An Analytical Introduction to Probability Theory

13. Optional Stopping

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Review: Consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. A sequence of sub-$\sigma$-algebras $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \ldots \subset \mathscr{F}$ is called a filtration. Suppose $M_n \in \mathscr{F}_n$. Then, $(M_n, \mathscr{F}_n)$ is a martingale if

$$\mathbb{E}|M_n| < \infty,\ \forall\, n \geq 0,$$
$$\mathbb{E}[M_n \,|\, \mathscr{F}_{n-1}] = M_{n-1},\ \forall\, n \geq 0.$$

13.1 Stopping Times

Definition 13.1. *The random variable τ is called a* stopping time *for $(\mathscr{F}_n)_{n\geq 0}$ if $\tau \in \{0, 1, \ldots\} \cup \{\infty\}$ and $\{\tau \leq n\} \in \mathscr{F}_n,\ \forall\, n \geq 0$.*

From the definition of σ-algebras, we have

$$\begin{aligned}
\{\tau > n\} &= \{\tau \leq n\}^{\mathsf{c}} \in \mathscr{F}_n,\\
\{\tau = n\} &= \{\tau \leq n\} \setminus \{\tau \leq n-1\} \in \mathscr{F}_n.
\end{aligned}$$

Consider a sequence $(X_n)_{n\geq 0}$. If $\tau < \infty$ a.s., we define

$$X_\tau = \sum_{k=0}^{\infty} \mathbf{1}_{\{\tau = k\}} X_k$$

and we call X_τ a "stopped process". Denote $n \wedge \tau = \min(n, \tau)$ and we have $n \wedge \tau < \infty$ a.s.

Theorem 13.1. *If $(M_n, \mathscr{F}_n)$ is a martingale, then $(M_{n\wedge\tau}, \mathscr{F}_n)$ is a martingale.*

Proof. WLOG, we assume $M_0 = 0$. Let $A_n = \mathbf{1}_{\{\tau \geq n\}} = \mathbf{1}_{\{\tau \leq n-1\}^{\mathsf{c}}}$, where $\{\tau \leq n-1\}^{\mathsf{c}} \in \mathscr{F}_{n-1}$. Therefore, A_n is predictable. The martingale transform $(\sum_{k=1}^{n} A_k(M_k - M_{k-1}), \mathscr{F}_n)$ is thus a martingale from Theorem 12.1. Furthermore, we have

$$\begin{aligned}
\sum_{k=1}^{n} A_k(M_k - M_{k-1}) &= \sum_{k=1}^{n} \mathbf{1}_{\{\tau \geq k\}}(M_k - M_{k-1})\\
&= \sum_{k=1}^{n} \mathbf{1}_{\{\tau \geq k\}} M_k - \sum_{k=0}^{n-1} \mathbf{1}_{\{\tau \geq k+1\}} M_k\\
&= \sum_{k=1}^{n-1} \mathbf{1}_{\{\tau = k\}} M_k + \mathbf{1}_{\{\tau \geq n\}} M_n\\
&= \mathbf{1}_{\{\tau \leq n-1\}} M_\tau + \mathbf{1}_{\{\tau \geq n\}} M_n\\
&= M_{n\wedge\tau},
\end{aligned}$$

and the theorem is proved. $\square$

Example 13.1 (Random walk)**.** *Let*

$$X_i = \begin{cases} 1 & w.p.\ \frac{1}{2}, \\ -1 & w.p.\ \frac{1}{2}, \end{cases}$$

for $i \geq 1$ be i.i.d. random variables. Let $S_n = \sum_{i=1}^n X_i$. Then, $(S_n, \mathscr{F}_n)$ is a martingale (see Example 12.3). Let $A, B \in \mathbb{Z}^+$ and $\tau = \inf\{n : S_n = A \text{ or } S_n = -B\}$ (the first time S_n hitting A or $-B$). From Theorem 13.1, $(S_{n\wedge\tau}, \mathscr{F}_n)$ is a martingale. We therefore have

$$\mathbb{E}S_{n\wedge\tau} = \mathbb{E}S_0 = 0.$$

Suppose S_n has not hit $-B$ yet. Then, a sequence of $A+B$ realizations of $X_i = 1$ will make the sum hit A. Let $E_k = \{X_i = 1 : i \in [k(A+B), (k+1)(A+B))\}$, which are independent for all $k \geq 0$. We have

$$\mathbb{P}(\tau > n(A+B)) \leq \mathbb{P}\left(\bigcap_{k=0}^{n-1} E_k^{\mathsf{c}}\right) = \left(1 - \frac{1}{2^{A+B}}\right)^n.$$

