Notes

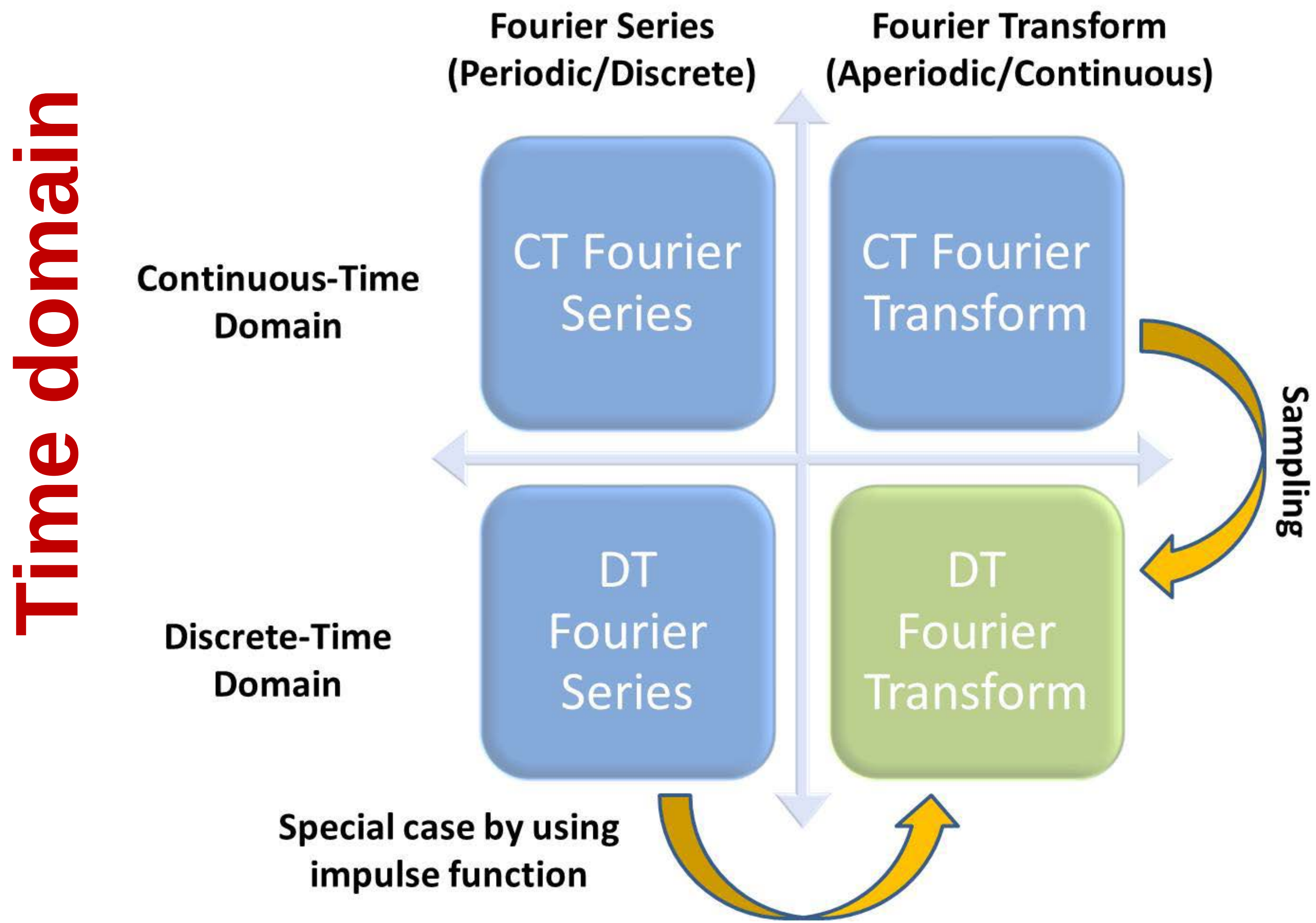
Assignments

- 5.23, 5.29, 5.33

Tutorial problems

- Property of DTFT: 5.24
- Difference equation of LTI systems: 5.36

Frequency domain



DTFT

- Therefore, we get the discrete-time Fourier transform pair

Discrete-Time Fourier Transform

$$\begin{aligned} \textit{Synthesis Equation:} \quad x[n] &= \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \\ \textit{Analysis Equation:} \quad X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \end{aligned}$$

- Observations
  - Continuous spectrum: Similar to CTFT
  - Periodic with period 2π: Different from CTFT
  - Low frequency: close to 0 and 2π; high frequency: close to π

Periodicity Properties of DT Complex Exponentials

- For DT complex exponentials, signals with frequencies ω_0 and $\omega_0 + k \bullet 2\pi$ are identical. $\boxed{e^{j(\omega_0 + k\cdot 2\pi)n} = e^{j\omega_0 n} \cdot e^{jk\cdot 2\pi n} = e^{j\omega_0 n}}$
  - We need only consider a frequency interval of length 2π, and on most cases, we use the interval: $0 \le \omega_0 < 2\pi$, or $-\pi \le \omega_0 < \pi$

- $e^{j\omega_0 n}$ does ***not*** have a continually increasing rate of oscillation as ω_0 is increased in magnitude.

