Algebraic Geometry note

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Preface

This is the note for Algebraic Geometry note refer to GTM 52.

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1 Varieties

1.1 Hilbert's Nullstellensatz

1.1.1 Affine varieties and ideals

Algebraic geometry is the study of zeros of polynomials, and Hilbert's Nullstellensatz theorem establishes the fundamental relationship between geometry and algebra, thus forming the basis of algebraic geometry. I have therefore written it down in a separate section, and it is also an exercise in Atiyah's commutative algebra.

Def1.1.1.1: A n-dimension affine space over k,$\mathbb{A}^n = \{(a_1, a_2, ..., a_n)\}$. Let $\mathbb{K}[x_1, x_2, ..., x_n]$ be a polynomial ring, and let $S \subset \mathbb{K}[x_1, x_2, ..., x_n]$be a polynomial set. Then the zero set of S is $Z(S) = \{p \in \mathbb{A}^n | f(p) = 0, \forall p \in S\}$. The sets which forms like $Z(S)$ called affine algebraic set.If a polynomial $f \in \mathbb{K}[x_1, x_2, ..., x_n]$ is a Non-constant polynomial,we called it hypersurface.

From GTM 52's **prop1.1**:The unions of two algebraic sets is an algebraic sets. The intersection of any family of algebraic sets is an algebraic set. The empty set and the whole space are algebraic sets. From this prop ,naturally, we can remind the definition of topological space, we can find that the props of algebraic sets is similar to the open sets in topological space, so if we let the algebraic sets to be open sets, we can get the definition of the *Zariski topology* on $\mathbb{A}^n$ as below.

Def1.1.1.2: *Zariski topology* is a topological space on $\mathbb{A}^n$ by taking the open subsets to be the complements of the algebraic sets. It's obviously Zariski is a topology according to the proposition.

For a sub set I of a Ring R, for additive, if I is a subgroup of R and $\forall f \in I, \forall g \in R, gf \in I$. We call I is a ideal of R. Let ideal $\langle S\rangle = \{\sum_{i=1}^{n} a_i s_i | n \geq 0, a_i \in R, s_i \in S\}$. We call $\langle S\rangle$ is an ideal generated by the subgroup S. If $I \neq R$, I is **proper ideal**, and an ideal M is not in any proper ideal $I \neq M$, we call it **Maximal ideal**.

It's easy to find that $Z(S) = Z(\langle S\rangle)$ i.e. every affine algebraic set is the ideal's zero set of the polynomial ring $\mathbb{K}[x_1, x_2, ..., x_n]$. In general, there can be many ideals corresponding to the same affine algebraic set, but for every given set X of zeros there exists a corresponding maximal ideal, which is the set $I(X)$ of all polynomials that are zero over X.

Def1.1.1.3: Let $X \subseteq \mathbb{A}^n$. The ideal of X is: $I(X) = \{f \in \mathbb{K}[x_1, x_2, ..., x_n] | f(p) = 0, \forall p \in X\}$

1.1.2 Noetherian ring and Hilbert basic theorem

We know that algebraic sets are the zero sets of polynomial rings, moreover, we can define the algebraic sets by infinite elements of the polynomial ring. To illustrate it, we need to introduce **Noetherian ring**.

Def1.1.2.1: If a ring R satisfies the following two equivalence conditions.
①: If every ideal $I \subset R$ is infinitely generated, i.e, $I = \langle f_1, f_2, ..., f_r \rangle$.
②: R fulfilling the ascending chain condition: every ideal chain in R are stationary. i,e, $I_1 \subseteq I_2 \subseteq ... \subseteq I_n, \exists N \in \mathbb{N}, I_N = I_N + 1 = ...$
Then this ring is **Noetherian Ring**.

Next, we will learn the Hilbert basic theorem, which means a polynomial ring over an Noetherian ring is also an Noetherian ring.

Hilbert basic theorem: If ring R is an Noetherian ring, then polynomial ring $R[x_1, x_2, ..., x_n]$ is also an Noetherian ring.
Proof: $\because R[x_1, x_2, ..., x_n] = R[x_1, x_2, ..., x_n - 1][x_n]$, we just need to prove, if R is an Noetherian ring, then $R[x_1, x_2, ..., x_n]$ is an Noetherian ring, which is equivalent to prove if $R[x]$ is not an Noetherian ring, then R is not an Noetherian ring.

Let $I \subset R$ is an infinite generate ideal. Select $f \in I \setminus \{0\}$ to be a polynomial with the lowest degree, and similarly we can find $f_n \in I \setminus \langle f_1, f_2, ..., f_r \rangle$. Let $n_k = deg(f_k)$, and let $x^(n_k)$'s coefficients are a_k. Then $n_1 \leq n_2 \leq n_3 ...$, and $\langle a_1 \rangle \subseteq \langle a_1, a_2 \rangle \subseteq ...$ is an ideal chain of R.

Suppose that the chain is stationary, then it exists k, such that $\langle a_1, a_2, ..., a_k \rangle = \langle a_1, a_2, ..., a_k - 1 \rangle$. Then $a_k + 1 = \sum_{i=1}^{n} b_i a_i, b_i \in \mathbb{R} . \because f_k + 1 \in I \setminus \langle f_1, f_2, ..., f_k + 1 \rangle$, $\therefore g = f_k + 1 - \sum_{i=1}^{k} (b_i x^(n_k + 1 - n_i) f_i \in f_i \in I \setminus \langle f_1, f_2, ..., f_k \rangle$. But $x^(n_k + 1)$ not in g, $deg(g) < n_k + 1 = deg(f_k + 1)$. It's a contradictory to $f_k + 1$ has the lowest degree.□

Because a field $\mathbb{K}$ only has two ideals, i.e. 0 and $\mathbb{K}$, so $\mathbb{K}$ is an Noetherian ring. According to the Hilbert basic theorem, $\mathbb{K}[x_1, ..., x_n]$ is also an Noetherian ring, so every ideal was finite generated. In other words, every affine algebraic set just need finite polynomials to define.

1.1.3 Irreducible and affine variety

We say a algebraic set is **irreducible** if it can't be written as an union of two smaller algebraic sets. In fact, for all algebraic set can be written as an union of different irreducible algebric sets, these irreducible algebric sets are irreducible component, and unique.

Def1.1.3.1: If a topological space X can be written as $X = X_1 \cup X_2$, and $X_1, X_2 \neq X$, which are the closed subsets of X, then X is reducible. Otherwise, X is irreducible. An irreducible affine algebraic set is **affine variety**, a open subset of affine variety is **locally closed subvariety**.

To show every algebraic set can be written as finite union of irreducible algebraic sets, we need to introduce Noetherian topological space.

Def1.1.3.2: An Noetherian topological space is a topological space meets the requirement: The descending chain of each closed subset $X \supset X_1 \supset X_2 \supset ...$ becomes stationary.

The next theorem will show that every Noetherian top-space can be uniquely written as an union of finite irreducible closed subsets.

Theorem: Every Notherian topological space X can be written as $X = X_1 \cup X_2 \cup ... \cup X_r$, where X_i is irreducible closed subset, and $X_i \nsubseteq X_j$. X_i called the irreducible component.

