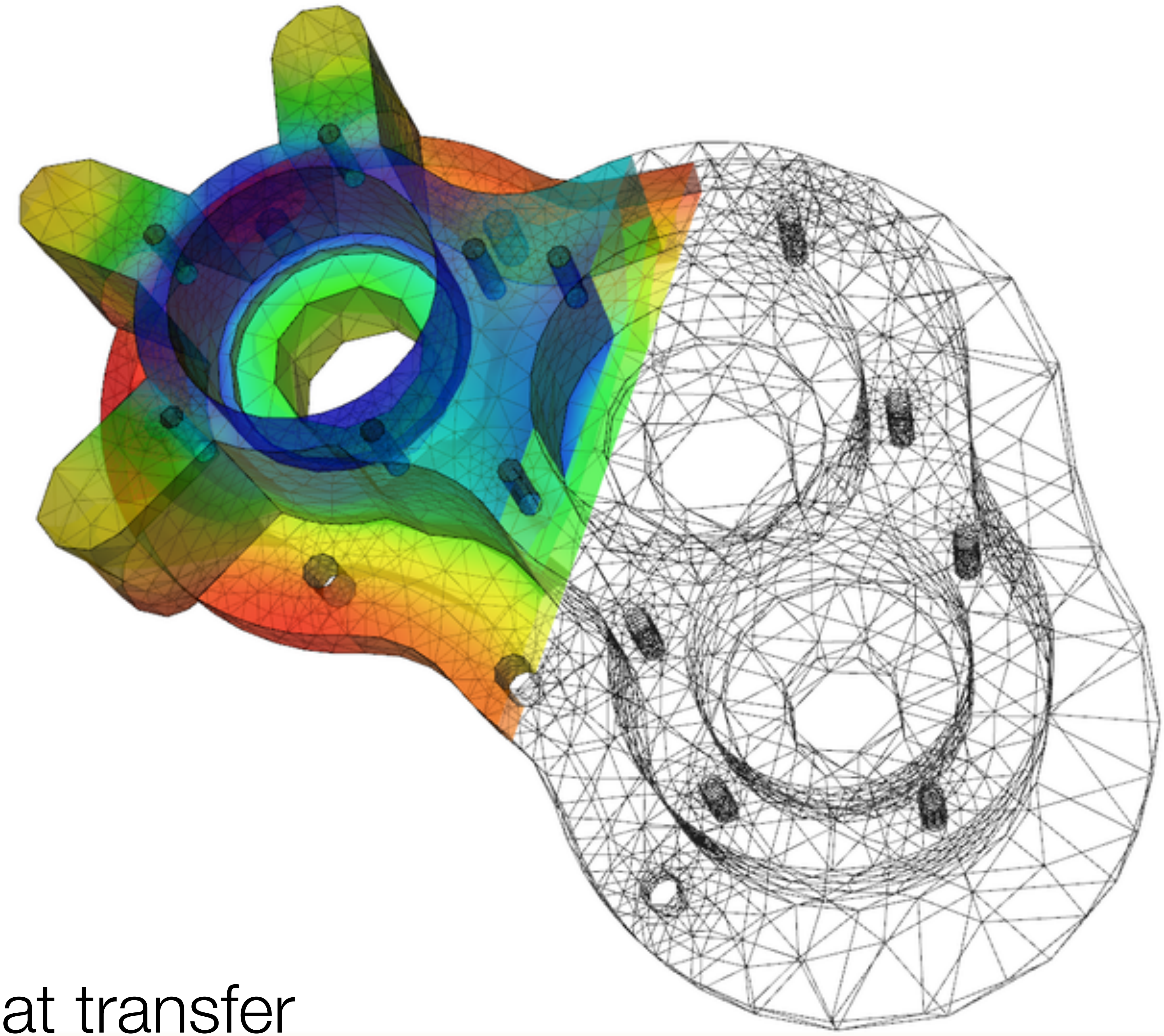
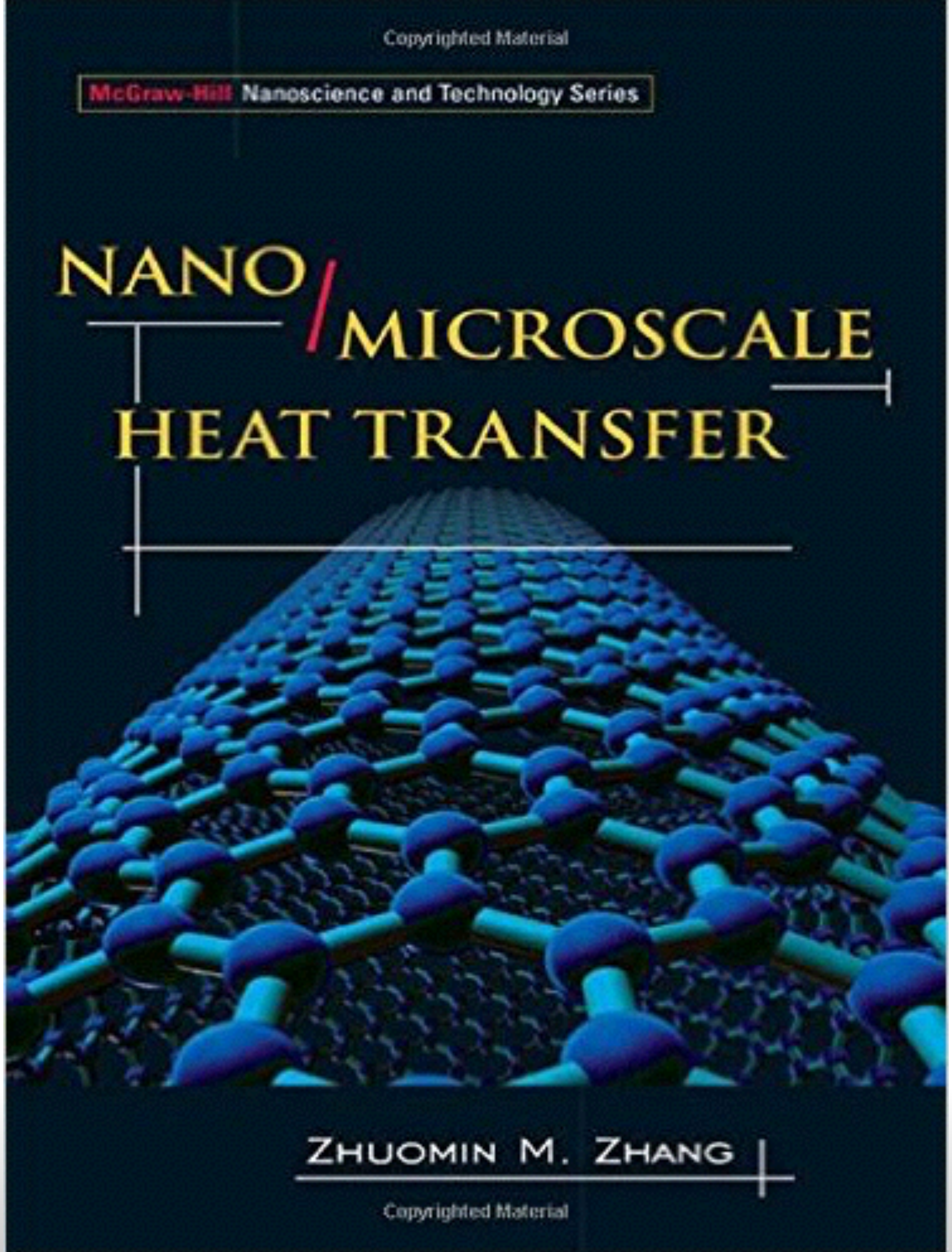
Finite-element methods

