Assignments for Chapter 1

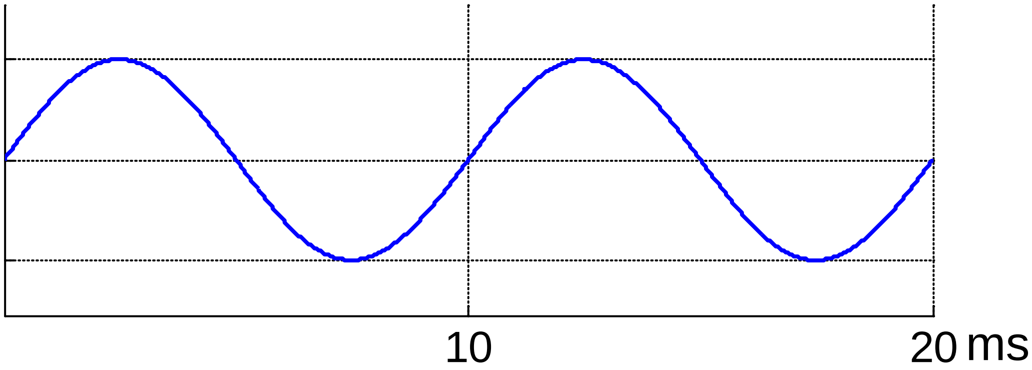
- 1.20
- 1.21 (c) (f)
- 1.24 (a)
- 1.26
- 1.27 (a) (f)
- 1.41

Tutorial Questions (Week 3)

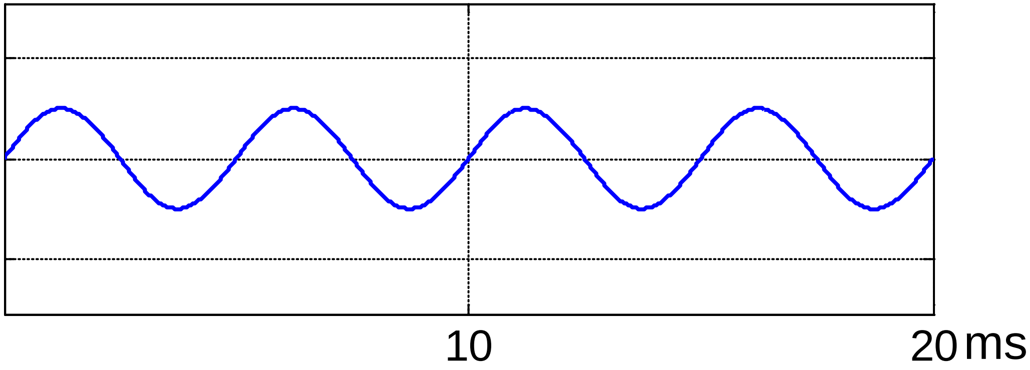
- Basic Problems with Answers 1.15, 1.18
- Basic Problems 1.29, 1.31
- Advanced Problems 1.33, 1.42

- 1教306
- 周1、2、3、4晚上9:00-10:00

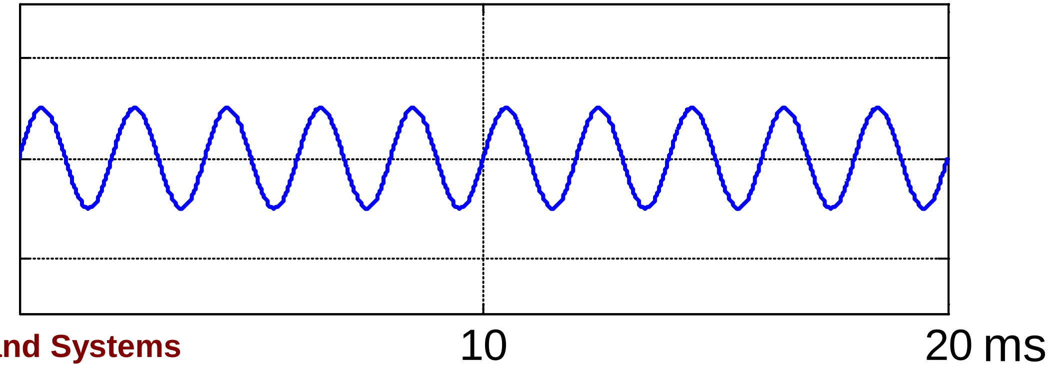
Harmonically Related Signal Sets



T_0=10 ms
f_0 =100 Hz



T_1=5 ms
f_1 =200 Hz



T_2=2 ms
f_2 =500 Hz

Harmonically Related Signal Sets (cont.)

- A set of periodic exponentials which have a common period T_0.

$$\{\phi_k(t) = e^{jk\omega_0 t}, k = 0, \pm 1, \pm 2, \ldots\}$$

must be integer multiple

fundamental frequency $|k\omega_0|$

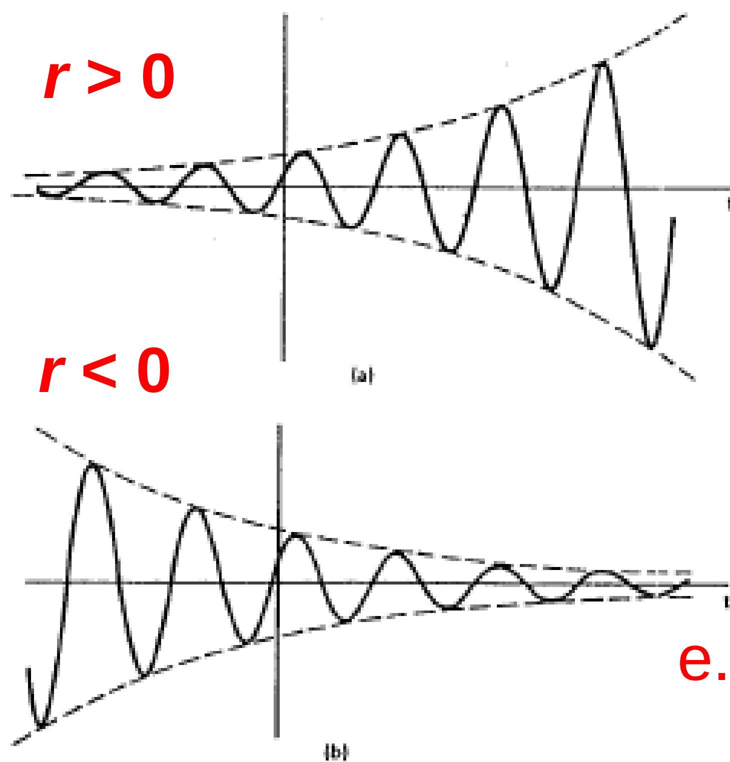
fundamental period $T_k = \dfrac{2\pi}{|k\omega_0|} = \dfrac{T_0}{|k|}, \quad T_0 = \dfrac{2\pi}{\omega_0}$

- The *k*th harmonic $\phi_k(t)$ is periodic with period T_0, as it goes through $|k|$ of its fundamental periods T_k in duration of length T_0.

General Complex Exponential Signals - CT

- General format (C and a are complex numbers)

$$x(t) = Ce^{at} = |C|\, e^{j\theta} \cdot e^{(r+j\omega_0)t} = |C|\, e^{rt} \cdot e^{j(\omega_0 t+\theta)}$$

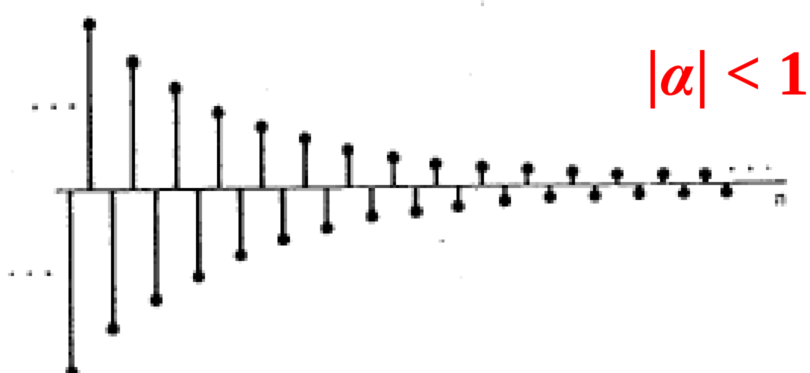


e.g., damped sinusoids

General Complex Exponential Signals - DT

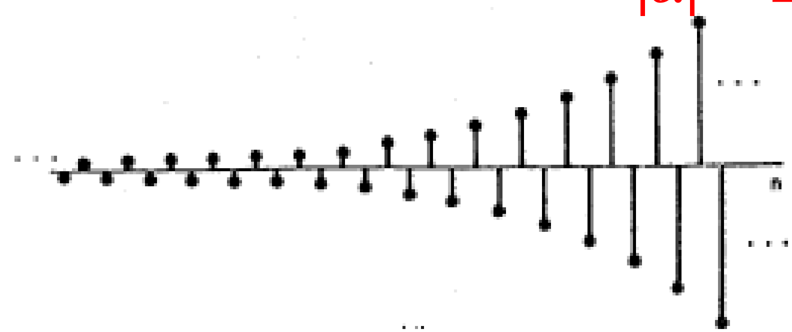
- General format (C and α are complex numbers)

$$x[n] = C\alpha^n = |C|e^{j\vartheta} \cdot |\alpha|^n e^{j\omega_0 n} = |C||\alpha|^n e^{j(\omega_0 n+\vartheta)}$$



