# 学习报告

姜钦瀚

2020 年 3 月 11 日

考虑如下的随机微分方程

$$dX_t = a\left(t, X_t\right) dt + b\left(t, X_t\right) dW_t, t_0 \leq t \leq T, X_{t_0} = X_0 \tag{1}$$

给定时间划分 $t_0 = \tau_0 < \tau_1 < \cdots < \tau_n < \cdots < \tau_N = T$，记 $\Delta_n = \tau_{n+1} - \tau_n$,$\delta = max\Delta_n$。

## 1 定义

**定义 1.1.** 称格式 $Y_\delta$ 在 $T$ 处强收敛于 $X$ 如果

$$\lim_{\delta \downarrow 0} E\left(\left|X_T - Y^\delta(T)\right|\right) = 0 \tag{2}$$

## 2 strong scheme

$\mathcal{A} = \Lambda_k := \left\{\alpha; \quad \alpha \in \mathcal{M}, l(\alpha) + n(\alpha) \leq 2k \text{ or } l(\alpha) = n(\alpha) = k + \frac{1}{2}\right\}$ ,K 阶强格式为

$$X^{n+1} = \sum_{\alpha \in \Lambda_k} I_\alpha \left[f_\alpha\left(t_n, X^n\right)\right]_{t_n, t_{n+1}} = \sum_{\alpha \in \Lambda_k} f_\alpha\left(t_n, X^n\right) I_{\alpha, t_n, t_{n+1}} \tag{3}$$

## 3 Euler 格式

Euler 格式是强 0.5 阶格式定义 $\mathcal{A}$ 为

$$\mathcal{A} = \{\phi, (0), (1)\} = \Lambda_{0.5} \tag{4}$$

由上面定义可知设 Brown 运动有 m 个分量，k 维的 Euler 格式为

$$Y_{n+1}^k = Y_n^k + a^k \Delta + \sum_{j=1}^{m} b^{k,j} \Delta W^j \tag{5}$$

### 3.1 数值模拟

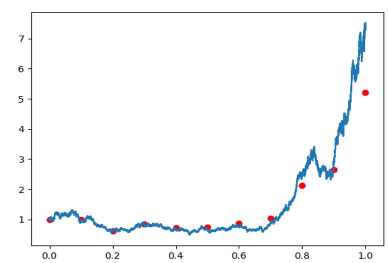
考虑如下的线性随机微分方程

$$dX(t) = \lambda X(t)dt + \mu X(t)dW(t), \quad X_{t_0} = X_0 \tag{6}$$

其真解为

$$X(t) = X_{t_0} \exp\left(\left(\lambda - \frac{1}{2}\mu^2\right)t + \mu W(t)\right) \tag{7}$$

取区间为 $[0,1]$ 将区间均匀剖分，$\delta = 10^{-5}$, 模拟出真解的情况，然后取 $\delta = 0.1$ 应用欧拉格式，数值模拟结果如下 分别取 $\delta = 10^{-i}, i = \{0,1,2,3,4\}$，各




重复实验 1000 次，取 $T=1$ 处的误差，并求平均值来模拟 $E\left(\left|X_T - Y^{\delta}(T)\right|\right)$ 结果见下面表格

表 1: 误差

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| 4.26748204 | 1.481642574 | 0.45718003 | 0.13571932 | 0.03757269 |

## 4 Milstein 格式

Milstein 格式是强 1 阶格式，

$$\mathcal{A} = \{\phi, (0), (1), (1,1)\} = \Lambda_1 \tag{8}$$

。代入 $Ito-Taylor$ 展开得

$$
\begin{aligned}
X^{n+1}=&\sum_{\alpha\in\Lambda_1} I_\alpha\left[f_\alpha\left(t_n,X^n\right)\right]_{t_n,t_{n+1}}=\sum_{\alpha\in\Lambda_1} f_\alpha\left(t_n,X^n\right) I_{\alpha,\mathrm{tn},t_{n+1}}\\
=&f_\phi\left(t_n,X^n\right) I_{\phi,t_n,t_{n+1}}+f_{(0)}\left(t_n,X^n\right) I_{(0),t_n,t_{n+1}}+f_{(1)}\left(t_n,X^n\right) I_{(1),t_n,t_{n+1}}\\
&+f_{(1,1)}\left(t_n,X^n\right) I_{(1,1),t_n,t_{n+1}}\\
=&X^n+a\left(t_n,X^n\right)\Delta t_n+b\left(t_n,X^n\right)\Delta W_{tn}+f_{(1,1)}\left(t_n,X^n\right)\int_{t_n}^{t_{n+1}}\int_{t_n}^{s} dW_\tau^1 dW_s^1
\end{aligned}
\tag{9}
$$

