学习报告

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1 Coefficient Functions

考虑如下的 d 维 sde

$$X_t = X_{t_0} + \int_{t_0}^{t} a\left(s, X_s\right) ds + \sum_{j=1}^{m} \int_{t_0}^{t} b^j\left(s, X_s\right) dW_s^j \tag{1}$$

定义

$$L^0 = \frac{\partial}{\partial t} + \sum_{k=1}^{d} a^k \frac{\partial}{\partial x^k} + \frac{1}{2} \sum_{k,l=1}^{d} \sum_{j=1}^{m} b^{k,j} b^{l,j} \frac{\partial^2}{\partial x^k \partial x^l} \tag{2}$$

对于 $j \in 1, ..., m$ 定义

$$L^j = \sum_{k=1}^{d} b^{k,j} \frac{\partial}{\partial x^k} \tag{3}$$

对于任意 $\alpha = (j_1, ..., j_l)$ 和任意函数 $f \in C^h(R^+ \times R^d, R), h = l(\alpha) + n(\alpha)$ 定义 Ito coefficient function

$$f_\alpha = \begin{cases} f & : \quad l = 0 \\ L^{j_1} f_{-\alpha} & : \quad l \geq 1 \end{cases} \tag{4}$$

2 Hierarchiacal and Remainder Sets

Hierarchical 集 $\mathcal{A}$ 的定义：$\mathcal{A} \in \mathcal{M}$ 满足 $\mathcal{A} \neq \emptyset$ 且 $\sup_{\alpha \in \mathcal{A}} l(\alpha) < \infty$，$-\alpha \in \mathcal{A} \quad \text{for each} \quad \alpha \in \mathcal{A} \backslash \{v\}$。
对于上述的 $\mathcal{A}$ 定义 remainder 集 $\mathcal{B}(\mathcal{A})$ 为

$$\mathcal{B}(\mathcal{A}) = \{\alpha \in \mathcal{M}\ \mathcal{A} : -\alpha \in \mathcal{A}\}$$

3 Ito-Taylor 展开

考虑 d 维的 Ito 过程

$$X_t = X_{t_0} + \int_{t_0}^{t} a\left(s, X_s\right) ds + \sum_{j=1}^{m} \int_{t_0}^{t} b^j\left(s, X_s\right) dW_s^j \tag{5}$$

这里 $t \in [t_0, T]$，m 代表 Brown 的分量数。

定理 3.1. 设 ρ 和 τ 是两个满足如下条件的停时

$$t_0 \leq \rho(\omega) \leq \tau(\omega) \leq T, a.s.$$

令 $\mathcal{A} \subseteq \mathcal{M}$ 是一个 *hierarchical* 集，$f : R^+ \times R^d \to R$，假设 f 足够光滑，且下列多重积分存在 *Ito-Taylor* 展开如下

$$f\left(\tau, X_\tau\right) = \sum_{\alpha \in \mathcal{A}} I_\alpha \left[f_\alpha\left(\rho, X_\rho\right)\right]_{\rho,\tau} + \sum_{\alpha \in B(\mathcal{A})} I_\alpha \left[f_\alpha\left(\cdot, X_\cdot\right)\right]_{\rho,\tau} \tag{6}$$

下面的两个例子体现了 Ito-Taylor 展开与 Taylor 公式和 Ito 公式的联系：
设 $\mathcal{A} = \{v\}$ 则 $\boldsymbol{B}(\{v\}) = \{(0), (1), \ldots, (m)\}$，则由 $Ito - Taylor$ 展开可知

$$\begin{aligned} f\left(\tau, X_\tau\right) &= I_v \left[f_v\left(\rho, X_\rho\right)\right]_{\rho,\tau} + \sum_{\alpha \in \mathcal{B}(\{v\})} I_\alpha \left[f_\alpha\left(\cdot, X_\cdot\right)\right]_{\rho,\tau} \\ &= f\left(\rho, X_\rho\right) + \int_\rho^\tau L^0 f\left(s, X_s\right) ds + \sum_{j=1}^{m} \int_\rho^\tau L^j f\left(s, X_s\right) dW_s^j \end{aligned} \tag{7}$$

回顾 L^i 算子的定义，上述公式便是 Ito 公式。接下来考虑 $d = 1, f(t,x) = f(x), a = 1, b = 0, \rho = 0, \tau = t$ 则显然 $X_t = t$, 取 $\Gamma_l = \{\alpha \in \mathcal{M} : l(\alpha) \leq l\}$ 对任意 $\alpha \in \Gamma_l$ 有

$$f_\alpha = \begin{cases} f & : \alpha = v \\ f^{(I)} : & l \geq 1 \text{ and } j_1 = \cdots = j_l = 0 \end{cases}$$

只要有一个 j_i 不为 0, 由于 $b = 0$,f_α 必为 0，且当 $j_1 = ... = j_l = 0$ 时

$$I_\alpha[f(X.)]_{t_0,t} = \int_{t_0}^{t} \cdots \int_{t_0}^{s_2} f\left(s_1\right) ds_1 \ldots ds_k$$

