Exercises: Multi-Factor Risk Metrics and Hedges

Fixed Income Securities - Tuckman and Serrat

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Problem 1. Using the following instructions, complete a spreadsheet to compute the two-year and five-year key-rate duration profiles of four-year bonds. For the purposes of this question, key-rate shifts are in terms of spot rates.

Problem 1 - Part A. In column A put the coupon payment dates in years, from 0.5 to 5 in increments of 0.5. Put a spot rate curve, flat ar 3%, in Column B. Put the discount factors corresponding to this spot rate curve in Column C. Now price a 3% and an 8% 4y coupon bond under this initial spot rate curve.

Solution The final table pertinent to this part should be:

Term	**Spot rate**	**Discount**
0.5	3%	0.985222
1.0	3%	0.970662
1.5	3%	0.956317
2.0	3%	0.942184
2.5	3%	0.928260
3.0	3%	0.914542
3.5	3%	0.901027
4.0	3%	0.887711
4.5	3%	0.874592
5.0	3%	0.861667

Table 1: Five year spot rate curve for a 3% spot rate.

We can consider first principles to compute the discount factor, by thinking of each term as the payment date of a zero-coupon bond (definition of spot rates). We know that a zero-coupon bond can always be discounted to \$1 at its start date, so for semiannual bonds we have

$$\left(1+\frac{r(t)}{2}\right)^{2t} d(t) = 1,$$

where $d(t)$ is the discount factor at time t, and $r(t)$ is the spot rate at time t. This gives

$$d(t) = \left(1+\frac{r(t)}{2}\right)^{-2t},$$

which is used to compute the discount factors for each term.

Now we can price the bonds by summing the discounted cash flows. The following table makes this clear:

3% coupon		**8% coupon**	
Cash flow	**Discounted**	**Cash flow**	**Discounted**
1.5	1.477833	4.0	3.940887
1.5	1.455993	4.0	3.882647
1.5	1.434475	4.0	3.825268
1.5	1.413276	4.0	3.768737
1.5	1.392390	4.0	3.713041
1.5	1.371813	4.0	3.658169
1.5	1.351540	4.0	3.604107
101.5	90.102679	104.0	92.321957
0.0	0.000000	0.0	0.000000
0.0	0.000000	0.0	0.000000

Table 2: Curves for a 4-year 3% coupon bond and a 4-year 8% coupon bond.

The prices can simply be determined by summing these discounted cash flows:

$$\begin{aligned} P^{(3\%)} &= 100.000000 \\ P^{(8\%)} &= 118.714813 \end{aligned} \tag{1}$$

Problem 1 - Part B. Create a new spot rate curve in column D by adding a 2-year key-rate shift of 10 basis points. Compute the new discount factors in Column E. What are the new bond prices?

Solution The new table now looks like this:

	Initial		**2y shift**	
Term	**Spot rate**	**Discount**	**Spot rate**	**Discount**
0.5	3%	0.985222	3.100%	0.984737
1.0	3%	0.970662	3.100%	0.969706
1.5	3%	0.956317	3.100%	0.954905
2.0	3%	0.942184	3.100%	0.940330
2.5	3%	0.928260	3.083%	0.926365
3.0	3%	0.914542	3.066%	0.912760
3.5	3%	0.901027	3.050%	0.899475
4.0	3%	0.887711	3.033%	0.886558
4.5	3%	0.874592	3.017%	0.873949
5.0	3%	0.861667	3.000%	0.861667

Table 3: Spot rate curve with a 10bp 2y key-rate shift.

How do we get the spot rates for the 2y shift? For the 2-year term, that is the point at which the 10bp shift occurs, so this is of course $3\% + 0.1\% = 3.1\%$. Under the assumptions of key-rate shifts, we know that the lowest key-rate term (2y) and the highest key-rate term (usually 30y) remain at their shift point for all lower terms and all higher terms, respectively. This is why all terms below 2y share the value of 3.1%. Under another assumption of key-rate shifts, each key-rate only impacts the adjacent key-rate, so the shift in spot rate decreases linearly until it reaches zero at the adjacent key-rate (5y in this case). That is how we get the values between 3.1% at the 2y term and 3.0% at the 5y term - it's a linearly decreasing line from 3.1% to 3.0%, which is a decrease in the shift from 10bp to 0bp.

The discount factors are computed in the same way as in Part A, described above.