Proof: Firstly, we prove the existence. Suppose that the X doesn't exist these decomposition, then X is reducible, otherwise $X = X$ is the decomposition we need.$\therefore X = X_1 \cup Y_1$, where $X_1, Y_1 \neq X$, and they are closed subsets of X. For one of these tow subsets,we can take X_1 firstly(is same as take Y_1)., $\because X$ doesn't exist the decomposition theorem said, $\therefore X_1 = X_2 \cup Y_2$, where $X_2, Y_2 \neq X_1$, and they are closed subsets of X_1. Repeat this operation, we can get a descending chain $X \supset X_1 \supset X_2 \supset ...$ with no stationary, which is contradictory to X is an Noetherian space.
Secondly, we will prove the unique. Assume that $X = X_1 \cup X_2 \cup ... \cup X_n = Y_1 \cup Y_2 \cup ... \cup Y_n$. Because $X_i = \cup_j (Y_j \cap X_i)$, and X_i is irreducible, then $\exists j' \in \{1, 2, ..., s\}, X_i \subseteq Y'_j$. Similarly, $Y'_j \subseteq X_k$. Then $X_i \subseteq X_k$, so $i = k$ and $X_i = Y'_j$. Thus, we have $r = s$, and $X_1, X_2, ..., X_r$ are equal to $Y_1, Y_2, ..., Y_s$.$\square$

We know that $\mathbb{K}[x_1, ..., x_n]$ is an Noetherian ring, so for all ascending chain $\emptyset \subset I_1 \subset I_2 \subset ...$ will be stationary, so for all descending chain $\mathbb{A}^n \supset Z(I_1) \supset Z(I_2) \supset ...$ will also be stationary. i.e, affine space $\mathbb{A}^n$ is an Noetherian space. Moreover, it's easy to find that the subspace of an Noetherian space is also an Noetherian space, so all the affine algebraic set is Noetherian, i.e, for all affine algebraic set can be uniquely decomposed as an union of finite irreducible closed subsets.

1.1.4 Strong Hilbert's Nullstellensatz

Now we can introduce Hilbert's Nullstellensatz. Firstly, it's easy to find that if we regard Z and I as maps, they are inverse map of each other. And we can induce from the definitions of algebraic set and ideal: $I \subseteq I(Z(I)), X \subseteq Z(I(X))$. And because $X = Z(I)$, where I is an ideal, so for all affine algebraic set $X \subseteq \mathbb{A}^n$, we have $Z(I(X)) = X$. Then we can determine the relations between algebraic sets in affine space and ideals of polynomial ring.

If we consider the simplest algebraic set,i.e, the algebraic set only have one element$\{p = (a_1, a_2, ..., a_n) \subseteq \mathbb{A}^n\}$, the corresponding ideal $I = \langle x_1 - a_1, x_2 - a_2, ..., x_n - a_n\rangle$ is maximal ideal, and weak Hilbert's Nullstellensatz means the ideals like I are all maximal ideal.

Theorem(Weak Hilbert's Nullstellensatz Theorem): The maximal ideals in $\mathbb{K}[x_1, x_2, ..., x_n]$ has the form of $I = \langle x_1 - a_1, x_2 - a_2, ..., x_n - a_n\rangle$, where $p = (a_1, a_2, ..., a_n) \in \mathbb{A}^n$ is a point in affine space.

We will prove it in the next section, now we need to know two propositions from weak Hilbert's Nullstellensatz Theorem. $p \longmapsto I(p), M \longmapsto Z(M)$ are bijection with mutual inverse mapping, and if $Z(I) = \emptyset$, then $I = \mathbb{K}[x_1, x_2, ..., x_n]$. But the one-to-one correspondence between the points in $mathbbA^n$ and $\mathbb{K}[x_1, x_2, ..., x_n]$ is not extendable to all algebraic sets. In generals, an ideal J couldn't meet $I(Z(J)) = J$, for example in $\mathbb{A}^1$, $I(Z(x^n)) = I(\{0\}) = \langle x\rangle$. So we need to introduce the definition of the radical ideal.

Def:If I is an ideal of A, then it's radical is $\sqrt{I} = \{a \in A | a^n \in I, n > 0\}$, which is also an ideal. If an ideal I satisfy $I = \sqrt{J}$, where J is an ideal, or we can say $I = \sqrt{I}$, then we call I is a radical ideal.

So we can know that for all algebraic set $X \subseteq \mathbb{A}^n$, $I(X)$ is a radical ideal. Strong Hilbert's Nullstellensatz gives relation between I and $I(Z(I))$.

Theorem(Strong Hilbert's Nullstellensatz Theorem): If $I \subset \mathbb{K}[x_1, x_2, ..., x_n]$ is an ideal, then $I(Z(I)) = \sqrt{I}$.
Proof:$\mathbb{K}[x_1, x_2, ..., x_n]$ is an Noetherian ring, so $I = \langle f_1.f_2, ..f_n\rangle, f_i \in I$. $Z(I)$ is an algebraic set, so $I(Z(I))$ is a radical ideal, and $I \subseteq I(Z(I))$. So we have $\sqrt{I} \subseteq I(Z(I))$.
Next we need to use Weak Hilbert's Nullstellensatz Theorem to prove $I(Z(I)) \subseteq \sqrt{I}$.,i.e, Select $f \in I(Z(I))$, We need to prove $\exists N \in \mathbb{N}, s.t f^N \in I = \langle f_1.f_2, ..f_n\rangle$. Let $J = \langle f_1.f_2, ..f_r, f_t - 1)\rangle \subset \mathbb{K}[x_1, x_2, ..., x_n, t]$.If $p \in \mathbb{A}^n, a \in \mathbb{K}$,then if and only if when $p \in Z(I), f(p)a = 1$,$(p, a) \in Z(J)$. But f is zero over $Z(I)$, so it's impossible, i.e, $Z(J) = \emptyset$. From Weak Hilbert's Nullstellensatz Theorem ,$J = \mathbb{K}[x_1, x_2, ..., x_n, t]$,so $1 \in J$. Then we have $1 = g_0(ft - 1) + \sum_{i=1}^{r} g_i f_i, g_i \in \mathbb{K}[x_1, x_2, ..., x_n, t], f_i \in I$. Next we define a ring homomorphism: $\phi : \mathbb{K}[x_1, x_2, ..., x_n, t] \longrightarrow \mathbb{K}[x_1, x_2, ..., x_n], g(x_1, x_2, ..., x_n, t) \longmapsto$

$g(x_1, x_2, ..., x_n, \frac{1}{f})$. $\because \phi(ft-1) = 0, \phi(f_i) = f_i$, so we have $1 = \sum_{i=1}^{r} \phi(g_i)f_i$ after apply ϕ in this equation. $\because t \longmapsto \frac{1}{f}, \phi(g_i) = \frac{G_i}{f^{n_i}}, G_i \in \mathbb{K}[x_1, x_2, ..., x_n], n_i \geqslant 0$. Let $N = max_{i\in\{1,2,3,...,r\}} n_i$, and multi f^N at both sides of the equation, we have $f^N = \sum_{i=1}^{r} G_i f^{N-n_i} f_i, G_i f^{N-n_i} \in \mathbb{K}[x_1, x_2, ..., x_n], f_i \in I$. Then we have $f^N \in I$, Thus $I(Z(I)) \subseteq I$.□

From Strong Hilbert's Nullstellensatz Theorem, we know that the maps I and Z form a bijection between the set of algebraic sets in $\mathbb{A}^n$ and the set of radical ideals in $\mathbb{K}[x_1, x_2, ..., x_n]$,and they are mutually inverse maps.So it build a relation between geometry and algebra. Moreover, Strong Hilbert's Nullstellensatz Theorem also has a proposition about irreducible.