(c)

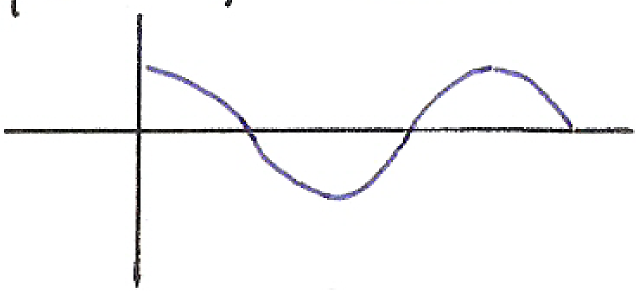
$|\alpha| > 1$



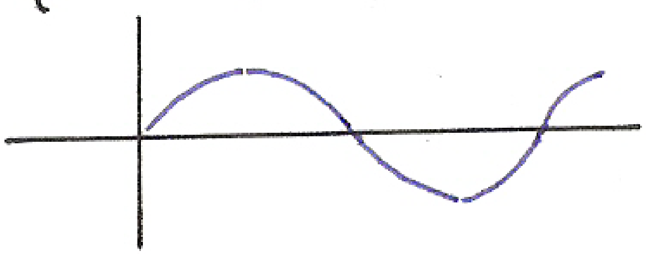
Periodic Complex Exponential/Sinusoidal Signals

$$x(t) = e^{j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t)$$

$$\mathrm{Re}\{e^{j\omega_0 t}\} = \cos\omega_0 t$$



$$\mathrm{Im}\{e^{j\omega_0 t}\} = \sin\omega_0 t$$



-Fundamental frequency: ω_0

-Fundamental period: $T_0 = \dfrac{2\pi}{\omega_0}$

-In CT, $e^{j\omega_0 t}$ **always periodic**

-Distinct signals for distinct values of ω_0**.**

-Rapid variation with large ω_0

Periodicity Properties of DT Complex Exponentials

Important Differences Between Continuous-time and Discrete-time Exponential/Sinusoidal Signals

– For discrete-time, signals with frequencies ω_0 and $\omega_0 + m \cdot 2\pi$ are identical. This is Not true for continuous-time.

Proof:

$$\boxed{e^{j(\omega_0 + m\cdot 2\pi)n} = e^{j\omega_0 n} \cdot e^{jm\cdot 2\pi n} = e^{j\omega_0 n}}$$

$$e^{j(\omega_0 + \mathrm{X})t} = e^{j(\omega_0 + \mathrm{X})t}$$

or $e^{j(\omega_0 + m\cdot 2\pi)t} \neq e^{j\omega_0 t}$

Periodicity Properties of DT Complex Exponentials (cont.)

Understanding:

- We need only consider a frequency interval of length 2π, and on most cases, we use the interval: $0 \le \omega_0 < 2\pi$, or $-\pi \le \omega_0 < \pi$

- $e^{j\omega_0 n}$ does ***not*** have a continually increasing rate of oscillation as ω_0 is increased in magnitude.

  low-frequency (slowly varying): ω_0 near 0, 2π, …, or 2k • π

  high-frequency (rapid variation): ω_0 near ± π, …, or (2k+1) • π

$$e^{j(2k+1)\pi n} = e^{j\pi n} = (e^{j\pi})^n = (-1)^n$$

$$e^{j2\pi n} = (e^{j2\pi})^n = (1)^n = 1$$

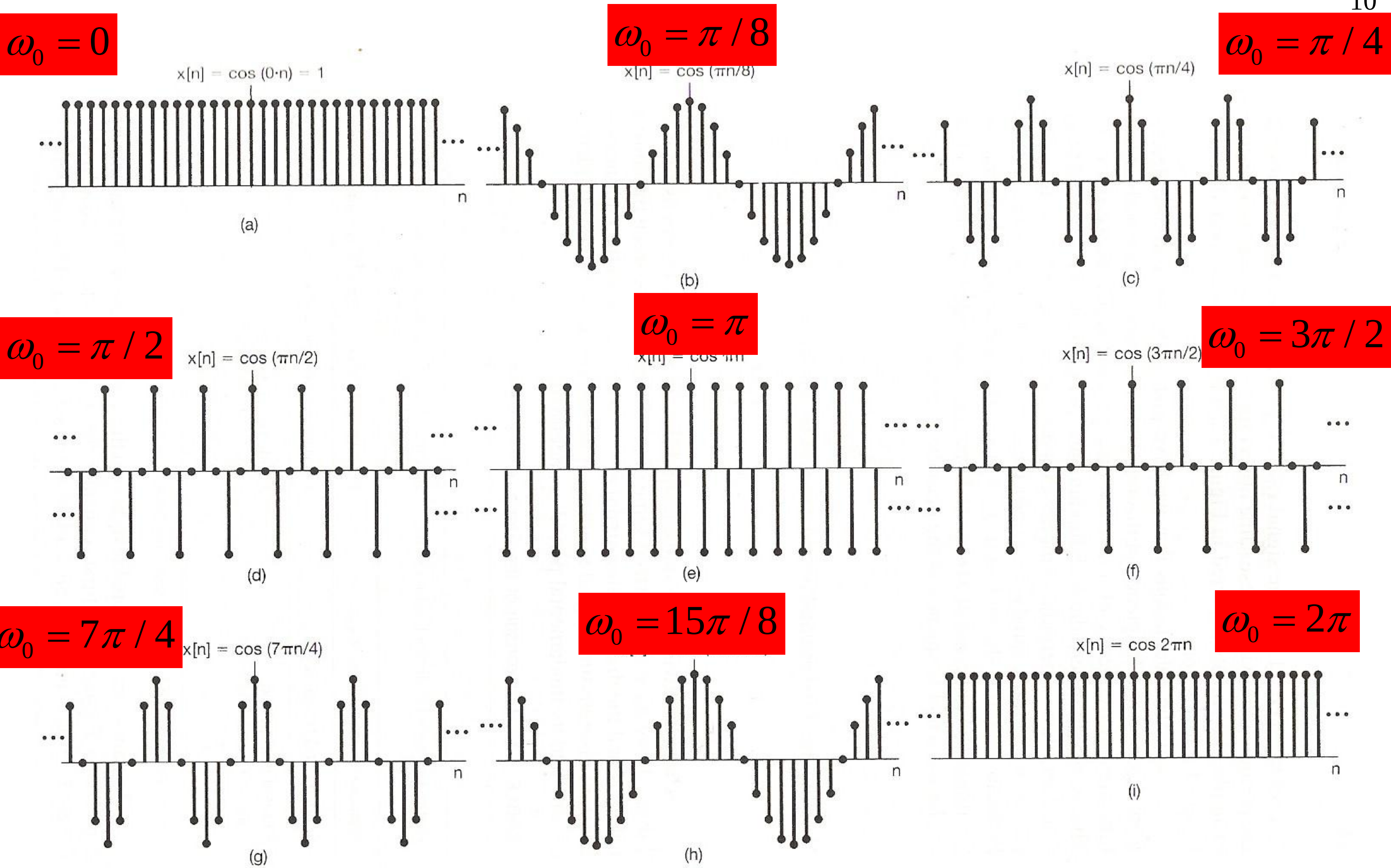


Figure 1.27 Discrete-time sinusoidal sequences for several different frequencies.

Periodicity Properties of DT Complex Exponentials (cont.)

