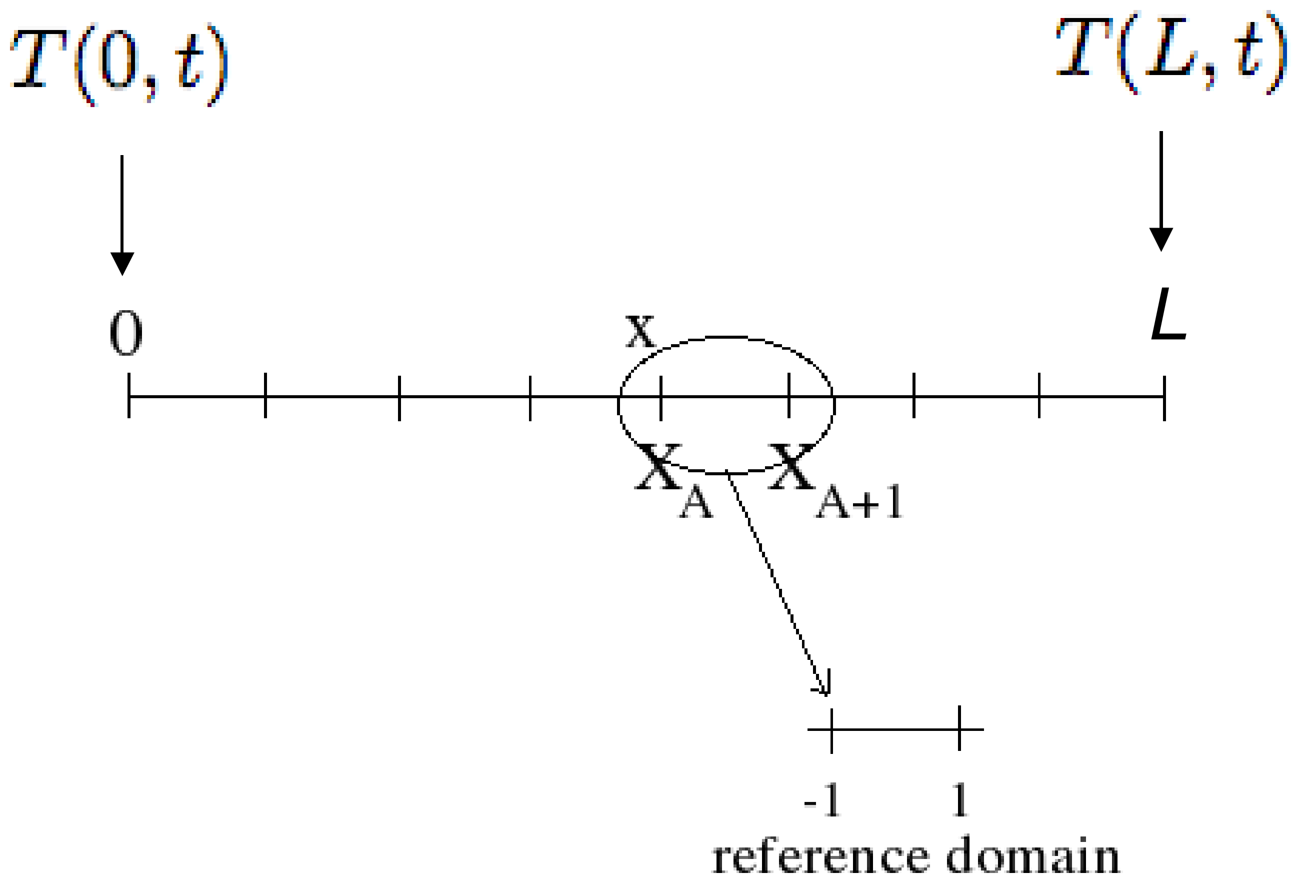
Spectral-element method

1D unsteady-state diffusion equation

Strong form: $\rho c_p \partial_t T - \partial_x(\kappa \partial_x T) = 0$

IC & BC:
$$\begin{cases} T(x,0) &= 0 \\ T(L,t) &= 0 \\ T(0,t) &= 10 \end{cases}$$



