Notes

Assignments

- 4.14, 4.25, 4.31, 4.33, 4.35

Tutorial problems

- Basic Problems wish Answers 4.10, 4.16
- Basic Problems 4.26, 4.30

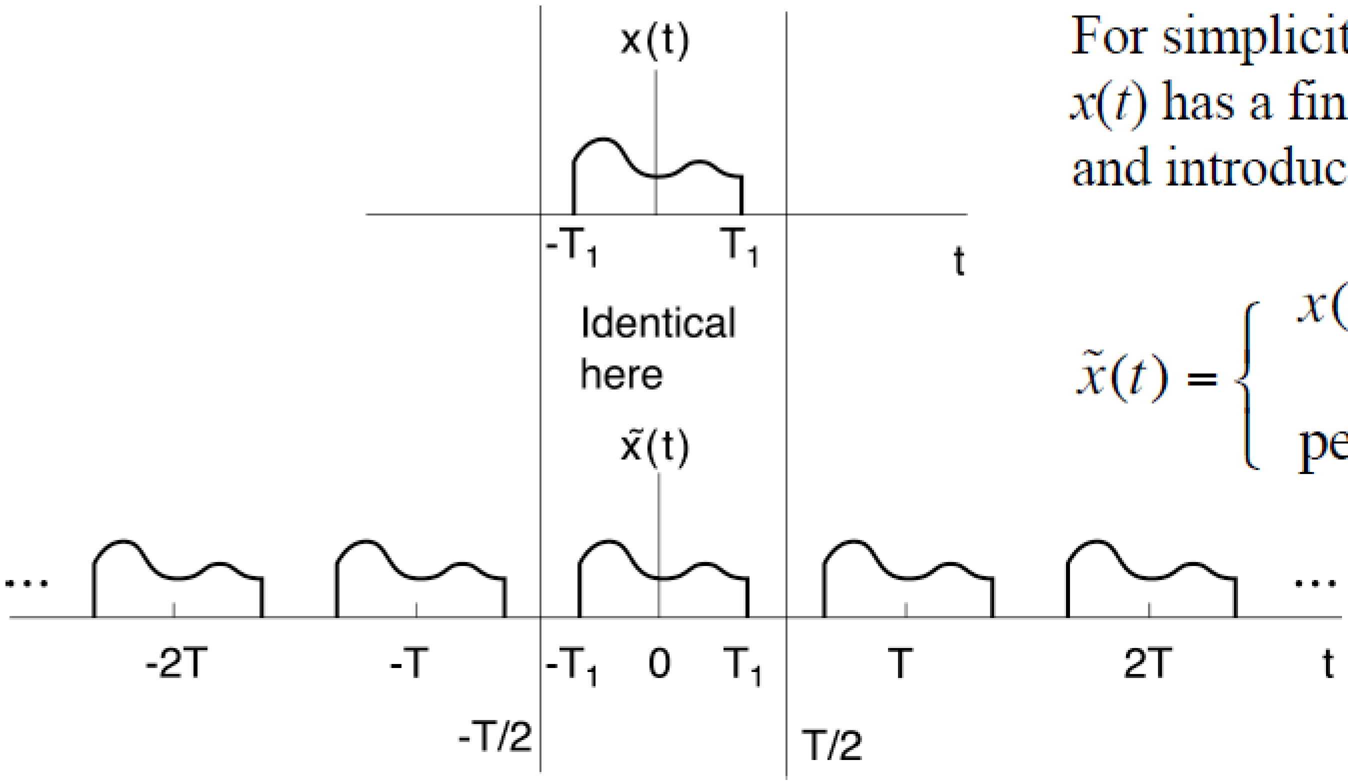
Mid-term examination

- **Time: Nov. 10 (Saturday) 7:00-9:00 pm**
- **Venue: TBD**
- **Range: Chapters 1-4**
- **Allow: one (A4) page note**
- **Problem language: English**
- **Final examination: 40% for Chapters 1-4**

Chapter 4
The Continuous-Time Fourier Transform

(cont.)

So, on the derivation of FT ...



For simplicity, assume $x(t)$ has a finite duration, and introduce a periodic $\tilde{x}(t)$

$$\tilde{x}(t) = \begin{cases} x(t) & \frac{-T}{2} < t < \frac{T}{2} \\ \text{periodic} & |t| > \frac{T}{2} \end{cases}$$

$$\text{As } T \to \infty, \qquad x(t) = \tilde{x}(t) \text{ for all } t$$

The CT Fourier Transform Pair

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

— *FT*

Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$

— Inverse *FT*

Inverse Fourier Transform

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$\mathcal{F}(x(t)) = X(j\omega)$$

$$x(t) = \mathcal{F}^{-1}(X(j\omega))$$

With CTFT, now the frequency response of an LTI system makes complete sense

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} H(jk\omega_0) a_k e^{jk\omega_0 t}$$

