Q_1: $f(\lambda) = \det(A - \lambda I_d)$

$$f(\lambda) = \det\left(\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} 5-\lambda & 1 \\ 1 & 5-\lambda \end{pmatrix} = \lambda^2 - 10\lambda + 24 = (\lambda-4)(\lambda-6)$$

Q_2: $f(\lambda) = \det\begin{pmatrix} 5-\lambda & 1 \\ 1 & 5-\lambda \end{pmatrix}$

$$= \lambda^2 - 10\lambda + 24$$

$$\therefore \begin{cases} \lambda_1 = 6 \\ \lambda_2 = 4 \end{cases}$$

Q_3: Since we know that a square matrix is invertible if and only if it does not have a zero eigenvalue, and $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, we conclude that matrix A is invertible.

Q_4: $A = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$

$\lambda_1 = 6$

$A - \lambda_1 I = \begin{pmatrix} 5-6 & 1 \\ 1 & 5-6 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

$\therefore E_{\lambda_1} = \text{span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$

$\therefore \hat{v}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\lambda_2 = 4$

$A - \lambda_2 I = \begin{pmatrix} 5-4 & 1 \\ 1 & 5-4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$\therefore E_{\lambda_2} = \text{span}\left(\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$

$\therefore \hat{v}_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Q_5: $V = \begin{pmatrix} | & | \\ \hat{v}_1 & \hat{v}_2 \\ | & | \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix}$

$V^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$\therefore VDV^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$= \begin{pmatrix} 3\sqrt{2} & -2\sqrt{2} \\ 3\sqrt{2} & 2\sqrt{2} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$= \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} = A$

$\therefore A = VDV^T$

Q6: $tr(A) = 5+5 = 10$

$\lambda_1 + \lambda_2 = 6+4 = 10$

$\therefore tr(A) = \lambda_1 + \lambda_2$

Q7: $det(A) = 5\times5 - 1\times1 = 24$

$\lambda_1 \times \lambda_2 = 6\times4 = 24$

$\therefore det(A) = \lambda_1 \times \lambda_2$

Q8: $f(0) = \lambda^2 - 10\lambda + 24$

$= 0^2 - 0 + 24$

$= 24$

$det(A) = 24$

$\therefore f(0) = det(A)$

I asked Qi Ma for Question 4.