5.1 Convergence

Definition 5.1 (Almost sure convergence)**.** *We say $X_n \to X$ almost surely (a.s.) or with probability 1 if*

$$\mathbb{P}\Big(\Big\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\Big\}\Big) = 1.$$

Recall that in the original MCT in Theorem 4.2, we suppose that $0 \leq X_n(\omega) \leq X_{n+1}(\omega)$ and $X_n(\omega) \to X(\omega)$ for *all* $\omega \in \Omega$. Here, we show that we can replace the condition "for all $\omega \in \Omega$" with "almost surely". Let

$$A = \{\omega \,:\, X_n(\omega) \to X(\omega),\, 0 \leq X_n(\omega) \leq X_{n+1}(\omega),\, n \geq 1\}$$

and suppose $\mathbb{P}(A) = 1$. We have $0 \leq X_n \mathbf{1}_A(\omega) \leq X_{n+1}\mathbf{1}_A(\omega)$, and $X_n\mathbf{1}_A(\omega) \to X\mathbf{1}_A(\omega)$ $\forall\, \omega \in \Omega$. Then from the MCT in Theorem 4.2, we have

$$\mathbb{E}[X_n \mathbf{1}_A] \to \mathbb{E}[X\mathbf{1}_A].$$

For any $Y \geq 0$, let $Y \wedge n = \min(Y, n)$ for $n \geq 1$. Observe that

$$\begin{aligned}
\mathbb{E}[Y\mathbf{1}_{A^c}] &= \mathbb{E}\Big[\lim_{n\to\infty} (Y \wedge n)\mathbf{1}_{A^c}\Big] \\
&= \lim_{n\to\infty} \mathbb{E}[(Y \wedge n)\mathbf{1}_{A^c}] \qquad (1) \\
&\leq \lim_{n\to\infty} n\mathbb{P}(A^c) \\
&= 0,
\end{aligned}$$

where we have use the MCT in (1). Therefore, $\mathbb{E}[X_n\mathbf{1}_A] = \mathbb{E}X_n$ and $\mathbb{E}[X\mathbf{1}_A] = \mathbb{E}X$, and the MCT holds if "$\forall\, \omega \in \Omega$" is replaced with "almost surely" in its theorem statement. The same thing applies for Fatou's Lemma and the DCT.

Definition 5.2 (Convergence in probability)**.** *If $\forall\, \epsilon > 0$,*

$$\lim_{n\to\infty} \mathbb{P}(|X_n - X| > \epsilon) = 0,$$

we say that X_n converges to X in probability and write $X_n \xrightarrow{\text{p}} X$.

Lemma 5.1. *If $X_n \to X$ a.s., then $X_n \xrightarrow{\text{p}} X$.*

Proof. Suppose $X_n \to X$ a.s., then $\mathbf{1}_{\{|X_n - X| > \epsilon\}} \to 0$ a.s. From DCT (since probability measure is finite), we have $\mathbb{P}(|X_n - X| > \epsilon) \to 0$. □

The converse is not true and here is an example.

Example 5.1. *Suppose $\Omega = [0, 1]$. Consider the random variables X_n shown in Fig. 1, where X_n follows a similar pattern for $n \geq 5$. We have $X_n \xrightarrow{\text{p}} 0$, but clearly $X_n \nrightarrow 0$ a.s. as $X_n(\omega) = 1$ for infinitely many values of n for every $\omega \in \Omega$.*

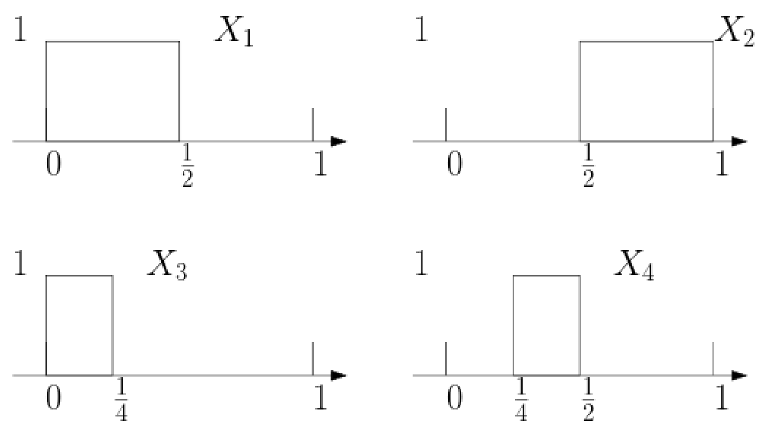

Figure 1: X_n converges in probability but not almost surely.