Furthermore,

$$\begin{aligned} \mathbb{E}\tau &= \sum_{k=1}^{\infty} \mathbb{P}(\tau \geq k) \\ &\leq \sum_{n=0}^{\infty} \mathbb{P}(\tau > n(A+B))(A+B) \\ &\leq (A+B) \sum_{n=0}^{\infty} \left(1 - \frac{1}{2^{A+B}}\right)^n \\ &< \infty. \end{aligned}$$

Therefore, $\tau < \infty$ a.s. We also have $|S_{n\wedge\tau}| \leq \max\{A, B\}$. From DCT, we then obtain

$$\begin{aligned} 0 = \lim_{n\to\infty} \mathbb{E}S_{n\wedge\tau} = \mathbb{E}S_\tau &= A\mathbb{P}(S_\tau = A) - B(1 - \mathbb{P}(S_\tau = A)) \\ \implies \mathbb{P}(S_\tau = A) &= \frac{B}{A+B}. \end{aligned}$$

We will see another way to prove $\tau < \infty$ a.s. in Example 15.2.

Example 13.2. *Let $M_n = S_n^2 - n$ and $(M_n, \mathscr{F}_n)$ is a martingale (see Example 12.4) and τ be the same stopping time as in Example 13.1. Then $(M_{n\wedge\tau}, \mathscr{F}_n)$ is a martingale with $\mathbb{E}M_{n\wedge\tau} = \mathbb{E}M_0 = 0$. We have*

$$|M_{n\wedge\tau}| \leq (A \vee B)^2 + \tau,$$

and since τ *is integrable (see Example 13.1), the DCT yields*

$$\mathbb{E}M_\tau = \lim_{n\to\infty} \mathbb{E}M_{n\wedge\tau} = 0.$$

Therefore, we have $0 = \mathbb{E}M_\tau = \mathbb{E}S_\tau^2 - \mathbb{E}\tau$*, and*

$$\begin{aligned}\mathbb{E}\tau &= \mathbb{E}S_\tau^2 \\ &= \frac{B}{A+B}A^2 + \frac{A}{A+B}B^2 \\ &= AB.\end{aligned}$$

Example 13.3 (Biased random walk)**.** *Let*

$$X_i = \begin{cases} 1 & w.p.\ p \neq \frac{1}{2}, \\ -1 & w.p.\ q = 1-p, \end{cases}$$

for $i \geq 1$ *be i.i.d. random variables and* $\phi(\lambda) = \mathbb{E}e^{\lambda X_1} = pe^\lambda + qe^{-\lambda}$*. Let*

$$M_n = \frac{e^{\lambda S_n}}{\phi(\lambda)^n}.$$

We choose λ *such that* $\phi(\lambda) = 1 \implies e^\lambda = \dfrac{q}{p} \implies M_n = e^{\lambda S_n} = \left(\dfrac{q}{p}\right)^{S_n}$*. Then* $(M_n, \mathscr{F}_n)$ *is a martingale (see Example 12.6), and so is* $(M_{n\wedge\tau}, \mathscr{F}_n)$*, where* τ *is the same stopping time as in Example 13.1. We have*

$$\begin{aligned}&\mathbb{E}M_{n\wedge\tau} = \mathbb{E}M_0 = 1, \\ &|M_{n\wedge\tau}| \leq (1 \vee q/p)^{A\vee B}.\end{aligned}$$

