Pseudo-Spectral method

1D wave equation

2nd-order partial differential equation:

$$\rho(x)\partial_t^2 u(x,t) = \partial_x[\kappa(x)\partial_x u(x,t)], \quad (x \in [0,L], t \in [0,+\infty))$$

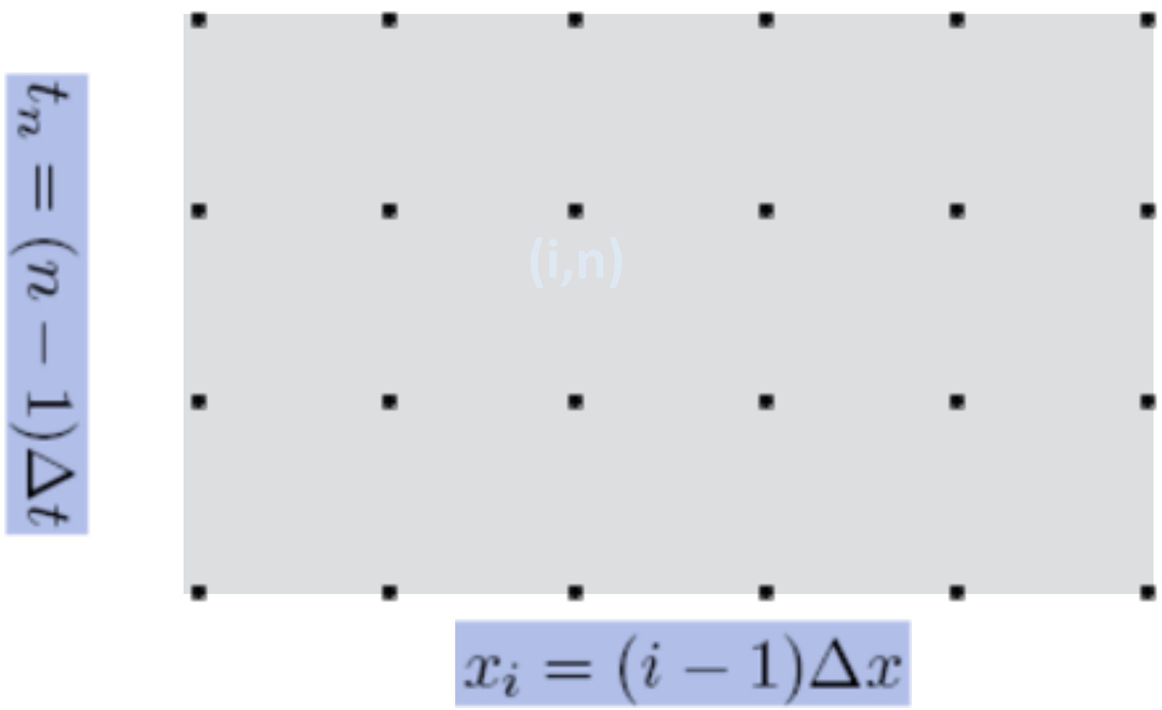
velocity-stress formulation

1st-order partial differential equations

$$\begin{aligned} \rho(x)\partial_t v(x,t) &= \partial_x T(x,t) \\ \partial_t T(x,t) &= \kappa(x)\partial_x v(x,t) \end{aligned} \qquad \begin{aligned} v(x,t) &= \partial_t u(x,t) \\ T(x,t) &= \kappa(x)\partial_x u(x,t) \end{aligned}$$

Pseudo-spectral method

Discretization $u_i^n = u(x_i, t_n)$



continuous form:

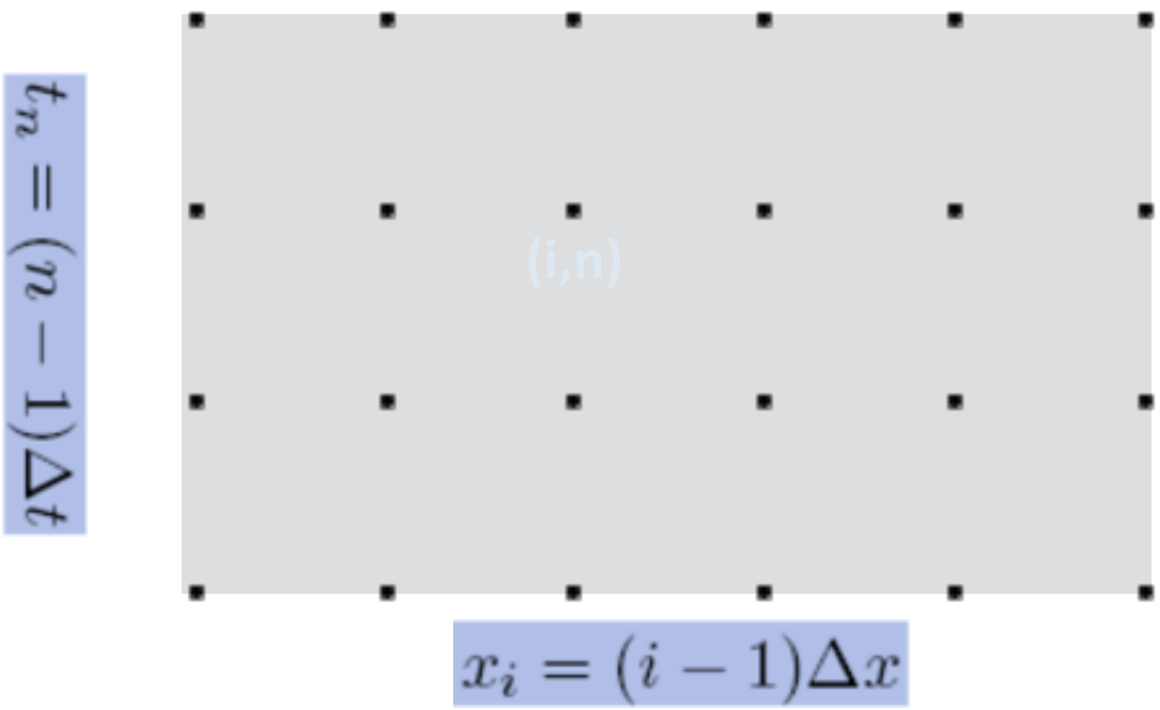
$$
\begin{aligned}
\rho(x)\partial_t v(x,t) &= \partial_x T(x,t) \\
\partial_t T(x,t) &= \kappa(x)\partial_x v(x,t)
\end{aligned}
$$

discretized form:

$$
\begin{aligned}
\rho(x_i)\partial_t v(x_i,t_n) &= \partial_x T(x_i,t_n) \\
\partial_t T(x_i,t_n) &= \kappa(x_i)\partial_x v(x_i,t_n)
\end{aligned}
$$

Pseudo-spectral method

Derivatives



temporal derivative: central differences

$$
\begin{aligned}
v_i^{n+1} &= v_i^{n-1} + \frac{2\Delta t}{\rho_i} \times (\frac{\partial T}{\partial x})_i^n \\
T_i^{n+1} &= T_i^{n-1} + 2\Delta t \times \kappa_i \times (\frac{\partial v}{\partial x})_i^n
\end{aligned}
$$

spatial derivative: Fourier transforms

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} f(x) &= \frac{\mathrm{d}}{\mathrm{d}x}[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{F}(k) e^{ikx} \mathrm{d}k] \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} ik\tilde{F}(k) e^{ikx} \mathrm{d}k
\end{aligned}
$$

$$
\begin{aligned}
\tilde{F}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} \mathrm{d}x \\
f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{F}(k) e^{ikx} \mathrm{d}k
\end{aligned}
$$

