15. Martingale Convergence

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15.1 Right-Closable Martingale

We can generalize our index set to any totally ordered set $(N, \leq)$ (if $n, m \in N$, then $n \leq m$ or $m \leq n$). We say that $(\mathscr{F}_n)_{n\in N}$ is a filtration if $\mathscr{F}_n \subset \mathscr{F}_m$ for all $n \leq m$ in N.

Definition 15.1. *A submartingale $(X_n, \mathscr{F}_n)_{n\in N}$ is* right-closable *if $X_n \leq \mathbb{E}[X \mid \mathscr{F}_n]$ for all $n \in N$, for some $X \in L^1$. (For a martingale, the inequality is replaced with equality.)*

If there exists $n_0 \in N$ such that $n \leq n_0$ for all $n \in N$, then $X_n \leq \mathbb{E}[X_{n_0} \mid \mathscr{F}_n]$ for all $n \in N$, and we say that the submartingale is right-closed.

Note that a right-closable submartingale or martingale can always be made right-closed by simply adding an upper bound n_0 to N and defining $X_{n_0} = X$.

Example 15.1 (Reverse martingale)**.** *Suppose X_k, $k \geq 1$ are i.i.d. Let $S_n = \sum_{k=1}^n X_k$. Take N to be the set of negative integers and for $n \geq 1$, let*

$$\mathscr{F}_{-n} = \sigma(S_n, X_{n+1}, X_{n+2}, \ldots).$$

Since $\mathscr{F}_{-(n+1)} \subset \mathscr{F}_{-n}$, $(\mathscr{F}_{-n})_{-n\in N}$ is a filtration. By symmetry, we have for all $k = 1, \ldots, n$,

$$\mathbb{E}[X_k \mid \mathscr{F}_{-n}] = \mathbb{E}[X_1 \mid \mathscr{F}_{-n}].$$

Therefore,

$$S_n = \mathbb{E}[S_n \mid \mathscr{F}_{-n}] = \sum_{k=1}^n \mathbb{E}[X_k \mid \mathscr{F}_{-n}] = n\mathbb{E}[X_1 \mid \mathscr{F}_{-n}].$$

Let $Z_{-n} = \dfrac{S_n}{n} = \mathbb{E}[X_1 \mid \mathscr{F}_{-n}]$. Then $(Z_{-n}, \mathscr{F}_{-n})_{-n\in N}$ is a right-closed martingale.

Lemma 15.1. *(i) If $(X_n, \mathscr{F}_n)_{n\in N}$ is a right-closable martingale, then $(X_n)_{n\in N}$ is uniformly integrable.*

(ii) If $(X_n, \mathscr{F}_n)_{n\in N}$ is a right-closable submartingale, then $(X_n \vee a)_{n\in N}$ is uniformly integrable for all $a \in \mathbb{R}$.

Proof. (i) Since the martingale is right-closable, $\exists\, X \in L^1$ such that $X_n = \mathbb{E}[X \mid \mathscr{F}_n]$ for all $n \in N$. We have $|X_n| \le \mathbb{E}[|X| \mid \mathscr{F}_n]$ and $\mathbb{E}|X_n| \le \mathbb{E}|X| < \infty$. Then,

$$\begin{aligned}
\mathbb{E}|X_n|\mathbf{1}_{\{|X_n|>K\}} &\le \mathbb{E}\big[\mathbb{E}[|X| \mid \mathscr{F}_n]\mathbf{1}_{\{|X_n|>K\}}\big] \\
&= \mathbb{E}|X|\mathbf{1}_{\{|X_n|>K\}} \quad \because \{|X_n| > K\} \in \mathscr{F}_n \\
&\le M\mathbb{P}(|X_n| > K) + \mathbb{E}|X|\mathbf{1}_{\{|X|>M\}} \quad \forall\, M > 0 \\
&\overset{\text{Markov}}{\le} \frac{M}{K}\mathbb{E}|X_n| + \mathbb{E}|X|\mathbf{1}_{\{|X|>M\}}
\end{aligned}$$

and

$$\limsup_{K\to\infty} \mathbb{E}|X_n|\mathbf{1}_{\{|X_n|>K\}} \le \mathbb{E}|X|\mathbf{1}_{\{|X|>M\}} \xrightarrow{M\to\infty} 0.$$

