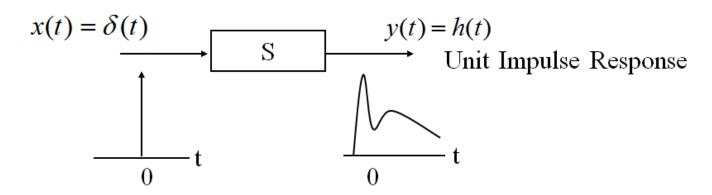
Notes

- Assignments
 - **♦** 3.3
 - **3.21**
 - 3.22 (a) -> Figs. (b) (d) (f)
 - **♦** 3.24
 - **3.25**
- Tutorial problems
 - Basic Problems wish Answers 3.8
 - Basic Problems 3.34
 - Advanced Problems 3.40

Review for Chapter 2

Why to introduce unit impulse response?



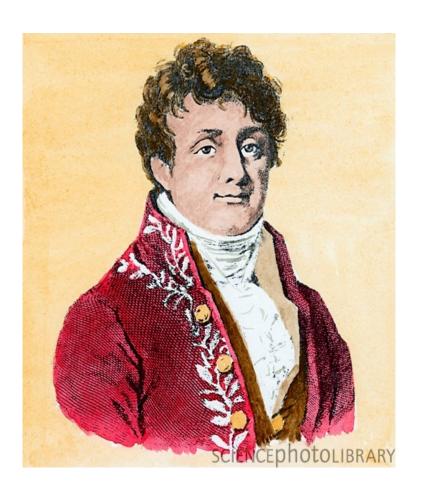
- Why to introduce convolution?
 - (DT or CT) Signal can be represented by a linear combination of unit impulse function
 - When it goes through the system, the output is computed via convolution of input signal and unit impulse response

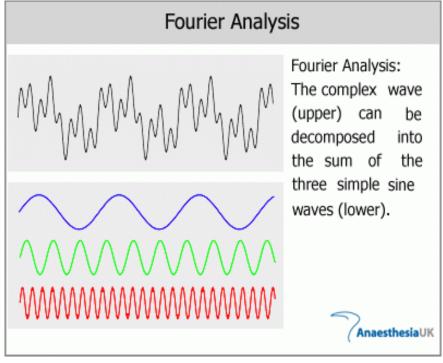
Chapter 3

Fourier Series Representation of Periodic Signals

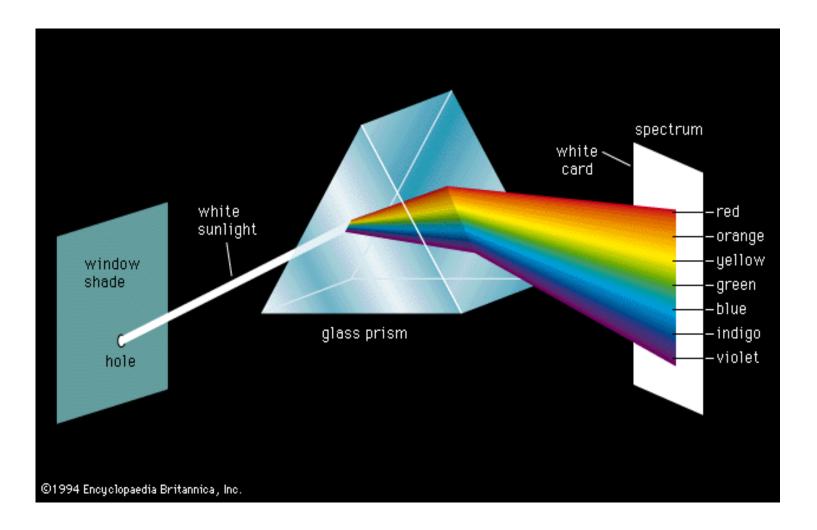
Fourier Series – A Time-Frequency Analysis Perspective