low-frequency (slowly varying): ω_0 near 0, 2π, …, or $2k \bullet \pi$

high-frequency (rapid variation): ω_0 near $\pm\pi$, …, or $(2k+1) \bullet \pi$

$$e^{j2\pi n} = (e^{j2\pi})^n = (1)^n = 1$$

$$e^{j(2k+1)\pi n} = e^{j\pi n} = (e^{j\pi})^n = (-1)^n$$

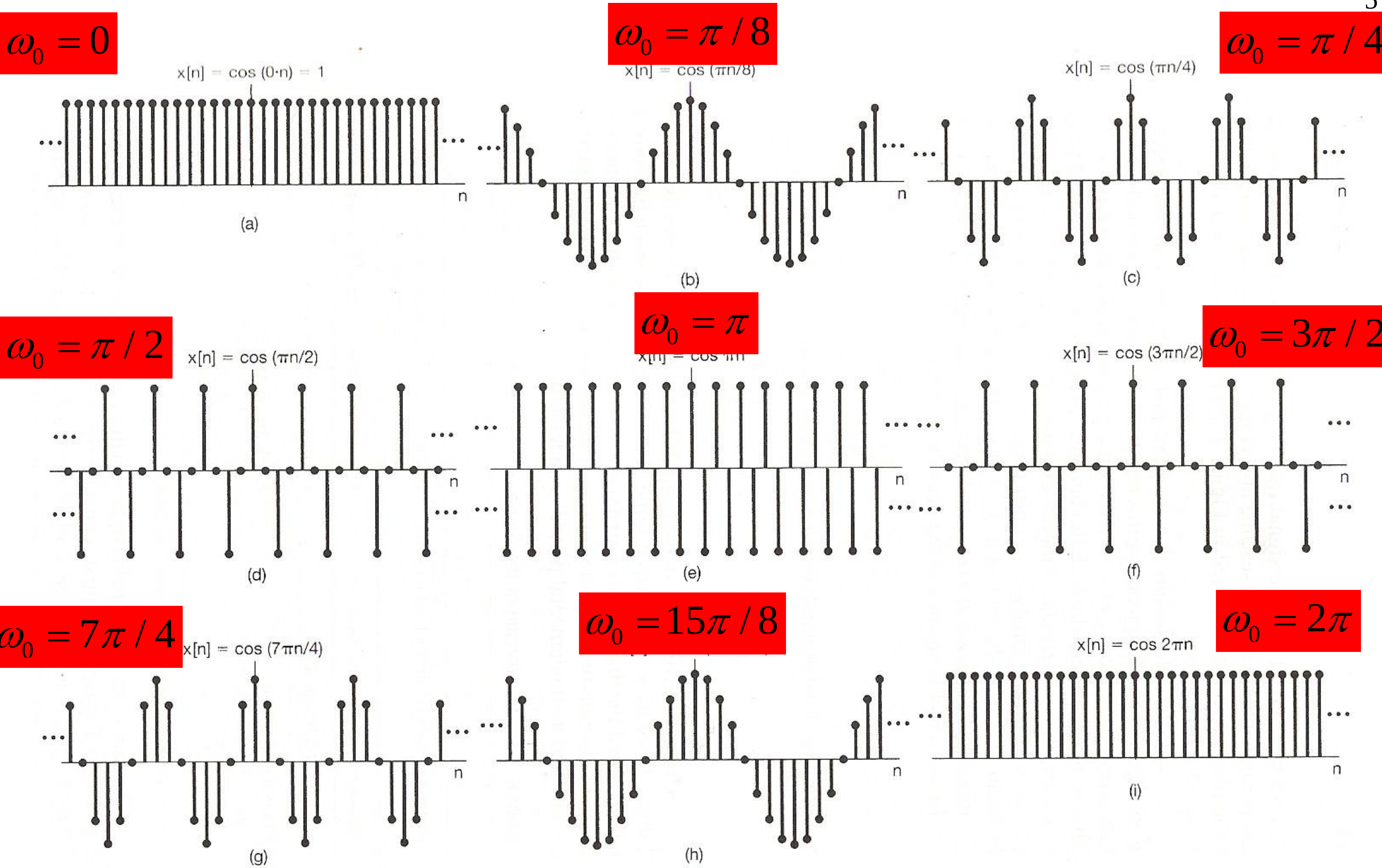


Figure 1.27 Discrete-time sinusoidal sequences for several different frequencies.

Discrete-Time Fourier Transform

$$\text{Synthesis Equation:} \quad x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$\text{Analysis Equation:} \quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

DT Fourier Series Pair $\left(\omega_o = \frac{2\pi}{N}\right)$

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \qquad \text{(Synthesis equation)}$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n} \qquad \text{(Analysis equation)}$$

With CTFT, now the frequency response of an LTI system makes complete sense

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} H(jk\omega_0) a_k e^{jk\omega_0 t}$$

$$a_k \longrightarrow \underbrace{H(jk\omega_0)}_{\text{"gain"}} a_k$$

$$\boxed{H(j\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt}$$

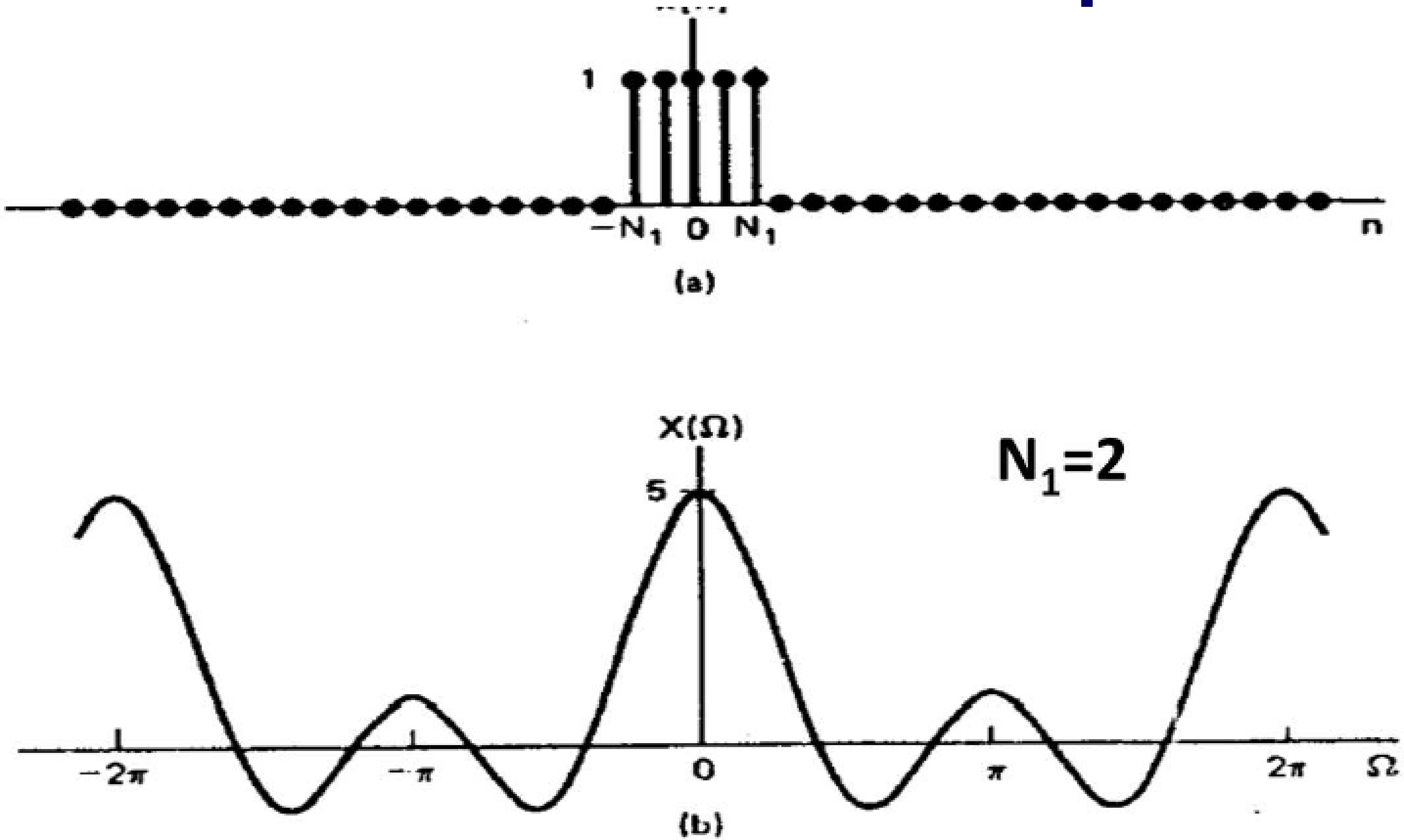
$$\boxed{\text{Impulse response} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \text{Frequency response}}$$

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} H(e^{jk\omega_o}) a_k e^{jk\omega_0 n}$$

$$a_k \longrightarrow \underbrace{H(e^{jk\omega_o})}_{\text{"gain"}} a_k$$

$$\boxed{H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n] e^{-j\omega n}}$$

