Projection method for Maxwell equation (Mixed finite element method)

Jiang Guo

Nankai University,China

E-mail:jiangliuer0705@gmail.com

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Outline

curl-curl problem

Equation

Let $\Omega \subset \mathbb{R}^d (d = 2, 3)$ be an open,bounded domain with boundary $\Gamma = \partial\Omega$. We consider the following Maxwell curl-curl problem:

$$\nabla \times \nabla \times \mathbf{u} - \omega^2 \mathbf{u} = \mathbf{f}, \quad in\ \Omega \tag{1}$$

$$\nabla \cdot \mathbf{u} = g, \quad in\ \Omega \tag{2}$$

$$\nabla \times \mathbf{n} = \mathbf{0}, \quad in\ \Gamma \tag{3}$$

Hilbert Space

$$H_0(\nabla\times;\Omega) = \{\mathbf{v} \in (L^2(\Omega))^d : \nabla \times \mathbf{v} \in (L^2(\Omega))^{2d-3}, \mathbf{v} \times \mathbf{n} = 0\}$$

$$\mathcal{U} = \mathbb{C}(\overline{\Omega})^d \cap H_0(\nabla\times;\Omega) \cap H(\nabla\cdot;\Omega)$$

curl-curl problem

Classical Variational Form

Find $\mathbf{u} \in \mathcal{U}$ such that

$$(\nabla\times\mathbf{u}, \nabla\times\mathbf{v})+(\nabla\cdot\mathbf{u}, \nabla\cdot\mathbf{v})-\omega^2(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})+(g, \nabla\cdot\mathbf{v}), \qquad \forall\, \mathbf{v} \in \mathcal{U}$$

Definition of Projection

$$Q_h = \{g \in H_0^1(\Omega) : q|_T \in P_1(T), \forall T \in \mathcal{T}_h\}$$

$$W_h = \{w \in (H^1(\Omega) \cap L_0^2(\Omega))^{2d-3} : w|_T \in (P_1(T))^{2d-3}, \forall T \in \mathcal{T}_h\}$$

$$\left\{ \begin{array}{c} (\breve{R}_h(\nabla\cdot\mathbf{v}), q)_{0,h} = -(\mathbf{v}, \nabla q), \quad \forall q \in Q_h \\ \breve{R}_h(\nabla\cdot\mathbf{v}) \in Q_h \end{array} \right. \tag{4}$$

$$\left\{ \begin{array}{c} (R_h(\nabla\times\mathbf{v}), w)_{0,h} = (\mathbf{v}, \nabla\times w), \quad \forall w \in W_h \\ R_h(\nabla\times\mathbf{v}) \in W_h \end{array} \right. \tag{5}$$

curl-curl problem

Projection FEM Variational Form

Find $\mathbf{u}_h \in \mathcal{U}_h$, such that

$$B_h(\mathbf{u}_h, \mathbf{v}_h) = L_h(\mathbf{v}_h), \forall\ \mathbf{v}_h \in \mathcal{U}_h \tag{6}$$

where

$$\mathcal{U}_h = \left\{\mathbf{v} \in \mathcal{U} : \mathbf{v}|_T \in \left(P_1^+(T)\right)^d, \forall T \in \mathcal{T}_h\right\}(Mini\ \ element)$$

$$\begin{aligned} B_h(\mathbf{u}_h, \mathbf{v}_h) &:= \left(R_h(\nabla \times \mathbf{u}_h), R_h(\nabla \times \mathbf{v}_h)\right) \\ &\qquad + \left(\check{R}_h(\nabla \cdot \mathbf{u}_h), \check{R}_h(\nabla \cdot \mathbf{v}_h)\right) - \ \omega^2(\mathbf{u}_h, \mathbf{v}_h) \\ L_h(\mathbf{v}_h) &:= (\mathbf{f}, \mathbf{v}_h) + \left(g, \check{R}_h(\nabla \cdot \mathbf{v}_h)\right) \end{aligned}$$

curl-curl problem

Implementation of Projection by Mixed FEM

Let $\mathcal{M}_h = \mathcal{U}_h \times Q_h \times W_h$, find $(\mathbf{u}_h, p_h, \xi_h) \in \mathcal{M}_h$ such that $\forall(\mathbf{v}_h, q_h, w_h) \in \mathcal{M}_h$, we have the following equations