$$a_k \longrightarrow \underbrace{H(jk\omega_0)}_{\text{"gain"}} a_k$$

$$\boxed{H(j\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt}$$

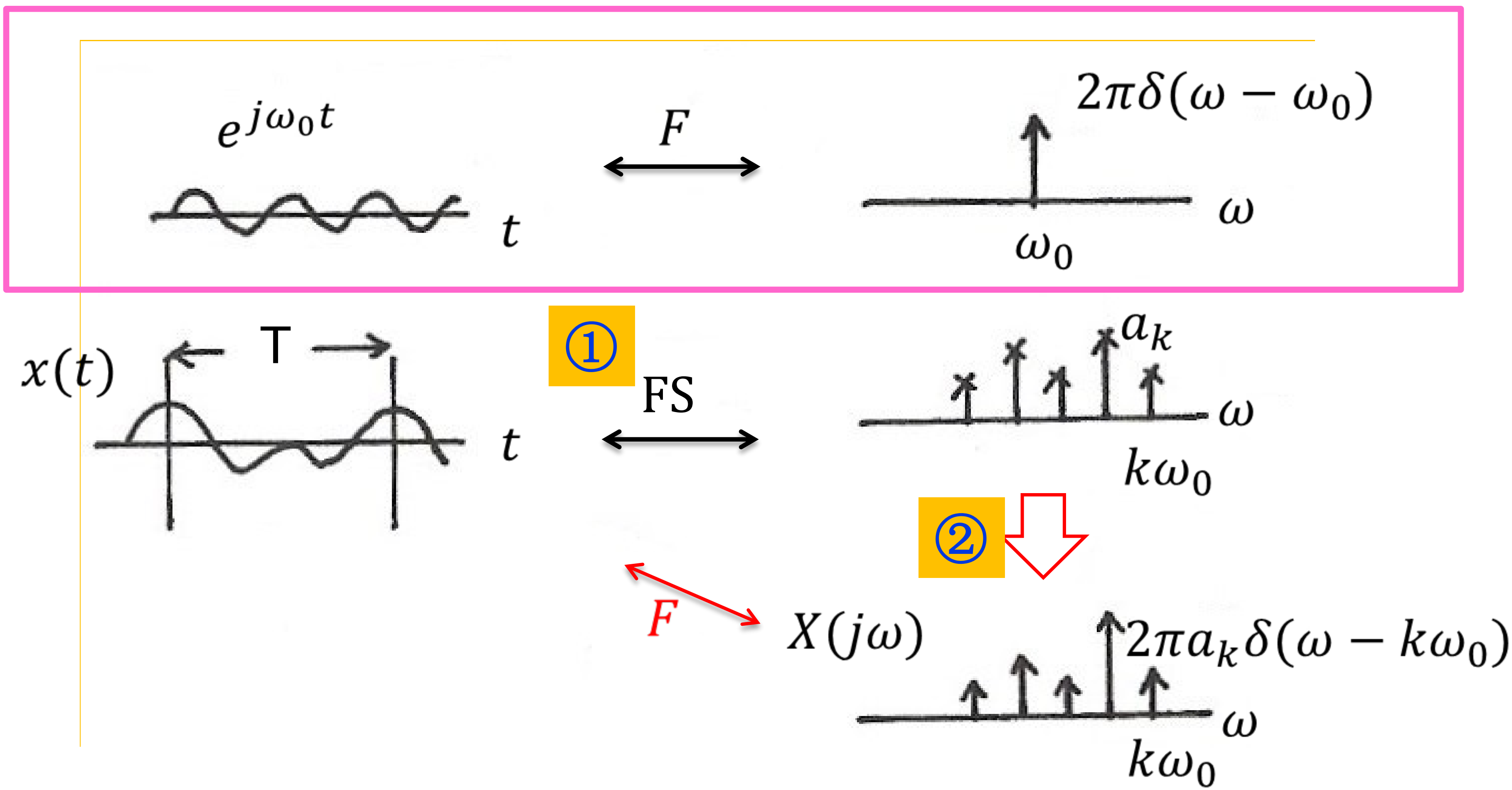
Impulse response $\overset{\mathcal{F}}{\longleftrightarrow}$ Frequency response

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} H(e^{jk\omega_o}) a_k e^{jk\omega_0 n}$$

$$a_k \longrightarrow \underbrace{H(e^{jk\omega_o})}_{\text{"gain"}} a_k$$

$$\boxed{H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n] e^{-j\omega n}}$$

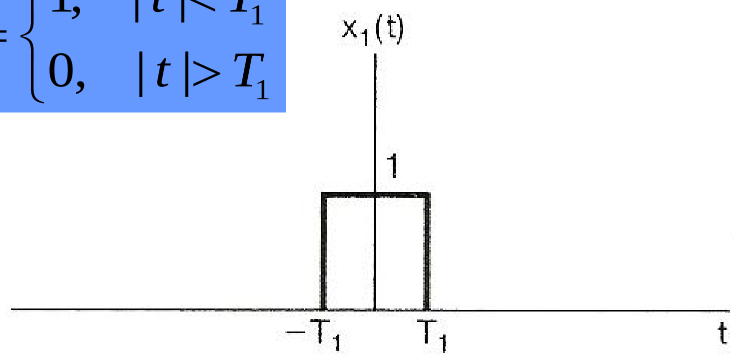
Fourier Transform for Periodic Signals – Unified Framework



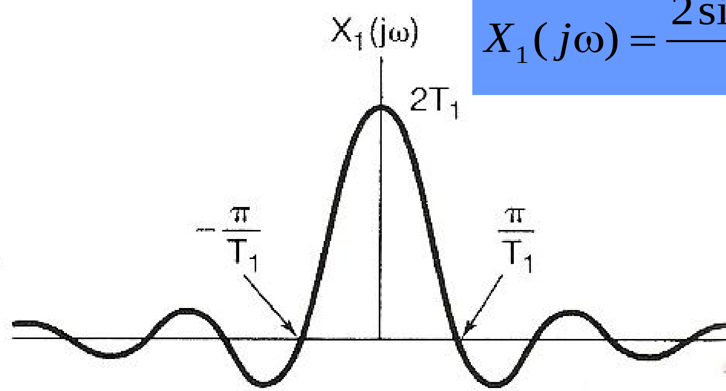
CTFT Properties (cont.)

6) Duality

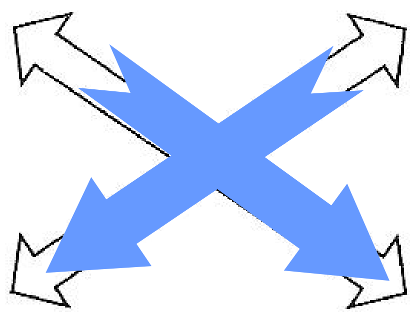
$$x_1(t)=\begin{cases}1, & |t|<T_1\\ 0, & |t|>T_1\end{cases}$$



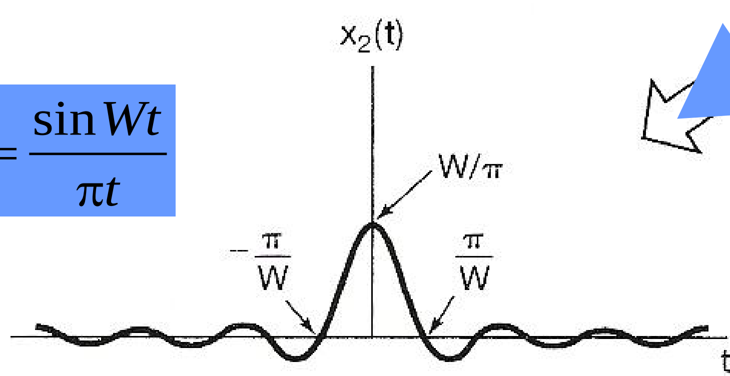
$\xleftrightarrow{\mathfrak{F}}$



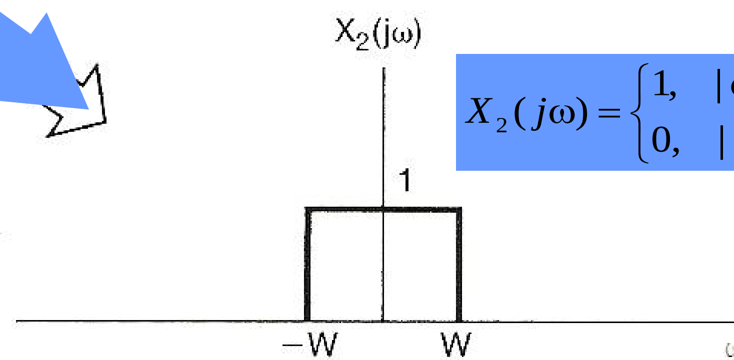
$$X_1(j\omega)=\frac{2\sin\omega T_1}{\omega}$$