Important Differences Between Continuous-time and Discrete-time Exponential/Sinusoidal Signals

- For discrete-time, ω_0 is usually defined only for $[-\pi, \pi]$ or $[0, 2\pi]$. For continuous-time, ω_0 is defined for $(-\infty, \infty)$
- For discrete-time, the signal is periodic only when $\omega_0 N = 2\pi m$

Why?

$$e^{j\omega_0 n} = e^{j\omega_0 (n+N)} \rightarrow e^{j\omega_0 N} = 1 \rightarrow \omega_0 N = 2\pi m$$

or $$\frac{\omega_0}{2\pi} = \frac{m}{N}$$

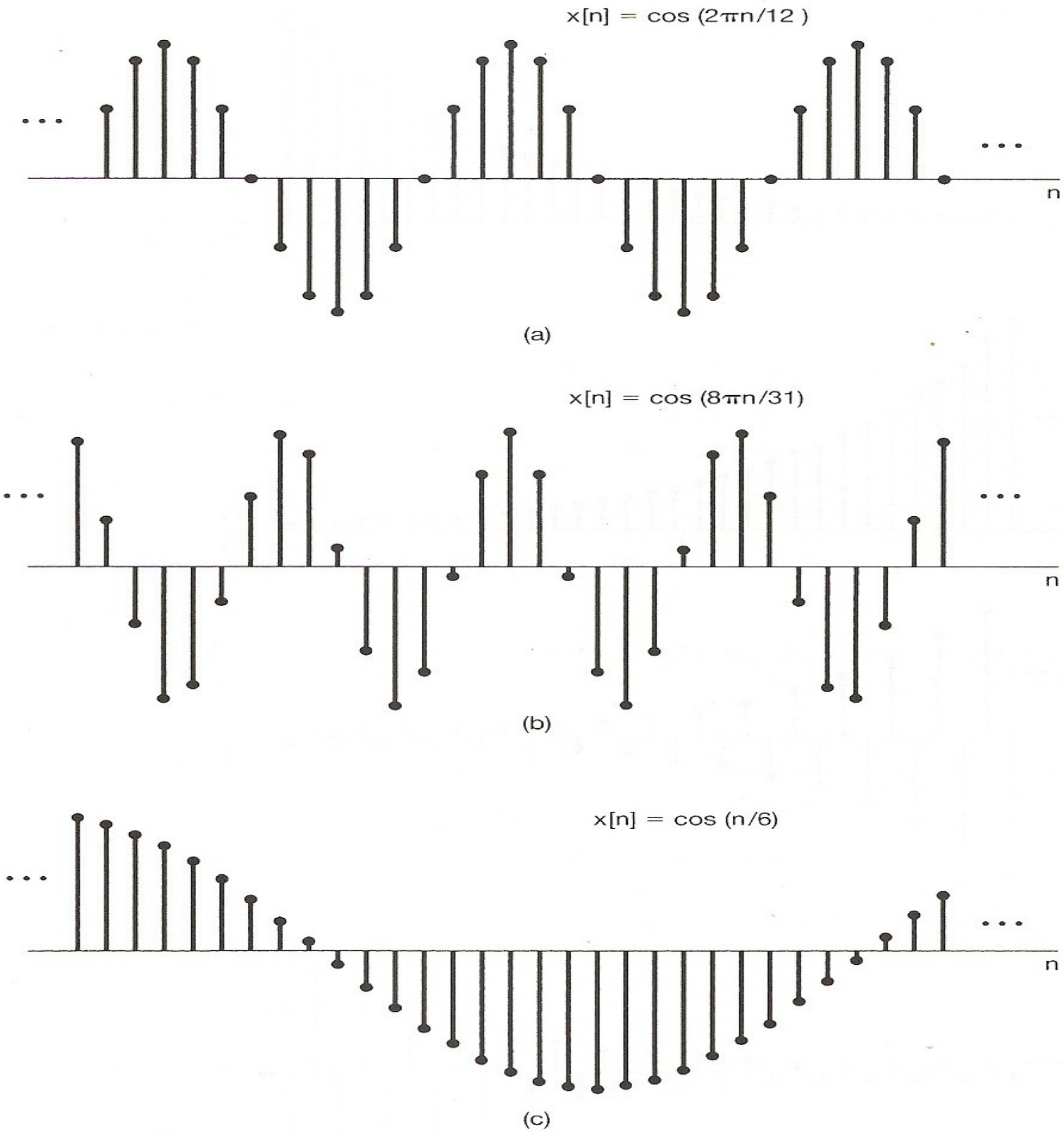


Figure 1.25 Discrete-time sinusoidal signals.

$$\frac{\omega_0}{2\pi} = \frac{2\pi/12}{2\pi} = \frac{1}{12}$$

$$\frac{\omega_0}{2\pi} = \frac{8\pi/31}{2\pi} = \frac{4}{31}$$

$$\frac{\omega_0}{2\pi} = \frac{1/6}{2\pi} = \frac{1}{12\pi}$$

Periodicity Properties of DT Complex Exponentials (cont.)

- How to determine the fundamental frequency and fundamental period of a periodic signal $e^{j\omega_0 n}$?
  - This is different from CT periodic signal $e^{j\omega_0 t}$
  - We have $\omega_0 N = 2\pi m$

    Fundamental frequency: $\dfrac{2\pi}{N} = \dfrac{\omega_0}{m}$

    Fundamental period: $N = m(\dfrac{2\pi}{\omega_0})$

See comparison between CT and DT signals in Table 1.1

Periodicity Properties of DT Complex Exponentials (cont.)

Harmonically related discrete-time signal sets

$$\{\phi_k[n] = e^{jk(\frac{2\pi}{N})n} \quad , \qquad \mathrm{k} = 0, \pm 1, \pm 2, \ldots\ldots\}$$

all with common period N

$$\phi_{k+N}[n] = \phi_k[n]$$

Proof:

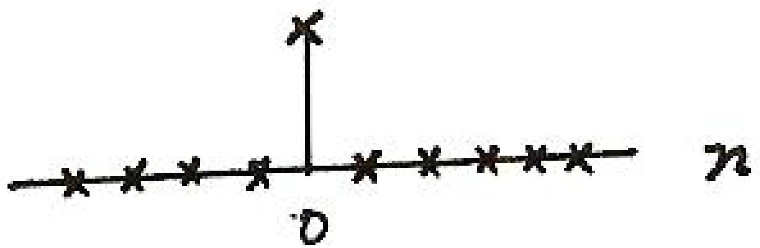
$$\boxed{\phi_{k+N}[n] = e^{j(k+N)(\frac{2\pi}{N})n} = e^{jk(\frac{2\pi}{N})n} \cdot e^{j2\pi n} = e^{jk(\frac{2\pi}{N})n} = \phi_k[n]}$$

This is different from continuous case. Only N distinct signals in this set.

Unit Impulse (or Unit Sample) Function

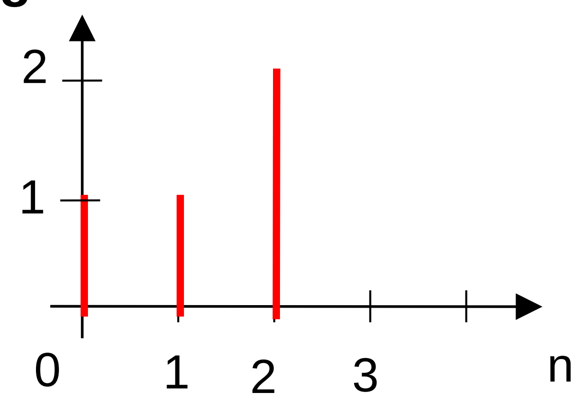
Discrete-time

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$



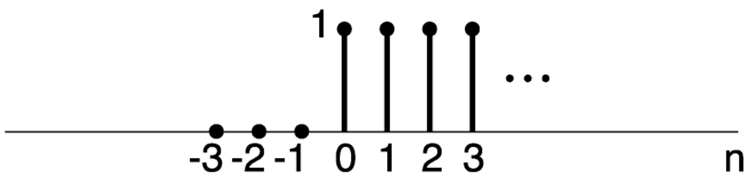
- As a basic building function, we can use unit impulse function to represent other different signals.

