

## 15. Martingale Convergence

### 15.1 Right-Closable Martingale

We can generalize our index set to any totally ordered set  $(N, \leq)$  (if  $n, m \in N$ , then  $n \leq m$  or  $m \leq n$ ). We say that  $(\mathcal{F}_n)_{n \in N}$  is a filtration if  $\mathcal{F}_n \subset \mathcal{F}_m$  for all  $n \leq m$  in  $N$ .

**Definition 15.1.** A submartingale  $(X_n, \mathcal{F}_n)_{n \in N}$  is right-closable if  $X_n \leq \mathbb{E}[X \mid \mathcal{F}_n]$  for all  $n \in N$ , for some  $X \in L^1$ . (For a martingale, the inequality is replaced with equality.)

If there exists  $n_0 \in N$  such that  $n \leq n_0$  for all  $n \in N$ , then  $X_n \leq \mathbb{E}[X_{n_0} \mid \mathcal{F}_n]$  for all  $n \in N$ , and we say that the submartingale is right-closed.

Note that a right-closable submartingale or martingale can always be made right-closed by simply adding an upper bound  $n_0$  to  $N$  and defining  $X_{n_0} = X$ .

**Example 15.1** (Reverse martingale). Suppose  $X_k, k \geq 1$  are i.i.d. Let  $S_n = \sum_{k=1}^n X_k$ . Take  $N$  to be the set of negative integers and for  $n \geq 1$ , let

$$\mathcal{F}_{-n} = \sigma(S_n, X_{n+1}, X_{n+2}, \dots).$$

Since  $\mathcal{F}_{-(n+1)} \subset \mathcal{F}_{-n}$ ,  $(\mathcal{F}_{-n})_{-n \in N}$  is a filtration. By symmetry, we have for all  $k = 1, \dots, n$ ,

$$\mathbb{E}[X_k \mid \mathcal{F}_{-n}] = \mathbb{E}[X_1 \mid \mathcal{F}_{-n}].$$

Therefore,

$$S_n = \mathbb{E}[S_n \mid \mathcal{F}_{-n}] = \sum_{k=1}^n \mathbb{E}[X_k \mid \mathcal{F}_{-n}] = n\mathbb{E}[X_1 \mid \mathcal{F}_{-n}].$$

Let  $Z_{-n} = \frac{S_n}{n} = \mathbb{E}[X_1 \mid \mathcal{F}_{-n}]$ . Then  $(Z_{-n}, \mathcal{F}_{-n})_{-n \in N}$  is a right-closed martingale.

**Lemma 15.1.** (i) If  $(X_n, \mathcal{F}_n)_{n \in N}$  is a right-closable martingale, then  $(X_n)_{n \in N}$  is uniformly integrable.

(ii) If  $(X_n, \mathcal{F}_n)_{n \in N}$  is a right-closable submartingale, then  $(X_n \vee a)_{n \in N}$  is uniformly integrable for all  $a \in \mathbb{R}$ .

*Proof.* (i) Since the martingale is right-closable,  $\exists X \in L^1$  such that  $X_n = \mathbb{E}[X | \mathcal{F}_n]$  for all  $n \in N$ . We have  $|X_n| \leq \mathbb{E}[|X| | \mathcal{F}_n]$  and  $\mathbb{E}|X_n| \leq \mathbb{E}|X| < \infty$ . Then,

$$\begin{aligned} \mathbb{E}|X_n| \mathbf{1}_{\{|X_n| > K\}} &\leq \mathbb{E}[\mathbb{E}[|X| | \mathcal{F}_n] \mathbf{1}_{\{|X_n| > K\}}] \\ &= \mathbb{E}|X| \mathbf{1}_{\{|X_n| > K\}} \quad \because \{|X_n| > K\} \in \mathcal{F}_n \\ &\leq M\mathbb{P}(|X_n| > K) + \mathbb{E}|X| \mathbf{1}_{\{|X| > M\}} \quad \forall M > 0 \\ &\stackrel{\text{Markov}}{\leq} \frac{M}{K} \mathbb{E}|X_n| + \mathbb{E}|X| \mathbf{1}_{\{|X| > M\}} \end{aligned}$$

and

$$\limsup_{K \rightarrow \infty} \mathbb{E}|X_n| \mathbf{1}_{\{|X_n| > K\}} \leq \mathbb{E}|X| \mathbf{1}_{\{|X| > M\}} \xrightarrow{M \rightarrow \infty} 0.$$

