# Computing Radon points

The Radon point forms affine combinations with the point set.

- $$\sum_{i=0}^{d+1} p_i \alpha_i = 0 \quad \textit{and} \quad \sum_{i=0}^{d+1} \alpha_i = 0, \quad \alpha_i \in \mathbb{R}$$

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- $$\begin{cases} p_0^{(x)}\alpha_0 + p_1^{(x)}\alpha_1 + \ldots + p_{d+1}^{(x)}\alpha_{d+1} = 0 \\ p_0^{(y)}\alpha_0 + p_1^{(y)}\alpha_1 + \ldots + p_{d+1}^{(y)}\alpha_{d+1} = 0 \\ p_0^{(w)}\alpha_0 + p_1^{(w)}\alpha_1 + \ldots + p_{d+1}^{(w)}\alpha_{d+1} = 0 \\ \alpha_0 + \alpha_1 + \ldots + \alpha_{d+1} = 0 \end{cases}$$

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- In $\mathbb{E}^d$, $(d+1)$ equations in $(d+2)$ unknowns $\boxtimes$

# Computing Radon points

$$\boxed{\alpha_{(d+1)} = -1}$$

$$\begin{cases} p_0^{(x)}\alpha_0 + \ldots + p_d^{(x)}\alpha_d = p_{(d+1)}^{(x)} \\ p_0^{(y)}\alpha_0 + \ldots + p_d^{(y)}\alpha_d = p_{(d+1)}^{(y)} \\ p_0^{(w)}\alpha_0 + \ldots + p_d^{(w)}\alpha_d = p_{(d+1)}^{(w)} \\ \alpha_1 + \ldots + \alpha_d = 1 \end{cases}$$

In $\mathbb{E}^3$, four equations in four unknowns ☑

# Solving systems of linear equations

- $$\begin{bmatrix} p_0^{(x)} & p_1^{(x)} & \cdots & p_d^{(x)} \\ p_0^{(y)} & p_1^{(y)} & \cdots & p_d^{(y)} \\ p_0^{(w)} & p_1^{(w)} & \cdots & p_d^{(w)} \\ \cdots & \cdots & \ddots & \cdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_d \end{bmatrix} = \begin{bmatrix} p_{d+1}^{(x)} \\ p_{d+1}^{(y)} \\ p_{d+1}^{(w)} \\ \cdots \\ 1 \end{bmatrix}$$

  In symbolic form

  $$\mathbf{P}\underline{\alpha} = \underline{b}$$

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  In symbolic form

  $$\mathbf{P}\underline{\alpha} = \underline{b}$$

- Build the *augmented* matrix

  $$\mathbf{A} = [\mathbf{P}|\underline{b}]$$

# Augmented matrix

- $$\mathbf{A} = \left[\begin{array}{cccc|c} p_0^{(x)} & p_1^{(x)} & \cdots & p_d^{(x)} & p_{d+1}^{(x)} \\ p_0^{(y)} & p_1^{(y)} & \cdots & p_d^{(y)} & p_{d+1}^{(y)} \\ p_0^{(w)} & p_1^{(w)} & \cdots & p_d^{(w)} & p_{d+1}^{(w)} \\ \cdots & \cdots & \ddots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 & 1 \end{array}\right]$$

# Augmented matrix

- $$\mathbf{A} = \left[\begin{array}{cccc|c} p_0^{(x)} & p_1^{(x)} & \cdots & p_d^{(x)} & p_{d+1}^{(x)} \\ p_0^{(y)} & p_1^{(y)} & \cdots & p_d^{(y)} & p_{d+1}^{(y)} \\ p_0^{(w)} & p_1^{(w)} & \cdots & p_d^{(w)} & p_{d+1}^{(w)} \\ \cdots & \cdots & \ddots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 & 1 \end{array}\right]$$
- Convert to an *upper triangular* matrix.

$$\mathbf{C} = \left[\begin{array}{cccc|c} c_{00} & c_{01} & \cdots & c_{0d} & d_0 \\ 0 & c_{11} & \cdots & c_{1d} & d_1 \\ 0 & 0 & \cdots & c_{2d} & d_2 \\ \cdots & \cdots & \ddots & \cdots & \cdots \\ 0 & 0 & \cdots & c_{dd} & d_d \end{array}\right]$$