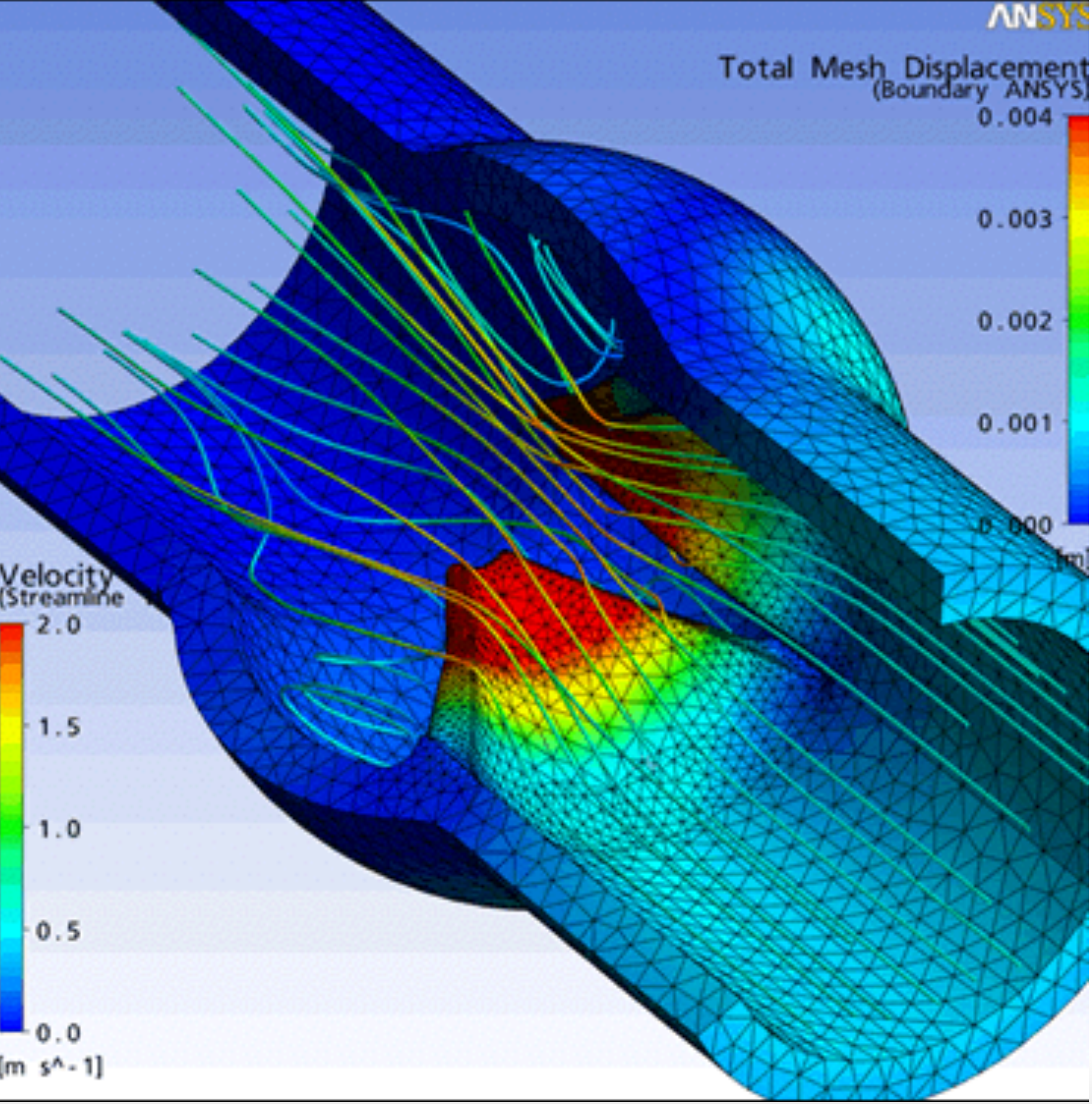


Heat transfer

civil engineering

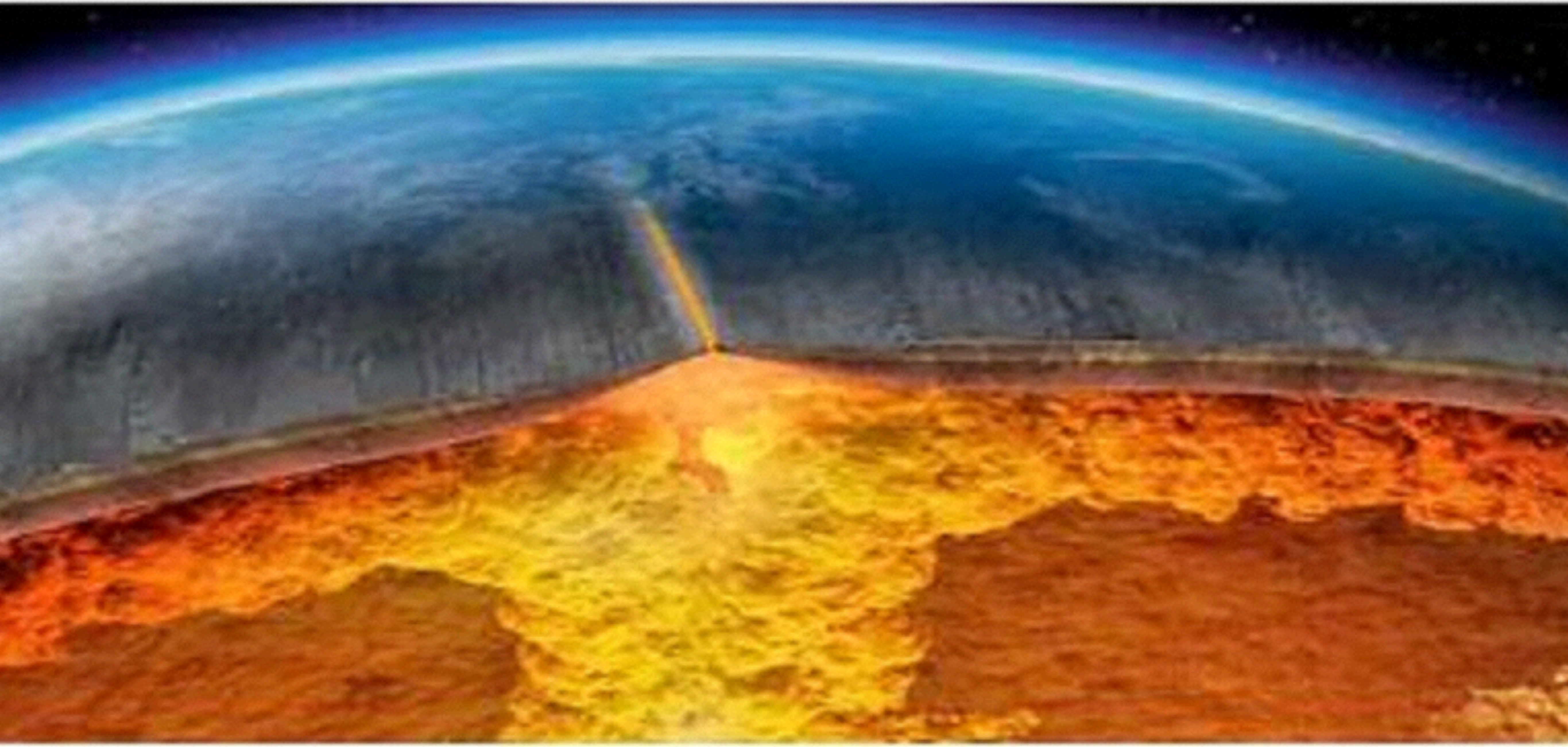


Heat transfer
nano-scales



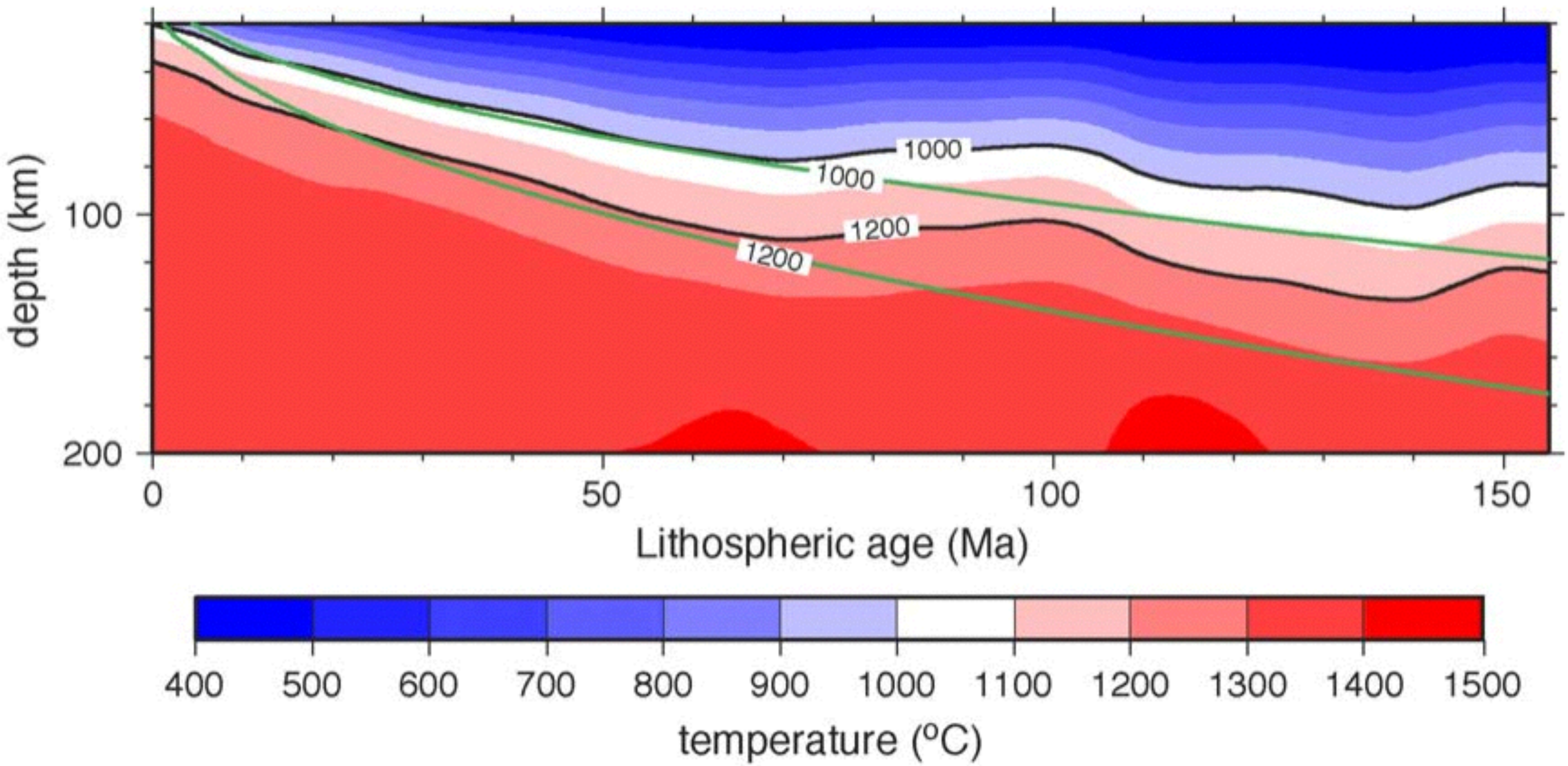
Heat transfer

biological tissue

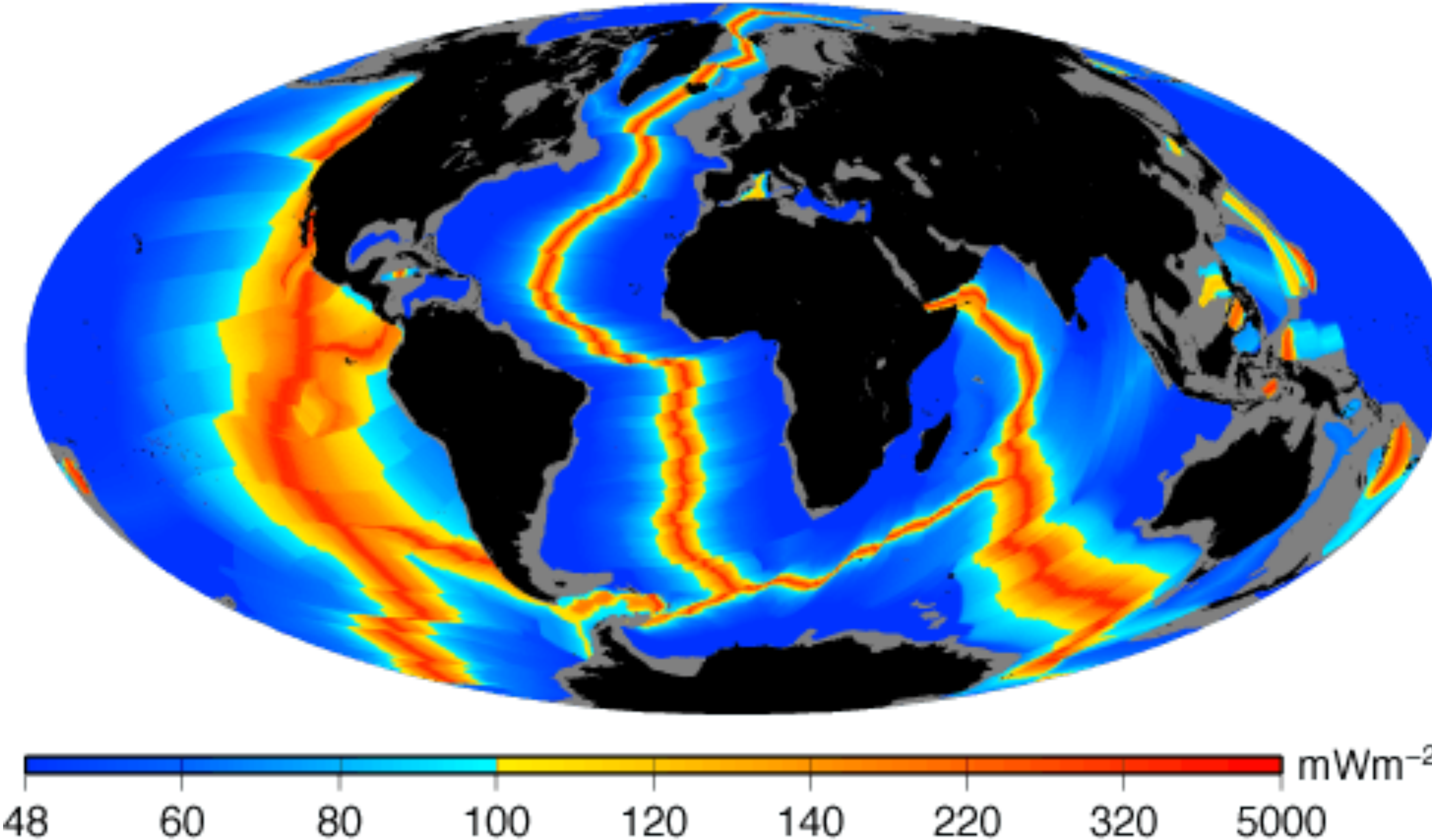


Heat transfer

