Notes

- **Assignment**
  - **2.10**
  - **2.11**
  - **2.22 (b) (e)**
  - **2.25**
  - **2.28 (a) (c) (e) (g)**

- **Tutorial questions (Week 5)**
  - **Basic Problems with Answers 2.20**
  - **Basic Problems 2.29**
  - **Advanced Problems 2.40, 2.43, 2.47**

That is ...

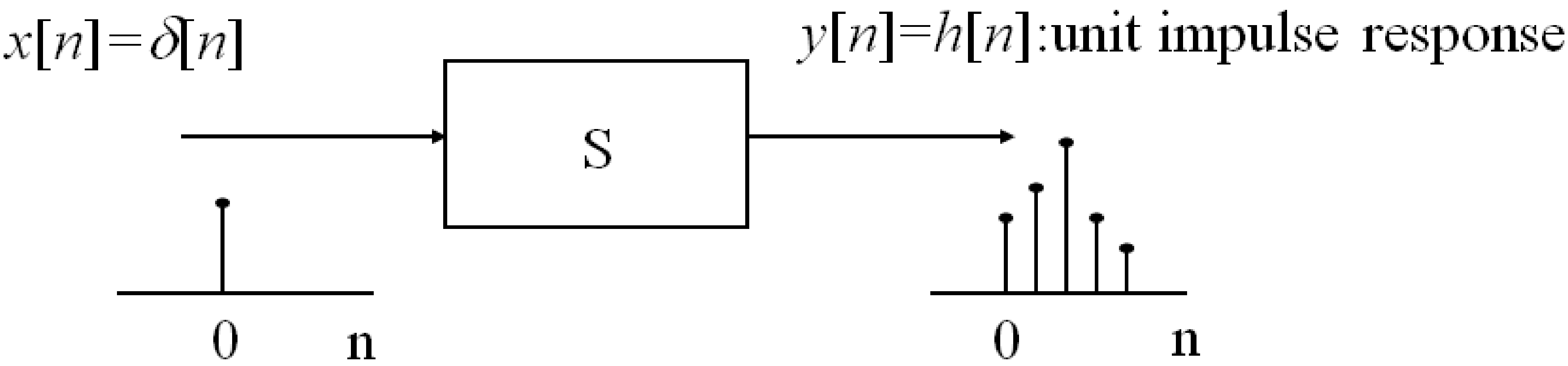
$$x[n] = \ldots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \ldots$$

$$\Downarrow$$

$$x[n] = \sum_{k=-\infty}^{+\infty} \underbrace{x[k]}_{\text{Coefficients}} \underbrace{\delta[n-k]}_{\text{Basic Signals}}$$

Important to note the "-" sign

Unit Impulse Response



Response of DT LTI Systems

$$x[n] \longrightarrow \boxed{\text{DT System}} \longrightarrow y[n]$$

- Now suppose the system is **LTI**, and define the *unit impulse response* $h[n]$:

$$\delta[n] \longrightarrow h[n]$$

$$\Downarrow$$

From Time-Invariance:

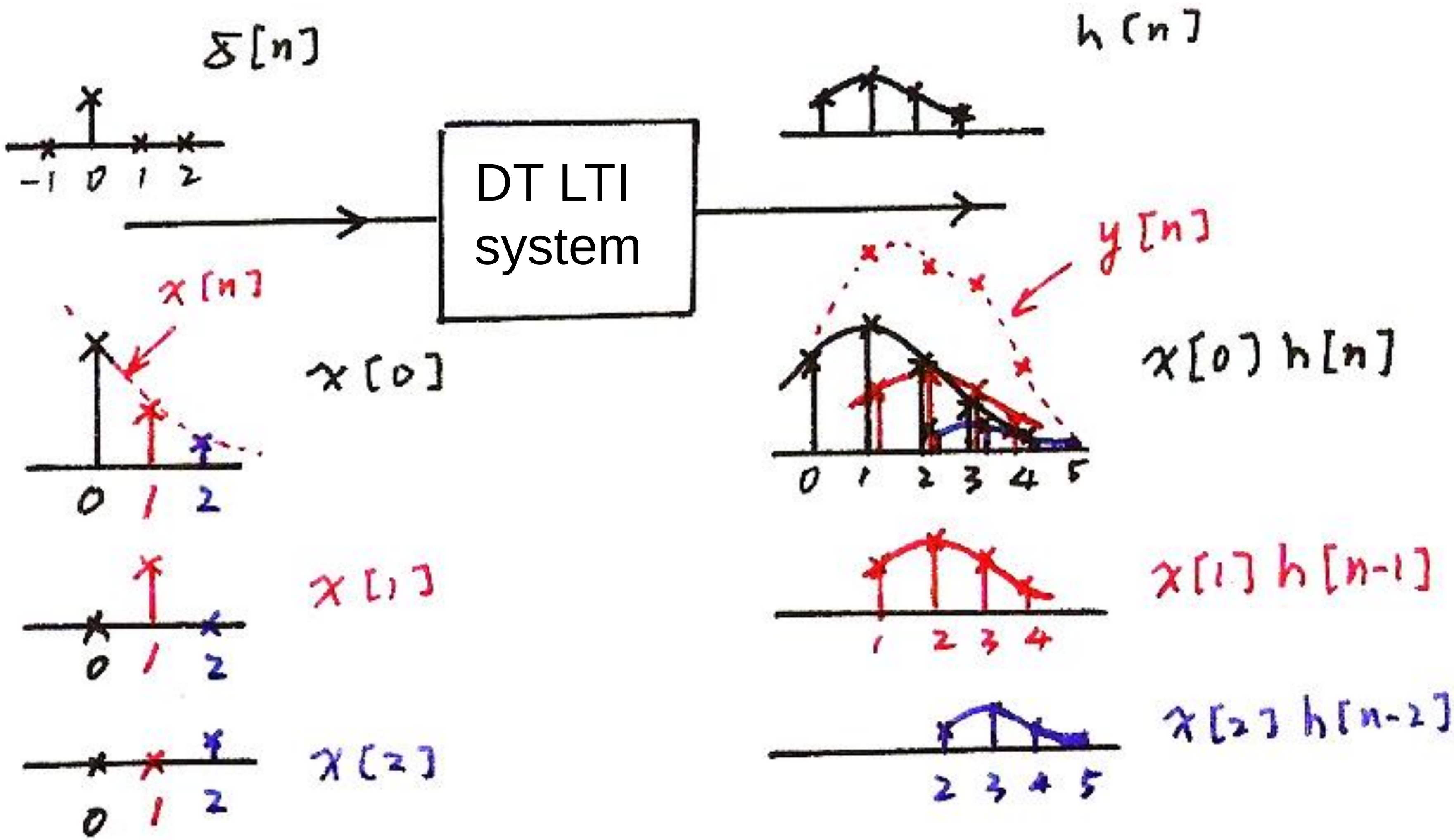
$$\delta[n-k] \longrightarrow h[n-k]$$

From Linearity:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k] \longrightarrow y[n] = \underbrace{\sum_{k=-\infty}^{+\infty} x[k]\,h[n-k]}_{\textit{convolution sum}} = x[n] * h[n]$$

The output for an input signal is the superposition of a series of "shifted, scaled unit impulse response"

Input/Output Relation



- A different way to visualize the convolution sum
  - looked at on the index k

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]$$

Contribution to the output signal at time n

input signal

flipped version of $h[k]$ located at $k = n$

Convolution operation procedure:

$$h[k] \xrightarrow{\text{Flip}} h[-k] \xrightarrow{\text{Slide}} h[n-k] \xrightarrow{\text{Multiply}} x[k]h[n-k]$$

$$\xrightarrow{\text{Sum}} \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

F S M S

The Unit-impulse function $\delta(t)$

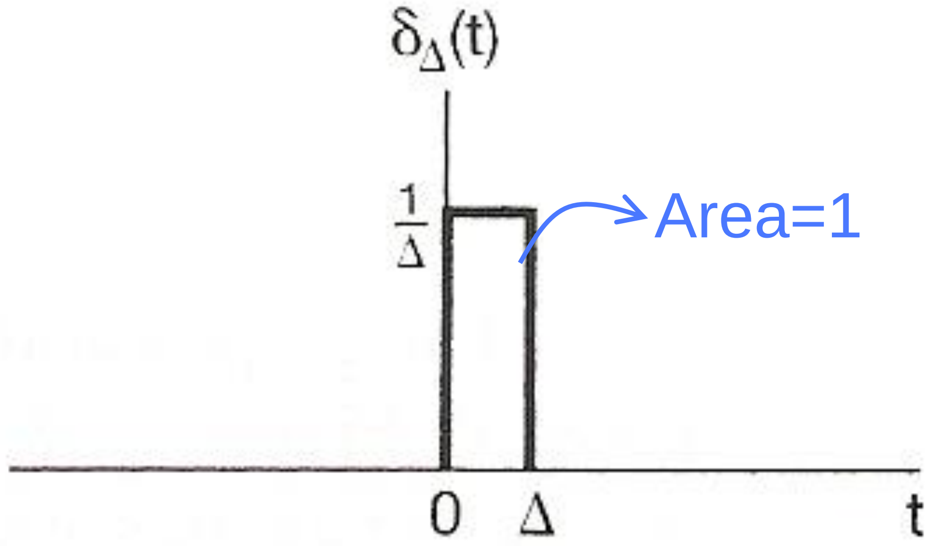
$$\left.\begin{aligned} \delta(t) &= 0 \quad \mathit{for}\ t \neq 0 \\ \delta(t) &= \infty \quad \mathit{for}\ t = 0 \end{aligned}\right\} \qquad (1)$$