Weak form

Weak form: $$\int_0^L \rho c_p w \partial_t T \mathrm{d}x = -\int_0^L \kappa \partial_x w \partial_x T \mathrm{d}x + [w\kappa\partial_x T]_0^L$$

Temperature field (and test function) expanded on basis functions:

$$T(x(\xi),t) = \sum_\alpha^N T^\alpha(t) l_\alpha^N(\xi)$$

unknowns

Weak form

Weak form: $$\int_0^L \rho c_p w \partial_t T \mathrm{d}x = -\int_0^L \kappa \partial_x w \partial_x T \mathrm{d}x + [w\kappa\partial_x T]_0^L$$

zero

$\rightarrow$ $\mathbf{M}\dot{\mathbf{d}} + \mathbf{K}\mathbf{d} = \mathbf{F}$ where $\dot{\mathbf{d}} = \partial_{\mathbf{t}}\mathbf{d}$

capacity matrix

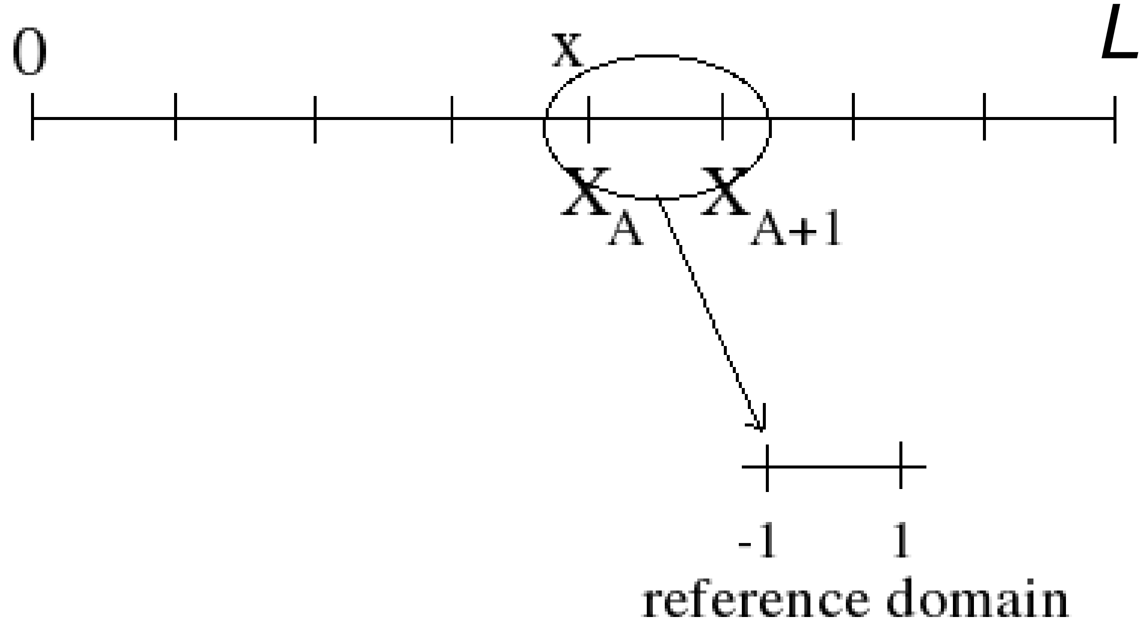
stiffness matrix

force vector

Reference domain

Definition of the reference domain:

Consider the mapping $\xi : [X_A, X_{A+1}] \to [\xi_1, \xi_2]$, such that



$$\begin{cases} \xi(X_A) & = \xi_1 = -1 \\ \xi(X_{A+1}) & = \xi_2 = 1 \end{cases}$$

$$x(\xi) = \sum_{a=1}^{2} X_a N_a(\xi)$$

with shape functions being degree-1 Lagrange polynomials

$$N_a(\xi) = \frac{1}{2}(1 + \xi_a \xi) \quad \text{a=1,2}$$

Jacobian: $J = \dfrac{\partial x}{\partial \xi}$

Reference domain

Interpolation:

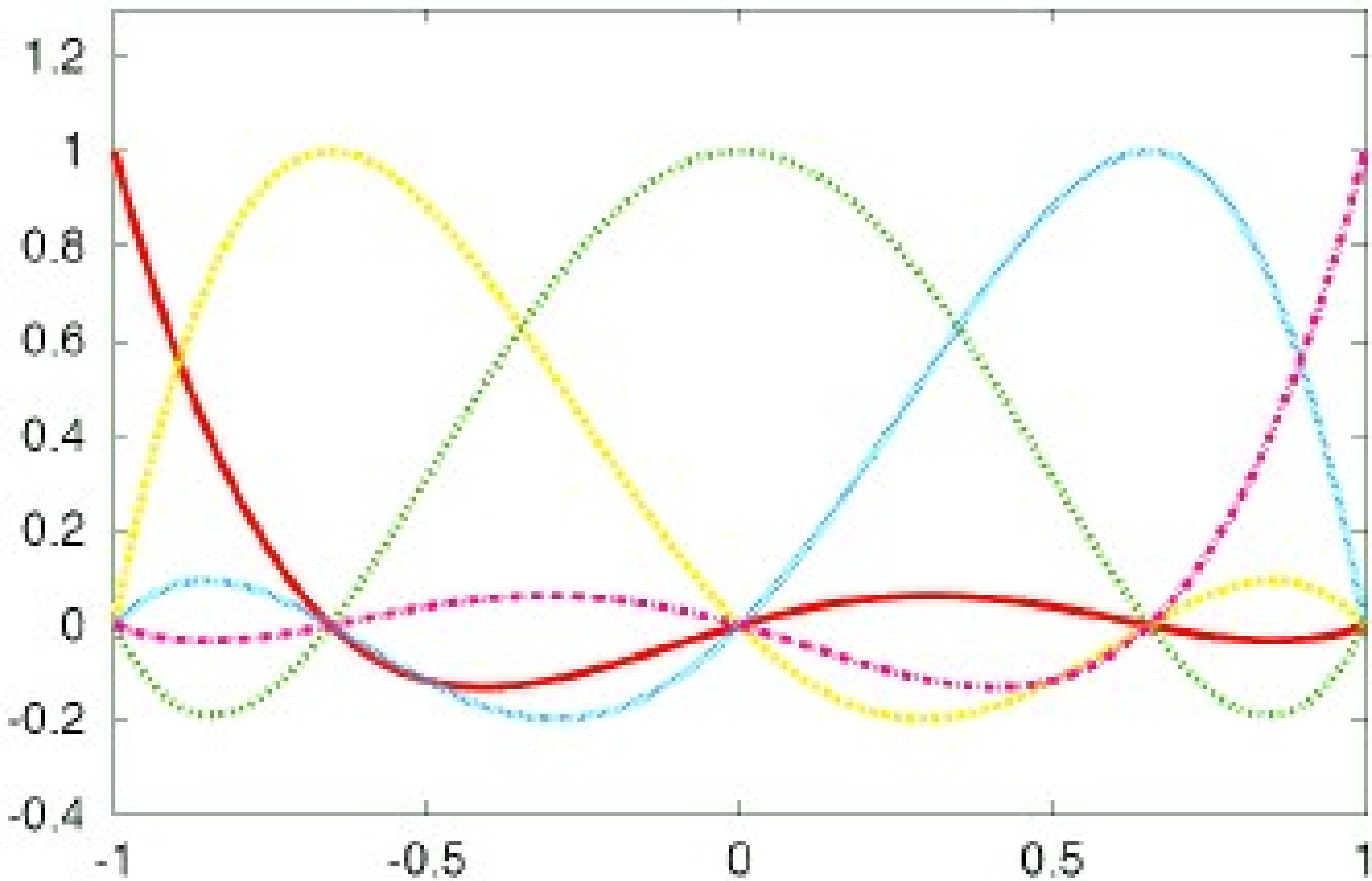
$$T(x(\xi), t) = \sum_{\alpha}^{N} T^{\alpha}(t) l_{\alpha}^{N}(\xi)$$

Gauss-Lobatto-Legendre quadrature integration rule:

$$\begin{aligned}\int_{\Omega_e} T(x,t)\mathrm{d}x &= \int_{-1}^{1} T(x(\xi),t) J^{\alpha}\mathrm{d}\xi \\ &\sim \sum_{\alpha}^{N} \omega_{\alpha} T^{\alpha}(t) J^{\alpha}\end{aligned}$$

