Notes

Assignments

4.50 & 4.51 attached

Tutorial problems

7.41, 7.44, 7.47, 7.49

Outline

- Sampling is a general procedure to generate DT signals from CT signals, where information of the original signals can be kept
- Core sampling theory:
  - Impulse train, zero-order hold, 1st-order hold, etc.
  - Analysis in frequency domain
  - Nyquist rate
- Undersampling: Aliasing
- Application: process continuous-time signals discretely
- More sampling techniques: decimation, downsampling and upsampling

Sampling theorem

Sampling Theorem

Let $x(t)$ be a band-limited signal with

$$X(j\omega) = 0 \textit{ for } |\omega| > \omega_M.$$

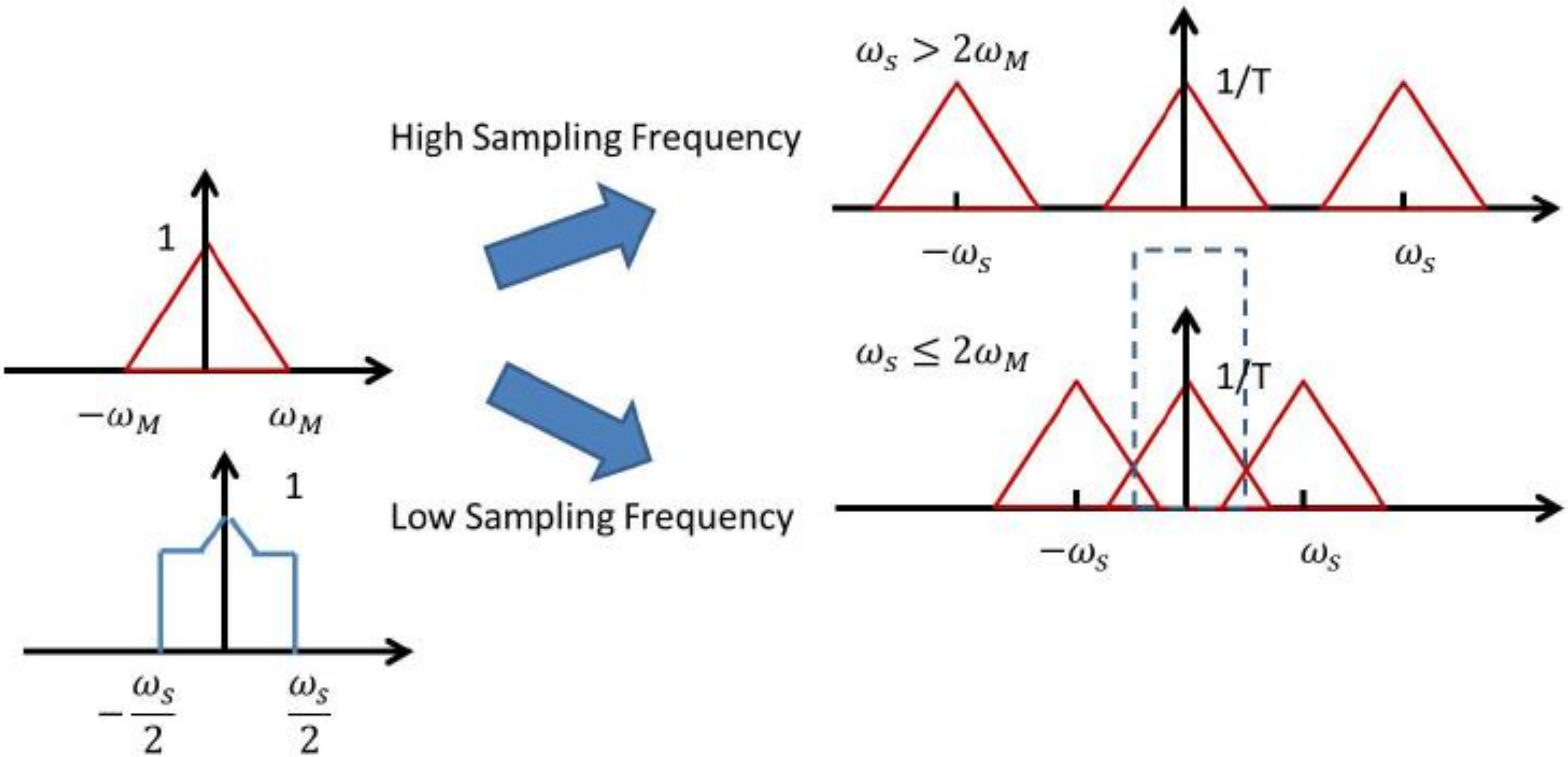
Then, $x(t)$ is uniquely determined by its samples $x(nT)$ or $x_p(t)$ if

$$\omega_s = \frac{2\pi}{T} > 2\omega_M,$$

where $2\omega_M$ is referred to as the Nyquist rate.

Undersampling & Aliasing

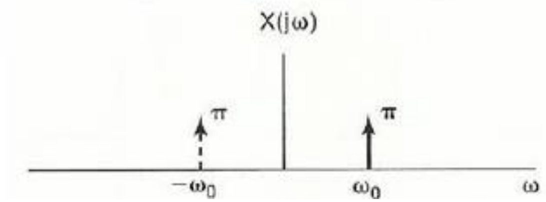
- Undersampling: insufficient sampling frequency $\omega_s < 2\omega_M$
- Perfect reconstruction is impossible with undersampling.
- Aliasing: distortion due to undersampling



Aliasing: Example

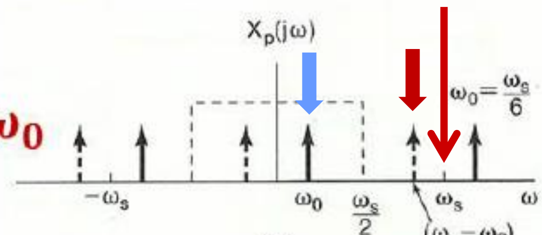
Signal before sampling: $\cos \omega_0 t$

Sampling rate: ω_s



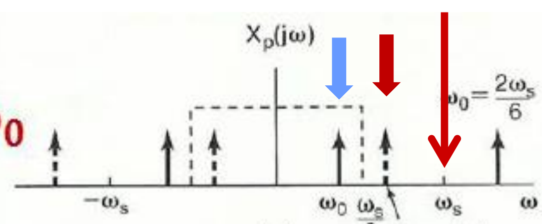
Lowpass Filter: $\frac{-\omega_s}{2} \sim \frac{\omega_s}{2}$

$\omega_s = 6\omega_0$



(b)

$\omega_s = 3\omega_0$



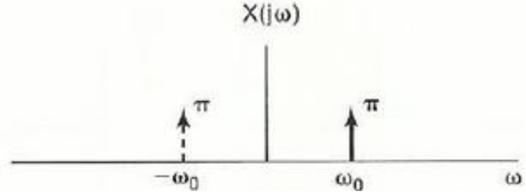
(c)

$\cos \omega_0 t$

Aliasing: Example

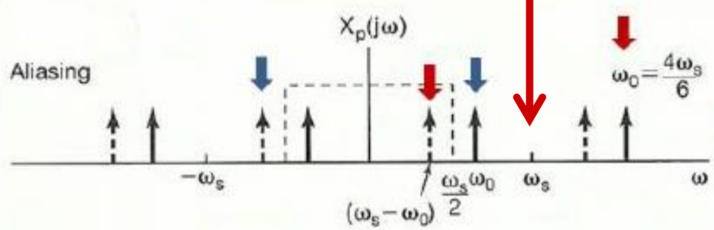
Signal before sampling: $\cos \omega_0 t$

Sampling rate: ω_s



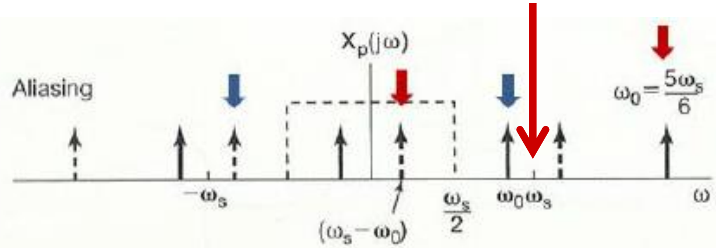
Lowpass Filter: $\frac{-\omega_s}{2} \sim \frac{\omega_s}{2}$

Undersampling



(d)

$$\omega_s = \frac{3}{2}\omega_0$$



(e)