$$\int_{0^-}^{0^+} \delta(t)\,dt = 1 \qquad (2)$$

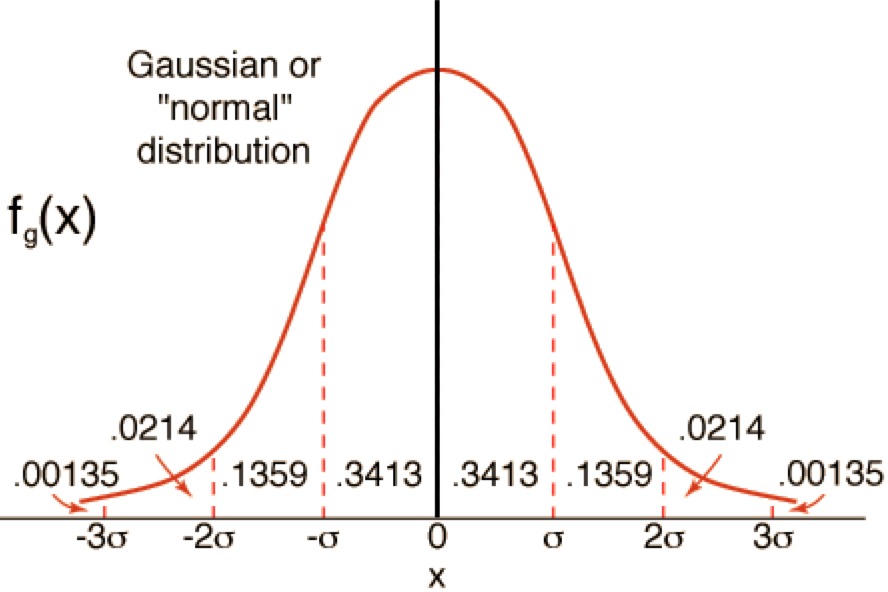
— an infinitesimally sharp pulse with an unity area.

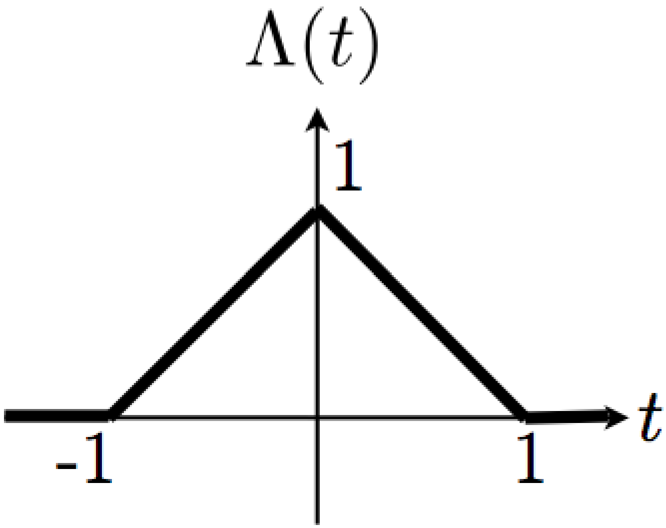
Construction of the Unit-impulse function δ(*t*)

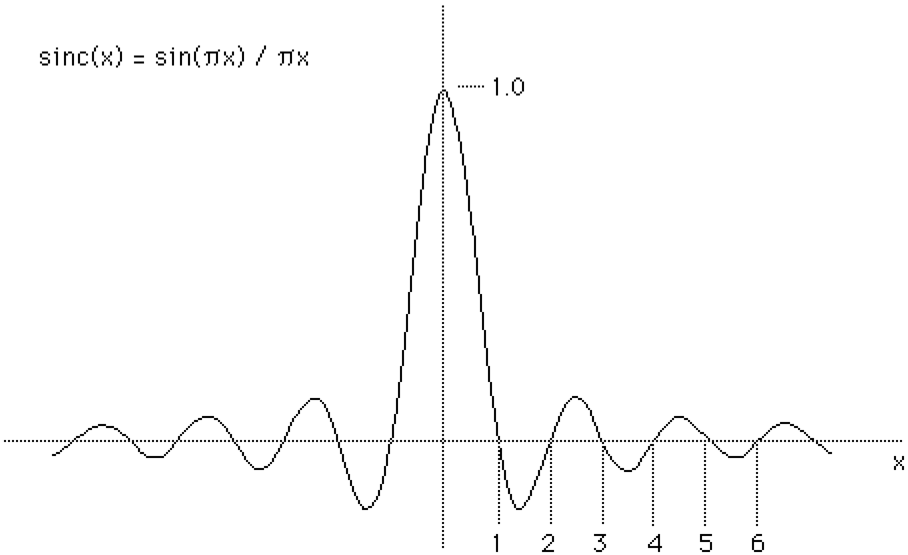
One of the simplest way — rectangular pulse, taking the limit $\Delta \rightarrow 0$.



But this is by no means the only way. One can construct a $\delta(t)$ function out of many other functions, *Eg.* Gaussian pulses, triangular pulses, sinc functions, *etc.*, as long as the pulses are short enough — much shorter than the characteristic time scale of the system.

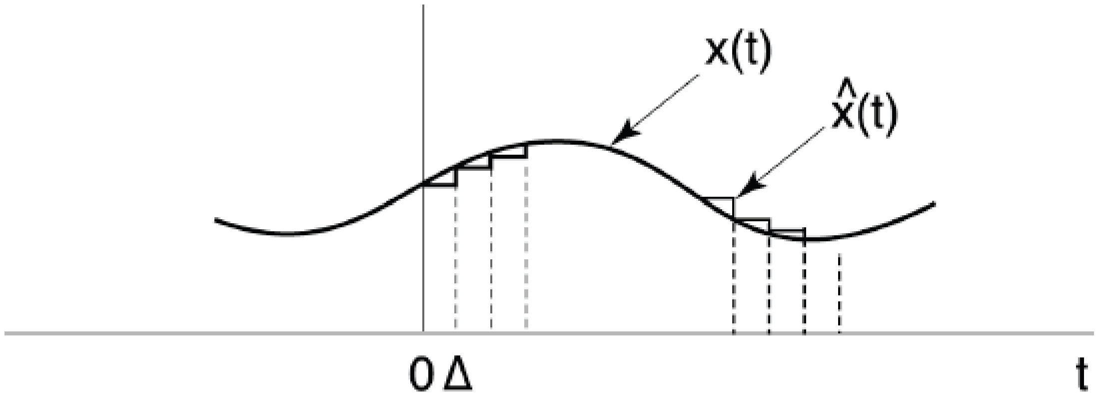

Gaussian or
"normal"
distribution
fg(x)
.0214
.0214
.00135
.1359
.3413
.3413
.1359
.00135
-3σ
-2σ
-σ
0
σ
2σ
3σ
x


Λ(t)
1
-1
1
t


sinc(x) = sin(πx) / πx
1.0
1
2
3
4
5
6
x

Representation of CT Signals

- Approximate any input $x(t)$ as a sum of shifted, scaled pulses (in fact, that is how we do integration)