Proposition: (1):For affine algebraic set $X \subseteq \mathbb{A}^n$, if and only if $I(X)$ is a prime ideal, i.e, if $fg \in I(X)$ or $g \in I(X)$, then $I(X) \neq \mathbb{K}[x_1, x_2, ..., x_n]$ X is an affine variety. Or we can equally say that if and only if $X = Z(I)$, where I is prime ideal and X is an affine variety.
(2): Supposed that $f \in \mathbb{K}[x_1, x_2, ..., x_n]$ is an irreducible polynomial, i.e, $f = gh$, then g or h is constant, then $Z(f)$ is irreducible.
Proof: These two conditions are equivalence, because every prime ideals are radical ideals, and from Strong Hilbert's Nullstellensatz Theorem, $I = I(Z(I))$. Firstly, supposed that X is irreducible, and let $fg \in I(X)$. Then $X = Z(I(X)) \subseteq Z(fg) = Z(f) \cup Z(g)$, so $X = (Z(f) \cap X) \cup (Z(g) \cap X)$. And because of X is irreducible, $X = Z(f) \cap X$ or $X = Z(g) \cap X$. Then $X \subseteq Z(f)$ or $X \subseteq Z(g)$, which means $f \in I(X)$ or $g \in I(X)$.
Then we suppose that X is reducible, $X = X_1 \cup X_2$,where $X_1, X_2 \neq X$ are closed subset of X. Then there exists $f_1 \in I(X_1) \setminus I(X)$ and $f_2 \in I(X_2) \setminus I(X)$. $\because$ every point in $f_1 f_2$ over the $X = X_1 \cup X_2$ are zero, $f_1 f_2 \in I(X)$, so $I(X)$ is not a prime ideal.□

Affine space and affine variety which are not single point sets are not compact, so we want to find a way to compactify them. So we will introduce projective space and projective variety to solve this problem in next section, and we also can get a Nullstellensatz Theorem in projective space.

1.1.5 Weak Hilbert's Nullstellensatz

1.1.6 Solutions for 1.1

1.1(a): $y = x^2 \to y - x^2 = 0$, then $A(Y) = f[x,y]/(y - x^2)$, we need to prove $\phi : k[t] \to f[x,y]/(y - x^2)$ is an isomorphic, then $\phi : f[x,y]/(y - x^2) \to k[x,x^2], \bar{x} \longmapsto x$, it's obviously that ϕ is a homomorphic, then it's also obvious that ϕ is surjective, and for $x \neq y \in k[x,x^2]$, we also can find that $\bar{x} \neq \bar{y}$, if $\bar{x} = \bar{y}$, $x = y$. So the map ϕ is also a injective, and it's a bijective map. We get $f[x,y]/(y - x^2) \cong k[x,x^2] \cong k[x]$, because we can get a isomorphic map $\psi : k[x,x^2] \to k[x], (x,x^2) \longmapsto x$.so $f[x,y]/(y - x^2) \cong k[x]$.

1.1(b): $A(Z) = k[x,y]/(xy - 1) \cong k[x, \frac{1}{x}]$, if we have a ring homomorphic map $\phi : k[x,y]/(xy - 1) \to k[t]$, then we also have $\psi : k[x, \frac{1}{x}] \to k[t]$, but ψ is not surjective, because we can find $0 \in k[t]$ but $0 \notin k[x, \frac{1}{x}]$.

1.1(c):?

1.2: This question is a promotion of 1.1, $A(Y) = f[x,y,z]/(y - x^2, z - x^3) \cong f(x,x^2,x^3)$, we can find a isomorphic $\phi : f(x,x^2,x^3) \to k, (x,x^2,x^3) \longmapsto x$, so $A(Y) = f[x,y,z]/(y - x^2, z - x^3) \cong f(x,x^2,x^3) \cong k$.For the second question, $I(Y) = (y - x^2, z - x^3)$, next, we prove this: $\forall f \in f[x,y,z]$, $f = h_1(y - x^2) + h_2(z - x^3) + r(x)$, for $r(x) \in k[x]$, we just need to prove $x^\alpha y^\beta z^\gamma = x^\alpha(x^2 + (y - x^2))^\beta(x^3 + (z - x^3)))^\gamma = x^\alpha(x^{2\beta} + \sum_{i=1}^{\beta} C_\beta^i(y - x^2)^i x^{2(\beta - i)})(x^3\gamma + \sum_{j=1}^{\gamma}(z - x^3)^j x^{3(\gamma - j)})) = h_1(y - x^2) + h_2(z - x^3) + r(x)$, where $h_1 = \sum_{i=1}^{\beta} C_\beta^i(y - x^2)^{i-1}x^{2(\beta - i)}$,$h_2 = \sum_{j=1}^{\gamma}(z - x^3)^{j-1}x^{3(\gamma - j)}))$, $r(x) = x^{\alpha + 2\beta + 3\gamma}$, $h_1, h_2 \in k[x,y,z]$, so $(y - x^2, z - x^3) \subseteq I(Y)$. Next, we need to show that $I(Y) \subseteq (y - x^2, z - x^3)$, let $f \in I(Y)$, suppose that $f = h_1(y - x^2) + h_2(z - x^3) + r(x)$, then $f(t,t^2,t^3) = 0 = 0 + 0 + r(x) \Rightarrow r(x) = 0$, so $f = h_1(y - x^2) + h_2(z - x^3) \subseteq (y - x^2, z - x^3)$, we have $I(Y) \subseteq (y - x^2, z - x^3)$.

1.3: $\begin{cases} x^2 - yz = 0 \\ xz - x = 0 \end{cases}$,

we have three situations, when $x = 0, y = 0$,z is free; when $x = 0, z = 0, y$ is free; and $x \neq 0, z = 1, y = x^2$. So $Y = Z(x^2 - y, z - 1) \cup Z(z) \cup Z(y)$, their prime ideals respectively are $(x^2 - y, z - 1), (z), (y)$, the prove just like 1.2.

1.4:
Hint: A top.space X is Hausdorff $\Leftrightarrow$ the diagonal $\triangle = \{(x,x) \in X \times X | x \in X\}$ is closed in $X \times X$ in the product topology.

Let $\mathbb{A}^2 = \mathbb{A} \times \mathbb{A}$, a basis for the closed set of $\mathbb{A}^1 \times \mathbb{A}^1$ is given by $\{X \times Y | X \subseteq \mathbb{A}^1 closed, Y \subseteq \mathbb{A}^1 closed\}$, so every closed set is finite. However, because $Z(x - y) \subseteq \mathbb{A}^2$ is closed and infinite, so the two topology are not equal.

1.5:
Hint: Finite generate algebra: a finite generated algebra is a commutative associate algebra A over a field k, $\exists$ a finite $a_1, ..., a_n$ of A ,s.t. every element of A can be represented as a polynomial by $a_1, ..., a_n$, i.e. $\exists a_1, ..., a_n \in A$, s.t. $\phi : k[x_1, ..., x_n] \to A$ is homomorphism and surjective, so we have $A \cong k[x_1, ..., x_n]/Ker\phi$.

$\Leftarrow$: if B is finitely generated k-algebra, then $B \cong k[x_1, ..., x_n]/a$, for some ideal a. Moreover, if B has no nilpotent elements. a is a radical ideal, $A(Z(a)) = k[x_1, ..., x_n]/I(Z(a)) = k[x_1, ..., x_n]/\sqrt{a} = k[x_1, ..., x_n]/a = B$.