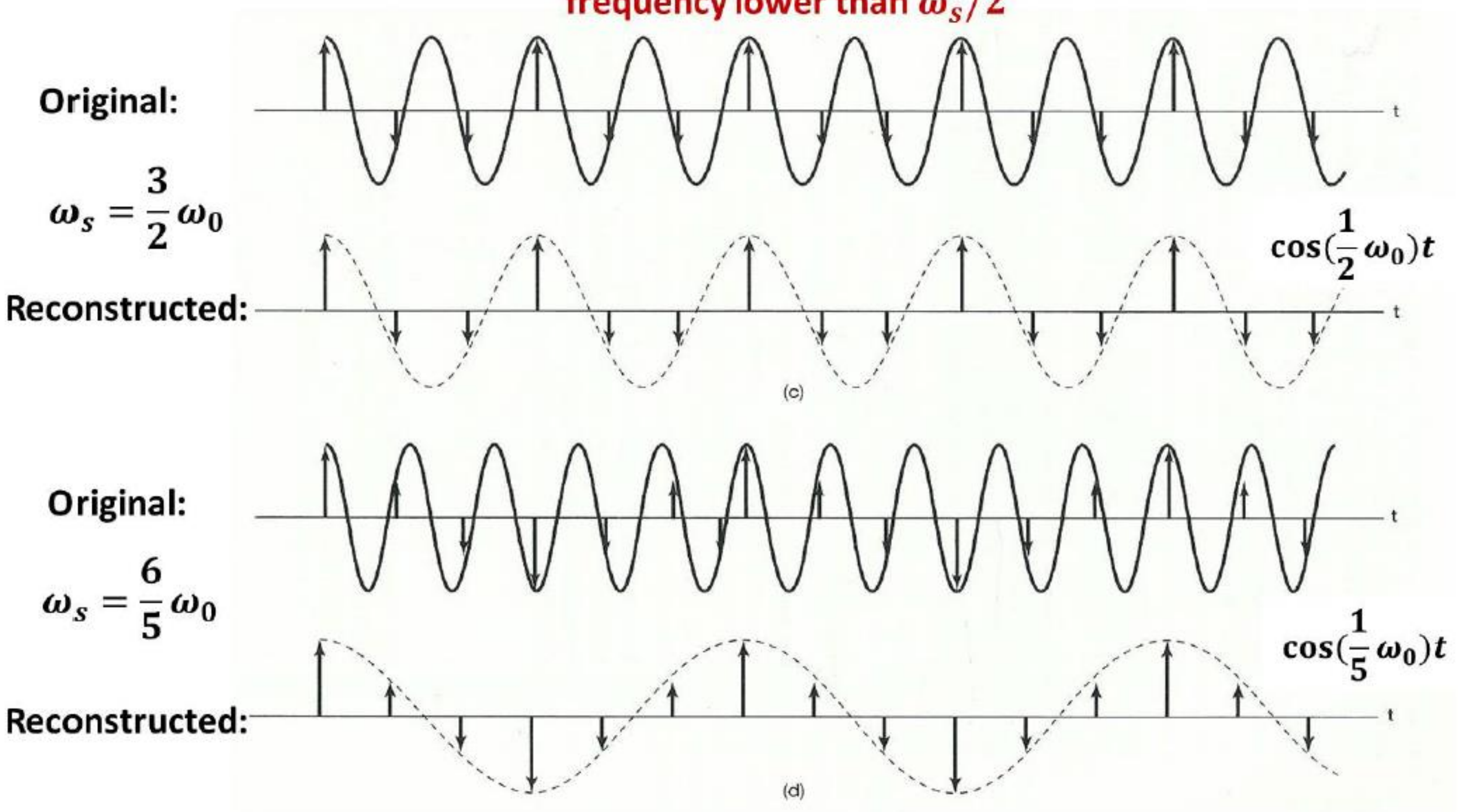
$$\omega_s = \frac{6}{5}\omega_0$$

Aliasing: $\cos(\omega_s - \omega_0)t$

Under fixed w_s, when w_0 is increasing, what is the frequency of recovered wave?

Aliasing: Example

Low-pass filtering: Interpret the samples by cosine function with frequency lower than $\omega_s/2$

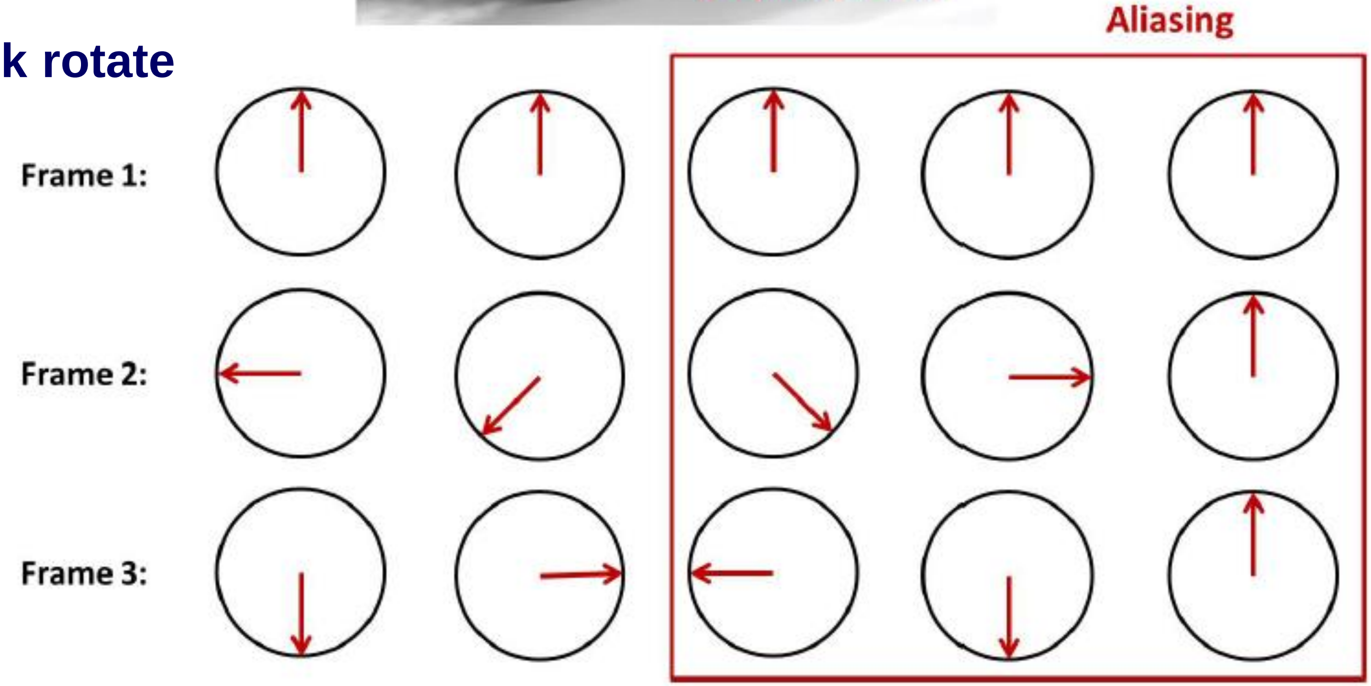


Aliasing: $\cos(\omega_s - \omega_0)t$

Aliasing in Movies

- Wheel's rotation in movies

Anti-clock rotate

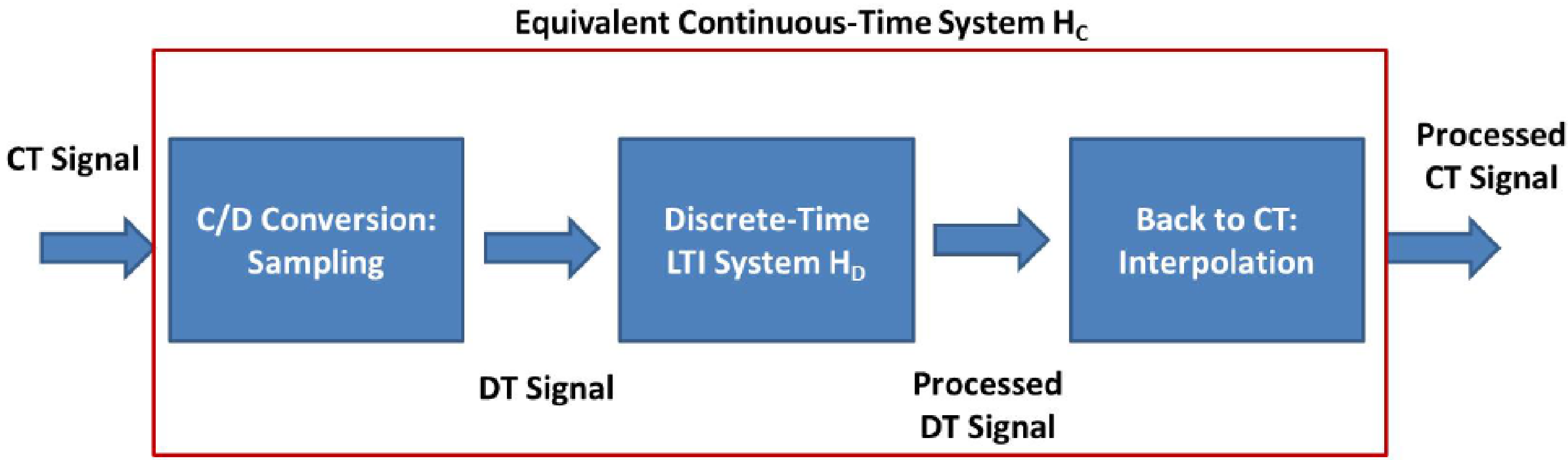


90^0 / frame 135^0 225^0 270^0 360^0

Process Continuous-Time Signals Discretely

- People would like to process continuous-time signal in discrete-time (digital) domain

Block Diagram



- It is much easier to design DT system.
- What's the relation between H_C and H_D?

or

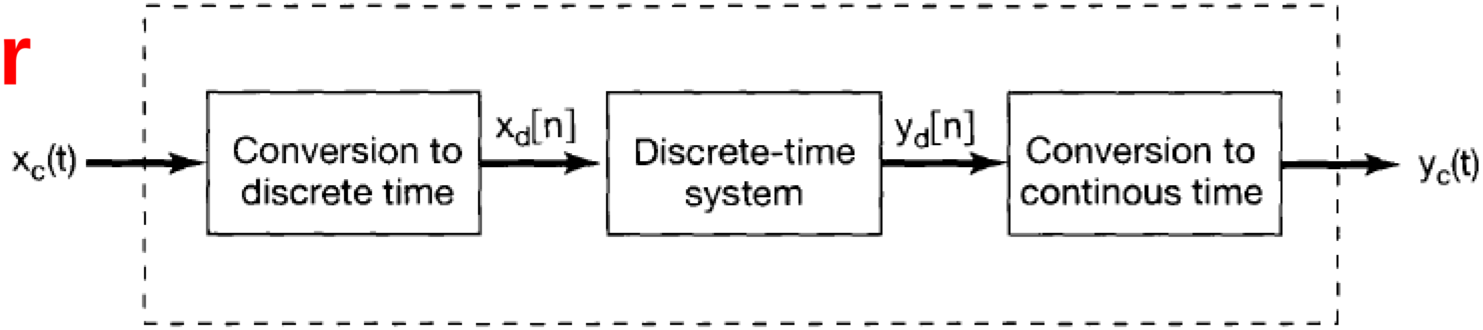


Figure 7.19 Discrete-time processing of continuous-time signals.