将上述式子代入 $f(X_t)$ 的 Ito-Taylor 展开中可得

$$
\begin{aligned}
f(t) = f\left(X_{t_0}\right) + \sum_{\alpha \in \Gamma_k \setminus \{v\}} I_\alpha\left[f_\alpha\left(X_{t_0}\right)\right]_{t_0,t} + \sum_{\alpha \in \mathcal{B}(\Gamma_k)} I_\alpha\left[f_\alpha(X.)\right]_{t_0,t} \\
= \sum_{i=0}^{k} \int_{t_0}^{t} \cdots \int_{t_0}^{s_2} f^{(i)}\left(X_{t_0}\right) ds_1 \ldots ds_i + \int_{t_0}^{t} \cdots \int_{t_0}^{s_2} f^{(k+1)}\left(X_{s_1}\right) ds_1 \ldots ds_{k+1} \\
= f\left(t_0\right) + \sum_{i=1}^{k} \frac{1}{i!} f^{(i)}\left(t_0\right)\left(t-t_0\right)^i + \int_{t_0}^{t} \cdots \int_{t_0}^{s_2} f^{(k+1)}\left(s_1\right) ds_1 \ldots ds_{k+1}
\end{aligned}
\tag{8}
$$

这就是普通的 Taylor 展开

引理 3.1. 设 ρ 和 τ 是两个满足如下条件的停时

$$t_0 \leq \rho(\omega) \leq \tau(\omega) \leq T, a.s.$$

函数 $f: R^+ \times R^d \to R$ 属于 $\mathcal{C}^{1,2}$ 则

$$f\left(\tau, X_\tau\right) = f\left(\rho, X_\rho\right) + \sum_{j=0}^{m} I_{(j)}\left[L^j f(\cdot, X.)\right]_{\rho,\tau} \tag{9}$$

当引理中的 $\rho = t_0$，$\tau = T$ 时上式为 Ito 公式。

引理 3.2. 设 ρ 和 τ 是两个满足如下条件的停时

$$t_0 \leq \rho(\omega) \leq \tau(\omega) \leq T, a.s.$$

$\alpha, \beta \in \mathcal{M}$, 且 $l(\beta) > 0$。$f: R^+ \times R^d \to R$，假设 f 足够光滑，且下列多重积分存在，则

$$I_\alpha\left[f_\beta(\cdot, X.)\right]_{\rho,\tau} = I_\alpha\left[f_\beta\left(\rho, X_\rho\right)\right]_{\rho,\tau} + \sum_{j=0}^{m} I_{(j)*\alpha}\left[f_{(j)*\beta}(\cdot, X.)\right]_{\rho,\tau} \tag{10}$$

证明. 关于 $l(\alpha)$ 使用数学归纳法，设 $l(\alpha) = 0$ 则 $\alpha = v$，由引理 3.1

$$
\begin{aligned}
I_\alpha\left[f_\beta(\cdot, X.)\right]_{\rho,\tau} &= f_\beta\left(\tau, X_\tau\right) \\
&= f_\beta\left(\rho, X_\rho\right) + \sum_{j=0}^{m} I_{(j)}\left[L^j f_\beta(\cdot, X.)\right]_{\rho,\tau} \\
&= I_\alpha\left[f_\beta\left(\rho, X_\rho\right)\right]_{\rho,\tau} + \sum_{j=0}^{m} I_{(j)*\alpha}\left[f_{(j)*\beta}(\cdot, X.)\right]_{\rho,\tau}
\end{aligned}
\tag{11}
$$

接下来设 $l(\alpha)=k\geq 1$,$\alpha=(j_1,...,j_k)$，对 $I_{(j_k)}\left[I_{\alpha-}\left[(f_\beta(\cdot,X.)]_{\rho\cdot}\right]_{\rho,\tau}$ 使用归纳假设得

$$
\begin{aligned}
I_\alpha\left[f_\beta(\cdot,X.)\right]_{\rho,\tau} =&I_{(j_k)}\left[I_{\alpha-}\left[(f_\beta(\cdot,X.)\right]_{\rho\cdot}\right]_{\rho,\tau}\\
=&I_{(j_k)}\left[I_{\alpha-}\left[(f_\beta(\rho,X_\rho)\right]_{\rho\cdot}\right]_{\rho,\tau}\\
&+\sum_{j=0}^{m}I_{(j_k)}\left[I_{(j)*\alpha-}\left[f_{(j)*\beta}(\cdot,X.)\right]_{\rho,\cdot}\right]_{\rho,\tau}\\
=&I_\alpha\left[f_\beta(\rho,X_\rho)\right]_{\rho,\tau}\\
&+\sum_{j=0}^{m}I_{(j)*\alpha}\left[f_{(j)\cdot\beta}(\cdot,X.)\right]_{\rho,\tau}
\end{aligned}
\tag{12}
$$