1.6: First we prove any nonempty open subset of an irreducible topological space is dense and irreducible. Let X be an irreducible top.space, $U \subseteq X$ is open, assume that U is not dense. Then we can find a nonempty open set $V \subseteq X$, s.t $V \cap U = \emptyset$. Then $X = U^c \cup V^c$, it's contradictory that X is irreducible. So U is dense.
If U is not irreducible, let $U = Y_1 \cup Y_2$, where each Y_i is closed of U. Then for two closed subset $X_1, X_2 \subseteq X$, s.t $Y_i = U \cap X_i$, $(X_1 \cup X_2) \cup U^c = X$, so X is reducible, contradiction.
Now we prove that closure of irreducible open subset Y is also irreducible. Suppose that Y is an irreducible subset of X and $\bar{Y} = Y_1 \cup Y_2$, then $Y = (Y_1 \cap Y) \cup (Y_2 \cap Y)$, because Y is irreducible, so Y just equal to $(Y_1 \cap Y)$, because $\bar{Y}$ is the smallest closed subset of X containing Y, so $\bar{Y} = Y$, so $\bar{Y}$ is irreducible.

1.7(a): $(i) \Rightarrow (ii)$: If X is noetherian top.space, then $\exists$ closed sets Y_i satisfying $Y_1 \supseteq Y_2 \supseteq ...$, $\exists m$, s.t $Y_m = Y_{m+1} = ...$, let $\sum$ be any nonempty family of closed subsets. Choose a $X_1 \subseteq \sum$, if X_1 is minimal, we prove it. if X_1 is not minimal, choose a $X_2 \subseteq \sum$, $X_2 \subseteq X_1$, if X_2 is minimal, we prove it. If X_2 is not minimal, choose a $X_3 \subseteq \sum$, $X_3 \subseteq X_2$,... We repeat this process, and we can always find a minimal element in $\sum$, because X is noetherian.
$(ii) \Rightarrow (i)$: If every nonempty family of closed subsets has a minimal element, then the closed sets of X meets the D.C.C., so we prove it.
$(i) \Rightarrow (iii)$: X is noetherian for closed set, $Y_1 \subseteq Y_2 \subseteq ... \subseteq Y_m \subseteq ...$, $Y_m = Y_m + 1 = ... \Rightarrow (Y_m^c) \subseteq .... \subseteq (Y_1^c)$, for open set, so satisfy the ascending chain.
$(iii) \Rightarrow (i)$: The prof is same as above.
$(iii) \Leftrightarrow (iv)$: as same as $(i) \Leftrightarrow (ii)$.
1.7(b): Suppose that X is a noetherian top.space $\Rightarrow$ the open set s of X have ascending chain condition. Let $x = \bigcup_\alpha U_i$, pick U_1, U_2, then $U_1 \subseteq (U_1 \cup U_2)$, then we can pick a U_3, s.t $U_2 \subseteq (U_1 \cup U_2 \cup U_3)$, because of the ascending chain conditions, we continous this operation, then we can get a finite cover of X.
1.7(c): Let $Y \subseteq X$ be the subset of a noetherian top.space X. Consider an open chain of subset $V_0 \subseteq V_1 \subseteq ...$ in Y. $U_i \subseteq X$ is open subset, s.t $V_i = U_i \cap Y$. Form the open sets $W_i = \cup_{i=1}^k U_i$, so $W_k \cap Y = \cup_{i=1}^k V_i = V_k$. The chain $V_0 \subseteq V_1 \subseteq ...$ stable in Y, because X is noetherian. so by (a), Y is noetherian.

1.7(d):This proof was by Joe Cutrone and Nick Marshburn.?
Let X be a noetherian top.space and is also a Hausdorff space. C be an irreducible closed set. If C were not a point, then any $x, y \in C$ have disjoint open sets, which are dense(the conclusion of 1.6). So C is a finite union of irreducible closed sets, i.e, a finite set of points.

1.8: Let Y be an affine variety, $dimY = r$, let H be a hypersurface, s.t $Y \nsubseteq X$ and $Y \cap H \neq \emptyset$, then $I(H) \nsubseteq I(Y)$. Let H be defined by $f(x_1, ..., x_n) = 0$, the irreducible components of $Y \cap H$ correspond to minimal prime ideals p in $k[Y]$ containing f. Because $Y \nsubseteq X$, f is not zero-divisor, every minimal prime ideal p containing f has height 1. Then from Th1.8A, every irreducible component of $Y \cap H$ is $dimY - dimp = r - 1$.

1.9: Let $a = (f_1, ..., f_r)$, then every $f_i = 0$ can define a hypersurface H_i, by applying 1.8 r times then we know that every irreducible component of $Z(a)$ has dimension $>= n - r$.

1.10(a): If we have a distinct irreducible closed subsets chain $Y_0 \subset Y_1 \subset ... \subset Y_n$ of Y, where $n = dimY$. Then Y_i are also irreducible closed subsets of X with induced topology. Let C_i be closed subsets of X and let $Y_i = C_i \cap Y$, then from 17.(c), we can find a strictly increasing chain $(C_0 \cap Y) \subset (C_1 \cap Y) \subset ... \subset (C_n \cap Y)$, then for any chain in Y, we can always find a chain C_i in X, so $dimY <= dimX$.
1.10(b): Let X be a topological space with open covering $\Cup U_i$. From (a), $dimU_i <= dimX$, so $dimsup_iU_i <= dimX$. For any chain of irreducible subset $C_0 \subset C_1 \subset ... \subset C_n$, we cam choose an open set $U_i \cap U_0 \neq \emptyset$, then $(U_i \cap C_0) \subset (U_i \cap C_1) \subset ... \subset (U_i \cap C_n) \subset ...$ and because X is not a noetherian space, so $dimU_i >= dimX$, then $sup_i dimU_i = dimX$.
1.10(c):Let $X = \{0, 1\}$ with $\tau = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$. Then $\{0\}$ is a open set and its closure is X, so $\{0\}$ is dense. And $dim\{0\} = 0$, but $\{1\} \subset \{0, 1\}$, so $dim\{0, 1\} = 1$, so with $U = \{0\}$, $dimU < dimX$.
1.10(d):
Notice: this prove need $dimX = dimY < \infty$.
Let Y be a closed subset of an irreducible finite-dim top.space X, s.t $dimX = dimY = n < \infty$. Let a chain of Y is $Y_0 \subset Y_1 \subset ... \subset Y_n$, which are a sequence of distinct closed irreducible subsets, and we can choose a sequence subsets U_i of X, and we can write Y_i as $Y \cap U_i$, from this ,Y_i is also closed, because Y and U_i are closed. So $\bar{Y}_i = Y_i$ and from (1.6) we know that Y_i is irreducible. So the chain in Y is also a distinct, irreducible and closed chain in X. So we can say that $Y_n = X$ because if $Y_n \neq X$, then we can find a chain $Y_0 \subset Y_1 \subset ... \subset Y_n \subset X$, the length of chain at least $n + 1$, so $X = Y_n \subseteq Y$, and Y is a closed subset of X, so $X = Y$
1.10(e):? this solution is also from Joe Cutrone and Nick Marshburn. But I am not totally understand.
For $n \in \mathbb{Z}_{>=0}$, let $U_n = \{n, n+1, ...\}, \tau = \{\emptyset, U_0, ..\}$ is a topology of open sets on $\mathbb{Z}_{>=0}$. In this space, if C and C' are closed sets, then it's easy to see that

either $C \subset C'$ or $C' \subset C$, that every nonempty closed set is irreducible, and that every closed set in $\mathbb{Z}_{>=0}$ is finite.
1.11:?
1.12: $2x^2 + y^2 + 1$, then $\mathbb{Z}_{\mathbb{A}^2_{\mathbb{R}}}(2x^2 + y^2 + 1) = \emptyset$, so it's not irreducible.