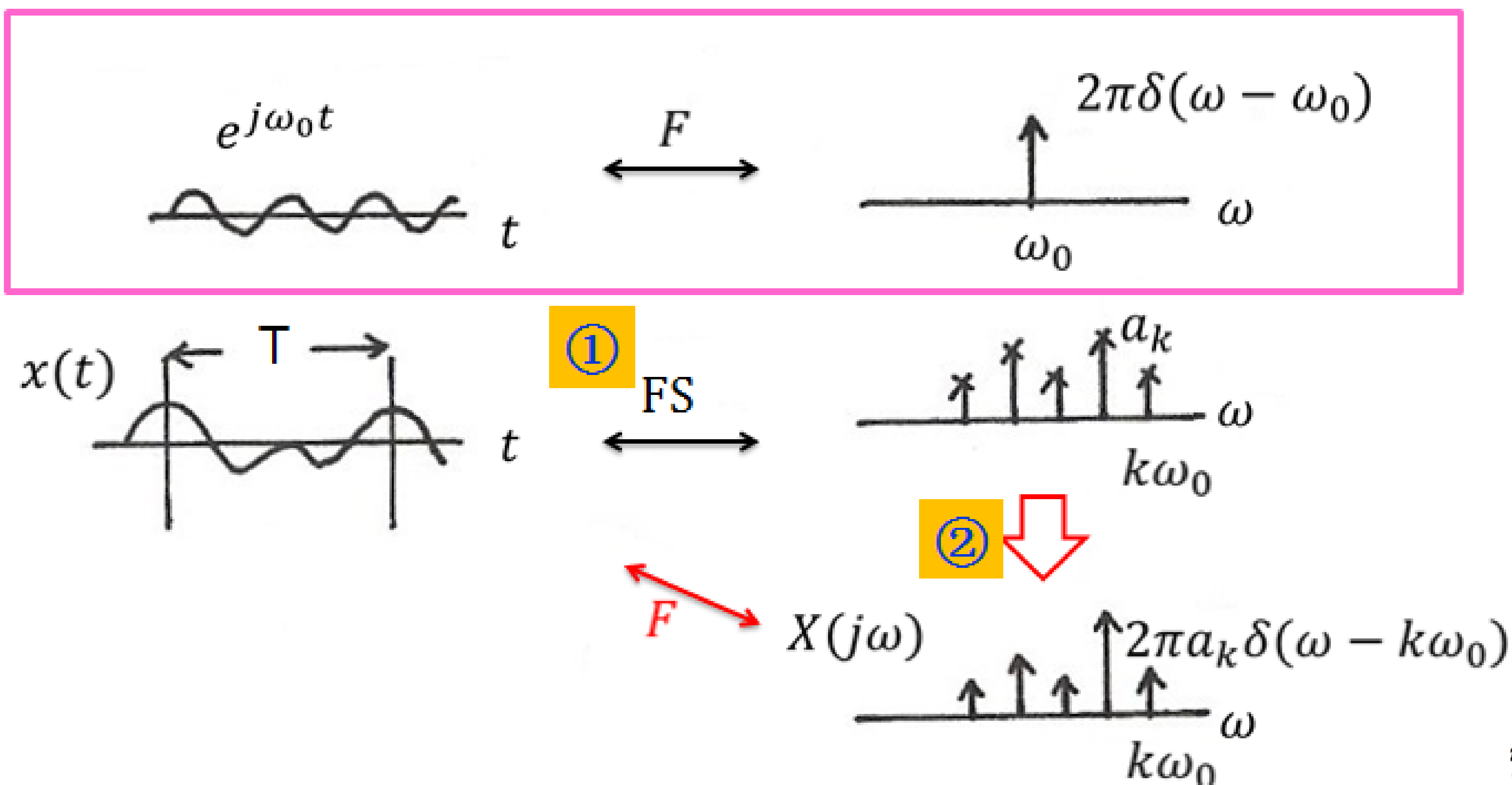
Fourier Transform Example



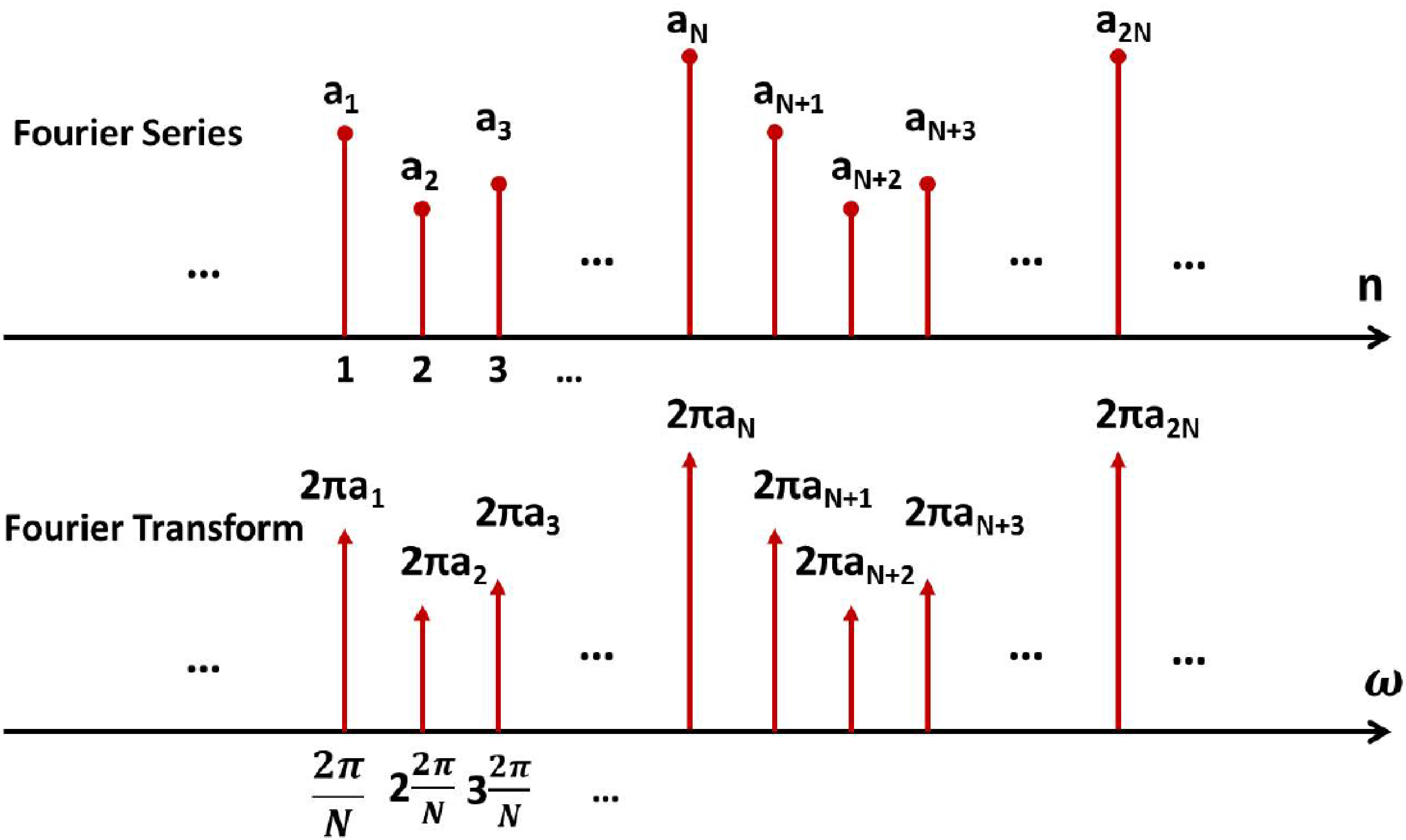
Can Periodic Signals Have DTFT?