Joseph Fourier

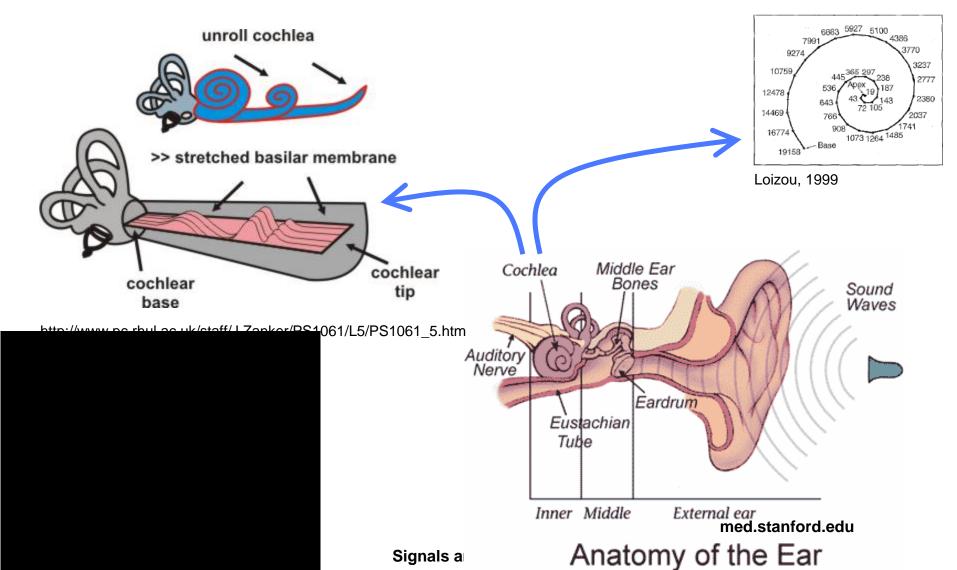




A Useful Analogy



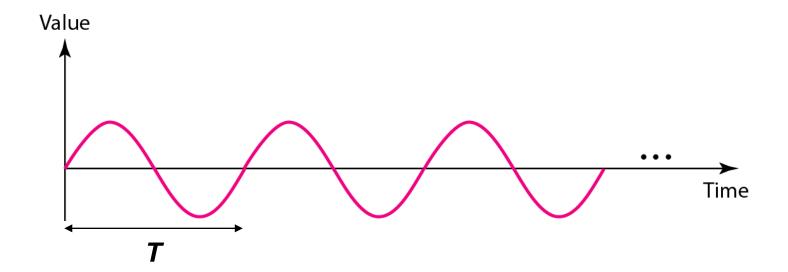
A Physiological Analogy



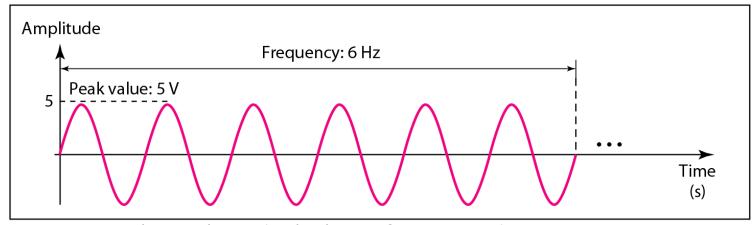
Note

Frequency and period are the inverse of each other.

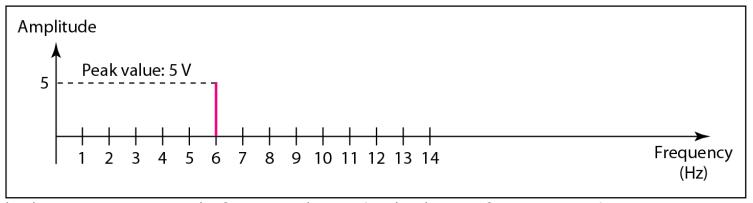
$$f = \frac{1}{T}$$
 and $T = \frac{1}{f}$



The time-domain and frequency-domain plots of a sine wave

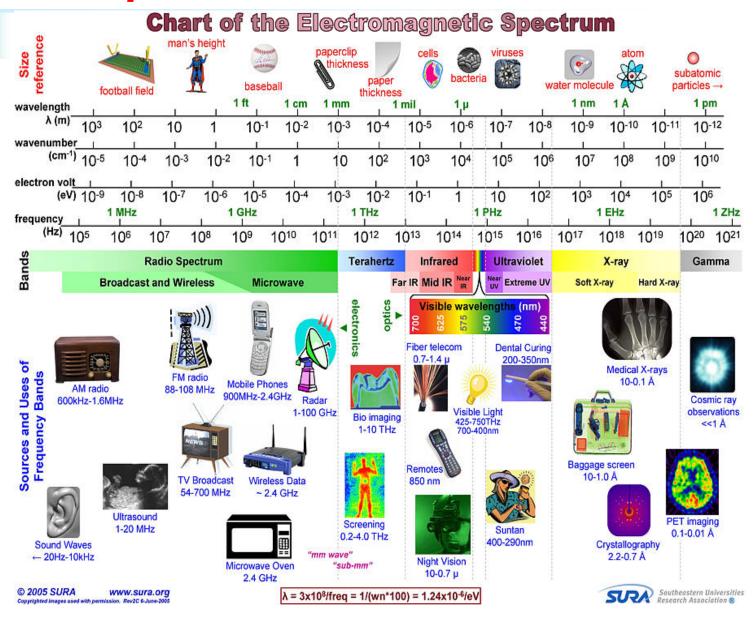


a. A sine wave in the time domain (peak value: 5 V, frequency: 6 Hz)



b. The same sine wave in the frequency domain (peak value: 5 V, frequency: 6 Hz)

Example



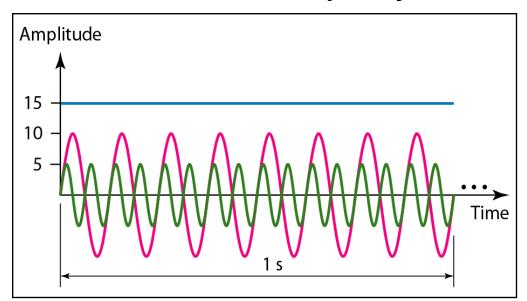
Note

A complete sine wave in the time domain can be represented by one single spike in the frequency domain.