(ii) Since the submartingale is right-closable,  $\exists X \in L^1$  such that  $X_n \leq \mathbb{E}[X | \mathcal{F}_n]$  for all  $n \in N$ . We have

$$\mathbb{E}[X_n \mathbf{1}_{\{X_n > K\}}] \leq \mathbb{E}[X \mathbf{1}_{\{X_n > K\}}] \tag{1}$$

$$\mathbb{E}[X_n \vee a] \leq \mathbb{E}[X \vee a] \quad \text{from Jensen's inequality.} \tag{2}$$

Take  $K > |a|$ . Then  $|X_n \vee a| > K$  iff  $X_n \vee a = X_n > K$ . Therefore,

$$\begin{aligned} \mathbb{E}[(X_n \vee a) \mathbf{1}_{\{|X_n \vee a| > K\}}] &= \mathbb{E}[X_n \mathbf{1}_{\{X_n > K\}}] \\ &\stackrel{(1)}{\leq} \mathbb{E}[X \mathbf{1}_{\{X_n > K\}}] \\ &\leq M\mathbb{P}(X_n > K) + \mathbb{E}[X \mathbf{1}_{\{X > M\}}] \quad \forall M > 0 \\ &\stackrel{\text{Markov}}{\leq} \frac{M}{K} \mathbb{E}X_n^+ + \mathbb{E}|X| \mathbf{1}_{\{|X| > M\}} \\ &\leq \frac{M}{K} \mathbb{E}X^+ + \mathbb{E}|X| \mathbf{1}_{\{|X| > M\}} \quad \text{from letting } a = 0 \text{ in (2).} \end{aligned}$$

Taking  $K \rightarrow \infty$  and then  $M \rightarrow \infty$ , we obtain the desired result. □

## 15.2 Doob's Upcrossing Inequality

Consider a submartingale  $(X_n, \mathcal{F}_n)_{n \geq 0}$ . Let  $a < b$ . The first times that  $X_n$  crosses  $a$  downwards and then  $b$  upwards are given by the stopping times

$$\tau_1 = \inf\{n \geq 0 : X_n \leq a\}, \quad \tau_2 = \inf\{n > \tau_1 : X_n \geq b\}.$$

We repeat this process and define by induction, for  $k \geq 2$ ,

$$\tau_{2k-1} = \inf\{n > \tau_{2k-2} : X_n \leq a\}, \quad \tau_{2k} = \inf\{n > \tau_{2k-1} : X_n \geq b\}.$$

Then

$$U_n(a, b) = \sup\{k : \tau_{2k} \leq n\}$$

is the number of upward crossings of the interval  $[a, b]$  up to time  $n$ .

**Theorem 15.1** (Doob's upcrossing inequality). *If  $(X_n, \mathcal{F}_n)_{n \geq 0}$  is a submartingale, then for any  $a < b$ ,*

$$(b - a)\mathbb{E}U_n(a, b) \leq \mathbb{E}(X_n - a)^+.$$

*Proof.* Let  $Y_n = (X_n - a)^+$ , which is a non-negative submartingale by Lemma 14.1. Then  $U_n(a, b)$  is equal to the number of upcrossings of  $[0, b - a]$  by  $Y_0, \dots, Y_n$ . For  $m \geq 1$ , let

$$A_m = \begin{cases} 1, & \text{if } \tau_{2k-1} < m \leq \tau_{2k} \text{ for some } k, \\ 0, & \text{otherwise.} \end{cases}$$

Since  $\{\tau_{2k-1} < m \leq \tau_{2k}\} = \{\tau_{2k-1} \leq m-1\} \cap \{\tau_{2k} \leq m-1\}^c \in \mathcal{F}_{m-1}$ ,  $(A_m)_{m \geq 1}$  is predictable. Consider the martingale transform

$$\tilde{Y}_n = Y_0 + \sum_{m=1}^n A_m(Y_m - Y_{m-1}) \geq (b - a)U_n(a, b), \quad (3)$$

where the inequality follows because each upcrossing of  $[0, b - a]$  by  $(Y_n)$  results in a gain of at least  $b - a$  on the left-hand side. We have

