An Analytical Introduction to Probability Theory

6. Borel-Cantelli Lemmas

DASN, NTU https://personal.ntu.edu.sg/wptay/

6.1 Introduction

Let $A_1, A_2, \ldots \in \mathcal{A}$ be an infinite sequence of events. Let $N(\omega) = \sum_{i=1}^{\infty} \mathbf{1}_{A_i}(\omega)$. The set of sample points $\omega \in \Omega$ that belong to events in $\{A_1, A_2, \ldots\}$ infinitely often (i.o.) is given by

$$\left\{A_n \text{ i.o.}\right\} \triangleq \left\{\omega : N(\omega) = \infty\right\} = \bigcap_{n\geq 1} \bigcup_{m\geq n} A_m \triangleq \limsup_{n\to\infty} A_n.$$

Note that $\mathbf{1}_{\{A_n \text{ i.o.}\}} = \limsup_{n\to\infty} \mathbf{1}_{A_n}$.

The set of sample points that belong finitely often (f.o.) to the events in the sequence is

$$\left\{A_n \text{ f.o.}\right\} \triangleq \left\{\omega : N(\omega) < \infty\right\} = \bigcup_{n\geq 1} \bigcap_{m\geq n} A_m^c \triangleq \liminf_{n\to\infty} A_n^c.$$

Similarly, $\mathbf{1}_{\{A_n \text{ f.o.}\}} = \liminf_{n\to\infty} \mathbf{1}_{A_n^c}$.

Notice that $\left\{A_n \text{ i.o.}\right\}$ and $\left\{A_n \text{ f.o.}\right\}$ are complements of each other. Thus, $\mathbb{P}(A_n \text{ i.o.}) + \mathbb{P}(A_n \text{ f.o.}) = 1$.

As an example, suppose $X_n(\omega) \to 0$ for all $\omega \in \Omega$, then

$$\exists\, n_0(\omega), \text{ s.t. } X_n(\omega) \leq 1, \forall\, n \geq n_0(\omega).$$

Therefore $\omega \in \{X_n \geq 1\}$ cannot be infinitely often (i.o.).

6.2 Borel-Cantelli Lemmas

Lemma 6.1 (Borel-Cantelli Lemmas)**.**

(i) If $\sum_{n\geq 1} \mathbb{P}(A_n) < \infty$, *then* $\mathbb{P}(A_n \text{ i.o.}) = 0$.

(ii) If $A_1, A_2, \ldots$ *are independent, and* $\sum_{n\geq 1} \mathbb{P}(A_n) = \infty$, *then* $\mathbb{P}(A_n \text{ i.o.}) = 1$.

Proof.

(i) Let $B_n = \bigcup_{m \geq n} A_m$, then $B_{n+1} \subset B_n$. We have

$$\mathbb{P}(A_n \text{ i.o.}) = \mathbb{P}\left(\bigcap_{n \geq 1} B_n\right) = \lim_{n \to \infty} \mathbb{P}(B_n) \leq \lim_{n \to \infty} \sum_{m \geq n} \mathbb{P}(A_m),$$

where the second equality and last inequality follow from Lemma 3.1 and Lemma 3.3, respectively. Since $\sum_{n \geq 1} \mathbb{P}(A_n) < \infty$, we have $\lim_{n \to \infty} \sum_{m \geq n} \mathbb{P}(A_m) = 0$.

(ii) Let $C_n = \bigcap_{m \geq n} A_m^c$. From independence, we have

$$\begin{aligned} \mathbb{P}(C_n) = \prod_{m \geq n} \mathbb{P}(A_m^c) &= \prod_{m \geq n} (1 - \mathbb{P}(A_m)) \\ &\leq \prod_{m \geq n} e^{-\mathbb{P}(A_m)} \quad \left(\text{using } 1 - p \leq e^{-p}\right) \\ &= \exp\left(-\sum_{m \geq n} \mathbb{P}(A_m)\right) = 0. \end{aligned}$$

Therefore, we obtain

$$\mathbb{P}(A_n \text{ f.o.}) = \mathbb{P}\left(\bigcup_{n \geq 1} C_n\right) \leq \sum_{n \geq 1} \mathbb{P}(C_n) = 0 \implies \mathbb{P}(A_n \text{ i.o.}) = 1.$$

□

We can strengthen the second Borel-Cantelli Lemma as follows.