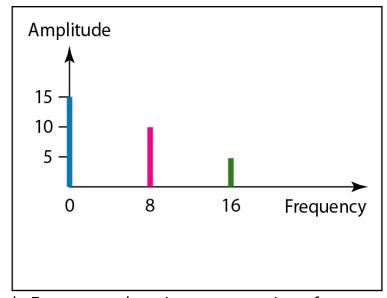
Example

The frequency domain is more compact and useful when we are dealing with more than one sine wave.

The time domain and frequency domain of three sine waves



a. Time-domain representation of three sine waves with frequencies 0, 8, and 16

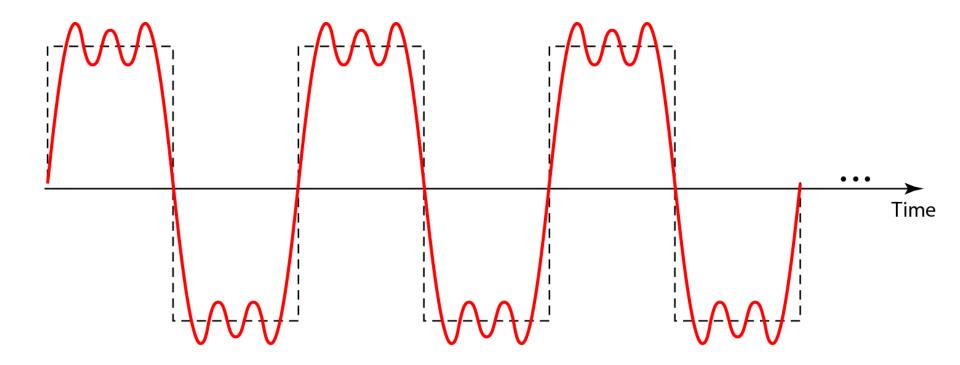


b. Frequency-domain representation of the same three signals

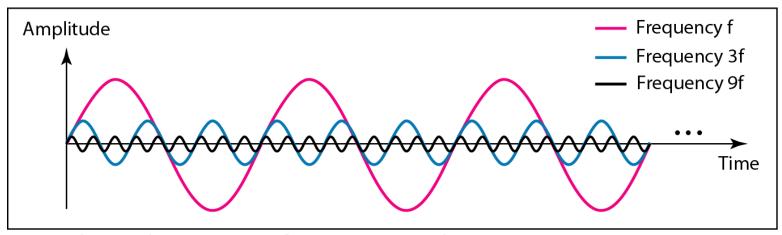
Composite Signals and Periodicity

- If the composite signal is periodic, the decomposition gives a series of signals with discrete frequencies.
- If the composite signal is aperiodic, the decomposition gives a combination of sine waves with continuous frequencies.

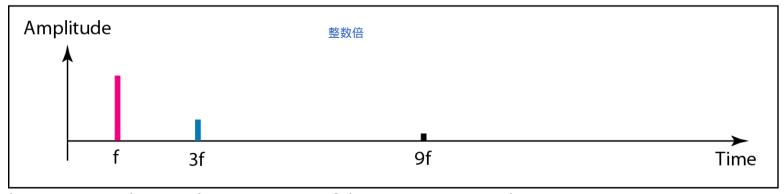
A composite periodic signal



Decomposition of a composite periodic signal in the time and frequency domains

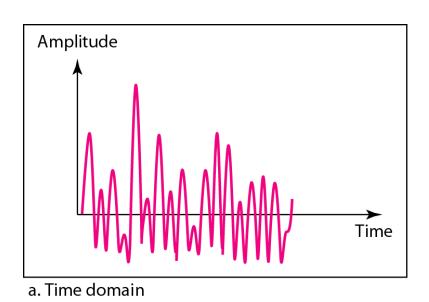


a. Time-domain decomposition of a composite signal

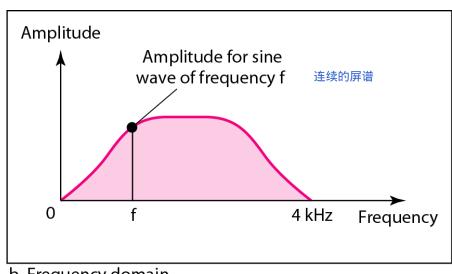


b. Frequency-domain decomposition of the composite signal

The time and frequency domains of an aperiodic signal



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b. Frequency domain

What is the difference between periodic and aperiodic signals?

Fourier Analysis



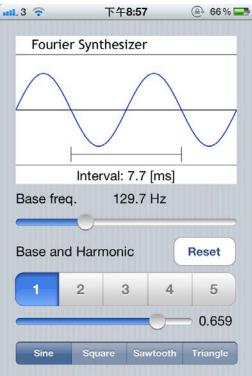
Fourier analysis is a tool that changes a time domain signal to a frequency domain signal and vice versa.

