Contents

MAXWELL EQUATIONS

J.Guo

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1 Maxwell equation

Let $\Omega \subset \mathbb{R}^d (d = 2, 3)$ be an open,bounded domain with boundary $\Gamma = \partial\Omega$. We consider the following Maxwell curl-curl problem: Find $\mathbf{u}$ such that

$$\nabla \times \nabla \times \mathbf{u} - \omega^2 \mathbf{u} = \mathbf{f}, \qquad in\ \Omega \tag{1.1}$$

and

$$\mathbf{u} \times \mathbf{n} = 0. \qquad on\ \partial\Omega \tag{1.2}$$

where $\mathbf{f} \in (L^2(\Omega))^d$ is a given vector function and $\omega \in \mathbb{R}(\omega \neq 0)$. $\mathbf{n}$ be the unit outward normal vector to Γ.

We also consider the following Maxwell curl-curl-grad-div problem: Find $\mathbf{u}$ such that

$$\nabla \times \nabla \times \mathbf{u} - \nabla \nabla \cdot \mathbf{u} - \omega^2 \mathbf{u} = \mathbf{f}, \qquad in\ \Omega \tag{1.3}$$

and boundary condition: $\mathbf{u} \times \mathbf{n} = 0$ and $\nabla \cdot \mathbf{u} = 0$.

2 Notation

Finite element space and $\times$-operator.

2.1 Finite element space

$$\mathcal{U}_h = \{\mathbf{v} \in \mathbb{C}(\overline{\Omega})^d \cap H_0(\nabla\times; \Omega) : \mathbf{v}|_T \in (P_k(T))^d, \forall T \in \mathcal{T}_h\} \tag{2.1}$$

$$\mathcal{U}_h^+ = \{\mathbf{v} \in \mathbb{C}(\overline{\Omega})^d \cap H_0(\nabla\times; \Omega) \cap H(\nabla\cdot; \Omega) : \mathbf{v}|_T \in (P_k(T))^d, \forall T \in \mathcal{T}_h\} \tag{2.2}$$

$$\mathcal{U}_h^0 = \{\mathbf{v} \in \mathbb{C}(\overline{\Omega})^d \cap H_0(\nabla\times; \Omega) \cap H_0(\nabla\cdot; \Omega) : \mathbf{v}|_T \in (P_k(T))^d, \forall T \in \mathcal{T}_h\} \tag{2.3}$$

where

$\mathcal{T}_h$ is a regular triangulation of $\Omega \subset \mathbb{R}^2$(tetrahedrons in $\mathbb{R}^3$),
h denotes the maximum of diameter of T, $\forall T \in \mathcal{T}$,
$P_k(T)$ denotes the space of polynomials of degree k on T,

$$H(\nabla\times; \Omega) = \{\mathbf{v} \in (L^2(\Omega))^d : \nabla \times \mathbf{v} \in (L^2(\Omega))^{2d-3}\} \tag{2.4}$$

$$\begin{aligned} H_0(\nabla\times; \Omega) &= \{\mathbf{v} \in (L^2(\Omega))^d : \nabla \times \mathbf{v} \in (L^2(\Omega))^{2d-3}, \mathbf{v} \times \mathbf{n}|_{\partial\Omega} = 0\} \\ &= \{\mathbf{v} \in H(\nabla\times; \Omega) : \mathbf{v} \times \mathbf{n}|_{\partial\Omega} = 0\} \end{aligned}$$

$$H(\nabla\cdot; \Omega) = \{\mathbf{v} \in (L^2(\Omega))^d : \nabla \cdot \mathbf{v} \in L^2(\Omega)\} \tag{2.5}$$

$$H(\nabla\cdot^0; \Omega) = \{\mathbf{v} \in H(\nabla\cdot; \Omega) : \nabla \cdot \mathbf{v} = 0\}$$

$$H_0(\nabla\cdot; \Omega) = \{\mathbf{v} \in H(\nabla\cdot; \Omega) : \nabla \cdot \mathbf{v}|_{\partial\Omega} = 0\}$$

2.2 Curl operator

Domain in 3D: $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{n} = (n_1, n_2, n_3)$, where $u_i, n_i \in \mathcal{C}^1(\Omega)(i = 1, 2, 3)$,$\nabla \times \mathbf{u}$ and $\mathbf{u} \times \mathbf{n}$ are defined as follows.

$$\nabla \times \mathbf{u} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ u_1 & u_2 & u_3 \end{vmatrix} = (\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}, \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}, \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y})$$

$$\mathbf{u} \times \mathbf{n} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = (u_2 n_3 - u_3 n_2, u_3 n_1 - u_1 n_3, u_1 n_2 - u_2 n_1)$$

Domain in 2D: $\mathbf{u} = (u_1, u_2)$ and $\mathbf{n} = (n_1, n_2)$, where $u_i, n_i, \phi \in \mathcal{C}^1(\Omega)(i = 1, 2)$,$\nabla \times \mathbf{u}$, $\mathbf{u} \times \mathbf{n}$,$\phi \times \mathbf{u}$ and $\mathbf{u} \times \phi$ are defined as follows.

$$\nabla \times \mathbf{u} = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}, \qquad \mathbf{u} \times \mathbf{n} = u_1 n_2 - u_2 n_1$$

$$\mathbf{u} \times \phi = (u_2 \phi, -u_1 \phi), \qquad \phi \times \mathbf{u} = (-u_2 \phi, u_1 \phi),$$

Note: $\phi \to (0, 0, \phi) := \widehat{\phi}, \mathbf{u} \to (u_1, u_2, 0) := \widehat{\mathbf{u}}$
$\Rightarrow \phi \times \mathbf{u} := \widehat{\phi} \times \widehat{\mathbf{u}}$ and $\mathbf{u} \times \phi := \widehat{\mathbf{u}} \times \widehat{\phi}$