$$\begin{aligned} \mathbb{E}[A_m(Y_m - Y_{m-1}) \mid \mathcal{F}_{m-1}] &= A_m \mathbb{E}[Y_m - Y_{m-1} \mid \mathcal{F}_{m-1}] \\ &\leq \mathbb{E}[Y_m - Y_{m-1} \mid \mathcal{F}_{m-1}], \end{aligned}$$

where the inequality follows because  $\mathbb{E}[Y_m \mid \mathcal{F}_{m-1}] \geq Y_{m-1}$ . Therefore,  $\mathbb{E}[A_m(Y_m - Y_{m-1})] \leq \mathbb{E}[Y_m - Y_{m-1}]$  and

$$\begin{aligned} \mathbb{E}\tilde{Y}_n - \mathbb{E}Y_0 &\leq \mathbb{E}Y_n - \mathbb{E}Y_0 \\ \mathbb{E}\tilde{Y}_n &\leq \mathbb{E}Y_n \end{aligned}$$

and from (3),

$$(b - a)\mathbb{E}U_n(a, b) \leq \mathbb{E}Y_n.$$

□

## 15.3 Submartingale Convergence

**Theorem 15.2** (Submartingale Convergence Theorem). *Suppose  $(X_n, \mathcal{F}_n)_{-\infty < n < \infty}$  is a submartingale.*

- (i)  $X_{-\infty} = \lim_{n \rightarrow -\infty} X_n$  exists a.s. and  $\mathbb{E}X_{-\infty}^+ < \infty$ . Let  $\mathcal{F}_{-\infty} = \bigcap_n \mathcal{F}_n$ . Then  $(X_n, \mathcal{F}_n)_{-\infty \leq n < \infty}$  is a submartingale, i.e.,  $X_{-\infty} \leq \mathbb{E}[X_m | \mathcal{F}_{-\infty}]$  for all  $m > -\infty$ .
- (ii) If  $\mathbb{E}X_n^+ \leq B < \infty$ , then  $X_{\infty} = \lim_{n \rightarrow \infty} X_n$  exists a.s. and  $\mathbb{E}X_{\infty}^+ \leq B$ .
- (iii) If  $(X_n^+)_{-\infty < n < \infty}$  is uniformly integrable, then for  $X_{\infty} = \lim_{n \rightarrow \infty} X_n$ , we have  $X_m \leq \mathbb{E}[X_{\infty} | \mathcal{F}_m]$  for all  $m < \infty$ . Let  $\mathcal{F}_{\infty} = \sigma(\bigcup_n \mathcal{F}_n)$ . Then  $(X_n, \mathcal{F}_n)_{-\infty < n \leq \infty}$  is right-closed.

*Proof.* Note that  $X_n$  converges as  $n \rightarrow \pm\infty$  iff  $\limsup X_n = \liminf X_n$ ,<sup>1</sup> i.e.,  $X_n$  diverges only on the event

$$\{\limsup X_n > \liminf X_n\} = \bigcup_{\substack{a < b, \\ a, b \in \mathbb{Q}}} A_{ab},$$

where

$$A_{ab} = \{\limsup X_n \geq b > a \geq \liminf X_n\} \subset \{\lim U_n(a, b) = \infty\}.$$

Therefore to show that  $\mathbb{P}(A_{ab}) = 0$ , it suffices to show that  $\mathbb{E}[\lim U_n(a, b)] < \infty$ .

- (i) For each  $n \geq 1$ ,  $U_n(a, b)$  is the number of upcrossings of the submartingale  $Y_1 = X_{-n}, Y_2 = X_{-n+1}, \dots, Y_n = X_{-1}$ . From Doob's upcrossing inequality (Theorem 15.1), we have

$$\mathbb{E}U_n(a, b) \leq \frac{\mathbb{E}(Y_n - a)^+}{b - a} = \frac{\mathbb{E}(X_{-1} - a)^+}{b - a} < \infty.$$

The MCT then gives  $\mathbb{E} \lim_{n \rightarrow \infty} U_n(a, b) = \lim_{n \rightarrow \infty} \mathbb{E}U_n(a, b) < \infty$ . Therefore  $X_{-\infty} = \lim_{n \rightarrow -\infty} X_n$  exists a.s. From Fatou's lemma,

$$\mathbb{E}X_{-\infty}^+ \leq \liminf_{n \rightarrow \infty} \mathbb{E}X_{-n}^+ \leq \mathbb{E}X_{-1}^+ < \infty.$$

