

2. Basics of Real Analysis - II

Let (\mathcal{X}, d) be a metric space.

2.1 Open and Closed Sets

Definition 2.1. The open ball of radius $\epsilon > 0$ is defined by

$$B(x, \epsilon) = \{y \in \mathcal{X} : d(x, y) < \epsilon\}.$$

Definition 2.2. A set U is open if $\forall x \in U, \exists \epsilon > 0$ such that $B(x, \epsilon) \subset U$. A set F is closed if $F^c = \mathcal{X} \setminus F$ is open.

A set can be both open and closed, e.g., \mathcal{X}, \emptyset .

Example 2.1. Let $\mathcal{X} = \{x_1, x_2, \dots\}$ be a discrete space. Consider the discrete metric

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

For any $A \subset \mathcal{X}$ and $\epsilon \in (0, 1)$, $B(a, \epsilon) = \{a\} \subset A$. Therefore A is open. This also proves that A is closed as A^c is open.

If $U_i, i \geq 1$ are open, then $\bigcup_{i=1}^{\infty} U_i$ is open, $\bigcap_{i=1}^n U_i$ is open but $\bigcap_{i=1}^{\infty} U_i$ may not be open.

Example 2.2. Suppose that $U_i = (\frac{1}{2} - \frac{1}{i}, \frac{1}{2} + \frac{1}{i})$, then U_i is open while $\bigcap_{i=1}^{\infty} U_i = \{1/2\}$ is closed.

If F_i is closed, then $\bigcup_{i=1}^{\infty} F_i$ is closed and $\bigcap_{i=1}^{\infty} F_i$ is closed.

Definition 2.3. x is a limit point of A if $\forall \epsilon > 0, \exists y \in B(x, \epsilon) \cap A$, and $y \neq x$.

Therefore, x is a limit point of A if $\exists y_1, y_2, \dots \in A, y_i \neq x$ for all $i \geq 1$, s.t. $y_i \rightarrow x$.

Lemma 2.1. A is closed if and only if all limit points of A are in A .

Proof. Suppose that A is closed, so that A^c is open. Suppose there exists a limit point x of A s.t. $x \in A^c$. Then $\exists \epsilon > 0$, s.t. $B(x, \epsilon) \subset A^c$, which is a contradiction to x being a limit point of A .

Suppose that all limit points of A belongs to A . Consider $x \in A^c$. There exists $\epsilon > 0$ s.t. $B(x, \epsilon) \subset A^c$ because otherwise there exists $(y_i) \subset A$ s.t. $y_i \rightarrow x$, which means that x is a limit point of A leading to a contradiction. This shows that A^c is open and hence A is closed. \square

2.2 Compact Spaces

Definition 2.4. \mathcal{X} is sequentially compact if every sequence in \mathcal{X} has a convergent subsequence in \mathcal{X} .

Definition 2.5. \mathcal{X} is totally bounded if $\forall \epsilon > 0$, there exists a finite collection $\{B(x_i, \epsilon) : i = 1, 2, \dots, N_\epsilon\}$ such that

$$\mathcal{X} \subset \bigcup_{i=1}^{N_\epsilon} B(x_i, \epsilon).$$

Note that if a set \mathcal{X} is totally bounded, then it is bounded. The converse is not true: consider the discrete space in Example 2.1, it is bounded but not totally bounded if it is infinite.

Theorem 2.1. For a metric space (\mathcal{X}, d) , the following are equivalent:

- (i) \mathcal{X} is sequentially compact.
- (ii) \mathcal{X} is complete and totally bounded.
- (iii) Every open cover of \mathcal{X} has a finite subcover. We say that \mathcal{X} is compact.

Proof.

1) (i) \Leftrightarrow (ii):

We first show that (i) \Rightarrow (ii). Since \mathcal{X} is sequentially compact, every Cauchy sequence in \mathcal{X} has a convergent subsequence. From Lemma 1.11, the Cauchy sequence also converges in \mathcal{X} , so \mathcal{X} is complete. Suppose that \mathcal{X} is not totally bounded. Then $\exists \epsilon > 0$ so that \mathcal{X} cannot be covered by a finite collection of open balls. Choose any $x_1 \in \mathcal{X}$. Then $\exists x_2 \notin B(x_1, \epsilon)$. Similarly, $\exists x_3 \notin B(x_1, \epsilon) \cup B(x_2, \epsilon)$, and so on. The sequence (x_n) does not contain any convergent subsequence since $d(x_i, x_j) \geq \epsilon$ for any $i \neq j$. This is a contradiction to (i).