3 Some theories

3.1 Curl-Curl Problem

Find $\mathbf{u} \in \mathcal{U}_h$, such that

$$\int_\Omega \nabla \times \mathbf{u}_h \cdot \nabla \times \mathbf{v}_h - \omega^2 \int_\Omega \mathbf{u}_h \cdot \mathbf{v}_h = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \qquad \forall \mathbf{v}_h \in \mathcal{U}_h \tag{3.1}$$

we can get variational form because of $\mathbf{u} \times \mathbf{n}|_\Gamma = \mathbf{0}$ and the following $\nabla \times$ Green's formula holds:

$$\int_\Omega \nabla \times \mathbf{u} \cdot \mathbf{\Phi} - \int_\Omega u \cdot \nabla \times \mathbf{\Phi} = \int_{\partial\Omega} \mathbf{n} \times \mathbf{u} \cdot \mathbf{\Phi} \tag{3.2}$$

where $\mathbf{u} = (u_1, u_2, u_3), \Phi = (\phi_1, \phi_2, \phi_3) \in H(\nabla \times; \Omega)$,
Proof of Green's formula:

$$\begin{aligned}
& \int_\Omega \nabla \times \mathbf{u} \cdot \mathbf{\Phi} - \int_\Omega \mathbf{u} \cdot \nabla \times \mathbf{\Phi} \\
= & \int_\Omega \phi_1(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}) + \phi_2(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}) + \phi_3(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}) \\
& - \int_\Omega u_1(\frac{\partial \phi_3}{\partial y} - \frac{\partial \phi_2}{\partial z}) + u_2(\frac{\partial \phi_1}{\partial z} - \frac{\partial \phi_3}{\partial x}) + u_3(\frac{\partial \phi_2}{\partial x} - \frac{\partial \phi_1}{\partial y}) \\
= & \int_\Omega (\phi_1 \frac{\partial u_3}{\partial y} + u_3 \frac{\partial \phi_1}{\partial y}) - (\phi_1 \frac{\partial u_2}{\partial z} + u_2 \frac{\partial \phi_1}{\partial z}) + (\phi_2 \frac{\partial u_1}{\partial z} + u_1 \frac{\partial \phi_2}{\partial z}) \\
& - \int_\Omega (\phi_2 \frac{\partial u_3}{\partial x} + u_3 \frac{\partial \phi_2}{\partial x}) + (\phi_3 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial \phi_3}{\partial x}) - (\phi_3 \frac{\partial u_1}{\partial y} + u_1 \frac{\partial \phi_3}{\partial y}) \\
= & \int_{\partial\Omega} \phi_1(u_3 n_2 - u_2 n_3) + \phi_2(u_1 n_3 - u_3 n_1) + \phi_3(u_2 n_1 - u_1 n_2) \\
= & \int_{\partial\Omega} \mathbf{n} \times \mathbf{u} \cdot \mathbf{\Phi}
\end{aligned}$$

Note: while in 2D,the conclusion is the same.

► $\forall\phi \in H_0^1(\Omega)$, we have $\nabla\phi \in H_0(\nabla\times;\Omega)$ and $\nabla\times\nabla\times(\nabla\phi) = \mathbf{0}$.

►**Helmholtz decomposition:** $\forall\mathbf{u} \in H_0(\nabla\times;\Omega)$, we have $\mathbf{u} = \mathring{\mathbf{u}} + \nabla\phi$ with $\mathring{\mathbf{u}} \in H(\nabla\cdot^0;\Omega)$ and $\phi \in H_0^1(\Omega)$.

►**Proposition 3.1** Maxwell's curl-curl problem $\Leftrightarrow$ Poisson equation + Reduced curl-curl problem.

Proof. Let curl-curl problem variational form is find $\mathbf{u} \in H_0(\nabla\times;\Omega)$ such that

$$(\nabla\times\mathbf{u}, \nabla\times\mathbf{v}) - \omega^2(\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v}), \qquad \forall\mathbf{v} \in H_0(\nabla\times;\Omega)$$

(1). Let $\eta \in H_0^1(\Omega)$, then $\mathbf{v} = \nabla\eta \in H_0(\nabla\times;\Omega)$, and

$$\begin{aligned}
(\nabla\times\mathbf{u}, \nabla\times(\nabla\eta)) - \omega^2(\mathbf{u},\nabla\eta) &= (\mathbf{f},\nabla\eta)\\
-\omega^2(\mathring{\mathbf{u}}+\nabla\phi,\nabla\eta) &= (\mathbf{f},\nabla\eta)\\
\omega^2(\nabla\cdot\mathring{\mathbf{u}},\eta) - \omega^2(\nabla\phi,\nabla\eta) &= (\mathbf{f},\nabla\eta) \ \text{(Green's formula)}
\end{aligned}$$

So $\phi \in H_0^1(\Omega)$ satisfies $-\omega^2(\nabla\phi,\nabla\eta) = (\mathbf{f},\nabla\eta) \quad \forall\eta \in H_0^1(\Omega)$, which is the variational form of the Possion problem.

(2). Let $\mathring{\mathbf{v}} \in H_0((\nabla\times;\Omega) \cap H(\nabla\cdot^0;\Omega)$, and

$$\begin{aligned}
(\nabla\times\mathbf{u}, \nabla\times\mathring{\mathbf{v}}) - \omega^2(\mathbf{u},\mathring{\mathbf{v}}) &= (\mathbf{f},\mathring{\mathbf{v}})\\
(\nabla\times(\mathring{\mathbf{u}}+\nabla\phi), \nabla\times\mathring{\mathbf{v}}) - \omega^2(\mathring{\mathbf{u}}+\nabla\phi,\mathring{\mathbf{v}}) &= (\mathbf{f},\mathring{\mathbf{v}})\\
(\nabla\times\mathring{\mathbf{u}}, \nabla\times\mathring{\mathbf{v}}) - \omega^2(\mathring{\mathbf{u}},\mathring{\mathbf{v}}) &= (\mathbf{f},\mathring{\mathbf{v}})
\end{aligned}$$