Lemma 6.2. *If $A_1, A_2, \ldots$ are pairwise independent, and $\sum_{n \geq 1} \mathbb{P}(A_n) = \infty$, then $\mathbb{P}(A_n \text{ i.o.}) = 1$.*

Proof. Let $N_n = \sum_{k=1}^{n} \mathbf{1}_{A_k}$. We have

$$\begin{aligned} \mathbb{E}N_n &= \sum_{k=1}^{n} \mathbb{P}(A_k), \\ \operatorname{var}(N_n) &= \sum_{k=1}^{n} \mathbb{P}(A_k)\,(1 - \mathbb{P}(A_k)) \\ &\leq \mathbb{E}N_n. \end{aligned} \tag{1}$$

Furthermore, we have

$$\begin{aligned} \mathbb{P}\Big(N_n \leq \frac{1}{2}\mathbb{E}N_n\Big) &\leq \mathbb{P}\Big(|N_n - \mathbb{E}N_n| \geq \frac{1}{2}\mathbb{E}N_n\Big) \\ &\leq \frac{4}{(\mathbb{E}N_n)^2} \operatorname{var}(N_n) \quad \text{from Chebyshev's inequality} \\ &\leq \frac{4}{\mathbb{E}N_n}. \end{aligned}$$

Since $N_n \le N = \sum_{k\ge1} \mathbf{1}_{A_k}$, we obtain

$$\mathbb{P}\Big(N \le \frac{1}{2}\mathbb{E}N_n\Big) \le \mathbb{P}\Big(N_n \le \frac{1}{2}\mathbb{E}N_n\Big) \le \frac{4}{\mathbb{E}N_n}.$$

Moreover, we have

$$\begin{aligned} \lim_{n\to\infty} \mathbb{E}N_n &= \lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{P}(A_k) = \infty, \\ \implies \lim_{n\to\infty} \mathbb{P}\Big(N \le \frac{1}{2}\mathbb{E}N_n\Big) &\le \lim_{n\to\infty} \frac{4}{\mathbb{E}N_n} = 0. \end{aligned}$$

We claim that

$$\mathbf{1}_{\{N\le\frac{1}{2}\mathbb{E}N_n\}}(\omega) \to \mathbf{1}_{\{N<\infty\}}(\omega),$$

since if $N(\omega) < \infty$, then for n sufficiently large, $\mathbf{1}_{\{N\le\frac{1}{2}\mathbb{E}N_n\}}(\omega) = 1$, otherwise $\mathbf{1}_{\{N\le\frac{1}{2}\mathbb{E}N_n\}}(\omega) = 0$ for all $n \ge 1$. Therefore from DCT, we have

$$\mathbb{P}(A_n \text{ f.o}) = \mathbb{P}(N < \infty) = \lim_{n\to\infty} \mathbb{P}\Big(N \le \frac{1}{2}\mathbb{E}N_n\Big) = 0,$$

and

$$\mathbb{P}(A_n \text{ i.o}) = 1.$$

$\square$

Remark 6.1. *The condition of pairwise independence in Lemma 6.2 can be strengthened to $\mathbb{P}(A_i \cap A_j) \le \mathbb{P}(A_i)\mathbb{P}(A_j), \forall\ i \ne j$ since* (1) *still holds under this condition.*

In the following, we first prove a bound that will be useful in further generalizing the second Borel-Cantelli Lemma.

