Notes

Assignments

7.21, 7.23

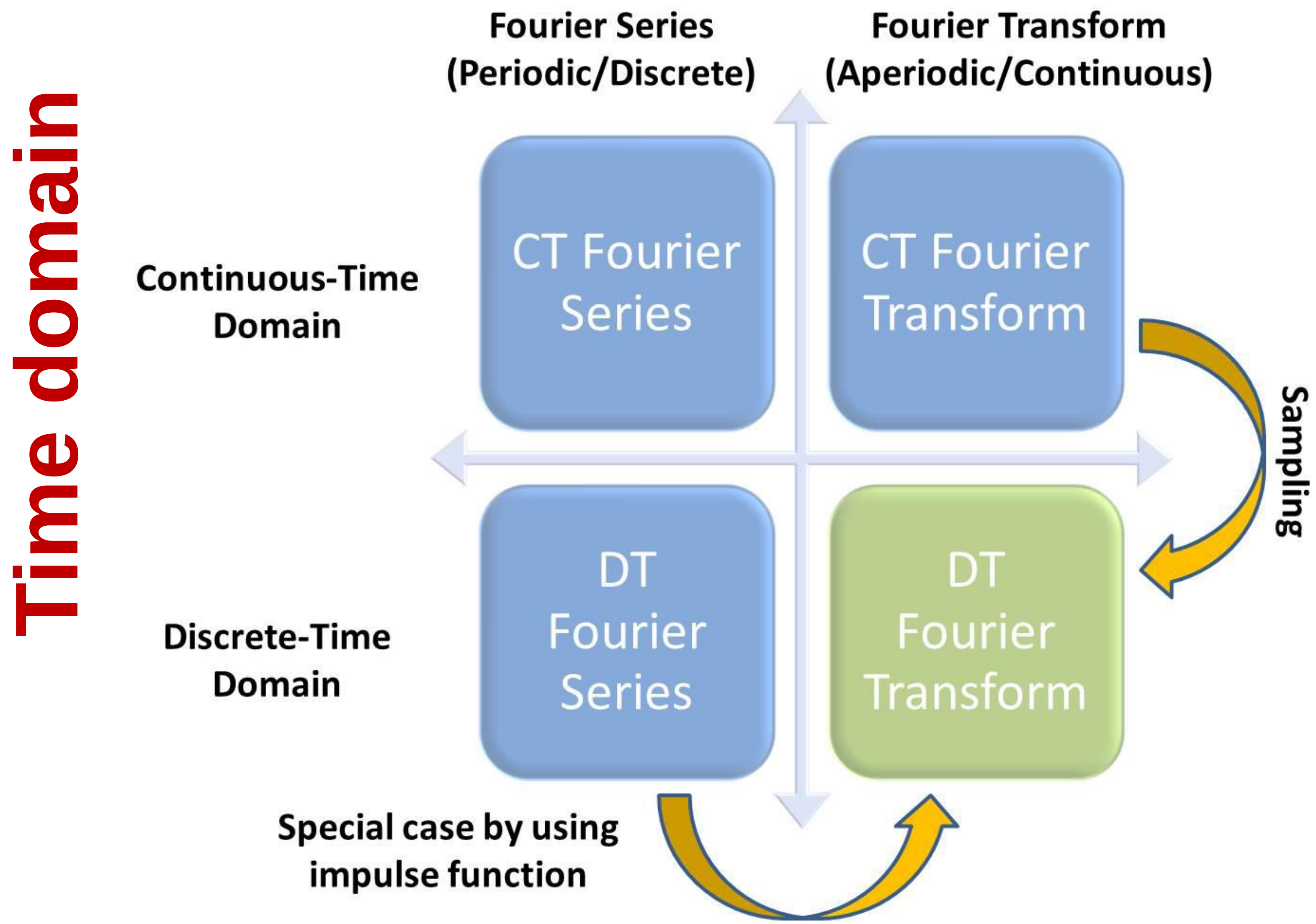
Tutorial problems

7.25, 7.37, 7.40

南方科技大学 SUSTC

Chapter 7: Sampling

Frequency domain



CTFT Properties: Multiplication Property

thus if $$x(t) * y(t) \longleftrightarrow X(j\omega) \cdot Y(j\omega)$$

then the other way around is also true

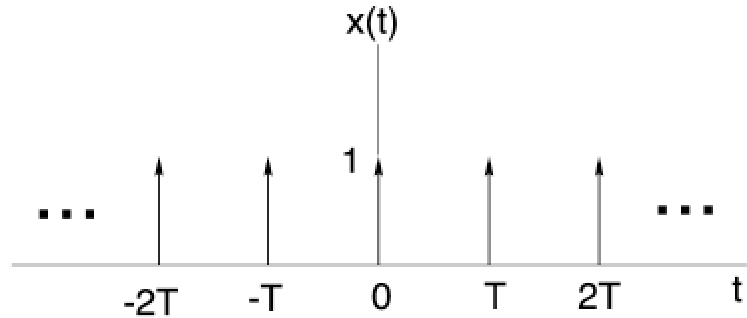
$$\boxed{\begin{aligned} x(t) \cdot y(t) &\longleftrightarrow \frac{1}{2\pi} X(j\omega) * Y(j\omega) \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\theta) Y(j(\omega - \theta)) d\theta \end{aligned}}$$

— A consequence of *Duality*

Review

Example 4.8

$$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT) \quad \text{— sampling function}$$



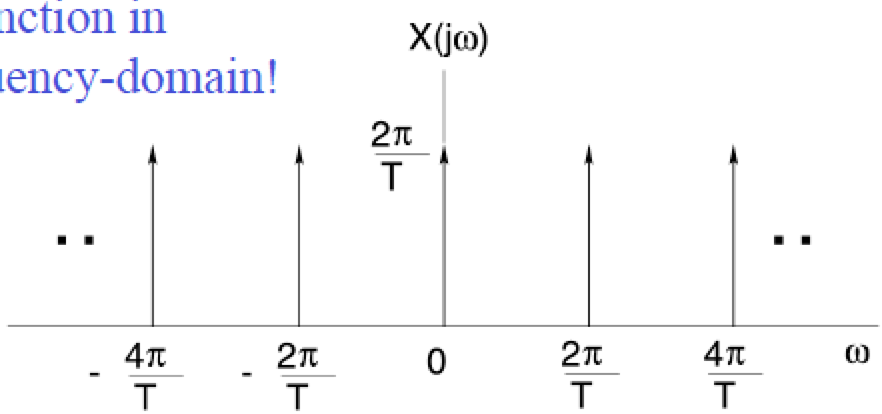
$$x(t) \overset{FS}{\longleftrightarrow} a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_o t}\,dt = \frac{1}{T}$$

$$\Downarrow\; x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_o t} = \sum_{k=-\infty}^{+\infty} \frac{1}{T} e^{jk\omega_o t}$$

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{2\pi}{T}\,\delta(\omega - \underbrace{\frac{k2\pi}{T}}_{k\omega_o})$$

$\omega_s = 2\pi / T$: sampling frequency

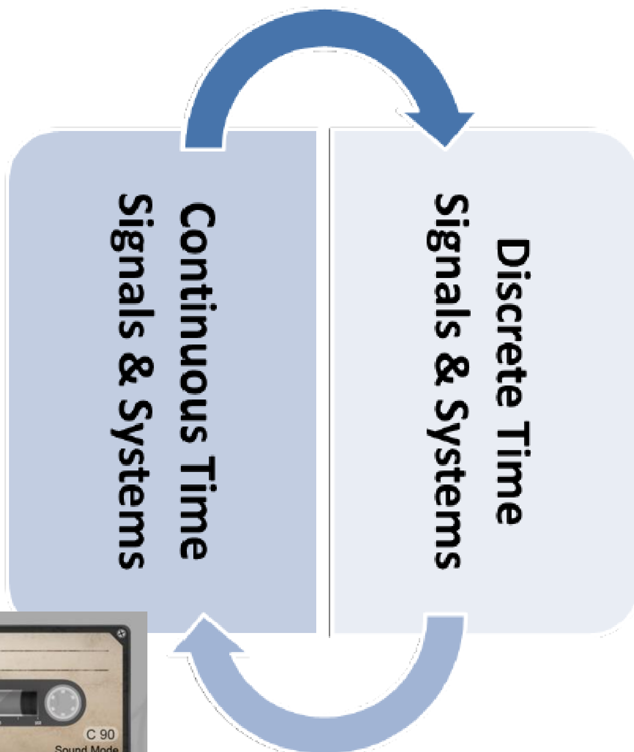
Same function in the frequency-domain!