5.2 Weak Law of Large Numbers

Markov's inequality: Suppose $a > 0$ and $X \geq 0$. Then,

$$a\mathbf{1}_{\{X \geq a\}}(\omega) \leq X\mathbf{1}_{\{X \geq a\}}(\omega) \leq X.$$

Taking expectations, we have

$$a\mathbb{P}(X \geq a) \leq \mathbb{E}X,$$
$$\mathbb{P}(X \geq a) \leq \frac{1}{a}\mathbb{E}X.$$

Chebyshev's inequality follows from Markov's inequality by replacing X with $|X - \mathbb{E}X|^2$ and setting $a = \epsilon^2$: for $\epsilon > 0$,

$$\mathbb{P}(|X - \mathbb{E}X| \geq \epsilon) \leq \frac{1}{\epsilon^2}\,\text{var}(X).$$

Theorem 5.1 (Weak Law of Large Numbers (WLLN))**.** *Suppose $X_1, X_2, \ldots$ are such that $\mathbb{E}X_i = 0$, $\mathbb{E}X_i^2 = \sigma^2 < \infty$ and $\mathbb{E}[X_i X_j] \leq 0$ for $i \neq j$. Then,*

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{\mathrm{p}} 0 \text{ as } n \to \infty.$$

Proof. We have

$$\begin{aligned}\mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2\right] &= \frac{1}{n^2}\mathbb{E}\left[\sum_{i}^{n} X_i^2 + 2\sum_{i<j} X_i X_j\right] \\ &\leq \frac{1}{n^2}\sum_{i}^{n}\mathbb{E}X_i^2 = \frac{\sigma^2}{n}.\end{aligned}$$

From Chebyshev's inequality, we then have for any $\epsilon > 0$,

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| \geq \epsilon\right) \leq \frac{1}{\epsilon^2}\frac{\sigma^2}{n} \to 0,$$

as $n \to \infty$. □

5.3 Product Measures

Given two measure spaces $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$, where both $\mu_1$and μ_2 are σ-finite, we can define a new measurable space $(\Omega, \mathcal{F})$, where $\Omega = \Omega_1 \times \Omega_2$ and

$$\mathcal{F} = \sigma\left\{A \times B : A \in \mathcal{A}_1, B \in \mathcal{A}_2\right\}.$$

We call $A \times B$ a rectangle. A natural measure μ for this measurable space satisfies

$$\mu(A \times B) = \mu_1(A)\mu_2(B) \tag{2}$$

for all rectanges $A \times B$. It can be checked that the collection of finite disjoint unions of rectangles is an algebra (exercise). To extend this measure μ to $\mathcal{F}$, we make use of Caratheodory's Extension Theorem (Theorem 3.1): we show that for $A \times B = \bigcup_{i\geq 1} A_i \times B_i$ a disjoint union of rectangles, we have $\mu(A \times B) = \sum_{i\geq 1}\mu_1(A_i)\mu_2(B_i)$.

For $x \in A$, let $I(x) = \{i : x \in A_i\}$ and $B = \bigcup_{i\in I(x)} B_i$ a disjoint union. We have

$$\begin{aligned}\mathbf{1}_A(x)\mu_2(B) &= \mathbf{1}_A(x)\mu_2\left(\bigcup_{i\in I(x)} B_i\right) \\ &= \sum_{i\in I(x)} \mathbf{1}_A(x)\mu_2(B_i) \\ &= \sum_{i\geq 1} \mathbf{1}_{A_i}(x)\mu_2(B_i).\end{aligned}$$

Integrating w.r.t. μ_1, we have

$$\begin{aligned}
\mu(A \times B) &= \int \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{1}_{A_i}(x)\mu_2(B_i)\,\mathrm{d}\mu_1 \\
&\overset{\text{MCT}}{=} \lim_{n\to\infty} \int \sum_{i=1}^{n} \mathbf{1}_{A_i}(x)\mu_2(B_i)\,\mathrm{d}\mu_1 \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \int \mathbf{1}_{A_i}(x)\mu_2(B_i)\,\mathrm{d}\mu_1 \\
&= \sum_{i\geq 1} \mu_1(A_i)\mu_2(B_i),
\end{aligned}$$

where the interchange of the sum and integral in the penultimate equality holds because the number of terms is finite. Therefore, Caratheodory's Extension Theorem shows that there is a unique extension of μ defined by (2) to $\mathcal{F}$. This is called a *product measure*. Notationally, we write $\mu = \mu_1 \times \mu_2$.

