Exercises: One-Factor Risk Metrics and Hedges

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Problem 1. The following tables give the prices of TYU0 and of TYU0C 120 as of May 2010 for a narrow range of the 7-year par rate. Please fill in the other columns, ignoring cells marked 'X'. Over the given range, which security's price-rate function is concave and which is convex? How can you tell?

Rate	**Price**	**DV01**	**Duration**	**Convexity**	**1st Deriv.**	**2nd Deriv.**
3.320%	115.5712	X	X	X	X	X
3.412%	114.8731			X		X
3.504%	114.1715					
3.596%	113.4668			X		X
3.688%	112.7591	X	X	X	X	X

Table 1: TYU0

Rate	**Price**	**DV01**	**Duration**	**Convexity**	**1st Deriv.**	**2nd Deriv.**
3.320%	.4564	X	X	X	X	X
3.412%	.3483			X		X
3.504%	.2619					
3.596%	.1940			X		X
3.688%	.1415	X	X	X	X	X

Table 2: TYU0C 120

Solution Start with the table for TYU0 (the futures security). The first derivative is an estimate, so is given as

$$\frac{dP}{dy} = \frac{\Delta P}{\Delta y}$$

for the surrounding rates. Similarly,

$$\frac{d^2P}{dy^2} = \frac{\left(\frac{dP}{dy}\right)_2 - \left(\frac{dP}{dy}\right)_1}{\Delta y}.$$

It is useful to begin by computing all of the derivatives, so that they can easily be used to calculate the DV01, duration and convexity.

$$
\begin{aligned}
\left.\frac{dP}{dy}\right|_{3.412\%} &= \frac{114.1715 - 115.5712}{3.504\% - 3.320\%} = -760.7065 \\
\left.\frac{dP}{dy}\right|_{3.504\%} &= \frac{113.4668 - 114.8731}{3.596\% - 3.412\%} = -764.2935 \\
\left.\frac{dP}{dy}\right|_{3.596\%} &= \frac{112.7591 - 114.1715}{3.688\% - 3.504\%} = -767.6087 \\
\left.\frac{d^2P}{y^2}\right|_{3.504\%} &= \frac{-767.6087 + 760.7065}{3.596\% - 3.412\%} = -3,751.1957
\end{aligned} \tag{1}
$$

Now that we have the 1st and 2nd derivatives, it is easy to compute the DV01, duration and convexity. The DV01 is given by

$$DV01 = -\frac{1}{10,000}\frac{\Delta P}{\Delta y}.$$

Calculating it for each one:

$$
\begin{aligned}
DV01(3.412\%) &= -\frac{1}{10,000}(-760.7065) = 0.0761 \\
DV01(3.504\%) &= -\frac{1}{10,000}(-764.2935) = 0.0764 \\
DV01(3.596\%) &= -\frac{1}{10,000}(-767.6087) = 0.0768
\end{aligned} \tag{2}
$$

The duration is given by

$$D = -\frac{1}{P}\frac{\Delta P}{\Delta y}.$$

Calculating it for each rate:

$$
\begin{aligned}
D(3.412\%) &= -\frac{1}{114.8731}(-760.7065) = 6.6221 \\
D(3.504\%) &= -\frac{1}{114.1715}(-764.2935) = 6.6943 \\
D(3.596\%) &= -\frac{1}{113.4668}(-767.6087) = 6.7651
\end{aligned} \tag{3}
$$

The convexity is given by

$$C = \frac{1}{P}\frac{d^2P}{y^2}.$$

Calculating it for the only rate it is appropriate for:

$$C(3.504\%) = \frac{1}{114.1715}(-3,751.1957) = -32.8558.$$

The calculations are identical for the TYU0C 120 security. The final tabulated results for each are below.

The TYU0 has a negative convexity, so is concave. The TYU0C 120 has a positive convexity, so it convex.

Rate	Price	DV01	Duration	Convexity	1st Deriv.	2nd Deriv.
3.320%	115.5712	X	X	X	X	X
3.412%	114.8731	0.0761	6.6221	X	-760.7065	X
3.504%	114.1715	0.0764	6.6943	-32.8558	-764.2935	-3,751.1957
3.596%	113.4668	0.0768	6.7651	X	-767.6087	X
3.688%	112.7591	X	X	X	X	X

Table 3: TYU0

Rate	Price	DV01	Duration	Convexity	1st Deriv.	2nd Deriv.
3.320%	.4564	X	X	X	X	X
3.412%	.3483	0.0106	303.4927	X	-105.7065	X
3.504%	.2619	0.0084	320.1936	83,569.2764	-83.8587	21,886.7935
3.596%	.1940	0.0065	337.2928	X	-65.4348	X
3.688%	.1415	X	X	X	X	X

Table 4: TYU0C 120

Problem 2. Using the data in Question 1, how would a market maker hedge the purchase of \$50m face amount of TYU0C with TYU0 when the 7-year par rate is 3.596%. Check how well this hedge works by computing the change in the value of the position should the rate move instantaneously from 3.596% to 3.668%. What if the rate falls to 3.320%? Is the P&L of the hedged position positive or negative? Why is this the case?

