**An Analytical Introduction to Probability Theory**

# 1. Basics of Real Analysis - I

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## 1.1 Introduction

This basic real analysis introduction covers topics in sequences, compact spaces and continuity. Recommended textbook is "Principles of Mathematical Analysis" by Walter Rudin.

## 1.2 Sequences

Inner product space $\subset$ normed space $\subset$ metric space $\subset$ topological space. Working with metric spaces suffices for our purposes.

**Definition 1.1.** *A metric space is an ordered pair* $(\mathcal{X}, d)$ *where* $\mathcal{X}$ *is a set and* $d$ *is a metric on* $\mathcal{X}$*, such that* $\forall\, x, y, z \in \mathcal{X}$*, the following holds:*

1) $d(x, y) \geq 0$*, where equality holds iff* $x = y$*.*
2) $d(x, y) = d(y, x)$*.*
3) *Triangle inequality:* $d(x, y) \leq d(x, z) + d(y, z)$*.*

**Example 1.1.** *Examples of metric space:*

- $(\mathbb{R}, |x - y|)$ *(normed space)*
- $\left(\mathbb{R}^k, \|x - y\|_p\right)$*, where* $p \geq 1$ *(normed space)*
- $\ell^p(\mathbb{R}) = \left\{x = (x_n)_{n\geq 1} \,:\, \|x\|_p = \left(\sum_{n\geq 1} |x_n|^p\right)^{1/p} < \infty\right\}$ *(normed space)*
- $(\mathcal{X}, d)$*, where* $d(x, y) = 0$ *if* $x = y$ *and* $d(x, y) = 1$ *otherwise.*

We write $(x_n)$ or $(x_n)_{n\geq 1}$ as shorthand for the sequence $x_1, x_2, \ldots$.

**Definition 1.2.** *For* $(x_n) \subset \mathcal{X}$*, we say that* $x_n \to x$ *as* $n \to \infty$ *or* $\lim_{n\to\infty} x_n = x$ *if*

$$\forall\, \epsilon > 0,\ \exists\, N \ \text{ s. t. } \ d(x_n, x) < \epsilon,\ \forall\, n \geq N.$$

**Definition 1.3.** *We say $x_n$ diverges to $\infty$ if there exists $x_0 \in \mathcal{X}$ and*

$$\forall K \geq 0, \ \exists N \ \text{s.t.} \ d(x_n, x_0) \geq K, \ \forall n \geq N.$$

We define extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. For $x \in \mathbb{R}$, we have:

1) $x + \infty = \infty$
2) $x - \infty = -\infty$
3) $x \cdot \pm\infty = \pm\infty, \forall x > 0$
4) $0 \cdot \pm\infty = 0$ (by convention)

The following lemmas are based on $(\mathbb{R}, |\cdot|)$ but most can be generalized to $(\mathcal{X}, d)$.

**Lemma 1.1.** *If $(a_n)$ converges, then its limit is unique.*

*Proof.* Supposing $a_n \to a$ and $a_n \to a'$, we have

$$\forall \epsilon > 0, \ \exists N \ \text{s.t.} \ |a_n - a| < \epsilon \text{ and } |a_n - a'| < \epsilon, \ \forall n \geq N.$$

By the triangle inequality, we obtain:

$$|a - a'| \leq |a_n - a| + |a_n - a'| < 2\epsilon.$$

Since $\epsilon$ is arbitrary, $a = a'$. □

**Lemma 1.2.** *If $(a_n)$ converges, $|a_n| < \infty$.*

*Proof.* Supposing $a_n \to a$, $\exists N$, s.t. $|a_n - a| < 1$, $\forall n \geq N$. Then we obtain

$$|a_n| \leq |a| + 1, \forall \ n \geq N,$$

and

$$|a_n| \leq \max\{a_1, \ldots, a_{N-1}, |a| + 1\} < \infty.$$

□

**Lemma 1.3.** *If $(a_n)$ is increasing (i.e., $a_{n+1} \geq a_n$ for all $n \geq 1$) and is bounded above, then $(a_n)$ converges.*

*Proof.* To prove the lemma, we need some definitions. We define $\sup A$ = least upper bound (LUB) of $A$ if $A \neq \emptyset$ and $A \subset \mathbb{R}$, i.e.,

1) $\forall\, a \in A$, $a \le \sup A$.