Finally, to show that  $(X_n, \mathcal{F}_n)_{-\infty \leq n < \infty}$  is a submartingale, we note that  $X_{-\infty} \in \mathcal{F}_n$  for all  $n$  (Exercise), which implies that  $X_{-\infty} \in \mathcal{F}_{-\infty}$ . Furthermore, to show  $X_{-\infty} \leq \mathbb{E}[X_m | \mathcal{F}_{-\infty}]$  for all  $m > -\infty$ , it suffices to show that for all  $A \in \mathcal{F}_{-\infty}$ , we have  $\mathbb{E}X_{-\infty} \mathbf{1}_A \leq \mathbb{E}X_m \mathbf{1}_A$  (cf. Lemma 12.1).

Consider the submartingale  $(X_n, \mathcal{F}_n)_{-\infty \leq n \leq 0}$ , which is right-closed. From Lemma 15.1,  $(X_n \vee a, \mathcal{F}_n)_{-\infty \leq n \leq 0}$  is uniformly integrable for all  $a \in \mathbb{R}$ . Furthermore,  $X_n \vee a \rightarrow X_{-\infty} \vee a$  a.s. as  $n \rightarrow \infty$ . From Lemma 14.4,  $X_n \vee a \rightarrow X_{-\infty} \vee a$  in  $L^1$ , hence

$$\mathbb{E}(X_{-\infty} \vee a) \mathbf{1}_A = \lim_{n \rightarrow -\infty} \mathbb{E}(X_n \vee a) \mathbf{1}_A \leq \mathbb{E}(X_m \vee a) \mathbf{1}_A,$$

for all  $m > -\infty$ . The last inequality follows because  $\mathbb{E}(X_n \vee a) \mathbf{1}_A \leq \mathbb{E}(X_m \vee a) \mathbf{1}_A$  for all  $n \leq m$ , which in turn is a consequence of the facts that  $(X_n \vee a)$  is a submartingale and  $A \in \mathcal{F}_m$ . Taking  $a \rightarrow -\infty$ , the MCT gives  $\mathbb{E}X_{-\infty} \mathbf{1}_A \leq \mathbb{E}X_m \mathbf{1}_A$ .

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<sup>1</sup>In this proof, if the convergence is not specified, it is taken to mean either  $n \rightarrow \infty$  or  $n \rightarrow -\infty$ .

(ii) From Doob's upcrossing inequality,

$$\mathbb{E}U_n(a, b) \leq \frac{\mathbb{E}(X_n - a)^+}{b - a} \leq \frac{|a| + \mathbb{E}X_n^+}{b - a} \leq \frac{|a| + B}{b - a} < \infty.$$

The same argument as in the proof of Item (i) completes the proof of Item (ii).

(iii) We want to show that for all  $m < \infty$  and  $A \in \mathcal{F}_m$ ,  $\mathbb{E}X_m \mathbf{1}_A \leq \mathbb{E}X_\infty \mathbf{1}_A$ . From Lemma 14.1,  $(X_n \vee a, \mathcal{F}_n)_{n < \infty}$  is a submartingale for all  $a \in \mathbb{R}$ . For  $K > 0$  and  $K > a$ ,  $X_n \vee a > K$  iff  $X_n \vee a = X_n^+ > K$ . Therefore, since  $(X_n^+)$  is uniformly integrable, so is  $(X_n \vee a)_{n < \infty}$ . As  $X_n \vee a \rightarrow X_\infty \vee a$  a.s. as  $n \rightarrow \infty$ , Lemma 14.4 again yields that  $X_n \vee a \rightarrow X_\infty \vee a$  in  $L^1$ , and hence

$$\mathbb{E}(X_\infty \vee a) \mathbf{1}_A = \lim_{n \rightarrow \infty} \mathbb{E}(X_n \vee a) \mathbf{1}_A \geq \mathbb{E}(X_m \vee a) \mathbf{1}_A,$$

for all  $m < \infty$ . Taking  $a \rightarrow -\infty$  and applying the MCT give the desired result. □