Basis functions

Lagrange polynomials:

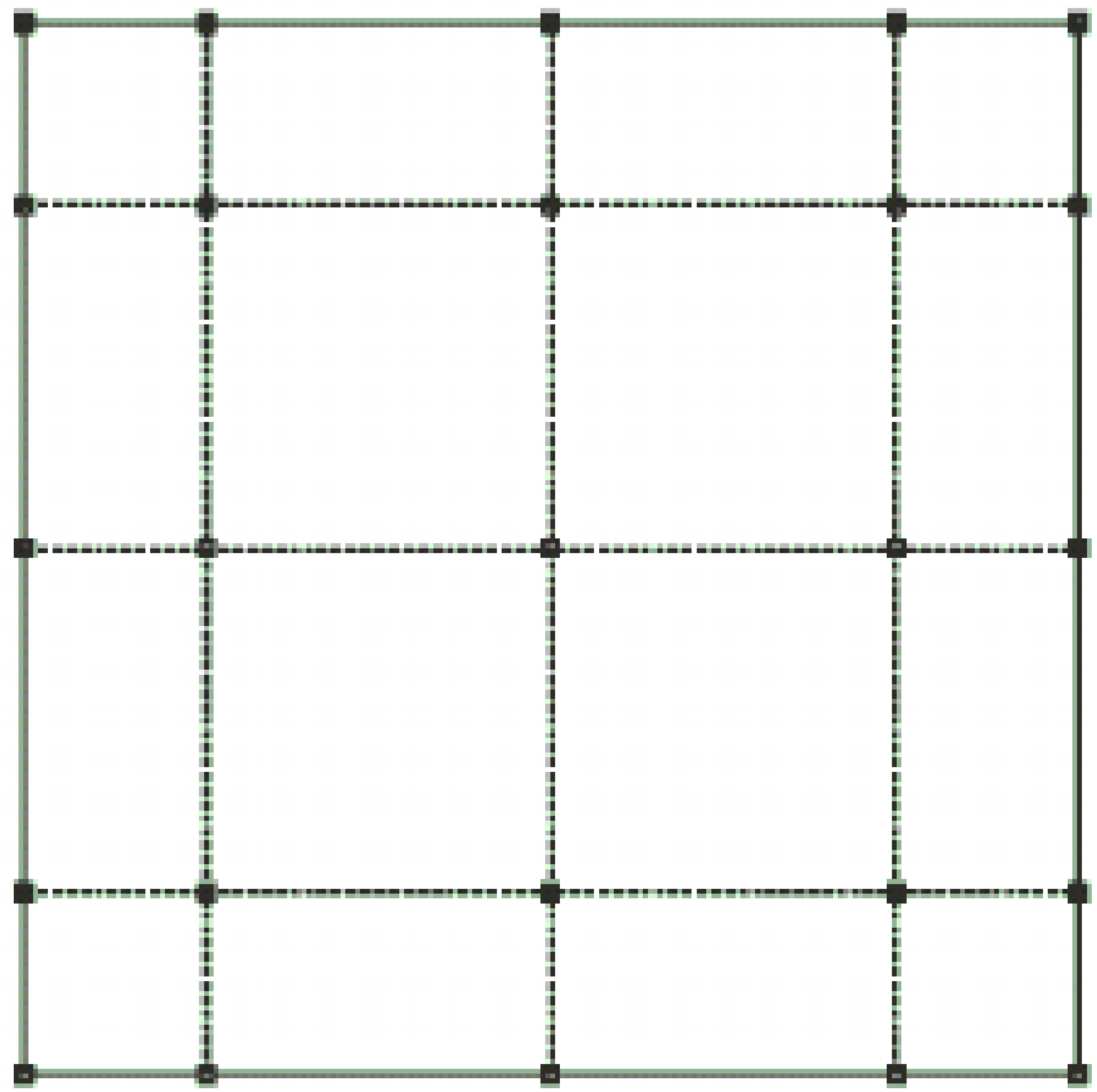


degree-4 polynomials

Lagrange polynomials property: $l_\alpha^N(\xi_\beta) = \delta_{\alpha\beta}$

Basis functions

Gauss-Lobatto-Legendre points:



degree-4 GLL points
(2D quad example)