e.g.1



e.g.2

$$x[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

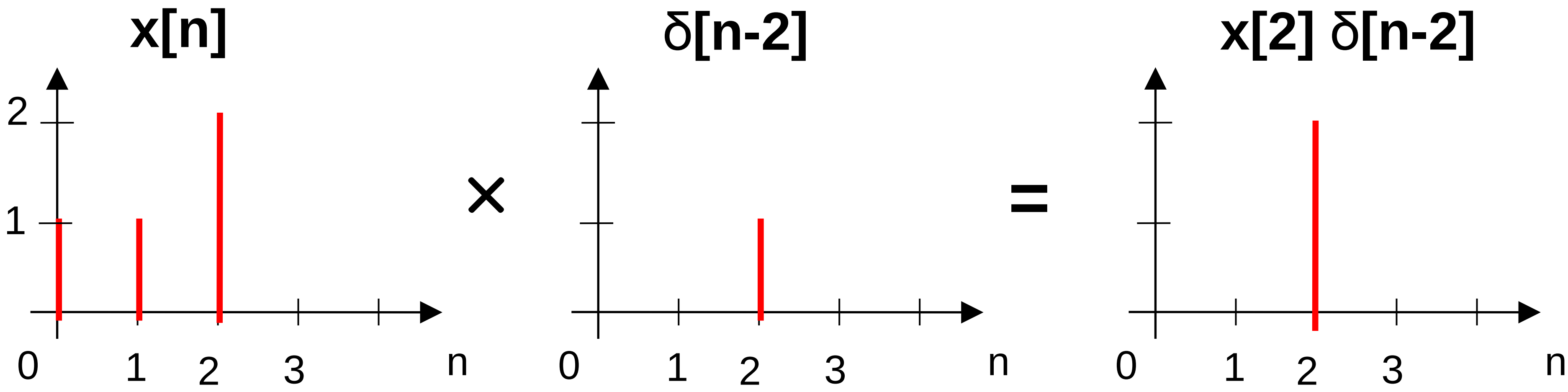


Unit Impulse Function (cont.)

- Sampling property

$$x[n]\ \delta[n] \quad = x[0]\ \delta[n]$$

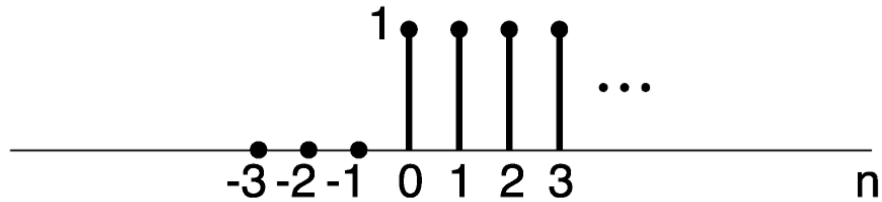
$$x[n]\ \delta[n-n_0] = x[n_0]\ \delta[n-n_0]$$



Unit Step Function

Discrete-time

$$x[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



Relation between unit impulse and unit step functions

– First difference

$$\delta[n] = u[n] - u[n-1]$$

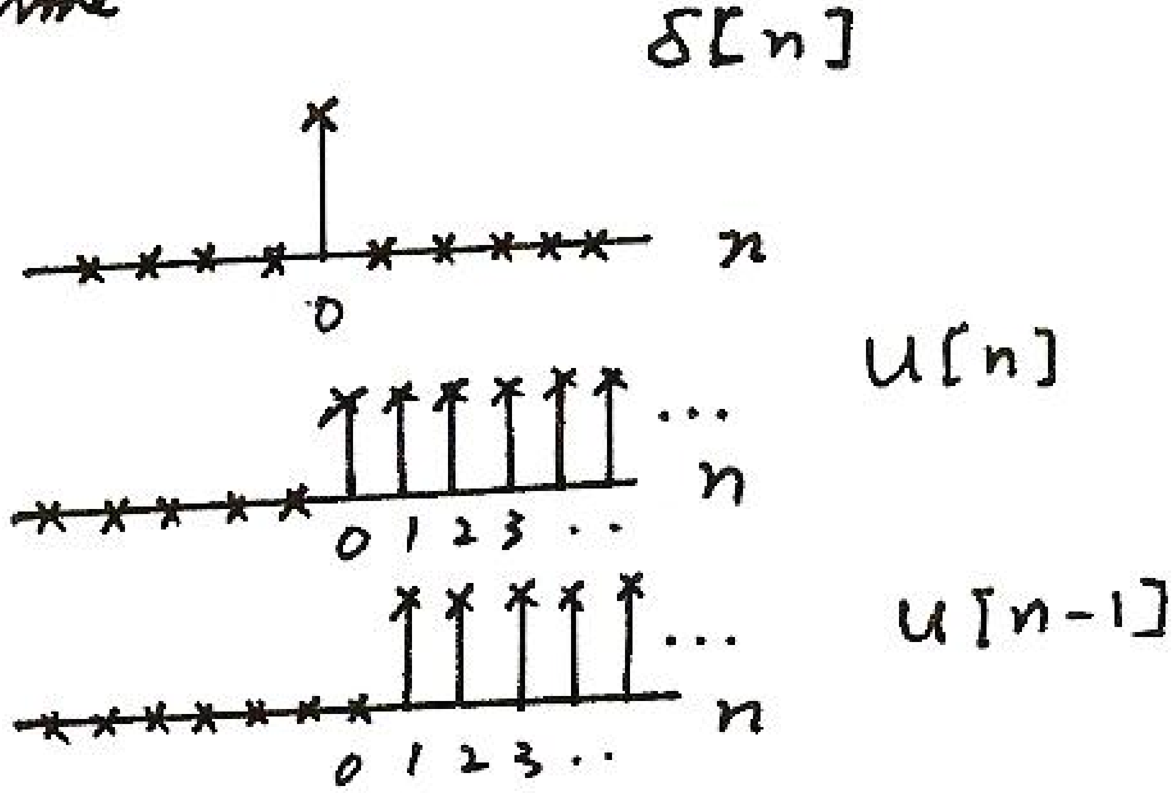
– Running Sum

$$u[n] = \sum_{m=-\infty}^{n} \delta[m] \begin{cases} =0, & n<0 \\ =1, & n\geq 0 \end{cases}$$