**Corollary 15.1** (Martingale Convergence Theorem). *Suppose  $(X_n, \mathcal{F}_n)_{n < \infty}$  is a martingale.*

- (i) *If  $\sup_n \mathbb{E}|X_n| < \infty$  (equivalently,  $\sup_n \mathbb{E}X_n^+ < \infty$  or  $\sup_n \mathbb{E}X_n^- < \infty$ ), then  $X_n \rightarrow X_\infty$  a.s. and  $\mathbb{E}|X_\infty| < \infty$ .*
- (ii) *If  $(X_n)_{n < \infty}$  is uniformly integrable, then  $(X_n, \mathcal{F}_n)_{n \leq \infty}$  is a right-closed martingale.*

Corollary 15.1 follows from Theorem 15.2 by simply observing that  $X_n$  and  $-X_n$  are both submartingales. Furthermore, Corollary 15.1 together with Lemma 15.1 says that a martingale is right-closable iff it is uniformly integrable.

**Example 15.2.** *Consider the simple random walk  $S_n$  in Example 13.1 with stopping time  $\tau$  when  $S_n$  hits the boundaries  $A$  or  $-B$ . Let  $M_n = S_{n \wedge \tau}$ , which is a bounded martingale. From the Martingale Convergence Theorem (Corollary 15.1),  $M_n$  converges a.s. For  $\omega \in \{\tau = \infty\}$ , we have  $|M_n(\omega) - M_{n+1}(\omega)| = |S_n(\omega) - S_{n+1}(\omega)| = 1$ , hence  $M_n(\omega)$  does not converge. Therefore  $\mathbb{P}(\tau = \infty) = 0$  and  $\mathbb{P}(\tau < \infty) = 1$ .*

**Corollary 15.2** (Supermartingale Convergence Theorem). *Suppose  $(X_n, \mathcal{F}_n)_{n \geq 0}$  is a supermartingale. If  $\sup_n \mathbb{E}X_n^- < \infty$ , then  $X_n \rightarrow X_\infty$  a.s., and  $\mathbb{E}X_\infty \leq \mathbb{E}X_0$ .*

*Proof.*  $-X_n$  is submartingale with  $\mathbb{E}(-X_n)^+ = \mathbb{E}X_n^-$ . □

**Theorem 15.3** (Levy's Convergence Theorem). *Given  $\mathbb{E}|X| < \infty$  and a filtration  $(\mathcal{F}_n)_{n \geq 0}$ , let  $\mathcal{F}_\infty = \sigma(\cup_n \mathcal{F}_n)$ . Then  $\mathbb{E}[X | \mathcal{F}_n] \rightarrow \mathbb{E}[X | \mathcal{F}_\infty]$  a.s.*

*Proof.* From Example 12.7,  $X_n = \mathbb{E}[X | \mathcal{F}_n]$  is a martingale and  $\mathbb{E}|X_n| \leq \mathbb{E}|X| < \infty$ . From Corollary 15.1,  $X_\infty = \lim_{n \rightarrow \infty} X_n$  exists a.s. We next show that  $X_\infty = \mathbb{E}[X | \mathcal{F}_\infty]$ .

Let  $A \in \bigcup_n \mathcal{F}_n$ . Then there exists  $m$  such that  $A \in \mathcal{F}_m$  and since  $X_n$  is a martingale, we have  $\mathbb{E}X_n \mathbf{1}_A = \mathbb{E}X_m \mathbf{1}_A$  for all  $n \geq m$ . As  $(X_n, \mathcal{F}_n)$  is right-closable, it is uniformly integrable, and Lemma 14.4 gives

$$\begin{aligned} \mathbb{E}X_\infty \mathbf{1}_A &= \lim_{n \rightarrow \infty} \mathbb{E}X_n \mathbf{1}_A \\ &= \mathbb{E}X_m \mathbf{1}_A \\ &= \mathbb{E}[\mathbb{E}[X | \mathcal{F}_m] \mathbf{1}_A] \\ &= \mathbb{E}X \mathbf{1}_A. \end{aligned}$$

Since  $\bigcup_n \mathcal{F}_n$  is an algebra (because  $(\mathcal{F}_n)$  is a filtration), the  $\pi$ - $\lambda$  theorem completes the proof.  $\square$