Now we can repeat the calculation for bond prices in Part A, which gives us the following:

3% coupon		**8% coupon**	
Cash flow	**Discounted**	**Cash flow**	**Discounted**
1.5	1.477105	4.0	3.938946
1.5	1.454559	4.0	3.878825
1.5	1.432358	4.0	3.819620
1.5	1.410495	4.0	3.761320
1.5	1.389547	4.0	3.705460
1.5	1.369140	4.0	3.651041
1.5	1.349212	4.0	3.597899
101.5	89.985587	104.0	92.201981
0.0	0.000000	0.0	0.000000
0.0	0.000000	0.0	0.000000

Table 4: Curves for the bonds from Table 2 under a 10bp 2y key-rate shift.

This gives the following prices under a 2y key-rate shift:

$$\begin{aligned} P^{(3\%)} &= 99.868004 \\ P^{(8\%)} &= 118.555092 \end{aligned} \tag{2}$$

Problem 1 - Part C. Create a new spot rate curve in column F by adding a 5-year key-rate shift of 10 basis points. Compute the new discount factors in Column F. What are the new bond prices?

Solution The new table now looks like this:

	Initial		2y shift		5y shift	
Term	**Spot rate**	**Discount**	**Spot rate**	**Discount**	**Spot rate**	**Discount**
0.5	3%	0.985222	3.100%	0.984737	3.000%	0.985222
1.0	3%	0.970662	3.100%	0.969706	3.000%	0.970662
1.5	3%	0.956317	3.100%	0.954905	3.000%	0.956317
2.0	3%	0.942184	3.100%	0.940330	3.000%	0.942184
2.5	3%	0.928260	3.083%	0.926365	3.017%	0.927881
3.0	3%	0.914542	3.066%	0.912760	3.033%	0.913651
3.5	3%	0.901027	3.050%	0.899475	3.050%	0.899475
4.0	3%	0.887711	3.033%	0.886558	3.066%	0.885406
4.5	3%	0.874592	3.017%	0.873949	3.083%	0.871369
5.0	3%	0.861667	3.000%	0.861667	3.100%	0.857434

Table 5: Spot rate curve with both a 10bp 2y key-rate shift and a 10bp 5y key-rate shift.

All calculations are analogous to those in Part B, and we get the following bond prices under the 5y key-rate shift:

$$\begin{aligned} P^{(3\%)} &= 99.761752 \\ P^{(8\%)} &= 118.463744 \end{aligned} \tag{3}$$

Problem 1 - Part D, E, F. Use the results in Part A through C to calculate the key-rate duration profiles of each of the bonds. Sum the key-rate durations for each bond to obtain the total duration attributed to each key-rate for the bond. Comment on the results. What would the key-rate duration profile of a 4y zero-coupon bond look like relative to those of these coupon bonds? How about a 5y zero-coupon bond?

Solution The results can be seen in the following table:

	Initial	2y shift		5y shift				
Coupon	**Value**	**Value**	**Duration**	**Value**	**Duration**	**Total duration**	**2y %**	**5y %**
3%	100.0000	99.8680	13.1996	99.7618	23.8248	37.0244	0.36	0.64
8%	118.7148	118.5551	13.4542	118.4637	21.1489	34.6030	0.39	0.61
4y 0%	88.7711	88.6558	12.9954	88.5406	25.9718	38.9673	0.33	0.67
5y 0%	86.1667	86.1667	0.0000	85.7400	49.1279	49.1279	0.0	1.0

Table 6: Key-rate duration profiles for the 4y 3% coupn bond, the 4y 8% coupon bond, a 4y zero-coupon bond, and a 5y zero-coupon bond.

The duration for a given key rate k is calculated by doing

$$D^{(k)} = -\frac{1}{P}\frac{\Delta P}{\Delta y^{(k)}},$$

where ΔP represents the change from the initial price before the key rate shift, and Δy is always 0.01%, because the duration is with respect to a 1bp change, by definition. Using the 2y key rate shift for the 8% coupon bond as an example:

$$D^{(2)} = -\frac{1}{100.0000}\frac{118.5551 - 118.7148}{0.01\%} = 13.4542.$$

All others are analogous to this.

Now we want to comment on the results of the proportion of duration attributed to each key-rate shift. For the 8% coupon bond we can see that the 2y key rate shift holds more duration than that of the 3% coupon bond. The reason for this is that bonds with larger coupons discount more of the bonds value early on, so a larger amount of the risk (key rate duration) is going to be concentrated at lower ends of the curve.

