Problem Set 7

SEM solution to 1D unsteady-state diffusion equation

Use the spectral-element method (SEM) to solve the 1D heat diffusion equation, that is to find the temperature $T(x,t)$ for $x \in [0, L = 3\ km]$ such that (strong form)

$$\rho c_p \partial_t T - \partial_x(\kappa \partial_x T) = 0$$

where ρ is the medium density, c_p is the the specific heat, and κ is the thermal conductivity, with the following initial & boundary conditions:

$$\begin{cases} T(x,0) &=& 0 \\ T(L,t) &=& 0 \\ T(0,t) &=& 10 \end{cases}$$

Problem:

Solve this problem for the following cases:

- evenly spaced mesh, constant $\rho = 2.5 \times 10^3 \; kg/m^3$, $c_p = 0.3 \times 10^3 \; cal/kg/K$ and $\kappa = 10.0 \times 10^{-1} \; cal/m/s/K$
- unevenly spaced mesh, constant ρ, c_p and κ
- variable conductivity

$$\kappa(x) = \begin{cases} \kappa_1 = 10.0 \times 10^{-1} \; cal/m/s/K & x \in [0, L/2] \\ \kappa_2 = 2.0 \times 10^{-1} \; cal/m/s/K & x \in [L/2, 0] \end{cases}$$

and compare your results. Plot several time steps, e.g.,

`t = 0.0 / 25.4 / 50.7 / 76.1 kyr`

and compare to the steady-state analytical solution.

Material:

In this homework directory, you will find the necessary fortran code in directory `code/` with following files:

`Makefile`: used for the compilation. By default it uses gfortran.

directory `obj/`, `bin/` : where you will find the ouputs of the compilation.

`constants.h`: a numbers of static constants, like the number of spectral elements (NSPEC=11), number of GLL points (NGLL=7), number of global points (`NGLOB = (NGLL-1)*NSPEC +1`), and two variables used to evaluate the GLL points & the weights.

`gll_library.f90`: library to compute the GLL points and the weights.

`lagrange_poly.f90`: library to compute the Lagrange interpolants based upon the GLL points as well as the first derivatives of these polynomials at any point $\xi \in [-1, 1]$.

`define_derivative_matrix.f90`: store the derivatives of the Lagrange polynomials as a NGLL×NGLL matrix.

`diffusion.f90`: the main program.

After compilation you will find the executable *xdiffusion* in the directory `bin/`.
You will see that the full structure of the code is given in the file `diffusion.f90`. You are asked to input your code in lieu of "put your code here".

Time scheme:

You will be using the predictor-corrector algorithm to march in time:

- Predictor:

$$
\begin{aligned}
T_{n+1} &= T_n + \frac{1}{2}\Delta t\, \dot{T}_n \\
\dot{T}_{n+1} &= 0 \quad \text{(initialization at the beginning of each time step)}
\end{aligned}
$$

- Solve:

$$
\begin{aligned}
rhs &= -KT_{n+1} \\
\delta\dot{T}_{n+1} &= M^{-1} rhs
\end{aligned}
$$

- Corrector:

$$
\begin{aligned}
T_{n+1} &= T_{n+1} + \frac{1}{2}\Delta t\, \delta\dot{T}_{n+1} \\
\dot{T}_{n+1} &= \dot{T}_{n+1} + \delta\dot{T}_{n+1}
\end{aligned}
$$