Lemma 6.3 (Second moment method)**.** *For $0 \le \rho < 1$ and $X \ge 0$ with $\mathbb{E}X < \infty$,*

$$\mathbb{P}(X > \rho\mathbb{E}X) \ge (1-\rho)^2 \frac{(\mathbb{E}X)^2}{\mathbb{E}X^2}.$$

Proof. Let $A = \{X > \rho\mathbb{E}X\}$. We have

$$\begin{aligned} \mathbb{E}X &= \mathbb{E}X\mathbf{1}_A + \mathbb{E}X\mathbf{1}_{A^c} \le \mathbb{E}X\mathbf{1}_A + \rho\mathbb{E}X, \\ (1-\rho)\mathbb{E}X &\le \mathbb{E}X\mathbf{1}_A. \end{aligned}$$

From the Cauchy-Schwarz inequality,

$$(1-\rho)^2(\mathbb{E}X)^2 \le (\mathbb{E}X\mathbf{1}_A)^2 \le \mathbb{E}X^2\mathbb{P}(A),$$

and the result follows. $\square$

Lemma 6.4 (Kochen-Stone)**.** *If* $\sum_{n\geq 1}\mathbb{P}(A_n)=\infty$, *then*

$$\mathbb{P}(A_n \text{ i.o.}) \geq \limsup_{n\to\infty} \frac{\left(\sum_{k=1}^n \mathbb{P}(A_k)\right)^2}{\sum_{i,j=1}^n \mathbb{P}(A_i\cap A_j)}.$$

Proof. Let $N_n=\sum_{k=1}^n \mathbf{1}_{A_k}$. We have

$$\mathbb{E}N_n = \sum_{i=1}^n \mathbb{P}(A_i),$$
$$\mathbb{E}N_n^2 = \sum_{i,j=1}^n P(A_i\cap A_j).$$

Let $0<\rho<1$. Because $\lim_{n\to\infty}\mathbb{E}N_n=\lim_{n\to\infty}\sum_{i=1}^n\mathbb{P}(A_i)=\infty$, we have $\{A_n \text{ f.o.}\}\subset\{N_n\leq \rho\mathbb{E}N_n,\ \forall\, n\geq n_0, \text{ for some } n_0\geq 1\}$. Therefore, $\{A_n \text{ i.o.}\}\supset\{N_n>\rho\mathbb{E}N_n \text{ i.o.}\}$ and

$$\begin{aligned}
\mathbb{P}(A_n \text{ i.o.}) &\geq \mathbb{P}(N_n>\rho\mathbb{E}N_n \text{ i.o.})\\
&\geq \limsup_{n\to\infty}\mathbb{P}(N_n>\rho\mathbb{E}N_n) \text{ (from Fatou's Lemma)}\\
&\geq \limsup_{n\to\infty}(1-\rho)^2\frac{(\mathbb{E}N_n)^2}{\mathbb{E}N_n^2} \text{ (from Lemma 6.3).}
\end{aligned}$$

Taking $\rho\to 0$, we obtain

$$\mathbb{P}(A_n \text{ i.o.}) \geq \limsup_{n\to\infty} \frac{\left(\sum_{k=1}^n \mathbb{P}(A_k)\right)^2}{\sum_{i,j=1}^n \mathbb{P}(A_i\cap A_j)}.$$

□

Lemma 6.5. *For* $X_1,X_2,\ldots,$ s. t. $\sum_{n\geq1}\mathbb{P}(|X_n|\geq\epsilon)<\infty, \forall\ \epsilon>0$, *we have*

$$X_n\to 0 \text{ a.s. as } n\to\infty.$$

Proof. Let

$$\begin{aligned}
F &= \left\{\omega\ :\ \limsup_{n\to\infty}|X_n|>0\right\}\\
&= \bigcup_{m\geq 1}\left\{\omega\ :\ \limsup_{n\to\infty}|X_n|>\frac{1}{m}\right\}.
\end{aligned}$$

Let $A_n=\{\omega:|X_n|>\frac{1}{m}\}$. We have $\sum_{n\geq1}\mathbb{P}(A_n)<\infty$. From the first Borel-Cantelli Lemma, we have

$$\mathbb{P}(A_n \text{ i.o.})=0.$$

Then, $\mathbb{P}\Big(\limsup_{n\to\infty}|X_n|>\frac{1}{m}\Big)\leq\mathbb{P}(A_n \text{ i.o.})=0.$

$$\begin{aligned}
&\implies \mathbb{P}(F)=0\\
&\implies \limsup_{n\to\infty}|X_n|=0 \text{ a.s.} \implies \lim_{n\to\infty}|X_n|=0 \text{ a.s..}
\end{aligned}$$

