(18 points) Question 1:

Suppose $E(X) = 2$, $Var(X) = 9$, $E(Y) = 0$, $Var(Y) = 4$, and $Corr(X,Y) = 0.25$. Find:
(a) $Var(X + Y)$.
(b) $Cov(X, X + Y)$.
(c) $Corr(X + Y, X - Y)$.

a) Using the definition of variance. we have:

$$Var(X+Y) = E[[X+Y-E(X+Y)]^2]$$
$$= E[(X-\mu_X+Y-\mu_Y)^2]$$
$$= E[(X-\mu_X)^2+(Y-\mu_Y)^2+2(X-\mu_X)(Y-\mu_Y)]$$

Now, applying the definition of variance and covarice, which is:

$$\begin{cases} Var(X) = E(X-\mu_X)^2 \\ Var(Y) = E(Y-\mu_Y)^2 \\ Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] \end{cases}$$

we have:

$$Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X,Y)$$

According to the Correlation function:

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}$$

we have :

$$Cov(X,Y) = Corr(X,Y) \cdot \sqrt{Var(X)\,Var(Y)}$$

Thus,

$$\begin{aligned} Var(X+Y) &= Var(X) + Var(Y) + 2 \cdot Corr(X,Y) \cdot \sqrt{Var(X)\,Var(Y)} \\ &= 9 + 4 + 2 \times 0.25 \times \sqrt{9 \times 4} \\ &= 16 \end{aligned}$$

b)

$$\begin{aligned} Cov(X, X+Y) &= E[(X-E(X))(X+Y-E(X+Y))] \\ &= E[(X-\mu_X)(X+Y-(\mu_X+\mu_Y))] \\ &= E[(X-\mu_X)(X-\mu_X+Y-\mu_Y)] \\ &= E[(X-\mu_X)^2 + (X-\mu_X)(Y-\mu_Y)] \\ &= E[(X-\mu_X)^2] + E[(X-\mu_X)(Y-\mu_Y)] \\ &= Var(X) + Cov(X,Y) \\ &= Var(X) + Corr(X,Y) \cdot \sqrt{Var(X)\,Var(Y)} \\ &= 9 + 0.25 \times \sqrt{9 \times 4} \\ &= 10.5 \end{aligned}$$

c)

$$Corr(X+Y, X-Y) = \frac{Cov(X+Y, X-Y) \ ①}{\sqrt{Var(X+Y)\,Var(X-Y)} \ ②}$$

①

$$\begin{aligned} Cov(X+Y, X-Y) &= E[(X+Y-(\mu_X+\mu_Y))(X-Y-(\mu_X-\mu_Y))] \\ &= E[[(X-\mu_X)+(Y-\mu_Y)][(X-\mu_X)-(Y-\mu_Y)]] \\ &= E[(X-\mu_X)^2 - (Y-\mu_Y)^2] \\ &= E[(X-\mu_X)^2] - E[(Y-\mu_Y)^2] \end{aligned}$$

$= Var(X) - Var(Y)$

$= 9 - 4$

$= 5$

② $Var(X+Y) = Var(X) + Var(Y) + 2 \cdot Corr(X,Y) \cdot \sqrt{Var(X)\,Var(Y)}$

(From a1)

$= 16$

Similarly,

$Var(X+Y) = Var(X) + Var(Y) - 2\,Cov(X,Y)$

$= Var(X) + Var(Y) - 2 \cdot Corr(X,Y) \cdot \sqrt{Var(X)\,Var(Y)}$

$= 9 + 4 - 2 \times 0.25 \times \sqrt{9 \times 4}$

$= 10$

Thus,

$$Corr(X+Y, X-Y) = \frac{5}{\sqrt{16 \times 10}}$$

$= 0.395$

(6 points) Question 2:

If X and Y are dependent but $Var(X) = Var(Y)$, find $Cov(X + Y, X - Y)$.

$$Cov(X+Y, X-Y) = E[(X+Y-(\mu_X+\mu_Y))(X-Y-(\mu_X-\mu_Y))]$$

$$= E[[(X-\mu_X)+(Y-\mu_Y)][(X-\mu_X)-(Y-\mu_Y)]]$$

$$= E[(X-\mu_X)^2 - (Y-\mu_Y)^2]$$

$$= E[(X-\mu_X)^2] - E[(Y-\mu_Y)^2]$$

$$= Var(X) - Var(Y)$$

$= 0$

(18 points) Question 3:

Suppose $Y_t = 5 + 2t + X_t$, where $\{X_t\}$ is a zero-mean stationary series with autocovariance function γ_k.