The next theorem tells us when we can interchange integrals in general.

Theorem 5.2 (Fubini or Fubini-Tonelli)**.** *Suppose* $(\Omega_1, \mathcal{A}_1, \mu_1)$ *and* $(\Omega_2, \mathcal{A}_2, \mu_2)$ *are* σ*-finite and* $\mu = \mu_1 \times \mu_2$ *is the product measure. Consider a measurable function* $f : \Omega = \Omega_1 \times \Omega_2 \mapsto \mathbb{R}$*. If* $f \geq 0$ *or* $\int |f|\,\mathrm{d}\mu < \infty$*, then*

$$\int_{\Omega_1}\int_{\Omega_2} f(x,y)\,\mathrm{d}\mu_2\,\mathrm{d}\mu_1 = \int_{\Omega} f\,\mathrm{d}\mu = \int_{\Omega_2}\int_{\Omega_1} f(x,y)\,\mathrm{d}\mu_1\,\mathrm{d}\mu_2.$$

Proof. Note that implicit in the theorem statement are the following that we need to prove:

(i) For each $x \in \Omega_1$, $y \mapsto f(x,y)$ is $\mathcal{A}_2$-measurable.

(ii) $x \mapsto \int_{\Omega_2} f(x,y)\,\mathrm{d}\mu_2$ is $\mathcal{A}_1$-measurable.

(iii) $\int_{\Omega_1}\int_{\Omega_2} f(x,y)\,\mathrm{d}\mu_2\,\mathrm{d}\mu_1 = \int_{\Omega} f\,\mathrm{d}\mu$.

Without loss of generality, we may assume $\mu_1, \mu_2 < \infty$ as the same proof is valid on each partition of Ω and then we can apply the MCT.

The proof follows the steps discussed at the end of Section 4.2. We first prove the theorem for the simplest case where $f = \mathbf{1}_E$, where $E \in \mathcal{F}$, the product σ-algebra. Let $E_x = \{y : (x,y) \in E\}$.

(i): Fix x, then $y \mapsto f(x,y) = \mathbf{1}_{E_x}(y)$. We need to show $E_x \in \mathcal{A}_2$. Let $\mathcal{E} = \{E \in \mathcal{F} : E_x \in \mathcal{A}_2\}$. We have $(E^c)_x = (E_x)^c$ since $y \in (E^c)_x \Leftrightarrow (x,y) \in E^c \Leftrightarrow y \in (E_x)^c$, and $\left(\bigcup_{i\geq 1} E_i\right)_x = \bigcup_{i\geq 1}(E_i)_x$. Therefore, $\mathcal{E}$ is a σ-algebra and it contains all rectangles $A \times B$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$, which implies that $\mathcal{F} \subset \mathcal{E}$.

(ii) & (iii): Let $\mathcal{L} = \{E \in \mathcal{F} : f = \mathbf{1}_E \text{ satisfies (ii) \&(iii)}\}$.

- $\Omega \in \mathcal{L}$.
- If $E \in \mathcal{L}$, $\mu_2((E^c)_x) = \mu_2((E_x)^c) = \mu_2(\Omega_2) - \mu_2(E_x)$. Since $\mu_2(\Omega_2) < \infty$ and $\mu_2(E_x)$ is $\mathcal{A}_1$-measurable, $\mu_2((E^c)_x)$ is $\mathcal{A}_1$-measurable. We also have

$$\begin{aligned}\int \mu_2((E^c)_x)\,\mathrm{d}\mu_1 &= \mu_2(\Omega_2)\mu_1(\Omega_1) - \int \mu_2(E_x)\,\mathrm{d}\mu_1\\ &= \mu(\Omega) - \mu(E)\\ &= \mu(E^c).\end{aligned}$$