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

Unit Step Function: First Difference

Discrete-Time

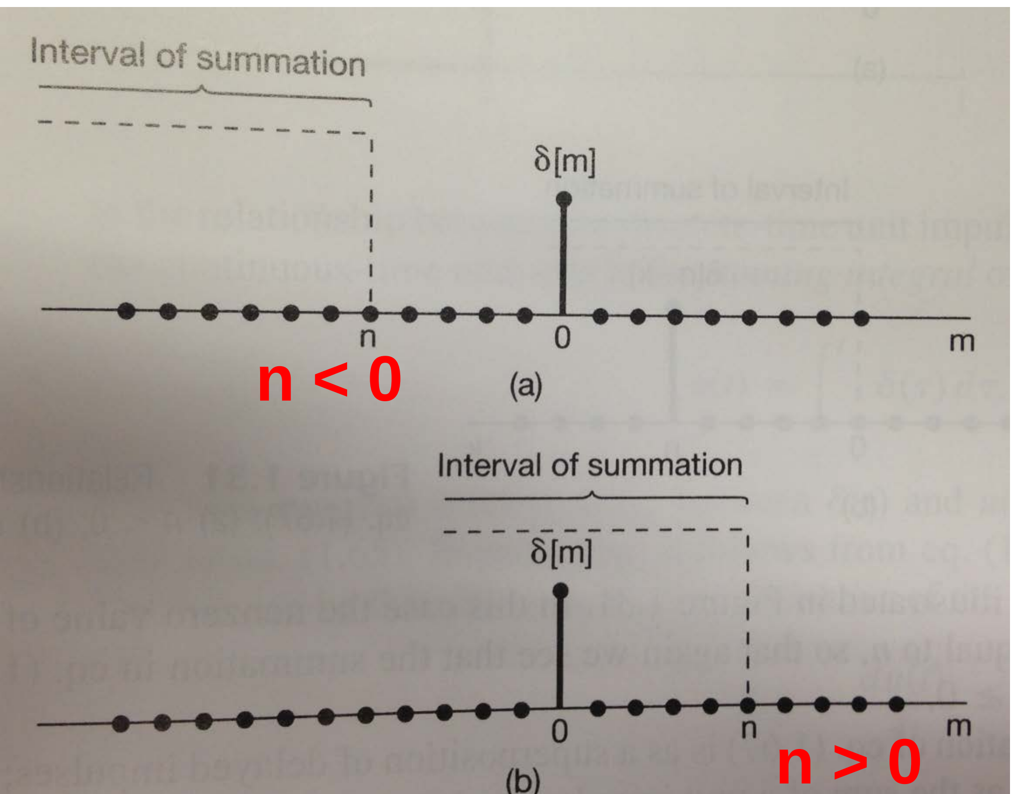


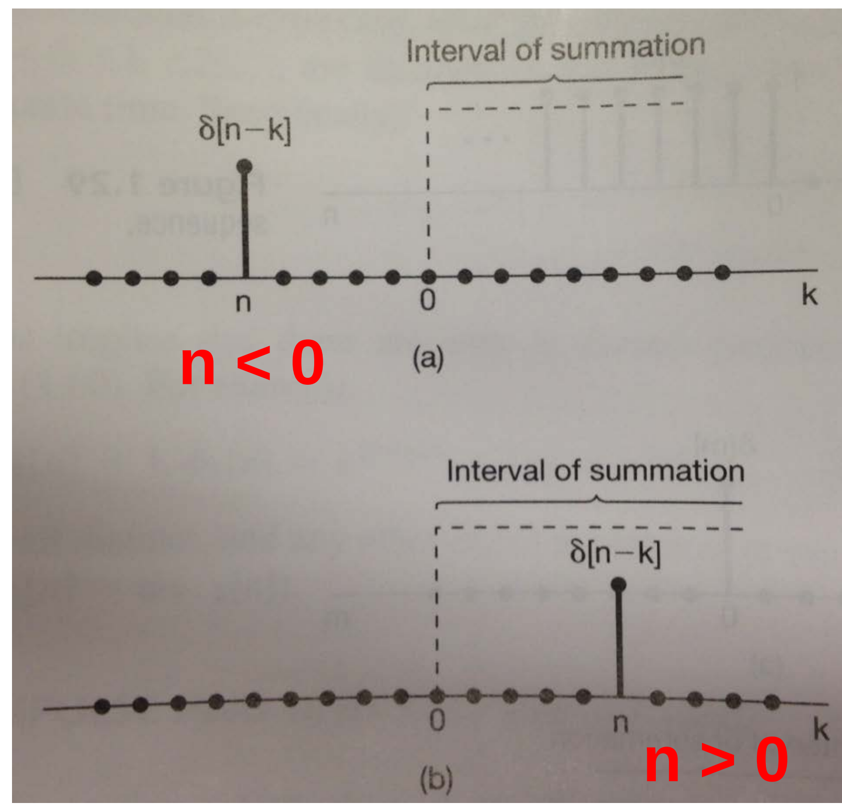
$$\delta[n] = u[n] - u[n-1]$$

Unit Step Function: Running Sum

$$u[n] = \sum_{m=-\infty}^{n} \delta[m] \begin{cases} =0, & n<0 \\ =1, & n\geq 0 \end{cases}$$

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

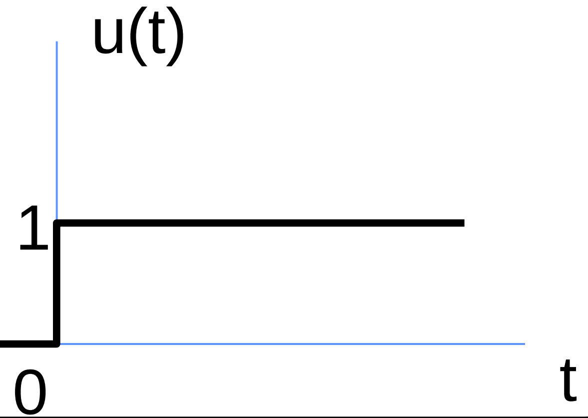


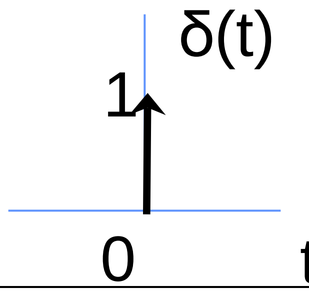


Unit Impulse and Unit Step Functions

Continuous-time

$$u(t)=\begin{cases}0, & t<0\\1, & t>0\end{cases}$$





Relation between unit impulse and unit step functions

– First Derivative

$$\delta(t)=\frac{du(t)}{dt}$$

– Running Integral

$$u(t)=\int_{-\infty}^{t}\delta(\tau)\,d\tau$$

Unit Impulse and Unit Step Functions (Cont.)

$$\delta_\Delta(t) = \frac{du_\Delta(t)}{dt}$$

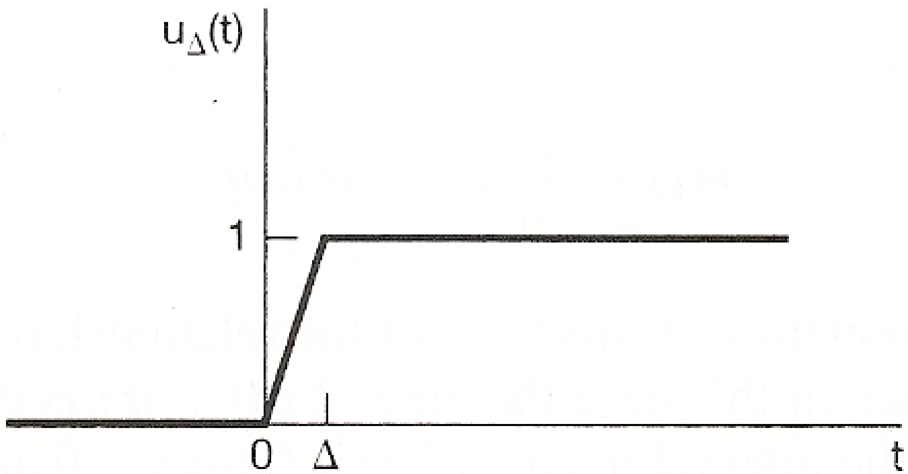


Figure 1.33 Continuous approximation to the unit step, $u_\Delta(t)$.

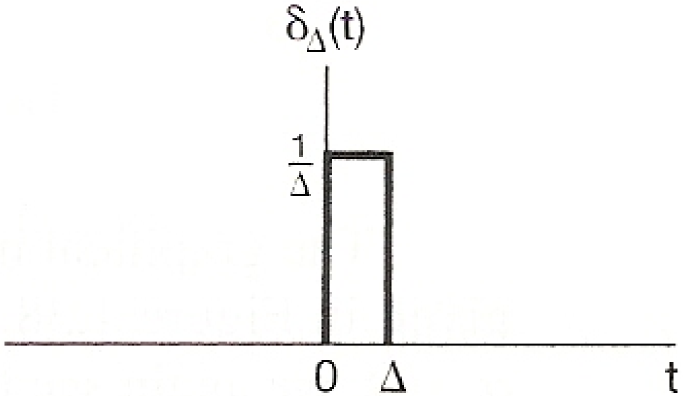
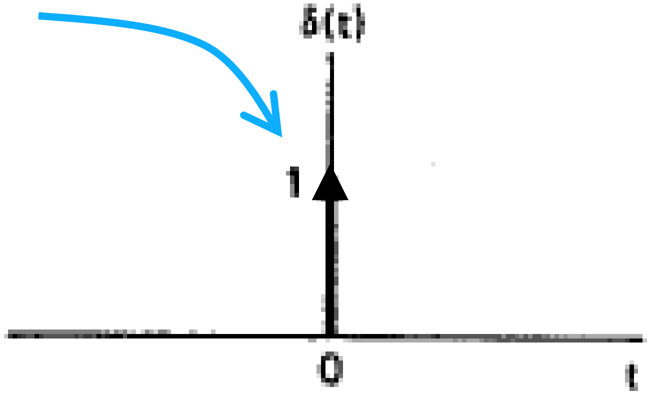


Figure 1.34 Derivative of $u_\Delta(t)$.