1.2 Projective variety

1.2.1 projective algebraic set and homogeneous coordinate

Def: In $\mathbb{K}^{n+1} \backslash \{0\}$, we can denfine an equivalence relation,$(a_0, ..., a_n) \sim (\lambda a_0, ..., \lambda a_n), \forall \lambda \in \mathbb{K}^{n+1} \setminus \{0\}$, and we can write $(a_0, ..., a_n)$'s equivalence class as $[a_0, ..., a_n]$. The quotient set $\mathbb{P}^n = \mathbb{K}^{n+1} \setminus \{0\} / \sim$ called **projective space**. a_i is the **homogeneous coordinate** of $[a_0, ..., a_n] \in \mathbb{P}^n$.

Next we will define the corresponding relationship between affine space $\mathbb{A}^n$ and projective space $\mathbb{P}^n$, we can regard $\mathbb{P}^n$ as $\mathbb{A}^n$ add a point at ∞.

Def: Let $U_i = \{[a_0, ..., a_n] \in \mathbb{P}^n | a_i \neq 0\}, i = 0, 1, ..., n.$. Define a map $\phi_i : U_i \to \mathbb{A}^n, [a_0, ..., a_n] \longmapsto (\frac{a_0}{a_i}, ..., \hat{1}, ..., \frac{a_n}{a_i})$, where $\hat{1}$ means remove the coordinate at this place.$\frac{a_j}{a_i}$ is the **affine coordinate** of $[a_0, ..., a_n] \in \mathbb{A}^n$. Obviously, ϕ_i is a bijection and we can define the inverse map of $u_i : \mathbb{A}^n \to U_i, (b_0, ..., \hat{1}, ..., b_n) \longmapsto [b_0, ..., 1, ..., b_n]$.

We can use u_i to match the point $(a_0, ..., a_n) \in \mathbb{A}^n$ and the point $[1, a_1, ..., a_n] \in U_0$, so we can see $\mathbb{A}^n$ as a subset of $\mathbb{P}^n$. In addition, $\mathbb{P}^n = \mathbb{A}^n \cup H_\infty$, where $H_\infty = \mathbb{P}^n \setminus U_0 = \{[0, a_1, ..., a_n] \in \mathbb{P}^n\}$, called hyperplane at infinity.

If we want to define **projective algebraic set** as zero point set in $\mathbb{K}[x_0, ..., x_n]$, we will meet a problem. Because generally $g(a_0, .., a_n) \neg (\lambda a_0, ..., \lambda a_n), g \in \mathbb{K}[x_0, ..., x_n]$ is not the function defined on the $\mathbb{P}^n$. So we need to use homogeneous polynomial, i.e the polynomials which satisfy $g(\lambda a_0, ..., \lambda a_n) = \lambda^d g(a_0, .., a_n), \forall \lambda \in \mathbb{K}$ to solve these problems.

Def: Let $g \in \mathbb{K}[x_0, ..., x_n]$ is a homogeneous polynomial with degree d. Iff $g(a_0, ..., a_n) = 0, p = [a_0, ..., a_n]$ are zero points of g, write as $g(p) = 0$. Let $S \subseteq \mathbb{K}[x_1, ..., x_n]$ is a homogeneous polynomial set. The **projective zero set** of S is $Z(S) = \{p \in \mathbb{P}^n | f(p) = 0, \forall p \in S\}$. To show the difference from affine algebraic set, we write projective algebraic set as $Z_p(S)$,write affine algebraic set as $Z_\alpha(S)$. If $f \in \mathbb{K}[x_0, ..., x_n]$ is a homogeneous polynomial with degree $d > 0$, then $Z(f)$ called **hypersurface** defined by f.

It's similar with affine algebraic set, every projective algebraic set are ideals' zero set.

Def: For all polynomial $f \in \mathbb{K}[x_0, ..., x_n]$ can be written as $f = f^{\{}(1)\} + ... + f^{\{}(d)\}$, where $f^{\{}(i)\}$ is a polynomial with degree i. $f^{\{}(i)\}$ are homogenous component of f. For a ideal $I \subseteq \mathbb{K}[x_0, ..., x_n]$,if for all $f \in I$, which homogenous component $f^{\{}(i)\} \in I, \forall i \in \{1, ..., d\}$, then I is **homogenous ideal**.

It's not hard to find that a ideal $I \subseteq \mathbb{K}[x_0, ..., x_n]$ is a homogenous ideal is equal to I generated by homogenous polynomial. So next we can give the definition of zero set of the homogenous ideal and the homogenous ideal of subset of $\mathbb{P}^n$.

Def: For a homogenous ideal $I \subseteq \mathbb{K}[x_0, ..., x_n]$, the zero set is $Z(I) = \{p \in \mathbb{P}^n | f(p) = 0, \forall f \in I, f homogenous.\}$. For $X \subseteq \mathbb{P}^n$,has homogenous ideal is $I(X) =< f \in \mathbb{K}[x_0, ..., x_n] homopoly | f(p) = 0, \forall p \in X >$, we usually write projective algebraic set's homogenous ideal as $I_H(X)$.

Similar with affine algebraic set, we also can define a topology on projective space $\mathbb{P}^n$ by defining projective algebraic set as closed set, this topology called Zariski topology on the $\mathbb{P}^n$. Moreover, $\mathbb{K}[x_0, ..., x_n]$ is a noether ring, and the closed set on $\mathbb{P}^n$ is the zero set of homogenous ideal, $\mathbb{P}^n$ and subspace are noetherian space. Especially, every open subset of a projective algebraic set can be seperated as irreducible component.

Def: A irreducible projective algebraic set is projective variety. A quasiprojective variety is a open subset of a projective variety.

1.2.2 Projective Nullstellensatz

Now we will give the relation between the projective algebraic set and homogenous ideals. We can apply affine nullstellensatz in the proof of projective nullstellensatz by using cone.

Def: For a non-empty affine algebraic set $X \subseteq \mathbb{A}^{n+1}$, if $\forall\lambda \in \mathbb{K}$,$\forall p = (a_0, ..., a_n) \in X, \lambda p = (\lambda a_0, ..., \lambda a_n) \in X$, then X is called cone. i.e, X contains all the lines which through 0 and p. For a projective algebraic set $X \subseteq \mathbb{P}^n$, its affine cone is $C(X) = \{(a_0, ..., a_n) \in \mathbb{A}^{n+1} | [a_0, ..., a_n] \in X\} \cup \{0\}$. Affine cone is obviously a cone.

From definition, if $X \subseteq \mathbb{P}^n$ is a non-empty projective algebraic set, then $I(C(X)) = I_H(X)$. In addition, if $X = Z_p(I)$, where I is a homogenous ideal, then $C(X) = Z_a(I)$. i.e, non-empty projective algebraic set has a one-to-one corresponding with its' affine cone, they connected by the same homogenous ideal.