half-space cooling



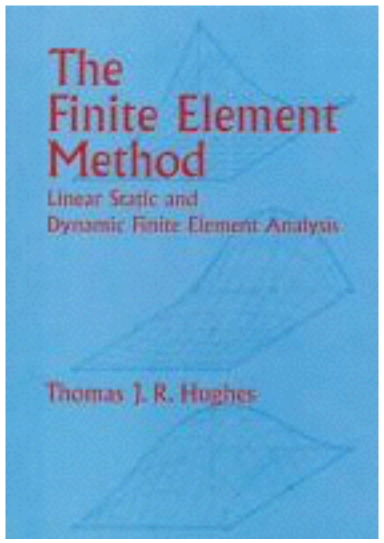
debate: half-space cooling of the Pacific plate
Ritzwoller et al. (EPSL, 2004)



debate: geothermal heat transfer into the oceans
Labrosse (2009)

Finite-element literature

books



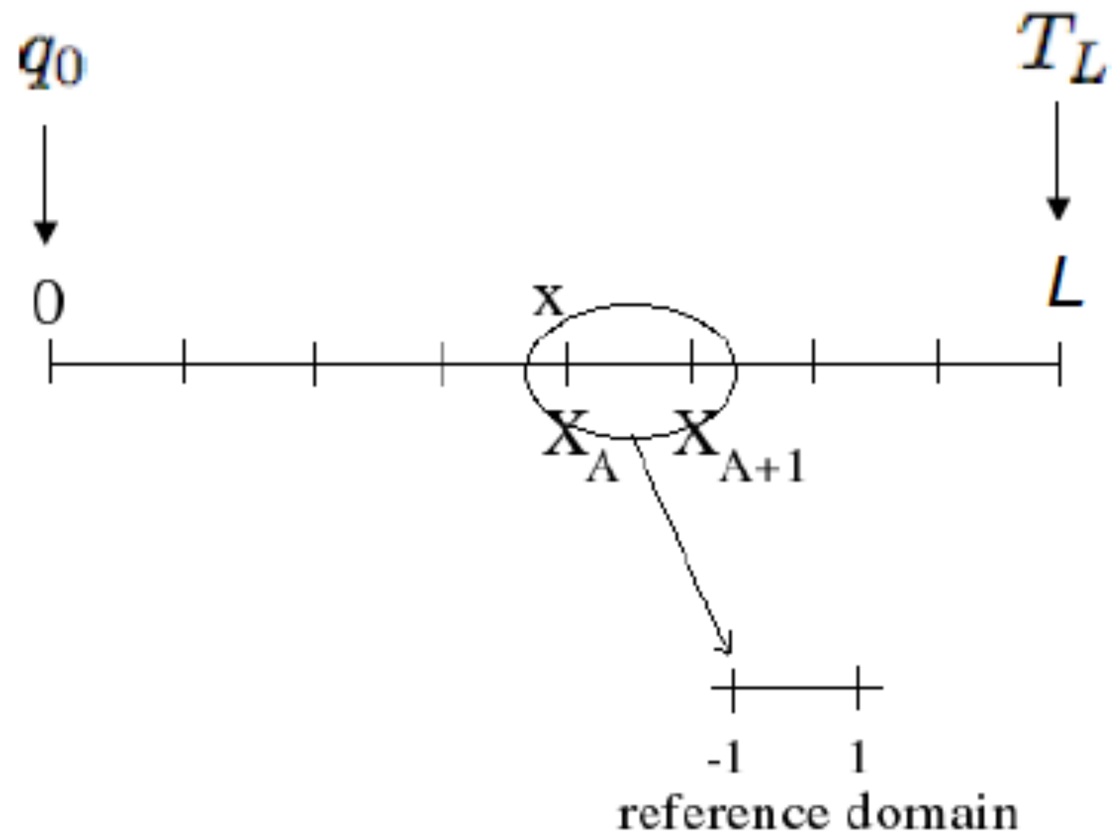
Thomas J.R. Hughes.
The Finite Element Method : linear static and dynamic finite element analysis,
Mineola, N.Y. : Dover, 2000.

O. C. Zienkiewicz, R. L. Taylor and J. Z. Zhu.
The Finite Element Method : its basis and fundamentals,
ISBN: 978-1-85617-633-0.

