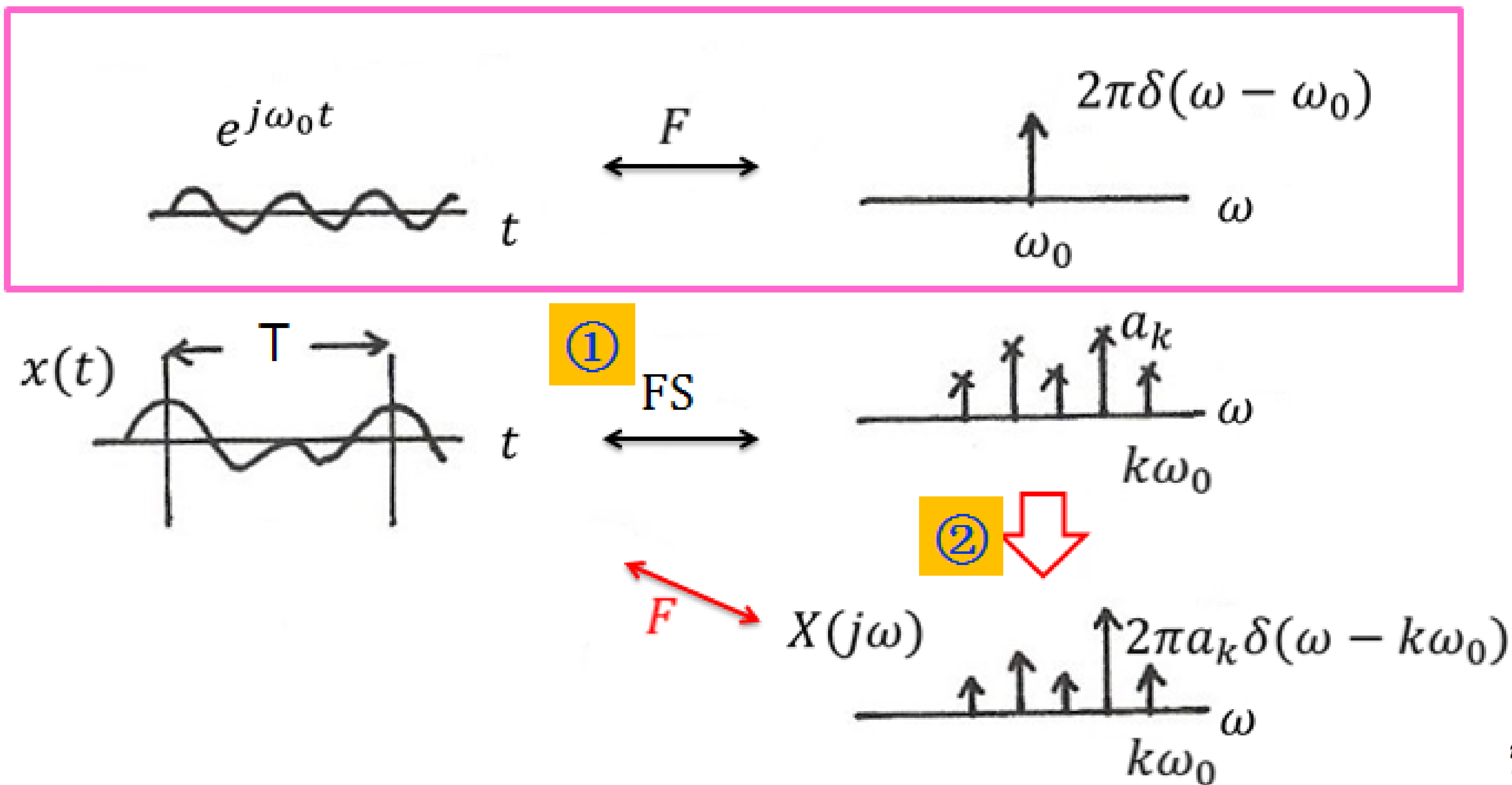
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- Justification of divergence:
  - Let $\omega = 2k\pi$, we have $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]$
  - Since $x[n]$ is periodic, the summation $\sum_{n=-\infty}^{\infty} x[n]$ will never converge unless $x[n] = 0$
- Conclusion: Most of periodic signals do NOT have DTFT according to the definition
- However, it's of significant engineering importance to extend Fourier transform to periodic signals

Can Periodic Signals Have DTFT?

Review

Fourier Transform for Periodic Signals – Unified Framework



DTFT with Periodic Signals (1/2)

Fourier Transform of $e^{j\omega n}$

The following transform pair is actually NOT rigorously defined:

$$x[n] = e^{j\omega_0 n} \longleftrightarrow X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$

- Synthesis:

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{2\pi} \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)e^{j\omega n}d\omega = e^{j\omega_0 n}$$

- Analysis:

$$\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} e^{j(\omega_0 - \omega)n} \quad \textit{converge??}$$

DTFT with Periodic Signals (2/2)

- According to the Fourier series, for a periodic signal with period N:

$$x[n] = a_0 + a_1 e^{j(2\pi/N)n} + \ldots + a_k e^{jk(2\pi/N)n} + \ldots + a_{N-1} e^{j(N-1)(2\pi/N)n}$$