Next we will introduce the Projective Nullstellensatz theorem and collary.

Theorem(Projective Nullstellensatz Theorem):Let $I \subseteq \mathbb{K}[x_0, ..., x_n]$ is a homogenous ideal.Then:
(1): $Z_p(I) = \emptyset$ is equivalence to I contains all the homogenous polynomial with degree N, where $N \in \mathbb{N}$.
(2): If $Z_p(I) \neq \emptyset$, then $I_H(Z_p(I)) = \sqrt{I}$.
Proof: (1):$Z_p(I) = \emptyset$ is equal to $C(Z_p(I)) = Z_a(I)$ is $\emptyset$ or 0, so $I = \mathbb{K}[x_0, ..., x_n]$. From Nullstellensatz Theorem, this means $\sqrt{I} = I(Z_a(I)) = I(C(Z_p(I))) \supseteq < x_0, ..., x_n >$. Then for every $i \in \{1, ..., n\}$, exist j_i, such that $x^{j_i} \in I$. If let $N = \sum_{i=0}^{n} j_i$, I contains all the homogenous polynomial with degree N.
(2):From Nullstellensatz Theorem, $\sqrt{I} = I(Z_a(I)) = I(C(Z_p(I))) = I_H(Z_p(I))$.$\square$

From the Projective Nullstellensatz Theorem, we know that the map I_H and Z_p can build a bijection that can be inverse each other between the set of algebraic set in $\mathbb{P}^n$ and the homogenous radical ideal which not equal to $< x_0, ..., x_n >$ in $\mathbb{K}[x_0, ..., x_n]$. In addition, it's same in the affine , there exists a corresponding relations between irreducible projective algebraic set and homogenous prime ideal.

Collary(1): For projective algebraic set $X \subseteq \mathbb{P}^n$, iff when $I_H(X)$ is a prime ideal, X is a projective variety. i,e, if and only if when $X = Z_p(I)$, where I is a prime ideal, X is a projective variety. **(2)**:If and only if when $f \in \mathbb{K}[x_0, ..., x_n]$ is a irreducible polynomial, hypersurface $Z_p(f)$ is irreducible.

From now, we introduced the corresponding relations between algebraic set and ideals in affine space and projective space, which gives the basic relationship between geometric and algebra.

1.2.3 Solutions for 1.2

2.1:Let $a \subset S$ be a homogenous ideal, $f \in S$ is a homogenous polynomial with $degf > 0$, s.t $f(P) = 0, \forall p \in Z(a)$, then $(a_0, ..., a_n) \in \mathbb{P}^n$ is a zero set of f iff $(a_0, ..., a_n) \in \mathbb{A}^{n+1}$ is a zero set of affine space's $f : \mathbb{A}^{n+1} \longmapsto k$. From the Nullstellensatz in affine space, we have $f \in \sqrt{a}$, i.e, $\exists q > 0$, s.t $f^q \in a$.

2.2: $(i) \Rightarrow (ii)$: Let $a \in k[x_0, ..., x_n]$ be a homogenous ideal, $Z(a) = \emptyset$, then for every $f \in S$ with $degf > 0$, we can induce $f \in \sqrt{a}$ and so $\oplus_{d>0} S_d \subseteq \sqrt{a}$. Then, if $a \cap S_0 \neq \emptyset$, we must have $a \cap S_0 = S_0$, since all non-zero elements of $S_0 = k$ are invertible and a is an ideal. Thus, we have either $\sqrt{a} = S$ or $\sqrt{a} = \oplus_{d>0} S_d$.
$(ii) \Rightarrow (iii)$: If $\sqrt{a} = S$, then $1 \in a$, then $a = S$ contains all the S_d. Suppose that $\sqrt{a} = S_+$, for $i = 0, ..., n$, we have that $x_i^{d_i} \in a$ for some $d_i > 0$. Let $d = max_i d_i$, we have $a \supseteq S_{nd}$, $\because x_0^{k_0}, ..., x_n^{k_n} \in S_{nd} \Rightarrow k_0 + ...k_n = nd \Rightarrow \exists i$, s.t $k_i >= d >= d_i$, so $x_i^{k_i} \in a$, then $a \supseteq S_n d$, so $a \supseteq S_d$, for some $d > 0$.
$(iii) \Rightarrow (i)$: Suppose that $a \subseteq S_d$, for some $d > 0$. And assume that exists a homogenous coordinate $P = (a_0, ..., a_n) \in Z(a)$, then every points in S_d must vanish at $(a_0, ..., a_n)$ as an ordinary function from $\mathbb{A}^{n+1}$ to k. Thus, each of $x_0^{k_0}, ..., x_n^{k_n} \in S_d$ vanish on $(a_0, ..., a_n)$, so that $a_0 = a_1 = a_2 = ... = a_n = 0$. But these are noe homogenous coordinate for any point in $\mathbb{P}^n$ so no such P can exist and so $Z(a) = \emptyset$.

2.3(a): If $T_1 \subseteq T_2$, if $P \in Z(T_2)$, then $P \in Z(T_1)$, because P vanish at T_2, so P is also vanish at T_1, so $Z(T_1) \subseteq Z(T_2)$.
2.3(b): If $Y_1 \subseteq Y_2$, then $f \in I(Y_1)$, $\because Y_1 \subseteq Y_2$, so f is vanish at every $P \in Y_1$, then $f \in I(Y_1)$, we have $I(Y_2) \subseteq I(Y_1)$.
2.3(c): $\forall f \in I(Y_1 \cup Y_2)$, then $P \in Y_1 \cup Y_2, f(P) = 0$, then $P \in Y_1, f(P) = 0$; $P \in Y_2, f(P) = 0$. So $f \in I(Y_1) \cap I(Y_2)$, $I(Y_1) \cap I(Y_2) \subseteq I(Y_1 \cup Y_2)$. it's similar to prove the inverse $I(Y_1 \cup Y_2) \subseteq I(Y_1) \cap I(Y_2)$, so we have $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.
2.3(d): let $a \subseteq S$ is a homogenous ideal with $Z(a) \neq \emptyset$, from (2.1), we can induce $I(Z(a)) \subseteq \sqrt{a}$.Next, to prove the inverse $I(Z(a)) \supseteq \sqrt{a}$, for some $r > 0, f \in a, f^r \in a$, then $\forall P \in Z(a)$, then $f(P) = 0$ and $f^r(P) = 0$, so, $f \in I(Z(A))$, $I(Z(a)) \supseteq \sqrt{a}$.
2.3(e): It's obviously to know that $Z(I(Y))$ ia a closed set containing Y, so $\bar{Y} \subseteq Z(I(Y))$, then to prove $\bar{Y} \supseteq Z(I(Y))$, let $P \notin \bar{Y}$, then $\bar{Y} \subseteq (\bar{Y} \cup \{P\})$, then $I(\bar{Y}) \subset I((\bar{Y} \cup \{P\})$. So there is a homogenous polynomial f vanishing on $\bar{Y}$, but not at P. Thus $P \notin Z(I(Y))$, so $Z(I(Y)) \subseteq Y$.