We next show that (ii) \Rightarrow (i). Since \mathcal{X} is totally bounded, for each $m \geq 1$, there exists a finite cover $\{B(x_{m,k}, 1/m) : k = 1, \dots, M_m\}$ of \mathcal{X} . Consider any infinite sequence (y_n) in \mathcal{X} . We assume that y_n are distinct because if there are infinitely many y_n that are the same, then there is a trivial convergent subsequence. Then there is a $B(x_{1,k_1}, 1)$ that contains a subsequence $(y_{1,n})$ of (y_n) . Similarly, there is a $B(x_{2,k_2}, 1/2)$ that contains a further subsequence $(y_{2,n})$ of $(y_{1,n})$, and so on. Consider the “diagonal” subsequence $(y_{m,m})_{m=1}^\infty$. This sequence is Cauchy and since \mathcal{X} is complete, it converges.

2) (i) \Leftrightarrow (iii):

We show (iii) \Rightarrow (i). To do that, we first prove the following facts:

- (a) Any compact $A \subset \mathcal{X}$ is closed. In particular, if \mathcal{X} is compact, then it is closed.
 Let $x \in A^c$ and $U_n = \{y : d(y, x) > 1/n\}$ for $n \geq 1$. Every $y \in \mathcal{X}$ with $y \neq x$ has $d(y, x) > 0$ so y belongs to some U_n . Therefore, $\{U_n : n \geq 1\}$ covers A and there must be a finite subcover. Let N to be the largest index in the subcover, i.e., every $y \in A$ lies in some U_n where $n \leq N$. Then $B(x, 1/N) \subset A^c$ and A is closed.
- (b) If \mathcal{X} is compact and $A \subset \mathcal{X}$ is closed, then A is compact.
 Let $\{U_n\}$ be an open cover of A . Then $\{U_n\} \cup \{A^c\}$ is an open cover of \mathcal{X} . There is a finite subcover, say, $\{U_1, \dots, U_N, A^c\}$ of \mathcal{X} . Then $\{U_1, \dots, U_N\}$ is a finite open cover of A .

Suppose \mathcal{X} is compact. Assume there is a sequence (x_n) that has no convergent subsequences. In particular, this sequence has infinitely distinct points y_1, y_2, \dots . Since there is no convergent subsequence, there is some open ball B_k containing each y_k and no other y_i . The set $A = \{y_1, y_2, \dots\}$ is closed as it has no limit points, so it is compact. But $\{B_k\}$ is an open cover of A and has no finite subcover, a contradiction. Therefore (x_n) has a convergent subsequence whose limit lies in \mathcal{X} as \mathcal{X} is closed.

We now show (i) \Rightarrow (iii). Suppose that \mathcal{X} is sequentially compact. Let $\{G_\alpha \subset \mathcal{X} : \alpha \in I\}$ be an open cover of \mathcal{X} . We claim that there exists $\epsilon > 0$ such that every ball $B(x, \epsilon)$ is contained in some G_α . Suppose not. Then for each positive integer n , $\exists y_n \in \mathcal{X}$ such that $B(y_n, 1/n)$ is not contained in any G_α . By hypothesis, there exists a subsequence $y_{n_i} \rightarrow y \in \mathcal{X}$. Since $\{G_\alpha \subset \mathcal{X} : \alpha \in I\}$ is an open cover of \mathcal{X} , there exists $G_{\alpha_0} \ni y$ and $\epsilon > 0$ such that $B(y, \epsilon) \subset G_{\alpha_0}$. Choose n_i sufficiently large so that $d(y_{n_i}, y) < \epsilon/2$ and $1/n_i < \epsilon/2$. Then $B(y_{n_i}, 1/n_i) \subset G_{\alpha_0}$, a contradiction.

Since \mathcal{X} is sequentially compact, it is totally bounded, so there exists a finite collection of balls of radius ϵ s.t. $\{B(x_i, \epsilon) : i = 1, 2, \dots, N\}$ covers \mathcal{X} . Choose $\alpha_i \in I$ s.t. $B(x_i, \epsilon) \subset G_{\alpha_i}$. Then $\{G_{\alpha_i} : i = 1, 2, \dots, N\}$ is a finite subcover of \mathcal{X} , which means \mathcal{X} is compact.

□

Theorem 2.2 (Heine-Borel). $A \subset \mathbb{R}^k$ is compact iff A is closed and bounded.