Note in this case, periodic in both time domain (with a period T) and frequency domain (with a period $2\pi/T$)

Introduction

Sampling: to facilitate digital processing via computers or chips



Any lossless conversion?

Process CT signals with DT systems?

Interpolation: to present the output of digital processing

Example: Video recording

- Signal to be sampled: real scene (continuous-time signals)
- Sampling: record by camera with a rate of 24, 25 or 30 frames per second
- Sampled signal: video tapes, mp4 files, avi files and etc. (discrete-time signals)
- Reconstruction: watch via eyes and interpret in the brain
- In our consciousness, the real scene can be reconstructed without information loss

Outline

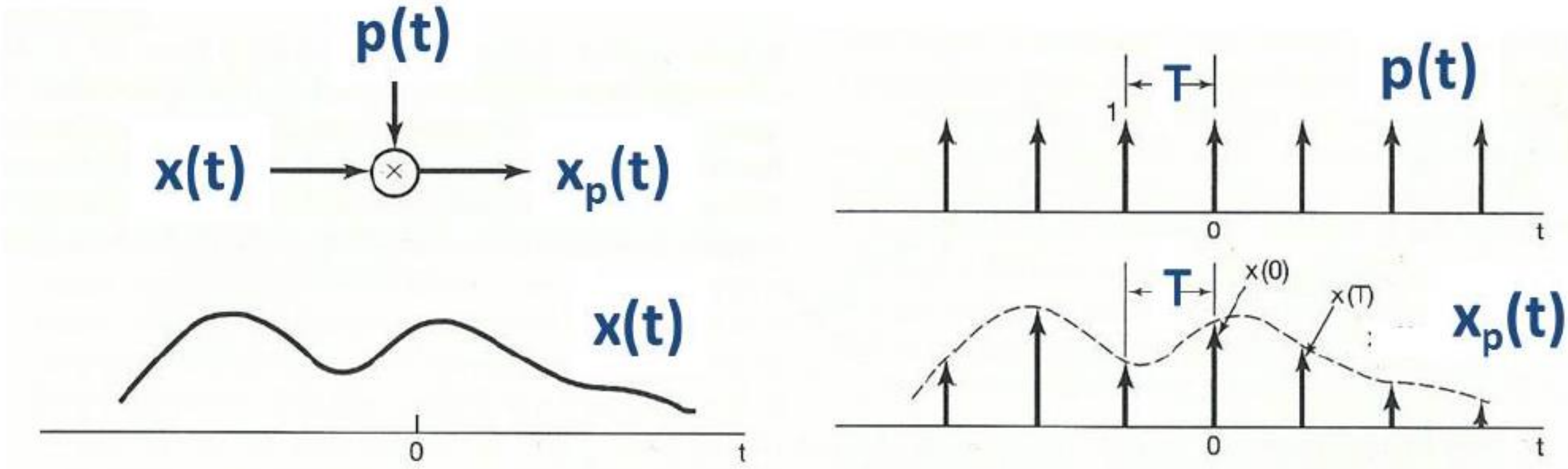
- Sampling is a general procedure to generate DT signals from CT signals, where information of the original signals can be kept
- Core sampling theory:
  - Impulse train, zero-order hold, 1st-order hold, etc.
  - Analysis in frequency domain
  - Nyquist rate
- Undersampling: Aliasing
- Application: process continuous-time signals discretely
- More sampling techniques: decimation, downsampling and upsampling

Note on digital signal

- Need be both discrete time (DT) and digital value
  - By sampling: sampling rate
  - By quantization: how many bits
  - Could be implemented by analogy-to-digital (A/D) converter

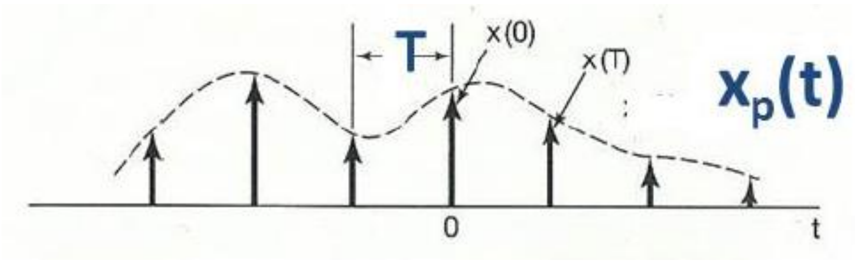
Impulse-train sampling

- Mathematically, sampling can be represented by multiplication



- Sampling function: $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$
- Sampling period: T
- Sampling:

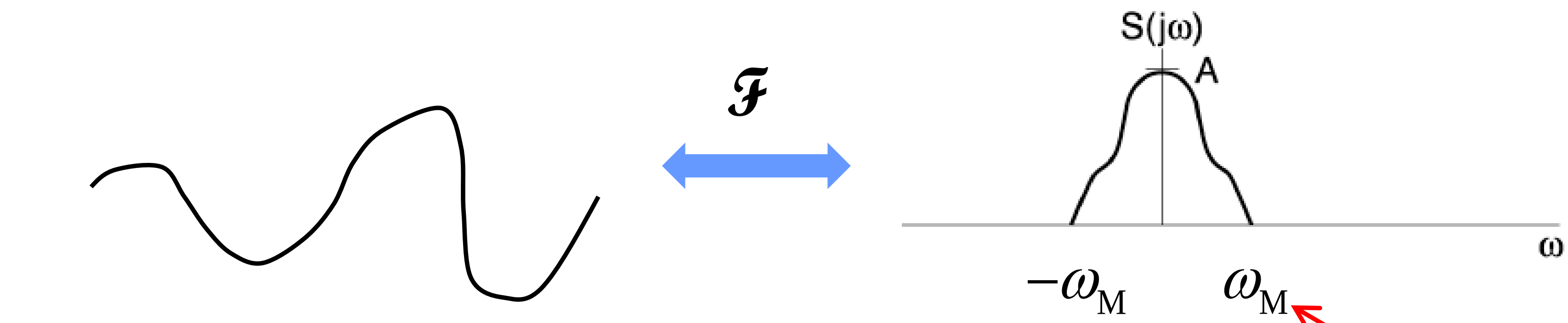
$$x_p(t) = x(t) \times p(t) = x(t) \times \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