1. $(p_h, q_h)_{0,h} + (\mathbf{u}_h, \nabla q_h) = 0$
2. $(\xi_h, w_h)_{0,h} - (\mathbf{u}_h, \nabla \times w) = 0$
3. $(\nabla \times p_h, \mathbf{v}_h) - (\nabla \xi_h, \mathbf{v}_h) - \omega^2(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) + (Sg, \nabla \cdot \mathbf{v}_h)$

where

Sg is the projection of g onto the space of Q_h, namely, find $Sg \in Q_h$ such that

$$(Sg, q_h)_{0,h} = (g, q_h), \quad \forall q_h \in Q_h$$

Note1: To simplify, we sometimes omit $(.,.)_{0,h}$.

curl-curl-grad-div problem

Equation

Let $\Omega \subset \mathbb{R}^d (d = 2, 3)$ be an open,bounded domain with boundary $\Gamma = \partial\Omega$. We consider the following Maxwell curl-curl-grad-div problem:

$$\begin{aligned} \nabla \times \nabla \times \mathbf{u} - \nabla\nabla \cdot \mathbf{u} - \omega^2 \mathbf{u} &= \mathbf{f}, \qquad \textit{in } \Omega && (7)\\ \nabla \times \mathbf{n} &= \mathbf{0}, \qquad \textit{in } \Gamma && (8)\\ \nabla \cdot \mathbf{u} &= 0, \qquad \textit{in } \Gamma && (9) \end{aligned}$$

Classical Variational Form

Find $\mathbf{u} \in \mathcal{U} \cap H_0(\nabla\cdot; \Omega) := \mathcal{U}^0$ such that

$$(\nabla \times \mathbf{u}, \nabla \times \mathbf{v}) + (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}) - \omega^2 (\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \qquad \forall\, \mathbf{v} \in \mathcal{U}^0$$

curl-curl-grad-div problem

Projection FEM Variational Form

Find $\mathbf{u}_h \in \mathcal{U}_h^0$, such that

$$B_h(\mathbf{u}_h, \mathbf{v}_h) = L_h(\mathbf{v}_h), \forall\ \mathbf{v}_h \in \mathcal{U}_h^0$$

where

$$\begin{aligned}
\mathcal{U}_h^0 &= \left\{\mathbf{v} \in \mathcal{U}^0 : \mathbf{v}|_T \in \left(P_1^+(T)\right)^d, \forall T \in \mathcal{T}_h\right\} \\
B_h(\mathbf{u}_h, \mathbf{v}_h) &:= \left(R_h(\nabla \times \mathbf{u}_h), R_h(\nabla \times \mathbf{v}_h)\right) \\
&\quad + (\check{R}_h(\nabla \cdot \mathbf{u}_h), \check{R}_h(\nabla \cdot \mathbf{v}_h)) - \omega^2(\mathbf{u}_h, \mathbf{v}_h) \\
L_h(\mathbf{v}_h) &:= (\mathbf{f}, \mathbf{v}_h)
\end{aligned}$$

Note2: The implementation of projection by mixed finite element method is similar as curl-curl problem.

Maxwell Eigenproblem

1. curlcurl eigenproblem
   Find $(\omega^2, \mathbf{u} \neq \mathbf{0}) \in \mathbb{R} \times \mathcal{U}^0$, such that

$$\begin{cases} \nabla \times \mu^{-1} \nabla \times \mathbf{u} = \omega^2 \epsilon \mathbf{u} & \mathit{in}\ \Omega \\ \nabla \cdot \epsilon \mathbf{u} = 0 & \mathit{on}\ \Gamma \\ \mathbf{u} \times \mathbf{n} = \mathbf{0} & \mathit{on}\ \Gamma \end{cases}$$

2. curlcurlgraddiv eigenproblem
   Find $(\omega^2, \mathbf{u} \neq \mathbf{0}) \in \mathbb{R} \times \mathcal{U}^0$, such that

$$\begin{cases} \nabla \times \mu^{-1} \nabla \times \mathbf{u} - \epsilon \nabla \nabla \cdot \epsilon \mathbf{u} = \omega^2 \epsilon \mathbf{u} & \mathit{in}\ \Omega \\ \nabla \cdot \epsilon \mathbf{u} = 0 & \mathit{on}\ \Gamma \\ \mathbf{u} \times \mathbf{n} = \mathbf{0} & \mathit{on}\ \Gamma \end{cases}$$

Note3: we only consider ccgd eigenproblem, cc is the same.