$$\hat{x}(t) = x(k\Delta), \quad k\Delta < t < (k+1)\Delta$$

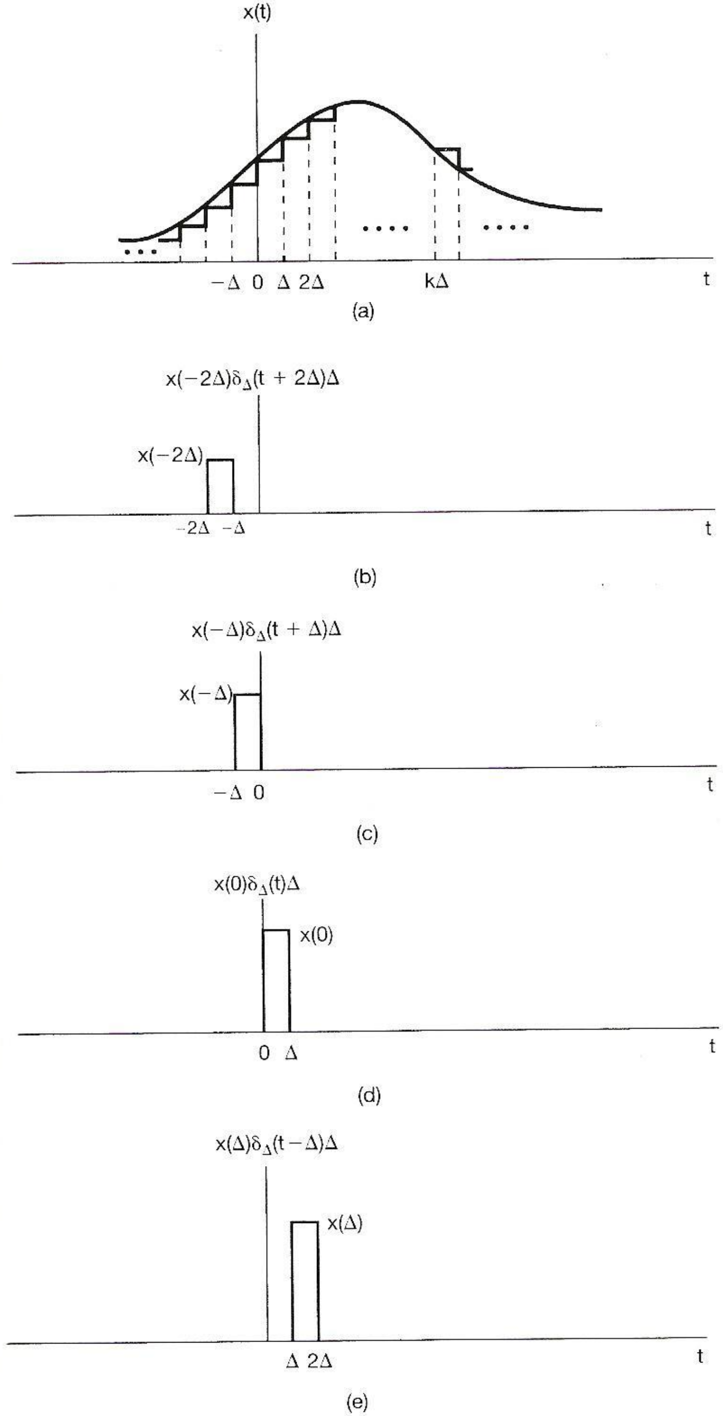
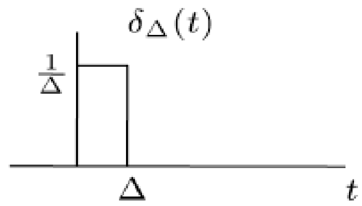
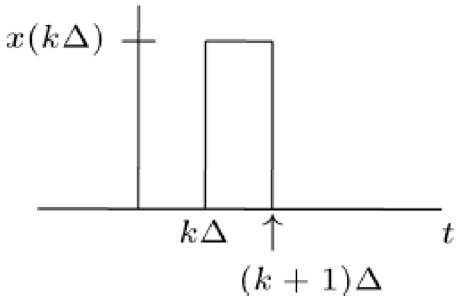


Figure 2.12 Staircase approximation to a continuous-time signal.

Representation of CT Signals (cont.)



$\delta_\Delta(t)$ has a unit area



$=$?

$$\Downarrow$$

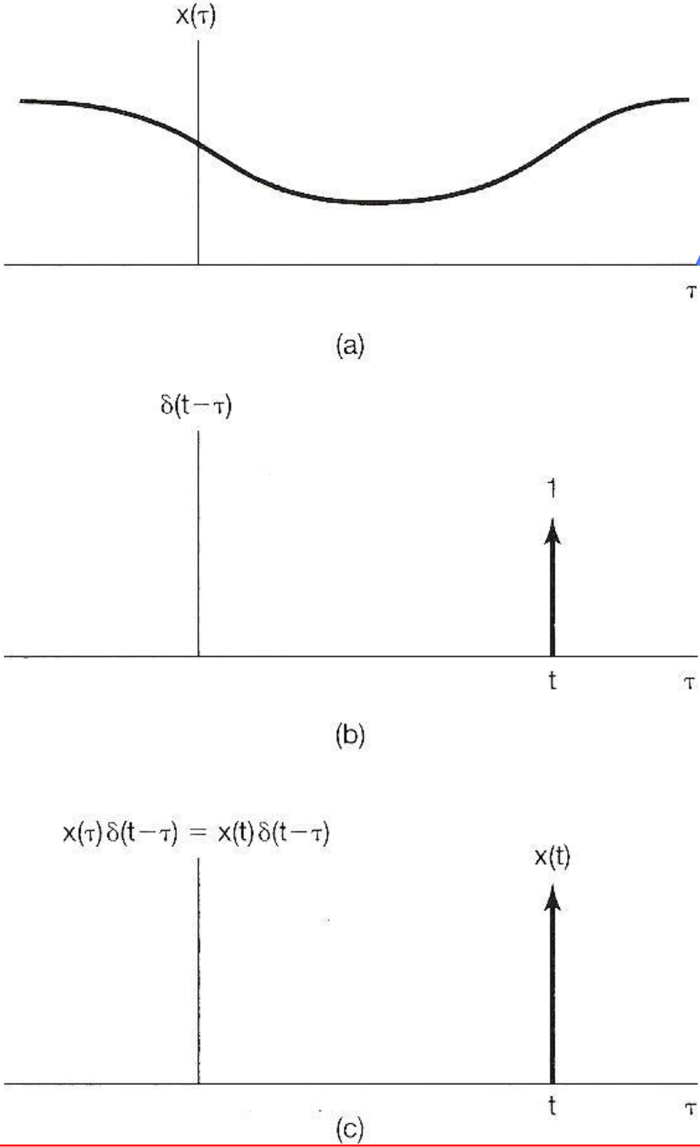
$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$$

$$\downarrow \quad \text{limit as } \Delta \to 0$$

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau}$$

Sifting property of the unit impulse

Recall, we have $x(\tau)\delta(t-\tau) = x(t)\delta(t-\tau)$, as a function of τ with *t* fixed

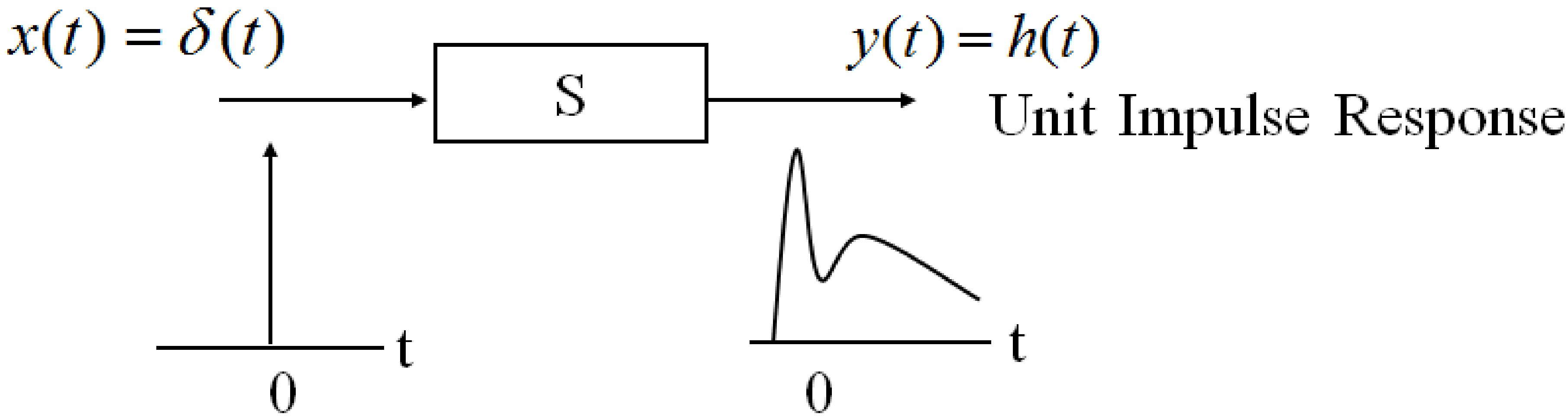


A function of τ with *t* fixed

Figure 2.14 (a) Arbitrary signal $x(\tau)$; (b) impulse $\delta(t-\tau)$ as a function of τ with t fixed; (c) product of these two signals.

$$\boxed{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t)\delta(t-\tau)d\tau = x(t)\int_{-\infty}^{\infty} \delta(t-\tau)d\tau = x(t)}$$

Unit Impulse Response



Response of a CT LTI System

$$x(t) \longrightarrow \boxed{\text{CT LTI}} \longrightarrow y(t)$$

- Now suppose the system is **LTI**, and define the *unit impulse response* $h(t)$:

$$\delta(t) \longrightarrow h(t)$$

$$\Downarrow$$

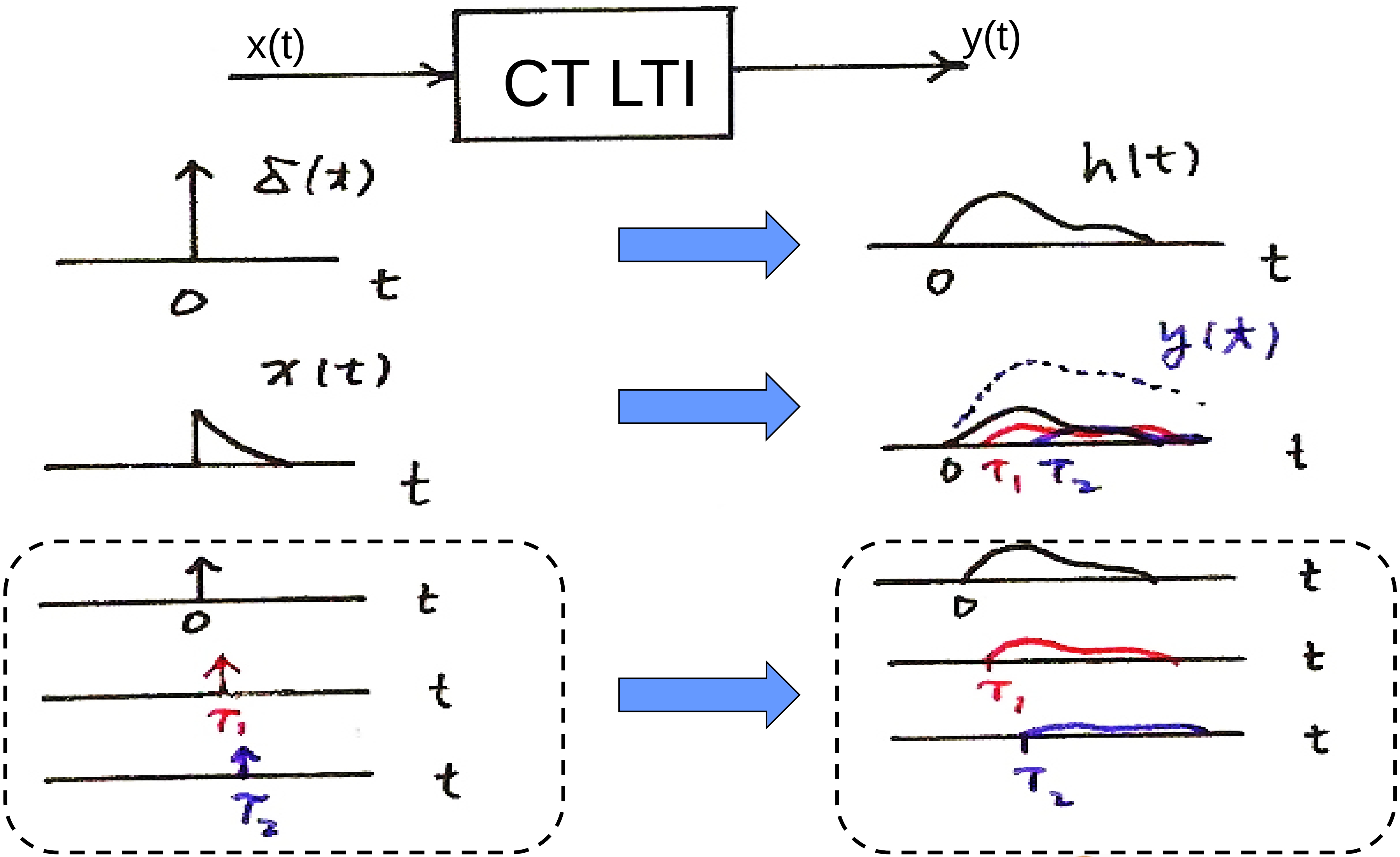
From Time-Invariance:

$$\delta(t-\tau) \longrightarrow h(t-\tau)$$

From Linearity:

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau \longrightarrow y(t) = \underbrace{\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau}_{\textit{Convolution Integration}} = x(t) * h(t)$$

Response of a CT LTI System (cont.)



Operation of CT Convolution

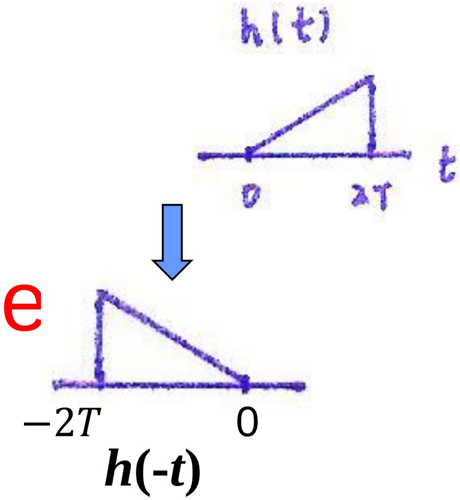
$$y(t) = x(t) * h(t) \equiv \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$ **Convolution Integral**

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$ **Convolution Sum**

- A different way to understand the convolution integral
  - ***y*(*t*) is a weighted integral of the input, where the weight on *x*(*τ*) is *h*(*t* - *τ*)**

$$h(\tau) \xrightarrow{\ Flip\ } h(-\tau) \xrightarrow{\ Slide\ } h(t-\tau) \xrightarrow{\ Multiply\ }$$

$$x(\tau)h(t-\tau) \xrightarrow{\ Integrate\ } \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$



flip, slide

⋮

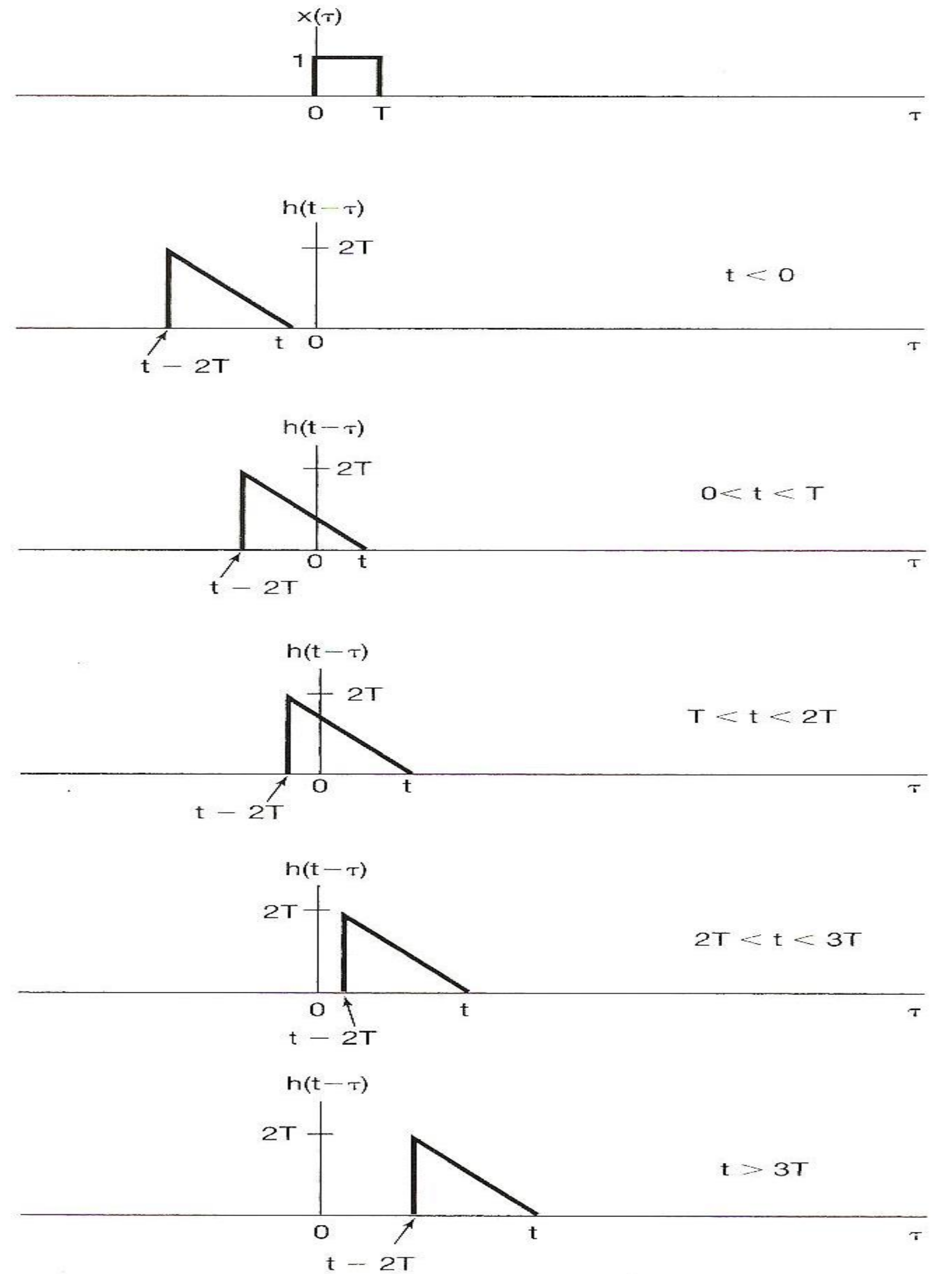


Figure 2.19 Signals $x(\tau)$ and $h(t-\tau)$ for different values of t for Example 2.7.