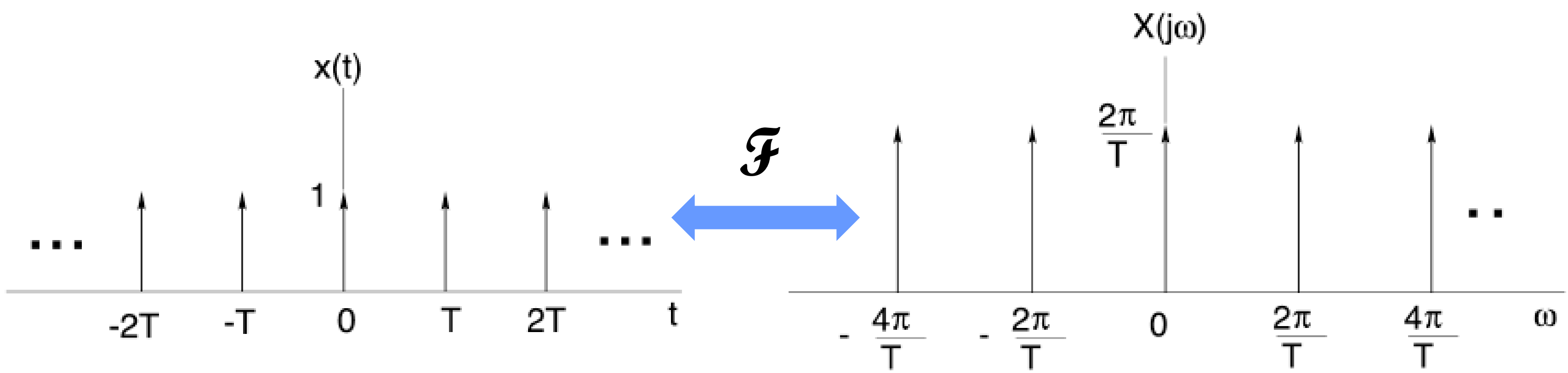


Sampling discards most of points in the original signals.

Is there any information loss in sampling?
Or can we perfectly reconstruct the original signal?

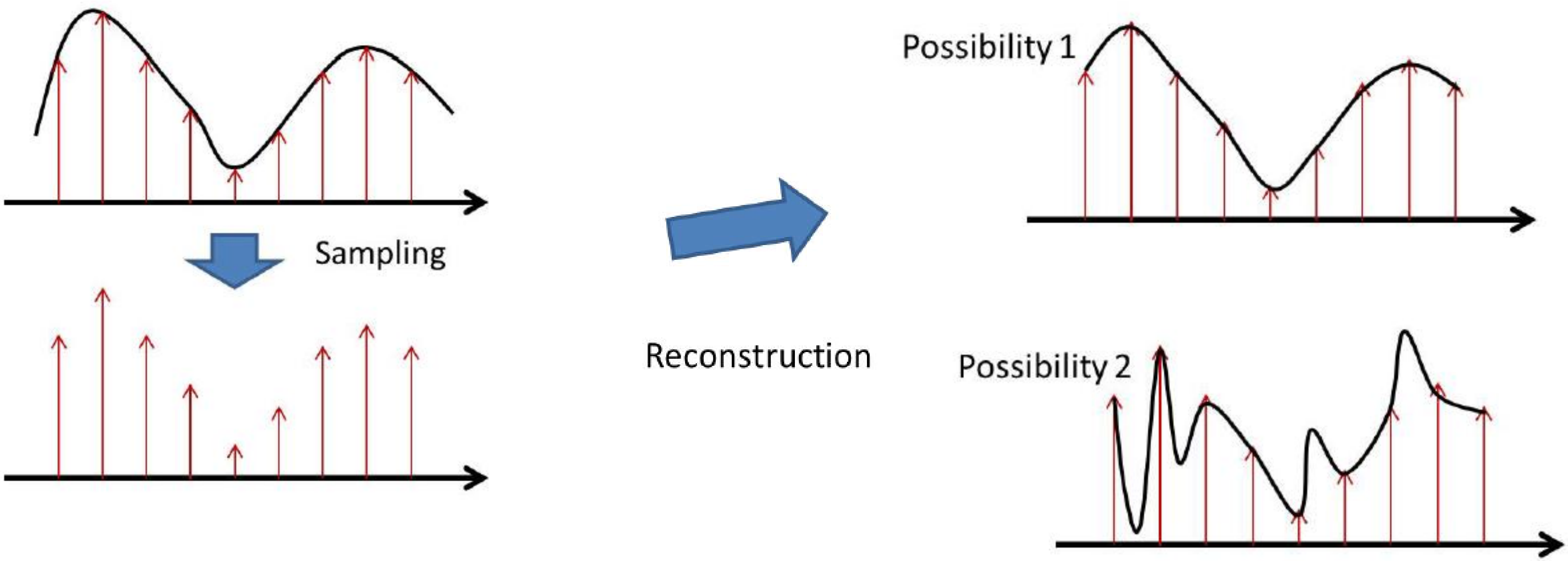
Two important frequencies





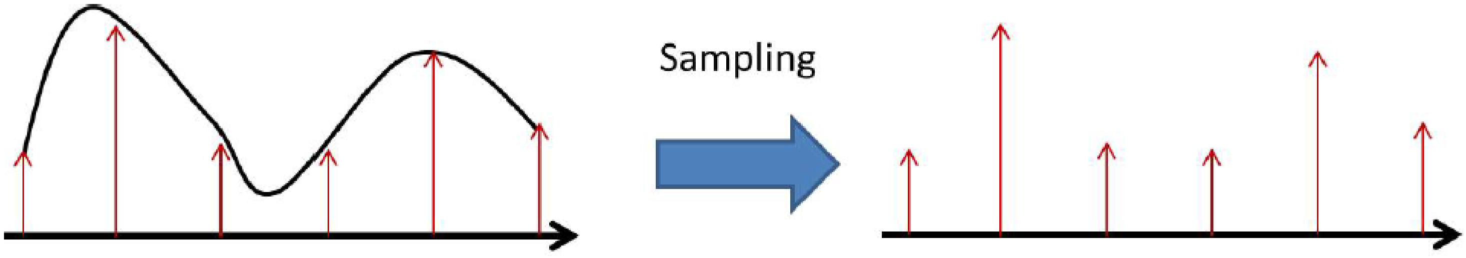
$$\omega_s = \frac{2\pi}{T}$$

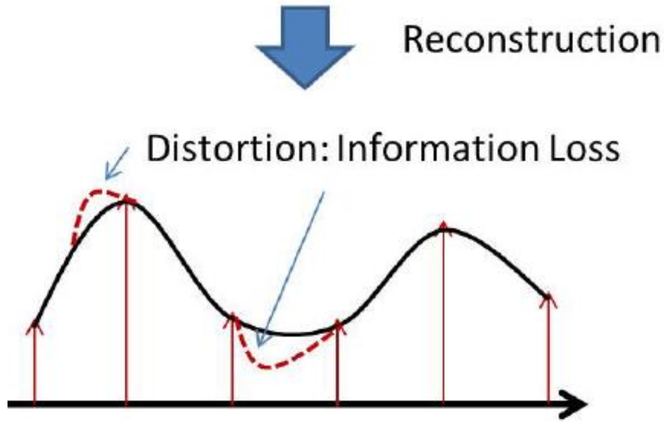
Observation (1/2)



Intuition 1: It seems that we need a smooth interpolation

Observation (2/2)





Intuition 2: It seems that the sampling frequency should be high enough, such that the detail fluctuation can be captured

- Sampling: the frequency should be high enough
- Reconstruction: the interpolation should be smooth enough

Frequency analysis (1/2)

- Theoretical tool: continuous-time Fourier transform
- Principle:

$$x(t) \times p(t) \rightleftarrows \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

- Fourier series of $p(t)$:

$$\begin{aligned} a_k &= \frac{1}{T}\int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} \delta(t-nT) e^{-jk\omega_s t} dt \quad \textit{where} \quad \omega_s = \frac{2\pi}{T} \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_s t} dt = \frac{1}{T} \end{aligned}$$

- Fourier Transform of $p(t)$:

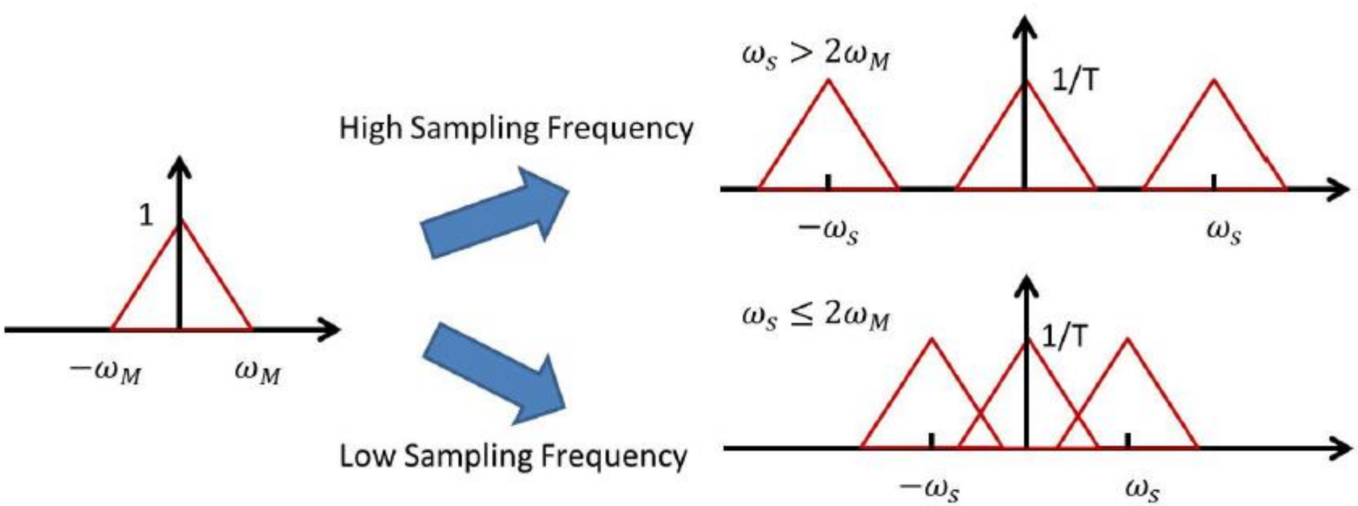
$$P(j\omega) = 2\pi a_k \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s)$$

Frequency analysis (2/2)

- Fourier transform of sampled signal $x_p(t)$:

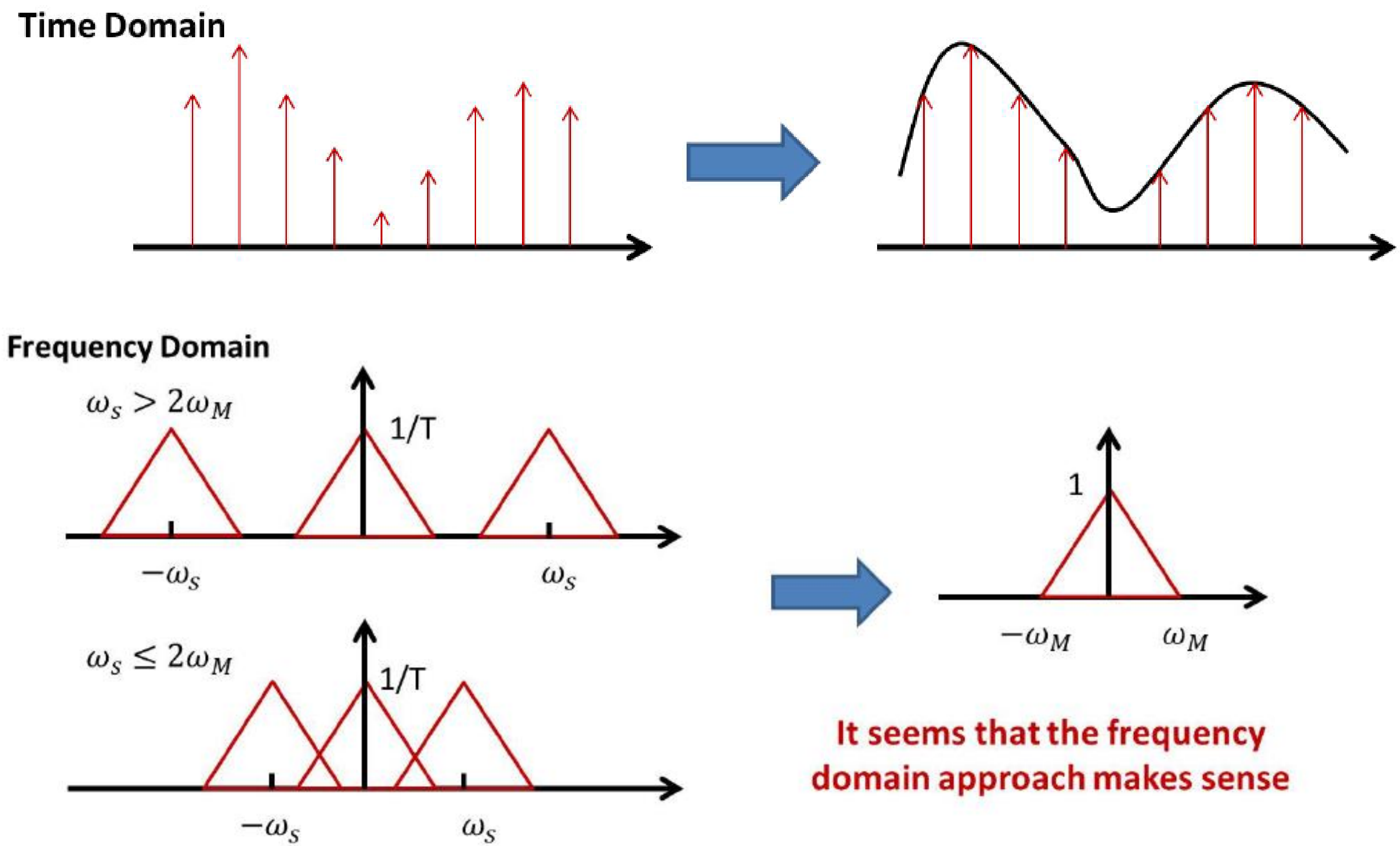
$$X_p(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)P(j(\omega-\theta))d\theta$$
$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

- Sampling: the Fourier transform of input signal is repeated with period ω_s



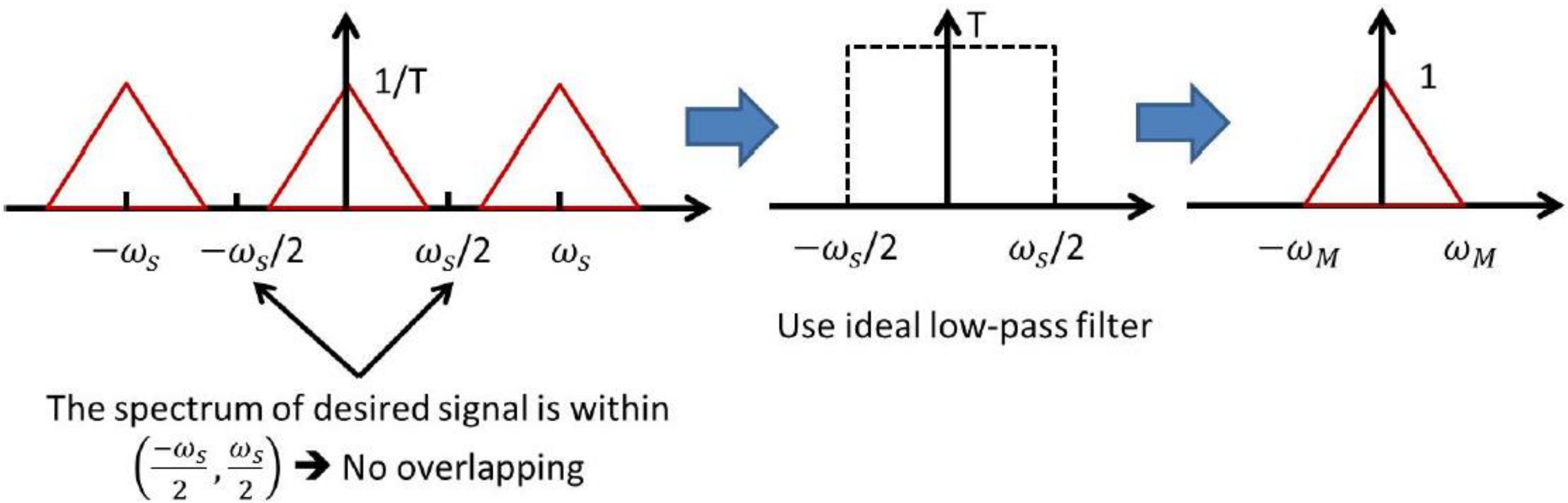
Reconstruction problem

- Given the sampled signal, can we perfectly reconstruct the signal before sampling?