# Augmented matrix - simple row operations

- Start with bottom line and first column

$$\mathbf{C} = \left[\begin{array}{cccc|c} \dots & \dots & \ddots & \dots & \dots \\ c_{(d-1)0} & c_{(d-1)1} & \cdots & c_{(d-1)d} & d_{d-1} \\ c_{d0} & c_{d1} & \cdots & c_{dd} & d_d \end{array}\right] \begin{array}{c} \dots \\ L_{d-1} \\ L_d \end{array}$$

# Augmented matrix - simple row operations

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- $L'_{d-1} = c_{d0} L_{d-1}$
  $L'_d = c_{(d-1)0} L_d$
  $L_{d-1} = L'_{d-1}$
  $L_d = L'_d - L'_{d-1}$

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- Go up one line and repeat until you reach the main diagonal

# Augmented matrix - simple row operations

- Start with bottom line and first column

$$\mathbf{C} = \left[\begin{array}{cccc|c} \cdots & \cdots & \ddots & \cdots & \cdots \\ c_{(d-1)0} & c_{(d-1)1} & \cdots & c_{(d-1)d} & d_{d-1} \\ c_{d0} & c_{d1} & \cdots & c_{dd} & d_d \end{array}\right] \begin{array}{c} \cdots \\ L_{d-1} \\ L_d \end{array}$$

- $L'_{d-1} = c_{d0} L_{d-1}$
  $L'_d = c_{(d-1)0} L_d$
  $L_{d-1} = L'_{d-1}$
  $L_d = L'_d - L'_{d-1}$
- Go up one line and repeat until you reach the main diagonal
- Then move to the next column and do the same

# Algorithm

```
//Build upper triangular matrix
for (char k=0; k<4; k++){
 for (char i=3; i>k; i--){
  //Store pivot
  double dPivL = aug[i][k];
  double dPivU = aug[i-1][k];
  for (char j=k; j<5; j++){
   //Multiply entire row by Upper pivot
   aug[i][j] = aug[i][j]*dPivU;
   //Multiply entire row by Lower pivot
   aug[i-1][j] = aug[i-1][j]*dPivL;
   //Subtract line above from curent line
   aug[i][j] = aug[i][j] - aug[i-1][j];
  }
 }
}
```

# Solve the system

- Get back to original system

$$\mathbf{C} = \begin{bmatrix} c_{00} & c_{01} & \cdots & c_{0d} \\ 0 & c_{11} & \cdots & c_{1d} \\ 0 & 0 & \cdots & c_{2d} \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \cdots & c_{dd} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \dots \\ \alpha_d \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \dots \\ d_d \end{bmatrix}$$

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- Calculate $\alpha_d = \dfrac{d_d}{c_{dd}}$

# Solve the system

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$$\mathbf{C} = \begin{bmatrix} c_{00} & c_{01} & \cdots & c_{0d} \\ 0 & c_{11} & \cdots & c_{1d} \\ 0 & 0 & \cdots & c_{2d} \\ \ldots & \ldots & \ddots & \ldots \\ 0 & 0 & \cdots & c_{dd} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \ldots \\ \alpha_d \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \ldots \\ d_d \end{bmatrix}$$

- Calculate $\alpha_d = \dfrac{d_d}{c_{dd}}$
- Calculate the rest using $\alpha_i = \dfrac{1}{c_{ii}}\left(d_i - \sum_{j=i+1}^{d} c_{ij}\alpha_j\right)$, where $i$ ranges from $d-1$ down to 0.

# Solver algorithm

# Compute the Radon point

$$\mathcal{S}_+ := \{i : \alpha_i \geqslant 0\} \text{ and } \mathcal{S}_- := \{j : \alpha_j < 0\}$$

$$c = \sum_{i \in \mathcal{S}_+} \alpha_i = -\sum_{j \in \mathcal{S}_-} \alpha_j$$

$$O_P = \sum_{i \in \mathcal{S}_+} \frac{p_i \alpha_i}{c} = -\sum_{j \in \mathcal{S}_-} \frac{p_j \alpha_j}{c}$$

### This is a Block

This is important information

### This is an Alert block

This is an important alert

### This is an Example block

This is an example