Math review

In our course, basically all geometries can be described by Cartesian coordinates in a Euclidean space. In the following, we give a very brief review of vector calculus.

Vectors, tensors

A **vector** $\mathbf{u}$ can be written as

$$\mathbf{u} = u_x\hat{\mathbf{x}} + u_y\hat{\mathbf{y}} + u_z\hat{\mathbf{z}}$$

where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ are the unit vectors in the x, y and z directions.

The **dot product** of two vectors is a scalar and defined as

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u_x v_x + u_y v_y + u_z v_z \\ &= \sum_i u_i v_i = u_i v_i \end{aligned}$$

where we use the index notation i assuming it takes the values 1, 2 and 3 for the x, y and z components, respectively. Note that we also use the *summation convention*, i.e., repeated indices in a product are summed over.

A **second-order tensor** $\mathbf{A}$ is a linear form that produces one vector from another, such as

$$\begin{aligned} \mathbf{u} &= \mathbf{A}\mathbf{v} \\ u_i &= \sum_j A_{ij} v_j = A_{ij} v_j \end{aligned}$$

where we sum again over indices $j = 1, 2, 3$.

The **double-dot product** of two second-order tensors is a scalar defined as

$$\mathbf{A} : \mathbf{B} = \sum_i \sum_j A_{ij} B_{ij} = A_{ij} B_{ij}$$

where we use the summation convention again over i and j. We will often use the double-dot product between a 4th-order tensor $\mathbf{c}$ and and a second-order tensor $\mathbf{A}$ like

$$(\mathbf{c} : \mathbf{A})_{ij} = c_{ijkl} A_{kl}$$

summing over indices k and l, resulting in a second-order tensor.

The **dyadic product** or **tensor product** of two vectors is a tensor defined as

$$\begin{aligned} \mathbf{u}\,\mathbf{v} &= \mathbf{u} \otimes \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{bmatrix} \\ (\mathbf{u}\,\mathbf{v})_{ij} &= u_i v_j \end{aligned}$$

Note that the product is not commutative, i.e., the ordering of the vectors is sensitive and in general $\mathbf{u}\,\mathbf{v} \neq \mathbf{v}\,\mathbf{u}$.

Gradients

The **gradient** of a scalar field $\nabla\lambda$ is a vector field defined by the partial derivatives in x, y and z directions, i.e.,

$$\begin{aligned}\nabla\lambda &= \frac{\partial\lambda}{\partial x}\hat{\mathbf{x}} + \frac{\partial\lambda}{\partial y}\hat{\mathbf{y}} + \frac{\partial\lambda}{\partial z}\hat{\mathbf{z}} \\ (\nabla\lambda)_i &= \partial_i\lambda\end{aligned}$$

where we use the notation ∂_i as shorthand for $\partial/\partial x$, $\partial/\partial y$ and $\partial/\partial z$ for $i = 1, 2, 3$ respectively.

The **gradient** of a vector field $\nabla\mathbf{v}$ is a tensor field defined by

$$\begin{aligned}\mathbf{A} &= \nabla\mathbf{v} \\ A_{ij} &= \partial_i v_j\end{aligned}$$

The **divergence** of a vector field, written as $\nabla\cdot\mathbf{v}$ is a scalar field defined by

$$\begin{aligned}\nabla\cdot\mathbf{v} &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \\ &= \partial_i v_i\end{aligned}$$

using the summation convection.

The **divergence** of a second-order tensor field $\nabla\cdot\mathbf{A}$ is a vector field

$$(\nabla\cdot\mathbf{A})_j = \partial_i A_{ij}$$

where we again sum over indices $i = 1, 2, 3$.

Gauss's theorem

The **divergence theorem**, also known as **Gauss's theorem**, equates the volume integral of a vector field to the surface integral of the orthogonal component of the vector field, i.e.,

$$\int_V \nabla\cdot\mathbf{v}\,\mathrm{d}V = \int_S \mathbf{v}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$$

where $\hat{\mathbf{n}}$ is the outward normal vector to the surface. Note that the same equality can be written for the divergence of a second-order tensor field.