Proof. Since the following proof can be repeated in every dimension, it suffices to prove only for $A \subset \mathbb{R}$.

We first prove sufficiency. Suppose A is closed and bounded and $(x_n) \subset A$. Then x_n is bounded since A is bounded. By Lemma 1.6, there exists convergent (x_{n_i}) such that $x_{n_i} \rightarrow x \in \mathbb{R}$. Since A is closed, $x \in A$. Therefore, A is compact.

Next, suppose that A is compact. Then from Theorem 2.1, A is complete and totally bounded, hence closed and bounded in \mathbb{R} .

□

2.3 Continuity

Definition 2.6. A function $f : (\mathcal{X}, d_{\mathcal{X}}) \mapsto (\mathcal{Y}, d_{\mathcal{Y}})$ is (pointwise) continuous at $x \in \mathcal{X}$ if $\forall \epsilon > 0, \exists \delta_x > 0$ s.t. $d_{\mathcal{Y}}(f(x), f(z)) < \epsilon$, whenever $d_{\mathcal{X}}(x, z) < \delta_x$.

Note that the definition is equivalent to saying that $\forall \epsilon > 0, \exists \delta_x > 0$ s.t. $B(x, \delta_x) \subset f^{-1}(B(f(x), \epsilon))$.

Lemma 2.2. $f : (\mathcal{X}, d_{\mathcal{X}}) \mapsto (\mathcal{Y}, d_{\mathcal{Y}})$ is continuous iff $f^{-1}(U)$ is open in \mathcal{X} for every open $U \subset \mathcal{Y}$.

Proof. Suppose f is continuous and U is open in \mathcal{Y} . For each $y \in U$, there exists an open ball $B(y, r) \subset U$. Since f is continuous, for each $x \in f^{-1}(\{y\})$, $\exists \delta_x > 0$ s.t. $B(x, \delta_x) \subset f^{-1}(B(y, r)) \subset f^{-1}(U)$. Therefore, $f^{-1}(U) = \bigcup_{x \in f^{-1}(U)} B(x, \delta_x)$ is open.

To prove the other direction, we have for every $x \in \mathcal{X}$ and $\epsilon > 0$, $f^{-1}(B(f(x), \epsilon))$ is open and contains x , so it contains an open ball around x . Thus, f is continuous at x . \square

Lemma 2.3. If f is continuous and $B \in \mathcal{X}$ is compact, then $f(B) \triangleq \{f(x) : x \in B\}$ is compact.

Proof. Consider $y_n = f(x_n)$. $\exists (x_{n_i})$ s.t. $x_{n_i} \rightarrow x \in B$. From the continuity of f , $y_{n_i} = f(x_{n_i}) \rightarrow f(x) \in f(B)$. \square

From Theorem 2.2, $f(B)$ is closed and bounded. Therefore, $\sup_{x \in B} f(x)$ and $\inf_{x \in B} f(x)$ are both achieved on B .

Definition 2.7. f is uniformly continuous if $\forall \epsilon > 0, \exists \delta > 0$ s.t. $d_{\mathcal{Y}}(f(x), f(z)) < \epsilon$, $\forall d_{\mathcal{X}}(x, z) < \delta$.

Example 2.3.

- (a) If $|f(x) - f(y)| < Ld(x, y)$ for some positive constant L and all $x, y \in \mathcal{X}$, we say that f is Lipschitz continuous. It is clear that f is uniformly continuous on \mathcal{X} .
- (b) If $f : \mathbb{R} \mapsto \mathbb{R}$ is differentiable with $\sup |f'| < \infty$, where f' is the derivative of f , then f is uniformly continuous from the mean value theorem.

Lemma 2.4. Suppose that $f : K \mapsto \mathcal{Y}$, where $K \subset \mathcal{X}$ is compact. If f is continuous, then f is uniformly continuous.

Proof. Fix $\epsilon > 0$. For each $x \in \mathcal{X}$, $\exists \delta_x > 0$, s.t. $d_{\mathcal{Y}}(f(x), f(y)) < \epsilon/2$ whenever $d_{\mathcal{X}}(x, y) < \delta_x$. We have $\{B(x, \delta_x/2) : x \in K\}$ is an open cover of K . Since K is compact, exists a finite subcover $\{B(x_i, \delta_{x_i}/2) : 1 \leq i \leq n\}$. Let $\delta = \min\{\delta_{x_1}, \dots, \delta_{x_n}\} > 0$.