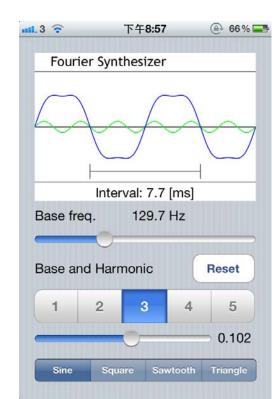
Apps



Fourier synthesizer:

Use pure-tone and/or harmonics to synthesize periodic signal





Fourier Series – An LTI System Analysis Perspective

How to represent signals in the study of LTI systems?

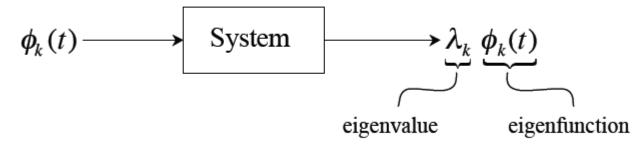
In chapter 2, based on unit impulse function and superposition property, we have:

- 1) The set of basic signals can be used to construct a broad and useful class of signals.
- 2) The response of an LTI system to each signal should be simple enough in structure, to provide us with a convenient representation for the response of the system to any signal constructed as a linear combination of the basic signals.

Any more basic function?

Eigenfunctions $\phi_k(t)$ and Their Properties

(Focus on CT systems now, but results apply to DT systems as well.)



Eigenfunction in \rightarrow same function out with a "gain"



From the superposition property of LTI systems:

$$x(t) = \sum_{k} a_k \phi_k(t) \longrightarrow \text{LTI} \longrightarrow y(t) = \sum_{k} \lambda_k a_k \phi_k(t)$$

Now the task of finding response of LTI systems is to determine λ_k . The solution is simple, general, and insightful.

Two Key Questions

1. What are the eigenfunctions of a general LTI system?

2. What kinds of signals can be expressed as superpositions of these eigenfunctions?

Various Eigenfunctions

Ex. #1: Identity system

$$x(t) \longrightarrow \delta(t) \longrightarrow x(t) * \delta(t) = x(t)$$

Any function is an eigenfunction for this LTI system.

Ex. #2: A delay
$$x(t) \longrightarrow \delta(t-T) \longrightarrow x(t-T)$$

Any periodic function x(t) = x(t+T) is an eigenfunction.

Ex. #3:
$$h(t)$$
 even
$$\cos \omega t \longrightarrow h(t) = h(-t) \longrightarrow y(t)$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau) \cos[\omega(t - \tau)] d\tau = \int_{-\infty}^{\infty} h(\tau) [\cos \omega t \cdot \cos \omega \tau + \sin \omega t \cdot \sin \omega \tau)] d\tau$$

$$= \cos \omega t \underbrace{\int_{-\infty}^{\infty} h(\tau) \cos \omega \tau d\tau}_{H(j\omega)} - \cos \omega t \text{ is an eigenfunction}$$

Fourier S

Complex Exponentials - The Only Eigenfunctions of *Any* **LTI Systems**

$$x(t) = e^{st} \longrightarrow h(t)$$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau$$

$$= \left[\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau\right]e^{st}$$

$$= H(s)e^{st}$$
eigenvalue eigenfunction

$$x[n] = z^{n}$$

$$h[n]$$

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]z^{n-m}$$

$$= \left[\sum_{m=-\infty}^{\infty} h[m]z^{-m}\right]z^{n}$$

$$= H(z)z^{n}$$

$$= igenvalue eigenfunction$$

System Functions H(s) or H(z)

CT:
$$x(t) \xrightarrow{e^{st}} h(t) \xrightarrow{H(s)e^{st}} y(t)$$

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

$$x(t) = \sum a_k e^{s_k t} \longrightarrow y(t) = \sum H(s_k)a_k e^{s_k t}$$

DT:

$$x[n] \xrightarrow{z^n} h[n] \xrightarrow{H(z)z^n} y[n]$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

$$x[n] = \sum_{n=-\infty}^{\infty} a_k z_k^n \longrightarrow y[n] = \sum_{n=-\infty}^{\infty} H(z_k) a_k z_k^n$$



Question 2: What kinds of signals can we represent as "sums" of complex exponentials?

For Now: Focus on restricted sets of complex exponentials

CT:

$$s = j\omega$$
 - purely imaginary,

i.e. signals of the form $e^{j\omega t}$

DT:

 $z = e^{j\omega}$, i.e. signals of the form $e^{j\omega n}$

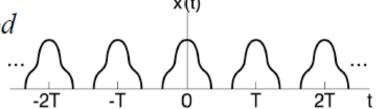


CT & DT Fourier Series and Transforms periodic aperiodic

Fourier Series Representation of CT Periodic Signals

$$x(t) = x(t+T)$$
 for all t

- smallest such T is the fundamental period
- $\omega_{\circ} = \frac{2\pi}{T}$ is the fundamental frequency $\frac{\dots}{-2T}$



 $e^{j\omega t}$ periodic with period $T \Leftrightarrow \omega = k\omega_o$



$$x(t) = \sum\nolimits_{k = -\infty}^{+\infty} {{a_k}{e^{jk{\omega _0}t}}} = \sum\nolimits_{k = -\infty}^{+\infty} {{a_k}{e^{jk2\pi t/T}}}$$

-periodic with period T

- $\{a_k\}$ are the Fourier (series) coefficients
- -k=0 DC
- $-k = \pm 1$ first harmonic
- $-k = \pm 2$ second harmonic

Question: How do we find the Fourier coefficients?