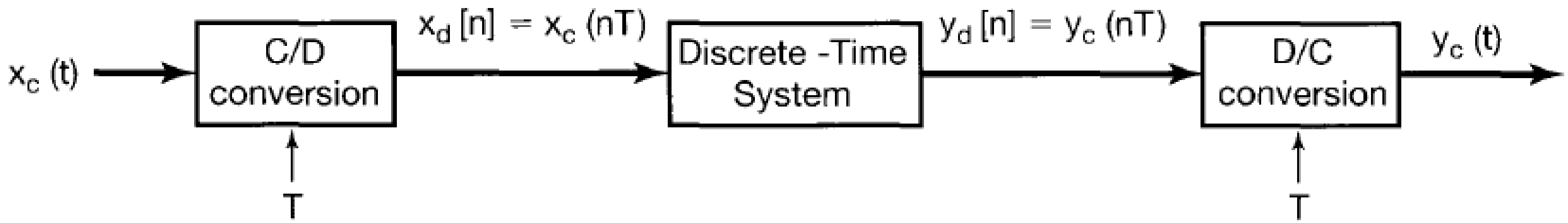
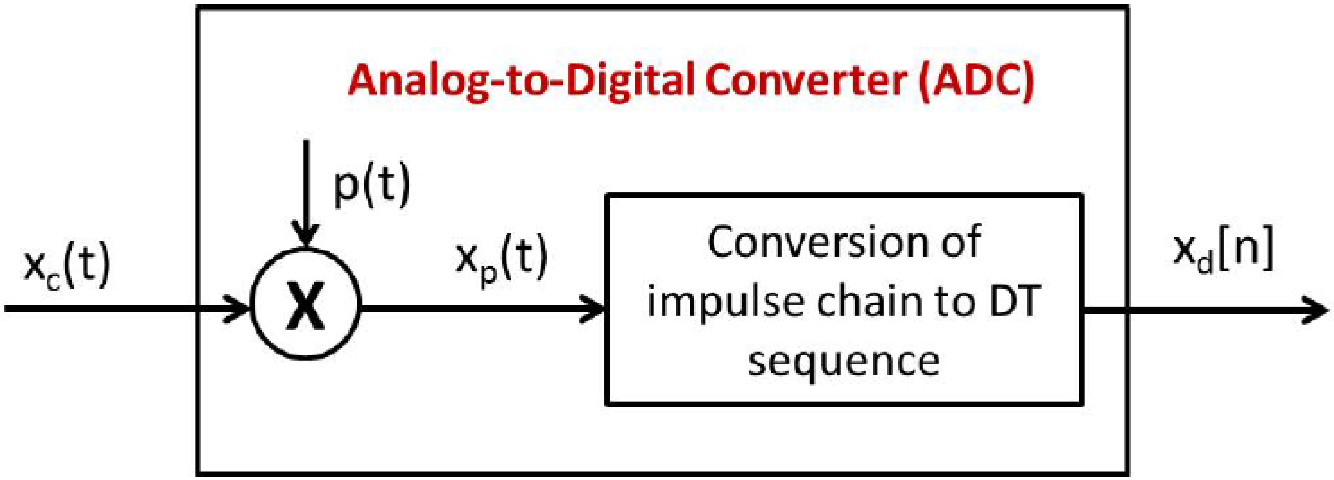


Figure 7.20 Notation for continuous-to-discrete-time conversion and discrete-to-continuous-time conversion. T represents the sampling period.

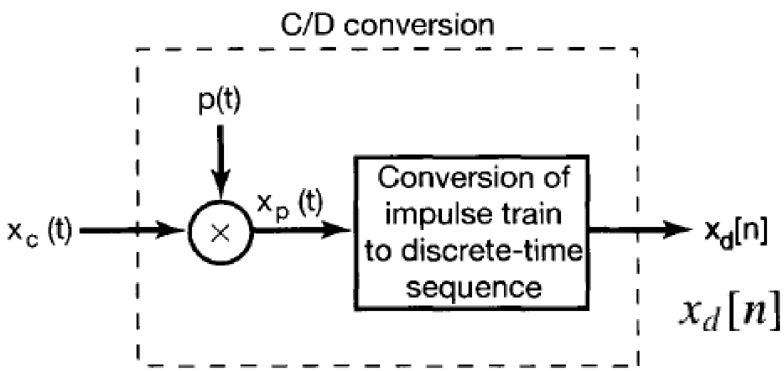
Discretization: C/D Conversion



- Mathematical Interpretation (Fourier Transform)

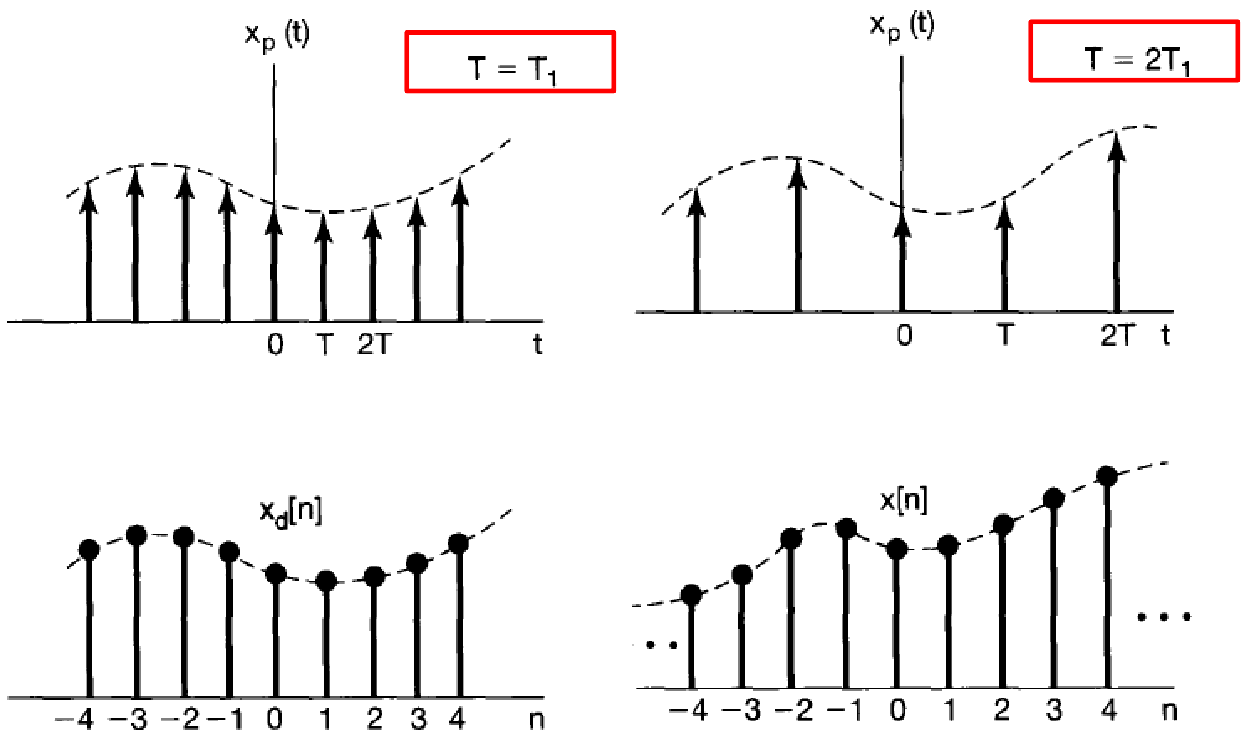
$$x_c(t) \quad \longleftrightarrow \quad X_c(j\omega)$$

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x_c(nT)\delta(t - nT) \quad \longleftrightarrow \quad X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$



$$x_d[n] = x_c(nT)$$

(a)



$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x_d[n]e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{+\infty} x_c(nT)e^{-j\Omega n}$$

$$X_p(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT)e^{-j\omega nT}$$

$$X_d(e^{j\Omega}) = X_p(j\Omega/T)$$

$$X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$X_d(e^{j\Omega}) = X_p(j\Omega/T)$$

