1.1 Introduction

This basic real analysis introduction covers topics in sequences, compact spaces and continuity. Recommended textbook is "Principles of Mathematical Analysis" by Walter Rudin.

1.2 Sequences

Inner product space $\subset$ normed space $\subset$ metric space $\subset$ topological space. Working with metric spaces suffices for our purposes.

Definition 1.1. *A metric space is an ordered pair $(\mathcal{X}, d)$ where $\mathcal{X}$ is a set and d is a metric on $\mathcal{X}$, such that $\forall\, x, y, z \in \mathcal{X}$, the following holds:*

1) $d(x, y) \geq 0$, where equality holds iff $x = y$.

2) $d(x, y) = d(y, x)$.

3) Triangle inequality: $d(x, y) \leq d(x, z) + d(y, z)$.

Example 1.1. *Examples of metric space:*

- $(\mathbb{R}, |x - y|)$ *(normed space)*

- $\left(\mathbb{R}^k, \|x - y\|_p\right)$, *where $p \geq 1$ (normed space)*

- $\ell^p(\mathbb{R}) = \left\{ x = (x_n)_{n \geq 1} \,:\, \|x\|_p = \left(\sum_{n \geq 1} |x_n|^p \right)^{1/p} < \infty \right\}$ *(normed space)*

- $(\mathcal{X}, d)$, *where $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ otherwise.*

We write (x_n) or $(x_n)_{n \geq 1}$ as shorthand for the sequence $x_1, x_2, \ldots$.

Definition 1.2. *For $(x_n) \subset \mathcal{X}$, we say that $x_n \to x$ as $n \to \infty$ or $\lim_{n \to \infty} x_n = x$ if*

$$\forall\, \epsilon > 0,\ \exists N \ \text{s.t.}\ d(x_n, x) < \epsilon,\ \forall\, n \geq N.$$

Definition 1.3. *We say x_n diverges to ∞ if there exists $x_0 \in \mathcal{X}$ and*

$$\forall K \geq 0, \ \exists N \ \text{s.t.} \ d(x_n, x_0) \geq K, \ \forall n \geq N.$$

We define extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. For $x \in \mathbb{R}$, we have:

1) $x + \infty = \infty$

2) $x - \infty = -\infty$

3) $x \cdot \pm\infty = \pm\infty, \forall\, x > 0$

4) $0 \cdot \pm\infty = 0$ (by convention)

The following lemmas are based on $(\mathbb{R}, |\cdot|)$ but most can be generalized to $(\mathcal{X}, d)$.

Lemma 1.1. *If (a_n) converges, then its limit is unique.*

Proof. Supposing $a_n \to a$ and $a_n \to a'$, we have

$$\forall \epsilon > 0, \ \exists N \ \text{s.t.} \ |a_n - a| < \epsilon \ \text{and} \ |a_n - a'| < \epsilon, \ \forall n \geq N.$$

By the triangle inequality, we obtain:

$$|a - a'| \leq |a_n - a| + |a_n - a'| < 2\epsilon.$$

Since ϵ is arbitrary, $a = a'$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 1.2. *If (a_n) converges, $|a_n| < \infty$.*

Proof. Supposing $a_n \to a$, $\exists N$, s.t. $|a_n - a| < 1, \forall n \geq N$. Then we obtain

$$|a_n| \leq |a| + 1, \forall \ n \geq N,$$

and

$$|a_n| \leq \max\{a_1, \ldots, a_{N-1}, |a| + 1\} < \infty.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 1.3. *If (a_n) is increasing (i.e., $a_{n+1} \geq a_n$ for all $n \geq 1$) and is bounded above, then (a_n) converges.*

Proof. To prove the lemma, we need some definitions. We define $\sup A = $ least upper bound (LUB) of A if $A \neq \emptyset$ and $A \subset \mathbb{R}$, i.e.,

1) $\forall a \in A$, $a \leq \sup A$.

2) if $\gamma < \sup A$, $\exists a \in A$, s.t. $\gamma < a$.