(ii) Since the submartingale is right-closable, $\exists\, X \in L^1$ such that $X_n \le \mathbb{E}[X \mid \mathscr{F}_n]$ for all $n \in N$. We have

$$\mathbb{E}\big[X_n\mathbf{1}_{\{X_n>K\}}\big] \le \mathbb{E}\big[X\mathbf{1}_{\{X_n>K\}}\big] \tag{1}$$
$$\mathbb{E}[X_n \vee a] \le E[X \vee a] \quad \text{from Jensen's inequality.} \tag{2}$$

Take $K > |a|$. Then $|X_n \vee a| > K$ iff $X_n \vee a = X_n > K$. Therefore,

$$\begin{aligned}
\mathbb{E}\big[(X_n \vee a)\mathbf{1}_{\{|X_n\vee a|>K\}}\big] &= \mathbb{E}\big[X_n\mathbf{1}_{\{X_n>K\}}\big] \\
&\overset{(1)}{\le} \mathbb{E}\big[X\mathbf{1}_{\{X_n>K\}}\big] \\
&\le M\mathbb{P}(X_n > K) + \mathbb{E}\big[X\mathbf{1}_{\{X>M\}}\big] \quad \forall\, M > 0 \\
&\overset{\text{Markov}}{\le} \frac{M}{K}\mathbb{E}X_n^+ + \mathbb{E}|X|\mathbf{1}_{\{|X|>M\}} \\
&\le \frac{M}{K}\mathbb{E}X^+ + \mathbb{E}|X|\mathbf{1}_{\{|X|>M\}} \quad \text{from letting } a = 0 \text{ in (2).}
\end{aligned}$$

Taking $K \to \infty$ and then $M \to \infty$, we obtain the desired result.

□

15.2 Doob's Upcrossing Inequality

Consider a submartingale $(X_n, \mathscr{F}_n)_{n\ge 0}$. Let $a < b$. The first times that X_n crosses a downwards and then b upwards are given by the stopping times

$$\tau_1 = \inf\{n \ge 0 \,:\, X_n \le a\},\ \tau_2 = \inf\{n > \tau_1 \,:\, X_n \ge b\}.$$

We repeat this process and define by induction, for $k \ge 2$,

$$\tau_{2k-1} = \inf\{n > \tau_{2k-2} \,:\, X_n \le a\},\ \tau_{2k} = \inf\{n > \tau_{2k-1} \,:\, X_n \ge b\}.$$

Then

$$U_n(a,b) = \sup\{k \,:\, \tau_{2k} \le n\}$$

is the number of upward crossings of the interval $[a,b]$ up to time n.

Theorem 15.1 (Doob's upcrossing inequality). *If $(X_n, \mathscr{F}_n)_{n\ge 0}$ is a submartingale, then for any $a < b$,*

$$(b-a)\mathbb{E}U_n(a,b) \le \mathbb{E}(X_n - a)^+.$$

Proof. Let $Y_n = (X_n - a)^+$, which is a non-negative submartingale by Lemma 14.1. Then $U_n(a,b)$ is equal to the number of upcrossings of $[0, b-a]$ by $Y_0, \ldots, Y_n$. For $m \ge 1$, let

$$A_m = \begin{cases} 1, & \text{if } \tau_{2k-1} < m \le \tau_{2k} \text{ for some } k, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\{\tau_{2k-1} < m \le \tau_{2k}\} = \{\tau_{2k-1} \le m-1\} \cap \{\tau_{2k} \le m-1\}^{\mathsf{c}} \in \mathscr{F}_{m-1}$, $(A_m)_{m\ge 1}$ is predictable. Consider the martingale transform

$$\tilde{Y}_n = Y_0 + \sum_{m=1}^{n} A_m (Y_m - Y_{m-1}) \ge (b-a) U_n(a,b), \tag{3}$$

where the inequality follows because each upcrossing of $[0, b-a]$ by (Y_n) results in a gain of at least $b-a$ on the left-hand side. We have