□

Corollary 6.1. *If* $X_n \xrightarrow{\text{p}} X$*, then* $\exists$ *subsequence* $(n(k))_{k\geq 1}$ *such that* $X_{n(k)} \to X$ *a.s.*

Proof. By the definition of convergence in probability, we can choose $(n(k))_{k\geq 1}$ such that $\forall\, \epsilon > 0$, we have

$$\mathbb{P}\Big(|X_{n(k)} - X| \geq \epsilon\Big) \leq \frac{1}{2^k},\ \forall\, k \geq 1.$$

Summing both sides over $k \geq 1$, we obtain

$$\sum_{k\geq 1} \mathbb{P}\Big(|X_{n(k)} - X| \geq \epsilon\Big) \leq 1.$$

By Lemma 6.5, we have $|X_{n(k)} - X| \to 0$ a.s. □

Lemma 6.6. $X_n \xrightarrow{\text{p}} X$ *iff for any subsequence* $(n(k))_{k\geq 1}$, $\exists$ *subsubsequence* $(n\,(k(r)))_{r\geq 1}$, s. t. $X_{n(k(r))} \to X$ *a.s.*

Proof. '$\Rightarrow$': It is obvious by Corollary 6.1.

'$\Leftarrow$': Suppose X_n does not converge in probability to X. Then, $\exists\, \epsilon > 0$ and subsequence $(n(k))_{k\geq 1}$, such that

$$\mathbb{P}\Big(|X_{n(k)} - X| \geq \epsilon\Big) \geq \epsilon,\ \forall\, k \geq 1.$$

Consequently, $\forall\ (n\,(k(r)))_{r\geq 1}$, $X_{n(k(r))} \not\to X$ a.s., which contradicts the claim. □

Note that Lemma 6.6 implies that the DCT holds with "almostly surely convergence" replaced by "convergence in probability".

6.3 SLLN with Finite 2nd Moments

Lemma 6.7. *Suppose* $X_1, X_2, \ldots$ *are pairwise independent,* $\mathbb{E}X_i = 0$, $\mathbb{E}X_i^2 \leq M < \infty$, $\forall\, i \geq 1$. *Let* $S_n = \sum_{i=1}^n X_i$. *Then* $\frac{S_n}{n} \to 0$ *a.s. as* $n \to \infty$.

Proof. From Lemma 6.5, it suffices to prove $\mathbb{P}\left(\left|\frac{S_n}{n}\right| > \epsilon \text{ i.o.}\right) = 0$, $\forall\, \epsilon > 0$. By applying Chebyshev's inequality, we obtain

$$\mathbb{P}\left(\frac{|S_n|}{n} \geq \epsilon\right) \leq \frac{\mathbb{E}S_n^2}{\epsilon^2 n^2} \leq \frac{M}{\epsilon^2 n}.$$

Unfortunately, $\sum_{n\geq 1} 1/n = \infty$ so we cannot obtain the desired conclusion immediately using the Boral Cantelli Lemma. Instead, we use a subsequence "trick" here. Letting $n(k) = k^2$ and summing both sides of above equation over $n(k)$ where $k \geq 1$, we obtain

$$\sum_{k=1}^{\infty} \mathbb{P}\left(\frac{|S_{n(k)}|}{n(k)} \geq \epsilon\right) \leq \frac{M}{\epsilon^2} \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

By applying Lemma 6.1, we obtain $\dfrac{|S_{n(k)}|}{n(k)} \to 0$ a.s. as $k \to \infty$.