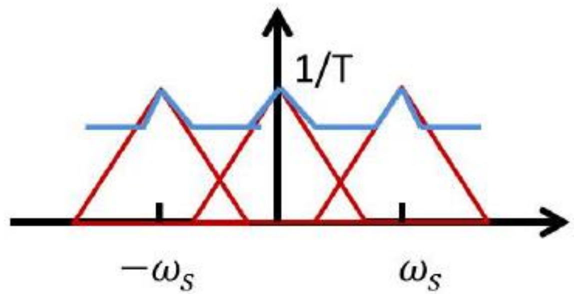
Reconstruction (1/2)

- Scenario of $\omega_s > 2\omega_M$



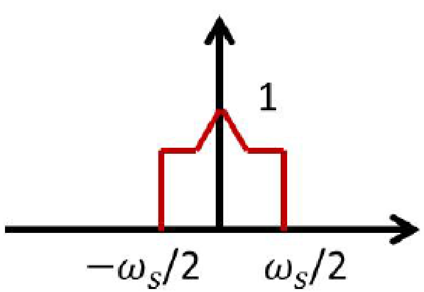
Reconstruction (2/2)

- Scenario of $\omega_s \leq 2\omega_M$

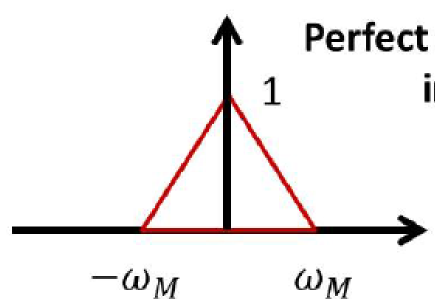


Since $\omega_s \leq 2\omega_M$, we don't know the frequency range of the desired signal

Sampling on the following signals can generate the same result:



OR



Perfect reconstruction is impossible

Observation: the original signal x(t) can be Uniquely and perfectly reconstructed from x(nT) only when $\omega_s > 2\omega_M$

Sampling theorem

Sampling Theorem

Let $x(t)$ be a band-limited signal with

$$X(j\omega) = 0 \textit{ for } |\omega| > \omega_M.$$

Then, $x(t)$ is uniquely determined by its samples $x(nT)$ or $x_p(t)$ if

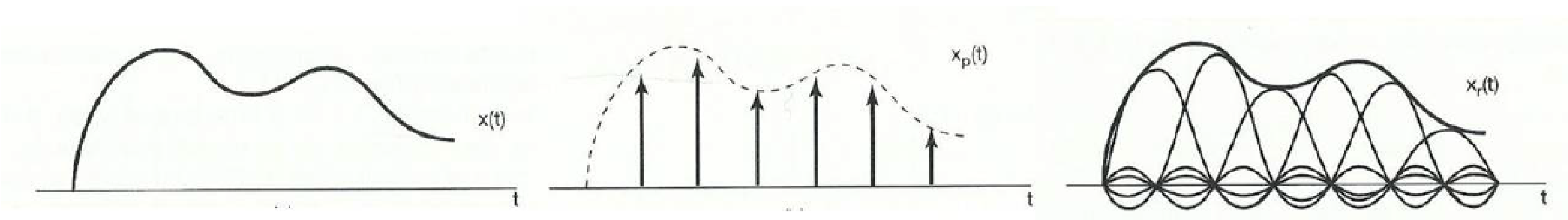
$$\omega_s = \frac{2\pi}{T} > 2\omega_M,$$

where $2\omega_M$ is referred to as the Nyquist rate.

- Questions:
  - How about $\omega_s = 2\omega_M$?
  - Sampling on band-pass signals

Signal reconstruction: Interpolation

- If $\omega_s > 2\omega_M$, original signal can be perfectly reconstructed by ideal low-pass filter.
- Time domain interpretation of lowpass filtering



$$
\begin{aligned}
x_r(t) &= x_p(t) * h(t) = \sum_{n=-\infty}^{+\infty} x(nT)h(t-nT) \\
&= \sum_{n=-\infty}^{+\infty} x(nT)\frac{\sin\frac{\omega_s}{2}(t-nT)}{\frac{\omega_s}{2}(t-nT)} = \sum_{n=-\infty}^{+\infty} x(nT)sinc(\frac{t-nT}{T})
\end{aligned}
$$

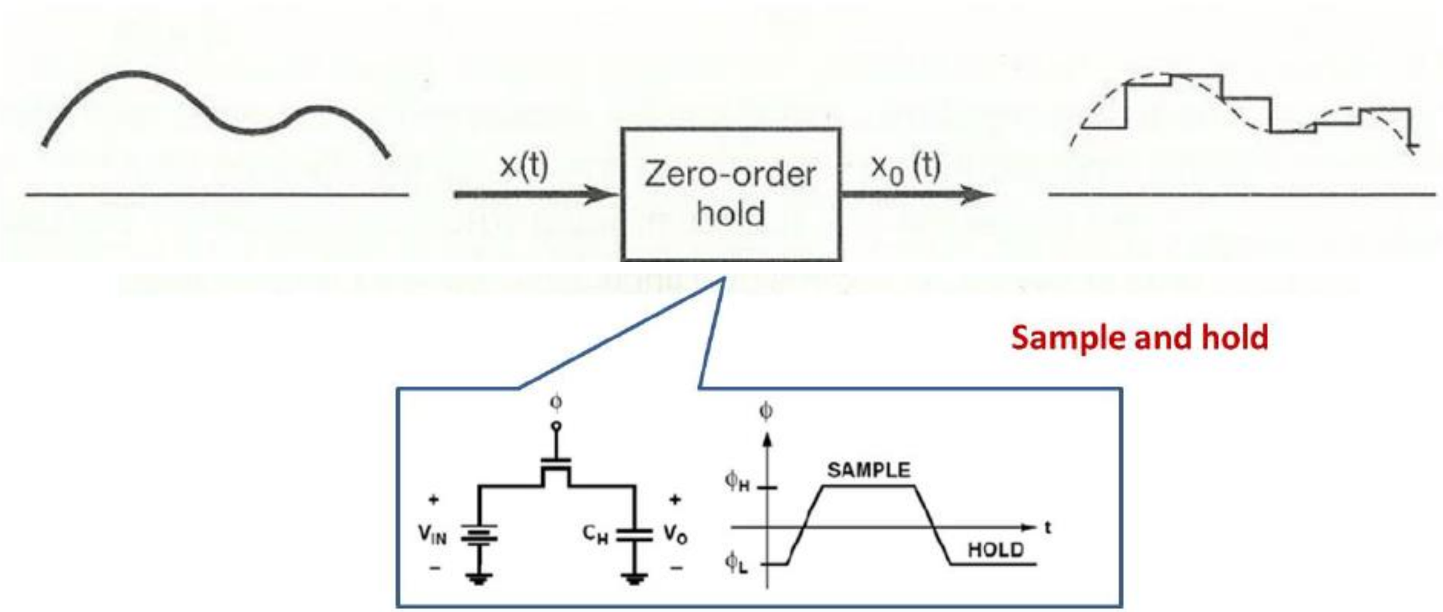
- Ideal lowpass filtering: interpolation with sinc function

Zero-order hold

- It's difficult to generate ideal impulse chain in practical implementation.

