

Finite-difference method



Heat equation

	symbols	SI units
Temperature	$T(x, t)$	K
Specific heat at constant pressure	$c_p(x)$	J/kg/K
density	$\rho(x)$	kg/m ³
thermal conductivity	$K(x)$	W/K/m
thermal diffusivity	$D(x)$	m ² /s

1-D heat energy conservation: $\rho(x)c_p(x)\partial_t T(x, t) = -\partial_x[q(x, t)]$

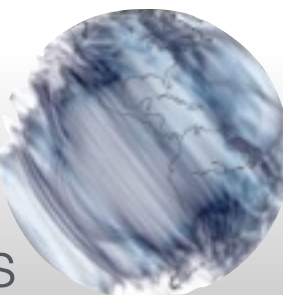
where $q(x, t) = -K(x)\partial_x T(x, t)$ is the heat flux (Fourier's law)

$$\Rightarrow \rho(x)c_p(x)\partial_t T(x, t) = \partial_x[K(x)\partial_x T(x, t)]$$

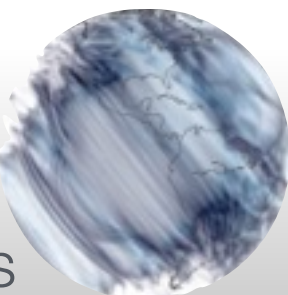
If considering **constant thermal properties** and introducing the thermal diffusivity:

$$D = \frac{K}{\rho c_p}$$

$$\Rightarrow \partial_t T(x, t) = D\partial_x^2 T(x, t)$$



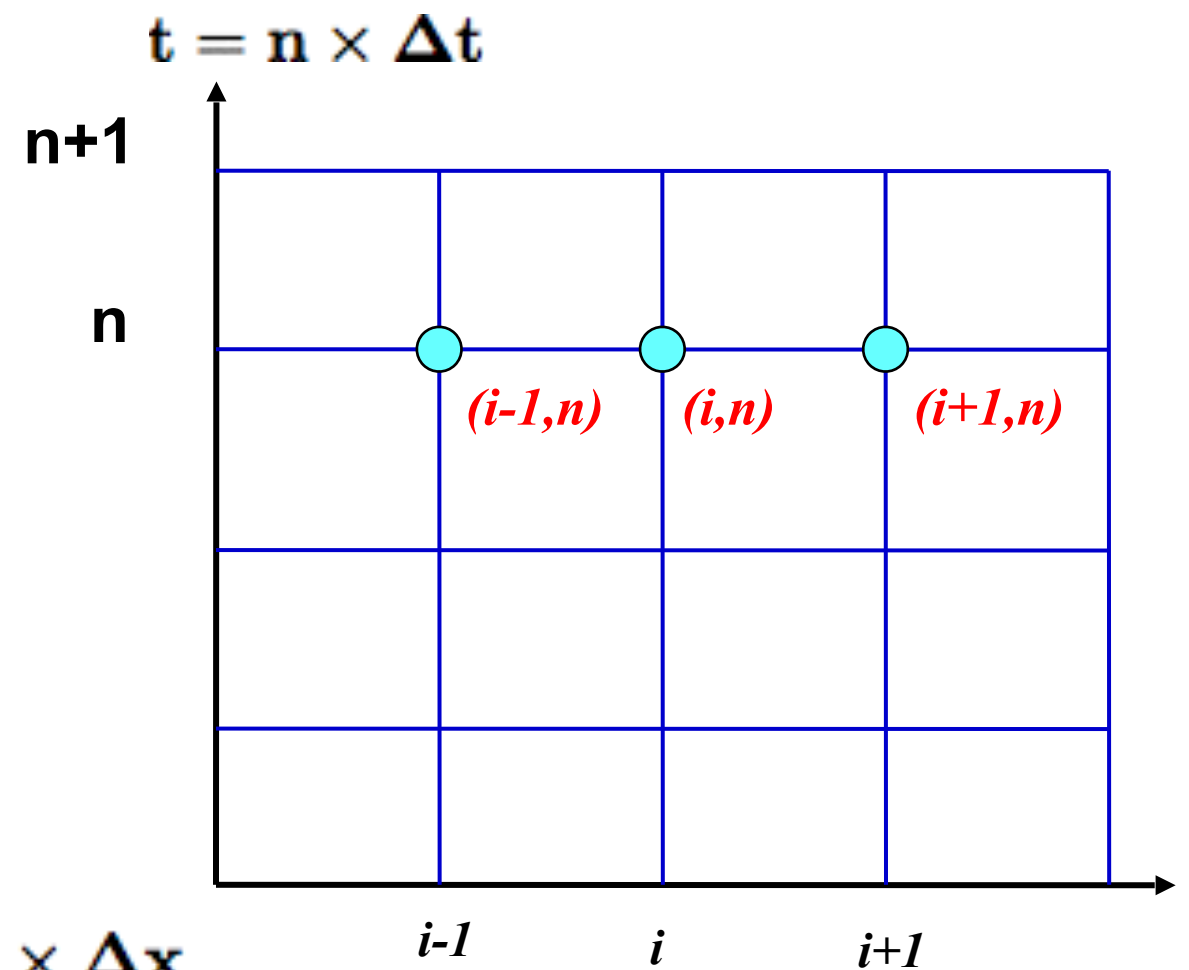
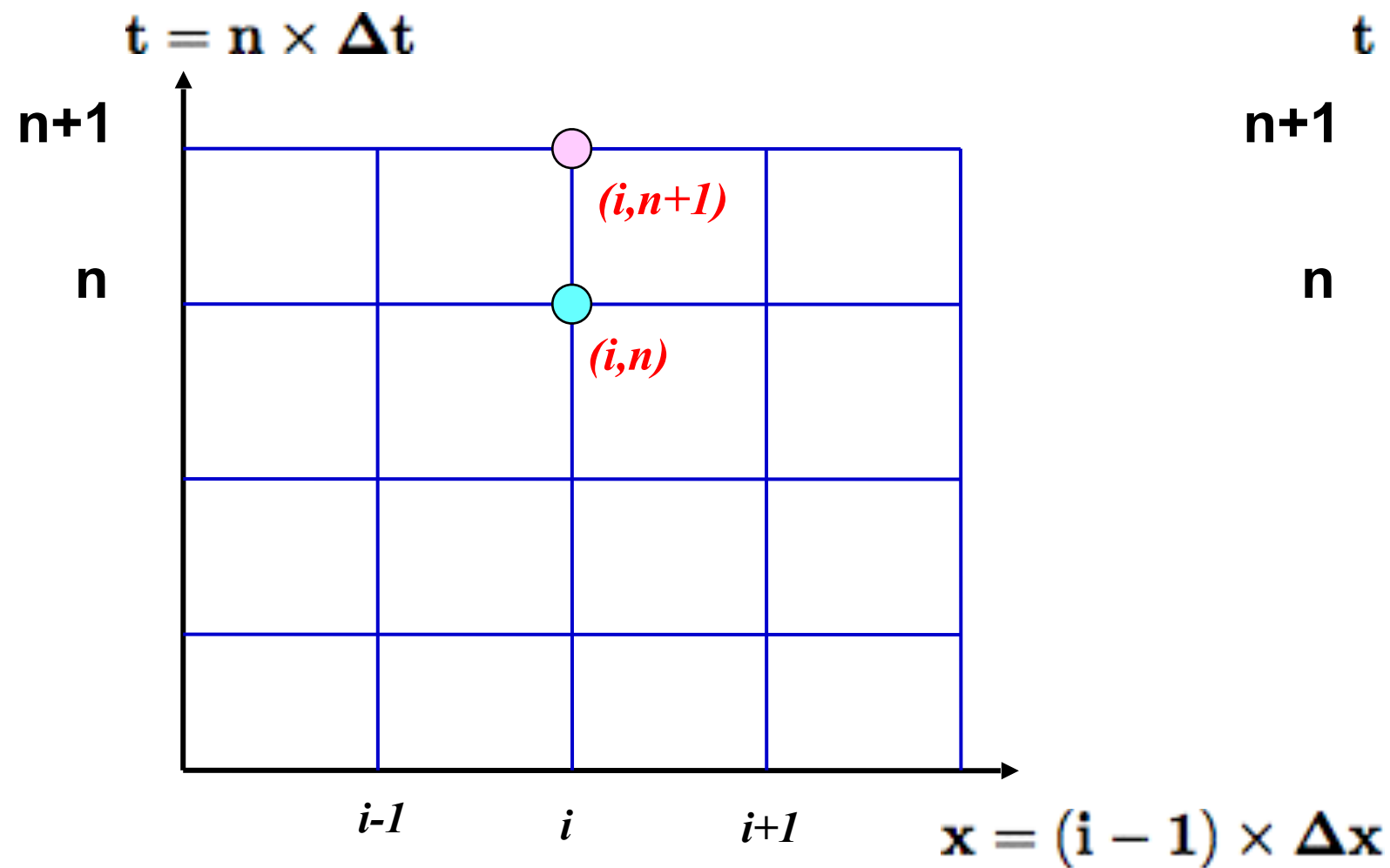
Forward scheme in time
&
central difference in space