□

对 Ito-Taylor 展开的证明如下：

证明. 对 $l_1(\mathcal{A})=\sup_{\alpha\in\mathcal{A}}l(\alpha)$ 进行归纳，当 $l_1(\mathcal{A})=0$ 时 $\mathcal{A}=\{v\}$, 则 $B(\mathcal{A})=\{(0),(1),\cdots,(m)\}$ 由引理 3.1 可知

$$
f(\tau,X_\tau)=\sum_{\alpha\in\mathcal{A}}I_\alpha\left[f_\alpha(\rho,X_\rho)\right]_{\rho,\tau}+\sum_{\alpha\in B(\mathcal{A})}I_\alpha\left[f_\alpha(\cdot,X.)\right]_{\rho,\tau}
\tag{13}
$$

接下来假设 $l_1(\mathcal{A})=k\geq 1$ 设

$$
\mathcal{E}=\{\alpha\in\mathcal{A}:l(\alpha)\leq k-1\}
$$

显然 $\mathcal{E}$ 是一个 hierarchical 集，由归纳假设

$$
f(\tau,X_\tau)=\sum_{\alpha\in\mathcal{E}}I_\alpha\left[f_\alpha(\rho,X_\rho)\right]_{\rho,\tau}+\sum_{\alpha\in\mathcal{B}(\mathcal{E})}I_\alpha\left[f_\alpha(\cdot,X.)\right]_{\rho,\tau}
\tag{14}
$$

显然 $\mathcal{A}\backslash\mathcal{E}\subseteq\mathcal{B}(\mathcal{E})$ 对任意 $\beta=\alpha\in\mathcal{A}\backslash\mathcal{E}$ 由引理 3.2 可得

$$
\begin{aligned}
f(\tau,X_\tau)=&\sum_{\alpha\in\mathcal{E}}I_\alpha\left[f_\alpha(\rho,X_\rho)\right]_{\rho,\tau}+\sum_{\alpha\in\mathcal{A}\backslash\varepsilon}I_\alpha\left[f_\alpha(\cdot,X.)\right]_{\rho,\tau}\\
&+\sum_{\alpha\in\mathcal{B}(\mathcal{E})\backslash(\mathcal{A}\backslash\varepsilon)}I_\alpha\left[f_\alpha(\cdot,X.)\right]_{\rho,\tau}
\end{aligned}
\tag{15}
$$

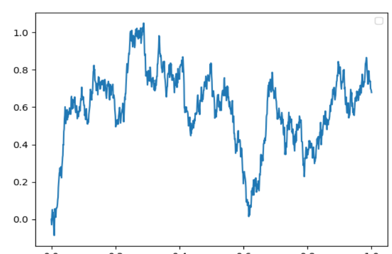
$$
\begin{aligned}
&= \sum_{\boldsymbol{\alpha} \in \mathcal{E}} I_\alpha \left[f_\alpha \left(\rho, X_\rho \right) \right]_{\rho,\tau} \\
&\quad + \sum_{\alpha \in \mathcal{A} \setminus \varepsilon} \left[I_\alpha \left[f_\alpha \left(\rho, X_\rho \right) \right]_{\rho,\tau} + \sum_{j=0}^{m} I_{(j)*\alpha} \left[f_{(j)*\alpha}(\cdot, X.) \right]_{\rho,\tau} \right] \\
&\quad + \sum_{\alpha \in \mathcal{B}(\mathcal{E}) \setminus (\mathcal{A} | \varepsilon)} I_\alpha \left[f_\alpha(\cdot, X.) \right]_{\rho,\tau} \\
&= \sum_{\alpha \in \mathcal{A}} I_\alpha \left[f_\alpha \left(\rho, X_\rho \right) \right]_{\rho,\tau} + \sum_{\alpha \in \mathcal{B}_1} I_\alpha \left[f_\alpha(\cdot, X) \right]_{\rho,\tau}
\end{aligned}
$$

显然 $\mathcal{B}_1 = \mathcal{B}(\mathcal{A})$ □

4 模拟随机过程的轨道

4.1 Brown 运动的轨道

对时间剖分 $0 = t_0 < t_1 < ... < t_n = T$ 求出 $\Delta t_i, i = 1, 2..n$，分别生成服从 $N(0, \Delta t_i)$ 的随机数 w_i，则 $Brown$ 运动在该轨道下 t_i 取值为 $\sum_{j=1}^{i} w_i$

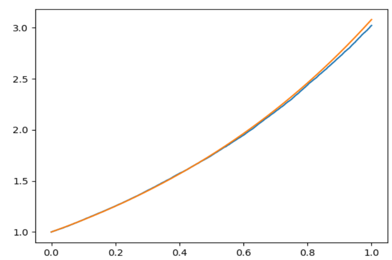


4.2 其他随机过程

模拟随机过程

$$
u(W(t)) = \exp\left(t + \frac{1}{2}W(t)\right) \tag{16}
$$

$u(W(t))$ 的均值为 $e^{9t/8}$，重复试验 4000 次求其均值，结果如下


3.0
2.5
2.0
1.5
1.0
0.0
0.2
0.4
0.6
0.8
1.0