For each $x \in K$, $\exists x_i$, s.t. $d_{\mathcal{X}}(x, x_i) < \delta_{x_i}/2$. Then for all y s.t. $d_{\mathcal{X}}(x, y) < \delta/2$, we have

$$d_{\mathcal{X}}(y, x_i) \leq d_{\mathcal{X}}(x, y) + d_{\mathcal{X}}(x, x_i) < \delta_{x_i}.$$

Therefore, we obtain

$$d_{\mathcal{Y}}(f(x), f(y)) \leq d_{\mathcal{Y}}(f(x), f(x_i)) + d_{\mathcal{Y}}(f(x_i), f(y)) \leq \epsilon,$$

which shows that f is uniformly continuous. \square

Note that in the above proof, y need not be in K . We obtain a slightly stronger result here.

Corollary 2.1. *Suppose $f : \mathcal{X} \mapsto \mathcal{Y}$ is continuous, and $K \subset \mathcal{X}$ is compact. Then $\forall \epsilon > 0$, $\exists \delta > 0$ such that $d_{\mathcal{Y}}(f(x), f(y)) < \epsilon$ whenever $x \in K$, $y \in \mathcal{X}$ and $d_{\mathcal{X}}(x, y) < \delta$.*

2.4 Riemann Integral

We consider a function $f : [a, b] \mapsto \mathbb{R}$ in this section. A partition $P = (x_0, \dots, x_N)$ is defined by $x_0 = a < x_1 < x_2 < \dots < x_N = b$. We say that Q is a refinement of P if $P \subset Q$. Let \mathcal{P} be the collection of all partitions. For $P \in \mathcal{P}$, define

$$U(f, P) = \sum_{i=1}^N \sup_{[x_{i-1}, x_i]} f \cdot (x_i - x_{i-1}),$$

$$L(f, P) = \sum_{i=1}^N \inf_{[x_{i-1}, x_i]} f \cdot (x_i - x_{i-1}).$$

Lemma 2.5. $L(f, P) \leq U(f, Q)$, $\forall P, Q \in \mathcal{P}$

Proof.

$$\begin{aligned} L(f, P) &\leq L(f, P \cup Q) \quad \text{since } \inf_{I_1} f \leq \inf_{I_2} f \text{ if } I_1 \supset I_2 \\ &\leq U(f, P \cup Q) \\ &\leq U(f, Q) \quad \text{since } \sup_{I_1} f \geq \sup_{I_2} f \text{ if } I_1 \supset I_2. \end{aligned}$$

\square

From the above lemma, we have

$$\sup_{P \in \mathcal{P}} L(f, P) \leq \inf_{P \in \mathcal{P}} U(f, P) \tag{1}$$

Definition 2.8. $f : [a, b] \mapsto \mathbb{R}$ is Riemann integrable if equality in (1) holds, i.e.,

$$\forall \epsilon > 0, \exists P \in \mathcal{P} \text{ s.t. } U(f, P) - L(f, P) < \epsilon.$$

Example 2.4. The following function is not Riemann integrable as $L(f, P) = 0$ and $U(f, P) = 1$ for all $P \in \mathcal{P}$:

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \cap [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.9. A set $A \subset \mathbb{R}$ has Lebesgue measure zero if $\forall \epsilon > 0$, there exists open intervals $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots$ s.t.

$$A \subset \bigcup_{i=1}^{\infty} (\alpha_i, \beta_i) \text{ and } \sum_{i=1}^{\infty} (\alpha_i - \beta_i) < \epsilon.$$

If a countable sequence of sets A_1, A_2, \dots each of which has Lebesgue measure zero, then the union $\bigcup_{i=1}^{\infty} A_i$ has Lebesgue measure zero. To see this, let $\epsilon > 0$ and A_j be covered by $\bigcup_i (\alpha_{ij}, \beta_{ij})$ with $\sum_i (\alpha_{ij} - \beta_{ij}) < \epsilon/2^j$.