Forward scheme in time & central difference in space

$$\begin{aligned}\partial_t T(x, t) &\approx \frac{T(x, t + \Delta t) - T(x, t)}{\Delta t} \\ &\equiv \frac{T_i^{n+1} - T_i^n}{\Delta t}\end{aligned}$$

$$\begin{aligned}\partial_x^2 T(x, t) &\approx \frac{T(x + \Delta x, t) - 2T(x, t) + T(x - \Delta x, t)}{\Delta x^2} \\ &\equiv \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2}\end{aligned}$$



Forward scheme in time & central difference in space

$$\partial_t T(x, t) \approx \frac{T(x, t + \Delta t) - T(x, t)}{\Delta t}$$

$$\partial_x^2 T(x, t) \approx \frac{T(x + \Delta x, t) - 2T(x, t) + T(x - \Delta x, t)}{\Delta x^2}$$

$$\partial_t T(x, t) = D \partial_x^2 T(x, t)$$

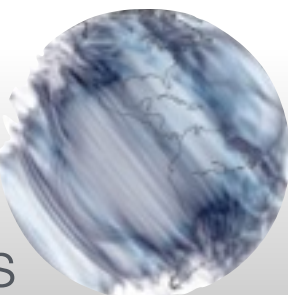


$$T_i^{n+1} = T_i^n + (T_{i+1}^n - 2T_i^n + T_{i-1}^n) \frac{D\Delta t}{\Delta x^2}$$

$$\begin{bmatrix} d_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & d_{N-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & d_N \end{bmatrix} \begin{bmatrix} T_1^n \\ T_2^n \\ T_3^n \\ \dots \\ T_{N-1}^n \\ T_N^n \end{bmatrix} = \begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 \\ 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & a_{N-1} & b_{N-1} & c_{N-1} \\ 0 & 0 & 0 & 0 & a_N & b_N \end{bmatrix} \begin{bmatrix} T_1^{n-1} \\ T_2^{n-1} \\ T_3^{n-1} \\ \dots \\ T_{N-1}^{n-1} \\ T_N^{n-1} \end{bmatrix}$$