(a) Find the mean function for $\{Y_t\}$.

(b) Find the autocovariance function for $\{Y_t\}$.

(c) Is $\{Y_t\}$ stationary? Why or why not?

a) $E(Y_t) = E(5+2t+X_t)$

$= E(5)+E(2t)+E(X_t)$

$= 5+2t+E(X_t)$

$= 5+2t+0$

$= 5+2t$

b) $Cov(Y_t, Y_{t-k}) = Cov(5+2t+X_t, 5+2(t-k)+X_{t-k})$

$= Cov(X_t, X_{t-k})$

$= \gamma_k$

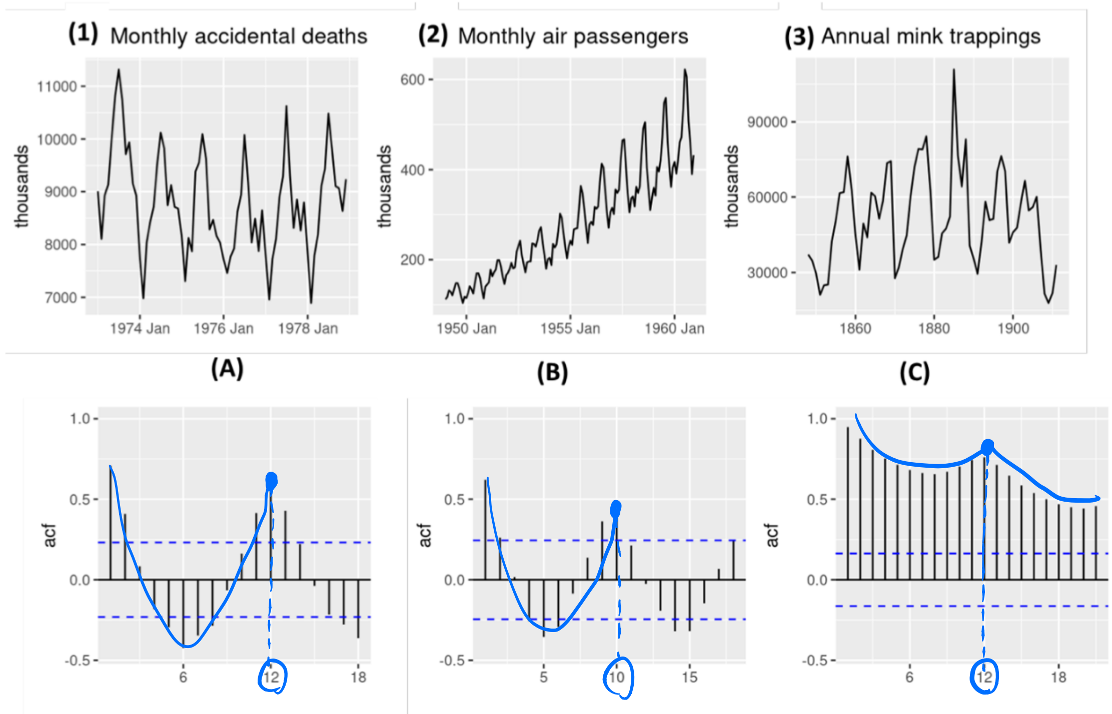
c) Stationary data can be defined as:

- mean is constant overtime
- for each k, the autocovariance $\gamma_Y(t+k, t)$ is independant of t.

Since $E(Y_t) = 5+2t$, which is not constant over time t, $\{Y_t\}$ is not stationary.

(18 points) Question 4:

The following time plots and ACF plots correspond to three different time series. Your task is to match each time plot in the first row with one of the ACF plots in the second row.



(1) - (A)

For monthly data, the ACF value will be seen at lag 12 and possibly also at lag 24, 36 ...
Also, it can be seen as a stationary data. Thus, the ACF decays quickly enough with increasing lags.

(2) - (C)

For seasonal monthly data, the ACF value will be seen at lag 12 and possibly also at lag 24, 36 ...
Also, (2) time plot shows clear trend and seasonality, which means it is the non-stationary data.
The slowly decaying and always positive ACF from (C) indicates this trend.

(3) - (B)

It represents annual data, thus the ACF value should not have clear patten every lag 12 as shown in A and C.