Let's first take a detour by studying a three-dimensional vector:

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z},$$

 $\hat{x}, \hat{y}, and \hat{z} are unit vectors.$

How do we find the coefficients A_x , A_y , and A_z ?

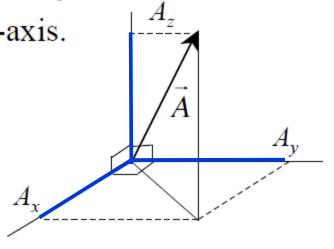
Easy, $A_x = \vec{A} \cdot \hat{x}$, $A_y = \vec{A} \cdot \hat{y}$, and $A_z = \vec{A} \cdot \hat{z}$.

Project the vector onto the x-, y-, and z-axis.

Why does it work this way?

Orthogonality:

$$\hat{y} \bullet \hat{x} = \hat{z} \bullet \hat{y} = \hat{x} \bullet \hat{z} \equiv 0$$



Back to Fourier Series

Now let's imagine that x(t) is a vector in a space of ∞ -dimensions and $e^{jk\omega_o t}$ ($k = 0, \pm 1, \pm 2, ...$) is a unit vector in this space. (Such a space does exist, it is called Hilbert space, just nobody lives there.) Then,

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_o t}, \quad \text{with the analog } y$$

$$x(t) \iff \vec{A}$$

$$e^{jk\omega_o t} \iff \hat{x}, \hat{y}, \hat{z}$$

$$a_k \iff A_x, A_y, A_z$$

Now all we need to do is to "project" x(t) onto the $e^{jk\omega_o t} - axis$, then we will obtain the coefficients a_k . How to do that? Again, need **orthogonality.**

Obtaining Fourier Series Coe orthogonality in the Hilbert space:

Orthogonality in the Hilbert space:

$$\frac{1}{T} \int_{T} e^{jk\omega_{o}t} \cdot e^{-jn\omega_{o}t} dt = \frac{1}{T} \int_{T} e^{j(k-n)\omega_{o}t} dt$$

 $(\int_T = \text{integral over } any \text{ interval of length } T, \text{ and the operation of } T$ $\frac{1}{T} \int_{T} e^{-jn\omega_{o}t} dt$ is to take an "inner product" with $e^{jn\omega_{o}t}$)

Now if we "project" x(t) onto $e^{jn\omega_o t}$ by taking the operation:

$$\frac{1}{T}\int_{T} x(t) \cdot e^{-jn\omega_{o}t} dt = \frac{1}{T}\int_{T} \sum_{k=-\infty}^{+\infty} a_{k} e^{j(k-n)\omega_{o}t} dt,$$

then only one term (a_n) will be nonzero. That is:

$$\frac{1}{T} \int_{T} x(t) \cdot e^{-jn\omega_{o}t} dt = a_{n}.$$



Finally...

$$\frac{1}{T} \int_{T} x(t) \cdot e^{-jn\omega_{o}t} dt = a_{n}.$$



CT Fourier Series Pair

$$(\omega_o = 2\pi/T)$$

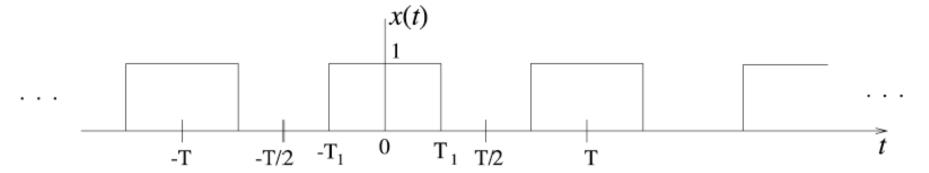
$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_o t}$$

(Synthesis equation)

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_o t} dt$$

(Analysis equation)

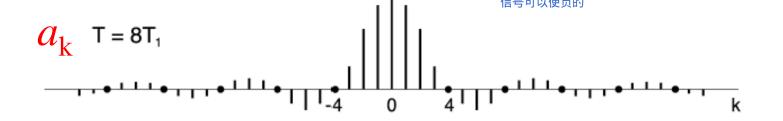
Example 3.5: Periodic Square Wave



$$a_{o} = \frac{1}{T} \int_{-T/2}^{T/2} x(t)dt = \frac{2T_{1}}{T}$$

$$k \neq 0 \qquad a_{k} = \frac{1}{T} \int_{-T/2}^{T/2} x(t)e^{-jk\omega_{0}t}dt = \frac{1}{T} \int_{-T_{1}}^{T_{1}} e^{-jk\omega_{0}t}dt$$

$$(\omega_{o} = \frac{2\pi}{T}): \qquad = -\frac{1}{ik\omega} e^{-jk\omega_{0}t} \Big|_{-T_{1}}^{T_{1}} = \frac{\sin(k\omega_{o}T_{1})}{k\pi}$$