  Therefore, $E^c \in \mathcal{L}$.
- If $E_i \in \mathcal{L}$, $i \geq 1$, are disjoint, then

$$\begin{aligned}\mu_2\left((\bigcup_{i\geq 1} E_i)_x\right) &= \mu_2\left(\bigcup_{i\geq 1}(E_i)_x\right)\\ &= \sum_{i\geq 1}\mu_2((E_i)_x)\\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\mu_2((E_i)_x),\end{aligned}$$

  which is $\mathcal{A}_1$-measurable from Lemma 4.2 since each $\mu_2((E_i)_x)$ is $\mathcal{A}_1$-measurable. From the MCT, we have

$$\begin{aligned}\int \mu_2\left((\bigcup_{i\geq 1} E_i)_x\right)\,\mathrm{d}\mu_1 &= \lim_{n\to\infty}\int\sum_{i=1}^{n}\mu_2((E_i)_x)\,\mathrm{d}\mu_1\\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\int\mu_2((E_i)_x)\,\mathrm{d}\mu_1\\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\mu(E_i)\\ &= \mu\left(\bigcup_{i\geq 1} E_i\right).\end{aligned}$$

Therefore, $\mathcal{L}$ is a λ-system containing the collection of rectangles, which is a π-system. From the π-λ Theorem, we obtain $\mathcal{F} \subset \mathcal{L}$. We have now shown that Fubini's Theorem holds for $f = \mathbf{1}_E$, $E \in \mathcal{F}$.

From the linearity of integrals, the theorem holds for all simple functions f.

For $f \geq 0$, $\exists$ simple $f_i \uparrow f$. Applying MCT gives Fubini's theorem for non-negative f.

Finally, for general $f = f^+ - f^-$, we note that $\int |f|\,\mathrm{d}\mu < \infty$ implies $\int f^+\,\mathrm{d}\mu, \int f^-\,\mathrm{d}\mu < \infty$, and

$$\begin{aligned}\int f^+\,\mathrm{d}\mu &= \int\int f^+(x,y)\,\mathrm{d}\mu_1\,\mathrm{d}\mu_2 \implies \int f^+(x,y)\,\mathrm{d}\mu_1 < \infty\ \mu_2\text{-a.e.}\\ &= \int\int f^+(x,y)\,\mathrm{d}\mu_2\,\mathrm{d}\mu_1 \implies \int f^+(x,y)\,\mathrm{d}\mu_2 < \infty\ \mu_1\text{-a.e.}\end{aligned}$$

Similarly for f^- so that

$$\int\int f^+(x,y)\,\mathrm{d}\mu_1\,\mathrm{d}\mu_2 - \int\int f^-(x,y)\,\mathrm{d}\mu_1\,\mathrm{d}\mu_2 = \int\int f(x,y)\,\mathrm{d}\mu_1\,\mathrm{d}\mu_2.$$

The proof is now complete. $\square$

Example 5.2.

$$\begin{array}{cc}
 & \begin{array}{ccccc} x & \longrightarrow & & & \end{array} \\
y \Big\uparrow & \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 1 & -1 & \dots \\ 0 & 1 & -1 & 0 & \dots \\ 1 & -1 & 0 & 0 & \dots \end{pmatrix}
\end{array} \tag{3}$$

We have

$$\sum_y \sum_x f(x,y) = \sum_y 0 = 0.$$
$$\sum_x \sum_y f(x,y) = 1 + 0 + 0 + \cdots = 1.$$

This example shows that the conditions in Fubini's Theorem are essentially necessary.

Example 5.3. *Suppose* $((0,1),\mathcal{B}(0,1),\lambda)\times((0,1),2^{(0,1)},\nu)$*, where* $\nu(A)=|A|$ *is the counting measure, which is not* σ*-finite. Let*

$$f(x,y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Then we have

$$\int f(x,y)\,\mathrm{d}\nu = 1 \text{ for each } x \implies \int\int f(x,y)\,\mathrm{d}\nu\,\mathrm{d}\lambda = 1,$$
$$\int f(x,y)\,\mathrm{d}\lambda = 0 \text{ for each } y \implies \int\int f(x,y)\,\mathrm{d}\lambda\,\mathrm{d}\nu = 0.$$

This example shows that σ*-finiteness of the measures is necessary.*

5.4 Independence

Throughout this section, we consider a probability space $(\Omega,\mathcal{A},\mathbb{P})$.

Definition 5.3. *Two events* A *and* B *are independent if* $\mathbb{P}(A\cap B)=\mathbb{P}(A)\mathbb{P}(B)$*. We write* $A \perp\!\!\!\perp B$*.*

If $A \perp\!\!\!\perp B$, then it is easy to verify the following:

(i) $A^c \perp\!\!\!\perp B$.

(ii) $A \perp\!\!\!\perp B^c$.

(iii) $A^c \perp\!\!\!\perp B^c$.