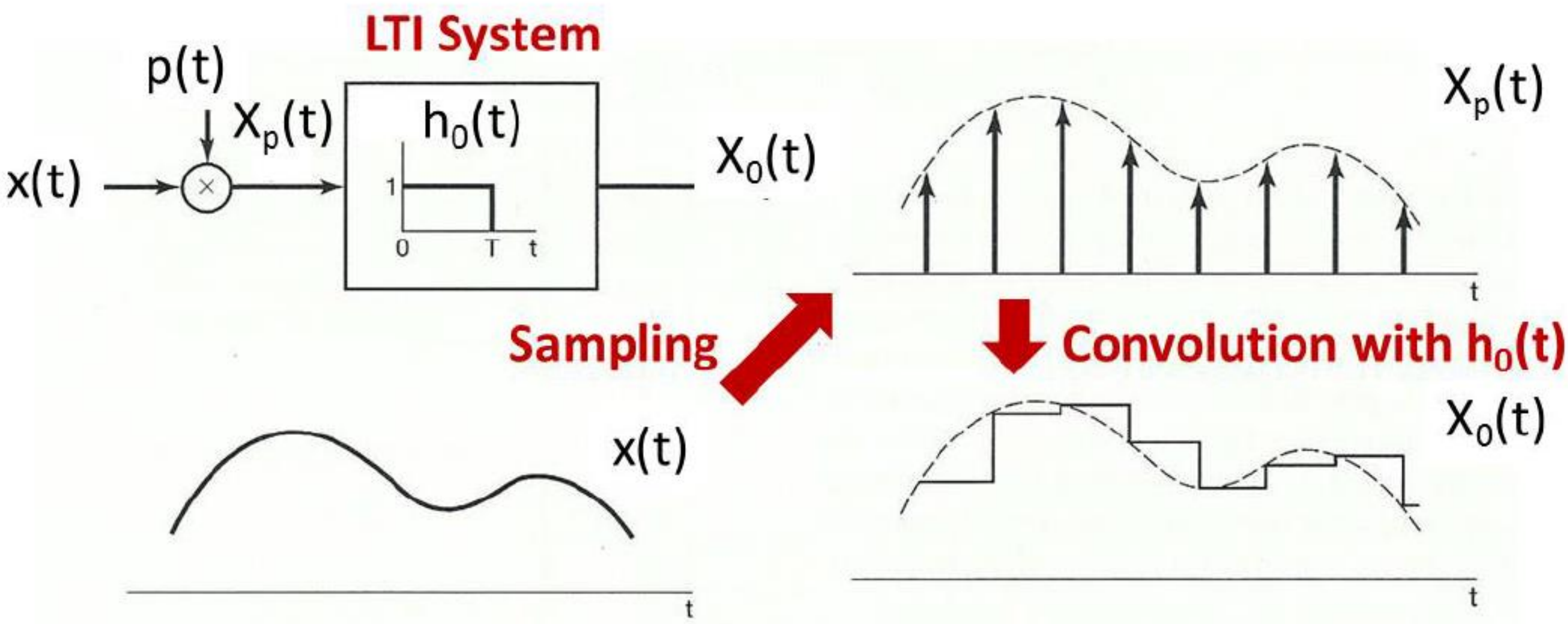
$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$$

- Alternative approach: zero-order hold



- How to interpret the system of "zero-order hold" mathematically?

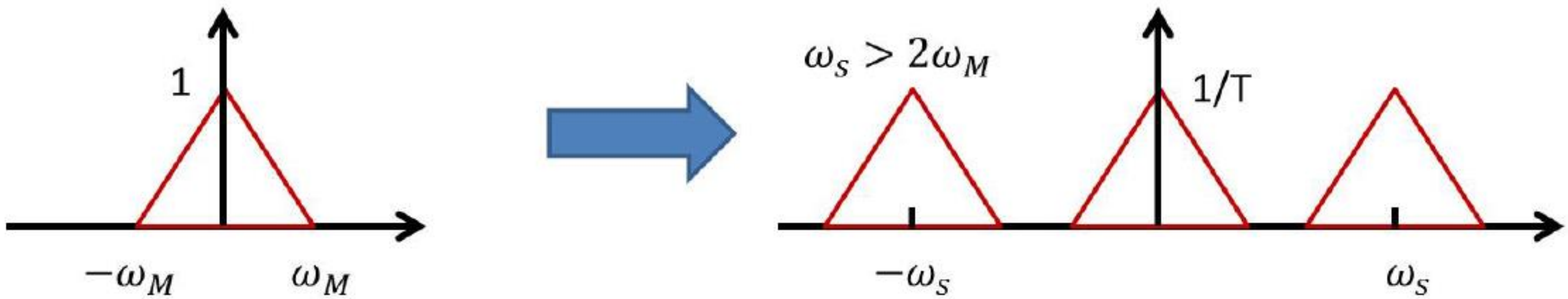
Interpretation of zero-order hold



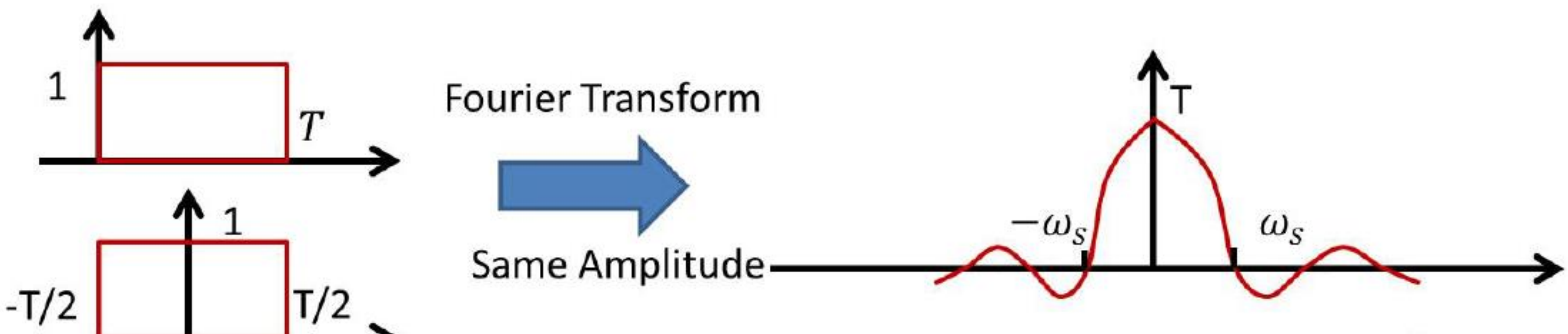
- Zero-order hold: sampling + interpolation with rectangular impulse response
- An approximation of the signal to be sampled.