Let $\Delta_k = \max\Big\{|S_n - S_{n(k)}| : n(k) < n < n(k+1)\Big\}$. For $n(k) \le n < n(k+1)$, we have

$$\begin{aligned}
&\frac{|S_n|}{n} \le \frac{|S_{n(k)}|}{n(k)} + \frac{\Delta_k}{n(k)}, \\
\implies &\limsup_{n\to\infty} \frac{|S_n|}{n} \le \limsup_{k\to\infty} \frac{|S_{n(k)}|}{n(k)} + \limsup_{k\to\infty} \frac{\Delta_k}{n(k)} = \limsup_{k\to\infty} \frac{\Delta_k}{n(k)}.
\end{aligned}$$

The proof is complete if we show $\dfrac{\Delta_k}{n(k)} \to 0$ a.s. as $k \to \infty$. Let $B_j = \Big\{\omega : |S_{n(k)+j} - S_{n(k)}| \ge \epsilon n(k)\Big\}$, for $1 \le j \le 2k$. We have

$$\begin{aligned}
\mathbb{P}(\Delta_k \ge \epsilon n(k)) &= \mathbb{P}\left(\bigcup_{j=1}^{2k} B_j\right) \\
&\le \sum_{j=1}^{2k} \mathbb{P}\Big(|S_{n(k)+j} - S_{n(k)}| \ge \epsilon n(k)\Big) \\
&\le \sum_{j=1}^{2k} \frac{jM}{\epsilon^2 n(k)^2} = \frac{M}{\epsilon^2 k^3}(2k+1).
\end{aligned}$$

Summing both sides over $k \ge 1$, we obtain

$$\sum_{k=1}^{\infty} \mathbb{P}\left(\frac{\Delta_k}{n(k)} \ge \epsilon\right) \le \frac{M}{\epsilon^2} \sum_{k=1}^{\infty} \frac{2k+1}{k^3} < \infty.$$

From Lemma 6.5, we obtain $\dfrac{\Delta_k}{n(k)} \to 0$ a.s. as $k \to \infty$, and the proof is complete. $\square$

For $X \ge 0$, we have

$$\sum_{k=1}^{\infty} \mathbf{1}_{\{X \ge k\}} \le X \le \sum_{k=0}^{\infty} \mathbf{1}_{\{X \ge k\}}.$$

Therefore, for any X, we obtain

$$\begin{aligned}
&\sum_{k=1}^{\infty} \mathbb{P}(|X| \ge k) \le \mathbb{E}|X| \le \sum_{k=0}^{\infty} \mathbb{P}(|X| \ge k), \\
&\sum_{k=1}^{\infty} \mathbb{P}(|X| \ge k) \le \infty \iff \mathbb{E}|X| < \infty.
\end{aligned}$$

As a side note, if $X \in \mathbb{Z}_+$, we have the following equality:

$$\begin{aligned}
X &= \sum_{k=1}^{\infty} \mathbf{1}_{\{X \ge k\}}, \\
\mathbb{E}X &= \sum_{k=1}^{\infty} \mathbb{P}(X \ge k).
\end{aligned}$$

Lemma 6.8. *Suppose $X_1, X_2, \ldots$ are i.i.d. Then,*

$$\lim_{n\to\infty} \frac{X_n}{n} = 0 \ \text{a.s.} \iff \mathbb{E}|X_1| < \infty.$$

Proof.

'$\Rightarrow$':

$$\lim_{n\to\infty} \frac{X_n}{n} = 0 \ \text{a.s.} \implies \mathbb{P}\left(\frac{|X_n|}{n} \geq 1 \text{ i.o.}\right) = 0.$$

From the second Borel-Cantelli Lemma, we have

$$\begin{aligned}
&\sum_{n\geq 1} \mathbb{P}\left(\frac{|X_n|}{n} \geq 1\right) < \infty \\
&\sum_{n\geq 1} \mathbb{P}(|X_1| \geq n) < \infty \\
&\mathbb{E}|X_1| < \infty.
\end{aligned}$$

'$\Leftarrow$':

$$\mathbb{E}\left|\frac{X_1}{\epsilon}\right| < \infty \implies \sum_{n\geq 1} \mathbb{P}(|X_n| \geq n\epsilon) < \infty.$$

The result then follows from Lemma 6.5. $\square$