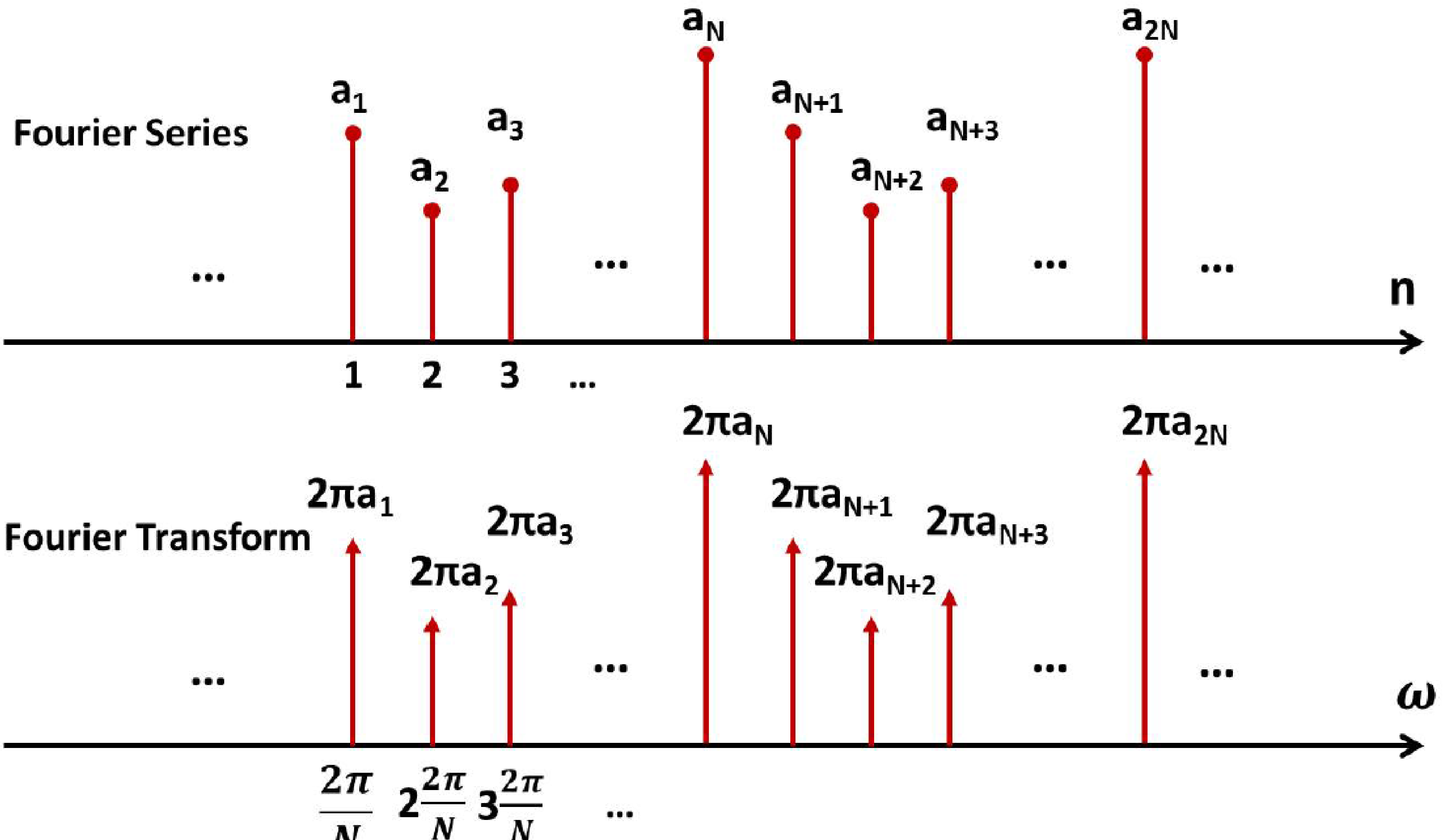
- $e^{jk(2\pi/N)n} \longleftrightarrow \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - k(2\pi/N) - 2\pi l)$

- Then, due to the linearity of Fourier transform

$$\begin{aligned} \mathcal{F}\left\{x[n]\right\} = \sum_{k=0}^{N-1} a_k \mathcal{F}\left\{e^{jk(2\pi/N)n}\right\} &= \sum_{k=0}^{N-1} a_k \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - k(2\pi/N) - 2\pi l) \\ &= \sum_{l=-\infty}^{\infty} \sum_{k=0}^{N-1} a_k 2\pi\delta(\omega - k(2\pi/N) - 2\pi l) \end{aligned}$$

- Fourier transform of a periodic signal is a periodic sequence of impulses
  - What's the period? How many impulses within one period?

Fourier Series v.s. Fourier Transform



Example: Discrete-Time Impulse Chain

- What's the Fourier transform of $x[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN]$?
- First of all, we calculate the Fourier series:

$$
\begin{aligned}
a_k &= \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\frac{2\pi}{N}n} \\
&= \frac{1}{N} \sum_{n=<N>} \sum_{k=-\infty}^{+\infty} \delta[n - kN] e^{-jk\frac{2\pi}{N}n} \\
&= \frac{1}{N} \sum_{n=<N>} \delta[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}
\end{aligned}
$$

Frequency domain