More on CT unit impulse function:

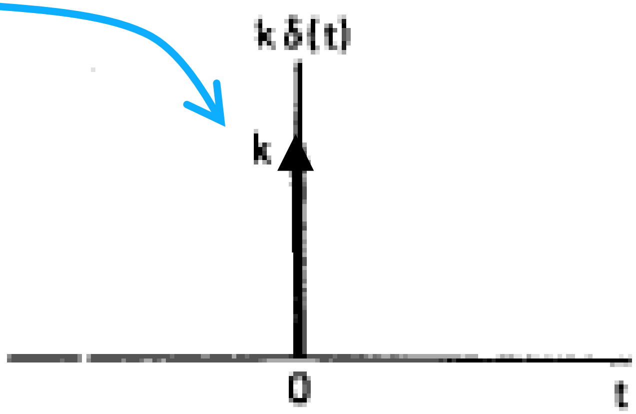
- **δ(t) has in effect no duration, but unit area.**

area = 1



CT unit impulse

area = k

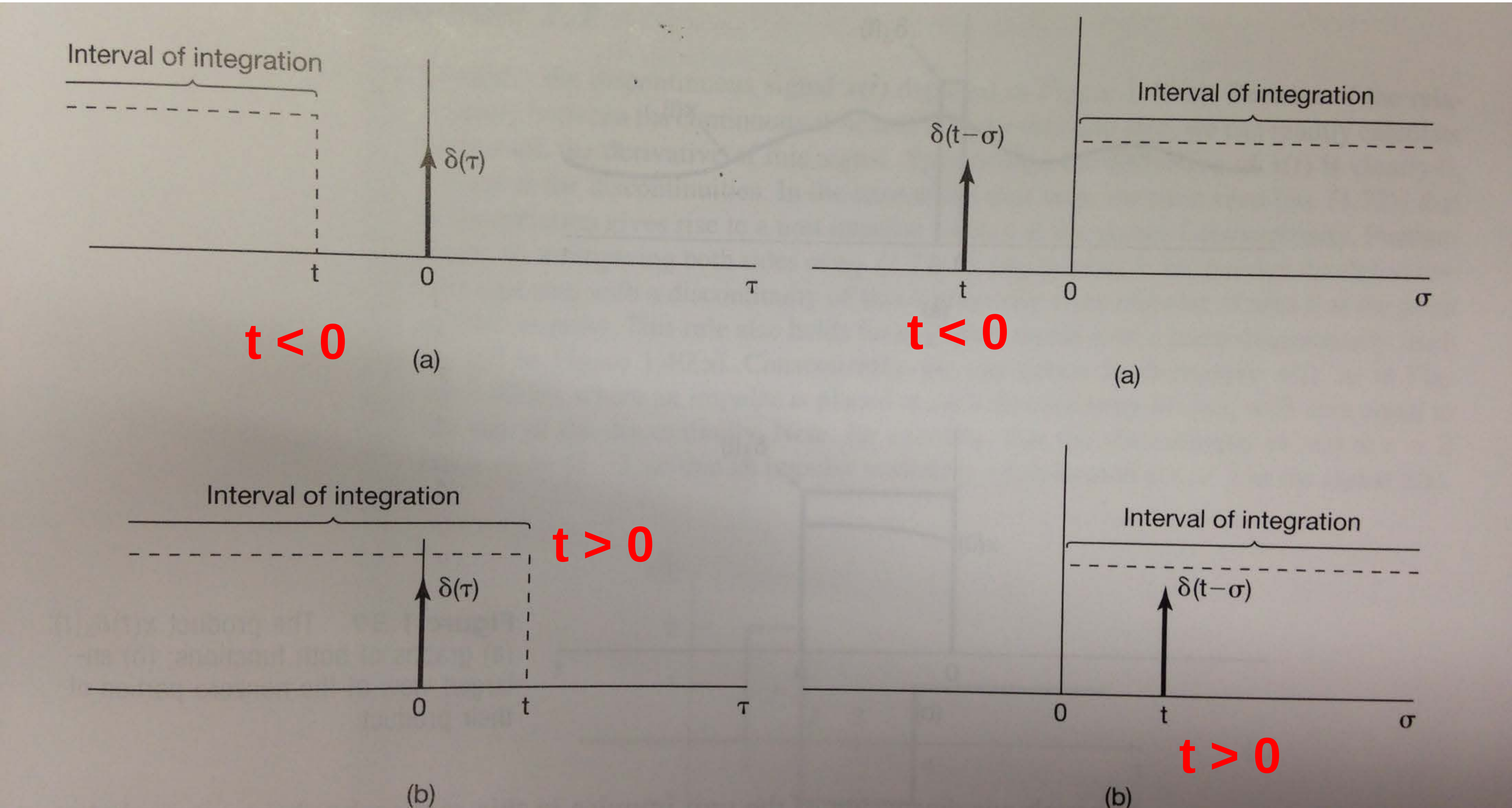


Scaled impulse

- **Or the integration of CT unit impulse function is unit.** $\int_{-\infty}^{\infty} \delta(t)dt = 1$

Running Integral

$$u(t)=\int_{-\infty}^{t}\delta(\tau)d\tau \qquad u(t)=\int_{0}^{\infty}\delta(t-\sigma)d\sigma$$



Sampling Property

– Sampling property

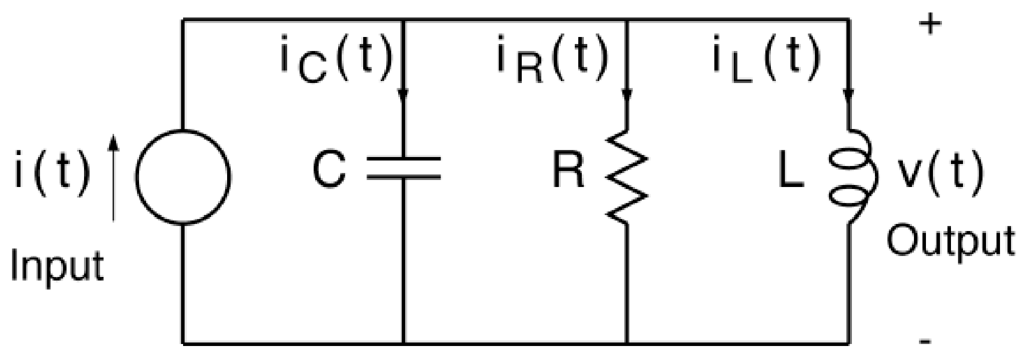
$$x(t)\,\delta(t-t_0)=x(t_0)\,\delta(t-t_0)$$

System Examples

- *Systems* are described from input/output perspective, that is, input to the system *x* causes the output *y*

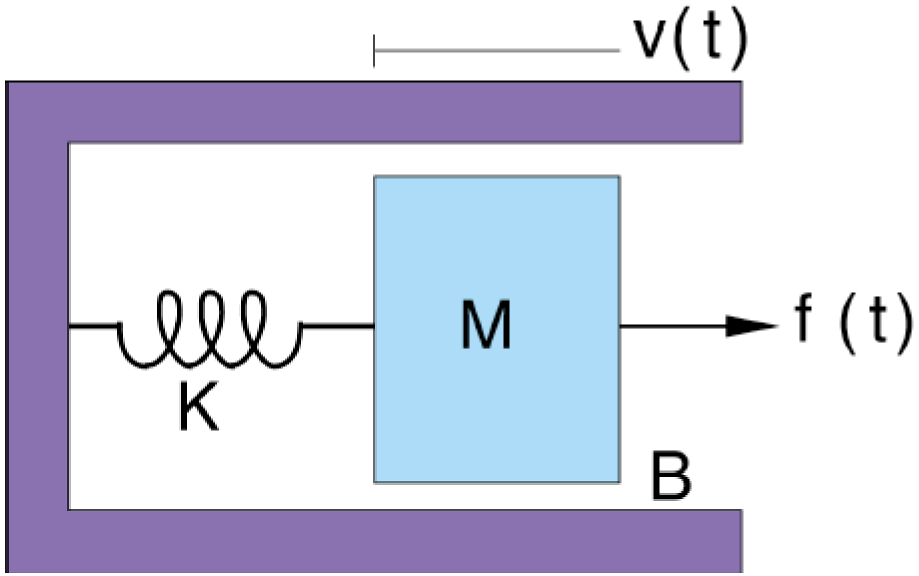
$$x(t) \longrightarrow \boxed{\text{CT System}} \longrightarrow y(t) \qquad x[n] \longrightarrow \boxed{\text{DT System}} \longrightarrow y[n]$$

Ex. #1 RLC circuit — an electrical system



$$i(t) = \underbrace{C\frac{dv(t)}{dt}}_{\text{capacitance}} + \underbrace{\frac{v(t)}{R}}_{\text{resistance}} + \underbrace{\frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau}_{\text{inductance}} .$$