Convergence of CT Fourier Series

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} ?$$

One useful notion for engineers: there is no energy in the difference

$$e(t) = x(t) - \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$\int_T |e(t)|^2 dt = 0$$

(just need x(t) to have finite energy per period) $\int_{T} |x(t)|^{2} dt < \infty$

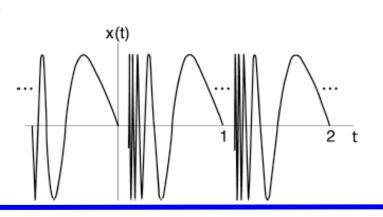
Dirichlet Conditions

Condition 1. x(t) is absolutely integrable over one period, i. e.

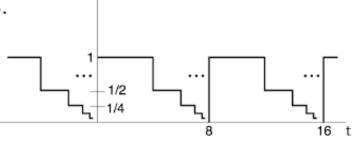
$$\int_T |x(t)| \, dt < \infty$$

- Condition 2. In a finite time interval, x(t) has a *finite* number of maxima and minima.
 - **Ex.** An example that violates Condition 2.

$$x(t) = \sin(2\pi/t) \quad 0 < t \le 1$$

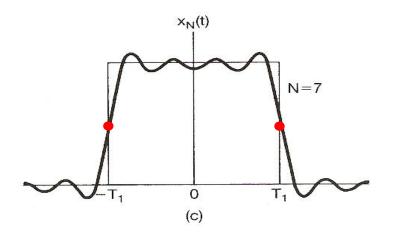


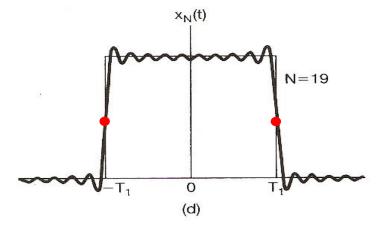
- **Condition 3.** In a finite time interval, x(t) has only a *finite* number of discontinuities.
 - Ex. An example that violates Condition 3.



Dirichlet Conditions (cont.)

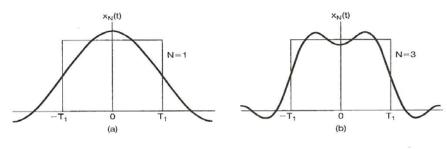
- How can the Fourier series (composed of continuous sine and cos functions) for the square wave (with many discontinuities) possibly make sense?
 - The Fourier series = x(t) at points where x(t) is continuous
 - The Fourier series = "midpoint" at points of discontinuity

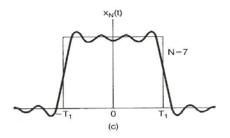


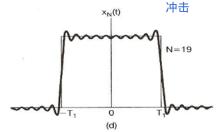


$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

- As $N \rightarrow \infty$, $x_N(t)$ exhibits *Gibbs*' phenomenon at points of discontinuity







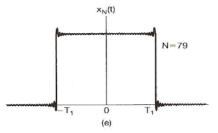


Figure 3.9 Convergence of the Fourier series representation of a square wave: an illustration of the Gibbs phenomenon. Here, we have depicted the finite series approximation $x_N(t) = \sum_{k=-N}^N a_k e^{jk\omega_0 t}$ for several values of N.

Gibbs Phenomenon:

The partial sum in the vicinity of the discontinuity exhibits ripples whose amplitude does not seem to decrease with increasing *N*

CT Fourier Series Pairs

$$(\omega_o = \frac{2\pi}{T})$$

Review:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{j2\pi kt/T}$$

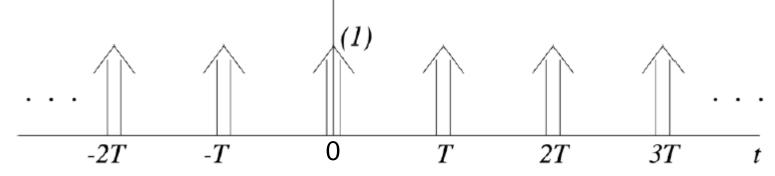
$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

Skip it in future

for shorthand

Periodic Impulse Train

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) - Sampling function, important for sampling later$$



$$a_{k} = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_{o}t} dt = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_{o}t} dt$$
$$= \frac{1}{T} \quad \text{for all } k!$$

$$\Downarrow$$

$$x(t) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} e^{jk\omega_0 t}$$
 (2) the same initial phase.