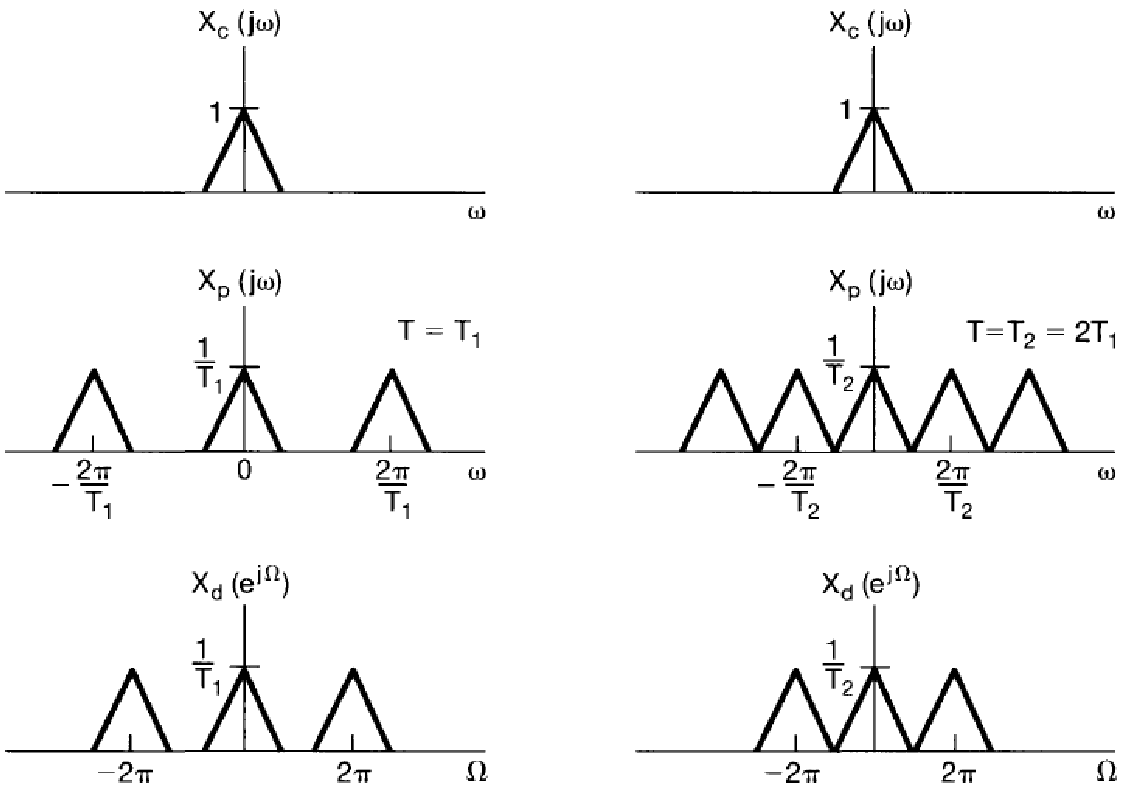


Figure 7.22 Relationship between $X_c(j\omega)$, $X_p(j\omega)$, and $X_d(e^{j\Omega})$ for two different sampling rates.

1. pulse-train sampling

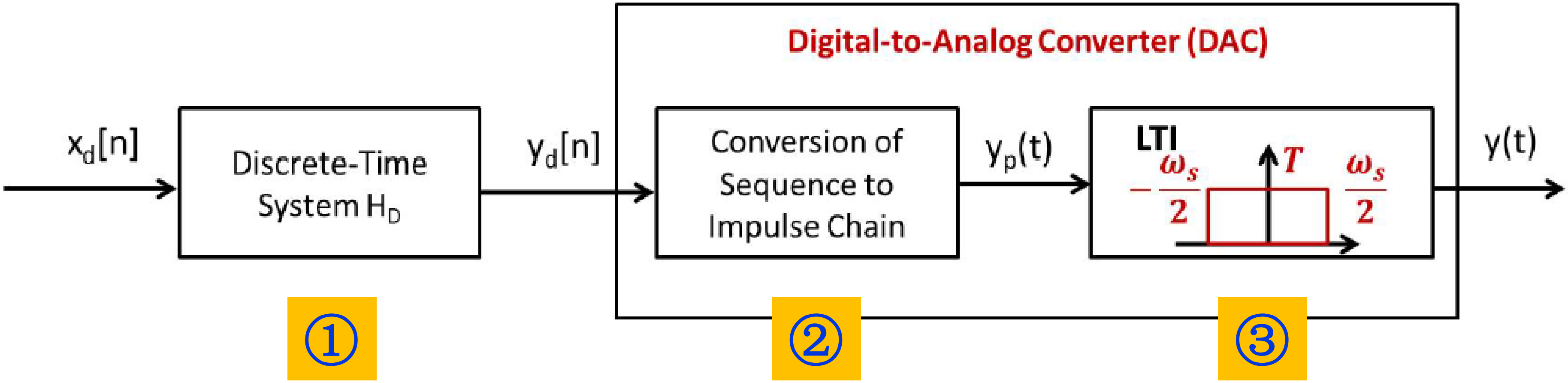
2. scaling or time-normalization

$$X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$X_d(e^{j\omega}) = X_p(j\omega / T)$$

$$\begin{aligned} X_d(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\frac{\omega}{T} - k\omega_s)) \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\frac{\omega - 2\pi k}{T})) \end{aligned}$$

DT Processing and Conversion



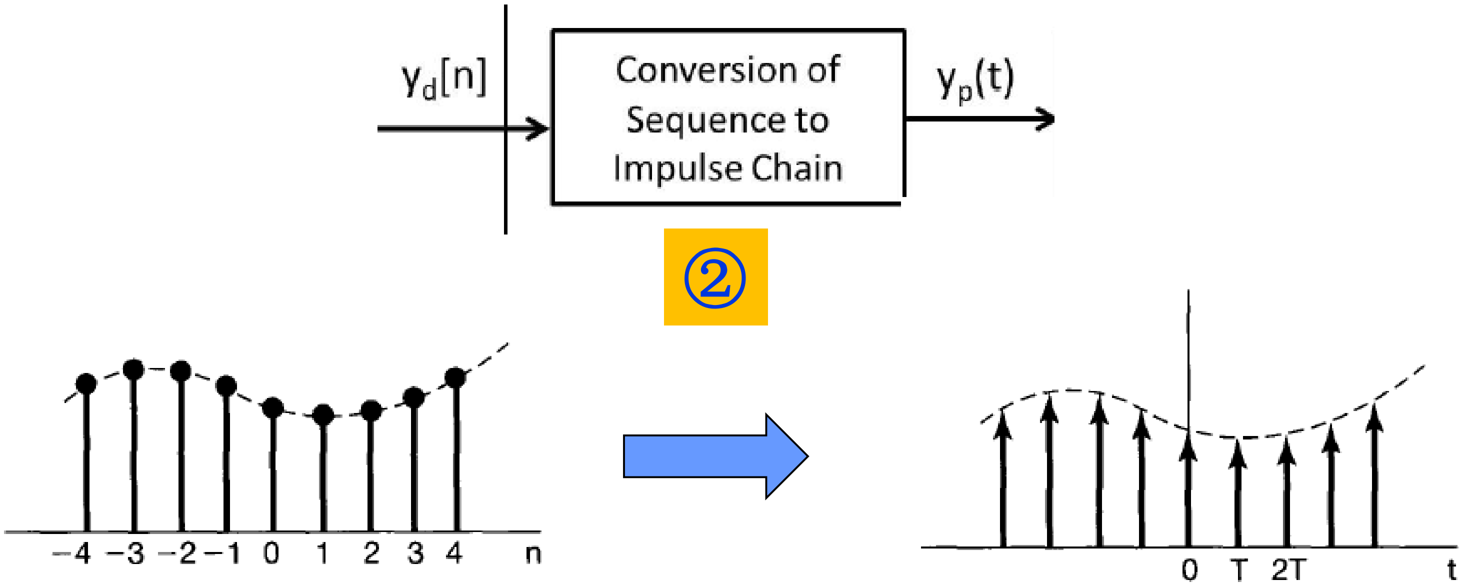
- Mathematical Interpretation (Fourier Transform)

① $$y_d[n] = x_d[n] * h_D[n] \quad \longleftrightarrow \quad Y_d(e^{j\omega}) = X_d(e^{j\omega})H_D(e^{j\omega})$$

② $$y_p(t) = \sum_{n=-\infty}^{\infty} y_d[n]\delta(t - nT) \quad \longleftrightarrow \quad Y_p(j\omega) = Y_d(e^{j\omega T})$$

③ $$y(t) = y_p(t) * h_{LP}(t) \quad \longleftrightarrow \quad Y(j\omega) = Y_p(j\omega)H_{LP}(j\omega)$$

$y_d[n] = y_c(nT)$

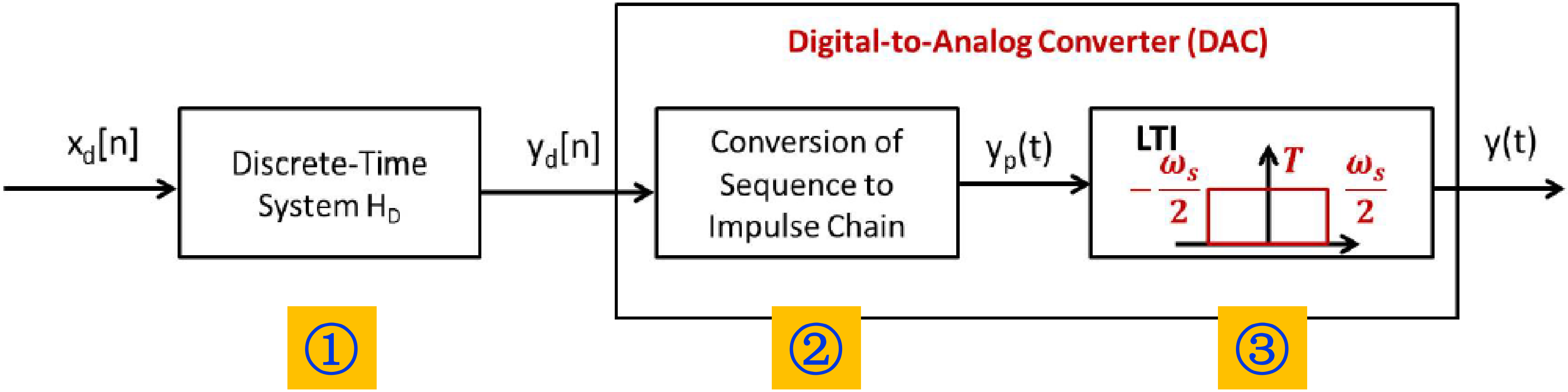


1. **n → t**
2. **time-scaling**

$$y_p(t) = \sum_{n=-\infty}^{\infty} y_d[n]\delta(t - nT) \quad \longleftrightarrow \quad Y_p(j\omega) = Y_d(e^{j\omega T})$$

Prove this !