I.e., the two σ-algebras $\{\emptyset, \Omega, A, A^c\}$ and $\{\emptyset, \Omega, B, B^c\}$ are "independent".

Definition 5.4. *The sub-σ-algebra $\mathcal{A}_i \subset \mathcal{A}$, $i = 1, 2, \ldots, n$, are independent if $\forall\, A_i \in \mathcal{A}_i$,*

$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} \mathbb{P}(A_i).$$

They are said to be pairwise independent *if $\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\mathbb{P}(A_j)\ \forall\, i \neq j$.*

We say that the random variables $X_1, X_2, \ldots, X_n$ are independent if $\sigma(X_i)$, $i = 1, \ldots, n$, are independent, i.e., $\forall\, B_i \in \mathcal{B}$, $i = 1, \ldots, n$,

$$\mathbb{P}(X_1 \in B_1, X_2 \in B_2, \ldots, X_n \in B_n) = \prod_{i=1}^{n} \mathbb{P}(X_i \in B_i).$$

Note that independence $\implies$ pairwise independence but the converse is false. Here are two counterexamples to show that pairwise independence does not imply independence.

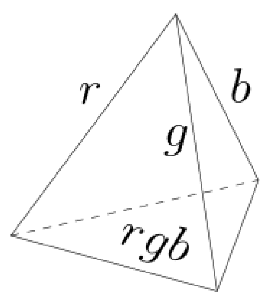

Figure 2: Dice example: pairwise independence does not imply independence.

Example 5.4. *Consider a fair tetrahedron dice that has one red edge, one green edge and one blue edge as shown in Fig. 2. The bottom has all three colors. Let r be the event that the dice when tossed lands on a face with a red boundary. The events b, g and rgb are defined similarly. Then, by symmetry, we have*

$$\mathbb{P}(r) = \mathbb{P}(g) = \mathbb{P}(b) = \frac{1}{2},$$
$$\mathbb{P}(rgb) = \mathbb{P}(rg) = \mathbb{P}(gb) = \mathbb{P}(rb) = \frac{1}{4}.$$

Therefore, these events are pairwise independent but not independent.

Example 5.5. *Consider two six-sided fair dice. Let*

$$\begin{aligned} A_1 &= \{\textit{1st dice is odd}\}, \\ A_2 &= \{\textit{2nd dice is odd}\}, \\ A_{sum} &= \{\textit{sum of the two dice is odd}\}. \end{aligned}$$

We have

$$\begin{aligned} &\mathbb{P}(A_1) = \mathbb{P}(A_1) = \mathbb{P}(A_{sum}) = \frac{1}{2}, \\ &\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1 \cap A_{sum}) = \mathbb{P}(A_2 \cap A_{sum}) = \frac{1}{4}, \\ &\mathbb{P}(A_1 \cap A_2 \cap A_{sum}) = 0. \end{aligned}$$

Therefore, the events are pairwise independent but not independent. In particular, $\sigma(A_1)$ is not independent with $\sigma(\{A_2, A_{sum}\})$.

Lemma 5.2. *Suppose the two collections of subsets $\mathcal{E}$ and $\mathcal{C}$ are π-systems and $\mathbb{P}(B \cap C) = \mathbb{P}(B)\mathbb{P}(C)$, $\forall B \in \mathcal{E}$, $C \in \mathcal{C}$. Then $\sigma(\mathcal{E})$ and $\sigma(\mathcal{C})$ are independent.*

Proof. Let $\mathcal{D}_1 = \{D \in \mathcal{A} : \mathbb{P}(D \cap C) = \mathbb{P}(D)\mathbb{P}(C), \forall C \in \mathcal{C}\}$. As an exercise, one can check that $\mathcal{D}_1$ is a λ-system. Since $\mathcal{E} \subset \mathcal{D}_1$, from the π-λ theorem we have $\sigma(\mathcal{E}) \subset \mathcal{D}_1$.

Let $\mathcal{D}_2 = \{D \in \mathcal{A} : \mathbb{P}(B \cap D) = \mathbb{P}(B)\mathbb{P}(D), \forall B \in \sigma(\mathcal{E})\}$. Similarly $\mathcal{D}_2$ is a λ-system. From above, $\mathcal{C} \subset \mathcal{D}_2$ and by the π-λ theorem, we have $\sigma(\mathcal{C}) \subset \mathcal{D}_2$. Therefore, $\sigma(\mathcal{E}) \perp\!\!\!\perp \sigma(\mathcal{C})$. $\square$

By induction, if $\mathcal{B}_i$ for $i = 1, \ldots, n$ are π-systems and are independent, then $\sigma(\mathcal{B}_i)$ are independent.