**Example 15.3** (Improved SLLN). *For a sequence of r.v.s  $(X_n)_{n \geq 1}$ , let  $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ ,  $n \geq 1$ , be a filtration. Suppose that  $\mathbb{E}[X_n | \mathcal{F}_{n-1}] = 0$  for all  $n \geq 1$ . In particular, the r.v.s are not necessarily pairwise independent. Then for  $n > m$ , we have  $\mathbb{E}X_n X_m = \mathbb{E}[\mathbb{E}[X_n X_m | \mathcal{F}_m]] = \mathbb{E}[X_m \mathbb{E}[X_n | \mathcal{F}_m]] = 0$ . Let  $S_n = \sum_{k=1}^n X_k$ .*

*Suppose that  $(b_n)_{n \geq 1}$  is a sequence of positive constants increasing to  $\infty$  and  $\sum_{n=1}^\infty \frac{\mathbb{E}X_n^2}{b_n^2} < \infty$ , then  $\frac{S_n}{b_n} \rightarrow 0$  a.s. To see this, let  $Y_n = \sum_{k=1}^n \frac{X_k}{b_n}$ . It is easy to verify that this is a martingale. For  $K > 0$ , we have*

$$\mathbb{E}|Y_n| \mathbf{1}_{\{|Y_n| > K\}} \leq \frac{1}{K} \mathbb{E}|Y_n|^2 = \frac{1}{K} \sum_{k=1}^n \frac{\mathbb{E}X_k^2}{b_k^2}.$$

*We then have*

$$\sup_n \mathbb{E}|Y_n| \mathbf{1}_{\{|Y_n| > K\}} \leq \frac{1}{K} \sum_{k=1}^\infty \frac{\mathbb{E}X_k^2}{b_k^2} \xrightarrow{K \rightarrow \infty} 0.$$

*Therefore,  $(Y_n)$  is uniformly integrable. From the Martingale Convergence Theorem (Corollary 15.1),  $Y_n$  converges a.s. The generalized Kronecker's lemma (Lemma 7.5) then implies that  $\frac{S_n}{b_n} \rightarrow 0$  a.s.*

**Example 15.4.** *Continuing from Example 15.1,  $(Z_{-n}, \mathcal{F}_{-n})_{n \geq 1}$  is a right-closed martingale. The Submartingale Convergence Theorem (Theorem 15.2) says that  $\lim_{n \rightarrow \infty} Z_{-n} = Z_{-\infty} \in \mathcal{F}_{-\infty} = \bigcap_{n \geq 1} \mathcal{F}_{-n}$ . Each event in  $\mathcal{F}_{-n}$  is invariant under finite permutations, therefore by the Hewitt-Savage 0-1 Law, it has probability 0 or 1. Hence,  $Z_{-\infty}$  is a constant a.s. But  $\mathbb{E}Z_{-n} = \mathbb{E}X_1$  for all  $n \geq 1$ , so  $Z_{-\infty} = \mathbb{E}X_1$  a.s., i.e.,  $\frac{S_n}{n} \rightarrow \mathbb{E}X_1$  a.s., which is Kolmogorov's SLLN.*

**Example 15.5** (Levy's 0-1 Law). *From Theorem 15.3, if  $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ , then for any  $A \in \mathcal{F}_\infty$ , we have  $\mathbb{E}[\mathbf{1}_A | \mathcal{F}_n] \rightarrow \mathbf{1}_A$  a.s. The famous probabilist K. L. Chung once commented that this result "is obvious or incredible".*

1. *It is obvious:*  $\mathbb{E}[\mathbf{1}_A | \mathcal{F}_\infty] = \mathbf{1}_A$ .
2. *It is incredible:* Consider  $X_i, i \geq 1$ , independent with  $\mathcal{F}_n = \sigma(X_1, \dots, X_n) \uparrow \mathcal{F}_\infty$ . An event  $A \in \mathcal{T} = \bigcap_{n \geq 1} \sigma(X_n, X_{n+1}, \dots)$  the tail  $\sigma$ -algebra is independent of  $\mathcal{F}_n$  for all  $n \geq 1$  (recall the grouping lemma (Lemma 8.3)). Therefore,  $\mathbb{P}(A) = \mathbb{E}[\mathbf{1}_A | \mathcal{F}_n] \xrightarrow{n \rightarrow \infty} \mathbf{1}_A \in \{0, 1\}$  a.s. from Levy's 0-1 Law, which recovers Kolmogorov's 0-1 Law!