DT Processing and Conversion



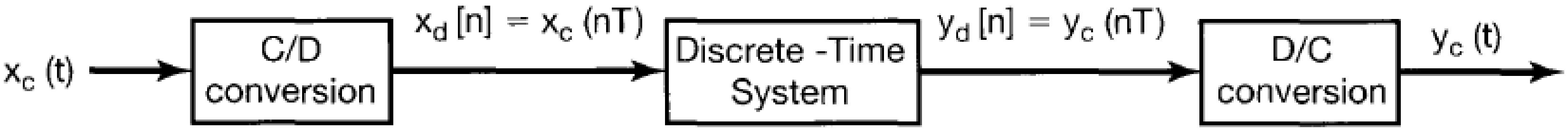
- Mathematical Interpretation (Fourier Transform)

① $$y_d[n] = x_d[n] * h_D[n] \quad \longleftrightarrow \quad Y_d(e^{j\omega}) = X_d(e^{j\omega})H_D(e^{j\omega})$$

② $$y_p(t) = \sum_{n=-\infty}^{\infty} y_d[n]\delta(t - nT) \quad \longleftrightarrow \quad Y_p(j\omega) = Y_d(e^{j\omega T})$$

③ $$y(t) = y_p(t) * h_{LP}(t) \quad \longleftrightarrow \quad Y(j\omega) = Y_p(j\omega)H_{LP}(j\omega)$$

$$Y(j\omega) = X_d(e^{j\omega T})H_D(e^{j\omega T})H_{LP}(j\omega)$$



$$Y(j\omega) = X_d(e^{j\omega T})H_D(e^{j\omega T})H_{LP}(j\omega)$$

$$X_d(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\frac{\omega}{T} - k\omega_s))$$

$$\begin{aligned}
Y(j\omega) &= \left[\frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))\right] H_D(e^{j\omega T})H_{LP}(j\omega) \\
&= X_c(j\omega)H_D(e^{j\omega T}) \\
&= X_c(j\omega)\widetilde{H}_D(e^{j\omega T}) \qquad (1)
\end{aligned}$$

where

$$\widetilde{H}_D(e^{j\omega T}) = \begin{cases} H_D(e^{j\omega T}) & |\omega| < \omega_s/2 \\ 0 & otherwise \end{cases} \qquad (2)$$

- It is equivalent to a continuous-time LTI system $H_C(j\omega) = \widetilde{H}_D(e^{j\omega T})$
- $H_D(e^{j\omega T})$ is a periodic extension of $\widetilde{H}_D(e^{j\omega T})$ with period $\omega_s = 2\pi/T$

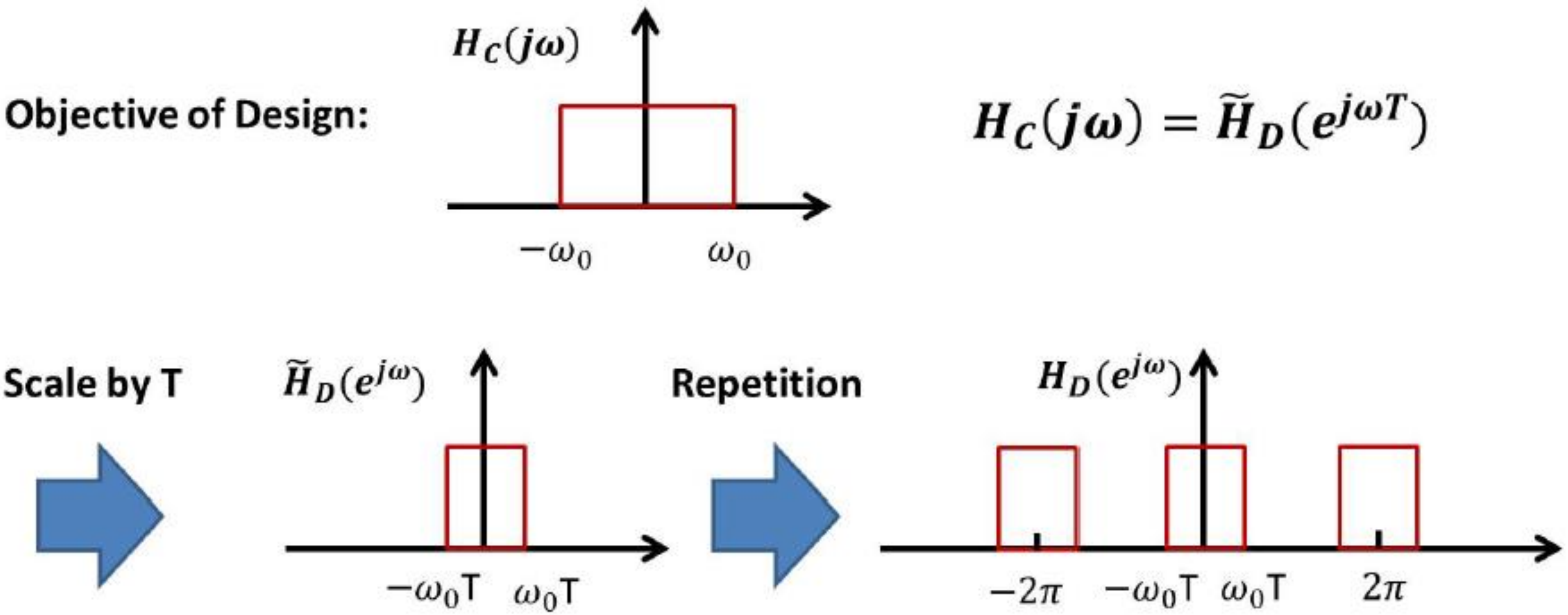
System Design

$$H_C(j\omega) = \widetilde{H}_D(e^{j\omega T})$$

- How can we design a CT LTI system with frequency response H_C via DT LTI system?
- Step 1: Sampling frequency ω_s or $2\pi/T$ should be larger than Nyquist rate
- Step 2: $\widetilde{H}_D(e^{j\omega T}) = H_C(j\omega)$
- Step 3: Frequency response of DT LTI system
  $H_D(e^{j\omega T}) = \sum_{k=-\infty}^{\infty} \widetilde{H}_D(e^{j(\omega - k\omega_s)T})$ or
  $H_D(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \widetilde{H}_D(e^{j(\omega - k\omega_s T)}) = \sum_{k=-\infty}^{\infty} H_C(j\frac{\omega - 2k\pi}{T})$

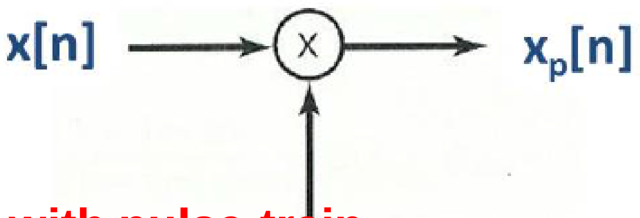
System Design Example

- How to implement an ideal CT lowpass filter?



Sampling on Discrete-Time Signals

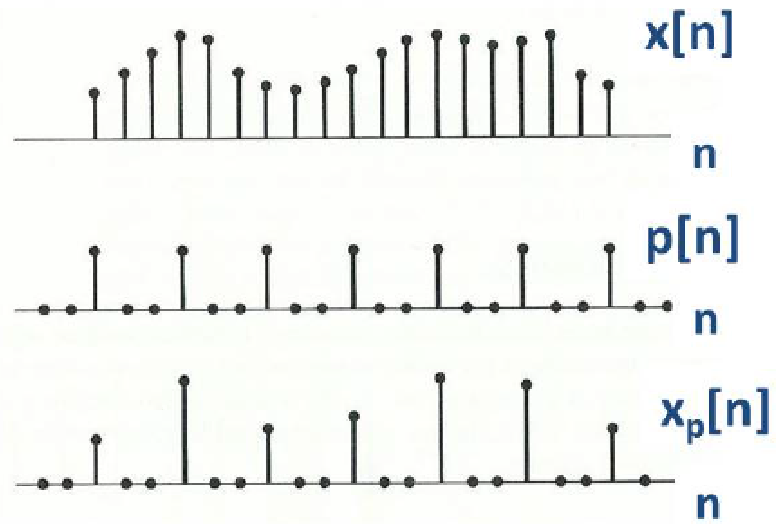
- In digital signal processing, we may sample the discrete-time signals
- Example: image, audio, video compression
- Its mathematical model is give below:



Different with pulse-train sampling

$$p[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN]$$

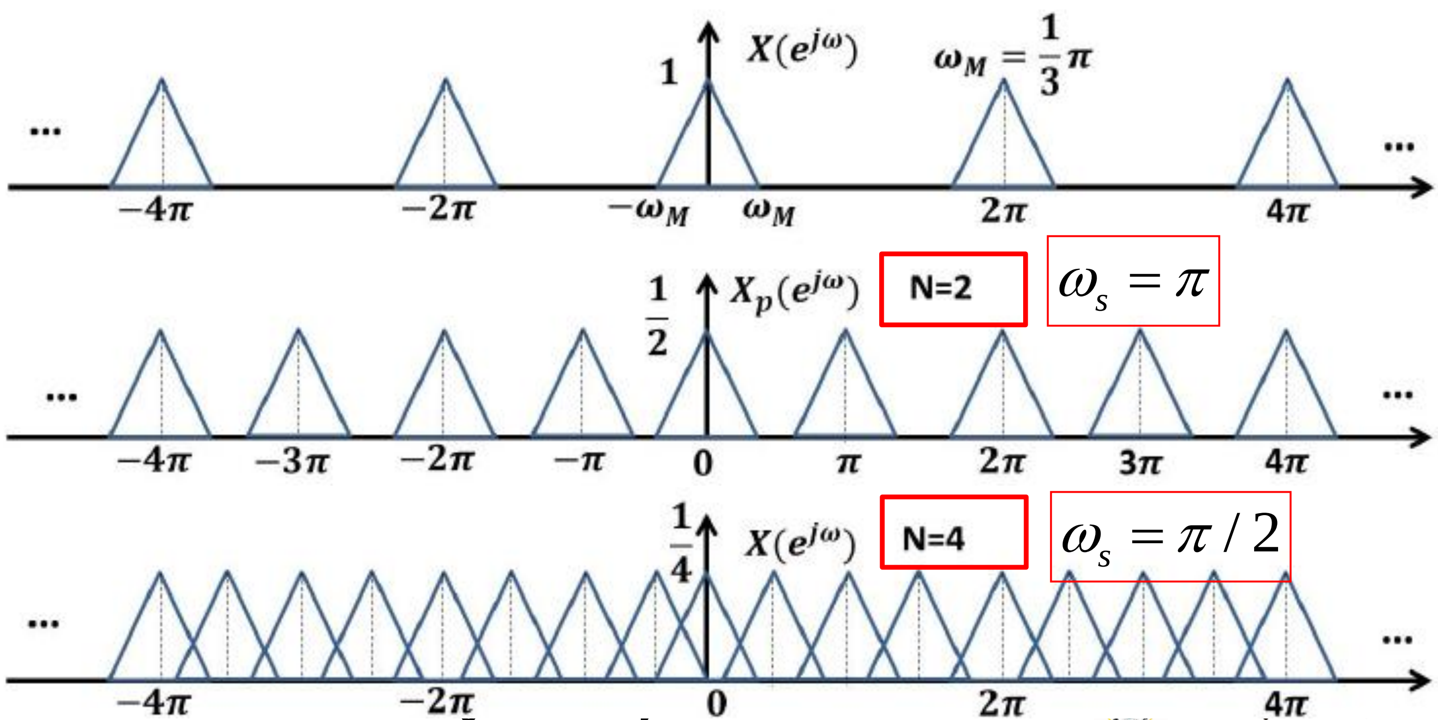
$$x_p[n] = \sum_{k=-\infty}^{+\infty} x[kN]\delta[n - kN]$$



Frequency Analysis

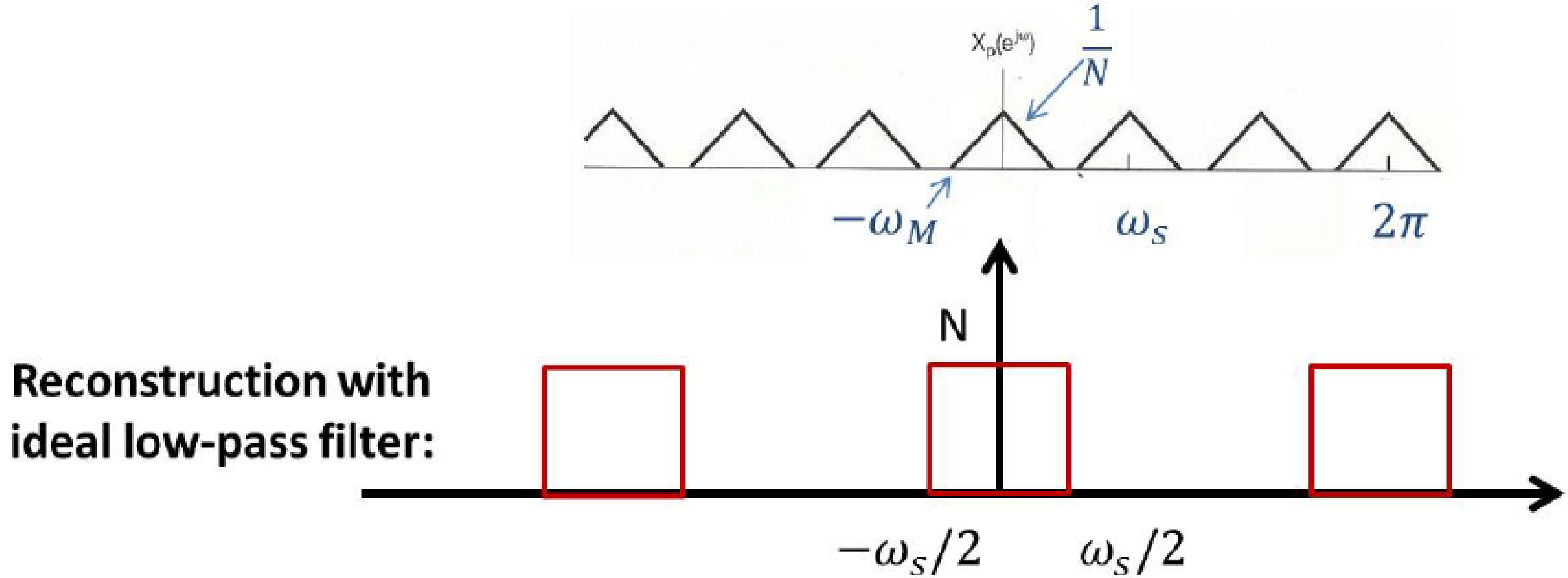
$$P(e^{j\omega}) \;=\; \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_s) \quad where \; \boxed{\omega_s = 2\pi/N}$$

$$X_p(e^{j\omega}) \;=\; \frac{1}{2\pi}\int_{2\pi}P(e^{j\theta})X(e^{j(\omega-\theta)})d\theta = \frac{1}{N}\sum_{k=0}^{N-1}X(e^{j(\omega-k\omega_s)})$$



Reconstruction

- Perfect reconstruction is applicable when $\omega_s > 2\omega_M \leftrightarrow N < \frac{\pi}{\omega_M}$



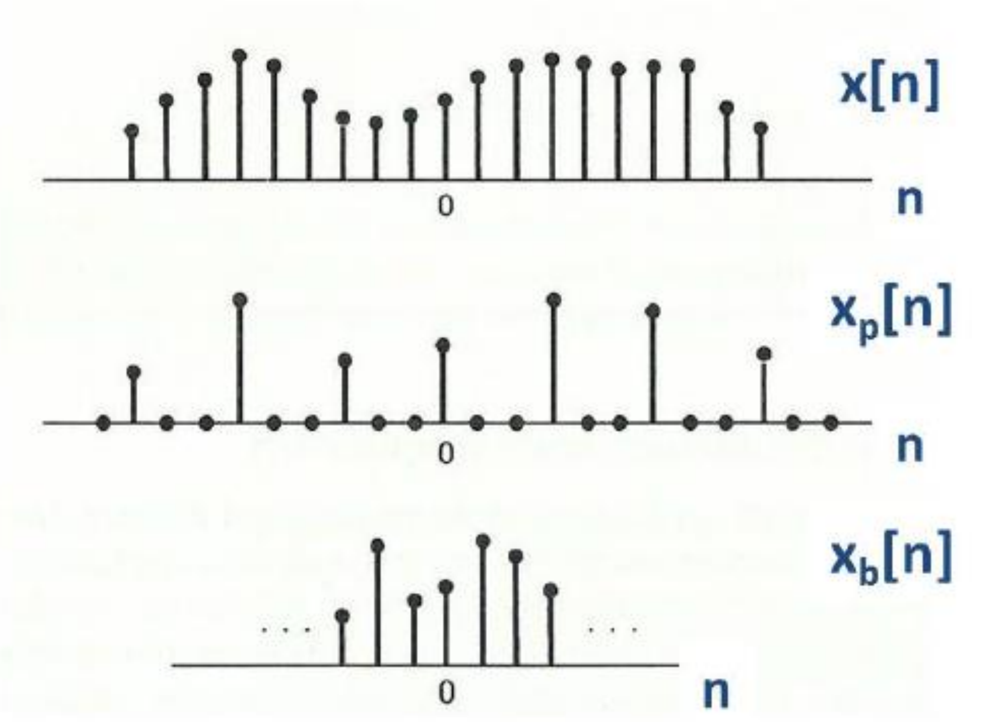
- Aliasing occurs when $\omega_s < 2\omega_M$

Decimation

- After sampling, there will be a great amount of redundancy
- Decimation: discrete-time sampling + remove zeros

Sampling:

Decimation:



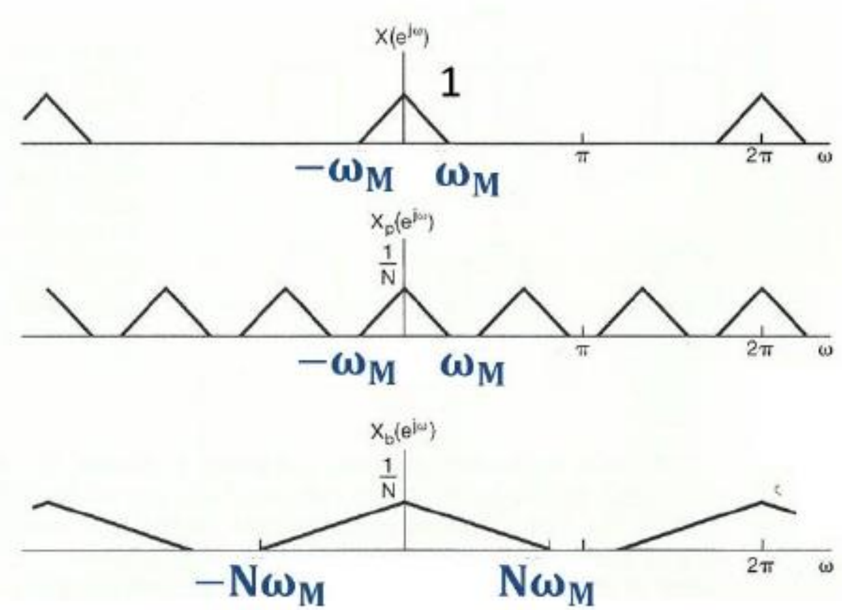
$$x_b[n] = x_p[nN] = x[nN]$$

Frequency Analysis

$$
\begin{aligned}
X_b(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x_b[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x_p[n]e^{-j\omega n/N} = X_p(e^{j\omega/N}) \\
&= \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j\omega/N - k\omega_s})
\end{aligned}
$$

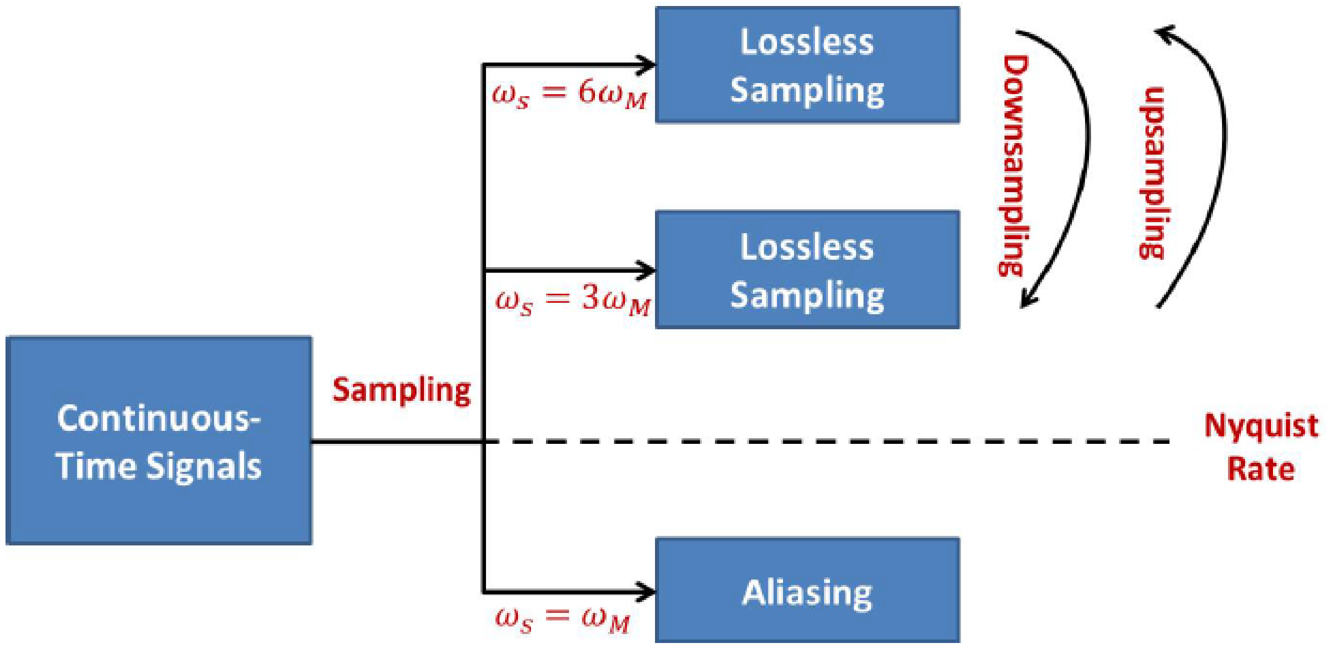
Sampling:

Decimation:



Condition for Perfect Reconstruction: $N\omega_M < \pi$

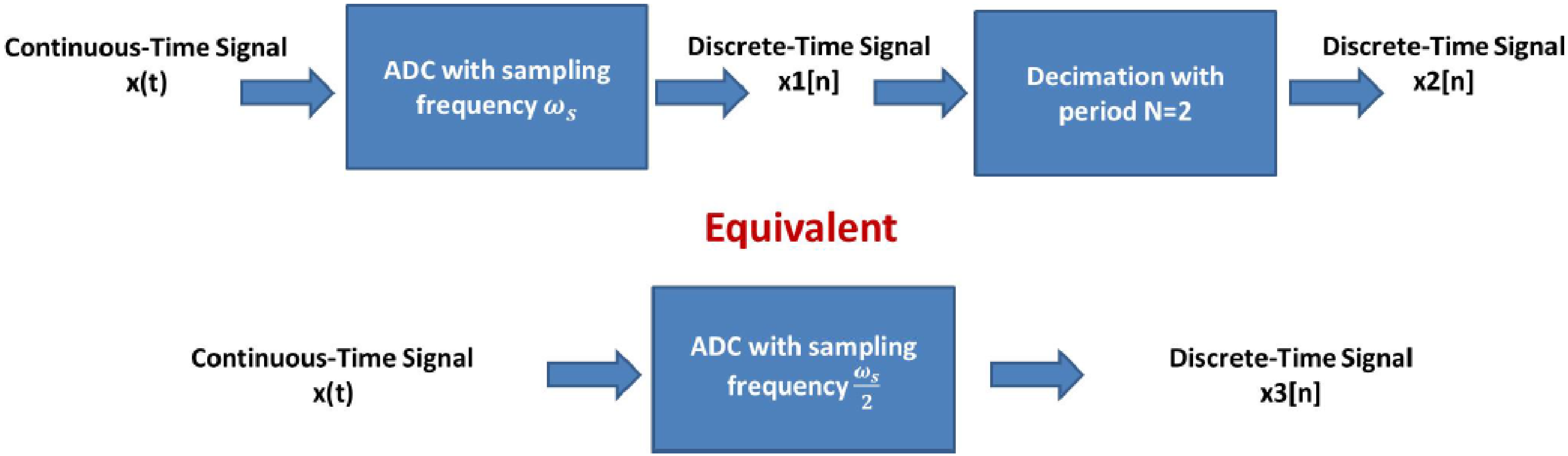
Anything Else?



- Downsampling: to reduce the sampling frequency (decimation)
- Upsampling: to generate a DT signal with higher sampling frequency
- As long as Nyquist rate is satisfied, the transform between low-sampling-frequency version and high-sampling-frequency versions is lossless.

Downsampling

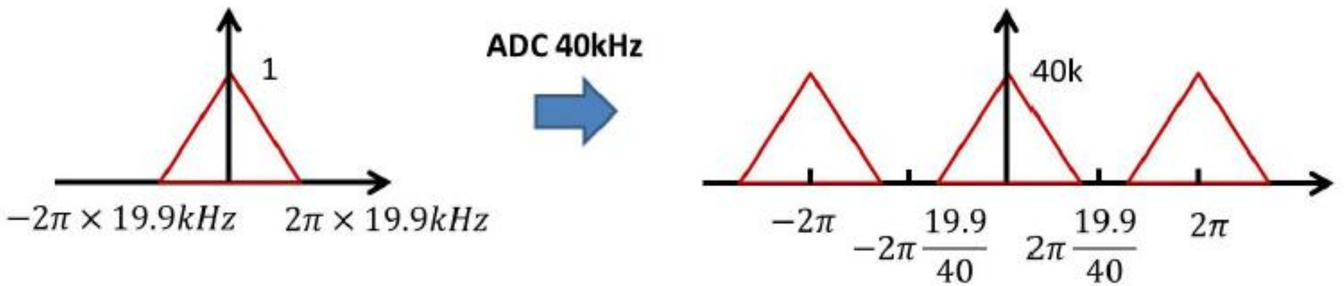
- Downsampling: a general procedure to reduce the sampling frequency



When do we need downsampling?

Downsampling Example (1/2)

- Suppose we have a clip of voice, $x(t)$, with bandwidth $=$19.9kHz
- It can be converted to DT signal with sampling frequency 40kHz, denoted as $x_1[n]$

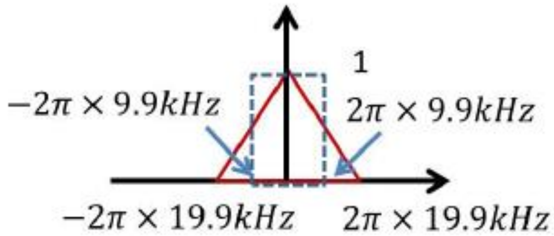


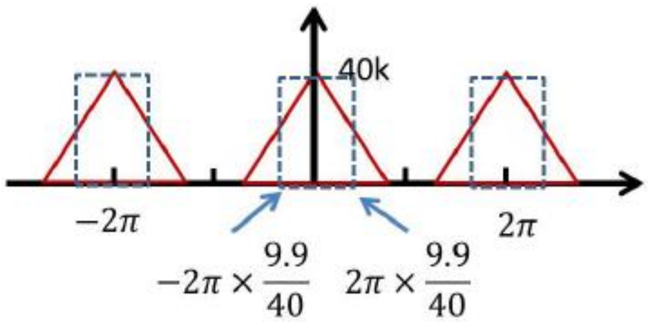
- Based on $x_1[n]$, if we want to save the voice information within 9.9kHz into another DT signal, what can we do?