— All components have:

- (1) the same amplitude,

(A Few) Properties of CT Fourier Series

- Linearity $x(t) \longleftrightarrow a_k, y(t) \longleftrightarrow b_k, \Rightarrow \alpha x(t) + \beta y(t) \longleftrightarrow \alpha a_k + \beta b_k$
- Conjugate Symmetry

$$x(t)$$
 real \Rightarrow $a_{-k} = a_k *$

Proof:
$$a_{-k} = \frac{1}{T} \int_{T} x(t) e^{jk\omega_{o}t} dt = \left[\frac{1}{T} \int_{T} x * (t) e^{-jk\omega_{o}t} dt\right]^{*} = a_{k}^{*}$$

$$\downarrow a_{k} = \operatorname{Re}\{a_{k}\} + j\operatorname{Im}\{a_{k}\}$$

$$= |a_{k}| e^{j\Delta a_{k}} \qquad \operatorname{Re}\{a_{k}\} - j\operatorname{Im}\{a_{k}\}$$

$$\operatorname{Re}\{a_{k}\} \text{ is even}, \quad \operatorname{Im}\{a_{k}\} \text{ is odd}$$

$$\operatorname{or} \qquad |a_{k}| \text{ is even}, \quad \Delta a_{k} \text{ is odd}$$

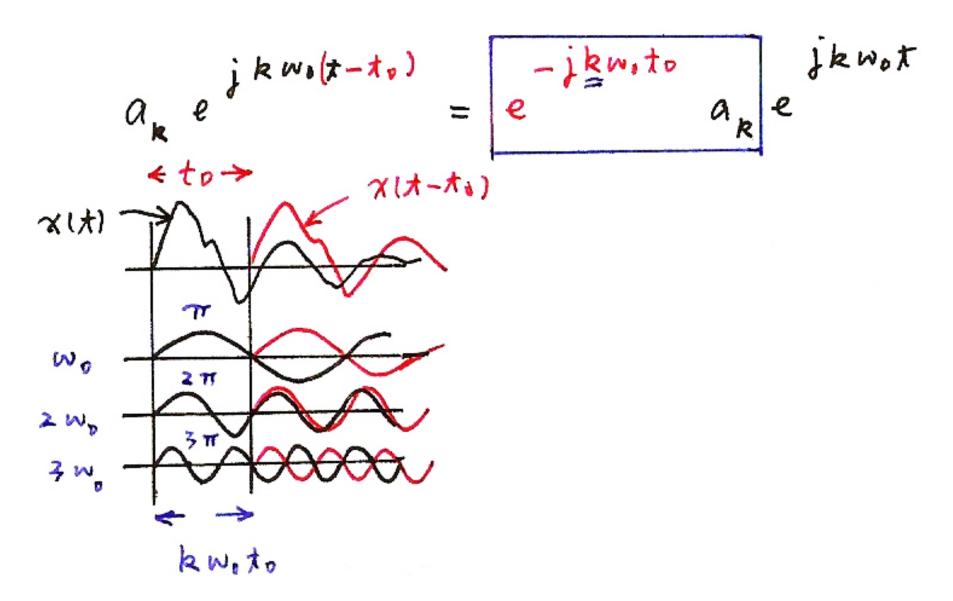
Time shift

$$x(t) \longleftrightarrow a_k$$

$$x(t-t_o) \longleftrightarrow a_{\nu}e^{-jk\omega_o t_o} = a_{\nu}e^{-jk2\pi t_o/T}$$

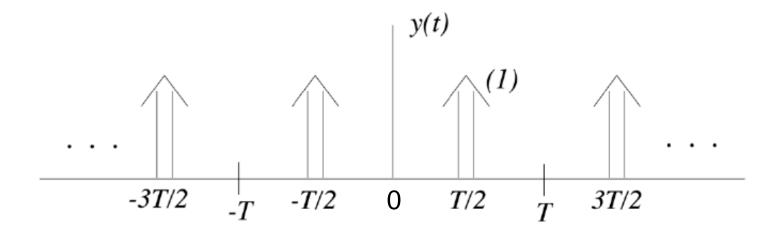
Introduce a linear phase shift $\propto t_0$

Time Shift



Example: Shift by Half Period

$$y(t) = x(t - T/2) \leftarrow \frac{e^{-jk\omega_0 T/2} = e^{-jk\pi}}{a_k e^{-jk\pi}} = (-1)^k a_k$$



$$y(t) \iff (-1)^k a_k \qquad (a_k = \frac{1}{T} = \text{F.C. of } \sum_{n=-\infty}^{+\infty} \delta(t - nT))$$

$$\parallel \frac{(-1)^k}{T}$$

Time Reversal

$$x(-t) \stackrel{FS}{\longleftrightarrow} a_{-k}$$

the effect of sign change for x(t) and a_k are identical

Example:
$$x(t)$$
: ... $a_{-2} a_{-1} a_0 a_1 a_2 ...$ $x(-t)$: ... $a_2 a_1 a_0 a_{-1} a_{-2} ...$

Time Scaling

 α : positive real number

 $x(\alpha t)$: periodic with period T/α and fundamental frequency αw_0

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha w_0)t}$$

 a_k unchanged, but $x(\alpha t)$ and each harmonic component are different

Multiplication Property

$$x(t) \longleftrightarrow a_k$$
, $y(t) \longleftrightarrow b_k$ (Both $x(t)$ and $y(t)$ are