Review 7

Fourier Transform for Periodic Signals – Unified Framework



Fourier Series v.s. Fourier Transform



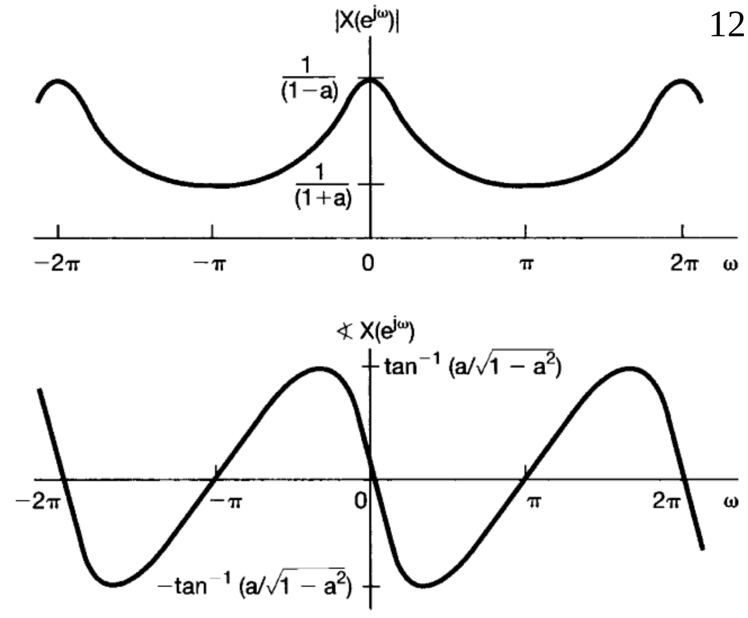
- **If x (t) is periodical, then X(*w*) is computed by Fourier ________, and its shape is ________?**

  Further more,
  - **If x (t) is CT, then a_k is _________?**
  - **If x (t) is DT, then a_k is _________?**
- **If x (t) is aperiodical, then X(*w*) is computed by Fourier ________, and its shape is ________?**

  Further more,
  - **If x (t) is CT, then X(*w*) is _________?**
  - **If x (t) is DT, then X(*w*) is _________?**

Example 5.1

a>0



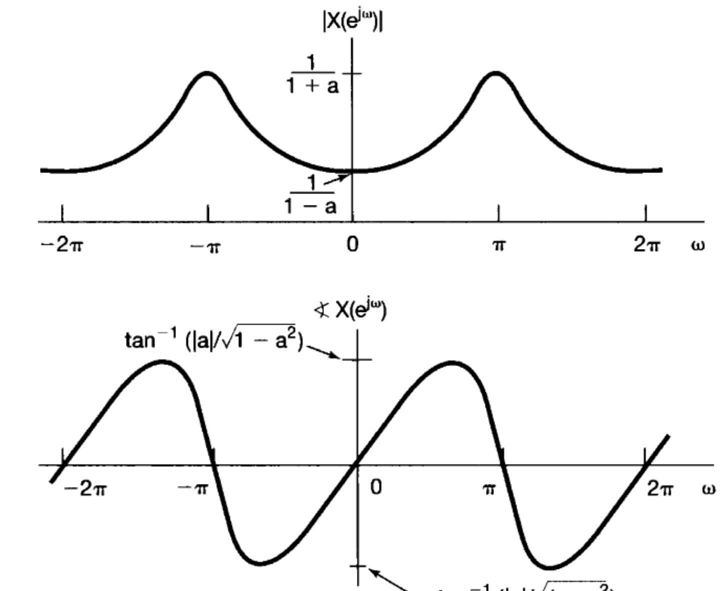
(a)

$$x[n] = a^n u[n], \qquad |a| < 1.$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}.$$

a<0



(b)

Example 5.2

$$x[n] = a^{|n|}, \qquad |a| < 1.$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^{|n|} e^{-j\omega n}$$

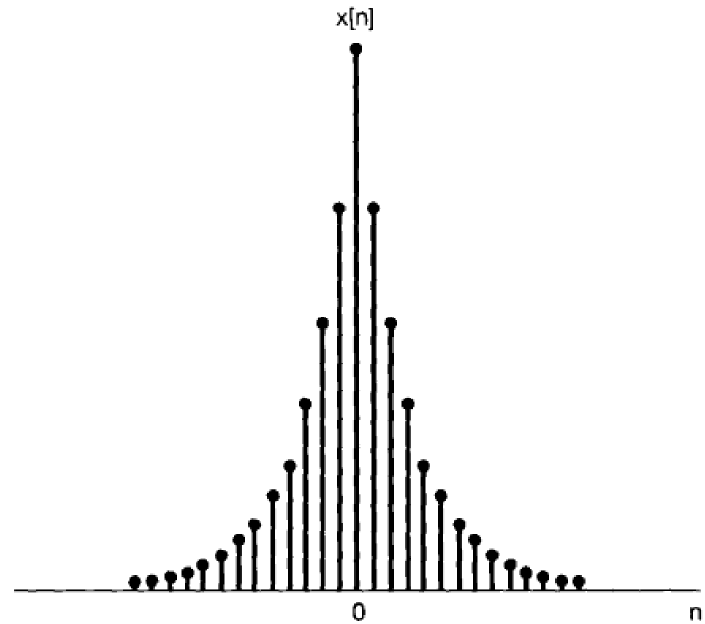
$$= \sum_{n=0}^{\infty} a^{n} e^{-j\omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n}.$$

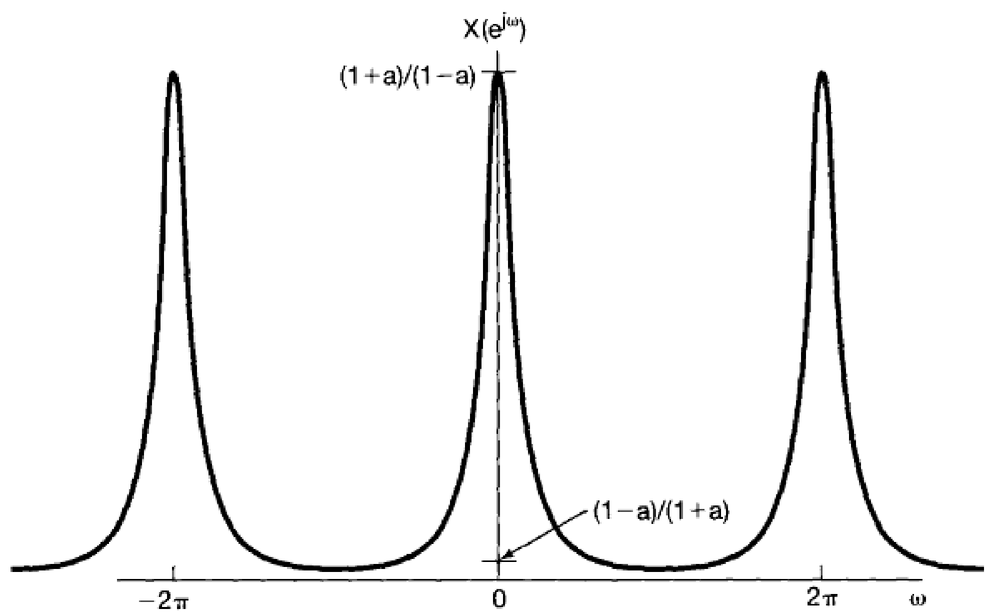
m= -n

$$= \sum_{n=0}^{\infty} (a e^{-j\omega})^{n} + \sum_{m=1}^{\infty} (a e^{j\omega})^{m}.$$

$$= \frac{1}{1 - a e^{-j\omega}} + \frac{a e^{j\omega}}{1 - a e^{j\omega}}$$

$$= \frac{1 - a^2}{1 - 2a\cos\omega + a^2}.$$





Example 5.4

$$x[n] = \delta[n].$$

$$\Downarrow$$

$$X(e^{j\omega}) = 1$$

Periodicity, Linearity and Shifting

- Periodicity

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

  - How about CTFT? Why?