$$\begin{aligned} \mathbb{E}[A_m(Y_m - Y_{m-1}) \,|\, \mathscr{F}_{m-1}] &= A_m \mathbb{E}[Y_m - Y_{m-1} \,|\, \mathscr{F}_{m-1}] \\ &\le \mathbb{E}[Y_m - Y_{m-1} \,|\, \mathscr{F}_{m-1}], \end{aligned}$$

where the inequality follows because $\mathbb{E}[Y_m \,|\, \mathscr{F}_{m-1}] \ge Y_{m-1}$. Therefore, $\mathbb{E}[A_m(Y_m - Y_{m-1})] \le \mathbb{E}[Y_m - Y_{m-1}]$ and

$$\begin{aligned} \mathbb{E}\tilde{Y}_n - \mathbb{E}Y_0 &\le \mathbb{E}Y_n - \mathbb{E}Y_0 \\ \mathbb{E}\tilde{Y}_n &\le \mathbb{E}Y_n \end{aligned}$$

and from (3),

$$(b-a)\mathbb{E}U_n(a,b) \le \mathbb{E}Y_n.$$

□

15.3 Submartingale Convergence

Theorem 15.2 (Submartingale Convergence Theorem). *Suppose $(X_n, \mathscr{F}_n)_{-\infty<n<\infty}$ is a submartingale.*

(i) $X_{-\infty} = \lim_{n\to-\infty} X_n$ *exists a.s. and* $\mathbb{E}X_{-\infty}^+ < \infty$. *Let* $\mathscr{F}_{-\infty} = \bigcap_n \mathscr{F}_n$. *Then* $(X_n, \mathscr{F}_n)_{-\infty\le n<\infty}$ *is a submartingale, i.e.,* $X_{-\infty} \le \mathbb{E}[X_m \,|\, \mathscr{F}_{-\infty}]$ *for all* $m > -\infty$.

(ii) *If* $\mathbb{E}X_n^+ \le B < \infty$, *then* $X_\infty = \lim_{n\to\infty} X_n$ *exists a.s. and* $\mathbb{E}X_\infty^+ \le B$.

(iii) *If* $(X_n^+)_{-\infty<n<\infty}$ *is uniformly integrable, then for* $X_\infty = \lim_{n\to\infty} X_n$, *we have* $X_m \le \mathbb{E}[X_\infty \,|\, \mathscr{F}_m]$ *for all* $m < \infty$. *Let* $\mathscr{F}_\infty = \sigma(\bigcup_n \mathscr{F}_n)$. *Then* $(X_n, \mathscr{F}_n)_{-\infty<n\le\infty}$ *is right-closed.*

Proof. Note that X_n converges as $n \to \pm\infty$ iff $\limsup X_n = \liminf X_n$,[1] i.e., X_n diverges only on the event

$$\{\limsup X_n > \liminf X_n\} = \bigcup_{\substack{a<b,\\ a,b\in\mathbb{Q}}} A_{ab},$$

where

$$A_{ab} = \{\limsup X_n \ge b > a \ge \liminf X_n\} \subset \{\lim U_n(a,b) = \infty\}.$$

Therefore to show that $\mathbb{P}(A_{ab}) = 0$, it suffices to show that $\mathbb{E}[\lim U_n(a,b)] < \infty$.

(i) For each $n \ge 1$, $U_n(a,b)$ is the number of upcrossings of the submartingale $Y_1 = X_{-n}, Y_2 = X_{-n+1}, \ldots, Y_n = X_{-1}$. From Doob's upcrossing inequality (Theorem 15.1), we have

$$\mathbb{E}U_n(a,b) \le \frac{\mathbb{E}(Y_n - a)^+}{b-a} = \frac{\mathbb{E}(X_{-1} - a)^+}{b-a} < \infty.$$