For the definition to be well-defined, we need the LUB/Completeness axiom: $\sup A$ exists in $\mathbb{R}$ for all $A \neq \emptyset$ bounded above. (Note that this axiom does not hold for $\mathbb{Q}$.) If A is not bounded above, we set $\sup A = \infty$. From Item 2), we obtain a useful property:

$$\forall \epsilon > 0, \ \exists a \in A, \ \text{s.t.} \ \sup A - \epsilon < a.$$

The greatest lower bound of A is written as $\inf A$.

Returning to the proof of the lemma, let $a = \sup a_n < \infty$ since (a_n) is bounded. We have

$$\forall \epsilon > 0, \ \exists N, \ \text{s.t.} \ a - \epsilon \leq a_N \leq a_n \leq a, \ \forall n \geq N.$$
$$\implies a_n \to a.$$

$\square$

Applying Lemma 1.3 to the negative of a sequence, we also conclude that any decreasing sequence that is bounded below converges.

Remark 1.1.

1. *If $a_n \in \overline{\mathbb{R}}_+$ and is increasing, then $\lim_{n \to \infty} a_n$ exists (maybe ∞).*

2. *If $a_n \in \mathbb{R}_+$, then from Lemma 1.3, $\sum_{k=1}^{\infty} a_k \triangleq \lim_{n \to \infty} \sum_{k=1}^{n} a_k$ exists.*

Definition 1.4. *A subsequence of (a_n) is a sequence (a_{n_i}), where $n_1 < n_2 < \ldots$ is an increasing sequence of indices.*

Lemma 1.4. *If $a_n \to a \iff a_{n_i} \to a$, for all subsequences (a_{n_i}).*

Proof. The direction "$\Leftarrow$" is obvious. To prove the other direction, we have $\forall \epsilon > 0$, $\exists N$ s.t. $|a_n - a| < \epsilon$, $\forall n \geq N$. Therefore, $\forall n_i \geq N$, $|a_{n_i} - a| < \epsilon$. $\square$

Lemma 1.5. *Every sequence (a_n) in $\mathbb{R}$ has a monotone subsequence.*

Proof. Suppose there are no increasing subsequences. Then, $\exists n_1$, s.t. $a_{n_1} \geq a_n$, $\forall n \geq n_1$. Similarly, $\exists n_2 > n_1$, s.t. $a_{n_2} \geq a_n$, $\forall n \geq n_2$, and so on. This constructs a decreasing subsequence $a_{n_1} \geq a_{n_2} \geq \ldots$. $\square$

Lemma 1.6. *Every bounded sequence in $\mathbb{R}$ contains a convergent subsequence.*

Proof. From Lemma 1.5, a bounded sequence has a monotone subsequence, which converges from Lemma 1.3. $\square$

Let $S =$ set of subsequence limits of $(a_n) \subset \overline{\mathbb{R}}$. We define $\limsup_{n\to\infty} a_n = \sup S$, and $\liminf_{n\to\infty} a_n = \inf S$. We have

$$\forall \, \epsilon > 0, \ \exists \, s \in S, \ \text{s.t.} \ \sup S - \epsilon/2 \le s \le \sup S.$$

Let $a_{n_i} \to s$ (can you see why such a subsequence must exist?) so that

$$\forall \, \epsilon > 0, \ \exists \, N, \ \text{s.t.} \ |a_{n_i} - s| < \epsilon/2, \ \forall \, n_i > N.$$

Therefore,

$$|a_{n_i} - \sup S| \le |a_{n_i} - s| + |s - \sup S| < \epsilon,$$

which implies that $\sup S \in S$ and $\sup S = \max S$, i.e., $\limsup_{n\to\infty} a_n$ exists. Similarly, we have $\inf S = \min S$.