- Linearity

$$ax_1[n] + bx_2[n] \longleftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

- Time Shifting and Frequency Shifting

$$\begin{aligned} x[n-n_0] &\longleftrightarrow e^{-j\omega n_0}X(e^{j\omega}) \\ e^{j\omega_0 n}x[n] &\longleftrightarrow X(e^{j(\omega-\omega_0)}) \end{aligned}$$

  - What's the physical meaning?

$$e^{j\theta} \cdot [a \cdot e^{j\beta}] = a \cdot e^{j(\beta+\theta)}$$

$$x[n - n_0] \quad \longleftrightarrow \quad e^{-j\omega n_0} X(e^{j\omega})$$

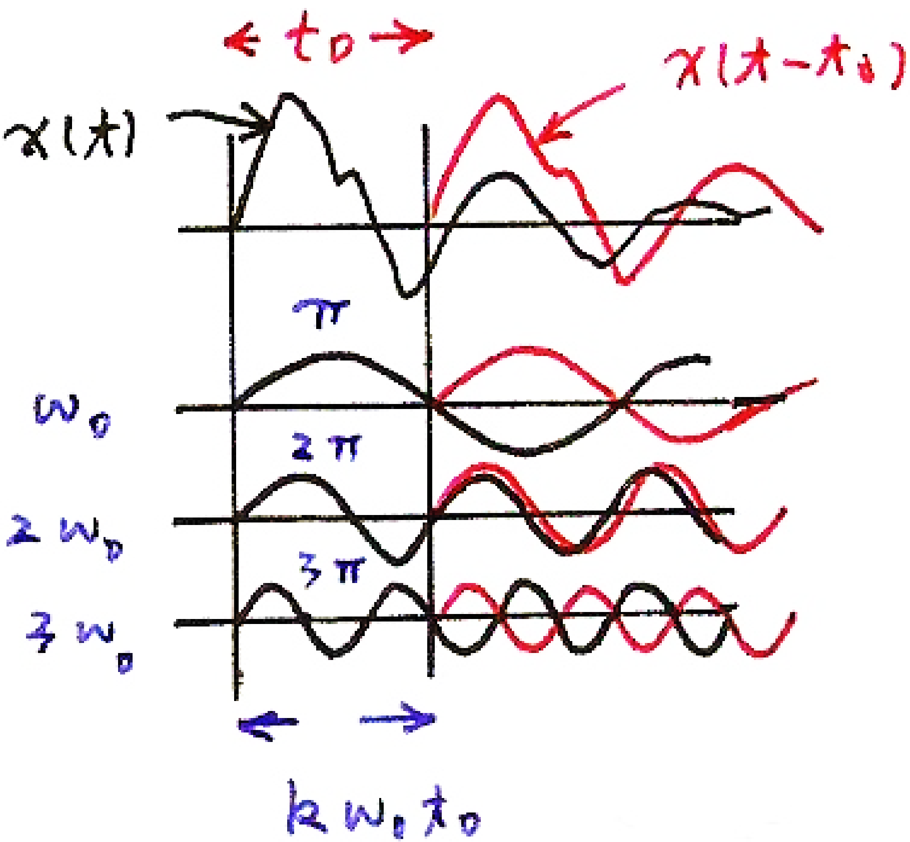
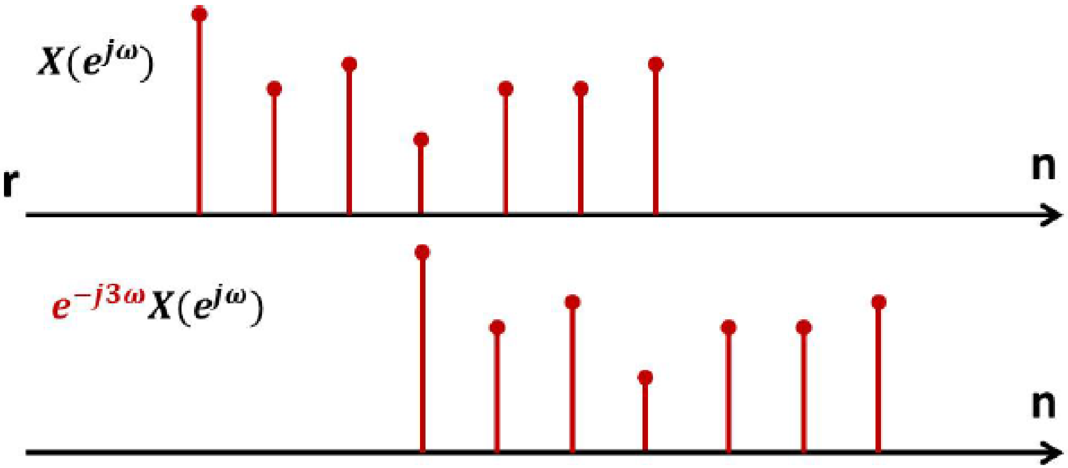


Illustration on Shifting

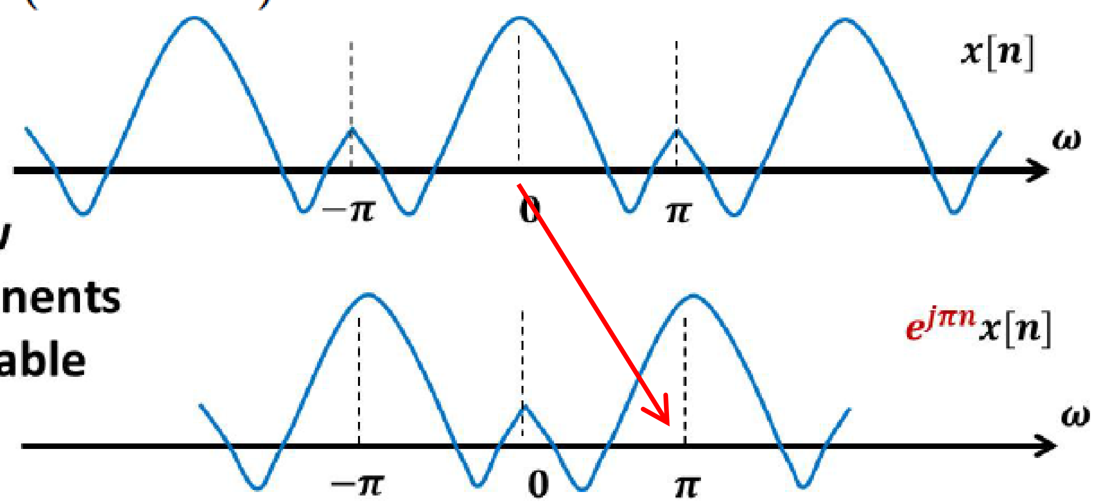
- $x[n - n_0] \longleftrightarrow e^{-j\omega n_0} X(e^{j\omega})$

