Exercises: Prices, Discount Factors and Arbitrage

Fixed Income Securities - Tuckman and Serrat

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Problem 1. What are the cash flow dates and the cash flows of \$1000 face value of the US Treasury $2\frac{3}{4}$s of May 31, 2017, issued on May 31, 2010?

Solution Assuming this bond has a semi-annual payment regime, we will receive the following coupon every 6 months:

$$C = \frac{2\frac{3}{4}}{2} \cdot \$1000 = \$13.75.$$

The final coupon payment, on May 31, 2017, will have the following cash flow:

$$F + C = \$1000 + \$13.75 = \$1013.75$$

Problem 2. Use Table 1 of the US Treasury bond prices for settlement on May 15, 2010, to derive the discount factors for cash flows to be received in 6 months, 1 year, and 1.5 years.

Solution We know that the price of a bond is the sum of its discounted cash flows. We also know that from the point of view of the current date (May 15, 2010), there is a different discount factor every 6 months due to the time value of money.

So, for each bond we can express its price as a function of its discounted cash flows. With $d(t)$ representing the discount factor t years after settlement, we have:

$$\begin{aligned} 102.15806 &= \left(100 + \frac{4\frac{1}{2}}{2}\right) d(0.5) \\ 99.60120 &= \frac{0}{2} d(0.5) + \left(100 + \frac{0}{2}\right) d(1) \\ 101.64355 &= \frac{1\frac{3}{4}}{2} d(0.5) + \frac{1\frac{3}{4}}{2} d(1) + \left(100 + \frac{1\frac{3}{4}}{2}\right) d(1.5) \end{aligned} \tag{1}$$

Bond	**Price**
$4\frac{1}{2}s$ of 15/11/2010	102.15806
0s of 15/05/2011	99.60120
$1\frac{3}{4}$s of 15/11/2011	101.64355

Table 1: US Treasury bond prices for settlement on May 15, 2010.

Solving for $d(t)$ gives

$$\begin{aligned} d(0.5) &= 0.9991008 \\ d(1) &= 0.996012 \\ d(1.5) &= 0.990313 \end{aligned} \tag{2}$$

Problem 3. Suppose there existed a Treasury issue with a coupon of 2% maturing on November 15, 2011. Using the discount factors derived from question 2, what would be the price of the 2s of November 15, 2011?

Solution The price of a bond is the sum of its discounted cash flows. We know that each semiannual cash flow is 2%/2 discounted, with the final cash flow being 100 + (2%/2) discounted. So the bond price is

$$\begin{aligned} B &= \frac{2\%}{2}d(0.5) + \frac{2\%}{2}d(1) + \Big(100 + \frac{2\%}{2}\Big)d(1.5) \\ &= \frac{2\%}{2}(0.9991008) + \frac{2\%}{2}(0.996012) + \Big(100 + \frac{2\%}{2}\Big)(0.990313) \\ &= 0.9991008 + 0.996012 + 101(0.990313) \\ B &= 102.0167 \end{aligned} \tag{3}$$

Problem 4. Say that the 2s of November 15, 2011 existed and traded at a price of 101 instead of the price derived from Question 3. How could an arbitrageur profit from this price difference using the bonds in the earlier table? What would the profit be?

Solution So we have a market price of 101, and a present value of 102.0167, calculated in the previous question. In order to capitalise on the arbitrage occurrence, we can construct a replicating portfolio of bonds that replicate the cash flows of the 2s Nov-11. Namely, the bonds used to calculate its present value, which are $4\frac{1}{2}$ Nov-10, 0 May-11, $1\frac{3}{4}$ Nov-11.

We can calculate the face values of each bond needed to do this using

$$F = C^{-1}\vec{c},$$

where F is a vector of the face values of each bond in the portfolio, C is a matrix of the cash flows of each of the bonds, and c is a vector of the cash flow of the bond we are replicating:

$$\begin{aligned} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} &= \begin{bmatrix} 100 + \frac{4\frac{1}{2}}{2} & \frac{0}{2} & \frac{1\frac{3}{4}}{2} \\ 0 & 100 + \frac{0}{2} & \frac{1\frac{3}{4}}{2} \\ 0 & 0 & 100 + \frac{1\frac{3}{4}}{2} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 100 + 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.001212 \\ 0.001239 \\ 1.001239 \end{bmatrix} \end{aligned} \tag{4}$$

Bond	Price
0s of 15/05/2020	69.21
$3\frac{1}{2}$s of 15/05/2020	?
$8\frac{3}{4}$s of 15/05/2020	145.67

Table 2: Caption

The profit will simply be the difference between the present value of the bond and the market price, so

$$\text{Profit} = \$102.0167 - \$101 = \$1.0167.$$

Problem 5. Given the prices of the two bonds in Table 2 as of May 15, 2010, find the price of the third by an arbitrage argument. Since the $3\frac{1}{2}$s of 15/05/2020 is the on-the-run 10-year, why might this arbitrage price not obtain in the market?

Solution We know that the price of a replicating portfolio based on arbitrage arguments is equivalent to the price of a bond through discounting cash flows (the present value). Since the 3 bonds in the table have the same maturity date, this means that the price of the $3\frac{1}{2}$ will lie between 69.21 and 145.67 in the same way in which $3\frac{1}{2}$ lies between 0 and $8\frac{3}{4}$.

It can be seen that $3\frac{1}{2} = 0.4 \times 8\frac{3}{4}$, so the price is

$$B = (0.4 \times 145.67) + (0.6 \times 69.21) = 99.794.$$

This arbitrage price would probably not obtain itself in the market because of the 'law of no arbitrage', meaning that a few arbitrageurs would very quickly capitalise on the price, meaning that natural (non-arbitrage) prices would form.