1D unsteady-state diffusion equation

Strong form: $\rho c_p \partial_t T - \partial_x(\kappa \partial_x T) = f$

Boundary conditions: $\begin{cases} T(L,t) & = & T_L \\ -\kappa\partial_x T(0,t) & = & q_0 \\ T(x,0) & = & T_0(x) \end{cases}$



FEM - 1D unsteady-state diffusion equation

Weak form: $$\int_0^L \rho c_p w \partial_t T \mathrm{d}x = -\int_0^L \kappa \partial_x w \partial_x T \mathrm{d}x + q_0 w(0) + \int_0^L w f \mathrm{d}x$$

Test function and temperature field expanded on some basis functions:

$$w(x) = \sum_{A=1}^{N} c_A N_A(x)$$

$$T(x) = \sum_{A=1}^{N_{el}} d_A N_A(x) + T_1 N_{n+1}(x)$$

unknown

FEM - 1D unsteady-state diffusion equation

Weak form: $$\int_0^L \rho c_p w \partial_t T \mathrm{d}x = -\int_0^L \kappa \partial_x w \partial_x T \mathrm{d}x + q_0 w(0) + \int_0^L w f \mathrm{d}x$$

We are solving: $$\boxed{\mathbf{M}\dot{\mathbf{d}} + \mathbf{K}\mathbf{d} = \mathbf{F}}$$

capacity matrix

stiffness matrix

force vector

FEM - 1D unsteady-state diffusion equation

Global level:

$$
\begin{aligned}
M_{AB} &= (N_A, \rho c_p N_B) \\
&= \int_0^L \rho c_p N_A N_B \mathrm{d}x
\end{aligned}
$$

$$
K_{AB} = a(N_A, N_B) = \int_0^L \kappa \partial_x N_A \partial_x N_B \mathrm{d}x
$$

and

$$
\begin{aligned}
F_A &= (N_A, f) + N_A(0) q_0 - a(N_A, N_{n+1}) T_L \\
&= \int_0^L N_A f \mathrm{d}x + N_A(0) q_0 - \int_0^L \partial_x N_A \partial_x N_{n+1} \mathrm{d}x
\end{aligned}
$$

FEM - 1D unsteady-state diffusion equation

Global level:

$$M_{AB} = \int_0^L \rho c_p N_A N_B \mathrm{d}x$$

$$K_{AB} = \int_0^L \kappa \partial_x N_A \partial_x N_B \mathrm{d}x$$

$$F_A = \int_0^L N_A f \mathrm{d}x + N_A(0) q_0 - \int_0^L \partial_x N_A \partial_x N_{n+1} \mathrm{d}x$$

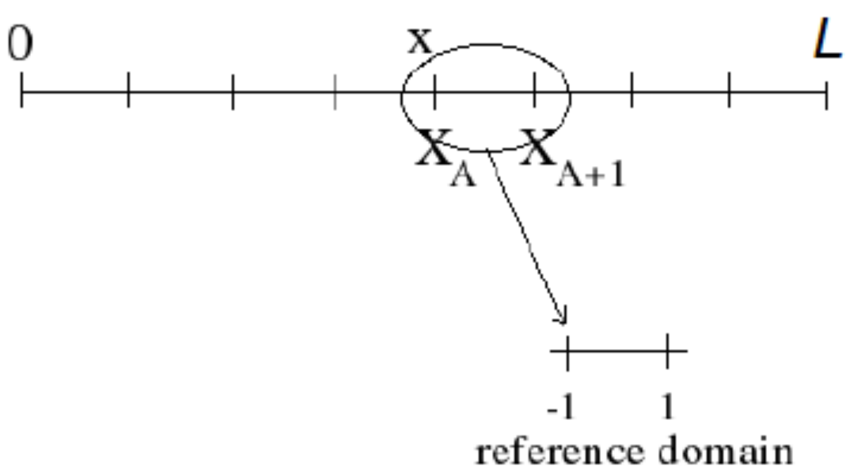
➔ $\mathbf{M}\dot{\mathbf{d}} + \mathbf{K}\mathbf{d} = \mathbf{F}$ where $\dot{\mathbf{d}} = \partial_\mathbf{t} \mathbf{d}$

mapping using Jacobian:

$$\int_0^1 g(x)dx = \sum_{\Omega_e} \int_{\Omega_e} g(x)dx = \sum_{\Omega_e} \int_{-1}^1 g(x(\xi)) J d\xi$$

FEM - 1D unsteady-state diffusion equation

Local level: Consider the mapping $\xi : [X_A, X_{A+1}] \rightarrow [\xi_1, \xi_2]$, such that



$$\begin{cases} \xi(X_A) & = \xi_1 = -1 \\ \xi(X_{A+1}) & = \xi_2 = 1 \end{cases}$$

=> Linear shape functions:

$$N_a(\xi) = \frac{1}{2}(1 + \xi_a \xi) \quad \text{a=1,2}$$

FEM - 1D unsteady-state diffusion equation

Local (element) resolution:

$$N_a(\xi) = \frac{1}{2}(1 + \xi_a \xi) \quad \text{a=1,2}$$

Capacity matrix:

$$m^e_{ab} = (N_a, \rho c_p N_b) = \int_{\Omega_e} \rho c_p N_a N_b \mathrm{d}x$$

$\Downarrow$ *Change of variables (reference domain)*

$$m^e_{ab} = \frac{h_e}{2} \int_{-1}^{1} \rho c_p N_a N_b \mathrm{d}\xi$$

$\Downarrow$ *Matricial form*

$$m^e = \frac{\rho c_p h_e}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

FEM - 1D unsteady-state diffusion equation

Local (element) resolution:

$$N_a(\xi) = \frac{1}{2}(1 + \xi_a \xi) \quad \text{a=1,2}$$

Stiffness matrix:

$$k^e_{ab} = a(N_a, \kappa N_b) = \int_{\Omega_e} \kappa \partial_x N_a \partial_x N_b \mathrm{d}x$$

$\Downarrow$ *Change of variables (reference domain)*

$$k^e_{ab} = \frac{2}{h_e} \int_{-1}^{1} \kappa \partial_\xi N_a \partial_\xi N_b \mathrm{d}\xi$$

$\Downarrow$ *Matricial form*

$$k^e = \frac{\kappa}{h_e} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

FEM - 1D unsteady-state diffusion equation

Local (element) resolution:

$$N_a(\xi) = \frac{1}{2}(1 + \xi_a \xi) \quad \text{a=1,2}$$

Force vector: $$f_a^e = \int_{\Omega_e} N_A f \mathrm{d}x + \begin{cases} \delta_{a1} q_0 & \text{for } e = 1 \\ -k_{a2}^e T_L & \text{for } e = N_{el} \\ 0 & \text{else} \end{cases}$$