The MCT then gives $\mathbb{E}\lim_{n\to\infty} U_n(a,b) = \lim_{n\to\infty} \mathbb{E}U_n(a,b) < \infty$. Therefore $X_{-\infty} = \lim_{n\to-\infty} X_n$ exists a.s. From Fatou's lemma,

$$\mathbb{E}X_{-\infty}^+ \le \liminf_{n\to\infty} \mathbb{E}X_{-n}^+ \le \mathbb{E}X_{-1}^+ < \infty.$$

Finally, to show that $(X_n, \mathscr{F}_n)_{-\infty\le n<\infty}$ is a submartingale, we note that $X_{-\infty} \in \mathscr{F}_n$ for all n (Exercise), which implies that $X_{-\infty} \in \mathscr{F}_{-\infty}$. Furthermore, to show $X_{-\infty} \le \mathbb{E}[X_m \,|\, \mathscr{F}_{-\infty}]$ for all $m > -\infty$, it suffices to show that for all $A \in \mathscr{F}_{-\infty}$, we have $\mathbb{E}X_{-\infty}\mathbf{1}_A \le \mathbb{E}X_m\mathbf{1}_A$ (cf. Lemma 12.1).

Consider the submartingale $(X_n, \mathscr{F}_n)_{-\infty\le n\le 0}$, which is right-closed. From Lemma 15.1, $(X_n \vee a, \mathscr{F}_n)_{-\infty\le n\le 0}$ is uniformly integrable for all $a \in \mathbb{R}$. Furthermore, $X_n \vee a \to X_{-\infty} \vee a$ a.s. as $n \to \infty$. From Lemma 14.4, $X_n \vee a \to X_{-\infty} \vee a$ in L^1, hence

$$\mathbb{E}(X_{-\infty} \vee a)\mathbf{1}_A = \lim_{n\to-\infty} \mathbb{E}(X_n \vee a)\mathbf{1}_A \le \mathbb{E}(X_m \vee a)\mathbf{1}_A,$$

for all $m > -\infty$. The last inequality follows because $\mathbb{E}(X_n \vee a)\mathbf{1}_A \le \mathbb{E}(X_m \vee a)\mathbf{1}_A$ for all $n \le m$, which in turn is a consequence of the facts that $(X_n \vee a)$ is a submartingale and $A \in \mathscr{F}_m$. Taking $a \to -\infty$, the MCT gives $\mathbb{E}X_{-\infty}\mathbf{1}_A \le \mathbb{E}X_m\mathbf{1}_A$.

[1] In this proof, if the convergence is not specified, it is taken to mean either $n \to \infty$ or $n \to -\infty$.

(ii) From Doob's upcrossing inequality,

$$\mathbb{E}U_n(a,b) \le \frac{\mathbb{E}(X_n - a)^+}{b-a} \le \frac{|a| + \mathbb{E}X_n^+}{b-a} \le \frac{|a| + B}{b-a} < \infty.$$

The same argument as in the proof of Item (i) completes the proof of Item (ii).

(iii) We want to show that for all $m < \infty$ and $A \in \mathscr{F}_m$, $\mathbb{E}X_m \mathbf{1}_A \le \mathbb{E}X_\infty \mathbf{1}_A$. From Lemma 14.1, $(X_n \vee a, \mathscr{F}_n)_{n<\infty}$ is a submartingale for all $a \in \mathbb{R}$. For $K > 0$ and $K > a$, $X_n \vee a > K$ iff $X_n \vee a = X_n^+ > K$. Therefore, since (X_n^+) is uniformly integrable, so is $(X_n \vee a)_{n<\infty}$. As $X_n \vee a \to X_\infty \vee a$ a.s. as $n \to \infty$, Lemma 14.4 again yields that $X_n \vee a \to X_\infty \vee a$ in L^1, and hence

$$\mathbb{E}(X_\infty \vee a)\mathbf{1}_A = \lim_{n\to\infty} \mathbb{E}(X_n \vee a)\mathbf{1}_A \ge \mathbb{E}(X_m \vee a)\mathbf{1}_A,$$

for all $m < \infty$. Taking $a \to -\infty$ and applying the MCT give the desired result.