Backward scheme in time
&
central difference in space



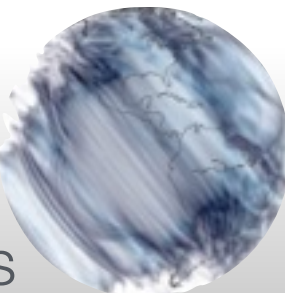
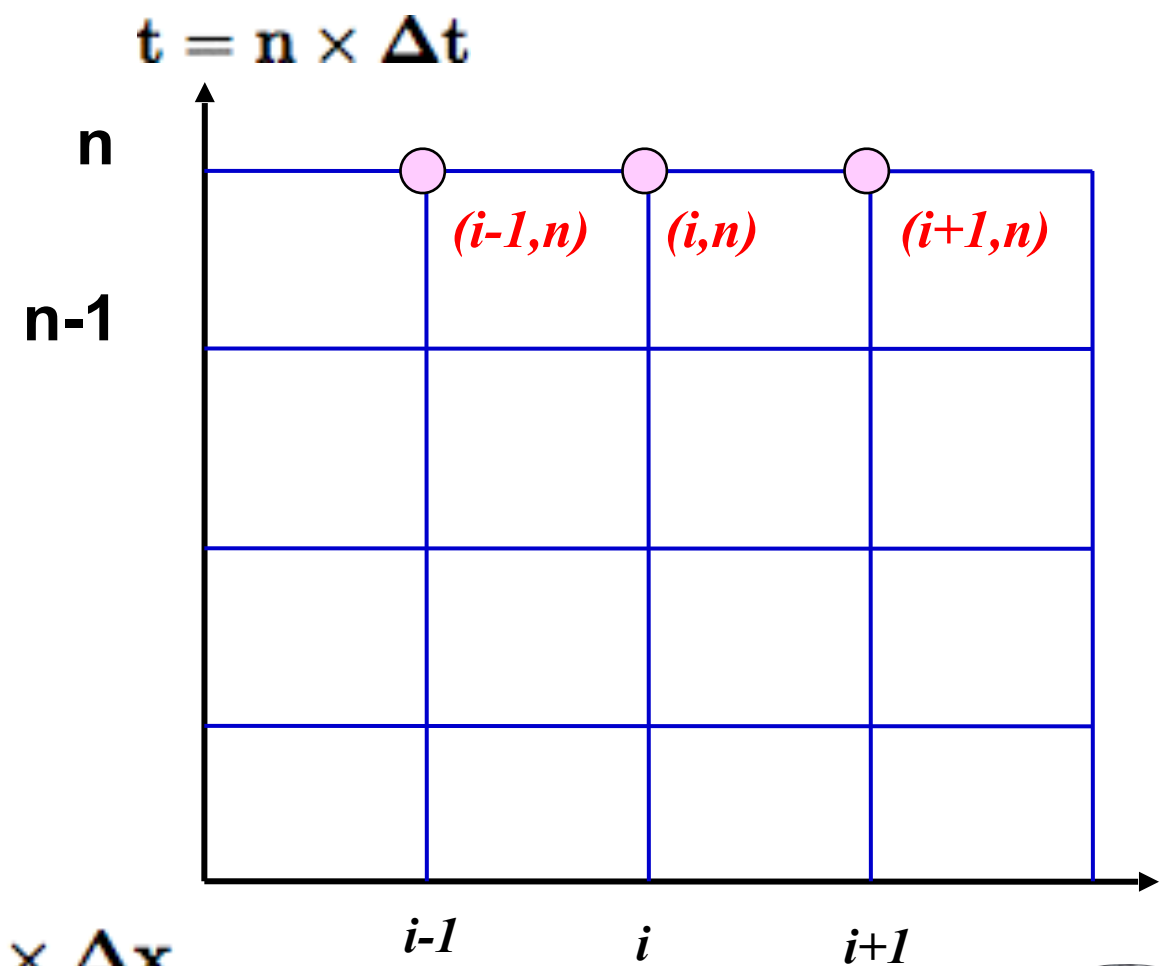
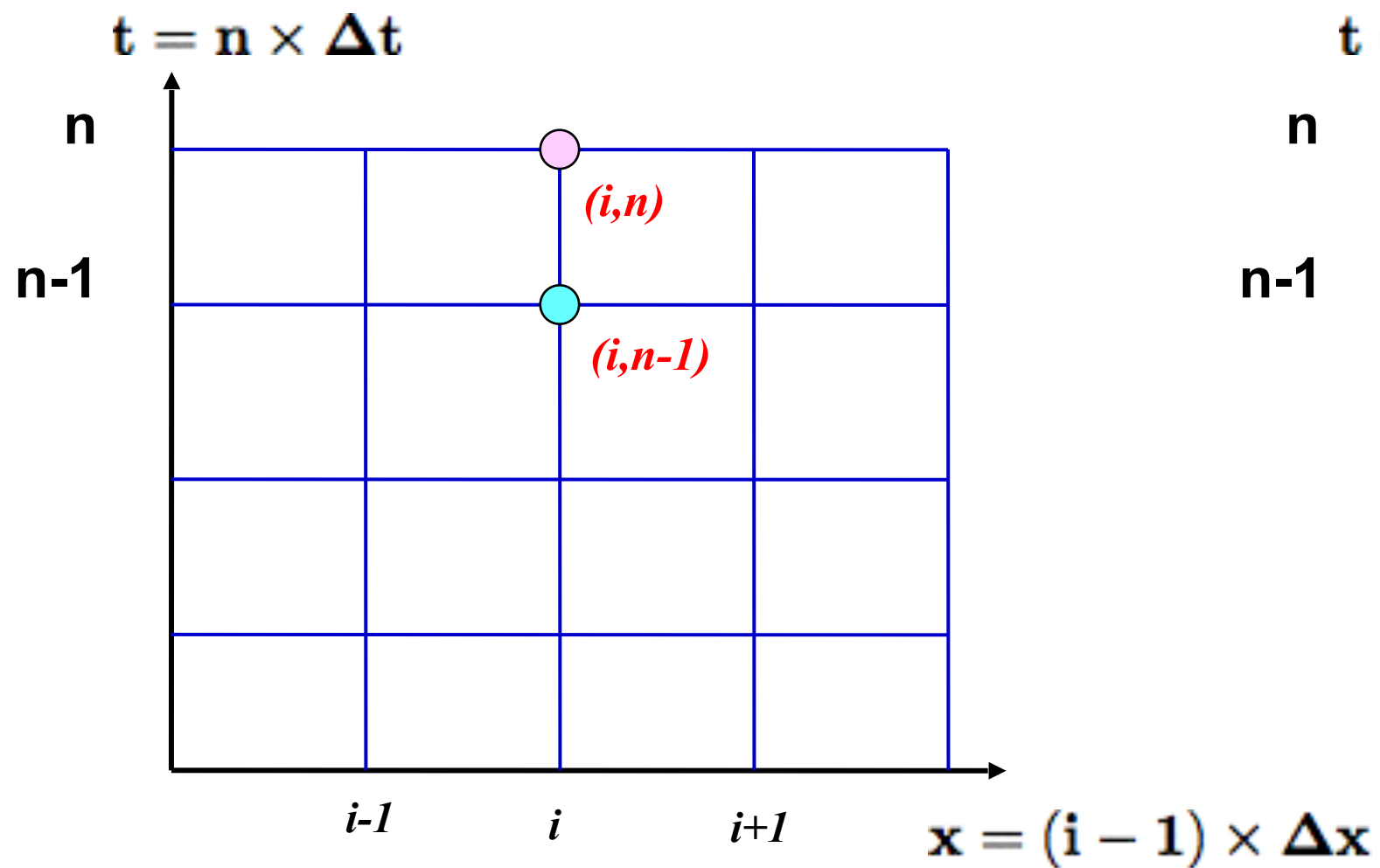
Backward scheme in time & central difference in space