$$x_2(t)=\frac{\sin Wt}{\pi t}$$



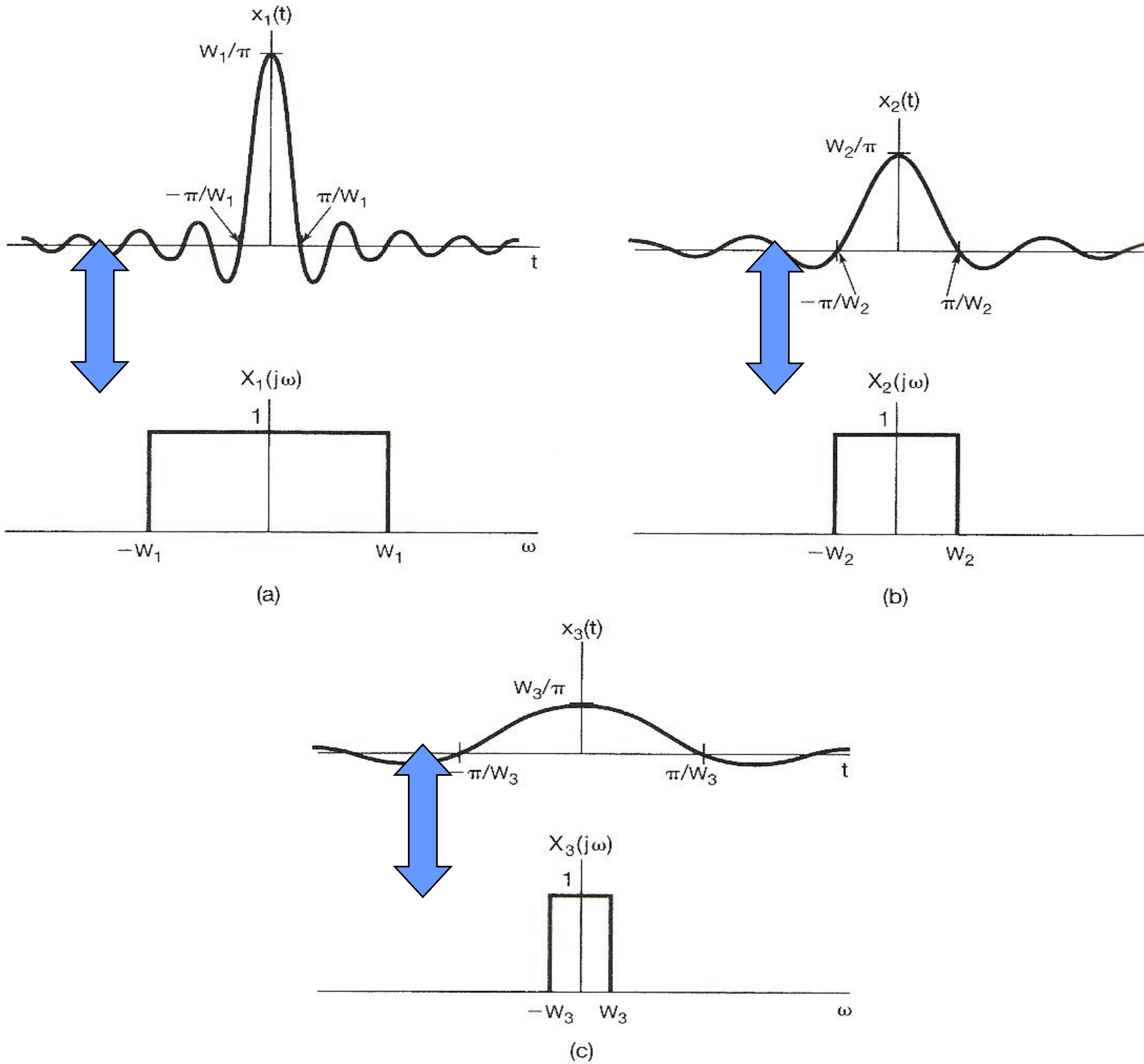
$\xleftrightarrow{\mathfrak{F}}$



$$X_2(j\omega)=\begin{cases}1, & |\omega|<W\\ 0, & |\omega|>W\end{cases}$$

Review

inverse relationship between signal "width" in time/frequency domains



$$x(t) \longleftrightarrow X(j\omega)$$

CTFT Properties (cont.)

- Time reversal $\quad x(-t) \longleftrightarrow X(-j\omega)$

- Conjugate Symmetry $\quad x(t)$ real $\longleftrightarrow X(-j\omega) = X^*(j\omega)$

$|X(-j\omega)| = |X(j\omega)|$ *Even* Or $\mathrm{Re}\{X(-j\omega)\} = \mathrm{Re}\{X(j\omega)\}$ *Even*

$\angle X(-j\omega) = -\angle X(j\omega)$ *Odd* $\quad \mathrm{Im}\{X(-j\omega)\} = -\mathrm{Im}\{X(j\omega)\}$ *Odd*

$$\Downarrow$$

a) $x(t)$ real and even $\quad x(t) = x(-t) = x^*(t)$

$$\Rightarrow X(j\omega) = X(-j\omega) = X^*(j\omega) \text{ — Real \& even}$$

b) $x(t)$ real and odd $\quad x(t) = -x(-t) = x^*(t)$

$$\Rightarrow X(j\omega) = -X(-j\omega) = -X^*(j\omega) \text{ — Purely imaginary \& odd}$$

c) $X(j\omega) = \mathrm{Re}\{X(j\omega)\} + j\,\mathrm{Im}\{X(j\omega)\}$

For real $x(t) = Ev\{x(t)\} + Od\{x(t)\}$

Table 4.2
Basic Fourier Transform Pairs

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \; k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k}\delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi}\operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t)\begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin\omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

$$e^{-at}u(t) \longleftrightarrow \frac{1}{a + j\omega}$$

$$te^{-at}u(t) \longleftrightarrow \frac{1}{(a + j\omega)^2}$$

Problem 4.15

- **Let $x(t)$ be a signal with Fourier transform** $X(j\omega)$.