Therefore, $\mathring{\mathbf{u}}$ satisfies the following reduced curl-curl problem:

Find $\mathring{\mathbf{u}} \in H_0(\nabla\times;\Omega) \cap H(\nabla\cdot^0;\Omega)$ such that

$$\begin{aligned}
(\nabla\times\mathring{\mathbf{u}}, \nabla\times\mathring{\mathbf{v}}) - \omega^2(\mathring{\mathbf{u}},\mathring{\mathbf{v}}) &= (\mathbf{f},\mathring{\mathbf{v}}),\\
\forall\mathring{\mathbf{v}} &\in H_0(\nabla\times;\Omega) \cap H(\nabla\cdot^0;\Omega)
\end{aligned} \tag{3.3}$$

►**Proposition 3.2** Given a Maxwell's curl-curl problem, we can get a corresponding Maxwell's curl-curl-grad-div problem.

Proof. Apply $\nabla\cdot$ to both sides of $\nabla\times\nabla\times\mathbf{u} - \omega^2\mathbf{u} = \mathbf{f}$, we have $\nabla\cdot\mathbf{u} = -\frac{1}{\omega^2}\nabla\cdot\mathbf{f}$, then $\mathbf{u}$ satisfies the following CCGD problem: Find $\mathbf{u} \in H_0(\nabla\times;\Omega)$, such that

$$\begin{cases}
\nabla\times\nabla\times\mathbf{u} - \nabla\nabla\cdot\mathbf{u} - \omega^2\mathbf{u} = \mathbf{f} + \frac{1}{\omega^2}\nabla\nabla\cdot\mathbf{f} & in\ \Omega\\
\mathbf{u}\times\mathbf{n} = \mathbf{0} & on\ \partial\Omega
\end{cases} \tag{3.4}$$

In order to to get CCGD variational form 3.6, which will introduce in next section, we need another boundary condition

$$\nabla\cdot\mathbf{u} = 0, \qquad on\ \partial\Omega \tag{3.5}$$

and now $\mathbf{u} \in H_0(\nabla\times;\Omega) \cap H_0(\nabla\cdot;\Omega)$.

3.2 Curl-Curl-Grad-Div problem

The Maxwell Curl-Curl-Grad-Div(CCGD) problem 1.3 finite element variational form: Find $\mathbf{u} \in \mathcal{U}_h^0$, such that

$$\int_\Omega \nabla\times\mathbf{u}_h\cdot\nabla\times\mathbf{v}_h + \int_\Omega \nabla\cdot\mathbf{u}_h\nabla\cdot\mathbf{v}_h - \omega^2\int_\Omega \mathbf{u}_h\cdot\mathbf{v}_h = \int_\Omega \mathbf{f}\cdot\mathbf{v}_h \qquad \forall\mathbf{v}_h \in \mathcal{U}_h^0$$

$$(\nabla\times\mathbf{u}_h, \nabla\times\mathbf{v}_h) + (\nabla\cdot\mathbf{u}_h, \nabla\cdot\mathbf{v}_h) - \omega^2(\mathbf{u}_h,\mathbf{v}_h) = (\mathbf{f},\mathbf{v}_h) \qquad \forall\mathbf{v}_h \in \mathcal{U}_h^0 \tag{3.6}$$

► $\nabla\cdot$ Green's formula: $\int_\Omega \mathbf{v}\cdot\nabla\phi + \int_\Omega \nabla\cdot\mathbf{v}\phi = \int_{\partial\Omega} \phi\mathbf{v}\cdot\mathbf{n}$

▶ When $\nabla \cdot \mathbf{f} = 0$, the solution $\mathbf{u}$ of CCDG, if exists, belongs to the space $H(\nabla \cdot^0; \Omega)$, then CCDG $\Leftrightarrow$ CC.

▶ [**Fredholm Theory**]The CCGD problem is well-posed as long as $\omega^2 \neq \lambda_i$, where $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ are eigenvalues defined by the following eigen-problem:

$$(\nabla \times \mathbf{u}, \nabla \times \mathbf{v}) - (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}) = \lambda_i(\mathbf{u}_h, \mathbf{v})$$

$$\forall \mathbf{v} \in H_0(\nabla \times; \Omega) \cap H(\nabla \cdot; \Omega)$$

▶ [**Regularity**]The regularity of the solution of CCGD problem is closely related to the regularity of the Laplace operator with homogeneous Dirichlet or Neumann boundary conditions.

Proof. For simplicity, we assume Ω is simply connected.
If $\mathbf{u} \in H_0(\nabla \times; \Omega) \cap H(\nabla \cdot; \Omega)$, there is an unique Helmholtz decomposition

$$\mathbf{u} = \mathring{\mathbf{u}} + \nabla \phi$$

where $\mathring{\mathbf{u}} \in H_0(\nabla \times; \Omega) \cap H(\nabla \cdot^0; \Omega)$ and $\phi \in H_0^1(\Omega)$

$$\begin{aligned} \nabla \cdot \mathring{\mathbf{u}} = 0 \quad &\Rightarrow \quad \exists \psi \in H^1(\Omega), s.t.\ \mathring{\mathbf{u}} = \nabla \times \psi \\ &\Rightarrow \quad \mathbf{u} = \nabla \times \psi + \nabla \phi \end{aligned}$$

(1). Apply $\nabla \cdot$ to both sides of $\mathbf{u} = \nabla \times \psi + \nabla \phi$, we have

$$\nabla \cdot \mathbf{u} = \nabla \cdot (\nabla \times \psi) + \nabla \cdot (\nabla \phi) = \triangle \phi$$

