Exercises: Spot, Forward and Par Rates

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Problem 1. You invest \$100 for two years at 2% compounded semiannually. How much do you have at the end of the two years?

Solution Using basic compounded rates of return, the return is

$$\begin{aligned} R &= F\Big(1+\frac{\hat{r}}{n}\Big)^{nT} \\ &= \$100\Big(1+\frac{0.02}{2}\Big)^{2\times 2} \\ &= \$104.06 \end{aligned} \tag{1}$$

Problem 2. You invested \$100 for three years and, at the end of those three years, your investment was worth \$107. What was your semiannually compounded rate of return?

Solution This is just the same as the previous question, except we are finding the rate of return instead of the actual return, so we just need to rearrange the equation:

$$\begin{aligned} R &= F\Big(1+\frac{\hat{r}}{n}\Big)^{nT} \\ \implies \$107 &= \$100\Big(1+\frac{\hat{r}}{2}\Big)^{6} \\ \implies \Big(\frac{\$107}{\$100}\Big)^{1/6} &= 1+\frac{\hat{r}}{2} \\ \implies \hat{r} &= 2\Big(\Big(\frac{\$107}{\$100}\Big)^{1/6}-1\Big) \\ \hat{r} &= 2.268\% \end{aligned} \tag{2}$$

Problem 3. Using the discount factors in the table, derive the corresponding spot and forward rates.

Term	**Discount factor**
0.5	0.998752
1	0.996758
1.5	0.993529

Solution We begin by deriving the spot rates. Spot rates are the rates earned on a single cash flow between settlement and time t. This means that any return using spot rates can be discounted to unit dollars. This gives us

$$\Big(1+\frac{\hat{r}(t)}{2}\Big)^{2t} d(t) = 1 \tag{3}$$

where we have used the fact that we are in a semiannual regime, and $d(t)$ is the discount factor between now and t. We can rearrange this to obtain the spot rates as a function of discount factor:

$$\hat{r}(t) = 2\Big(\Big(\frac{1}{d(t)}\Big)^{\frac{1}{2t}} - 1\Big) \tag{4}$$

So we have

$$\begin{aligned}\hat{r}(0.5) &= 2\Big(\Big(\frac{1}{0.998752}\Big)^{\frac{1}{2\times 0.5}} - 1\Big) = 0.250\% \\ \hat{r}(1) &= 2\Big(\Big(\frac{1}{0.996758}\Big)^{\frac{1}{2\times 1}} - 1\Big) = 0.325\% \\ \hat{r}(1.5) &= 2\Big(\Big(\frac{1}{0.993529}\Big)^{\frac{1}{2\times 1.5}} - 1\Big) = 0.433\%\end{aligned} \tag{5}$$

We also know that a spot rate over time t is equivalent to combining a spot rate over time $t - t'$ and a forward rate from t' to t. Since we are in a semiannual regime, this can be expressed as

$$\Big(1+\frac{\hat{r}(t)}{2}\Big)^{2t} = \Big(1+\frac{\hat{r}(t-\frac{1}{2})}{2}\Big)^{2(t-\frac{1}{2})}\Big(1+\frac{f(t)}{2}\Big), \tag{6}$$

where f is the forward rate from time $t - \frac{1}{2}$ to time t. From Equation 3 we can see that this is equivalent to

$$\begin{aligned}\frac{1}{d(t)} &= \frac{1}{d(t-\frac{1}{2})}\Big(1+\frac{f(t)}{2}\Big) \\ \implies f(t) &= 2\Big(\frac{d(t-\frac{1}{2})}{d(t)} - 1\Big)\end{aligned} \tag{7}$$

So we have:

$$
\begin{aligned}
f(0.5) &= 2\Big(\frac{1}{0.998752} - 1\Big) = 0.250\% \\
f(1) &= 2\Big(\frac{0.998752}{0.996758} - 1\Big) = 0.400\% \\
f(1.5) &= 2\Big(\frac{0.996758}{0.993529} - 1\Big) = 0.650\%
\end{aligned} \tag{8}
$$

So in summary, we have:

Term	**Discount factor**	**Spot rate**	**Forward rate**
0.5	0.998752	0.250%	0.250%
1	0.996758	0.325%	0.400%
1.5	0.993529	0.433%	0.650%

Problem 4. Are the forward rates above or below the spot rates in the answer to the previous question? Why is this the case?

Solution The forward rates are above the spot rates (with the first one as an exception, which is equal to the corresponding spot rate).

Considering the nature of the relationship between spot and forward rates, it is logical that a spot rate is the average of the preceding forward rates. For example, we have

$$0.325 = \hat{r}(1) = \frac{f(0.5) + f(1)}{2} = \frac{0.250 + 0.400}{2} = 0.325.$$

Then it follows that, if spot rates are rising across the curve (which they are in our case), forward rates should be above spot rates in order for this relationship to hold.

Problem 5. Using the discount factors from Problem 3, price a 1.5-year bond with a coupon of 0.5%. If over the subsequent 6 months the term structure remains unchanged, will the price of the 0.5% bond increase, decrease, or stay the same?

Solution From par rates, we know that a bond can be priced using only discount factors as

$$B = \frac{c}{2} \sum_{t=1}^{2T} d\Big(\frac{t}{2}\Big) + d(T),$$

where c is the coupon rate, and $d(t)$ is the discount factor between now and time t. This gives us

$$B = \frac{0.005}{2}(0.998752 + 0.996758 + 0.993529) + 0.993529 = 1.001.$$

From the table above, after 6 months we can see that the coupon rate (0.5%) is above the market forward rate (0.4%). Since our receiving coupon is above the market rate, the present value of the bond will increase under a constant term structure.