multiply

$$x(\tau)h(t-\tau)$$

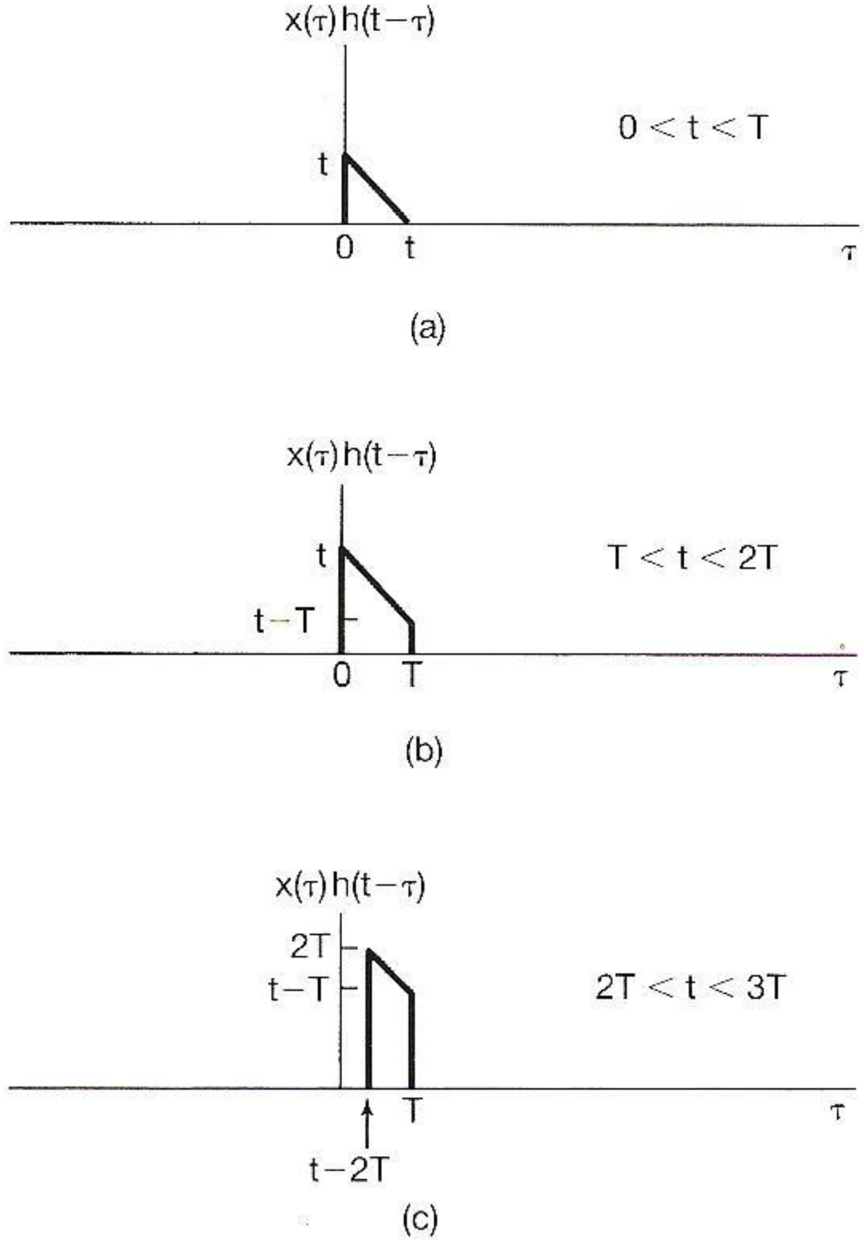


Figure 2.20 Product $x(\tau)h(t-\tau)$ for Example 2.7 for the three ranges of values of t for which this product is not identically zero. (See Figure 2.19.)

integrate

$$\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

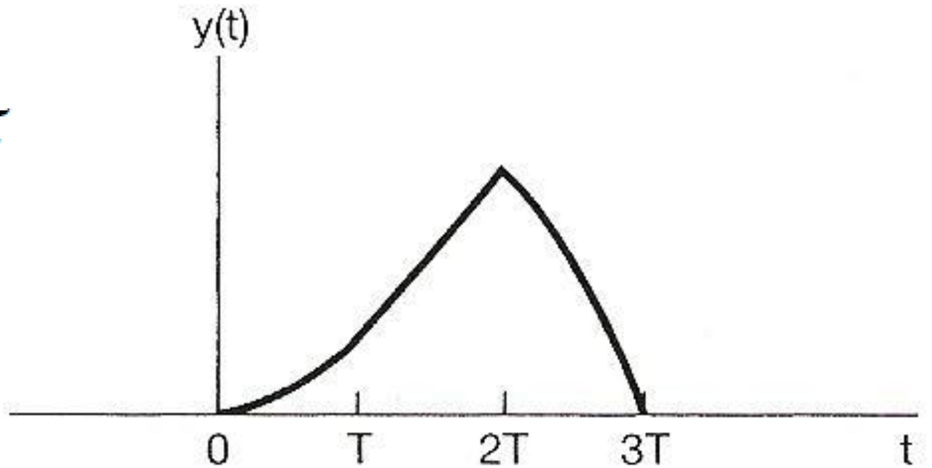
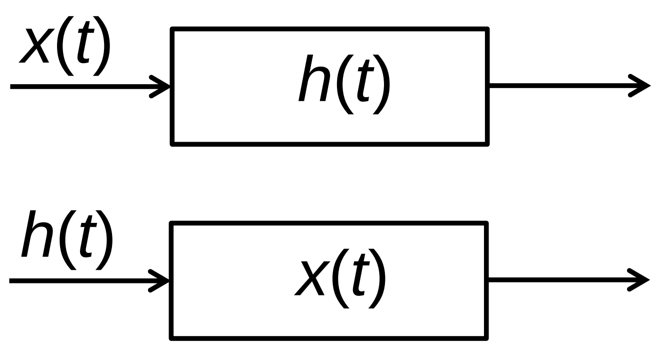


Figure 2.21 Signal $y(t) = x(t) * h(t)$ for Example 2.7.

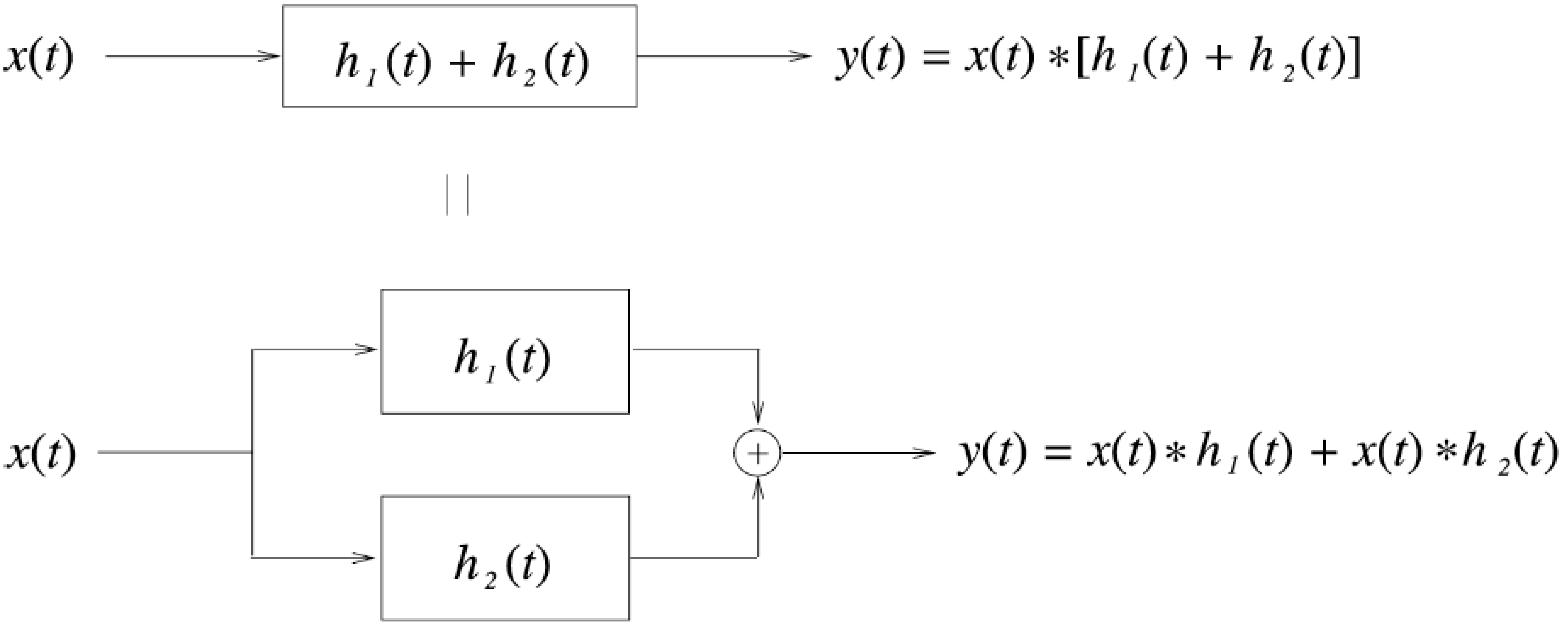
Property: Commutative

$$x(t) * h(t) = h(t) * x(t)$$

- The role of input signal and unit impulse response is interchangeable, giving the same output signal



Property: Distributive



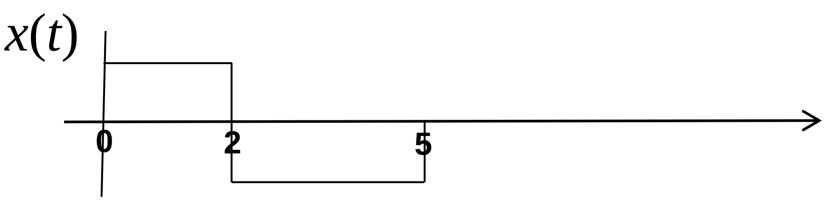
Property: Distributive (Cont.)