Delay raises linear phase shifting



- $e^{j\omega_0 n} x[n] \longleftrightarrow X(e^{j(\omega - \omega_0)})$

High and low frequency components are interchangeable



Conjugation, Differencing and Accumulation

- Conjugation

$$x^*[n] \longleftrightarrow X^*(e^{-j\omega})$$

  - $X(e^{j\omega}) = X^*(e^{-j\omega}) \Leftrightarrow x[n]$ is real
  - If $x[n]$ is real, then $\Re\{X(e^{j\omega})\}$ is even, $\Im\{X(e^{j\omega})\}$ is odd

- Differencing

$$x[n] - x[n-1] \quad \longleftrightarrow \quad (1 - e^{-j\omega})X(e^{j\omega})$$

  - High-pass or low-pass?

- Accumulation

$$\sum_{m=-\infty}^{n} x[m] \longleftrightarrow \frac{1}{1 - e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

  - How to derive it via differencing?

Effect of Differencing

- $J(i,j) = |M(i,j) - M(i+1,j+1)| + |M(i+1,j) - M(i,j+1)|$

Fourier Transform of $u[n]$

- How to derive the Fourier transform of u[n]?
- Option 1: From definition of Fourier transform

$$\mathcal{F}\{u[n]\} = \sum_{n=-\infty}^{+\infty} u[n]e^{-j\omega n} = \sum_{n=0}^{+\infty} e^{-j\omega n} \quad Converge??$$

- Option 2: Since $u[n] = \sum_{m=-\infty}^{n} \delta[m]$, according to the property of accumulation

$$\mathcal{F}\{u[n]\} = \frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

- Observation: Fourier transform of $u[n]$ does not exist according to the definition; however, it can be expressed in terms of $\delta(\cdot)$

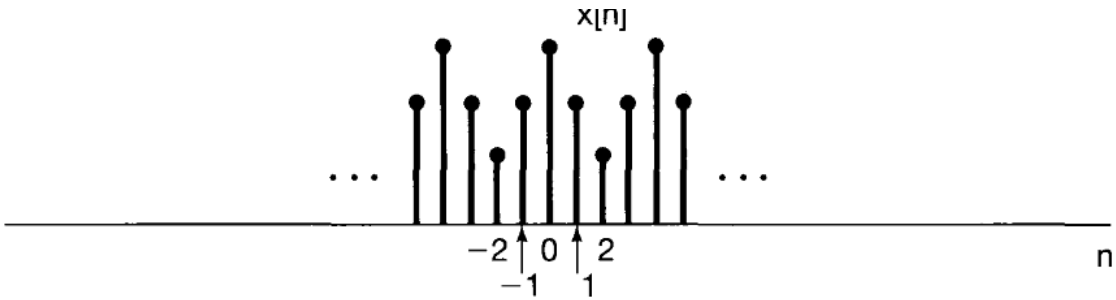
Time Reversal and Expansion

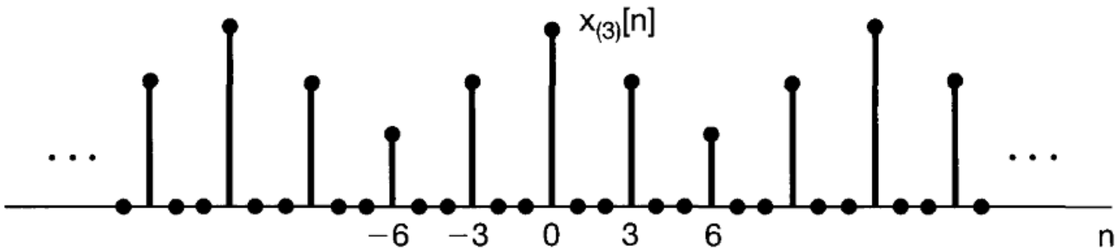
- Time Reversal

$$x[-n] \longleftrightarrow X(e^{-j\omega})$$

- Time Expansion
  - Define $x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of k} \\ 0, & \text{Otherwise} \end{cases}$, then

$$x_{(k)}[n] \longleftrightarrow X(e^{jk\omega})$$





Differentiation and Parseval

- Differentiation in Frequency

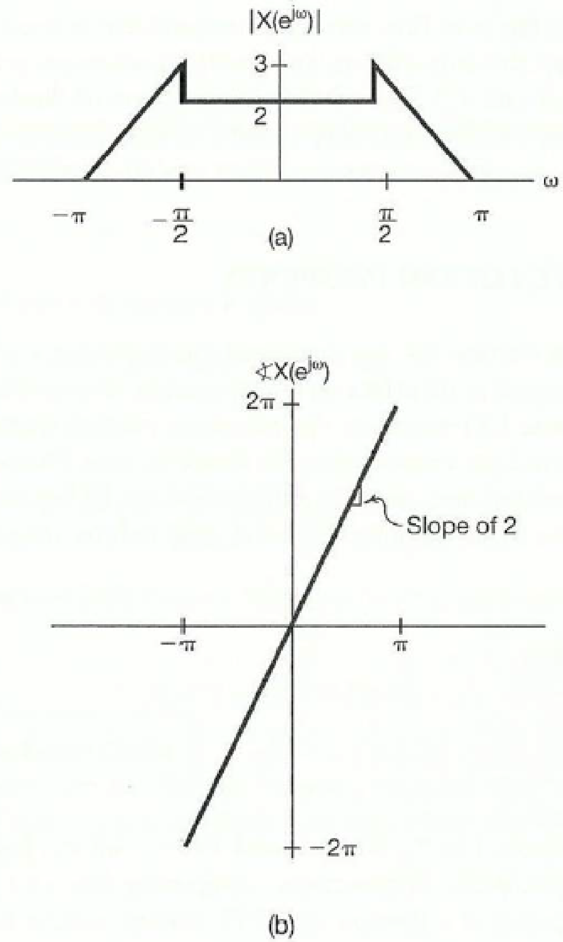
$$nx[n] \longleftrightarrow j\frac{dX(e^{j\omega})}{d\omega}$$

- Parseval's Relation

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

  - Energy density spectrum — $\frac{|X(e^{j\omega})|^2}{2\pi}$

Example



- See textbook, Example 5.10
- Spectrum within $[-\pi, \pi]$
- Is it periodic, real, even, of finite energy?

