Problem Set 5

FEM solution of 1D steady-state diffusion equation

Write a finite-element program using linear shape functions, to find the temperature $T = T(x)$ in $[0, 1]$ such that (strong form)

$$\partial_x^2 T + f = 0$$

where f is a source or sink, with the following boundary conditions:

$$\begin{aligned} T(1) &= T_1 \\ \partial_x T(0) &= -q_0 \end{aligned}$$

The function $f = f(x)$ can be an arbitrary function. The initial temperature T_1 at location $x = 1$ and heat flux q_0 are scalar constants.

Note that the analytical solution found for this problem was:

$$T(x) = T_1 + (1 - x)q_0 + \int_x^1 \left(\int_0^y f(z)dz \right) dy \tag{1}$$

for any y.

Problem:

Address this FE problem as follows:

1. Write the weak form of the equation by introducing the linear shape functions N_A & $N_{N_{el}+1}$.
2. Define the local (element level) stiffness matrix and right-hand-side vector.
3. Assemble these local matrices into global matrices (global level).
4. Prescribe the number of elements (N_{el}). Choose a couple of cases, for example, $N_{el} = 10$ and $N_{el} = 2$.
5. Explore two sets of boundary conditions:
   - (i) $T_1 = 1$, $q_0 = 1$, and $f(x) = 0$
   - (ii) $T_1 = 1$, $q_0 = 1$, and $f(x) = 1$

Compare the FEM solution to the exact solution to the strong form, by plotting the temperature T versus x.