$$[x_1(t)+x_2(t)]*h(t)=x_1(t)*h(t)+x_2(t)*h(t)$$

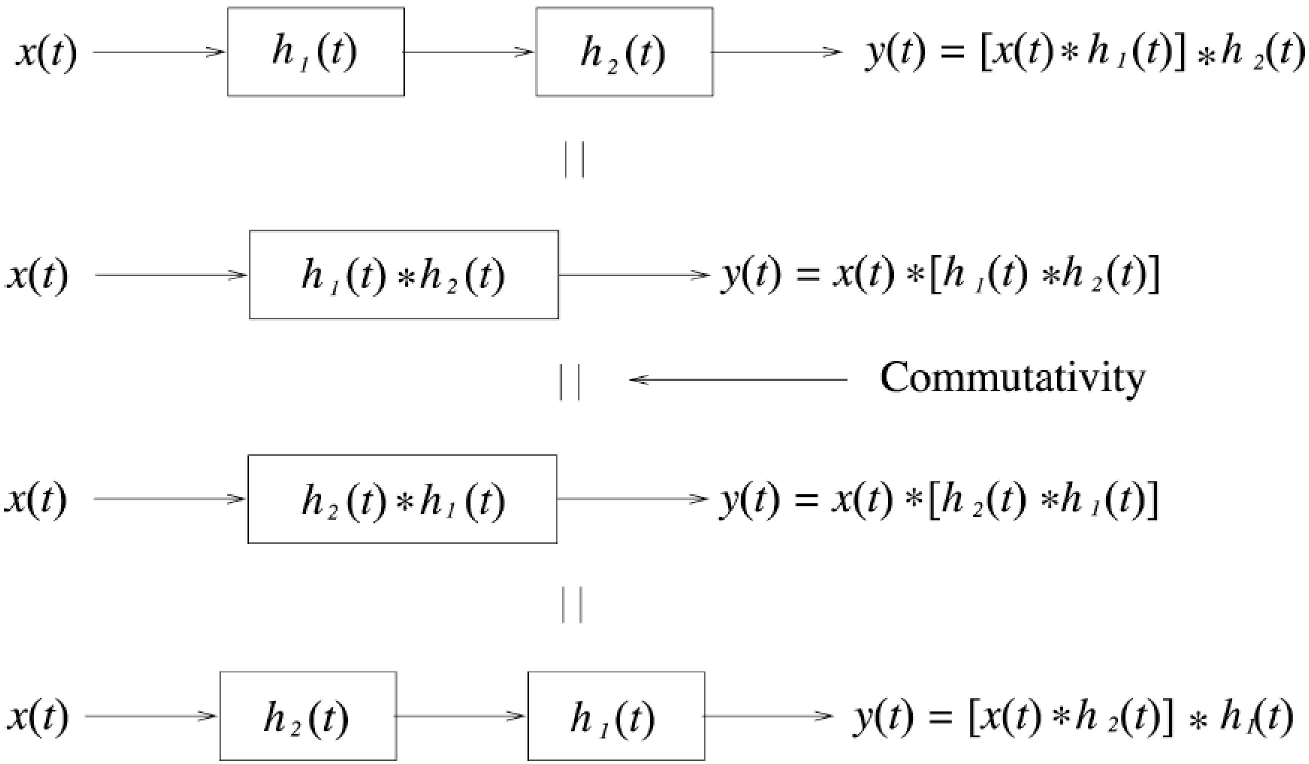
Problem 2.22 (b)

$$x(t)=u(t)-2u(t-2)+u(t-5)$$

$$h(t)=e^{2t}u(1-t)$$



Properties: Associative

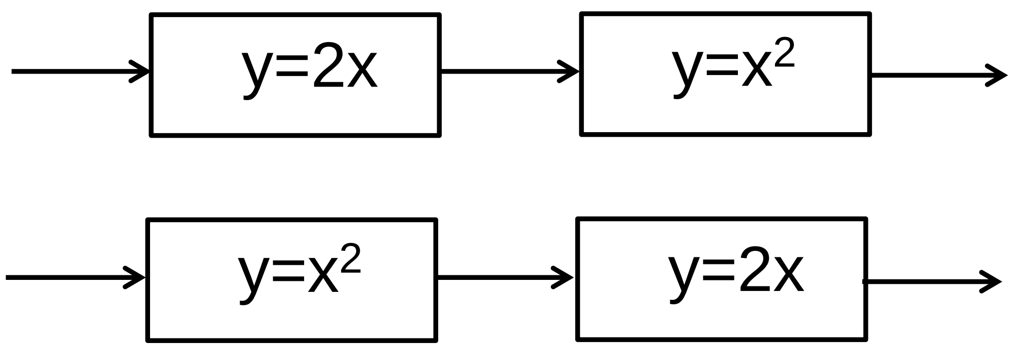


– Cascade of two systems gives an unit impulse response which is the convolution of the unit impulse responses of the two individual systems

– The behavior of a cascade of two systems is independent of the order in which the two systems are cascaded

Properties: Associative (Cont.)

- The order in which non-linear systems are cascaded cannot be changed.
- e.g.



Property: Memory/Memoryless

– A linear, time-invariant, causal system is memoryless only

if $h[n] = K\delta[n]$ $\qquad$ $h(t) = K\delta(t)$

$y[n] = Kx[n]$ $\qquad$ $y(t) = Kx(t)$

if k=1 further, they are identity systems

$$y[n] = x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = x[n] * \delta[n]$$

$$y(t) = x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t) * \delta(t)$$

Property: Invertibility

2.50. Consider the cascade of two systems shown in Figure P2.50. The first system, A, is known to be LTI. The second system, B, is known to be the inverse of system A. Let $y_1(t)$ denote the response of system A to $x_1(t)$, and let $y_2(t)$ denote the response of system A to $x_2(t)$.

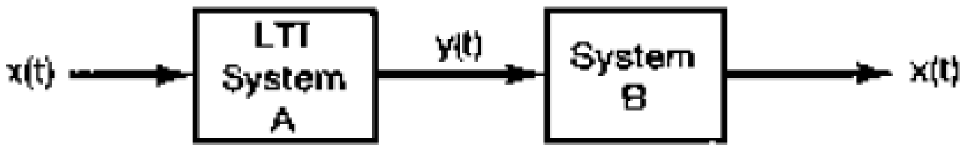


Figure P2.50

(a) What is the response of system B to the input $ay_1(t) + by_2(t)$, where a and b are constants?

(b) What is the response of system B to the input $y_1(t - \tau)$?

Property: Causality

Causality: CT LTI system is causal $\leftrightarrow h(t) = 0$, at $t < 0$

- This is because that the input unit impulse function δ(t)=0 at t<0

$t-\tau \geq 0$, or $\tau \leq t$

As a result:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

$y(t)$ only depends on $x(\tau < t)$.

Property: Stability

BIBO Stability: CT LTI system is stable $\Leftrightarrow \int_{-\infty}^{+\infty} |h(\tau)|\, d\tau < \infty$

➔ Sufficient condition: For $|x(t)| \le x_{\max} < \infty$.

$$|y(t)| = \left| \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau \right| \le x_{\max} \left| \int_{-\infty}^{+\infty} h(t-\tau) d\tau \right| < \infty.$$

➔ Necessary condition: Suppose $\int_{-\infty}^{+\infty} |h(\tau)| d\tau = \infty$

Let $x(t) = h^*(-t)/|h^*(-t)|$, then $|x(t)| \equiv 1$ bounded

$$\text{But } y(0) = \int_{-\infty}^{+\infty} x(\tau) h(-\tau) d\tau = \int_{-\infty}^{+\infty} \frac{h^*(-\tau) h(-\tau)}{|h(-\tau)|} d\tau = \int_{-\infty}^{+\infty} |h(-\tau)| d\tau = \infty$$

Property: Unit Step Response

unit step function → unit step response

Step response

$$s(t) = u(t) * h(t) = h(t) * u(t) = \int_{-\infty}^{t} h(\tau)\,d\tau$$

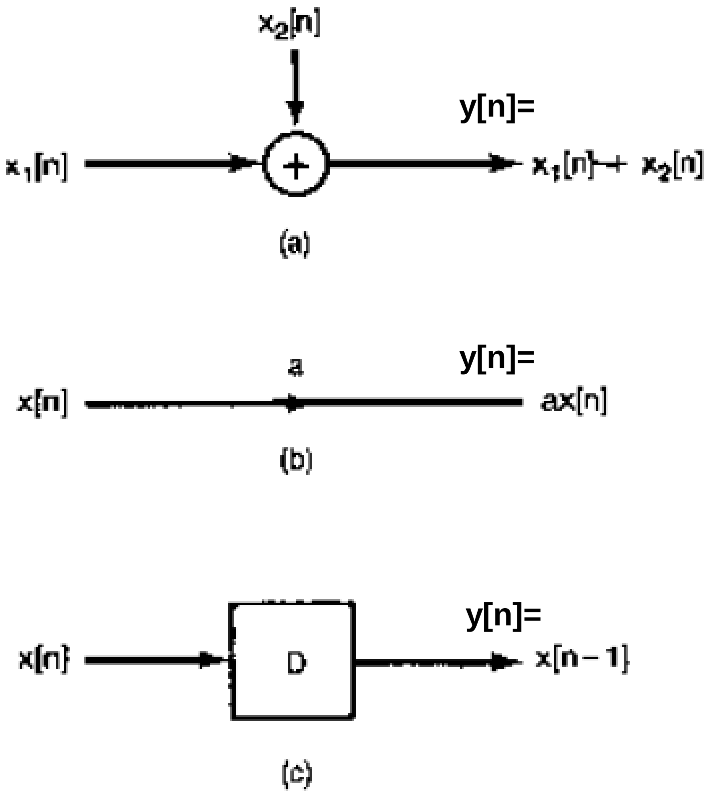
$x(t)=\delta(t) \longrightarrow \boxed{h(t)} \longrightarrow$ $h(t)$: unit impulse response