Lemma 1.7. *Suppose* $s = \limsup_{n\to\infty} a_n > -\infty$. *Then* $\forall \, \epsilon > 0$, *$\exists \, N$ s.t. $a_n \le s + \epsilon$, $\forall \, n \ge N$.*

Proof. If $s = \infty$, the lemma obviously holds. Suppose $s < \infty$. We prove by contradiction. Suppose $\exists \, \epsilon > 0$ and subsequence (a_{n_i}) s.t. $a_{n_i} > s + \epsilon$. Since $s < \infty$, (a_{n_i}) is bounded. Then from Lemma 1.6, $\exists$ subsubsequence $(a_{n_i'})$, s.t. $a_{n_i'} \to a' \ge s + \epsilon > s$, a contradiction to the definition of s. $\qquad\square$

Lemma 1.8. $\limsup\limits_{n\to\infty} a_n = \inf\limits_{n\ge 1} \sup\limits_{m\ge n} a_m$ *and* $\liminf\limits_{n\to\infty} a_n = \sup\limits_{n\ge 1} \inf\limits_{m\ge n} a_m$.

Proof. If $\limsup_{n\to\infty} a_n = \pm\infty$, then the result clearly holds. Suppose $-\infty < \limsup_{n\to\infty} a_n = \sup S < \infty$, where S is the set of subsequential limits.

For any $\epsilon > 0$, there exists a subsequence (a_{n_i}) s.t. $\sup S - \epsilon \le a_{n_i}$. Then,

$$\sup S - \epsilon \le \sup_{m\ge n} a_m, \ \forall \, n \ge n_1.$$

Letting $n \to \infty$, we have

$$\sup S - \epsilon \le \lim_{n\to\infty} \sup_{m\ge n} a_m = \inf_{n\ge 1} \sup_{m\ge n} a_m,$$

since $\sup_{m\ge n} a_m$ is a decreasing sequence in n. As ϵ is arbitrary, we have

$$\sup S \le \inf_{n\ge 1} \sup_{m\ge n} a_m.$$

On the other hand, for each $n \ge 1$, $\exists \, a_{m_n} \ge \sup_{m\ge n} a_m - 1/n$. Then we have

$$\sup S \ge \lim_{n\to\infty} a_{m_n} \ge \inf_{n\ge 1} \sup_{m\ge n} a_m.$$

The other claim is proved similarly and the proof is now complete. $\qquad\square$

1.3   Cauchy Sequences

Definition 1.5. *A sequence (a_n) is a Cauchy sequence if $\forall \epsilon > 0$, $\exists N$ s.t. $d(a_n, a_m) < \epsilon$, $\forall n, m \geq N$.*

Lemma 1.9. *If (a_n) converges, then it is Cauchy.*

Proof. Suppose $a_n \to a$. Then $\forall \epsilon > 0$, $\exists N$ s.t. $d(a_n, a) < \epsilon/2 \; \forall n \geq N$. Then $\forall n, m \geq N$, we have

$$d(a_n, a_m) \leq d(a_n, a) + d(a_m, a) < \epsilon.$$

$\square$

Definition 1.6. *A metric space $(\mathcal{X}, d)$ in which every Cauchy sequence converges in $\mathcal{X}$ is complete.*

Example of an incomplete space is $\mathbb{Q}$, the set of rational numbers: for every irrational number, one can construct a sequence of rational numbers that converges to it.

Theorem 1.1. $\mathbb{R}^k$ *is complete.*

Proof. WLOG, we prove for $\mathbb{R}$. We first show 2 lemmas.

Lemma 1.10. *A Cauchy sequence is bounded.*

Proof. Similar to Lemma 1.2. $\square$

Lemma 1.11. *If (a_n) is Cauchy and there is a subsequence $a_{n_i} \to a$, then $a_n \to a$.*

Proof. For any $\epsilon > 0$, $\exists N$ such that $\forall n, n_i \geq N$,

$$d(a_n, a) \leq d(a_n, a_{n_i}) + d(a_{n_i}, a) \leq \epsilon.$$

$\square$

We now return to the proof of Theorem 1.1. Let (a_n) be a Cauchy sequence in $\mathbb{R}$. Then (a_n) is bounded from Lemma 1.10. From Lemma 1.6, there exists a convergent subsequence. Lemma 1.11 then implies (a_n) converges and the proof is complete. $\square$