GLL points are the n+1 roots of $(1-\xi^2)P_n'(\xi) = 0$

P_n : Legendre polynomial of degree n

Mass matrix

Local (element) resolution: Capacity/mass matrix

$$
\begin{aligned}
\int_{\Omega_e} w\rho c_p \partial_t T \mathrm{d}x &= \int_{-1}^{1} \rho(x(\xi)) c_p(x(\xi)) w(x(\xi)) \partial_t T(x(\xi), t) J \mathrm{d}\xi \\
&\sim \sum_{\alpha=0}^{N} \omega_\alpha \rho^\alpha c_p^\alpha J^\alpha \sum_{\beta}^{N} w^\beta l_\beta^N(\xi_\alpha) \sum_{\gamma}^{N} \partial_t T^\gamma l_\gamma^N(\xi_\alpha) \\
&= \sum_{\alpha=0}^{N} \boxed{\omega_\alpha \rho^\alpha c_p^\alpha J^\alpha w^\alpha} \partial_t T^\alpha
\end{aligned}
$$

diagonal matrix

Stiffness matrix

Local (element) resolution: Stiffness matrix

$$\int_{\Omega_e} \kappa \partial_x w \partial_x T \mathrm{d}x = \int_{-1}^{1} \kappa(x(\xi))[\partial_x w(x(\xi))][\partial_x T(x(\xi),t)] J \mathrm{d}\xi$$

$$\sim \sum_{\alpha=0}^{N} \boxed{\omega_\alpha \kappa^\alpha [\sum_{\beta}^{N} w^\beta l_\beta'^N(\xi_\alpha)\partial_x \xi(\xi_\alpha)][\sum_{\gamma}^{N} T^\gamma l_\gamma'^N(\xi_\alpha)\partial_x \xi(\xi_\alpha)] J^\alpha}$$

=> $$M_\alpha \partial_t T^\alpha(t) = \sum_{\gamma}^{N} K_{\alpha,\gamma} T^\gamma(t)$$

Matricial form:

$$\mathbf{M}\partial_t \mathbf{T} = \mathbf{K}\mathbf{T}$$

Boundaries

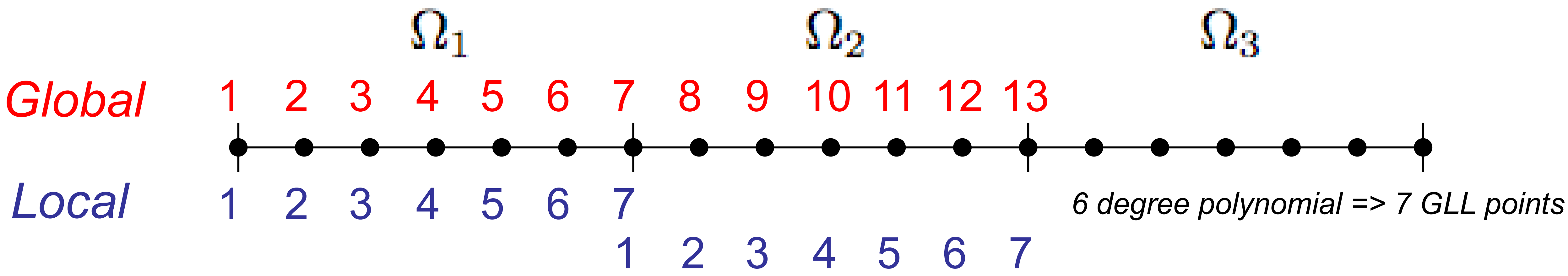
Boundary conditions:

$$w\kappa\partial_x T = \omega^N \kappa^N \sum_{\alpha=0}^{N} T^\alpha l_\alpha'^N(\xi_N)\partial_x\xi(\xi_N) \quad \text{for } x = 0$$

$$w\kappa\partial_x T = \omega^0 \kappa^0 \sum_{\alpha=0}^{N} T^\alpha l_\alpha'^N(\xi_0)\partial_x\xi(\xi_0) \quad \text{for } x = L$$