Convolution Property & LTI Systems (1/2)

- Let $h[n]$ be the impulse response of certain LTI system
- The output $y[n]$ of input $x[n]$ is given by $y[n] = x[n] * h[n]$
- For input signal $x[n] = e^{j\omega n}$,

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)} = e^{j\omega n} \underbrace{\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}}_{\text{Frequency Response } H(e^{j\omega})}$$

- For periodic input signal $x[n] = \sum_{k=<N>} a_k e^{jk(\frac{2\pi}{N})n}$:

$$y[n] = \sum_{k=<N>} a_k H(e^{jk(\frac{2\pi}{N})}) e^{jk(\frac{2\pi}{N})n}$$

$$b_k = a_k H(e^{jk(\frac{2\pi}{N})})$$

Convolution Property & LTI Systems (2/2)

Convolution Property

If $y[n] = x_1[n] * x_2[n]$, *then*

$$Y(e^{j\omega}) = X_1(e^{j\omega})X_2(e^{j\omega})$$

- For general input signal $x[n]$:

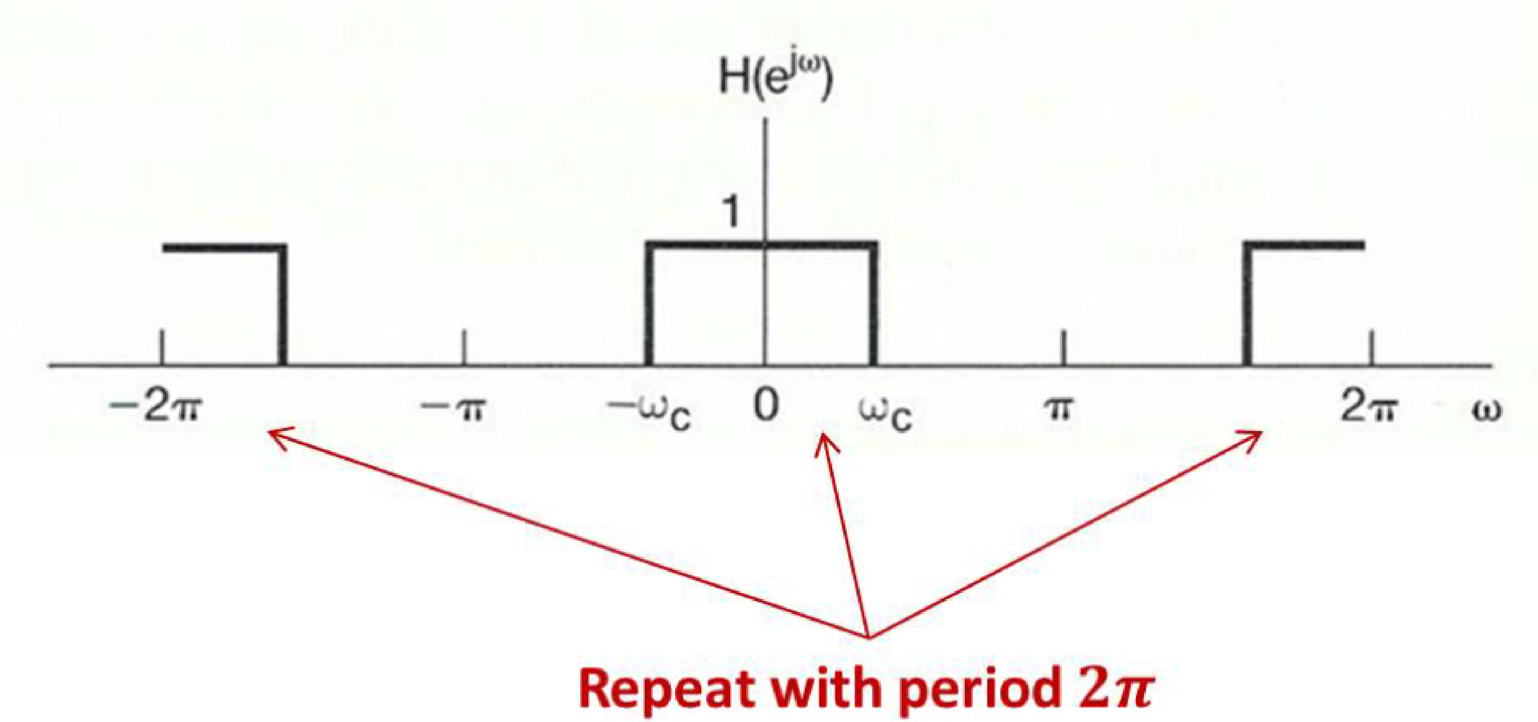
$$y[n] = x[n] * h[n] \longleftrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

- Observation: It's easier to evaluate LTI systems in frequency domain
- Drawback: Not every LTI system has frequency response
  - $h[n] = a^n u[n]$ $(a > 1)$
  - Stable LTI system has frequency response, because

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Example: Ideal Low-Pass Filter (1/2)

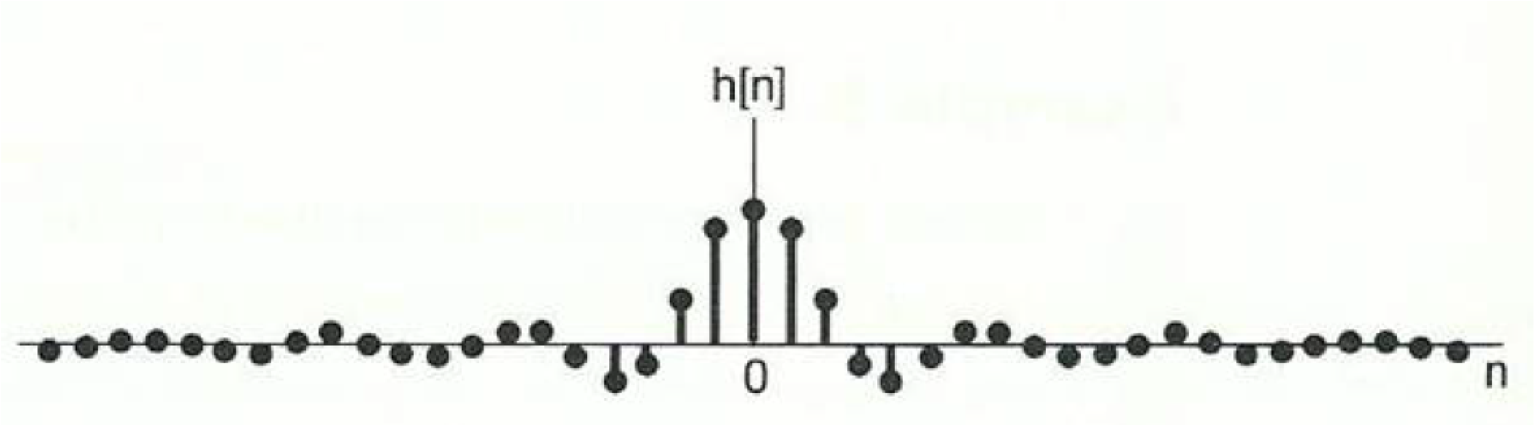
- What's ideal low-pass filter?
- Perfectly maintain the low-frequency component
- Perfectly cancel the high-frequency component



Example: Ideal Low-Pass Filter (2/2)

- Impulse response:

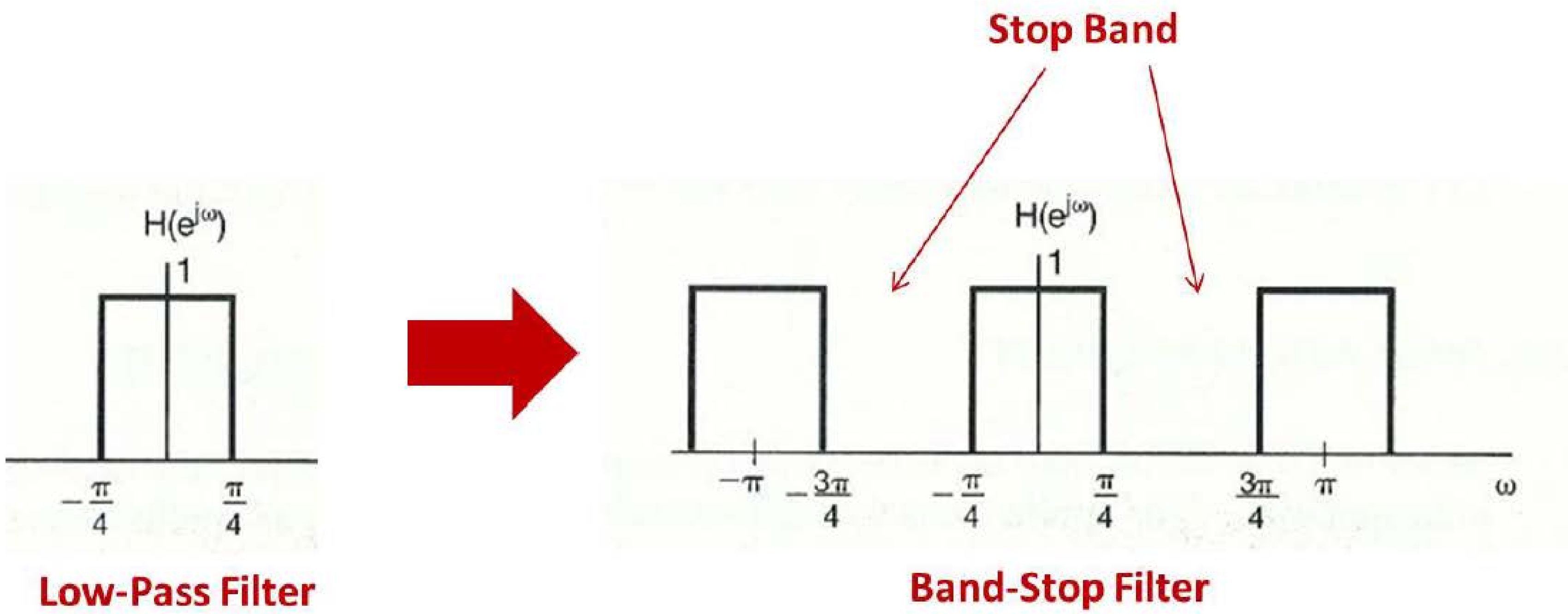
$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega = \frac{sin\omega_c n}{\pi n}$$



- Pros: no distortion in frequency domain
- Cons: non-causal
- See textbook, Example 5.12

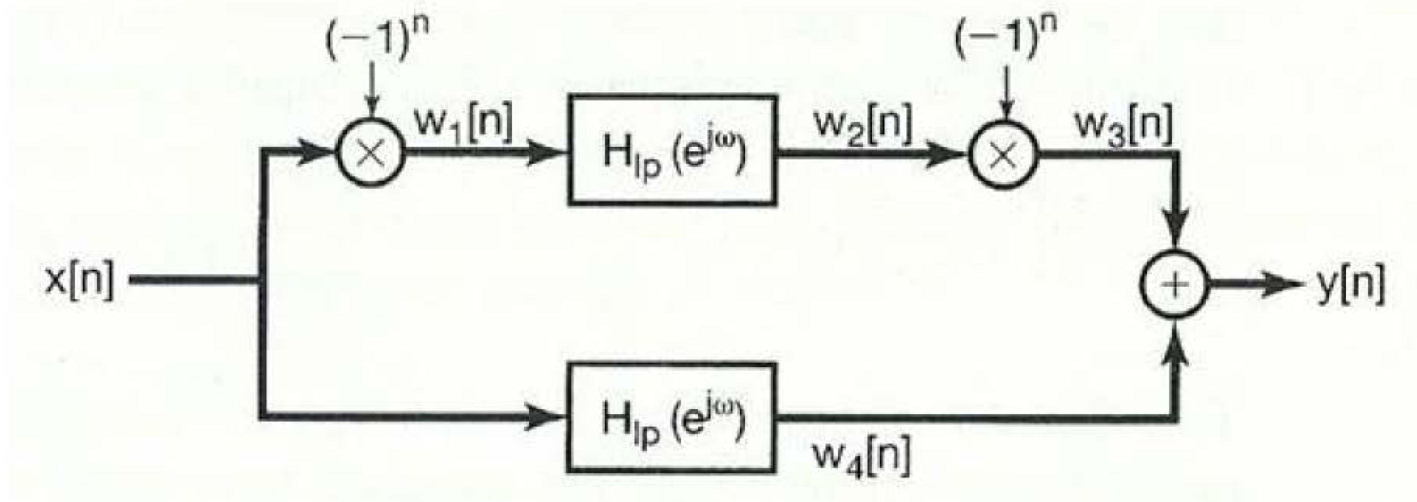
Example: Band-Stop Filter (1/2)

- Given low-pass filter $H_{lp}(e^{j\omega})$, how to generate band-stop effect from low-pass filter?



Example: Band-Stop Filter (2/2)

- Two branches: low-pass + high-pass
- See textbook, Example 5.14



- $(-1)^n = e^{j\pi n} \Rightarrow W_1(e^{j\omega}) = X(e^{j(\omega-\pi)})$

$$\Rightarrow W_2(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j(\omega-\pi)})$$

$$\Rightarrow W_3(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)})X(e^{j(\omega-2\pi)}) = H_{lp}(e^{j(\omega-\pi)})X(e^{j\omega})$$

Multiplication Property

Multiplication Property

Let $y[n] = x_1[n]x_2[n]$, *then*

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta,$$

which is periodic convolution of $X_1(e^{j\omega})$ *and* $X_2(e^{j\omega})$

- Comparison: Multiplication of periodic signals

$$x_1[n] \longleftrightarrow a_k \quad \text{and} \quad x_2[n] \longleftrightarrow b_k$$

$$\Rightarrow x_1[n]x_2[n] \longleftrightarrow \sum_{k=<N>} a_k b_{n-k} \quad \text{discrete-time periodic convolution}$$

Example #8: LTI Systems Described by LCCDE's

(Linear-constant-coefficient differential equations)

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Using the Differentiation Property

$$\frac{d^k x(t)}{dt^k} \longleftrightarrow (j\omega)^k X(j\omega)$$

$\Downarrow$ Transform both sides of the equation

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

$$\Downarrow$$

$$Y(j\omega) = \underbrace{\left[\frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}\right]}_{H(j\omega)} X(j\omega)$$

$$\boxed{H(j\omega) = \left[\frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}\right]}$$

Linear constant-coefficient difference equations (LCCDE)

$$nx[n] \overset{\mathfrak{F}}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].$$

$$\Downarrow$$

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega})$$

$$\Downarrow$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}.$$

Example 5.18

Consider the causal LTI system that is characterized by the difference equation with $|a| < 1$

$$y[n] - ay[n-1] = x[n],$$

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}.$$

$$h[n] = a^n u[n].$$

Example 5.19

Consider a causal LTI system that is characterized by the difference equation

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n].$$

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}}.$$

$$H(e^{j\omega}) = \frac{2}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})} = \frac{4}{1 - \frac{1}{2}e^{-j\omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\omega}}$$

$$h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]$$