2) if $\gamma < \sup A$, $\exists\, a \in A$, s. t. $\gamma < a$.

For the definition to be well-defined, we need the LUB/Completeness axiom: $\sup A$ exists in $\mathbb{R}$ for all $A \neq \emptyset$ bounded above. (Note that this axiom does not hold for $\mathbb{Q}$.) If $A$ is not bounded above, we set $\sup A = \infty$. From Item 2), we obtain a useful property:

$$\forall\, \epsilon > 0,\ \exists\, a \in A,\ \text{s. t. } \sup A - \epsilon < a.$$

The greatest lower bound of $A$ is written as $\inf A$.

Returning to the proof of the lemma, let $a = \sup a_n < \infty$ since $(a_n)$ is bounded. We have

$$\begin{aligned} &\forall\, \epsilon > 0,\ \exists\, N,\ \text{s. t. } a - \epsilon \le a_N \le a_n \le a,\ \forall\, n \ge N. \\ &\implies a_n \to a. \end{aligned}$$

□

Applying Lemma 1.3 to the negative of a sequence, we also conclude that any decreasing sequence that is bounded below converges.

**Remark 1.1.**

1. *If $a_n \in \overline{\mathbb{R}}_+$ and is increasing, then $\lim_{n\to\infty} a_n$ exists (maybe $\infty$).*
2. *If $a_n \in \mathbb{R}_+$, then from Lemma 1.3, $\sum_{k=1}^{\infty} a_k \triangleq \lim_{n\to\infty} \sum_{k=1}^{n} a_k$ exists.*

**Definition 1.4.** *A subsequence of $(a_n)$ is a sequence $(a_{n_i})$, where $n_1 < n_2 < \dots$ is an increasing sequence of indices.*

**Lemma 1.4.** *If $a_n \to a \iff a_{n_i} \to a$, for all subsequences $(a_{n_i})$.*

*Proof.* The direction "$\Leftarrow$" is obvious. To prove the other direction, we have $\forall\, \epsilon > 0$, $\exists\, N$ s. t. $|a_n - a| < \epsilon$, $\forall\, n \ge N$. Therefore, $\forall\, n_i \ge N$, $|a_{n_i} - a| < \epsilon$. □

**Lemma 1.5.** *Every sequence $(a_n)$ in $\mathbb{R}$ has a monotone subsequence.*

*Proof.* Suppose there are no increasing subsequences. Then, $\exists\, n_1$, s. t. $a_{n_1} \ge a_n$, $\forall\, n \ge n_1$. Similarly, $\exists\, n_2 > n_1$, s. t. $a_{n_2} \ge a_n$, $\forall\, n \ge n_2$, and so on. This constructs a decreasing subsequence $a_{n_1} \ge a_{n_2} \ge \dots$. □

**Lemma 1.6.** *Every bounded sequence in $\mathbb{R}$ contains a convergent subsequence.*

*Proof.* From Lemma 1.5, a bounded sequence has a monotone subsequence, which converges from Lemma 1.3. □

Let $S$ = set of subsequence limits of $(a_n) \subset \overline{\mathbb{R}}$. We define $\limsup_{n\to\infty} a_n = \sup S$, and $\liminf_{n\to\infty} a_n = \inf S$. We have

$$\forall\, \epsilon > 0,\ \exists\, s \in S,\ \text{s. t.}\ \sup S - \epsilon/2 \le s \le \sup S.$$

Let $a_{n_i} \to s$ (can you see why such a subsequence must exist?) so that

$$\forall\, \epsilon > 0,\ \exists\, N,\ \text{s. t.}\ |a_{n_i} - s| < \epsilon/2,\ \forall\, n_i > N.$$

Therefore,

$$|a_{n_i} - \sup S| \le |a_{n_i} - s| + |s - \sup S| < \epsilon,$$

which implies that $\sup S \in S$ and $\sup S = \max S$, i.e., $\limsup_{n\to\infty} a_n$ exists. Similarly, we have $\inf S = \min S$.

**Lemma 1.7.** *Suppose* $s = \limsup_{n\to\infty} a_n > -\infty$. *Then* $\forall\, \epsilon > 0$, $\exists\, N$ s. t. $a_n \le s + \epsilon$, $\forall\, n \ge N$.