Assembly

Local to Global:



You need an array which links Local *(where the calculation is done)* to Global *(where you want to know the results which are marched in time):*

$$\mathrm{iglob}(\mathrm{ielmn}, i) = j$$

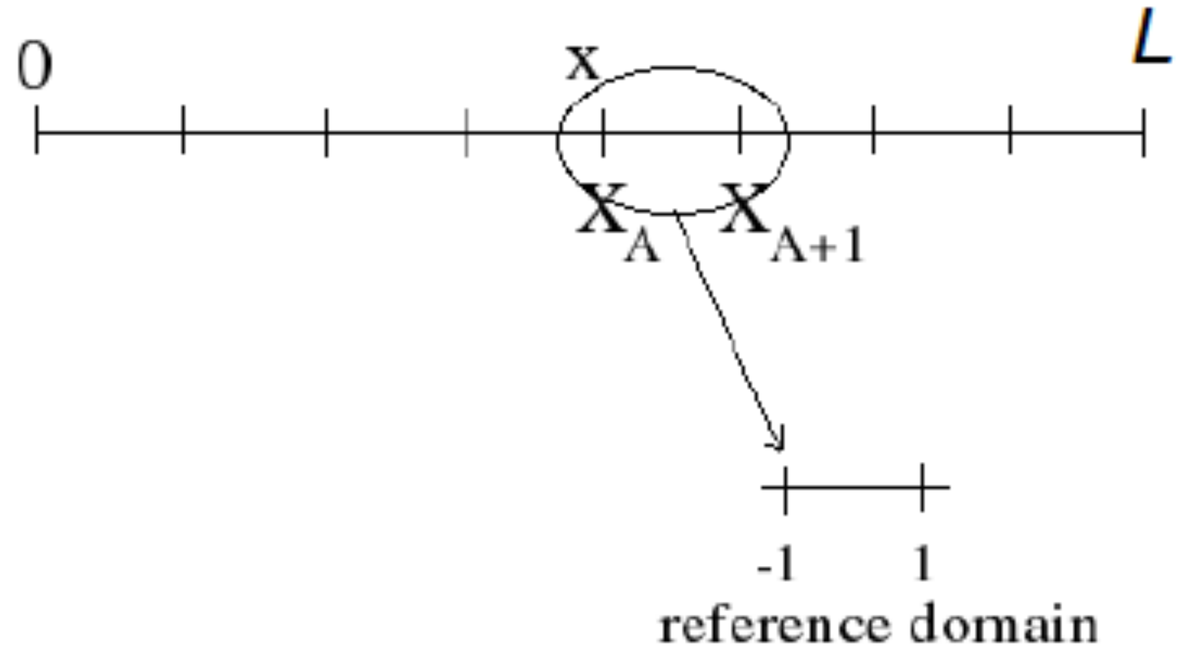
The element number (2)

The local number (4)

The global number (10)

Assembly

Assembling: back to global level



$M_{local}(\text{ielmn}, i) = \omega_\alpha \rho^\alpha c_p^\alpha J^\alpha$
$M_{global}(:) = 0$
!loop over the elements
do ielmn=1,Nel
!loop over the GLL points
    do i=1,NGLL
!get the global index
    j=iglob(ielmn,i) $M_{global}(j) = M_{global}(j) + M_{local}(\text{ielmn}, i)$
    enddo
enddo

Time stepping

Time scheme: Predictor-Corrector algorithm

- Predictor:

$$
\begin{aligned}
T_{n+1} &= T_n + \frac{1}{2}\Delta t \dot{T}_n \\
\dot{T}_{n+1} &= 0 \quad \text{(initialization at the beginning of each time step)}
\end{aligned}
$$

- Solve:

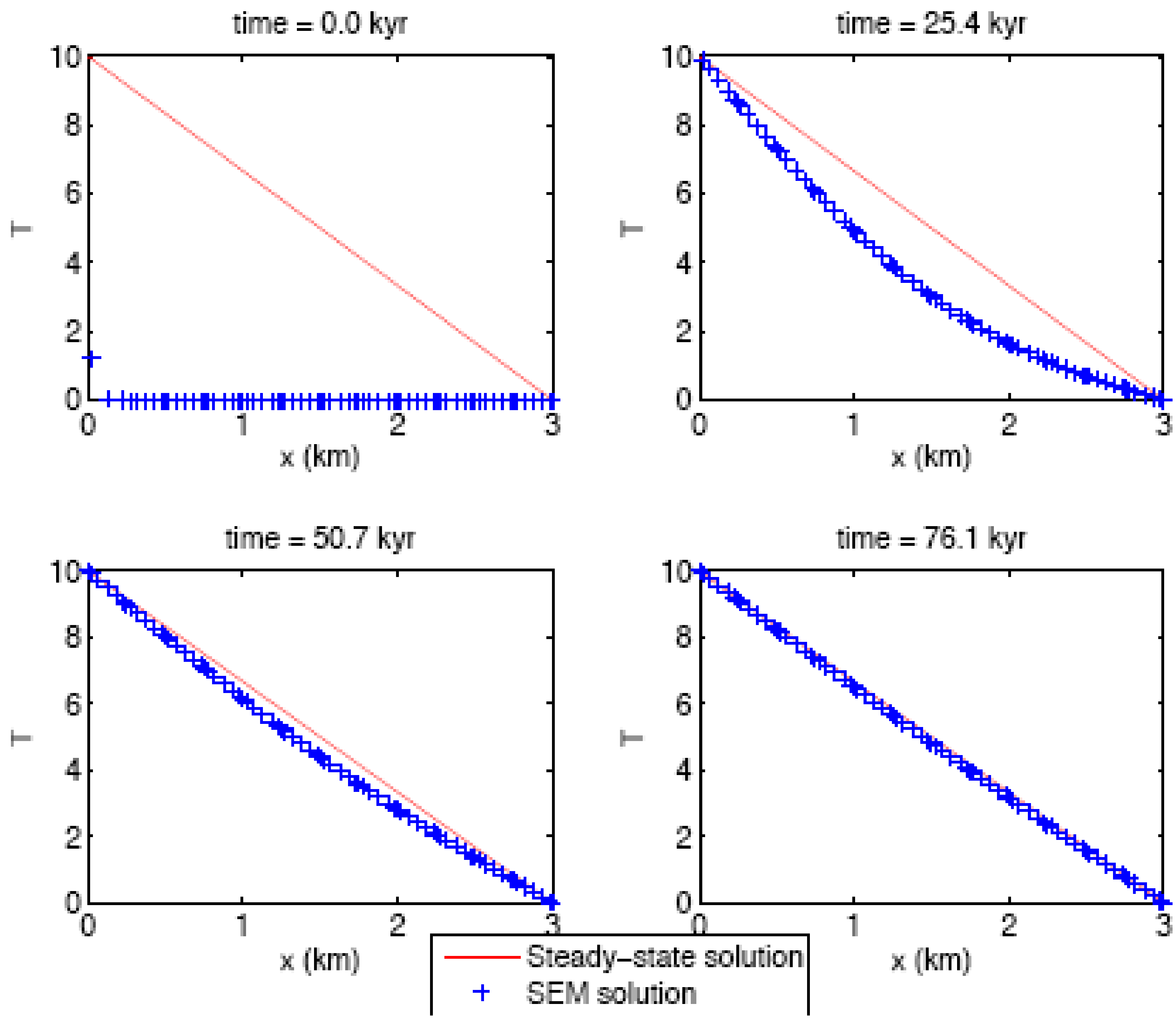
$$
\begin{aligned}
rhs &= -KT_{n+1} \\
\delta\dot{T}_{n+1} &= M^{-1} rhs
\end{aligned}
$$

- Corrector:

$$
\begin{aligned}
T_{n+1} &= T_{n+1} + \frac{1}{2}\Delta t \delta\dot{T}_{n+1} \\
\dot{T}_{n+1} &= \dot{T}_{n+1} + \delta\dot{T}_{n+1}
\end{aligned}
$$

SEM - 1D unsteady-state diffusion equation

Results: *even element spacing*



SEM - 1D unsteady-state diffusion equation

Results: *variable conductivity*