So the function $\phi \in H_0^1(\Omega)$ satisfies the following Dirichlet boundary value problem:

$$\left\{ \begin{array}{ll} \triangle \phi = \nabla \cdot \mathbf{u} & in\ \Omega \\ \phi = 0 & on\ \partial\Omega \end{array} \right. \tag{3.7}$$

(2). Apply $\nabla \times$ to both sides of $\mathbf{u} = \nabla \times \psi + \nabla \phi$, we have

$$\nabla \times \mathbf{u} = \nabla \times (\nabla \times \psi) + \nabla \times (\nabla \phi) = -\triangle \psi$$

[since $\triangle \omega = -\nabla \times \nabla \times \omega$, $\triangle \mathbf{w} = -\nabla \times \nabla \times \mathbf{w} + \nabla \nabla \cdot \mathbf{w}$]
So the function $\psi \in H^1(\Omega)$ satisfies the following Neumann boundary value problem:

$$\left\{ \begin{array}{ll} \triangle \psi = -\nabla \times \mathbf{u} & in\ \Omega \\ \frac{\partial \psi}{\partial \mathbf{n}} = 0 & on\ \partial\Omega \end{array} \right. \tag{3.8}$$

[since on $\partial\Omega$ $0 = \mathring{\mathbf{u}} \times \mathbf{n} = (\nabla \times \psi) \times \mathbf{n} = -\frac{\partial \psi}{\partial \mathbf{n}}$]

Since $\mathbf{u} = \nabla \times \psi + \nabla \phi$, the regularity of $\mathbf{u}$ can be derived through the elliptic regularity theory on polygonal domains. (P.Grisvard. 1985.)

4 Numerical experiments

4.1 Structure numerical solution

Step 1: find $\mathbf{u} \in H(\nabla\times;\Omega)$, such that $\mathbf{u} \times \mathbf{n}|_{\partial\Omega} = \mathbf{0}$;
(e.g. find $\phi|_{\partial\Omega} = 0$, let $\mathbf{u} = \nabla\phi$, then $\mathbf{f} = -\omega^2\nabla\phi$)
Step 2: find $\mathbf{u}^*$ depend on $\mathbf{u}$,such that $\nabla \cdot \mathbf{f}^* = \mathbf{0}$.
(e.g. fine θ, such that $-\omega^2\triangle\theta = \nabla \cdot \mathbf{f}$ and $\theta|_{\partial\Omega} = 0$. Let $\mathbf{u}^* = \mathbf{u} - \nabla\theta$, then $\mathbf{f}^* = \mathbf{f} + \omega^2\nabla\theta$ satisfied $\nabla \cdot \mathbf{f}^* = \mathbf{0}$)
Corollary 4.1 When $\nabla\cdot\mathbf{f} = 0$, the solution $\mathbf{u}$ of CCDG, if exists, belongs to the space $H(\nabla\cdot^0;\Omega)$ and CCDG Maxwell problem $\Leftrightarrow$ CC Maxwell problem.

4.2 Smooth solution domain

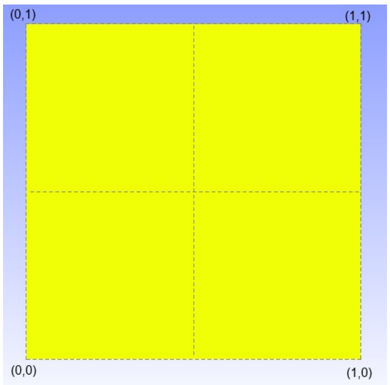

Figure 4.1: $\Omega = [0, 1]^2$.

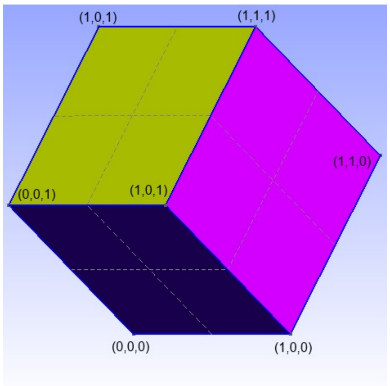

Figure 4.2: $\Omega = [0, 1]^3$.

4.3 Non-smooth solution domain

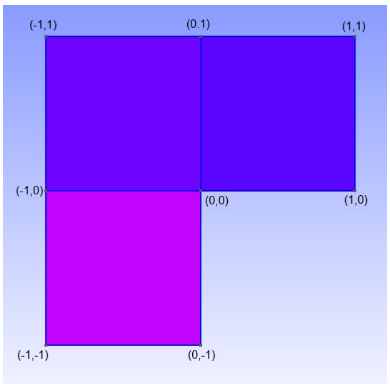

Figure 4.3: $\Omega = [-1, 1]^2/[0, 1] \times [-1, 0]$.

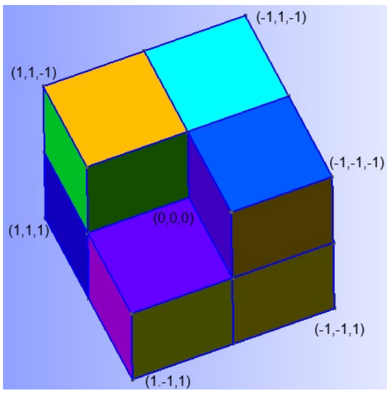

Figure 4.4: $\Omega = [-1, 1]^3/[0, 1]^3$

4.4 Exact Solution

Put $\omega = 1$, The exact solution of Figure4.1 and Figure4.3 as following

S2DI Let $p(x,y) = e^{x+y}\sin(\pi x)\sin(\pi y)$, we defined exact solution $\mathbf{u} = \nabla\big(p(x,y)\big)$

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} e^{x+y}\sin(\pi y)(\sin(\pi x) + \pi\sin(\pi x)) \\ e^{x+y}\sin(\pi x)(\sin(\pi y) + \pi\sin(\pi y)) \end{pmatrix} \tag{4.1}$$

$$\mathbf{u} \in H_0(\nabla\times;\Omega),\ \nabla\cdot\mathbf{u}|_{\partial\Omega} \neq 0$$
$$\mathbf{f} = -\mathbf{u}.$$

S2DII CC problem (1.1 and 1.2) CCGD variational form(3.4 and 3.5).