*Proof.* If $s = \infty$, the lemma obviously holds. Suppose $s < \infty$. We prove by contradiction. Suppose $\exists\, \epsilon > 0$ and subsequence $(a_{n_i})$ s. t. $a_{n_i} > s + \epsilon$. Since $s < \infty$, $(a_{n_i})$ is bounded. Then from Lemma 1.6, $\exists$ subsubsequence $(a_{n_i'})$, s. t. $a_{n_i'} \to a' \ge s + \epsilon > s$, a contradiction to the definition of $s$. □

**Lemma 1.8.** $\limsup\limits_{n\to\infty} a_n = \inf\limits_{n\ge 1} \sup\limits_{m\ge n} a_m$ *and* $\liminf\limits_{n\to\infty} a_n = \sup\limits_{n\ge 1} \inf\limits_{m\ge n} a_m$.

*Proof.* If $\limsup_{n\to\infty} a_n = \pm\infty$, then the result clearly holds. Suppose $-\infty < \limsup_{n\to\infty} a_n = \sup S < \infty$, where $S$ is the set of subsequential limits.

For any $\epsilon > 0$, there exists a subsequence $(a_{n_i})$ s. t. $\sup S - \epsilon \le a_{n_i}$. Then,

$$\sup S - \epsilon \le \sup_{m\ge n} a_m,\ \forall\, n \ge n_1.$$

Letting $n \to \infty$, we have

$$\sup S - \epsilon \le \lim_{n\to\infty} \sup_{m\ge n} a_m = \inf_{n\ge 1} \sup_{m\ge n} a_m,$$

since $\sup_{m\ge n} a_m$ is a decreasing sequence in $n$. As $\epsilon$ is arbitrary, we have

$$\sup S \le \inf_{n\ge 1} \sup_{m\ge n} a_m.$$

On the other hand, for each $n \ge 1$, $\exists\, a_{m_n} \ge \sup_{m\ge n} a_m - 1/n$. Then we have

$$\sup S \ge \lim_{n\to\infty} a_{m_n} \ge \inf_{n\ge 1} \sup_{m\ge n} a_m.$$

The other claim is proved similarly and the proof is now complete. □

## 1.3 Cauchy Sequences

**Definition 1.5.** *A sequence* $(a_n)$ *is a Cauchy sequence if* $\forall\, \epsilon > 0$, $\exists\, N$ s. t. $d(a_n, a_m) < \epsilon$, $\forall\, n, m \geq N$.

**Lemma 1.9.** *If* $(a_n)$ *converges, then it is Cauchy.*

*Proof.* Suppose $a_n \to a$. Then $\forall\, \epsilon > 0$, $\exists\, N$ s. t. $d(a_n, a) < \epsilon/2$ $\forall\, n \geq N$. Then $\forall\, n, m \geq N$, we have

$$d(a_n, a_m) \leq d(a_n, a) + d(a_m, a) < \epsilon.$$

□

**Definition 1.6.** *A metric space* $(\mathcal{X}, d)$ *in which every Cauchy sequence converges in* $\mathcal{X}$ *is complete.*

Example of an incomplete space is $\mathbb{Q}$, the set of rational numbers: for every irrational number, one can construct a sequence of rational numbers that converges to it.

**Theorem 1.1.** $\mathbb{R}^k$ *is complete.*

*Proof.* WLOG, we prove for $\mathbb{R}$. We first show 2 lemmas.

**Lemma 1.10.** *A Cauchy sequence is bounded.*

*Proof.* Similar to Lemma 1.2. □

**Lemma 1.11.** *If* $(a_n)$ *is Cauchy and there is a subsequence* $a_{n_i} \to a$*, then* $a_n \to a$.

*Proof.* For any $\epsilon > 0$, $\exists\, N$ such that $\forall\, n, n_i \geq N$,

$$d(a_n, a) \leq d(a_n, a_{n_i}) + d(a_{n_i}, a) \leq \epsilon.$$

□

We now return to the proof of Theorem 1.1. Let $(a_n)$ be a Cauchy sequence in $\mathbb{R}$. Then $(a_n)$ is bounded from Lemma 1.10. From Lemma 1.6, there exists a convergent subsequence. Lemma 1.11 then implies $(a_n)$ converges and the proof is complete. □