Suppose we are given the following facts:

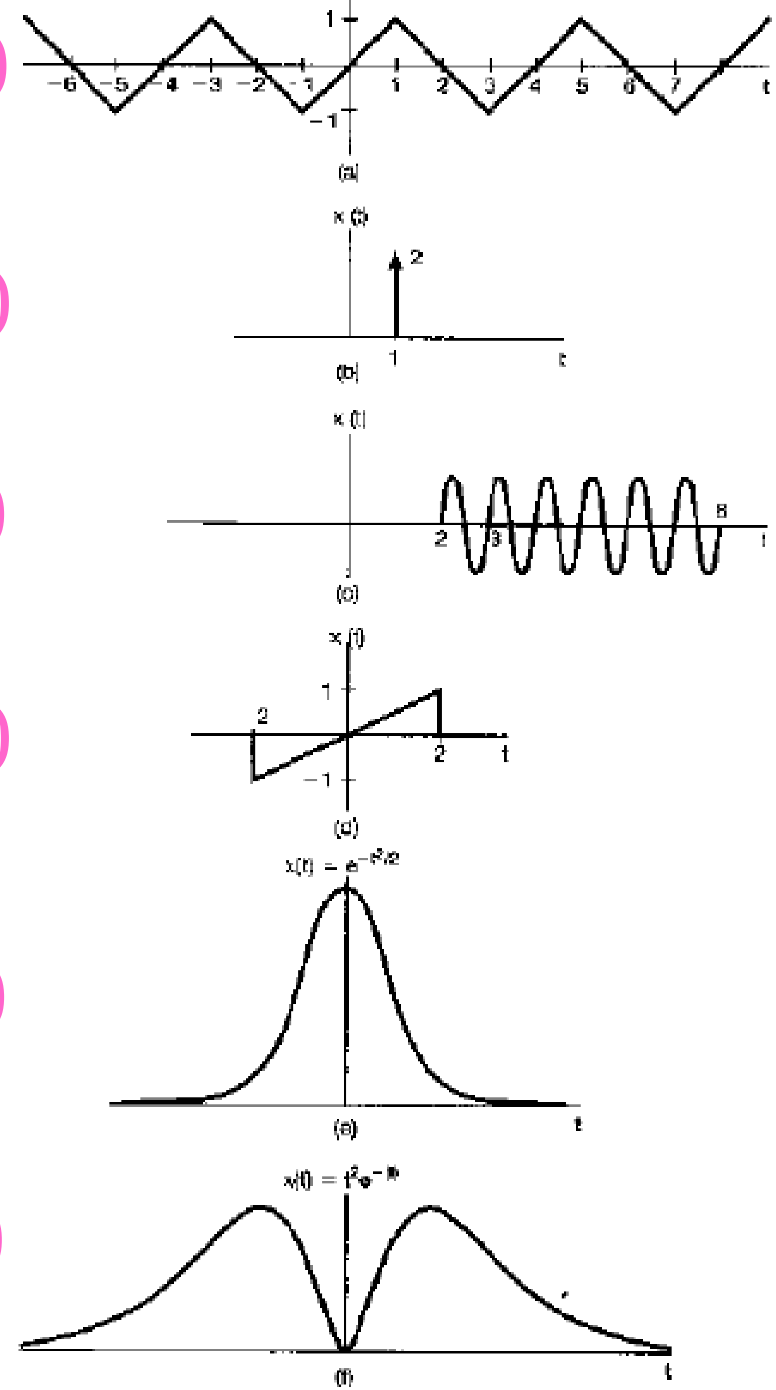
- $x(t)$ **is real**
- $x(t)=0$ **for** $t \le 0$
- $\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Re}\{X(j\omega)\} e^{j\omega t} d\omega = |t| e^{-|t|}$

Determine a closed-form expression for $x(t)$.

Problem 4.24 (a)

- Determine which, if any, of the real signals in (a)-(f) have Fourier transforms that satisfy each of the following condition:
  - $\mathrm{Re}\{X(j\omega)\}=0$
  - $\mathrm{Im}\{X(j\omega)\}=0$
  - There exists a real a such that $e^{ja\omega}X(j\omega)$ is real
  - $\int_{-\infty}^{\infty} X(j\omega)d\omega=0$
  - $\int_{-\infty}^{\infty} \omega X(j\omega)d\omega=0$
  - $X(j\omega)$ is periodic

(a)
(b)
(c)
(d)
(e)
(f)



CTFT Properties
8) Convolution Property

$$\boxed{\begin{array}{l} y(t) = h(t) * x(t) \longleftrightarrow Y(j\omega) = H(j\omega)X(j\omega) \\ \text{where} \quad h(t) \longleftrightarrow H(j\omega) \quad x(t) \longleftrightarrow X(j\omega) \end{array}}$$

Basically a consequence of the eigenfunction property

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega \Rightarrow x(t) = \int_{-\infty}^{+\infty} \underbrace{\left(\frac{1}{2\pi} X(j\omega) d\omega \right)}_{\text{coefficient}} e^{j\omega t}$$

$$a\, e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow H(j\omega) a\, e^{j\omega t}$$

$\Downarrow$ superposition

$$y(t) = \int_{-\infty}^{+\infty} \underbrace{\left(H(j\omega) \cdot \frac{1}{2\pi} X(j\omega) d\omega \right)}_{\text{New coefficient}} e^{j\omega t}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underbrace{H(j\omega) X(j\omega)}_{Y(j\omega)} e^{j\omega t}\, d\omega$$

The Frequency Response Revisited

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = h(t) * x(t)$$

$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$\Downarrow$$

The frequency response $H(j\omega)$ of a CT LTI system is simply the Fourier transform of its impulse response $h(t)$

Example #1:

$$x(t) = e^{j\omega_o t} \longrightarrow \boxed{H(j\omega)} \longrightarrow y(t)$$

Recall $\quad e^{j\omega_o t} \longleftrightarrow 2\pi\delta(\omega - \omega_o)$

$$Y(j\omega) = H(j\omega)X(j\omega) = H(j\omega)2\pi\delta(\omega - \omega_0) = 2\pi H(j\omega_0)\delta(\omega - \omega_0)$$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi H(j\omega_0)\delta(\omega - \omega_0)e^{j\omega t}d\omega$$