Maxwell Eigenproblem

The Classical Variational Form(ccgd)

Find $(\omega^2, \mathbf{u} \neq \mathbf{0}) \in \mathbb{R} \times \mathcal{U}_\epsilon^0$, such that

$$(\mu^{-1}\nabla \times \mathbf{u}, \nabla \times \mathbf{v}) + \ s(\nabla \cdot \epsilon\mathbf{u}, \nabla \cdot \epsilon\mathbf{v}) = \ \omega^2(\epsilon\mathbf{u}, \mathbf{v}), \quad \forall\ \mathbf{v} \in \mathcal{U}_\epsilon^0$$

where

$$\mathcal{U}_\epsilon^0 = H_0(\nabla\times;\Omega) \cap H_0(\nabla\cdot\epsilon;\Omega)$$

s is used to distinguish between true and false eigenvalue.

Both sides plus $(\epsilon\mathbf{u}, \mathbf{v})$, we have the following equivalent variational form: Find $(\lambda, \mathbf{u} \neq \mathbf{0}) \in \mathbb{R} \times \mathcal{U}_\epsilon^0$ $(\lambda = 1 + \omega^2)$, such that

$$(\mu^{-1}\nabla\times\mathbf{u}, \nabla\times\mathbf{v}) + \ s(\nabla\cdot\epsilon\mathbf{u}, \nabla\cdot\epsilon\mathbf{v}) + (\epsilon\mathbf{u}, \mathbf{v}) = \ \lambda(\epsilon\mathbf{u}, \mathbf{v}), \quad \forall\ \mathbf{v} \in \mathcal{U}_\epsilon^0$$

Projection FEM for Maxwell Eigenproblem

For any given $\mathbf{v} \in \left(L^2(\Omega)\right)^d$, define $\breve{R}_h(\nabla \cdot \epsilon \mathbf{v}) \in Q_h$ and $R_h(\mu^{-1}\nabla \times \mathbf{v}) \in W_h$ as follows:

$$\begin{aligned} (\breve{R}_h(\nabla \cdot \epsilon \mathbf{v}), q) &= -(\mathbf{v}, \epsilon \nabla q), \quad \forall q \in Q_h \\ (R_h(\mu^{-1}\nabla \times \mathbf{v}), w)_\mu &= (\mathbf{v}, \nabla \times w), \quad \forall w \in W_h \end{aligned}$$

Projection FEM:

Find $(\lambda_h, \mathbf{u}_h \neq \mathbf{0}) \in \mathbb{R} \times \mathcal{U}^0_{\epsilon h}$ such that

$$B_h(\mathbf{u_h}, \mathbf{v}_h) = \lambda_h(\epsilon \mathbf{u}_h, \mathbf{v}_h), \ \forall \ \mathbf{v_h} \in \mathcal{U}^0_{\epsilon h}$$

where: $\mathcal{U}^0_{\epsilon h} = \left\{\mathbf{v} \in \mathcal{U}^0_\epsilon : \mathbf{v}|_T \in \left(P_1^+(T)\right)^d, \forall T \in \mathcal{T}_h\right\}$

$$\begin{aligned} B_h(\mathbf{u_h}, \mathbf{v}_h) := &\left(R_h(\mu^{-1}\nabla \times \mathbf{u}_h), R_h(\mu^{-1}\nabla \times \mathbf{v}_h)\right)_\mu \\ &+ s\left(\breve{R}_h(\nabla \cdot \epsilon \mathbf{u}_h), \breve{R}_h(\nabla \cdot \epsilon \mathbf{v}_h)\right) + (\epsilon \mathbf{u}_h, \mathbf{v}_h) \end{aligned}$$

