Problem Set 6

FEM solution of 1D unsteady-state diffusion equation

Write a finite-element program, using linear shape functions, to find the temperature $T(x,t)$ in $[0,L]$ such that (strong form)

$$\rho\, c_p\, \partial_t T - \partial_x(\kappa\, \partial_x T) = f$$

where ρ is the medium density, c_p is the the specific heat, κ is the conductivity, and f is a source or sink, with the following initial & boundary conditions:

$$\begin{aligned} T(L,t) &= T_L \\ -\kappa\, \partial_x T(0,t) &= q_0 \\ T(x,0) &= T_0(x) \end{aligned}$$

Recipe:

Address this problem as follows:

1. Write the weak form of the equation, which translates into solving

$$\mathbf{M}\dot{\mathbf{d}} + \mathbf{K}\mathbf{d} = \mathbf{F}$$

   where $\mathbf{M}$ is the capacity matrix, defined as

$$\begin{aligned} M_{AB} &= (N_A, \rho\, c_p\, N_B) \\ &= \int_0^L \rho\, c_p\, N_A N_B \,\mathrm{d}x \end{aligned}$$

   and $\mathbf{K}$ is the conductivity (or stiffness) matrix, defined as

$$\begin{aligned} K_{AB} &= a(N_A, N_B) \\ &= \int_0^L \kappa\, \partial_x N_A \partial_x N_B \,\mathrm{d}x \end{aligned}$$

   and $\mathbf{F}$ is the right-hand-side (force) vector , defined as

$$F_A = (N_A, f) + N_A(0) q_0 - a(N_A, N_{n+1}) T_L - (N_A, \rho\, c_p\, N_{n+1}) \dot{T}_L$$

   finally, $\mathbf{d}$ are the unknown temperatures and $\dot{\mathbf{d}} = \partial_t \mathbf{d}$.

2. Define the local (element level) capacity & stiffness matrices and the right-hand-side vector.

3. Assemble these local matrices into global matrices (global level) using the Location Matrix (LM).

4. March in time, using the predictor-corrector algorithm:

- Predictor:

$$
\begin{aligned}
d_{n+1} &= d_n + (1-\alpha)\Delta t \dot{d}_n \\
\dot{d}_{n+1} &= 0 \quad \text{(initialization at the beginning of each time step)}
\end{aligned}
$$

- Solve:

$$
\begin{aligned}
rhs &= F - M\dot{d}_{n+1} - K d_{n+1} \\
\delta\dot{d}_{n+1} &= M^{-1} rhs
\end{aligned}
$$

- Corrector:

$$
\begin{aligned}
d_{n+1} &= d_{n+1} + \alpha\Delta t \dot{d}_{n+1} \\
\dot{d}_{n+1} &= \dot{d}_{n+1} + \delta\dot{d}_{n+1}
\end{aligned}
$$

where Δt is the time step.

The parameter α can take several values, leading to different methods with different level of stability (e.g., see Hughes, "The Finite Element Method - Linear Static and Dynamic Finite Element Analysis", page 459, for more details) such that

$\alpha = 0$: Forward differences,

$\alpha = 1/2$: Midpoint rule, Crank-Nicolson,

$\alpha = 1$: Backward differences.

For $\alpha \geq 1/2$ the scheme is unconditionally stable.

Use $\alpha = 1/2$.

Compare the FEM solution to the exact solution to the strong form, by plotting the temperature T versus x at different time steps. Solve the following two cases:

Problem A: Simple harmonic function

Single harmonic $T_0(x)$ as

$$T_0(x) = 1 + cos(x) \quad \text{in [0,L]}$$

where $L = \frac{\pi}{2}$ and the following conditions & parameters:

$$f = 0$$

$$q_0 = 0$$

$$T_L = 1$$

$$\kappa = 1$$

$$\rho\, c_p = 1$$

Try $N_{el} = 10$.

The exact solution to this problem is $T(x, t) = 1 + e^{-t}\cos(x)$.

Problem B: Half-space cooling

Here we consider the uniformly hot half-space cooling (e.g., oceanic lithospheric cooling). The differential equation for this problem is

$$\rho c_p \partial_t \theta \;=\; \partial_x(\kappa \partial_x \theta) \quad \text{in } 0 < x < \infty,$$

where $\theta = \frac{T - T_m}{T_0 - T_m}$ is a non-dimensional variable, using the following boundary conditions:

$$\begin{aligned} T(0,t) &= T_0 \quad \text{(surface temperature)} \\ T(x \to \infty, t) &\to T_m \\ T(x,0) &= T_m \quad \text{(initial temperature)} \end{aligned}$$

The exact solution to this problem is

$$\begin{aligned} \theta &= \frac{T - T_m}{T_0 - T_m} \\ &= \mathrm{erfc} \frac{x}{2\sqrt{\frac{\kappa}{\rho c_p} t}} \end{aligned}$$

where erfc(x) is the complementary error function defined as $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} \mathrm{d}t$. Notice that you can use the same algorithm developed for (a), by changing variable $y = L - x$, such that you solve

$$\rho \; c_p \partial_t T - \partial_y(\kappa \partial_y T) \;=\; f$$

with the boundary conditions

$$\begin{aligned} T(L,t) &= T_0 \\ -\kappa \partial_y T(0,t) &= q_0 \\ T(y,0) &= T_m \end{aligned}$$

Let $L = 20$, $\kappa = 1$, $q_0 = 0$, $\rho c_p = 1$, $f = 0$, $T_0 = 0$, and $T_m = 1$.

Please be aware that:

- The initial condition of this problem is not compatible with the boundary condition at

$y = L$. To handle that you can set the temperature value at $y = L$ to be T_0, and all the other nodes to be T_m. You can also try and use mode elements, which will make the transition from T_0 to T_m smoother.

- We are solving the problem for large L, that is, such that $L >> 2\sqrt{\frac{\kappa}{\rho c_p} t}$, say $L = 20$ and $t = (0, 9)$, therefore, we can approximately use the exact solution for the half-space problem as the analytical solution for our problem.