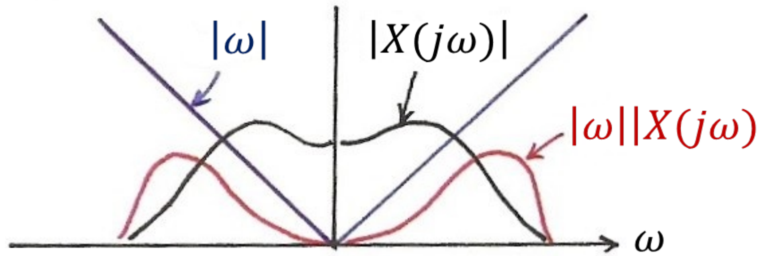
$$y(t) = H(j\omega_0)e^{j\omega_0 t}$$

Frequency Response Examples

Example 4.16 A differentiator

$$y(t) = \frac{dx(t)}{dt} \qquad \text{— an LTI system}$$

From differentiation property $\Rightarrow$ $\frac{d}{dt} \xleftrightarrow{FT} j\omega$



$$\Downarrow$$

$$H(j\omega) = j\omega$$

1) Amplifies high frequencies (enhances sharp edges)

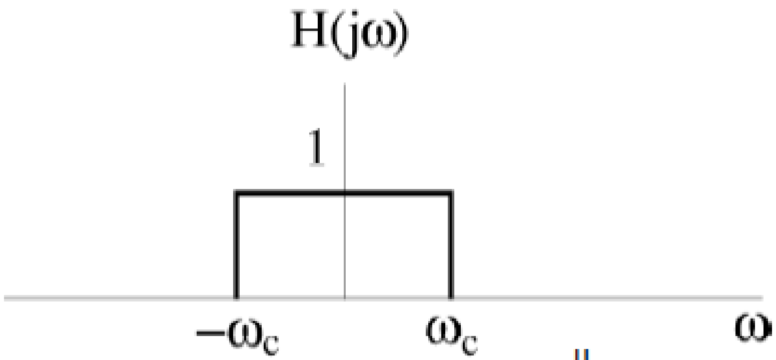
2) $+\pi/2$ phase shift ($j = e^{j\pi/2}$)

Larger at high ω_o phase shift

$$\frac{d}{dt}\sin\omega_0 t = \omega_0 \cos\omega_0 t = \omega_0 \sin(\omega_0 t + \pi/2)$$

$$\frac{d}{dt}\cos\omega_0 t = -\omega_0 \sin\omega_0 t = \omega_0 \cos(\omega_0 t + \pi/2)$$

Example 4.18： Impulse Response of an *Ideal* Lowpass Filter

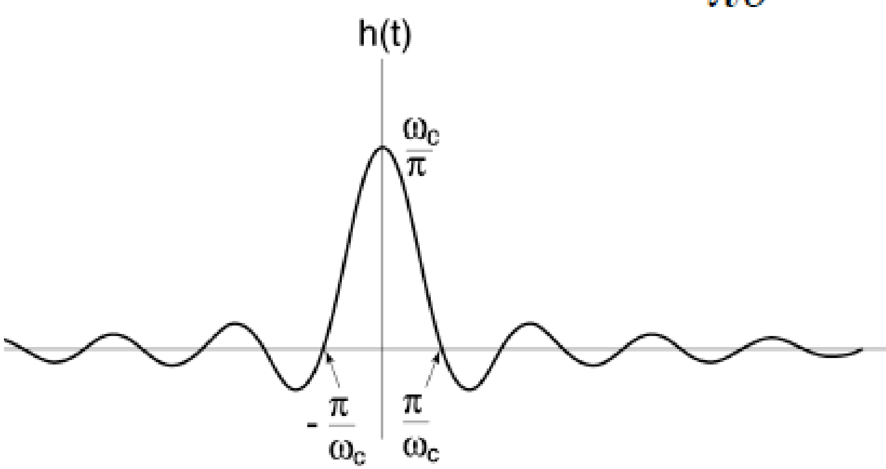


$$\boxed{h(t) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t}\,d\omega = \frac{\sin\omega_c t}{\pi t}} = \frac{\omega_c}{\pi}\operatorname{sinc}\left(\frac{\omega_c t}{\pi}\right)$$

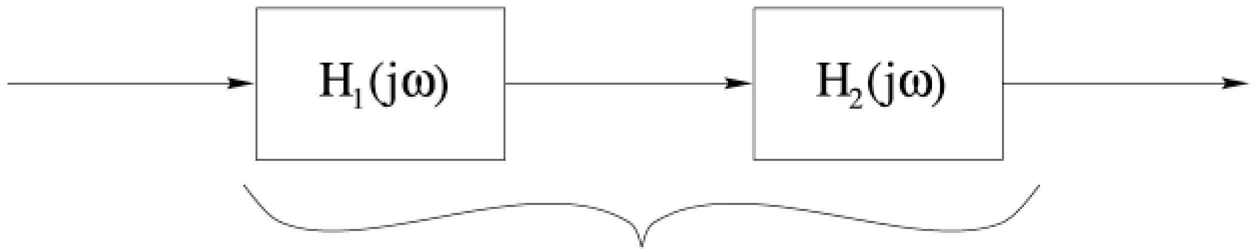
$$\operatorname{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$

Questions:

1) Is this a causal system?
2) What is $h(0)$?

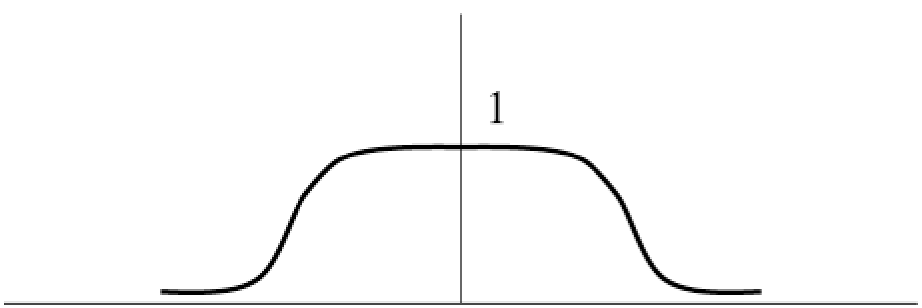


Example #4: Cascading filtering operations



$$H(j\omega) = H_1(j\omega)\, H_2(j\omega)$$

e.g. $H_1(j\omega) = H_2(j\omega)$

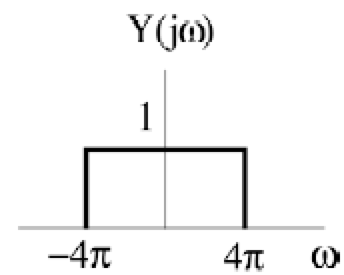


$\Longrightarrow$ $H(j\omega) = H_1^2(j\omega)$ has a sharper frequency selectivity

Example 4.20

$$\underbrace{\frac{\sin 4\pi t}{\pi t}}_{x(t)} * \underbrace{\frac{\sin 8\pi t}{\pi t}}_{h(t)} = ?$$

$$Y(j\omega) = X(j\omega)$$
$$\Rightarrow y(t) = x(t)$$



Example #6:

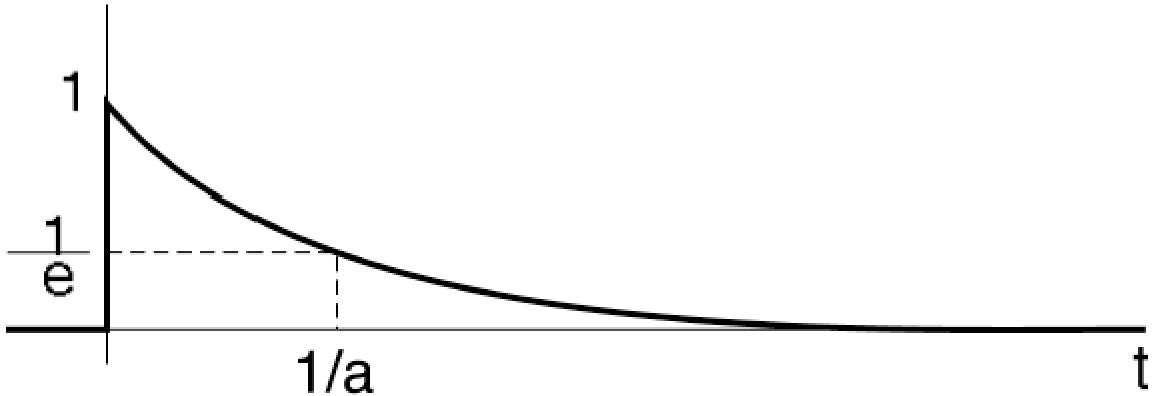
$$e^{-at^2} * e^{-bt^2} = ?$$

$$\updownarrow$$

$$\sqrt{\frac{\pi}{a}}e^{-\frac{\omega^2}{4a}} \times \sqrt{\frac{\pi}{b}}e^{-\frac{\omega^2}{4b}} = \frac{\pi}{\sqrt{ab}}e^{-\frac{\omega^2}{4}\left(\frac{1}{a}+\frac{1}{b}\right)}$$

Gaussian $\times$ Gaussian = Gaussian, Gaussian $*$ Gaussian = Gaussian

Review from the last lecture, right-sided exponential



$$x(t) = e^{-at}u(t)\ ,\quad a > 0$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt = \int_{0}^{+\infty} \underbrace{e^{-at}e^{-j\omega t}}_{e^{-(a+j\omega)t}}dt$$

$$= -\left(\frac{1}{a+j\omega}\right)e^{-(a+j\omega)t}\Big|_{0}^{\infty} = \boxed{\frac{1}{a+j\omega}}$$

Example 4.19

$$h(t) = e^{-t}u(t) \ , \ x(t) = e^{-2t}u(t)$$

$$y(t) = h(t) * x(t) = ?$$

$$\Downarrow$$

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{1}{(1+j\omega)} \cdot \frac{1}{(2+j\omega)}$$

$$\Downarrow \boxed{\text{Partial fraction expansion}}$$

$$Y(j\omega) = \frac{1}{1+j\omega}\ {\color{magenta}a=1} - \frac{1}{2+j\omega}\ {\color{magenta}a=2}$$

$$\Downarrow \text{inverse } FT$$

$$y(t) = [e^{-t} - e^{-2t}]u(t)$$

CTFT Properties
9) Multiplication Property

Since *FT* is highly symmetric,

$$x(t) = \mathfrak{F}^{-1}\{X(j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega, \quad X(j\omega) = \mathfrak{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

thus if $\quad x(t) * y(t) \longleftrightarrow X(j\omega)\cdot Y(j\omega)$

then the other way around is also true

$$\boxed{\begin{aligned} x(t)\cdot y(t) &\longleftrightarrow \frac{1}{2\pi} X(j\omega) * Y(j\omega) \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega-\theta))\, d\theta \end{aligned}}$$

$\frac{1}{2\pi}$ — A consequence of *Duality*

Examples of the Multiplication Property: Modulation Property

$\boxed{\text{Frequency shift}}$

$$e^{j\omega_0 t}x(t) \xleftrightarrow{F} X\left(j(\omega-\omega_0)\right)$$

$$e^{j\omega_0 t}\cdot x(t) \leftrightarrow \frac{1}{2\pi}[2\pi\delta(\omega-\omega_0) * X(j\omega)]$$
$$= X(j(\omega-\omega_0))$$

Example 4.21

$$r(t) = s(t)\cdot p(t) \longleftrightarrow R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j\omega)]$$

For $\quad p(t) = \cos\omega_0 t \longleftrightarrow P(j\omega) = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$

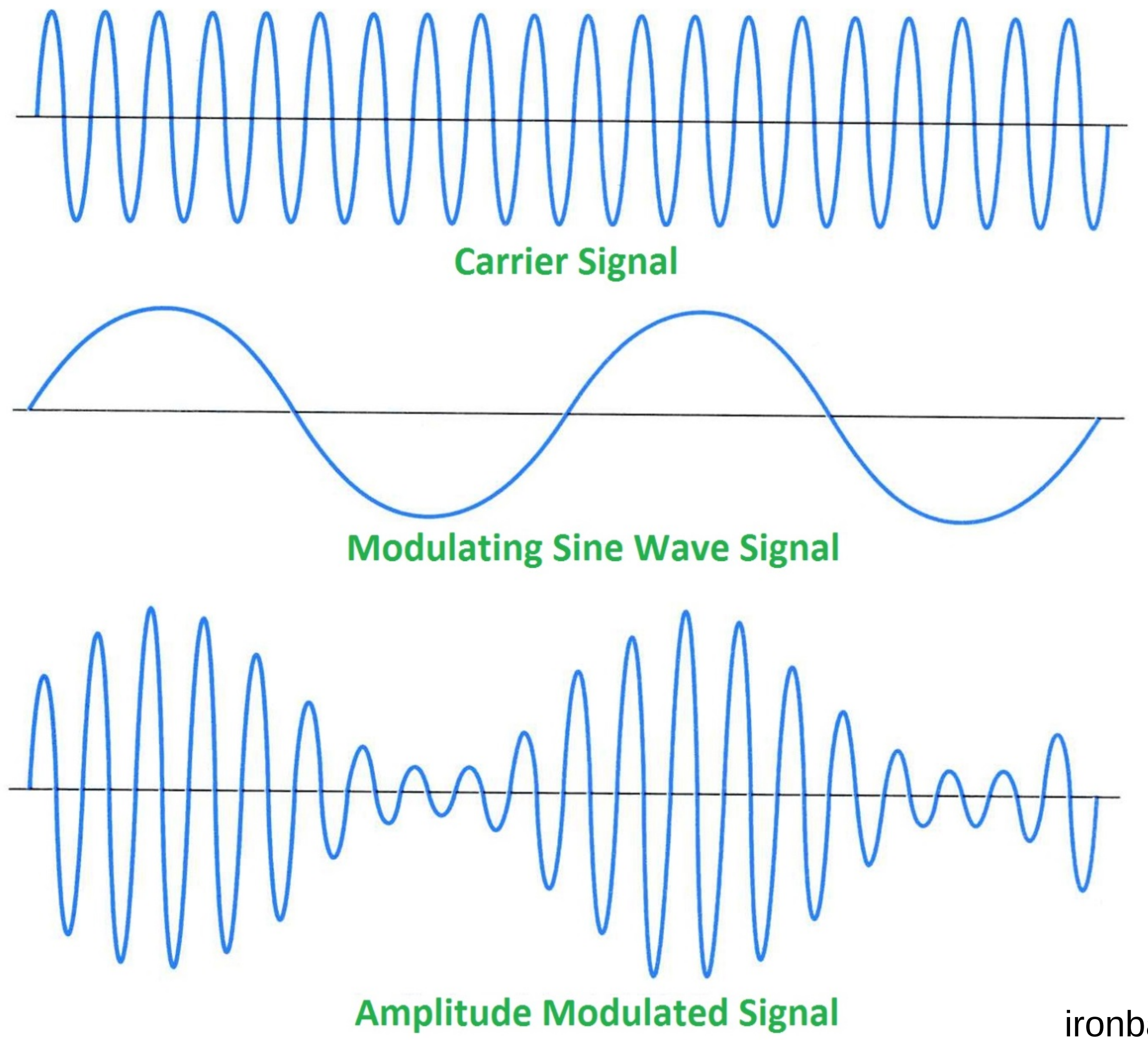
$$R(j\omega) = \frac{1}{2}S(j(\omega-\omega_0)) + \frac{1}{2}S(j(\omega+\omega_0))$$