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} e^{y}y(y-1) \\ e^{x}sin(\pi x) \end{pmatrix} \tag{4.2}$$

$$\mathbf{u} \in H_0(\nabla\times;\Omega) \cap H_0(\nabla\cdot;\Omega),$$

$$\mathbf{f} = \nabla\times\nabla\times\mathbf{u} - \mathbf{u} = \begin{pmatrix} -2e^{y}y(y+1) \\ e^{x}\big((\pi^2-2)\sin(\pi x) - 2\pi\cos(\pi x)\big) \end{pmatrix} \tag{4.3}$$

N2DI Let $\psi = r^{\frac{2}{3}}\sin(\frac{2}{3}\theta)\phi(r)$, where (r,θ) are the polar coordinates at the origin and the cut-off function $\varphi(r)$ is given by

$$\phi(r) = \begin{cases} 1, & r \le 0.25 \\ -16(r-0.75)^3[5 + 15(r-0.75) + 12(r-0.75)^2], & 0.25 \le r \le 0.75 \\ 0, & r \ge 0.75 \end{cases}$$

The exact solution is chosen to be $\mathbf{u} = \nabla\times\psi = (u_1, u_2)'$, then we have

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3}r^{\frac{2}{3}}\sin(\frac{1}{3}\theta)\phi(r) + r^{\frac{5}{3}}\cos(\frac{2}{3}\theta)\sin(\theta)\varphi'(r) \\ -\frac{2}{3}r^{\frac{2}{3}}\cos(\frac{1}{3}\theta)\phi(r) - r^{\frac{5}{3}}\cos(\frac{2}{3}\theta)\cos(\theta)\varphi'(r) \end{pmatrix} \tag{4.4}$$

$$\mathbf{u} \in H_0(\nabla\times;\Omega) \cap H_0(\nabla\cdot;\Omega),$$

$$\mathbf{f} = -\mathbf{u} - \nabla\times\nabla\times\mathbf{u}$$

N2DII Similar to N2DI let $\varphi = r^{\frac{2}{3}}\sin(\frac{2}{3}\theta)(x^2-1)(y^2-1)$, the exact solution is given by

$$\mathbf{u} = \nabla\varphi = \begin{pmatrix} -\frac{2}{3}r^{\frac{2}{3}}\sin(\frac{1}{3}\theta)(x^2-1)(y^2-1) + 2xr^{\frac{2}{3}}\sin(\frac{2}{3}\theta)(y^2-1) \\ \frac{2}{3}r^{\frac{2}{3}}\cos(\frac{1}{3}\theta)(x^2-1)(y^2-1) + 2yr^{\frac{2}{3}}\sin(\frac{2}{3}\theta)(x^2-1) \end{pmatrix} \tag{4.5}$$

$$\mathbf{u} \in H_0(\nabla\times;\Omega), \nabla\cdot\mathbf{u}|_{\partial\Omega} \neq 0$$
$$\mathbf{f} = -\mathbf{u} - \nabla\nabla\cdot\mathbf{u}.$$

S3D

N3D

Remark: In order to get the correct error rate, we give the following remarks:

1.while $\nabla\cdot\mathbf{u}|_{\Omega} = \mathbf{0}$, we consider ccgd problem(eq 3.4 and eq 3.5),instead of cc problem(eq 1.1 and eq 1.2). Variational form is eq 3.6.

2.while $\nabla\cdot\mathbf{u}|_{\Omega} \neq \mathbf{0}$, it is necessary to add $\nabla\cdot\mathbf{u} = g\ (in\ \Omega)$ to cc problem(eq 1.1 and eq 1.2), and finite element variational form is: Find $\mathbf{u}_h \in U_h^+$ such that

$$(\nabla\times\mathbf{u}_h, \nabla\times\mathbf{v}_h) + (\nabla\cdot\mathbf{u}_h, \nabla\cdot\mathbf{v}_h) - \omega^2(\mathbf{u}_h, \mathbf{v}_h) = (f, \mathbf{v}_h) + (g, \nabla\cdot\mathbf{v}_h),\ \forall\mathbf{v}_h \in U_h^+ \tag{4.6}$$

4.5 Error

Table 4.1: Relative error in L^2-norm for curl-curl P_1 approximation

$\|\mathbf{u}-\mathbf{u}_h\|_0$	S2DI		S2DII		N2D	
Mesh	u1 Error	Ratio	u1 Error	Ratio	Error	Ratio
1/8	7.297e-02		2.359e-02		0.589423	
1/16	1.886e-02	3.87	5.943e-03	3.97	0.635463	
1/32	4.754e-03	3.97	1.489e-03	3.99	0.655565	
1/64	1.191e-03	3.99	3.723e-04	4.00	0.665602	
1/128	2.979e-04	4.00	9.310e-05	4.00		
1/256	7.448e-05	4.00	2.328e-05	4.00		

Table 4.2: Relative error in $\nabla \times L^2$-norm for curl-curl P_1 approximation

$\|\nabla\times(\mathbf{u}-\mathbf{u}_h)\|_0+\|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_0$						
P_1	S2DI		S2DII		N2D	
Mesh	u1 Error	Ratio	u1 Error	Ratio	Error	Ratio
1/8	2.224e-01		1.653e-01		0.5894231	
1/16	1.116e-01	1.99	8.297e-02	1.99	0.635463	
1/32	5.581e-02	2.00	4.152e-02	2.00	0.655565	
1/64	2.791e-02	2.00	2.077e-02	2.00	0.665602	
1/128	1.396e-02	2.00	1.038e-02	2.00		
1/256	6.978e-03	2.00	5.192e-03	2.00		