↓ *Change of variables (reference domain)*

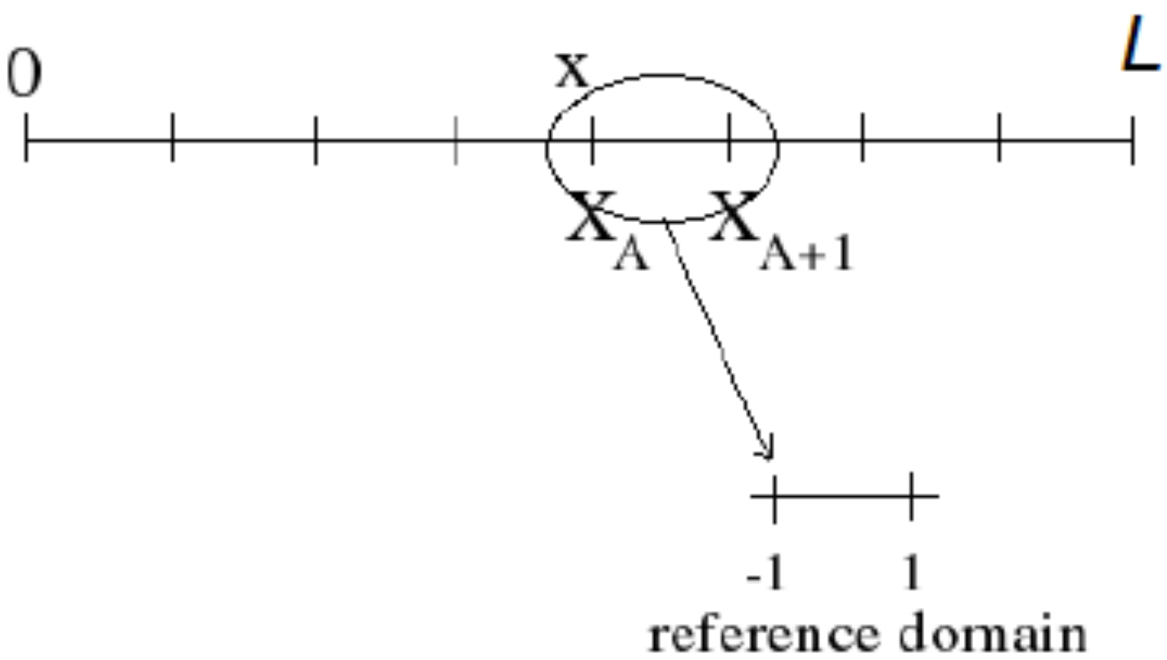
$$f_a^e = \frac{h_e}{2} \int_{-1}^{1} N_a f \mathrm{d}\xi + \text{boundary terms}$$

↓ *Matricial form*

$$f^e = \frac{h_e}{6} \begin{pmatrix} 2f_1 + f_2 \\ f_1 + 2f_2 \end{pmatrix} + \text{boundary terms}$$

FEM - 1D unsteady-state diffusion equation

Assembling: back to global level



$$\text{iglob(i,ielem)} = \begin{cases} \text{ielem} & \text{if i=1} \\ \text{ielem+1} & \text{if i=2} \end{cases}$$

code example:

```
do ielem = 1, nelem
    do i = 1,2
        u(iglobe(i,ielem)) = u(iglobe(i,ielem)) + u_local(i,ielem)
    end
end
```

FEM - 1D unsteady-state diffusion equation

Time scheme: Predictor-Corrector algorithm

- Predictor:

$$
\begin{aligned}
d_{n+1} &= d_n + (1-\alpha)\Delta t \dot{d}_n \\
\dot{d}_{n+1} &= 0 \quad \text{(initialization at the beginning of each time step)}
\end{aligned}
$$

- Solve:

$$
\begin{aligned}
rhs &= F - M\dot{d}_{n+1} - Kd_{n+1} \\
\delta\dot{d}_{n+1} &= M^{-1} rhs
\end{aligned}
$$

- Corrector:

$$
\begin{aligned}
d_{n+1} &= d_{n+1} + \alpha\Delta t \dot{d}_{n+1} \\
\dot{d}_{n+1} &= \dot{d}_{n+1} + \delta\dot{d}_{n+1}
\end{aligned}
$$

where Δt is the time step.

$$
\begin{cases}
\alpha = 0 & \text{forward differences} \\
\boxed{\alpha = 1/2 \quad \text{midpoint rule}} & \\
\alpha = 1 & \text{backward differences}
\end{cases}
$$

FEM - 1D unsteady-state diffusion equation

FEM solution: *initial, simple harmonic function*

$$\rho c_p \partial_t T - \partial_x(\kappa \partial_x T) = f = 0$$

Boundary conditions:

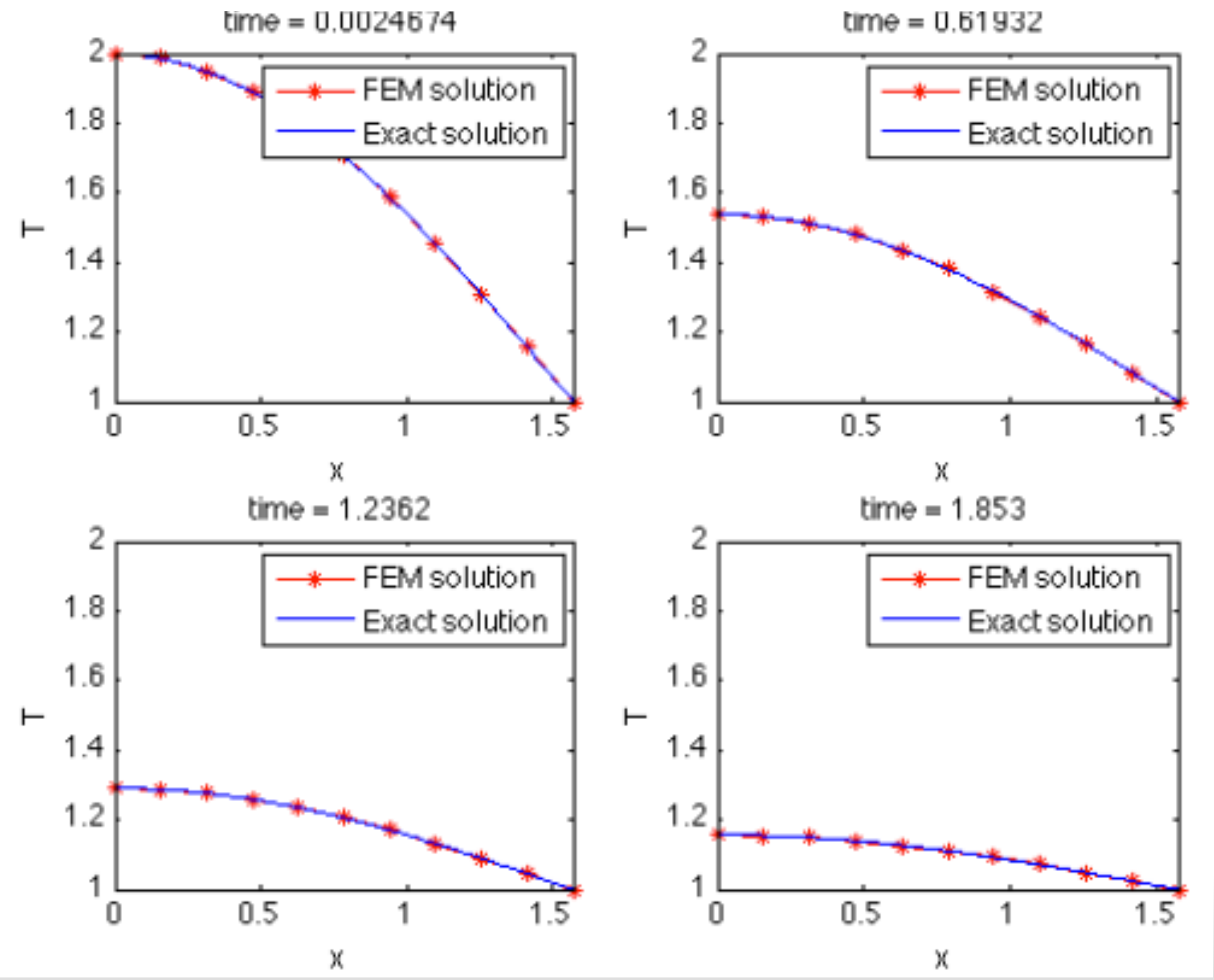
$$\begin{cases} T(L,t) & = & T_L = 1 \\ -\kappa\partial_x T(0,t) & = & q_0 = 0 \\ T(x,0) & = & T_0(x) = 1 + cos(x) \quad \text{in } [0,\text{L}] \text{ and } L = \dfrac{\pi}{2} \end{cases}$$