(cont.)

$p(t)$

$s(t)$

$r(t)$



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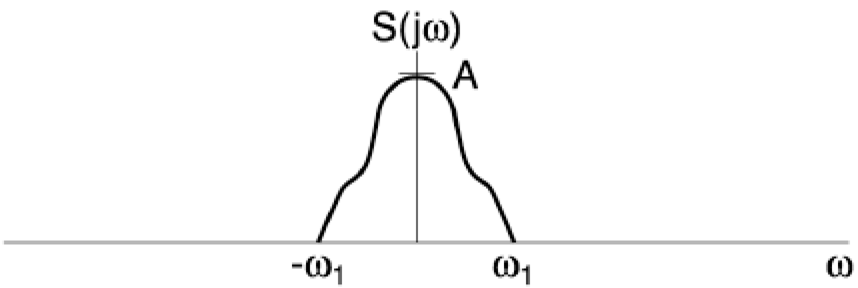
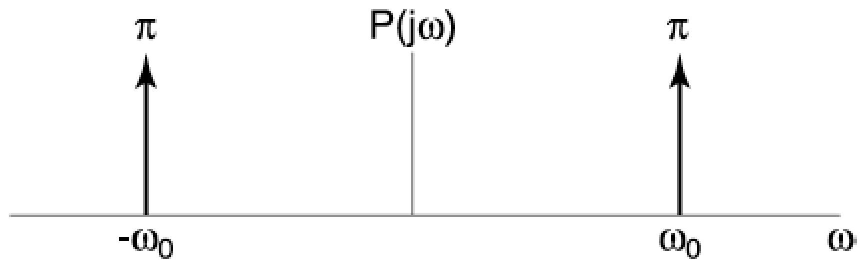
(cont.)

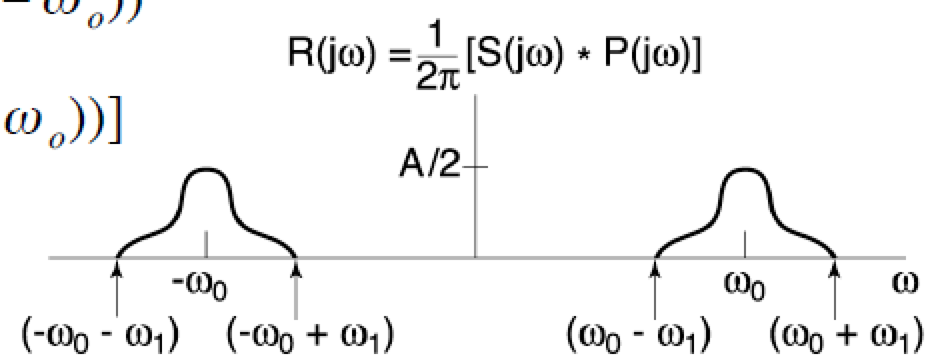
ω_1 : bandwidth



$r(t) = s(t) \cdot \cos(\omega_o t)$
Amplitude modulation (*AM*)