Ex. #2 A shock absorber – a mechanical system

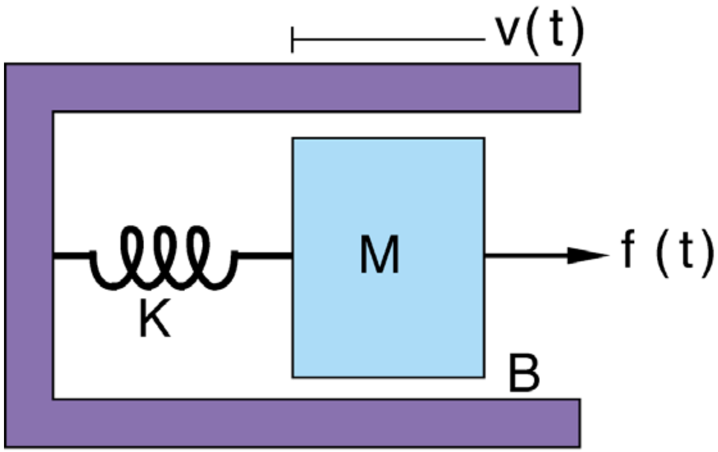


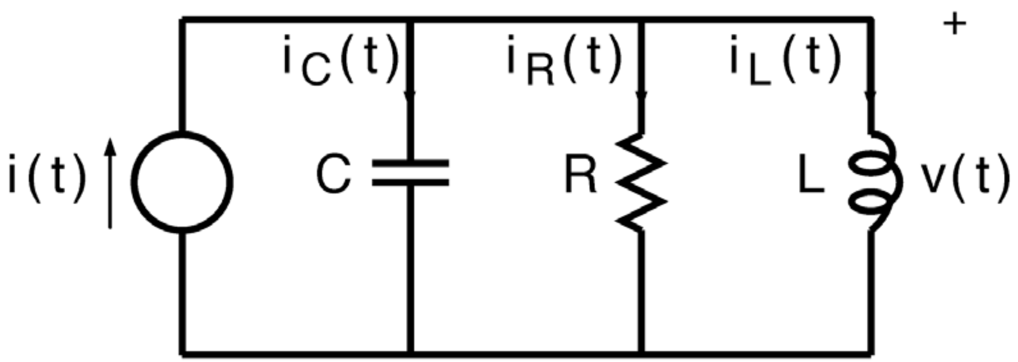
Force Balance:

$$f(t) = \underbrace{M\frac{dv(t)}{dt}}_{inertial\ force} + \underbrace{Bv(t)}_{friction} + \underbrace{K\int_{-\infty}^{t} v(\tau)d\tau}_{spring\ force}.$$

This equation looks quite familiar, we just saw it earlier.

- **Observation: different systems could be described by the same input/output relations**



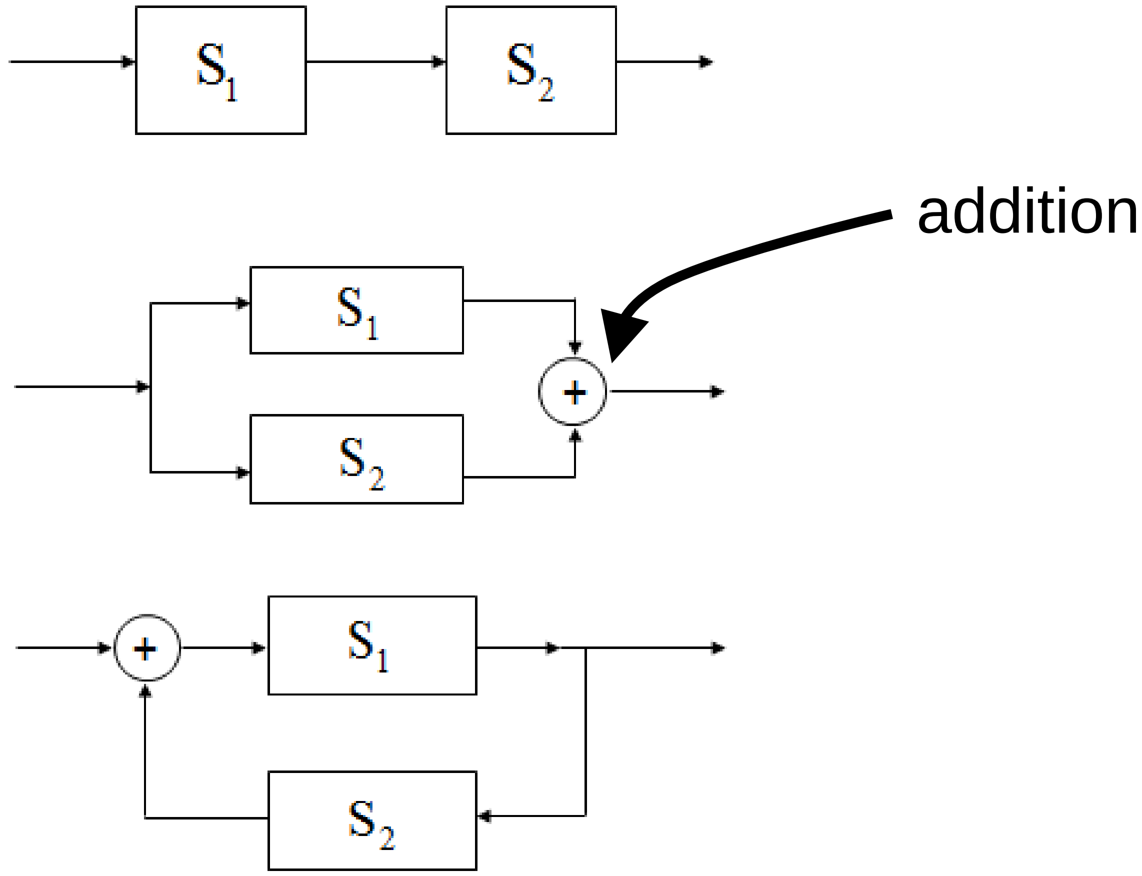


Observations

- A very rich class of systems are described by differential/difference equations.
- Such an equation, by itself, does not completely describe the input-output behaviour of a system: we need auxiliary conditions (initial conditions, boundary conditions).
- Very different physical systems may have very similar or same mathematical descriptions.

Interconnection of Systems

- **Series (cascade)**
- **Parallel**
- **Feedback**



System Properties (Causality, Linearity, Time-invariance, etc.)

- Why bother with such general properties?
  - Important practical/physical implications. We can make many important predictions of the system behaviours without having to do any mathematical derivations.
  - They allow us to develop powerful tools (*transformations*, more on this later) for analysis and design.

1) Memoryless or With Memory

Memoryless : output at a given time depends only on the input at the same time

eg. $$y[n] = (ax[n] - x^2[n])^2$$

With Memory

eg. $$y[n] = \sum_{k=-\infty}^{n} x[k]$$

summer or accumulator

Memoryless or With Memory (cont.)

- In physical system, memory is associated with the storage of energy, e.g., capacitor in electric circuit.

2) Invertability

invertible : distinct inputs lead to distinct outputs, i.e. an inverse system exits

$$x \rightarrow \boxed{S} \rightarrow y \qquad\qquad y \rightarrow \boxed{\text{Inverse system}} \rightarrow x$$

eg.

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$z[n] = y[n] - y[n-1] = x[n]$$

3) Causality

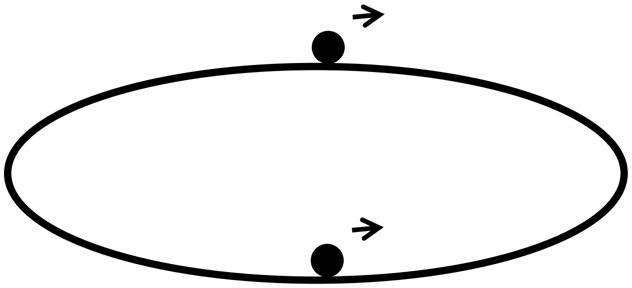
- **Causality**: A system is causal if the output does **not** anticipate future values of the input, i.e., if the output at any time depends only on values of the input up to that time.
- All real-time physical systems are **causal**, because time only moves forward, and effect occurs after cause. (Imagine if you own a noncausal system whose output depends on tomorrow's stock price.)
  - Do **not** apply to spatially varying signals. (We can move both left and right, up and down.)
  - Do **not** apply to systems processing *recorded (or non-realtime)* signals, e.g. taped sports games vs. live broadcast.

Causal or Non-causal?

- $y(t) = x^2(t-1)$
- $y(t)=x(t+1)$
- $y(t)=x(t) \cos(t+1)$
- $y[n]=x[-n]$
- $y[n]=(1/2)^{n+1} x^3[n-1]$

4) Stability

- Small input leads to response that does not diverge.



  - If the input to a stable system is bounded, the output must also be bounded.
  - e.g.: S_1: $y(t) = t\,x(t)$

    S_2: $y(t) = e^{x(t)}$

Stability (cont.)

- Stability of physical systems are due to the presence of mechanisms that dissipate energy, e.g., resistor, friction.
- Stable or not (Example 1.13 in textbook)?
  - Find specific example, e.g, for S: $y(t)=t\,x(t)$
  - To prove that all bounded inputs lead to bounded output.