$x(t)=u(t) \longrightarrow \boxed{h(t)} \longrightarrow$ $s(t)$: unit step response

The relation between unit step function and unit impulse function

$$h(t) = \frac{ds(t)}{d(t)} = s'(t)$$

Block diagram representation of 1-st order system



Problem 2.38

$$y[n]=\frac{1}{3}y[n-1]+\frac{1}{2}x[n]$$

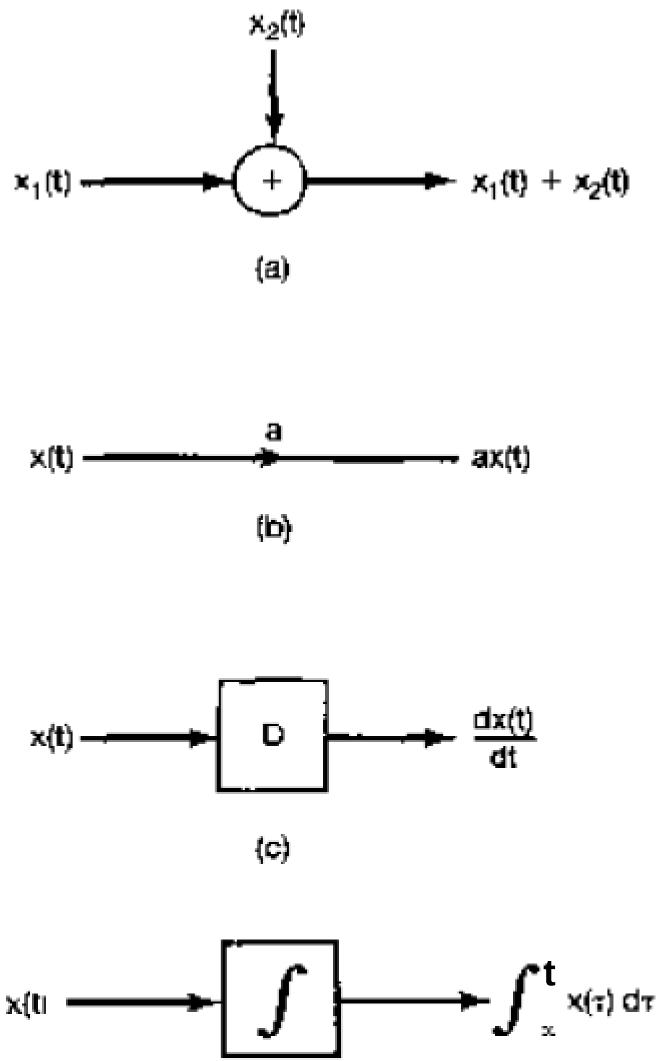
$$y[n]=\frac{1}{3}y[n-1]+x[n-1]$$

How about

$$y[n]=\frac{1}{3}y[n-1]+2y[n-2]+x[n-1]+3x[n-2]$$

$$\sum_{k=0}^{N}a_k y[n-k]=\sum_{k=0}^{N}b_k x[n-k]$$

Class practice

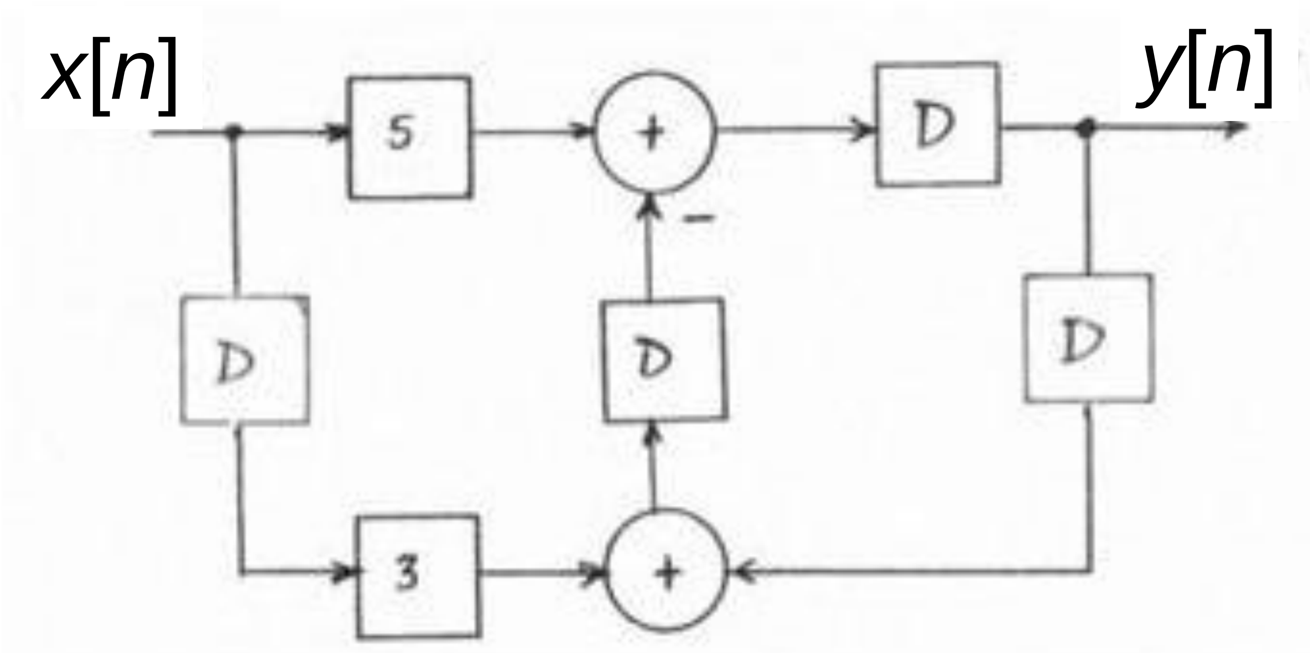


Problem 2.39

$$y(t) = -\frac{1}{2} dy(t)/dt + 4x(t)$$

$$dy(t)/dt + 3y(t) = x(t)$$

From block diagram to difference equation



More about $\delta(t)$: Operational Definition

A function can be defined by

- what it is at each value of the independent variable, or
- what it does under some mathematical operation (such as an integral or a convolution), or how it behaves with a system, or some mathematical constraints: Singularity Function

Operational Definition of Unit Impulse

- $\delta(t)$ can be defined as
  - $x(t) = x(t) * \delta(t)$ for any $x(t)$, $x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$
  - a signal which, when applied to a system, yields the impulse response $h(t) = h(t) * \delta(t)$
  - such definition leads to, or is equivalent to, other properties of $\delta(t)$,

    $$\int_{-\infty}^{\infty} \delta(\tau)d\tau = 1$$

    $$\int_{-\infty}^{\infty} g(\tau)\delta(\tau)d\tau = g(0)$$

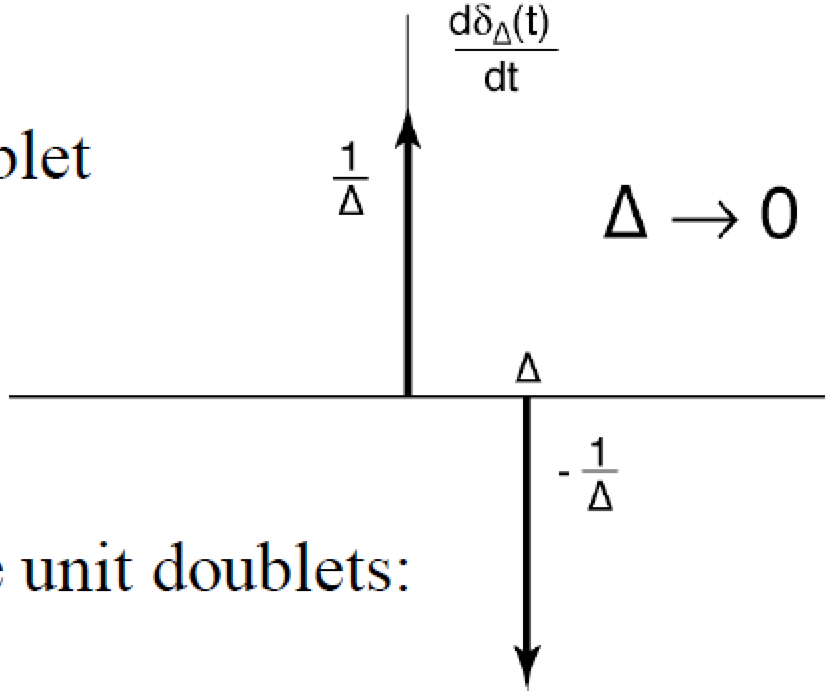
    they are also "operational definition" of $\delta(t)$
  - Such definition also leads to the sampling property $\mathrm{f}(t)\delta(t) = \mathrm{f}(0)\delta(t)$

The Unit Doublet — Differentiator

$$x(t) \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow y(t) = \frac{dx(t)}{dt}$$

Impulse response = unit doublet

$$\boxed{u_1(t) = \frac{d\delta(t)}{dt}}$$



The operational definitions of the unit doublets:

$$x(t) * u_1(t) = \frac{dx(t)}{dt}$$

Triplets and beyond!