### 4.1 数值模拟

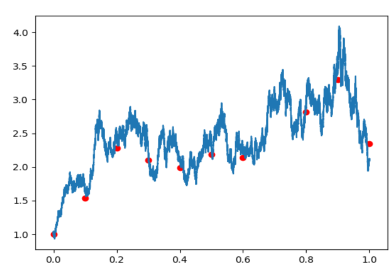
考虑如下的线性随机微分方程

$$dX(t)=\lambda X(t)dt+\mu X(t)dW(t),\quad X_{t_0}=X_0 \tag{10}$$

其真解为

$$X(t)=X_{t_0}\exp\left(\left(\lambda-\frac{1}{2}\mu^2\right)t+\mu W(t)\right) \tag{11}$$

取区间为 $[0,1]$ 将区间均匀剖分，$\delta=10^{-5}$, 模拟出真解的情况，然后取 $\delta=0.1$ 应用 Milstein 格式，数值模拟结果如下 分别取 $\delta=10^{-i},i=$




$\{0,1,2,3,4\}$，各重复实验 1000 次，取 $T=1$ 处的误差，并求平均值来模拟 $E\left(\left|X_T-Y^\delta(T)\right|\right)$ 结果见下面表格

# 5 强 1.5 阶格式

$$\mathcal{A}=\left\{\begin{array}{cccc}\emptyset, & (0), & (1), & (1,1)\\ (0,0), & (1,0), & (0,1), & (1,1,1)\end{array}\right\}=\Lambda_{1,5} \tag{12}$$

表 2: 误差变化

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| 4.576793014 | 1.481642574 | 0.16355771 | 0.01529428 | 0.00159989 |

$\alpha = (0,0)$ 对应

$$\begin{aligned} f_{(0,0)} &= L^0 L^0 f = L^0 a \\ &= \frac{\partial a}{\partial t} + a\frac{\partial a}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 a}{\partial x^2} \\ I_{(0,0),t_n,t_{n+1}} &= \int_{t_n}^{t_{n+1}}\int_{t_n}^{t} ds dt = \frac{1}{2}\Delta t_n \end{aligned} \tag{13}$$

$\alpha = (1,0)$ 对应

$$\begin{aligned} f_{(1,0)} &= L^1 L^0 f = L^1 a = b\frac{\partial a}{\partial x} \\ I_{(1,0),t_n,t_{n+1}} &= \int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW_s dt = \Delta Z_n \end{aligned} \tag{14}$$

$\alpha = (0,1)$ 对应

$$\begin{aligned} f_{(0,1)} &= L^0 L^1 f = L^0 b \\ &= \frac{\partial b}{\partial t} + a\frac{\partial b}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 b}{\partial x^2} \\ I_{(0,1)}, t_n, t_{n+1} &= \int^{t_{n+1}}\int_{t}^{t} ds dW_t = \overline{\Delta Z_n} \end{aligned} \tag{15}$$

$\alpha = (1,1,1)$ 对应

$$\begin{aligned} f_{(1,1,1)} &= L^1 L^1 L^1 f = L^1 L^1 b = L^1\left(b\frac{\partial b}{\partial x}\right) \\ &= b\frac{\partial}{\partial x}\left(b\frac{\partial b}{\partial x}\right) = b\left(\frac{\partial b}{\partial x}\right)^2 + b^2\frac{\partial^2 b}{\partial x^2} \\ &= \int_{tn}^{t_{n+1}}\int_{t_n}^{t}\int_{t_n}^{s} dW_r dW_s dW_t \\ I_{(1,1,1)} t_n, t_{n+1} &= \frac{1}{2}\left(\frac{1}{3}\left(\Delta W_{t_n}\right)^2 - \Delta t_n\right)\Delta W_{t_n} \end{aligned} \tag{16}$$