Table 4.3: Relative error in L^2-norm for curl-curl P_2 approximation

$\|\mathbf{u}-\mathbf{u}_h\|_0$	S2DI		S2DII		N2D	
Mesh	u1 Error	Ratio	u1 Error	Ratio	Error	Ratio
1/8	1.414e-03		4.419e-04		0.636119	
1/16	1.766e-04	8.01	5.564e-05	7.94	0.658902	
1/32	2.2137e-05	8.00	6.981e-06	7.97	0.670074	
1/64	2.7723e-06	7.98	8.732e-07	7.99		
1/128	3.4380e-07	8.06	1.021e-07	8.56		

Table 4.4: Relative error in $\nabla \times L^2$-norm for curl-curl P_2 approximation

$\|\nabla\times(\mathbf{u}-\mathbf{u}_h)\|_0+\|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_0$						
P_2	S2DI		S2DII		N2D	
Mesh	u1 Error	Ratio	u1 Error	Ratio	Error	Ratio
1/8	1.7082e-02		6.832E-03		0.960385	
1/16	4.3392e-03	3.94	1.721E-03	3.97	0.98068	
1/32	1.0914e-03	3.98	4.317E-04	3.99	0.990626	
1/64	2.7354e-04	3.99	1.081E-04	3.99		
1/128	6.8460e-05	4.00	2.704E-05	4.00		
1/256	1.7124e-05	4.00				

Table 4.5: Relative error in L^2-norm for curl-curl P_3 approximation

$\|\mathbf{u}-\mathbf{u}_h\|_0$	S2DI		S2DII		N2D	
Mesh	u1 Error	Ratio	u1 Error	Ratio	Error	Ratio
1/8	3.615e-05		3.335e-06			
1/16	2.260e-06	16.00	2.043e-07	16.32		
1/32	1.411e-07	16.01	1.288e-08	15.86		
1/64	8.807e-09	16.02	3.094e-09	4.16		
1/128	5.956e-10	14.79	1.638e-09	1.89		

Table 4.6: Relative error in $\nabla \times L^2$-norm for curl-curl P_3 approximation

$\|\nabla \times (\mathbf{u}-\mathbf{u}_h)\|_0 + \|\nabla \cdot (\mathbf{u}-\mathbf{u}_h)\|_0$						
P_3	S2DI		S2DII		N2D	
Mesh	u1 Error	Ratio	u1 Error	Ratio	Error	Ratio
1/8	5.061e-04		1.353e-04			
1/16	6.429e-05	7.87	1.702e-05	7.95		
1/32	8.086e-06	7.95	2.133e-06	7.98		
1/64	1.013e-06	7.98	2.669e-07	7.99		
1/128	1.267e-07	7.99	3.338e-08	8.00		

4.6 Rate

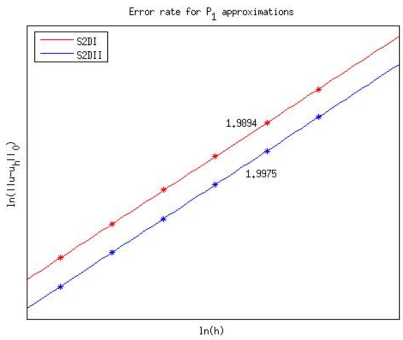

Figure 4.5: Rate of S2DI,S2DII in $\|\mathbf{u}-\mathbf{u}_h\|_0$ norm.

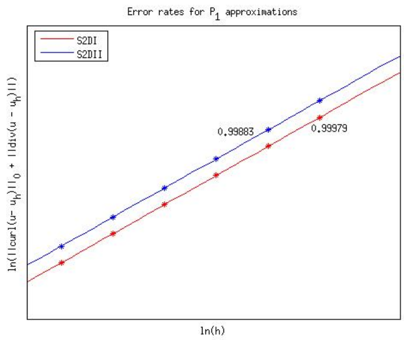

Figure 4.6: Rate of S2DI,S2DII in $\|\nabla\times(\mathbf{u}-\mathbf{u}_h)\|_0 + \|\nabla\cdot(\mathbf{u}-\mathbf{u_h})\|_0$ norm.

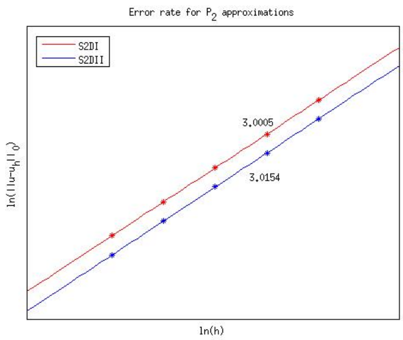

Figure 4.7: Rate of S2DI,S2DII in $\|\mathbf{u}-\mathbf{u}_h\|_0$ norm.

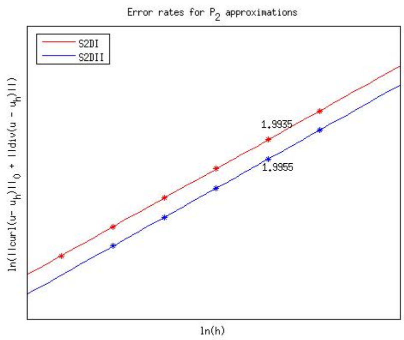

Figure 4.8: Rate of S2DI,S2DII in $\|\nabla\times(\mathbf{u}-\mathbf{u}_h)\|_0 + \|\nabla\cdot(\mathbf{u}-\mathbf{u_h})\|_0$ norm.

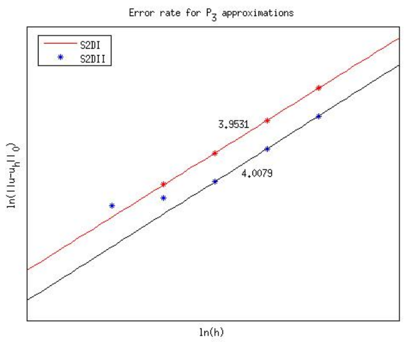

Figure 4.9: Rate of S2DI,S2DII in $\|\mathbf{u}-\mathbf{u}_h\|_0$ norm.