$n > 0$

$$u_n(t) = \underbrace{u_1(t) * \cdots * u_1(t)}_{n\ times}$$

Operational definitions:

$$x(t) * u_n(t) = \frac{d^n x(t)}{dt^n} \qquad (n > 0)$$

Integrators

$$x(t) \longrightarrow \boxed{\int} \longrightarrow y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

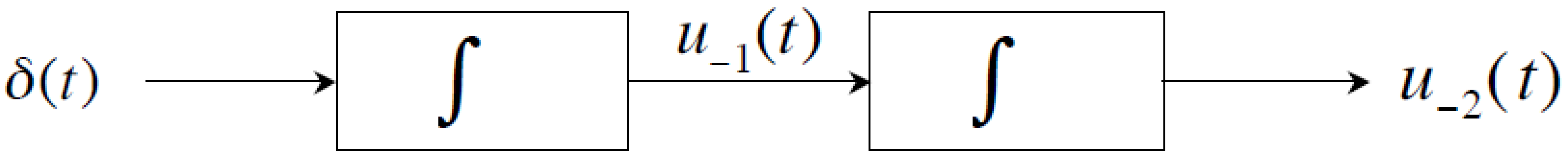
Impulse response: $\boxed{u_{-1}(t) \equiv u(t)}$

Operational definition: $x(t) * u_{-1}(t) = \int_{-\infty}^{t} x(\tau)d\tau$

Cascade of n integrators:

$$u_{-n}(t) = \underbrace{u_{-1}(t) * \cdots * u_{-1}(t)}_{n\ times} \qquad (n > 0)$$

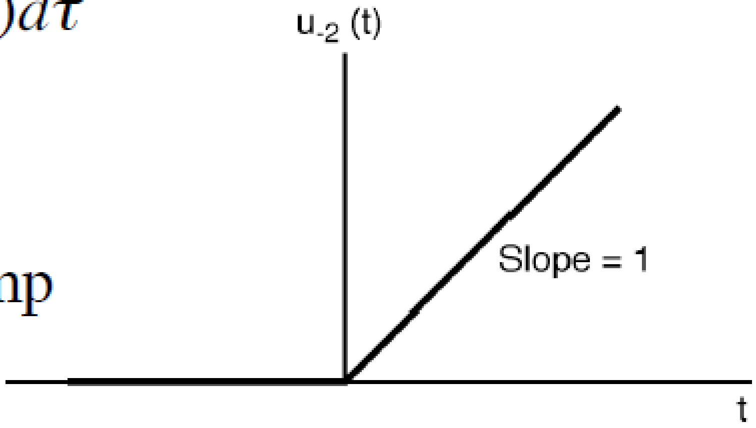
Integrators (Cont.)



$$u_{-2}(t) = \int_{-\infty}^{t} u_{-1}(\tau)\,d\tau = \int_{-\infty}^{t} u(\tau)\,d\tau$$

$$= u(t)\int_{0}^{t} d\tau$$

$$= t \cdot u(t) \qquad \text{the unit ramp}$$



More generally, for $n > 0$

$$u_{-n}(t) = \frac{t^{(n-1)}}{(n-1)!}u(t)$$

Notation

Define $\boxed{u_0(t) = \delta(t)}$

Then $$u_n(t) * u_m(t) = u_{n+m}(t)$$

n and m can be $\pm$

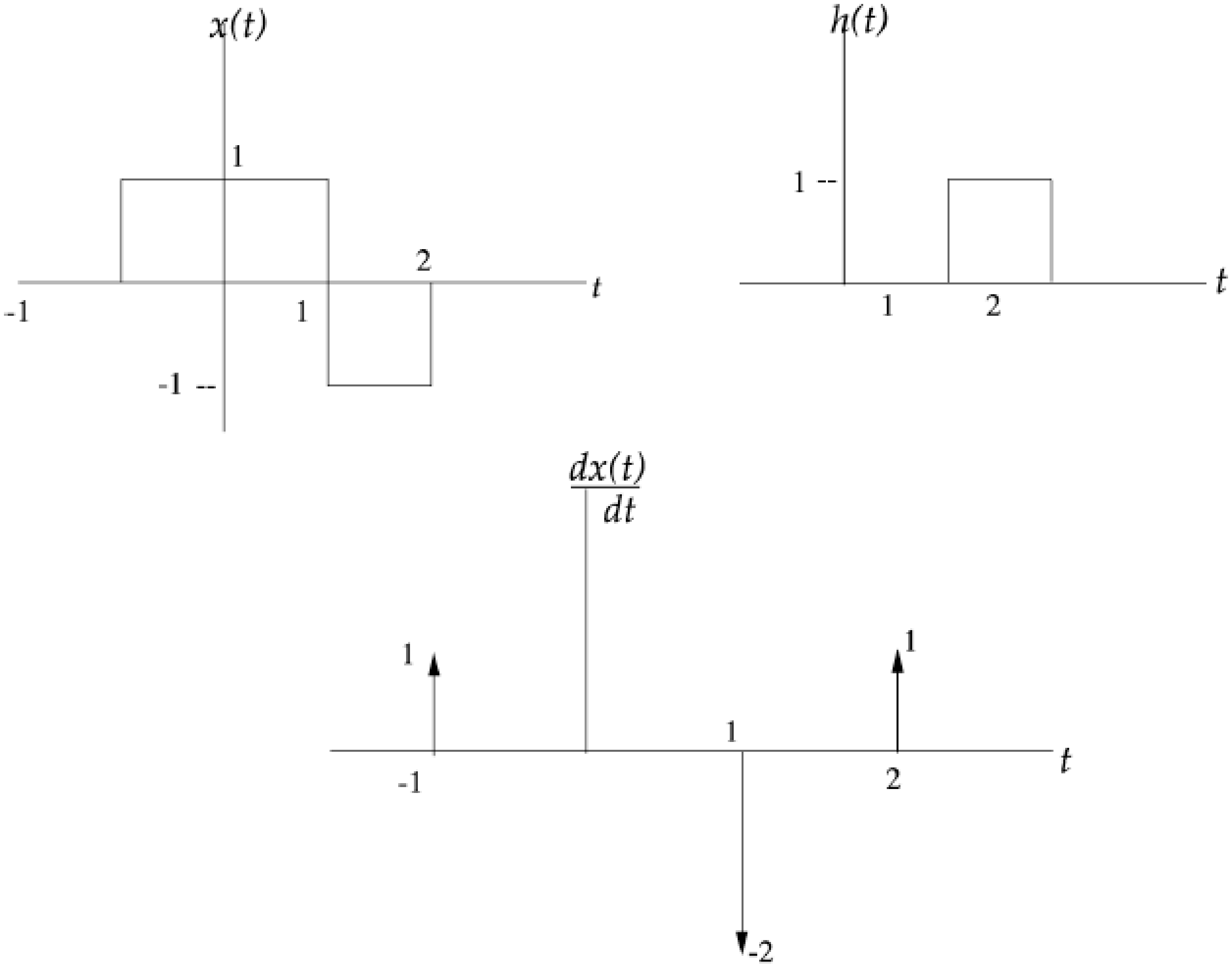
E.g.

$$u_1(t) * u_{-1}(t) = u_0(t)$$

$$||$$

$$\left(\frac{d}{dt}u(t)\right) = \delta(t)$$

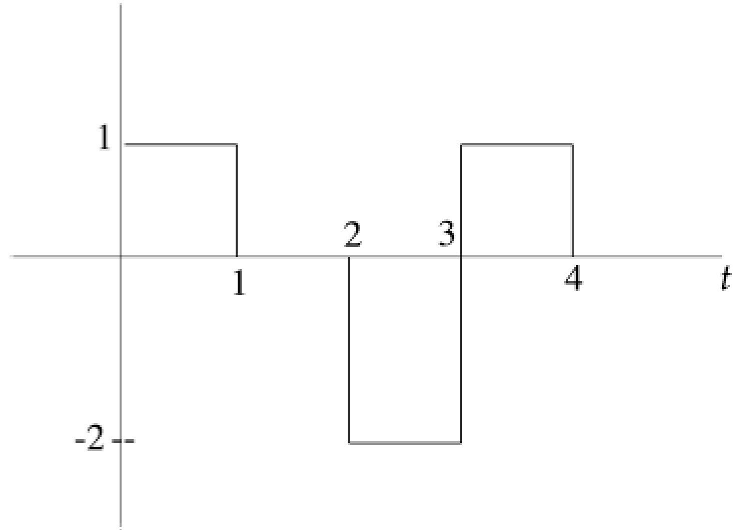
Example: Calculating $x(t)*h(t)$



$$\frac{dx(t)}{dt} = \delta(t+1) - 2\delta(t-1) + \delta(t-2)$$

Example (Cont.)

$$\frac{dx(t)}{dt} * h(t) = h(t+1) - 2h(t-1) + h(t-2)$$



$$x(t) * h(t) = u_{-1}(t) * \{[u_1(t) * x(t)] * h(t)]\}$$

$$= \int_{-\infty}^{t} \left[\frac{dx(\tau)}{d\tau} * h(\tau) \right] d\tau$$