Solution In summary, we have the following relevant values:

7y par rate	TYU0	DV01	TYU0C	DV01
3.504%	114.1715		0.2619	
3.596%	113.4668	0.0768	0.1940	0.0065
3.688%	112.7591		0.1415	

Having gone long the TYU0C, the table shows that money will be lost if rates rise, so we want to hedge this exposure to rising rates. We can hedge using the TYU0 futures such that

$$\Delta P_{\text{TYU0}} + \Delta P_{\text{TYU0C}} = 0.$$

For a face value of the hedge of F, this gives

$$F(DV01_{\text{TYU0}}) - \$50,000,000(DV01_{\text{TYU0C}}) = 0$$

$$\begin{aligned} \implies F &= -\$50,000,000\frac{DV01_{\text{TYU0C}}}{DV01_{\text{TYU0}}} \\ &= -\$50,000,000 \times \frac{0.0065}{0.0768} \\ &= -\$4,231,771 \end{aligned} \tag{4}$$

So in order to hedge the purchase of TYU0C, the market maker should short $4,231,771 face of the TYU0.

Now we want to check the performance of the hedge. The total value of the hedged position is

$$
\begin{aligned}
V_1 &= F_{\text{TYU0C}} \frac{P_{\text{TYU0C}}}{100} + F_{\text{TYU0}} \frac{P_{\text{TYU0}}}{100} \\
&= \$50,000,000 \Big(\frac{0.1940}{100} \Big) - \$4,231,770 \Big(\frac{113.4668}{100} \Big) \\
&= -\$4,704,654
\end{aligned}
\tag{5}
$$

After rates move from 3.596% to 3.688%, this value becomes

$$
\begin{aligned}
V_2 &= \$50,000,000 \Big(\frac{0.1415}{100} \Big) - \$4,231,770 \Big(\frac{112.7591}{100} \Big) \\
&= -\$4,700,956
\end{aligned}
\tag{6}
$$

This means that there is a change in the position of

$$\Delta V = V_2 - V_1 = \$3,677$$

Relative to the size of the position, its change under the change in rates is very small, demonstrating that this is an effective hedge.

Under the change from 3.596% to 3.320%, the value of the position becomes

$$
\begin{aligned}
V_2 &= \$50,000,000 \Big(\frac{0.4564}{100} \Big) - \$4,231,770 \Big(\frac{115.5712}{100} \Big) \\
&= -\$4,662,507,
\end{aligned}
\tag{7}
$$

which gives a change in the value of

$$\Delta V = V_2 - V_1 = \$42,147.$$

In both the case of rates moving up and the case of rates moving down, the P&L is positive ($3,677 and $42,147). The reason for this is that the convexity of the option is greater than the convexity of the future, meaning that it changes more in value for any change in rates. Since we are long the option, this translates into positive P&L.

Problem 3. Using the data from the answer to Question 1, how much would an investment manager make from \$100m of TYU0C if rates instantaneously fell from 3.504% to 3.404%. Use a duration estimate.

Solution We have seen in the book that, using a second order Taylor expansion, the relative change in price of a position can be estimated as

$$\frac{\Delta P}{P} \approx -D\Delta y + \frac{1}{2} C \Delta y^2,$$

where D is duration and C is convexity. Since $\Delta y^2 << \Delta y$, it is fair to further approximate this as

$$\frac{\Delta P}{P} \approx -D\Delta y.$$

Hence the investment managers return is

$$\begin{aligned} \Delta P &= -PD\Delta y \\ &= -\$100,000,000 \times 320.1936 \times (3.504\% - 3.404\%) \\ &= -\$32,019,360 \end{aligned} \tag{8}$$

Problem 4. Using the data in Question 1, provide a second order estimate of the price of TYU0C should the 7-year par rate be 3.75%.

Solution The second order Taylor expansion of the price-rate function for a small change in rates is

$$P(y + \Delta y) \approx +\frac{dP}{dy}\Delta y + \frac{1}{2}\frac{d^2P}{dy^2}\Delta y^2.$$

From Question 1, we know the first derivative and second derivative of the price-rate function at rates of 3.504%, so we can use this to compute Δy. Using the fact that $y + \Delta y = 3.75\%$ we have

$$\begin{aligned} P(3.75\%) &= P(3.504\%) - 83.8587(3.75\% - 3.504\%) + \frac{1}{2}(21,886.7935)(3.75\% - 3.504\%)^2 \\ &= 0.2619 - 83.8587(0.246) + 0.5(21,886.7935)(0.246)^2 \\ &= 0.1218. \end{aligned} \tag{9}$$

Problem 5. The table below gives the prices, durations, and convexities of three bonds. A) What is the duration and convexity of a portfolio that is long \$50m each of the 5- and 10-year bonds? B) What portfolio of the 5- and 30-year bonds has the same price and duration a the portfolio in part A. C) Which of the two portfolios has the greater convexity and why?