$$R(j\omega) = \frac{1}{2}[S(j(\omega - \omega_o)) + S(j(\omega + \omega_o))]$$



Drawn assume

$\omega_o - \omega_1 > 0$

i.e. $\omega_o > \omega_1$

Frequency-Selective Filtering with Variable Center Frequency

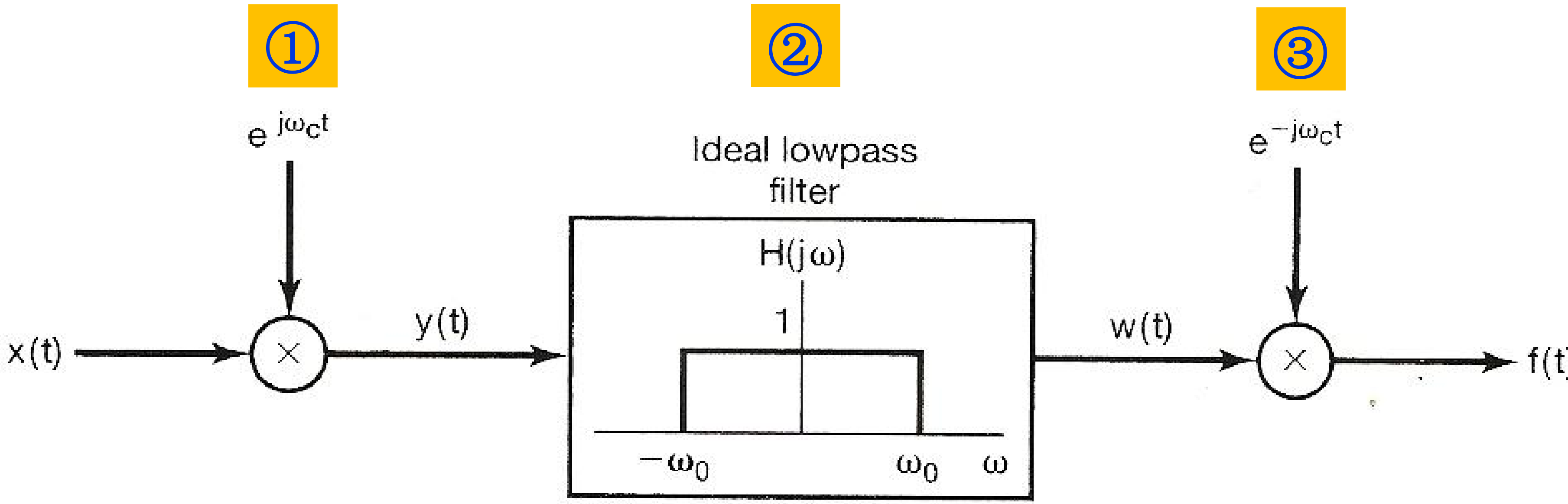


Figure 4.26 Implementation of a bandpass filter using amplitude modulation with a complex exponential carrier.

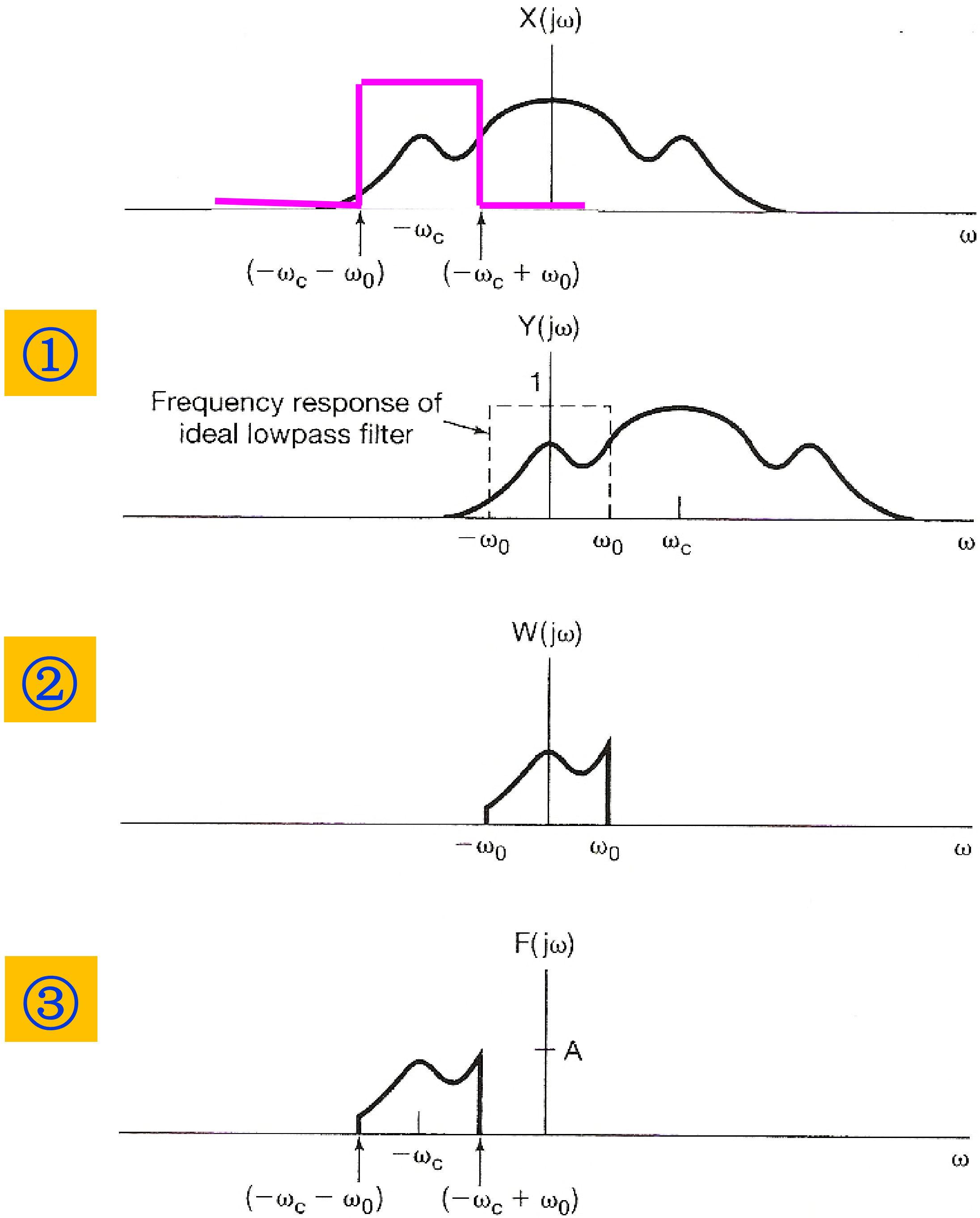


Figure 4.27 Spectra of the signals in the system of Figure 4.26.

Table 4.1 Properties of the Fourier Transform

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$	$X(j\omega)$
		$y(t)$	$Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ $[x(t)$ real$]$ $x_o(t) = \mathcal{O}d\{x(t)\}$ $[x(t)$ real$]$	$\mathcal{R}e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \lvert X(j\omega)\rvert^2 d\omega$$

Example #8: LTI Systems Described by LCCDE's

(Linear-constant-coefficient differential equations)

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Using the Differentiation Property

$$\frac{d^k x(t)}{dt^k} \longleftrightarrow (j\omega)^k X(j\omega)$$

$\Downarrow$ Transform both sides of the equation

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

$$\Downarrow$$

$$Y(j\omega) = \underbrace{\left[\frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}\right]}_{H(j\omega)} X(j\omega)$$

$$\boxed{H(j\omega) = \left[\frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}\right]}$$

Problem 4.34

- A causal and stable LTI system S has the frequency response

$$H(j\omega) = \frac{j\omega + 4}{6 - \omega^2 + 5j\omega}$$

  - Determine a differential equation relating the input $x(t)$ and output $y(t)$ of S
  - Determine the impulse response $h(t)$ of S
  - What is the output of S when the input is $x(t) = e^{-4t}u(t) - te^{-4t}u(t)$

$f_1(t) = u(t), f_2(t) = e^{-at}u(t)$ 计算 $f_1(t) * f_2(t)$

2.

$f_1(t)=(1+t)[u(t)-u(t-1)]$, $f_2(t)=u(t-1)-u(t-2)$，计算 $f_1(t)*f_2(t)$