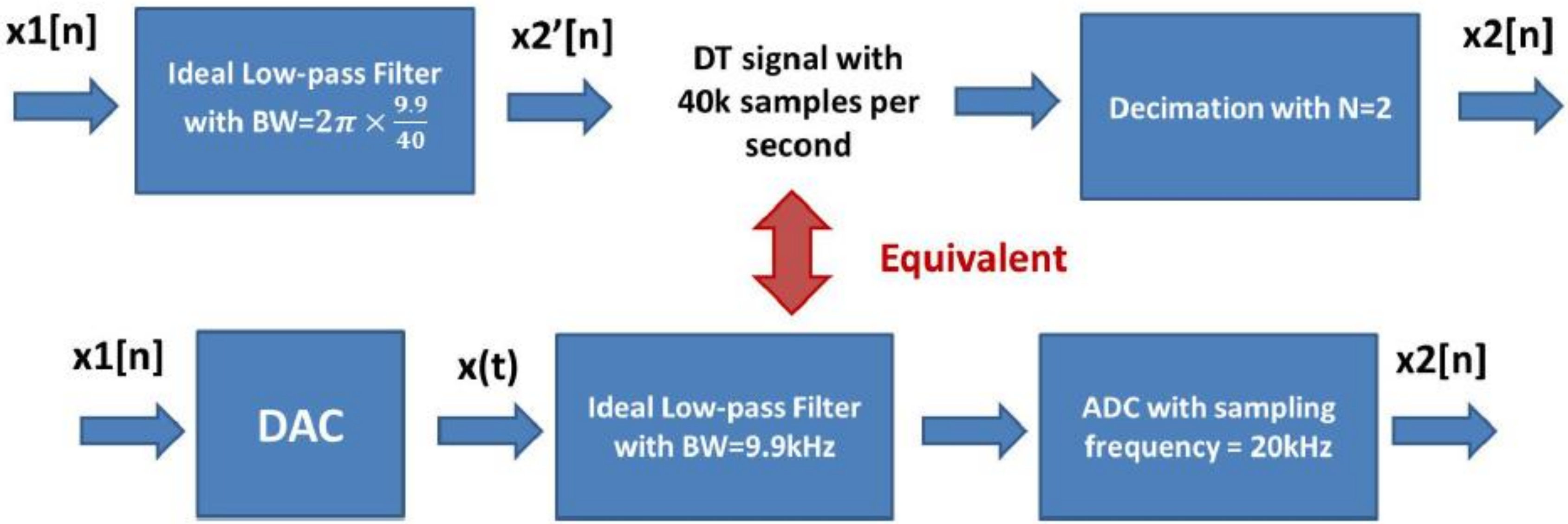
One Choice:

How can we generate x2[n] in discrete-time domain?

Downsampling Example (2/2)

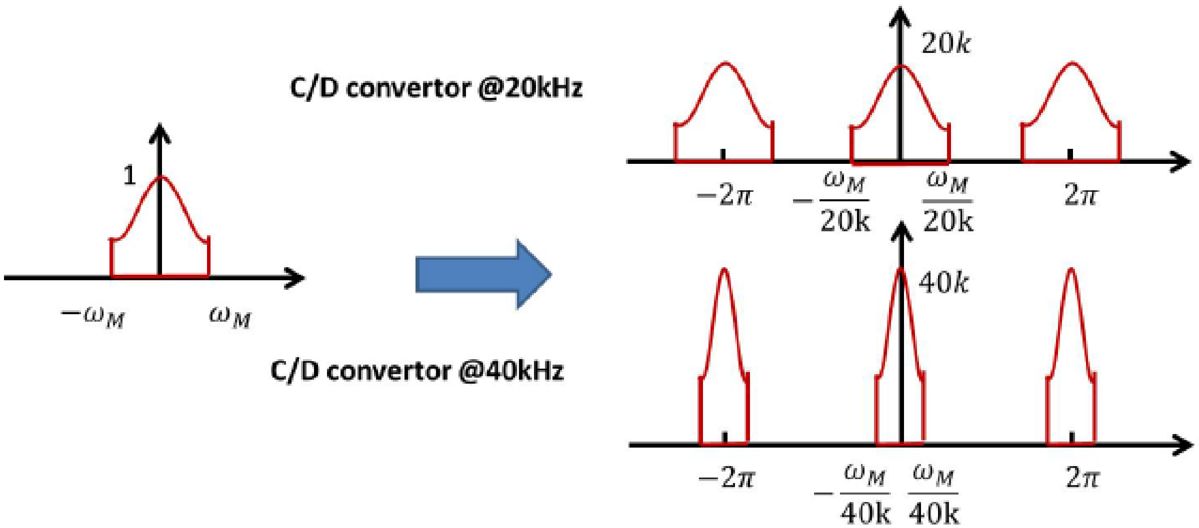






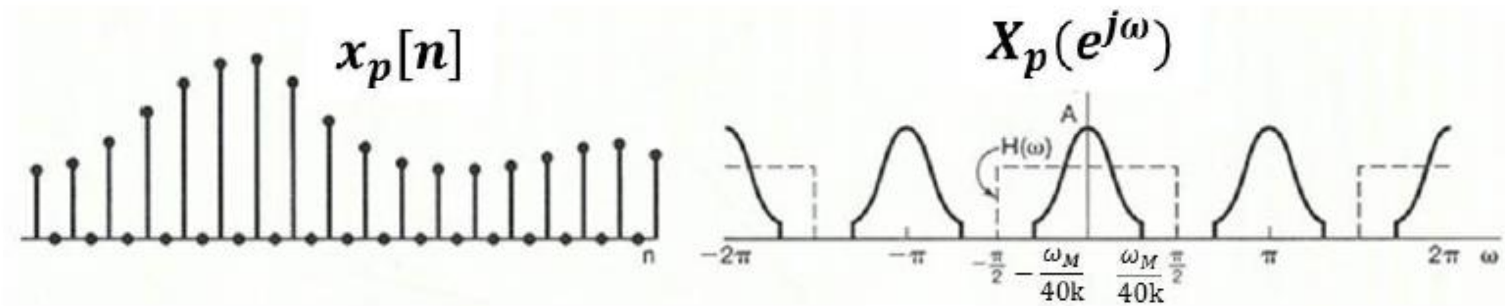
Upsampling

- Upsampling: a procedure to generate a sequence with higher sampling frequency
- Superpose the following two digital sound clips
  - Audio clip 1: Bandwidth= 19.9kHz, sampled at 40kHz
  - Audio clip 2: Bandwidth= 9.9kHz, sampled at 20kHz
- Double the sampling frequency of audio clip 2 (40kHz)
- How to do upsampling in discrete-time domain?



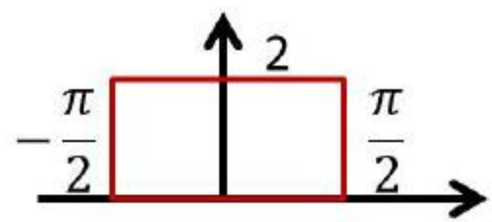
- Time expansion (Insert zeros):

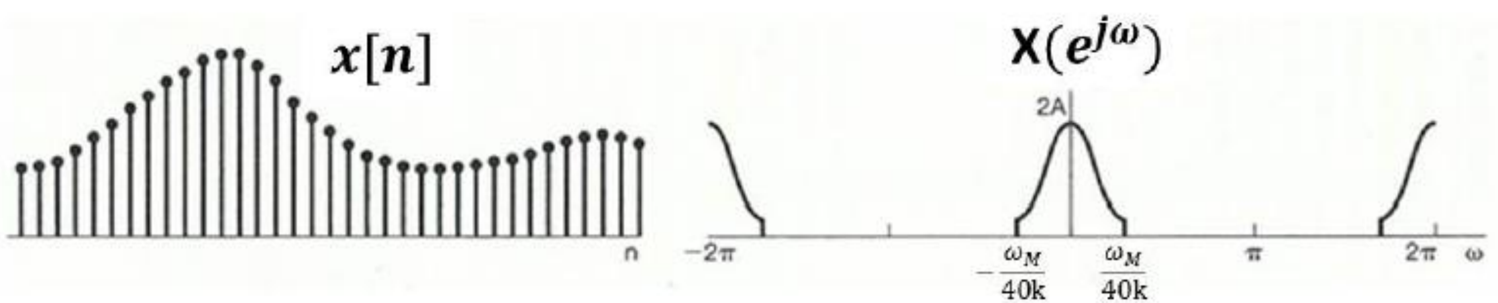
$$x_p[n] = x_{b(k)}[n] \longleftrightarrow X_p(e^{j\omega}) = X_b(e^{jk\omega})$$



- Low-pass filtering:

Ideal Low-Pass Filter:





Summary

- Undersampling
  - Aliasing
- How to process CT signal with DT system?
- Sampling on DT signal
  - Downsampling, and upsampling