Coupon	Maturity	Price	Duration	Convexity
2.50%	5y	102.248	4.687	25.052
2.75%	10y	100.000	8.691	86.130
3.00%	30y	95.232	19.393	495.423

Table 5: Caption

Solution (Part A)

The face value in the portfolio is \$50m for each bond. Given that the price of the 5y is 102.248, the value of the 5y in the portfolio is

$$\$1,000,000 \times \frac{102.248}{2} = \$51.124\text{m}.$$

The price of the 10y is 100.000, so this is still just \$50m. This gives a total portfolio value of \$101.124m.

The convexity of the portfolio is

$$\begin{aligned} C &= \sum_i \frac{P_i}{P} C_i \\ &= \frac{51.124}{101.124}(25.052) + \frac{50}{101.124}(86.130) \\ &= 55.2516, \end{aligned} \tag{10}$$

and the duration of the portfolio is

$$\begin{aligned} D &= \sum_i \frac{P_i}{P} D_i \\ &= \frac{51.124}{101.124}(4.687) + \frac{50}{101.124}(8.691) \\ &= 6.6667. \end{aligned} \tag{11}$$

Solution (Part B)

We can use the total portfolio value and the total portfolio duration to generate simultaneous equations to find the values of the 5y and the 30y in a portfolio with the same value and duration. The total value of the portfolio is

$$V_5 + V_{30} = \$101.124\text{m}$$

and the total duration of the portfolio is

$$\frac{V_5}{101.124}(4.687) + \frac{V_{30}}{101.124}(19.393) = 6.6667.$$

Solving for V_5 and V_30 gives us a portfolio composition of

$$\begin{aligned} V_5 &= \$13.606\text{m} \\ V_{30} &= \$87.518\text{m} \end{aligned} \tag{12}$$

Solution (Part C)

The convexity of the portfolio of 5y and 10y bonds is 55.2516. The convexity in the portfolio of 5y and 30y bonds is

$$C = \frac{87.518}{101.124}(25.052) + \frac{13.606}{101.124}(495.423) = 88.339.$$

So the convexity of the 5y/30y portfolio is larger. Both portfolios have the same duration, and yield-based convexity is proportional to the square of maturities (the cash flows in the yield-based convexity are weighted by the time of the cash flow squared). Since the 5y/30y portfolio has a greater total maturity than the 5y/10y, it inevitably has a greater convexity due to more squared time values.

Problem 6. The following table gives yield, DV01, and duration for three 15y bonds. The three coupon rates are 0%, 3.5%, and 7%. Which coupon rate belongs to which bond? What is the shape of the term structure of spot rates underlying the valuation of these bonds?

Bond	Yield	DV01	Duration
1	3.50%	0.1159	11.59
2	3.50%	0.0876	14.75
3	3.50%	0.1443	10.26

Table 6: Caption

Solution The yield-based DV01 can be expressed in the following formula:

$$\begin{aligned} DV01 &= \frac{1}{10,000}\left[\frac{1000}{y^2}\left(1 - \frac{1}{\left(1+\frac{y}{2}\right)^{2T}}\right) + T\left(1 - \frac{c}{y}\right)\frac{100}{\left(1+\frac{y}{2}\right)^{2T+1}}\right] \\ &= \frac{100c}{10,000y^2} - \frac{100c}{10,000y^2\left(1+\frac{y}{2}\right)^{2T}} + \frac{100T}{10,000\left(1+\frac{y}{2}\right)^{2T+1}} - \frac{100Tc}{10,000y\left(1+\frac{y}{2}\right)^{2T+1}} \\ 100(DV01) &= \frac{c}{y^2} - \frac{c}{y^2\left(1+\frac{y}{2}\right)^{2T}} + \frac{T}{\left(1+\frac{y}{2}\right)^{2T+1}} - \frac{Tc}{y\left(1+\frac{y}{2}\right)^{2T+1}} \end{aligned} \tag{13}$$

$$\implies c\Bigg[\frac{1}{y^2} - \frac{1}{y^2\left(1+\frac{y}{2}\right)^{2T}} - \frac{T}{y\left(1+\frac{y}{2}\right)^{2T+1}}\Bigg] = 100(DV01) - \frac{T}{\left(1+\frac{y}{2}\right)^{2T+1}}.$$

Now we can just take each bond and substitute in the DV01 (the yield and time to maturity are the same for each bond, so everything other than DV01 in the above equation is the same), and then rearrange for c, which will give us the coupon of each bond.

For bond 1, we get

$$\begin{aligned} 80.929c &= 2.82959 \\ c &= 3.5\% \end{aligned} \tag{14}$$

For bond 2, we get

$$\begin{aligned} 80.929c &= 0.000407 \\ c &= 0\% \end{aligned} \tag{15}$$

For bond 3, we get

$$\begin{aligned} 80.929c &= 5.6696 \\ c &= 7\% \end{aligned} \tag{16}$$

The only way in which it is possible for bonds of different coupons and equal maturity to have the same yield is if the term structure is flat. Hence, the term structure of the underlying spot rates is flat.