 $\downarrow \downarrow$ periodic with the same period T)

$$x(t) \cdot y(t) \longleftrightarrow c_k = \sum_{l=-\infty}^{+\infty} a_l b_{k-l} = a_k * b_k$$

$$\text{Proof:} \qquad \underbrace{\sum_{t} a_{l} e^{jl\omega_{o}t}}_{X(t)} \cdot \underbrace{\sum_{m} b_{m} e^{jm\omega_{o}t}}_{Y(t)} = \sum_{l,m} a_{l} b_{m} e^{j(l+m)\omega_{o}t} \xrightarrow{l+m=k} \underbrace{\sum_{k} \left[\sum_{l} a_{l} b_{k-l}\right]}_{C_{k}} e^{jk\omega_{o}t}$$

Parseval Relation

$$\frac{1}{T} \int_{T} |x(t)|^{2} dt = \sum_{k=-\infty}^{+\infty} |a_{k}|^{2}$$
Average signal power
$$\sum_{k=-\infty}^{+\infty} |a_{k}|^{2}$$
Power in the k_{th} harmonic

Power is the same whether measured in the time-domain or the frequency-domain

More

Frequency shifting

$$e^{jM\omega_0 t}x(t)\longleftrightarrow a_{k-M}$$

Note:

x(t) is periodic with fundamental frequency ω_0

$$x(t) \longleftrightarrow a_k$$

Differentiation

$$\frac{dx}{dt} \longleftrightarrow jk\omega_0 a_k$$

Integration

$$\int_{-\infty}^{t} x(t)dt \longleftrightarrow (\frac{1}{jk\omega_0})a_k$$

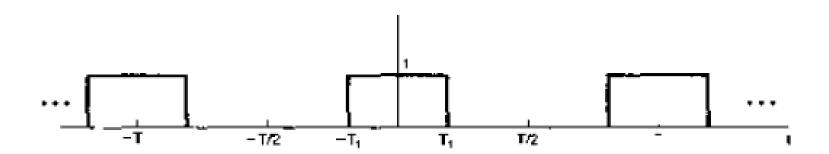
Signals and Systems

Property	Section	Periodic Signal	Fourier Series Coefficients
		$x(t)$ Periodic with period T and $y(t)$ fundamental frequency $\omega_0 = 2\pi/T$	a _t b _t
Linearity Time Shifting Frequency Shifting Conjugation Time Reversal	3.5.1 3.5.2 3.5.6 3.5.3	$Ax(t) + By(t)$ $x(t - t_0)$ $e^{tM\omega_0 t} = e^{tM(2\pi tT)t}x(t)$ $x^*(t)$ $x(-t)$	$Aa_k + Bb_k$ $a_k e^{-jk\omega_{ak_0}} = a_k e^{-jk(2\omega t T)t_0}$ $a_{k-\mu}$ a_k^* a_{-k}
Time Scaling Periodic Convolution	3.5 4	$x(\alpha t)$, $\alpha > 0$ (periodic with period T/α) $\int x(\tau)y(t-\tau)d\tau$	a_k Ta_kb_t
Multiplication	3.5.5	x(t)y(t)	$\sum_{k=-n}^{+\infty} a_k b_{k-1}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk \frac{2\pi}{T} u_k$
Integration		$\int_{-\infty}^{t} x(t) dt \frac{\text{(finite valued and})}{\text{periodic only if } a_0 = 0)}$	$\left(\frac{1}{jk\omega_0}\right)\alpha_k = \left(\frac{1}{jk(2\pi/T)}\right)\alpha_k$
Conjugate Symmetry for Real Signals	3:5.6	x(r) real	$\left\{egin{aligned} a_k &= a^*, \ ext{Re}\{a_k\} &= ext{Re}\{a_{-i}\} \ ext{Sm}\{a_k\} &= - ext{Sm}[a_{-k}\} \ a_k &= a_{-k} \ ext{Test} &= - ext{Test} \end{array} ight.$
Real and Even Signals Real and Odd Signals Even-Odd Decomposition of Real Signals	3.5.6 3.5.6	x(t) real and even x(t) real and odd $\begin{cases} x_r(t) = \delta v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \Theta d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	a_k real and even a_k purely imaginary and ock $\Re e\{a_k\}$ $j \Im m\{a_k\}$

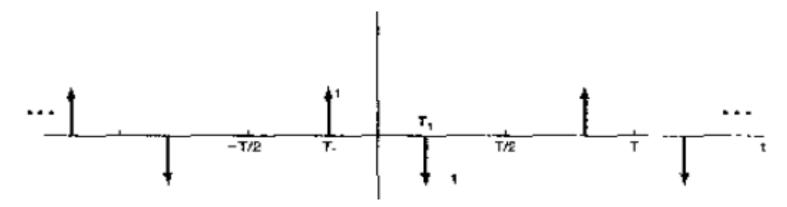
Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_{T}|x(t)|^{2}dt=\sum_{k=-\infty}^{+\infty}|a_{k}|^{2}$$

Example 3.8



1. by definition



2. using properties of time shift, differentiation, and integration

Summary

- Difference between periodic and aperiodic signals
- Eigenfunction and eigenvalue
- Fourier series for periodic signal
 - Analysis and synthesis equations
 - Properties
 - More: Gibbs phenomenon, Dirichlet conditions