From DCT, we have

$$\mathbb{E}M_\tau = \lim_{n\to\infty} \mathbb{E}M_{n\wedge\tau} = 1.$$

Denoting $\alpha = \mathbb{P}(S_\tau = A)$*, we have*

$$\mathbb{E}M_\tau = \alpha\left(\frac{q}{p}\right)^A + (1-\alpha)\left(\frac{q}{p}\right)^{-B}.$$

Therefore, we obtain

$$\alpha = \frac{\left(\frac{q}{p}\right)^B}{\left(\frac{q}{p}\right)^{A+B} - 1}.$$

13.2 Doob's Optional Stopping Theorem

In the previous examples, we have $\mathbb{E}M_\tau = \mathbb{E}M_0$ using the DCT and special properties of the stopping time τ. We want to know when this is true in general. But first, a counterexample.

Example 13.4 (Counterexample)**.** *Let*

$$X_n = \begin{cases} 2^n & w.p.\ \frac{1}{2}, \\ -2^n & w.p.\ \frac{1}{2}, \end{cases}$$

for $n \geq 0$ be independent random variables. Then $(S_n, \mathscr{F}_n)$ is a martingale. Let $\tau = \min\{k \geq 0 : S_k > 0\}$. When $\tau = k$, we have $S_\tau = S_k = -1 - 2 - 2^2 - \ldots - 2^{k-1} + 2^k = 1$. Therefore, $\mathbb{E}S_\tau = 1 \neq \mathbb{E}S_0 = \mathbb{E}X_0 = 0$.

Let τ be a stopping time w.r.t. $(\mathscr{F}_n)$. The σ-algebra (exercise: verify that it is a σ-algebra)

$$\mathscr{F}_\tau = \{A \in \mathscr{F} \ : \ \{\tau \leq n\} \cap A \in \mathscr{F}_n, \forall\, n \geq 0\}$$

consists of all events that depend on information up to the stopping time τ. We have the following properties:

(i) $\tau \in \mathscr{F}_\tau$ (τ is measurable w.r.t. $\mathscr{F}_\tau$). This is because $\{\tau \leq n\} \cap \{\tau \leq k\} = \{\tau \leq n \wedge k\} \in \mathscr{F}_{n\wedge k} \subset \mathscr{F}_n$ since τ is a stopping time.

(ii) If (M_n) is adapted to $(\mathscr{F}_n)$ and $\tau < \infty$ a.s., then $M_\tau \in \mathscr{F}_\tau$. Proof: $\forall\, B \in \mathcal{B}, \{\tau \leq n\} \cap \{M_\tau \in B\} = \bigcup_{k \leq n} \{\tau = k\} \cap \{M_k \in B\} \in \mathscr{F}_n$.

(iii) Let τ, ν be stopping times w.r.t. $(\mathscr{F}_n)$. Then $\{\tau < \nu\}, \{\tau = \nu\}, \{\tau > \nu\} \in \mathscr{F}_\tau, \mathscr{F}_\nu$.

(iv) If $A \in \mathscr{F}_\nu$, then $A \cap \{\tau \geq \nu\}$, $A \cap \{\tau > \nu\} \in \mathscr{F}_\tau$.

(v) $\tau(\omega) \leq \nu(\omega), \forall\, \omega \in \Omega \implies \mathscr{F}_\tau \subset \mathscr{F}_\nu$.

Theorem 13.2 (Optional Stopping Theorem)**.** *Let $(M_n, \mathscr{F}_n)$ be a martingale. Suppose*

(i) $\tau, \nu < \infty$ *are stopping times w.r.t.* $(\mathscr{F}_n)$,

(ii) $\mathbb{E}|M_\tau| < \infty$,

(iii) $\lim_{n\to\infty} \mathbb{E}|M_n| \mathbf{1}_{\{\tau \geq n\}} = 0$.