2.4(a): In fact, (2.3) has proved this.
2.4(b): $\rightarrow$: Y is an irreducible algebraic set in $\mathbb{P}^n$ and $fg \in I(Y)$ for two homogenous elements $f, g \in S$. Then we have $Y \subseteq Z(fg) = Z(f) \cup Z(g)$, we can implies $Y = (Z(f) \cap Y) \cup (Z(g) \cap Y)$, because Y is irreducible, we just have $Y = Z(f) \cap Y$ or $Y = Z(g) \cap Y$, so $Y \subseteq Z(f)$ or $Y \subseteq Z(g)$. Thus $f \in I(Y)$ or $g \in I(Y)$.
:$\leftarrow$: let p be a homogenous prime ideal and suppose $Z(p) = Y_1 \cup Y_2$, for two closed set Y_1, Y_2. Because p is prome, so $p \subseteq I(Y_1)$ or $p \subseteq I(Y_2)$, which means

that either $Z(p) = Y_1$,or $Z(p) = Y_2$.
2.4(c): $Z(0) = \mathbb{P}^n$, because (0) is prime ideal, so from (b), $Z(0) = \mathbb{P}^n$ is irreducible.

2.5(a): If $Y_1 \subseteq Y_2 \subseteq ...$ is a descending chain of closed sets in $\mathbb{P}^n$ then we have $I(Y_1) \supseteq I(Y_2) \subseteq ...$ is an ascending chain of ideals in S, which is a Noetherian ring, so it's stabilizes and it's also stable in $\mathbb{P}^n$.
2.5(b): The prove is as same as (1.5)

2.6:
2.7(a): $\mathbb{P}^n$ is irreducible, and $\mathbb{P}^n$ is covered by the open sets $U_I = \mathbb{P}^n - H_i$, H_i is the zero set of $X_i \in S$. Therefore, from (1.10b) $dim\mathbb{P}^n = sup_i dimU_i$, because $dimU_i = dim\mathbb{A}^n$, for all i, and because ϕ_i is a homeomorphism, for each i, $sup_i dimU_i = dimU_i$. From prop1.9, we have $dim\mathbb{P}^n = n$.
2.7(b): Let $Y \subseteq \mathbb{P}^n$ ba a quasi-projective variety and let $Y_i = \phi_i(Y \cap U_i)$, which is not empty for some i. Then there is a projective variety W, s.t $Y \subseteq W$, hence Y_i is quasi-projective variety. Because $Y_i = \phi(Y \cap U_i) \subseteq \phi_i(Y \cap U_i)$, the latter being an affine variety as in (2.6), the former an open set since ϕ_i is a homeomorphism. Therefore, we have $dimY = sup_i dimY \cap U_i = sup_i dimY_i = sup_i \bar{Y} \cap U_i$. And let $\bar{Y} = \phi_i(\bar{Y} \cap U_i)$, because ϕ_i is a homeomorphism, so $dim\bar{Y}_i = dim\bar{Y} \cap U_i$, so $dim\bar{Y}_i = dim\bar{Y} \cap U_i = dim\bar{Y}$. Finally, we have $dimY = dimS(Y) - 1 = dimA(Y_i) = dimY_i = dim\bar{Y}_i = dim\bar{Y} \cap U_i = dim\bar{Y}$.

2.8: $\Rightarrow$: Let $Y \subseteq \mathbb{P}^n$ with $dimY = n - 1$, then $P = I(Y)$ is prime and $n = dimY + 1 = dimS(Y) = n + 1 - ht(p) \Rightarrow ht(p) = 1$, from (1.12A), $p = (f)$, so f is irreducible.
$\Leftarrow$: $Y = Z(f)$, f is an irreducible homogenous polynomial with positive degree. $\because S$ is a UFD we can implies that (f) is prime, from (2.4C), $Y = Z(F)$ is irreducible and closed in Zarisky top., which means $Y = Z(F)$ is a projective variety. By (2.6), $dimY + 1 = dimS(Y) = dimS/(f) = dimS - ht(f) = n + 1 - ht(f)$, from Th11A, $ht(f) = 1$, so $dimY = n + 1 - 1 - 1 = n - 1$.

2.9(a): Let $F[x_0, ..., x_n] \in I(\bar{Y})$, then $f_i = \phi_i(F) = F(1, x_1, ..., x_n)$ vanishes on $Y \subseteq \mathbb{A}^n$, the affine piece of $\mathbb{P}^n$ defined by $x_0 = 1$, so $f \in I(Y)$ and clearly $\beta(f) = F$, so $F \in \beta(I(Y))$, so $(\beta(I(Y))) \subseteq I(\bar{Y})$ and similarly, we can have $I(\bar{Y}) \subseteq (\beta(I(Y)))$. So, $I(\bar{Y}) = (\beta(I(Y)))$.
2.9(b): From(1.2), $I(Y) = (y - x^2, z - x^3)$ and $\beta(y - x^2) = wy - x^2$,$\beta(z - x^3) = wz - x^3$, $I(\bar{Y}) = (wy - x^2, wz - x^3, wz^2 - y^3)$. Note that $wz^2 - y^3$ not generated by the first two generators because we couldn't get y^3.

2.10(a):
2.10(b):
2.10(c):

2.11(a):
2.11(b):

2.11(c):

1.3 Morphism

1.3.1 Regular function

In the last two sections we have discussed the affine and projective varieties, now we will discuss the map between them. In this section we define a regular functions on a variety and then define a morphism of varieties, which need we have a good category.

Let Y be a quasi-affine variety in $\mathbb{A}^n$, we will consider functions f from Y to k.

Def: A function $f : Y \longmapsto k$ is regular at a point $P \in Y$ if there is an open neighborhood U with $P \in U \subseteq Y$, and polynomials $g, h \in A = k[x_1, ..., x_n]$, s.t h is nowhere zero on U, and $f = \frac{g}{h}$ on U. (Here of course we interpret the polynomials as functions on $\mathbb{A}^n$, hence on Y). We say that f is regular on Y if it is regular at every point of Y.
f can be written as g/h.

Def: A regular funtion is continous, when k is identified with A_1^k in its Zariski topology.
k is a Zariski topology with one dim over k. Hint: we need to show that $f^{-1}(a)$ is closed in Y.

Now let us consider a quasi-projective variety $Y \subseteq \mathbb{P}^n$. It's similar with affine, we just use homogenous polynomial to replace polynomials in definition.

Def: A function $f : Y \longmapsto k$ is regular at a point $P \in Y$ if there is an open neighborhood U with $P \in U \subseteq Y$, and homogenous polynomials $g, h \in A = k[x_1, ..., x_n]$, of the same degree, s.t h is nowhere zero on U, and $f = \frac{g}{h}$ on U. (Note that in this case, even though g and h are not functions on $\mathbb{P}^n$, their quotient is a well-defined function whenever $h \neq 0$, since they are homogenous of the same degree.) We say that f is regular on Y if it is regular at every point of Y.

Remark: As in the quasi-affine case, a regular function is necessarily continuous. An important consequence of this is the fact that if f and g are regular functions on a variety X, and if $f = g$ on some nonempty open subset $U \subseteq X$, then $f = g$ everywhere. Indeed, the set of points where $f - g = 0$ is closed and dense, hence equal to X.

Next we can define the category of varieties.

Def: Let k be a fixed algebraically closed field. A variety over k(or simply variety) is any affine, quasi-affine,projective, or quasi-projective variety as defined above. If X, Y are two varieties, a morphism $\phi : X \to Y$ is a continuous map such that for every open set $V \subseteq Y$, and for every regular function $f : V \to k$, the function $f \circ \phi : \phi^{-1}(V) \to k$ is regular.