Theorem 2.3 (Henri Lebesgue). Suppose $f : [a, b] \mapsto \mathbb{R}$ is bounded. Then f is Riemann integrable iff $\exists A \subset [a, b]$ of Lebesgue measure zero s.t. f is continuous on $[a, b] \setminus A$.¹

Proof. We first show that if f is Riemann integrable, then its set of discontinuities has Lebesgue measure zero. Observe that $y \in (a, b)$ is a point of discontinuity of f iff $\exists j \in \mathbb{Z}_+$ s.t. $\sup_I f - \inf_I f \geq 1/j$ for all open intervals $I \subset (a, b)$ containing y . Let

$$S_j = \left\{ y \in (a, b) : \sup_I f - \inf_I f \geq \frac{1}{j} \forall \text{ open intervals } I \subset (a, b) \text{ with } y \in I \right\}.$$

Then, the set of discontinuities of f in (a, b) is $\bigcup_{j=1}^{\infty} S_j$. For $\epsilon > 0$, since f is Riemann integrable, there exists some partition $P = (x_0, \dots, x_N)$ s.t.

$$U(f, P) - L(f, P) = \sum_{i=1}^N \left(\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f \right) \cdot (x_i - x_{i-1}) < \frac{\epsilon}{j}. \quad (2)$$

Let $B = \{i : (x_{i-1}, x_i) \cap S_j \neq \emptyset\}$. Then from (2), we have

$$\sum_B \left(\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f \right) \cdot (x_i - x_{i-1}) + \sum_{B^c} \left(\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f \right) \cdot (x_i - x_{i-1}) < \frac{\epsilon}{j} \quad (3)$$

Since

$$\sum_{B^c} \left(\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f \right) \cdot (x_i - x_{i-1}) \geq 0$$

and

$$\sum_B \left(\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f \right) \cdot (x_i - x_{i-1}) \geq \frac{1}{j} \sum_{i \in B} (x_i - x_{i-1}),$$

¹ f is also said to be continuous almost everywhere on $[a, b]$.

we obtain from (3),

$$\sum_{i \in B} (x_i - x_{i-1}) < \epsilon.$$

We have

$$S_j \subset \bigcup_{i \in B} (x_{i-1}, x_i) \bigcup \{x_0, x_1, \dots, x_N\},$$

therefore S_j has Lebesgue measure zero.

We next prove the converse. Fix an $\epsilon > 0$. Assume that we have A with Lebesgue measure zero, which means that there is a cover $\bigcup_{j=1}^{\infty} (\alpha_j, \beta_j) \supset A$ s.t. $\sum_{j=1}^{\infty} (\beta_j - \alpha_j) < \epsilon$. Let $K = [a, b] \setminus \bigcup_{j \geq 1} (\alpha_j, \beta_j)$, which is closed and bounded and therefore compact by Theorem 2.2. Since f is continuous, from Corollary 2.1, $\exists \delta > 0$, s.t. $|f(x) - f(y)| < \epsilon$ whenever $x \in K$, $y \in [a, b]$ and $|x - y| < \delta$.

We choose a partition P with $a = x_0 < x_1 < x_2 < \dots < x_N = b$ s.t. $\max_{1 \leq i \leq N} (x_i - x_{i-1}) < \delta$. If $[x_{i-1}, x_i] \cap K = \emptyset$, then $[x_{i-1}, x_i] \subset \bigcup_j (\alpha_j, \beta_j)$ and

$$\sum_{i: [x_{i-1}, x_i] \cap K = \emptyset} \left(\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f \right) \cdot (x_i - x_{i-1}) \leq \left(\sup_{[a, b]} f - \inf_{[a, b]} f \right) \cdot \sum_j (\beta_j - \alpha_j) < M\epsilon,$$

where $M = \sup_{[a, b]} f - \inf_{[a, b]} f < \infty$. Suppose $[x_{i-1}, x_i] \cap K \neq \emptyset$. Then for any $y, z \in [x_{i-1}, x_i]$ and $y_i \in [x_{i-1}, x_i] \cap K$, we have

$$\begin{aligned} |f(y) - f(z)| &\leq |f(y) - f(y_i)| + |f(y_i) - f(z)| \\ &< \epsilon + \epsilon \\ &= 2\epsilon, \end{aligned}$$

where the last inequality follows because $|y - y_i|, |z - y_i| < \delta$. Therefore,

$$\sum_{i: [x_{i-1}, x_i] \cap K \neq \emptyset} \left(\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f \right) \cdot (x_i - x_{i-1}) \leq 2\epsilon(b - a).$$

We finally obtain

$$U(f, P) - L(f, P) \leq M\epsilon + 2\epsilon(b - a) = (M + 2(b - a))\epsilon,$$

and the proof is complete. \square

From Theorem 2.3, we see that only a very limited class of functions f is Riemann integrable. This is not sufficient to model many practical applications. Therefore, Henri Lebesgue, a French mathematician in the 17th century, embarked on a program to introduce a much more versatile integral known as the Lebesgue integral. We will introduce this in the coming week as part of the theory of probability.