Frequency analysis (1/2)

- Step 1: Impulse-train sampling

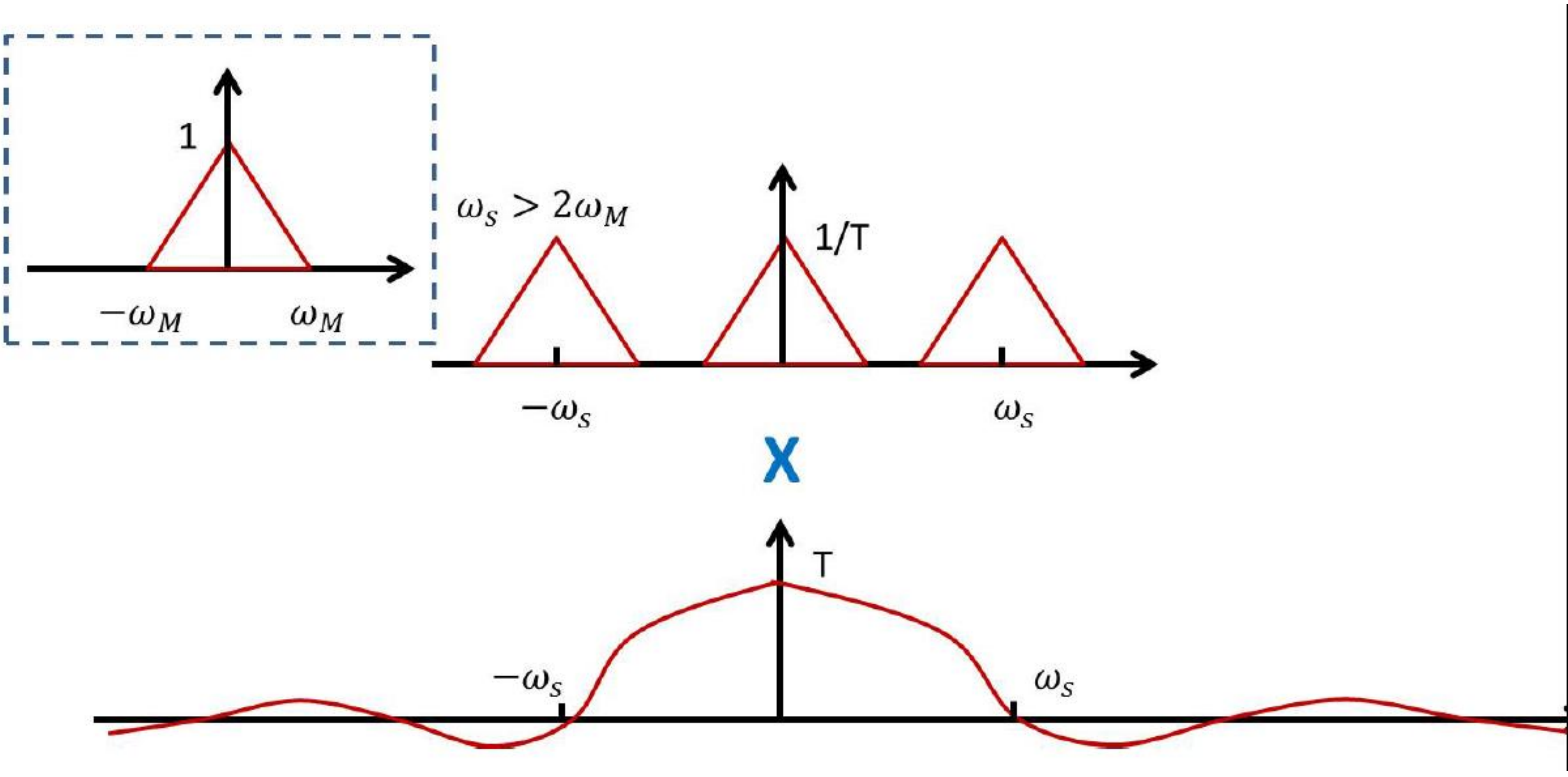


- Step 2: Frequency response of $h_0(t)$



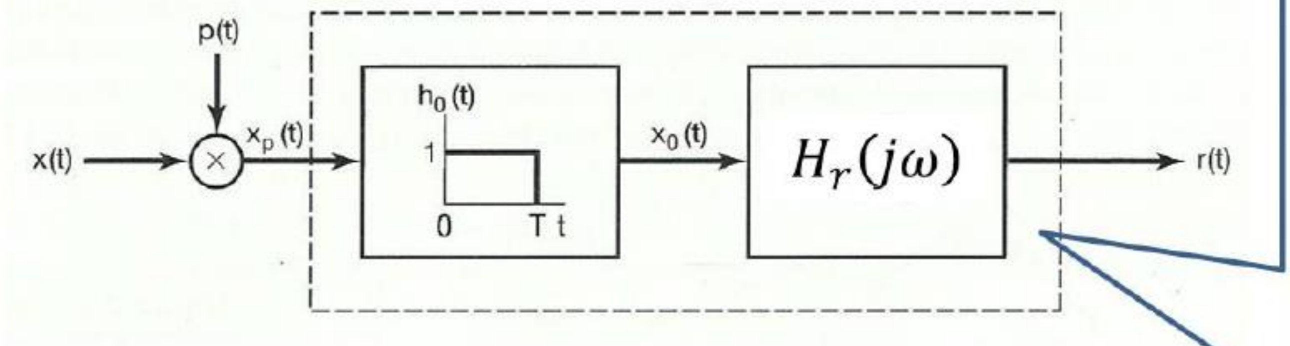
$$H(j\omega) = 2e^{-j\omega T/2}\frac{sin\omega T/2}{\omega}$$

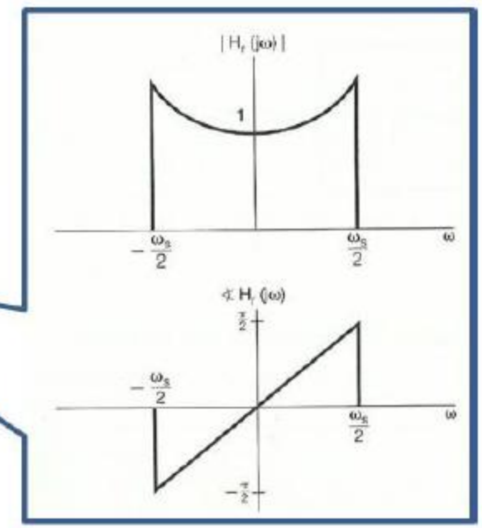
Frequency analysis (2/2)