Clearly the composition of two morphism is a morphism, so we have a category. In particular, we have the notion of isomorphism: an isomorphism $\phi : X \to Y$ of two varieties is a morphism which admits an inverse morphism $\psi : Y \to X$ with $\psi \circ \phi = id_X$ and $\phi \circ \psi = id_Y$. Note that an isomorphism is necessarily bijective and bicontinuous, but a bijective bicontinuous morphism need not be an isomorphism.

1.3.2 $\mathcal{O}(Y)$, $\mathcal{O}_P$, and $K(Y)$

Now we introduce some rings of functions associated with any variety.

Def: Let Y be a variety. We denote by $\mathcal{O}(Y)$ the ring of all regular functions on Y. If P is a point of Y, we define the local ring of P on Y, $\mathcal{O}_{P,Y}$(or simply $\mathcal{O}_P$ to be the ring of germs of regular functions on Y near P. In other words, an element of O_P is a pair of $< U, f >$ where U is an open subset of Y containing P, and f is regular function on U, and where we identify two such pairs $< U, f >$ and $< V, g >$ if $f = g$ on $U \cap V$.

Note that $\mathcal{O}_P$ is indeed a local ring: its maximal ideal m is the set of germs of regular functions which vanish at P. For if $f(P) \neq 0$, then $1/f$ is regular in some neighborhood of P. The residue field $\mathcal{O}_P/m$ is isomorphic to k.

Def: If Y is a variety, we define the function field $K(Y)$ of Y as follows: an element of $K(Y)$ is an equivalence class of pair $< U, f >$ where U is a nonempty open subset of Y, f is a regular function on U, and where we identify two pairs $< U, f >$ and $< V, g >$ if $f = g$ on $U \cap V$. The elements of $K(Y)$ are called rational functions on Y.
From the definition of the $\mathcal{O}_P$, We can naturally give the definition of function field $K(Y)$ by the equivalence of the $< U, f >$ and $< V, g >$.(germ)

The inverse of $< U, f >$ in $K(Y)$: Note that $K(Y)$ is in fact a field. Since Y is irreducible, any two non-empty open sets have a nonempty intersection. Hence we can define addition and multiplication in $K(Y)$, making it a ring. Then if $< U, f >\in K(Y)$ with $f \neq 0$, we can restrict f to the open set $V = U - U \cap Z(f)$ where it never vanishes, so that $1/f$ is regular on V, hence $< V, 1/f >$ is an inverse for $< U, f >$.

Now we have defined, for any variety Y, the ring of global functions $\mathcal{O}(Y)$, the local ring $\mathcal{O}_P$ at a point of Y, and the function field $K(Y)$. By restricting functions we obtain natural maps $\mathcal{O}(Y) \to \mathcal{O}_P \to K(Y)$ which in fact are injective.(ex3.1.1). Hence we will usually $\mathcal{O}(Y)$ and $\mathcal{O}_P$ as subrings of $K(Y)$.

If we replace Y by an isomorphic variety, then the corresponding rings are isomorphic. Thus we can say that $\mathcal{O}(Y)$, $\mathcal{O}_P$, and $K(Y)$ are **invariants** of the variety Y (and the point P) up to isomorphism.

Our next task is to relate $\mathcal{O}(Y)$, $\mathcal{O}_P$, and $K(Y)$ to the affine coordinate ring $A(Y)$ of an affine variety, and the homogenous coordinate ring $S(Y)$ of a projective variety, which were introduced earlier. We will find that for an affine variety Y, $A(Y) = \mathcal{O}(Y)$, so it is an invariant up to isomorphism. However, for a projective variety Y, $S(Y)$ is not an invariant: it depends on the embedding of Y in projective space.

Theorem: Let $Y \subseteq \mathbb{A}^n$ be an affine variety with affine coordinate ring $A(Y)$. Then:
(a): $\mathcal{O}(Y) \cong A(Y)$;
(b): For each point $P \in Y$, let $m_p \subseteq A(Y)$ be the ideal of functions vanishing at P. Then $P \longmapsto m_p$ gives a 1-1 correspondence between the points of Y and the maximal ideals of $A(Y)$;
(c): For each P, $\mathcal{O}_p \cong A(Y)_{m_p}$, and dim$\mathcal{O}_p$ =dim$Y$;
(d): $K(Y)$ is isomorphic to the quotient field of $A(Y)$, and hence $K(Y)$ is a finitely generated extension field of k, of transcendence degree = dim$Y$.

Proposition: Let $U_i \subseteq \mathbb{P}^n$ be the open set defined by the equation $x_i \neq 0$. Then the mapping $\phi_i : U_i \longmapsto \mathbb{A}^n$ of (2.2) above is an isomorphism of varieties.

Before stating the next result, we introduce some notation. If S is a graded ring, and p a homogenous prime ideal in S, then we denote by $S_(p)$ the subring of elements of degree 0 in the localization of S with respect to the multiplicative subset T consisting of the homogenous elements of S not in p. Note that $T^{-1}S$ has a natural grading given by deg(f/g)=$degf - degg$ for f homogenous in S and $g \in T$. $S_(p)$ is a local ring, with maximal ideal $(p\dot{T}^{-1}S) \cap S_(p)$. Similarly, if $f \in S$ is a homogenous element, we denote by $S_(f)$ the subring of elements of degree in the localized ring S_f.

Theorem: Ley $Y \subseteq \mathbb{P}^n$ be a projective variety with homogenous coordinate ring $S(Y)$. Then:
(a): $\mathcal{O}(Y) = k$;
(b): For any point $P \in Y$, let $m_p \subseteq S(Y)$ be the ideal generated by the set of homogenous $f \in S(Y)$ such that $f(P) = 0$. Then $\mathcal{O}_p = S(Y)_{m_p}$;
(c): $K(Y) \cong S(Y)_{((0))}$

Our next result shows that if X and Y are affine varieties, then X is also morphic to Y if and only if $A(X)$ is isomorphic to $A(Y)$ as a k-algebra. Actually the proof gives more, so we state the stronger result.

Before we consider the proposition, we need to know a lemma, which shows that the map ψ below is a morphism.
Lemma: Let X be any variet, and let $Y \subseteq \mathbb{A}^n$ be an affine variety. A map of sets $\psi : X \longmapsto Y$ is a morphism if and only if $x_i \circ \psi$ is a regular function on X for each i, where $x_1, ..., x_n$ are the coordinate functions on $\mathbb{A}^n$.

Proposition: Let X be any variety and let Y be an affine variety. Then there is a natural bijection mapping of sets α : Hom$(X, Y) \xrightarrow{\sim}$ Hom$(A(Y), \mathcal{O}(X))$ where the left Hom means morphisms of varieties, and the right Hom means homomorphisms of k-algebras.

Corollary: If X, Y are two affine varieties, then X and Y are isomorphic if and only if $A(X)$ and $A(Y)$ are isomorphic as k-algebras.

In the language of categories, we can express the above results as follows:

Corollary: The functor $X \longmapsto A(X)$ induces an arrow-reversing equivalence of categories between the category of affine varieties over k and the category of finitely generated integral domains over k.

We include have an algebraic result which will be used in the exercises.

Theorem(Finiteness of Integral Closure): Let A be an integral domain which is a finitely generated algebra over a field k. Let K be the quotient field of A, and let L be a finite algebraic extension of K. Then the integral closure A' of A in L is a finitely generatrd $A - module$, and is also a finitely generated k-algebra.

1.3.3 Solutions for 1.3

1.4 Rational Maps