Example

L-shaped Domain (cc - projection - mixed fem)

L-shape domain: $\Omega = [-1,1]^2/[0,1]\times[-1,0]$ Let $\omega = 1$ and $\varphi(x,y) = r^{\frac{2}{3}}\sin(\frac{2}{3}\theta)$, the exact solution is given by

$$\mathbf{u}(x,y) = \nabla\big((x^2-1)(y^2-1)\varphi(x,y)\big)$$

where $x = r\cos(\theta)$, $y = r\sin(\theta)$ and (r,θ) is polar coordinates.

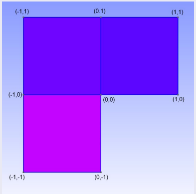


Figure: L-shape domain

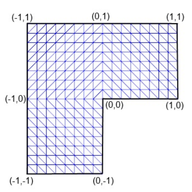


Figure: Mesh

Example

Cracked Square Domain (cc - projection - mixed fem)

Cracked square domain: $\Omega = [-1,1]^2/[0,1] \times [0]$ Let $\omega = 1$ and $\varphi(x,y) = r^{\frac{1}{2}} \sin(\frac{1}{2}\theta)$, the exact solution is given by

$$\mathbf{u}(x,y) = \nabla\left((x^2-1)(y^2-1)\varphi(x,y)\right)$$

where $x = r\cos(\theta)$, $y = r\sin(\theta)$ and (r,θ) is polar coordinates.

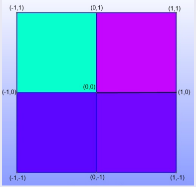


Figure: Cracked square domain

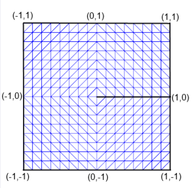


Figure: Mesh

Results

Error

L-shape Relative error of $\mathbf{u}$ by projection-mixed-fem

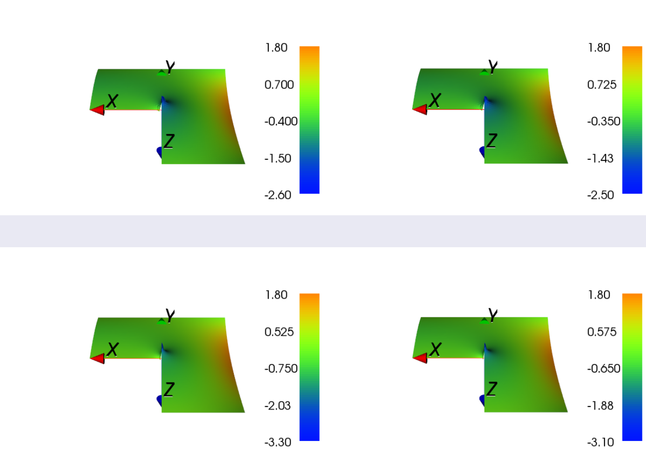
mesh	1/4	1/8	1/16	1/32	1/64
$\frac{\lVert u_1 - u_{1h}\rVert_0}{\lVert u_1\rVert_0}$	0.08746	0.04818	0.02870	0.01767	0.01103
ratio		1.81	1.68	1.62	1.60
$\frac{\lVert u_1 - u_{1h}\rVert_0}{\lVert u_1\rVert_0}$	0.13403	0.07837	0.04795	0.02987	0.01874
ratio		1.71	1.63	1.61	1.59

Cracked-square Relative error of $\mathbf{u}$ by projection-mixed-fem

mesh	1/4	1/8	1/16	1/32	1/64
$\frac{\lVert u_1 - u_{1h}\rVert_0}{\lVert u_1\rVert_0}$	0.08746	0.04818	0.02870	0.01767	0.01103
ratio		1.81	1.68	1.62	1.60
$\frac{\lVert u_1 - u_{1h}\rVert_0}{\lVert u_1\rVert_0}$	0.13403	0.07837	0.04795	0.02987	0.01874
ratio		1.71	1.63	1.61	1.59

Results

L-shape



Results

Cracked-square