Reconstruction

$$H(j\omega) = \mathrm{H}_0(\mathrm{j}\omega)H_r(j\omega)$$



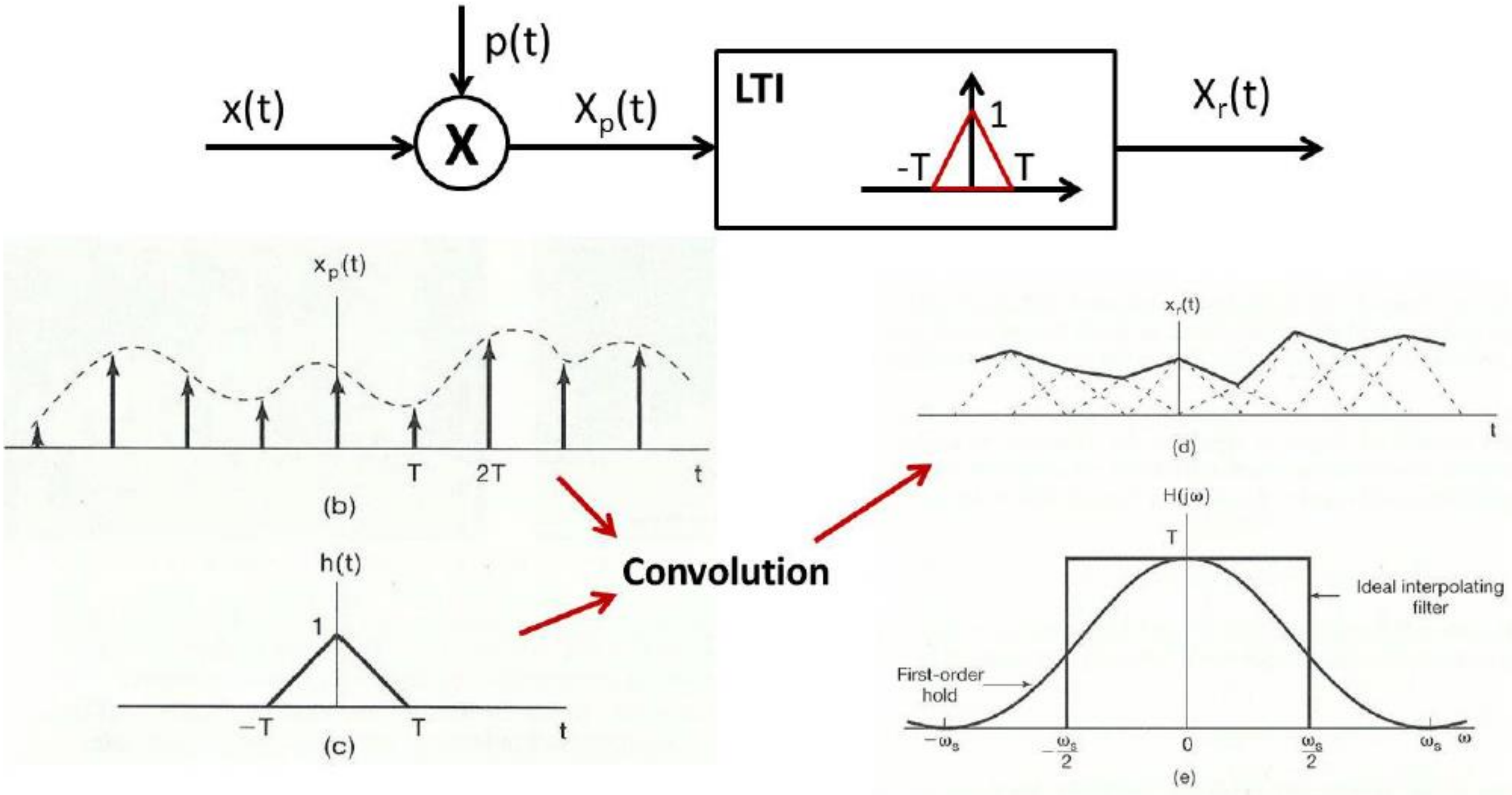


Compensate the frequency response of $h_0(t)$, such that H works as ideal lowpass filter

$$H_r(j\omega) = e^{j\omega T/2}\frac{\omega T}{2sin\omega T/2} \qquad -\frac{\omega_s}{2} \leq \omega \leq \frac{\omega_s}{2}$$

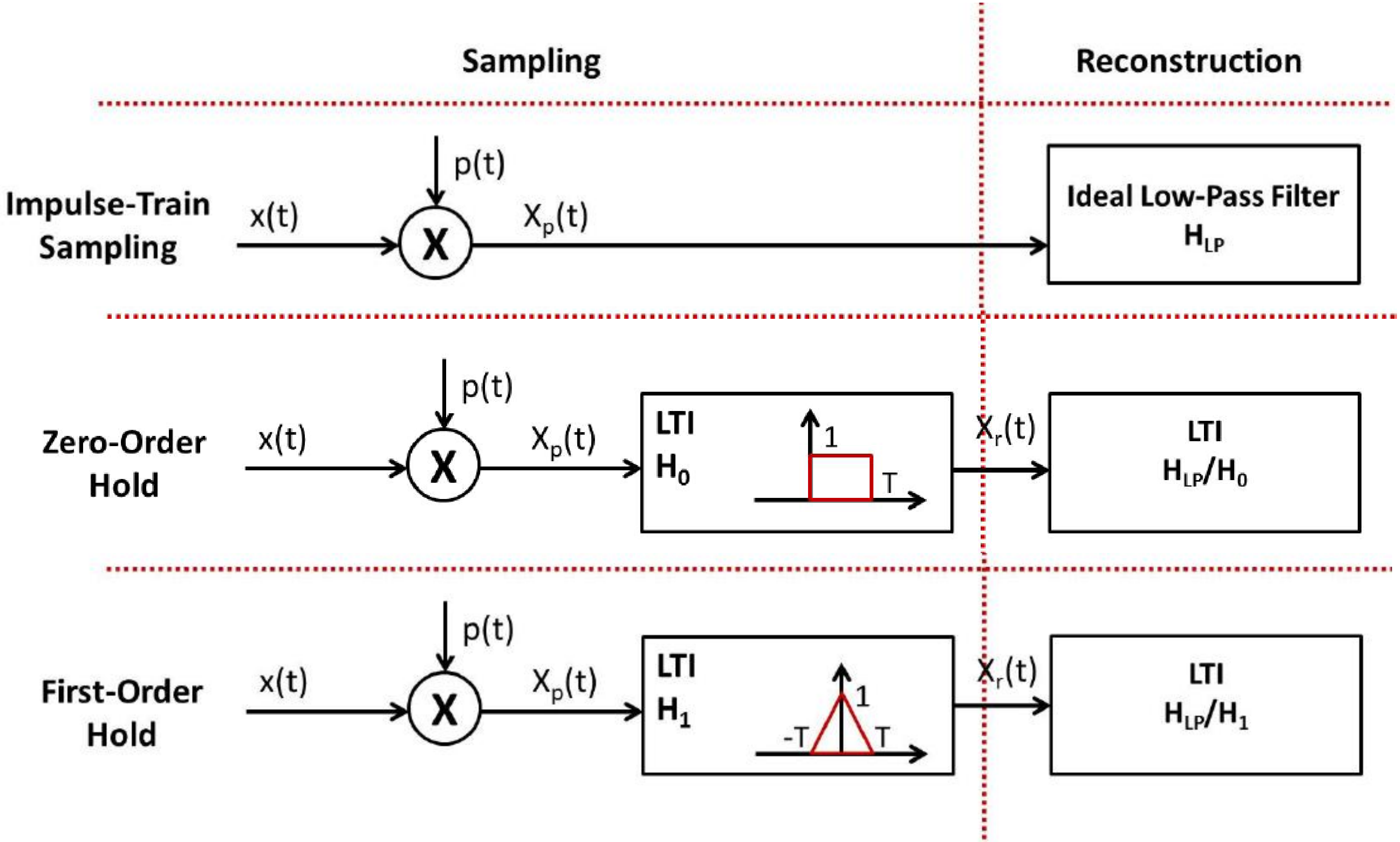
- $H(j\omega)$ should be a idea low-pass filter from $-\omega_s/2$ to ω_s

First-order hold



- First-order hold: sampling + interpolation with triangular wave
- How to reconstruct?

Summary: Sampling approaches



Problem

Problem (7.36)

Let $x(t)$ *be a band-limited signal such that* $X(j\omega) = 0$ *for* $|\omega| \geq \pi/T$.
(a) If $x(t)$ *is sampled using a sampling period* T, *determine an interpolating function* $g(t)$ *such that*

$$\frac{dx(t)}{dt} = \sum_{n=-\infty}^{\infty} x(nT)g(t - nT).$$

(b) Is the function $g(t)$ *unique?*

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

$$x(t) = x_p(t) * h(t) \qquad h(t) = \frac{\sin(\pi t / T)}{\pi t / T}$$

$$\frac{dx(t)}{dt} = x_p(t) * \frac{dh(t)}{dt} = x_p(t) * g(t) = \sum_{n=-\infty}^{\infty} x(nT) g(t - nT)$$

Therefore:

$$g(t) = \frac{dh(t)}{dt} = \frac{\cos(\pi t / T)}{t} - \frac{T \sin(\pi t / T)}{\pi t^2}$$