$$
\int_{-\infty}^{+\infty} |f(x)| \mathrm{d}x < \infty
$$

Pseudo-spectral method

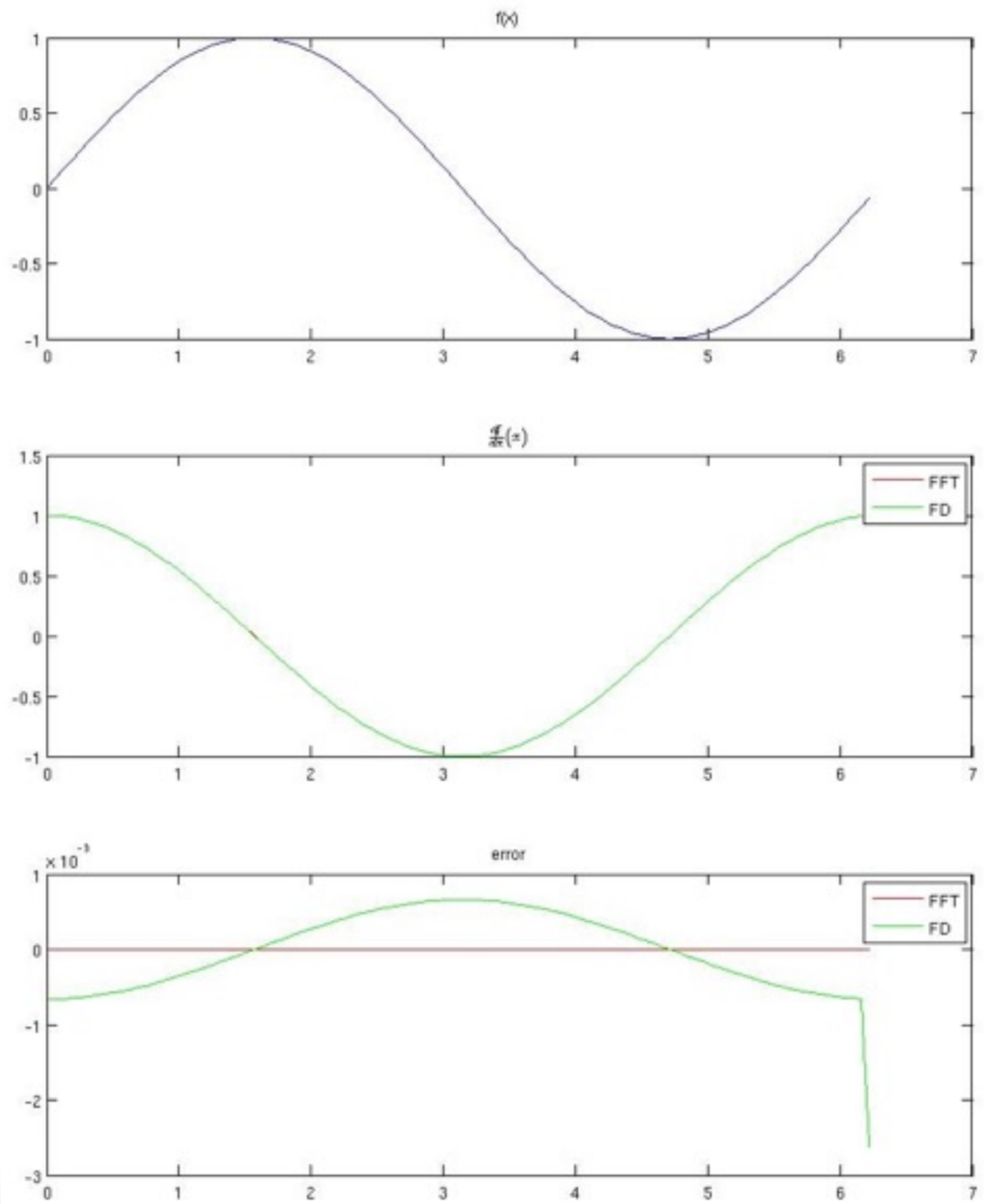
Recipe

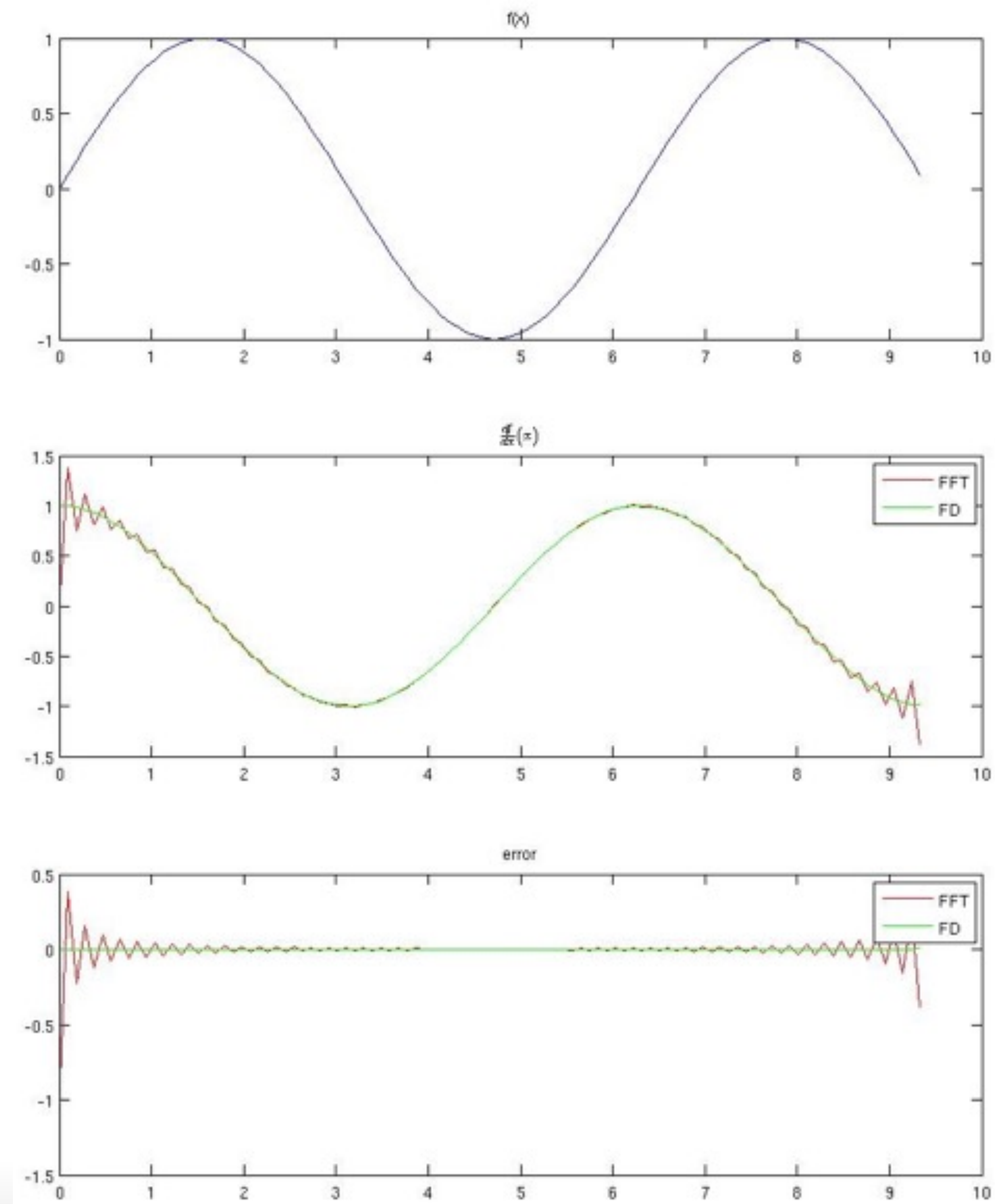
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}f(x) &= \frac{\mathrm{d}}{\mathrm{d}x}[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\tilde{F}(k)e^{ikx}\mathrm{d}k] \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}ik\tilde{F}(k)e^{ikx}\mathrm{d}k\end{aligned}$$

- First, Fourier Transform whatever field $f(x)$ we need to differentiate.
- Second, multiply each Fourier coefficient $\tilde{F}(k)$ by ik.
- Finally, carry out inverse Fourier Transform to get desired derivatives.