has exact solution:

$$T(x,t) = 1 + e^{-t}\cos(x)$$

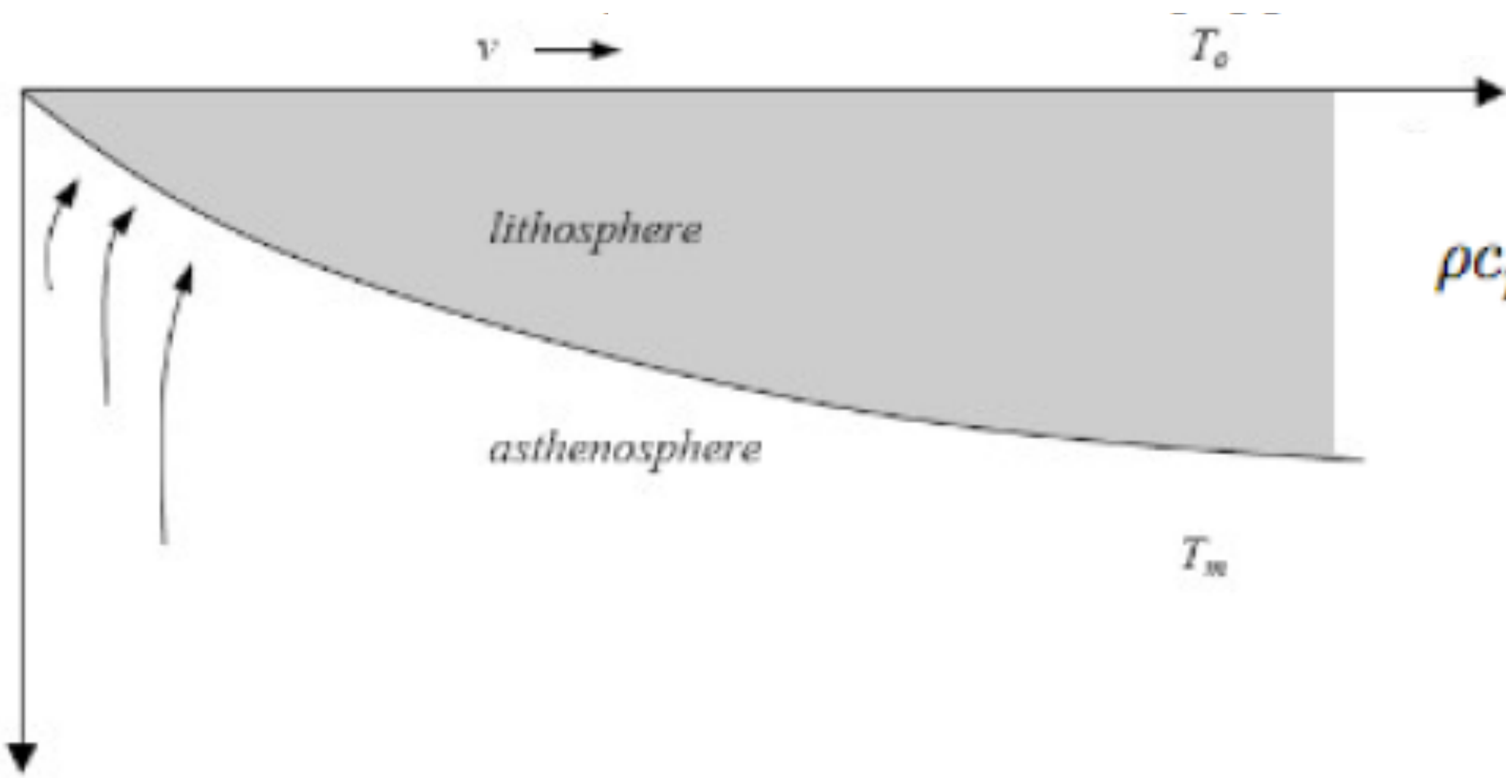
FEM - 1D unsteady-state diffusion equation

FEM solution: *simple harmonic function*



FEM - 1D unsteady-state diffusion equation

FEM solution: *half-space cooling*



$$
\begin{aligned}
\rho c_p \partial_t \theta &= \partial_x(\kappa \partial_x \theta) \quad \text{in } 0 < x < \infty, \\
&\quad \text{where } \theta = \frac{T - T_m}{T_0 - T_m} \\
&\quad \text{non-dimensional variable}
\end{aligned}
$$

Boundary conditions:

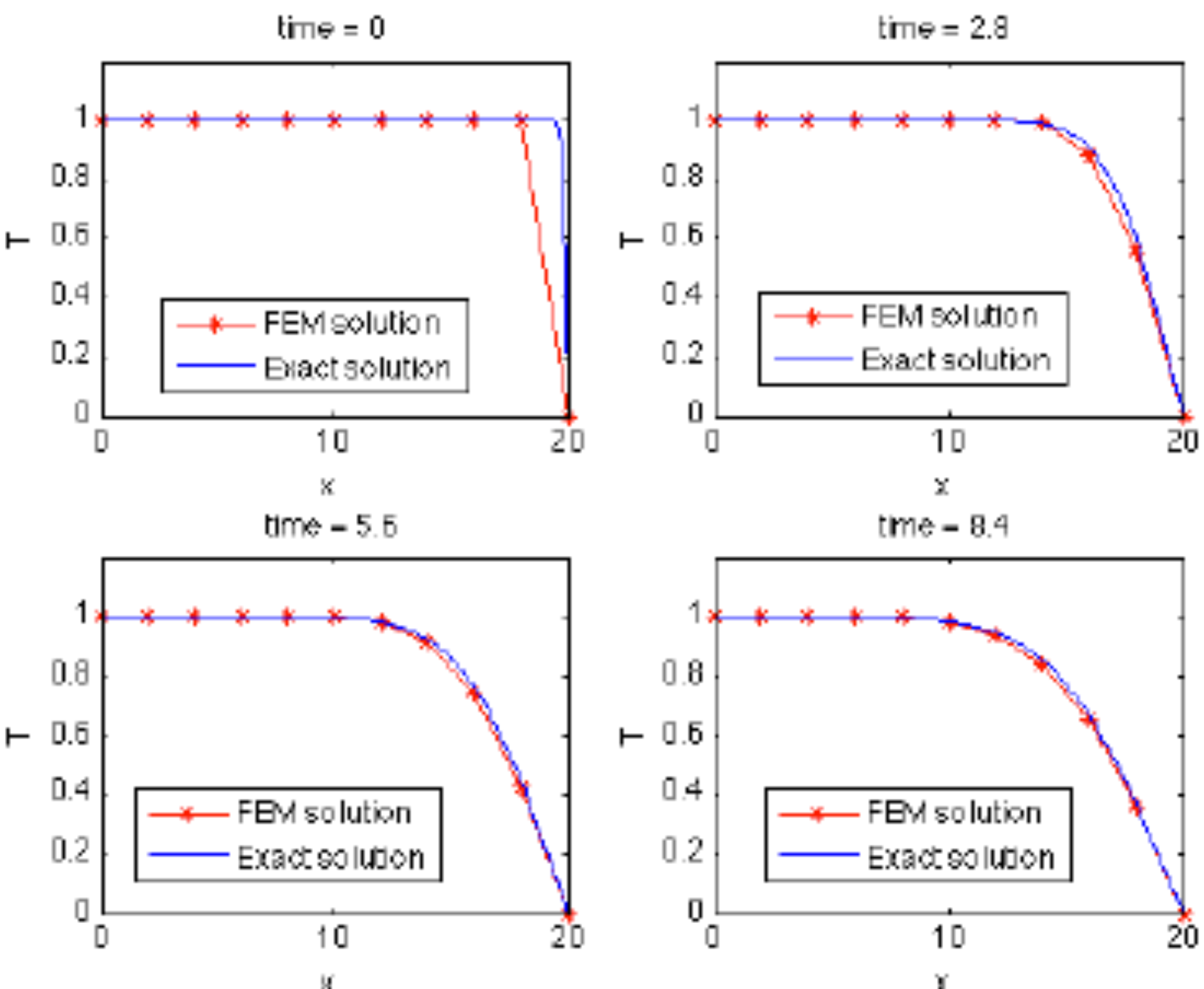
$$
\begin{aligned}
T(0,t) &= T_0 \quad \text{(surface temperature)} \\
T(x \to \infty, t) &\to T_m \\
T(x,0) &= T_m \quad \text{(initial temperature)}
\end{aligned}
$$

has exact solution:

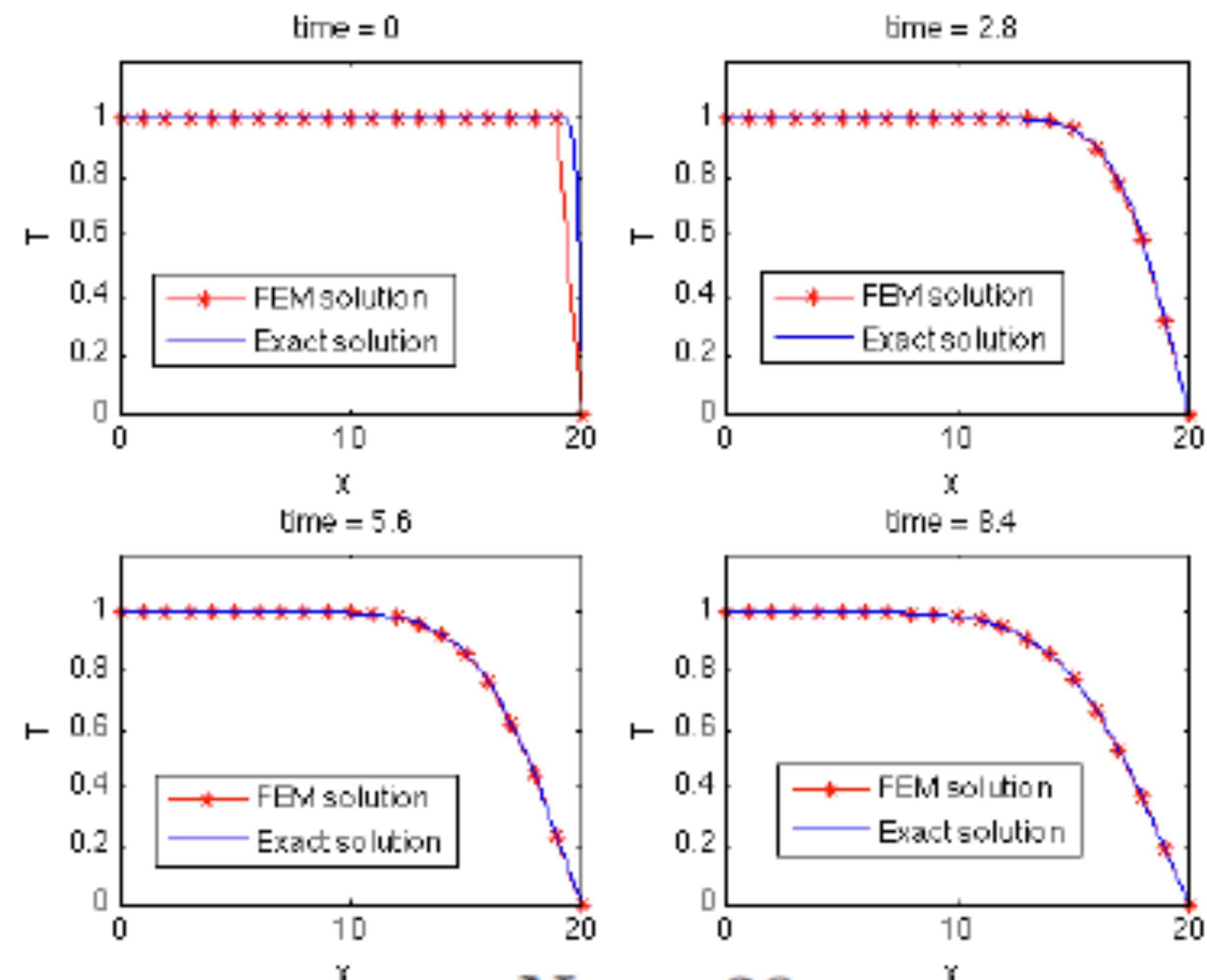
$$
\theta = \operatorname{erfc} \frac{x}{2\sqrt{\frac{\kappa}{\rho c_p} t}}
$$

FEM - 1D unsteady-state diffusion equation

FEM solution: *half-space cooling*



$N_{el} = 10$



$N_{el} = 20$