□

Corollary 15.1 (Martingale Convergence Theorem)**.** *Suppose $(X_n, \mathscr{F}_n)_{n<\infty}$ is a martingale.*

(i) If $\sup_n \mathbb{E}|X_n| < \infty$ (equivalently, $\sup_n \mathbb{E}X_n^+ < \infty$ or $\sup_n \mathbb{E}X_n^- < \infty$), then $X_n \to X_\infty$ a.s. and $\mathbb{E}|X_\infty| < \infty$.

(ii) If $(X_n)_{n<\infty}$ is uniformly integrable, then $(X_n, \mathscr{F}_n)_{n\le\infty}$ is a right-closed martingale.

Corollary 15.1 follows from Theorem 15.2 by simply observing that X_n and $-X_n$ are both submartingales. Furthermore, Corollary 15.1 together with Lemma 15.1 says that a martingale is right-closable iff it is uniformly integrable.

Example 15.2. *Consider the simple random walk S_n in Example 13.1 with stopping time τ when S_n hits the boundaries A or $-B$. Let $M_n = S_{n\wedge\tau}$, which is a bounded martingale. From the Martingale Convergence Theorem (Corollary 15.1), M_n converges a.s. For $\omega \in \{\tau = \infty\}$, we have $|M_n(\omega) - M_{n+1}(\omega)| = |S_n(\omega) - S_{n+1}(\omega)| = 1$, hence $M_n(\omega)$ does not converge. Therefore $\mathbb{P}(\tau = \infty) = 0$ and $\mathbb{P}(\tau < \infty) = 1$.*

Corollary 15.2 (Supermartingale Convergence Theorem)**.** *Suppose $(X_n, \mathscr{F}_n)_{n\ge0}$ is a supermartingale. If $\sup_n \mathbb{E}X_n^- < \infty$, then $X_n \to X_\infty$ a.s., and $\mathbb{E}X_\infty \le \mathbb{E}X_0$.*

Proof. $-X_n$ is submartingale with $\mathbb{E}(-X_n)^+ = \mathbb{E}X_n^-$. □

Theorem 15.3 (Levy's Convergence Theorem)**.** *Given $\mathbb{E}|X| < \infty$ and a filtration $(\mathscr{F}_n)_{n\ge0}$, let $\mathscr{F}_\infty = \sigma(\bigcup_n \mathscr{F}_n)$. Then $\mathbb{E}[X \mid \mathscr{F}_n] \to \mathbb{E}[X \mid \mathscr{F}_\infty]$ a.s.*

Proof. From Example 12.7, $X_n = \mathbb{E}[X \mid \mathscr{F}_n]$ is a martingale and $\mathbb{E}|X_n| \le \mathbb{E}|X| < \infty$. From Corollary 15.1, $X_\infty = \lim_{n\to\infty} X_n$ exists a.s. We next show that $X_\infty = \mathbb{E}[X \mid \mathscr{F}_\infty]$.

Let $A \in \bigcup_n \mathscr{F}_n$. Then there exists m such that $A \in \mathscr{F}_m$ and since X_n is a martingale, we have $\mathbb{E}X_n\mathbf{1}_A = \mathbb{E}X_m\mathbf{1}_A$ for all $n \ge m$. As $(X_n, \mathscr{F}_n)$ is right-closable, it is uniformly integrable, and Lemma 14.4 gives

$$\begin{aligned}\mathbb{E}X_\infty\mathbf{1}_A &= \lim_{n\to\infty} \mathbb{E}X_n\mathbf{1}_A \\ &= \mathbb{E}X_m\mathbf{1}_A \\ &= \mathbb{E}[\mathbb{E}[X \mid \mathscr{F}_m]\mathbf{1}_A] \\ &= \mathbb{E}X\mathbf{1}_A.\end{aligned}$$

Since $\bigcup_n \mathscr{F}_n$ is an algebra (because $(\mathscr{F}_n)$ is a filtration), the π-λ theorem completes the proof. □