Corollary 5.1. *The random variables $X_1, X_2, \ldots, X_n$ are independent if*

$$\mathbb{P}(X_1 \leq t_1, X_2 \leq t_2, \ldots, X_n \leq t_n) = \prod_{i=1}^{n} \mathbb{P}(X_i \leq t_i).$$

Proof. Since $\{(-\infty, t] : t \in \mathbb{R}\}$ is a π-system that generates $\mathcal{B}(\mathbb{R})$, the corollary follows from Lemma 5.2. $\square$

Lemma 5.3. *Suppose that each random variable X_i has pdf f_i, $i = 1, \ldots, n$. Then $X_1, \ldots, X_n$ are independent iff $\exists$ a joint pdf $f(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_i(X_i)$.*

Proof. '$\Leftarrow$':

$$\begin{aligned}\mathbb{P}\left(\bigcap_{i=1}^{n}\{X_i \in A_i\}\right) &= \int_{A_1\times\cdots\times A_n} f(x_1,\ldots,x_n)\,\mathrm{d}x_1\cdots\,\mathrm{d}x_n\\ &= \int_{A_1\times\cdots\times A_n}\prod_{i=1}^{n} f(x_i)\,\mathrm{d}x_1\cdots\,\mathrm{d}x_n\\ &= \prod_{i=1}^{n}\int_{A_i} f(x_i)\,\mathrm{d}x_i\\ &= \prod_{i=1}^{n}\mathbb{P}(X_i\in A_i).\end{aligned}$$

'$\Rightarrow$': Let $X = (X_1,\ldots,X_n)$. For $A_i \in \mathcal{B}$, $i = 1,\ldots n$, we are given

$$\begin{aligned}\mathbb{P}(X \in A_1\times\cdots\times A_n) &= \prod_{i=1}^{n}\mathbb{P}(X_i\in A_i)\\ &= \prod_{i=1}^{n}\int_{A_i} f_i(x_i)\,\mathrm{d}x_i\\ &= \int_{A_1\times\cdots\times A_n}\prod_{i=1}^{n} f_i(x_i)\,\mathrm{d}x_1\cdots\,\mathrm{d}x_n. \qquad (5)\end{aligned}$$

We want to show that

$$\mathbb{P}(X\in A) = \int_A \prod_{i=1}^{n} f_i(x_i)\,\mathrm{d}x_1\cdots\,\mathrm{d}x_n$$

for all A in the product σ-algebra $\mathcal{B}(\mathbb{R}^n) = \sigma\{A_1\times\cdots\times A_n : A_i\in\mathcal{B}\}$.

Let $\mathcal{L} = \{A\in\mathcal{E} : \mathbb{P}(X\in A) = \int_A\prod_{i=1}^n f_i(x_i)\,\mathrm{d}x_1\cdots\,\mathrm{d}x_n\}$. It can be checked that $\mathcal{L}$ is a λ-system and $\mathcal{P} = \{A_1\times\cdots\times A_n : A_i\subset\mathcal{B}\}$ is a π-system and generates $\mathcal{B}(\mathbb{R}^n)$. Since $\mathcal{P}\subset\mathcal{L}$ from (5), by the π-λ Theorem, we have $\sigma(\mathcal{P})\subset\mathcal{L}$ and the proof is complete. □

Lemma 5.4. *If $X \perp\!\!\!\perp Y$ and are integrable, then $\mathbb{E}[XY] = \mathbb{E}X\mathbb{E}Y$.*

Proof. The distribution of (X,Y) on $\mathbb{R}^2$ is the product measure $\mathbb{P}_X\times\mathbb{P}_Y$. From Fubini's theorem, we have

$$\mathbb{E}[XY] = \int_{\mathbb{R}^2} xy\,\mathrm{d}\mathbb{P}_X\times\mathbb{P}_Y(x,y) = \int_{\mathbb{R}} x\,\mathrm{d}\mathbb{P}_X(x)\int_{\mathbb{R}} y\,\mathrm{d}\mathbb{P}_Y(y) = \mathbb{E}X\mathbb{E}Y.$$

□

Suppose $X\perp\!\!\!\perp Y$. For any measurable functions f and g, $f(X)\perp\!\!\!\perp g(Y)$ since $\sigma(f(X))\subset\sigma(X)$.