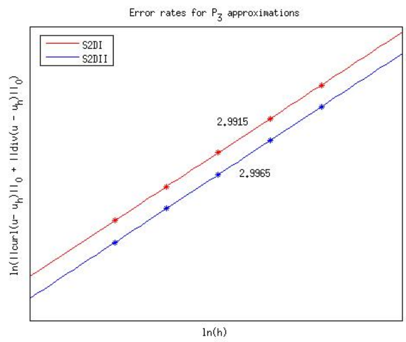

Figure 4.10: Rate of S2DI,S2DII in $\|\nabla \times (\mathbf{u}-\mathbf{u}_h)\|_0 + \|\nabla \cdot (\mathbf{u}-\mathbf{u_h})\|_0$.

4.7 Figure

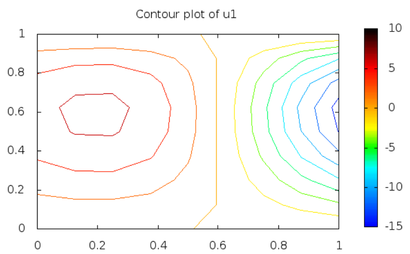

Figure 4.11: u1: Approximate solution, mesh: 8x8, P2, S2DI

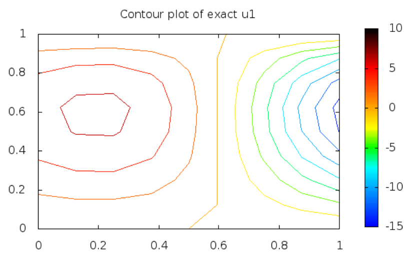

Figure 4.12: u1: Exact solution, mesh: 8x8

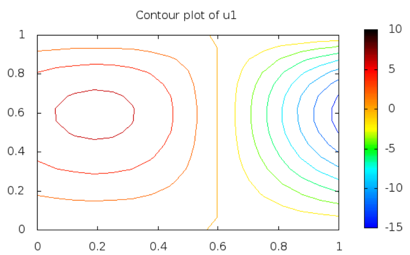

Figure 4.13: u1: Approximate solution, mesh: 16x16, P2, , S2DI

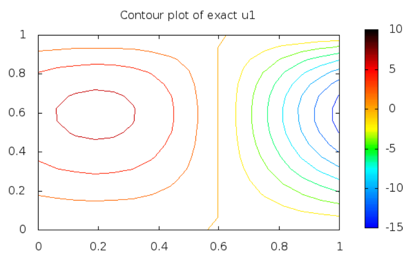

Figure 4.14: u1: Exact solution, mesh: 16x16

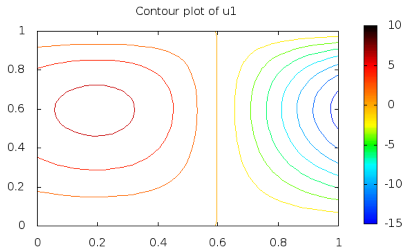

Figure 4.15: u1: Approximate solution, mesh: 32x32, P2, S2DI

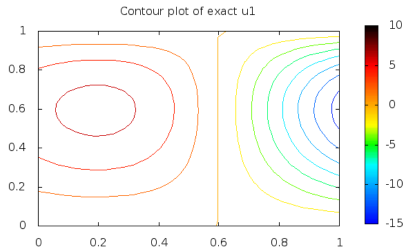

Figure 4.16: u1: Exact solution, mesh: 32x32

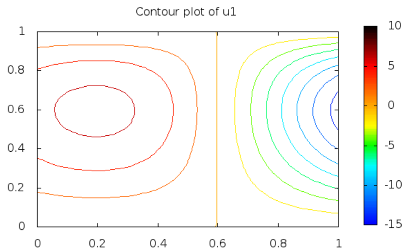

Figure 4.17: u1: Approximate solution, mesh: 64x64, P2, S2DI

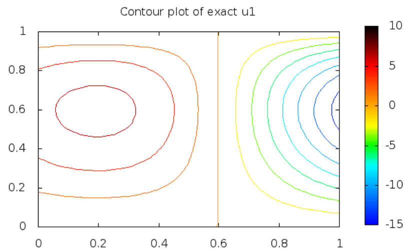

Figure 4.18: u1: Exact solution, mesh: 64x64

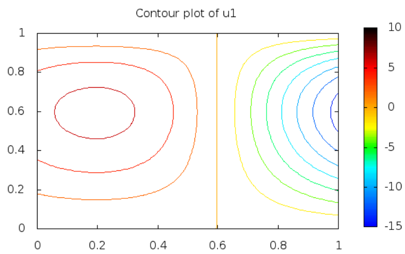

Figure 4.19: u1: Approximate solution, mesh: 128x128, P2, S2DI

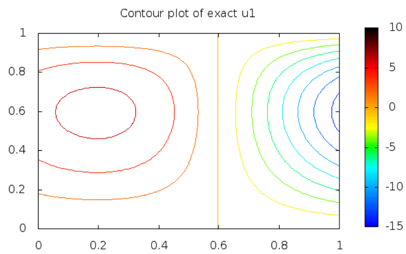

Figure 4.20: u1: Exact solution, mesh: 128x128

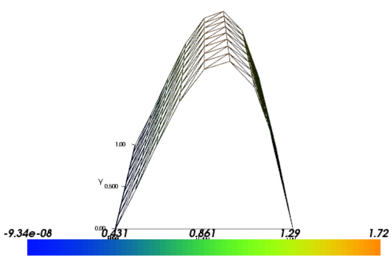

Figure 4.21: u2: Approximate solution, mesh: 8x8, P1, S2DII

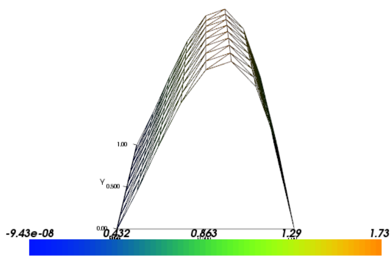

Figure 4.22: u2: Exact solution, mesh: 8x8

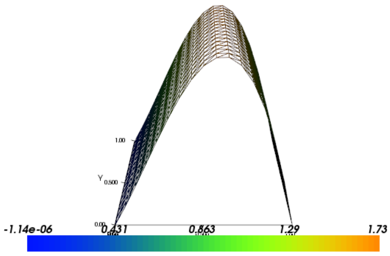

Figure 4.23: u2: Approximate solution, mesh: 16x16, P1, , S2DII