For Part F, we want the key-rate duration profile of a 4y and 5y 0-coupon bond. The values can be determined by referring to Table 5 and looking for the 4y and 5y terms and discounting from \$100 (or \$1 if you prefer). All other calculations are analogous to previous ones. One interesting aspect to note is that all of the 5y zero-coupon bond's duration is in the 5y key rate shift. This makes sense, because there is only discounting at the 5y term, and key rate assumptions state that the key rate shift drops to zero at adjacent terms, meaning that all of the key rate shift risk will be concentrated at the 5y key rate for a 5y zero-coupon bond.

Problem 2. Continue with the setting and results of Problem 1. Verify that a 3% 2y bond has a duration of 1.925 that is completely concentrated as a 2y key-rate duration. How would one hedge the key-rate risk profile of the 8% 4y bond with the 3% 2y bond and the 3% 4y bond. Note that the total value of the 8% bond and of the hedge need not be the same. Comment on the result.

Solution We can add a new row to Table 6 for the new 3% 2y bond, which will look like this:

	Initial	**2y shift**		**5y shift**	
Coupon	**Value**	**Value**	**Duration**	**Value**	**Duration**
3%	100.0000	99.8075	19.2461	100.0000	0.0000

Table 7: Key-rate duration profile for a 2y 3% coupon bond.

Everything is calculated analogously to Problem 1. Showing explicitly the key-rate duration calculation:

$$D^{(2)} = -\frac{1}{100.0000}\frac{99.8075 - 100.0000}{0.01\%} = 19.2461.$$

It can clearly be seen that the entirety of the duration is concentrated in the 2y key-rate shift, because it is zero for the 5y key-rate shift.

Consider a purchase of the 8% 4y bond. Let $F^{(2)}$ be the face amount of the 3% 2y bond hedge, and $F^{(4)}$ be the face amount of the 3% 4y bond hedge. To maintain duration neutrality in the position (i.e. getting rid of any duration risk), we let the total duration of the hedge be equal to the total duration of the original position. For the 2y key-rate shift we have

$$19.2461F^{(2)} + 13.1996F^{(4)} = 118.7148 \times 13.4542,$$

and for the 5y key-rate shift we have

$$0 \times F^{(2)} + 23.8248F^{(4)} = 118.7148 \times 21.1489.$$

Rearranging for $F^{(4)}$ and then using that to compute $F^{(2)}$ we get

$$\begin{aligned} F^{(2)} &= 10.7150 \\ F^{(4)} &= 105.7150 \end{aligned} \tag{4}$$

This makes sense. If we purchase a 4y bond it is intuitive that a there is more risk at the 4y end of the curve, so we need to purchase more of the 3% 4y bond to hedge it. The smaller purchase of the 2y bond is required to offset the risk from the discounting at the lower end of the curve required in the 8% 4y bond.

Problem 3. A trader constructs a butterfly portfolio that is short EUR 100m of the 10y swap and long 50% of the 10y swap's total '01 in 5y swaps and 50% of the 10y swap's total '01 in 15y swaps. What are the forward-bucket exposures of the resulting portfolio?

Solution We have the following table (Table 5.6 from the book) for forward-bucket exposures:

Security	0-2	2-5	5-10	10-15	All
5y swap	.0196	.0276	.0000	.0000	.0472
10y swap	.0194	.0269	.0394	.0000	.0857
15y swap	.0194	.0265	.0383	.0323	.1164

Table 8: General forward-bucket exposures for a 5y, 10y, and 15y swap.

We need to begin by computing the face amount of each position. We know the face amount of the short 10y swap, and that is $-\$100$m. So the total forward-bucket size of this position is:

$$\text{Total exposure} = \$100\text{m} \times \frac{0.0857}{100} = \$85,700.$$

Since the long positions are 50% of the short position, they each have a total size of $\$42,850$. Using these values, we can compute the face amounts of the 5y swap and 15y swap long positions. We have

$$\$42,850 = \frac{0.0472}{100} F^{(5)} \implies F^{(5)} = \$90.784\text{m}$$

and

$$\$42,850 = \frac{0.1164}{100} F^{(15)} \implies F^{(15)} = \$36.813\text{m}.$$

We can multiply these face values by the corresponding forward-bucket exposures from Table 8 and then divide by 100 to get the net forward-bucket exposures in dollar amounts. We get the following table:

Security	**Face**	**0-2**	**2-5**	**5-10**	**10-15**	**Total**
5y swap	90.784m	17,790	25,060	0	0	42,850
10y swap	-100m	-19,400	-26,900	-39,400	0	-85,700
15y swap	36.813m	7,142	9,755	14,099	11,891	42,850

Table 9: Dollar-amount forward-bucket exposures under the trade positions in Problem 3.