Example 15.3 (Improved SLLN)**.** *For a sequence of r.v.s $(X_n)_{n\ge1}$, let $\mathscr{F}_n = \sigma(X_1, \dots, X_n)$, $n \ge 1$, be a filtration. Suppose that $\mathbb{E}[X_n \mid \mathscr{F}_{n-1}] = 0$ for all $n \ge 1$. In particular, the r.v.s are not necessarily pairwise independent. Then for $n > m$, we have $\mathbb{E}X_nX_m = \mathbb{E}[\mathbb{E}[X_nX_m \mid \mathscr{F}_m]] = \mathbb{E}[X_m\mathbb{E}[X_n \mid \mathscr{F}_m]] = 0$. Let $S_n = \sum_{k=1}^n X_k$.*

Suppose that $(b_n)_{n\ge1}$ is a sequence of positive constants increasing to ∞ and $\sum_{n=1}^\infty \dfrac{\mathbb{E}X_n^2}{b_n^2} < \infty$, then $\dfrac{S_n}{b_n} \to 0$ a.s. To see this, let $Y_n = \sum_{k=1}^n \dfrac{X_k}{b_n}$. It is easy to verify that this is a martingale. For $K > 0$, we have

$$\mathbb{E}|Y_n|\mathbf{1}_{\{|Y_n|>K\}} \le \frac{1}{K}\mathbb{E}|Y_n|^2 = \frac{1}{K}\sum_{k=1}^n \frac{\mathbb{E}X_k^2}{b_k^2}.$$

We then have

$$\sup_n \mathbb{E}|Y_n|\mathbf{1}_{\{|Y_n|>K\}} \le \frac{1}{K}\sum_{k=1}^\infty \frac{\mathbb{E}X_k^2}{b_k^2} \xrightarrow{K\to\infty} 0.$$

Therefore, (Y_n) is uniformly integrable. From the Martingale Convergence Theorem (Corollary 15.1), Y_n converges a.s. The generalized Kronecker's lemma (Lemma 7.5) then implies that $\dfrac{S_n}{b_n} \to 0$ a.s.

Example 15.4. *Continuing from Example 15.1, $(Z_{-n}, \mathscr{F}_{-n})_{-n\le1}$ is a right-closed martingale. The Submartingale Convergence Theorem (Theorem 15.2) says that $\lim_{n\to\infty} Z_{-n} = Z_{-\infty} \in \mathscr{F}_{-\infty} = \bigcap_{n\ge1} \mathscr{F}_{-n}$. Each event in $\mathscr{F}_{-n}$ is invariant under finite permutations, therefore by the Hewitt-Savage 0-1 Law, it has probability 0 or 1. Hence, $Z_{-\infty}$ is a constant a.s. But $\mathbb{E}Z_{-n} = \mathbb{E}X_1$ for all $n \ge 1$, so $Z_{-\infty} = \mathbb{E}X_1$ a.s., i.e., $\dfrac{S_n}{n} \to \mathbb{E}X_1$ a.s., which is Kolmogorov's SLLN.*

Example 15.5 (Levy's 0-1 Law). *From Theorem 15.3, if $\mathscr{F}_n \uparrow \mathscr{F}_\infty$, then for any $A \in \mathscr{F}_\infty$, we have $\mathbb{E}[\mathbf{1}_A \,|\, \mathscr{F}_n] \to \mathbf{1}_A$ a.s. The famous probabilist K. L. Chung once commented that this result "is obvious or incredible".*

1. *It is obvious: $\mathbb{E}[\mathbf{1}_A \,|\, \mathscr{F}_\infty] = \mathbf{1}_A$.*
2. *It is incredible: Consider X_i, $i \geq 1$, independent with $\mathscr{F}_n = \sigma(X_1, \ldots, X_n) \uparrow \mathscr{F}_\infty$. An event $A \in \mathcal{T} = \bigcap_{n\geq 1} \sigma(X_n, X_{n+1}, \ldots)$ the tail σ-algebra is independent of $\mathscr{F}_n$ for all $n \geq 1$ (recall the grouping lemma (Lemma 8.3)). Therefore, $\mathbb{P}(A) = \mathbb{E}[\mathbf{1}_A \,|\, \mathscr{F}_n] \xrightarrow{n\to\infty} \mathbf{1}_A \in \{0, 1\}$ a.s. from Levy's 0-1 Law, which recovers Kolmogorov's 0-1 Law!*