Then $\forall\, A \in \mathscr{F}_\nu$,

$$\mathbb{E}M_\tau \mathbf{1}_{A \cap \{\tau \geq \nu\}} = \mathbb{E}M_\nu \mathbf{1}_{A \cap \{\tau \geq \nu\}}.$$

An intuitive interpretation is as follows: A martingale $(M_n, \mathscr{F}_n)$ represents a fair game (e.g., M_n is stock price under an efficient market). A stopping time τ represents a strategy to stop the game based only on information up to a given time (e.g., to sell off a stock, execute an option, etc.). By taking $A = \Omega$ and $\nu = 0$, the theorem says that $\mathbb{E}M_\tau = \mathbb{E}M_0$ for any τ satisfying the theorem conditions. Thus, there is no "winning strategy" in an efficient market! Furthermore, given any two competing strategies τ and ν, there is no advantage ν can gain over τ in terms of expected wealth even on events it has full information about.

In Example 13.4, we can check that

$$\mathbb{E}|S_n|\mathbf{1}_{\{\tau \geq n\}} = \mathbb{P}(\tau = n) + (2^{n+1} - 1)\mathbb{P}(\tau \geq n+1) = 1,$$

which violates the third condition in Theorem 13.2.

Proof. For $A \in \mathscr{F}_\nu$, let $A_n = A \cap \{\nu = n\} \in \mathscr{F}_n$. We first show that

$$\mathbb{E}M_\tau \mathbf{1}_{A_n \cap \{\tau \geq n\}} = \mathbb{E}M_n \mathbf{1}_{A_n \cap \{\tau \geq n\}}. \tag{1}$$

We have

$$\begin{aligned}
\mathbb{E}M_n \mathbf{1}_{A_n \cap \{\tau \geq n\}} &= \mathbb{E}M_n \mathbf{1}_{A_n \cap \{\tau = n\}} + \mathbb{E}M_n \mathbf{1}_{A_n \cap \{\tau > n\}} \\
&= \mathbb{E}M_\tau \mathbf{1}_{A_n \cap \{\tau = n\}} + \mathbb{E}\big[\mathbb{E}[M_{n+1} \,|\, \mathscr{F}_n]\, \mathbf{1}_{A_n \cap \{\tau > n\}}\big] \\
&= \mathbb{E}M_\tau \mathbf{1}_{A_n \cap \{\tau = n\}} + \mathbb{E}\big[\mathbb{E}\big[M_{n+1} \mathbf{1}_{A_n \cap \{\tau > n\}} \,\big|\, \mathscr{F}_n\big]\big] \quad \because \mathbf{1}_{A_n \cap \{\tau > n\}} \in \mathscr{F}_n \\
&= \mathbb{E}M_\tau \mathbf{1}_{A_n \cap \{\tau = n\}} + \mathbb{E}M_{n+1} \mathbf{1}_{A_n \cap \{\tau \geq n+1\}}.
\end{aligned}$$

By induction, we have for any $m > n$,

$$\mathbb{E}M_n \mathbf{1}_{A_n \cap \{\tau \geq n\}} = \mathbb{E}M_\tau \mathbf{1}_{A_n \cap \{n \leq \tau < m\}} + \mathbb{E}M_m \mathbf{1}_{A_n \cap \{\tau \geq m\}}. \tag{2}$$

We have

$$\left|\mathbb{E}M_m \mathbf{1}_{\{A_n \cap \{\tau \geq m\}\}}\right| \leq \mathbb{E}|M_m| \mathbf{1}_{\tau \geq m} \xrightarrow{m \to \infty} 0.$$

Since $\mathbb{E}|M_\tau| < \infty$, the DCT yields

$$\lim_{m \to \infty} \mathbb{E}M_\tau \mathbf{1}_{A_n \cap \{n \leq \tau < m\}} = \mathbb{E}M_\tau \mathbf{1}_{A_n \cap \{\tau \geq n\}}.$$

Therefore by letting $m \to \infty$ in (2), we obtain (1). We finally have

$$\begin{aligned}
\mathbb{E}M_\tau \mathbf{1}_{A \cap \{\tau \geq \nu\}} &= \sum_{n \geq 0} \mathbb{E}M_\tau \mathbf{1}_{A \cap \{\nu = n\} \cap \{\tau \geq n\}} \\
&= \sum_{n \geq 0} \mathbb{E}M_\tau \mathbf{1}_{A_n \cap \{\tau \geq n\}} \\
&\overset{(1)}{=} \sum_{n \geq 0} \mathbb{E}M_n \mathbf{1}_{A_n \cap \{\tau \geq n\}} \\
&= \mathbb{E}M_\nu \mathbf{1}_{A \cap \{\tau \geq \nu\}},
\end{aligned}$$

which completes the proof. $\square$

Some commonly used special cases of Theorem 13.2 are the following:

1. Let $\nu = 0$ and $A = \Omega$. Then from Theorem 13.2, we have $\mathbb{E}M_\tau = \mathbb{E}M_0$.
2. If $\tau \le B < \infty$ is bounded a.s., then the conditions in Theorem 13.2 are satisfied.
3. Suppose $\tau < \infty$. If $|M_n| \le K < \infty$ is bounded a.s., then the conditions in Theorem 13.2 are satisfied.

If we have stronger conditions on the martingale, we can obtain similar optional stopping results without appealing to Theorem 13.2. One such result is assuming *bounded* martingale differences.

Lemma 13.1. *Let $(M_n, \mathscr{F}_n)$ be a martingale. If $|M_n - M_{n-1}| \le K < \infty$ a.s. and $\mathbb{E}\tau < \infty$, then $\mathbb{E}M_\tau = \mathbb{E}M_0$.*

Proof. We have

$$|M_{n\wedge\tau} - M_0| = \left|\sum_{k=1}^{n\wedge\tau}(M_k - M_{k-1})\right| \le K\tau.$$

Then, from DCT, we obtain

$$\mathbb{E}M_\tau = \lim_{n\to\infty} \mathbb{E}M_{n\wedge\tau} = \mathbb{E}M_0,$$

where the last inequality follows from Theorem 13.1. □

Example 13.5. *At each time $n = 1, 2, \ldots$, a monkey chooses one letter randomly and uniformly from the 26 English letters. What is the expected time for the monkey to produce the sequence "ABRACADABRA"?*

Solution: Let T be the first time the monkey produces the desired sequence. One can show that $\mathbb{E}T < \infty$ and hence $T < \infty$ a.s. (hint: see Example 13.1). Suppose at each time j, a new gambler arrives, bet 1 that the j-th letter is "A". If he loses, he leaves. If he wins, he receives a reward of 26, bet all of that on the $(j+1)$-th letter being "B", and so on according to the sequence "ABRACADABRA". When the end of the sequence is reached, he also leaves. Denote M_n^j as the payoff of gambler j at time n. We let $M_n^j = 0$ if $n < j$. we have

$$\mathbb{E}M_n^j \le 26^{n-j+1} < \infty.$$

If the gambler loses before time n, we have

$$\mathbb{E}\big[M_n^j \,\big|\, \mathscr{F}_{n-1}\big] = 0 = M_{n-1}^j.$$

Otherwise, he wins $n - j$ times from j till time $n-1$ and we have

$$\mathbb{E}\big[M_n^j \,\big|\, \mathscr{F}_{n-1}\big] = 26^{n-j+1} \cdot \frac{1}{26} + 0 \cdot \frac{25}{26} = 26^{n-j} = M_{n-1}^j.$$

Therefore, $(M_n^j, \mathscr{F}_n)$ *is a martingale. Let* $X_0 = 0$ *and* $X_n = \sum_{j=1}^n (M_n^j - 1)$ *for* $n \geq 1$*, which is the total payoff of all gamblers at time* n*. It is clear that* $(X_n, \mathscr{F}_n)$ *is a martingale and* $|X_n - X_{n-1}| \leq 26^{11} + 26^4 + 26 < \infty$ *(consider the maximum increase or decrease in payoff going from time* $n-1$ *to* n*). From Lemma 13.1 (for the case where the martingale has bounded differences), we have*

$$\begin{aligned} &\mathbb{E}\big[26^{11} + 26^4 + 26 - T\big] = \mathbb{E}X_T = \mathbb{E}X_0 = 0. \\ \implies &\mathbb{E}T = 26^{11} + 26^4 + 26. \end{aligned}$$