5) Time Invariance (TI)

- Informally, a system is time-invariant (TI) if its behavior does not depend on the choice of *t = 0*. Then two identical experiments will yield the same results, regardless the starting time.
  - Mathematically (in DT): A system $x[n] \to y[n]$ is TI if for *any* input $x[n]$ and *any* time shift n_0

$$\begin{array}{ll} \text{If} & x[n] \to y[n] \\ \text{then} & x[n - n_0] \to y[n - n_0] \; . \end{array}$$

  - Similarly for CT time-invariant system

$$\begin{array}{ll} \text{If} & x(t) \to y(t) \\ \text{then} & x(t - t_0) \to y(t - t_0) \; . \end{array}$$

Time-invariant or Time-varying?

- **Steps:**

1) **Calculate $y_1(t) \leftarrow x_1(t)$**
2) **Calculate $y_2(t) \leftarrow x_2(t)=x_1(t-t_0)$**
3) **Does $y_1(t-t_0)$ equal $y_2(t)$?**

e.g.: $y[n] = \left(\frac{1}{2}\right)^{n+1} x^3[n-1]$

① $y_1[n] = \left(\frac{1}{2}\right)^{n+1} x_1^3[n-1]$

② $x_2[n] = x_1[n-n_0]$

$y_2[n] = \left(\frac{1}{2}\right)^{n+1} x_2^3[n-1]$

$= \left(\frac{1}{2}\right)^{n+1} x_1^3[n-n_0-1]$

③ $y_1[n-n_0] = \left(\frac{1}{2}\right)^{n-n_0+1} \cdot x_1^3[n-n_0-1]$

$\therefore$ $y_1[n-n_0] \neq y_2[n]$

$\therefore$ Time-varying

Now we can deduce something:

- If the input to a TI system is periodic, then the output is also periodic with the same period (Problem 1.43 (a)).

Proof: Suppose $x(t+T) = x(t)$

and $x(t) \quad \rightarrow \quad y(t)$

Then by TI

$$x(t+T) \quad \rightarrow \quad y(t+T)$$

$$\uparrow \qquad\qquad\qquad \uparrow$$

But these are the same input!

So these must be the same output, *i.e.*, $y(t) = y(t+T)$

6) Linear and Nonlinear Systems

- Many systems are nonlinear.
  - **e.g.:** economic system with

    input: fiscal and monetary policies, labor, resources, etc.

  → output: GDP, inflation, etc.
  - System behaviour is very unpredictable, because it is highly nonlinear.
- We will deal with only linear systems, which are good approximations of nonlinear systems in certain ranges.
  - **e.g.:** small-signal conductance of a nonlinear diode.
- Linear systems can be analysed accurately.

Linearity

We have $x_1(t) \rightarrow y_1(t)$, and $x_2(t) \rightarrow y_2(t)$.

The system is linear, if:

1) **Additivity property:** $x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t)$
2) **Scaling (or homogeneity) property:**

$$a\, x_1(t) \rightarrow a\, y_1(t)$$

where *a* is a complex number

e.g.: $y(t) = 2\, x(t)$ $\qquad$ $y(t) = x^2(t)$

Linearity (cont.)

A (CT) system is linear if it has the superposition property:

If $\quad x_1(t) \rightarrow y_1(t)$ and $x_2(t) \rightarrow y_2(t)$

then $\quad ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$

- For linear systems, zero input $\rightarrow$ zero output

"Proof" $\quad 0 = 0 \cdot x[n] \rightarrow 0 \cdot y[n] = 0$

Question: Is the system $y = A + x$ linear?

- *Incrementally linear system!*

Linear system or not?

- **Steps**

1) Have $y_1(t)$ and $y_2(t)$ as output signals to $x_1(t)$ and $x_2(t)$

2) Have $y_3(t)$ as output signal to $x_3(t) = a\, x_1(t) + b\, x_2(t)$

3) Does $y_3(t)$ equal "$a\, y_1(t) + b\, y_2(t)$"?

More examples on textbook

Read Example 1.17 ~ 1.20

Linearity (cont.)

- Superposition

If $$x_k[n] \rightarrow y_k[n]$$

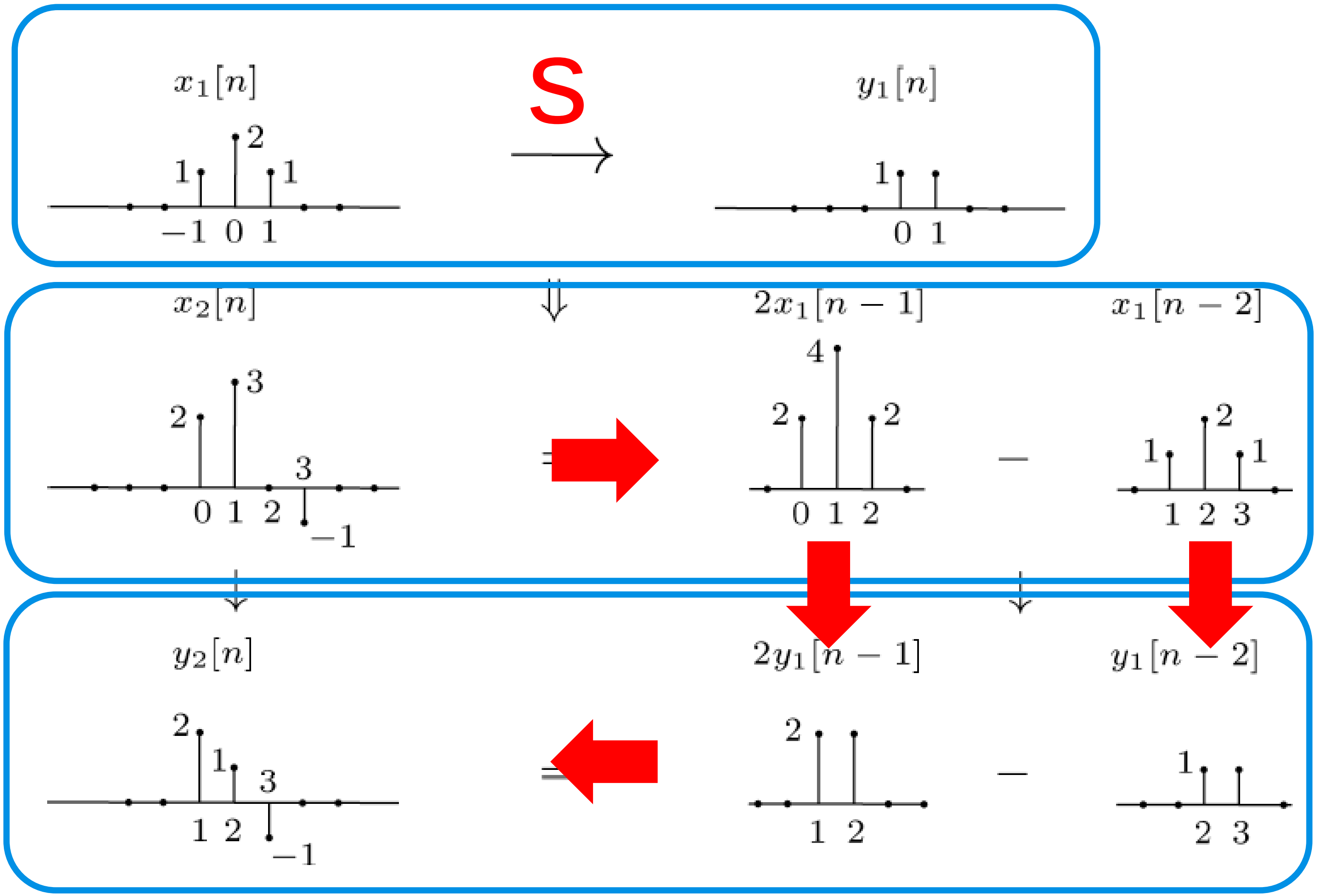
Then $$\boxed{\sum_k a_k x_k[n] \rightarrow \sum_k a_k y_k[n]}$$

- This property seems to be almost trivial now, but it is one of the most important ones

Linear Time-invariant (LTI) Systems

- Focus of most of this course
  - Practical importance (**e.g.** #1-2 earlier this lecture are both LTI systems.)
  - The powerful analysis tools (*transformation*) associated with LTI systems
- A basic fact: If we know the response of an LTI system to some inputs, we actually know the response to many inputs.

Example: DT LTI System



$x_1 \rightarrow y_1$

$x_2 = f(x_1)$