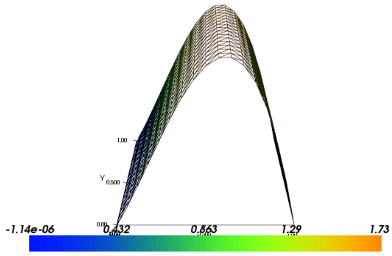

Figure 4.24: u2: Exact solution, mesh: 16x16

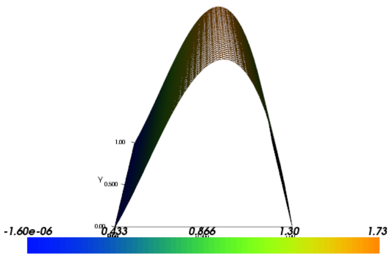

Figure 4.25: u2: Approximate solution, mesh: 32x32, P1, S2DII

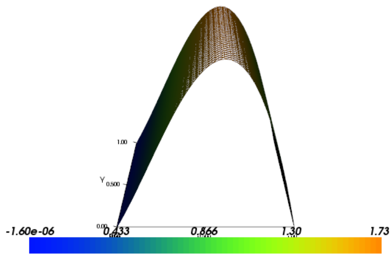

Figure 4.26: u2: Exact solution, mesh: 32x32

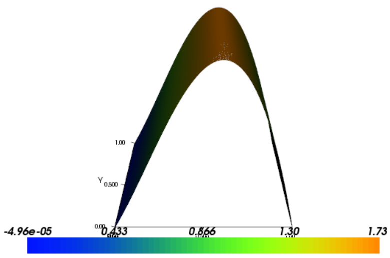

Figure 4.27: u2: Approximate solution, mesh: 64x64, P1, S2DII

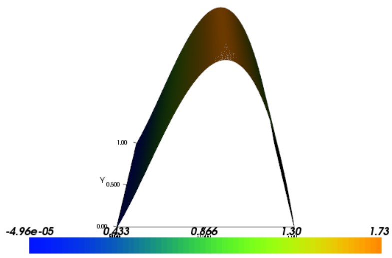

Figure 4.28: u2: Exact solution, mesh: 64x64

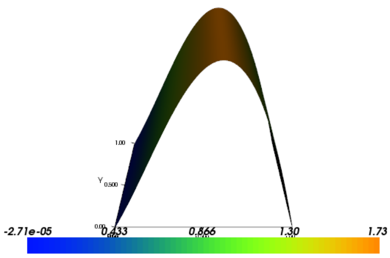

Figure 4.29: u2: Approximate solution, mesh: 128x128, P1, S2DII

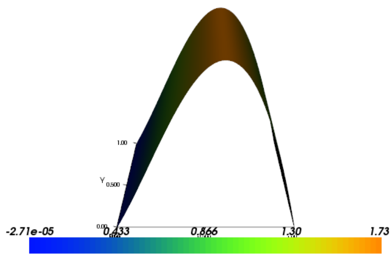

Figure 4.30: u2: Exact solution, mesh: 128x128

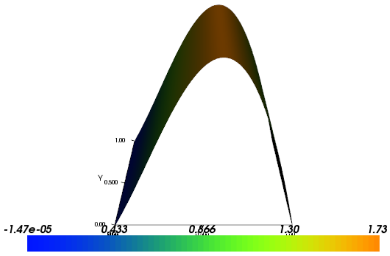

Figure 4.31: u2: Approximate solution, mesh: 256x256, P1, S2DII

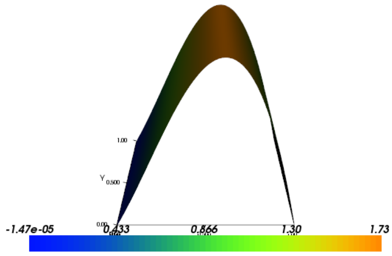

Figure 4.32: u2: Exact solution, mesh: 256x256

N2DII

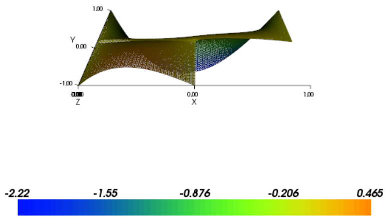

Figure 4.33: u2: Approximate solution, mesh: 32x32, P2, N2DII

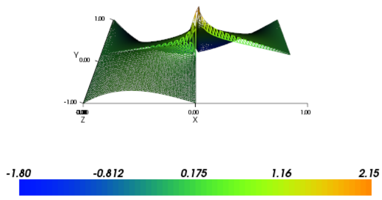

Figure 4.34: u2: Exact solution, mesh: 32x32

4.8 Conclusion

- S2DI,S2DII has exact error rate.
- N2DI,N2DII has not exact error rate while in classical finite element method.

4.9 Do next

- Finite Element Methods: Learning and implementing projection finite element method.
- Finite Element Problem: non-smoothing solution, Maxwell eigenproblem.
- Finite Element Domain: Cracked square, multi-materials(square for transmission problems), 3D.