Pseudo-spectral method

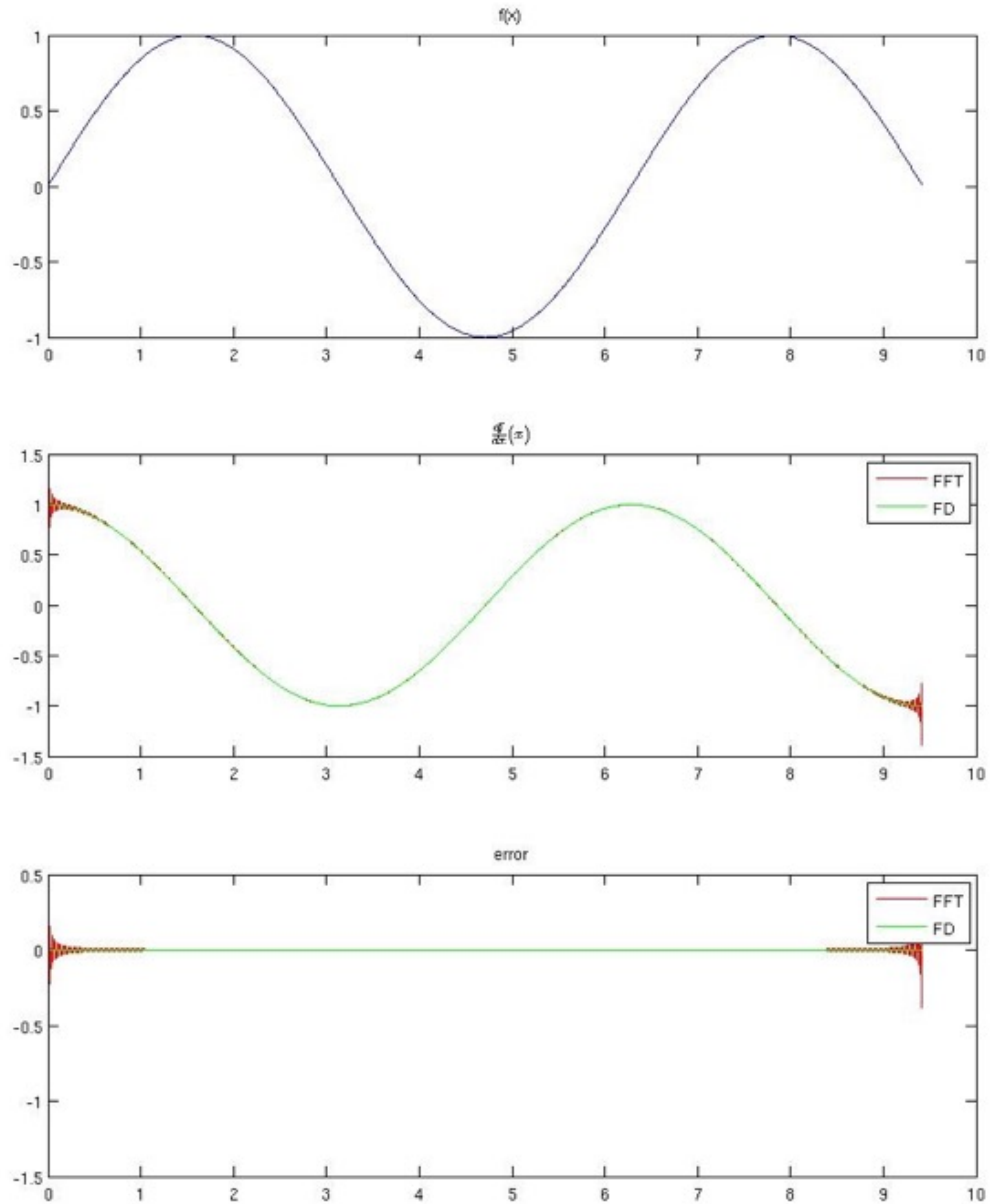
Results





Pseudo-spectral method

Results



Pseudo-spectral method

Results