$$\partial_t T(x, t) \approx \frac{T(x, t) - T(x, t - \Delta t)}{\Delta t}$$

$$\equiv \frac{T_i^n - T_i^{n-1}}{\Delta t}$$

$$\partial_x^2 T(x, t) \approx \frac{T(x + \Delta x, t) - 2T(x, t) + T(x - \Delta x, t)}{\Delta x^2}$$

$$\equiv \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2}$$



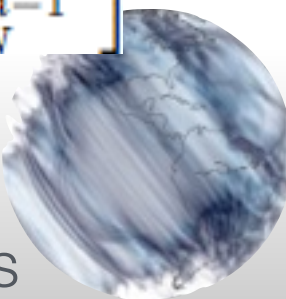
Backward scheme in time & central difference in space

$$\begin{aligned}\partial_t T(x, t) &\approx \frac{T(x, t) - T(x, t - \Delta t)}{\Delta t} &&\equiv \frac{T_i^n - T_i^{n-1}}{\Delta t} \\ \partial_x^2 T(x, t) &\approx \frac{T(x + \Delta x, t) - 2T(x, t) + T(x - \Delta x, t)}{\Delta x^2} &&\equiv \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2}\end{aligned}$$

$$\partial_t T(x, t) = D \partial_x^2 T(x, t)$$

$$-D \frac{\Delta t}{\Delta x^2} T_{i-1}^n + \left[1 + D \frac{2\Delta t}{\Delta x^2} \right] T_i^n - D \frac{\Delta t}{\Delta x^2} T_{i+1}^n = T_i^{n-1}$$

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 \\ 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & a_{N-1} & b_{N-1} & c_{N-1} \\ 0 & 0 & 0 & 0 & a_N & b_N \end{bmatrix} \begin{bmatrix} T_1^n \\ T_2^n \\ T_3^n \\ \dots \\ T_{N-1}^n \\ T_N^n \end{bmatrix} = \begin{bmatrix} d_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & d_{N-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & d_N \end{bmatrix} \begin{bmatrix} T_1^{n-1} \\ T_2^{n-1} \\ T_3^{n-1} \\ \dots \\ T_{N-1}^{n-1} \\ T_N^{n-1} \end{bmatrix}$$



Crank-Nicolson scheme



Crank-Nicolson scheme

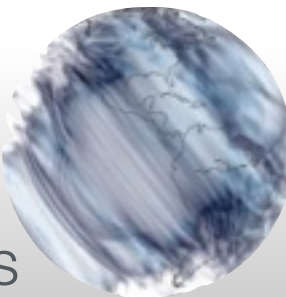
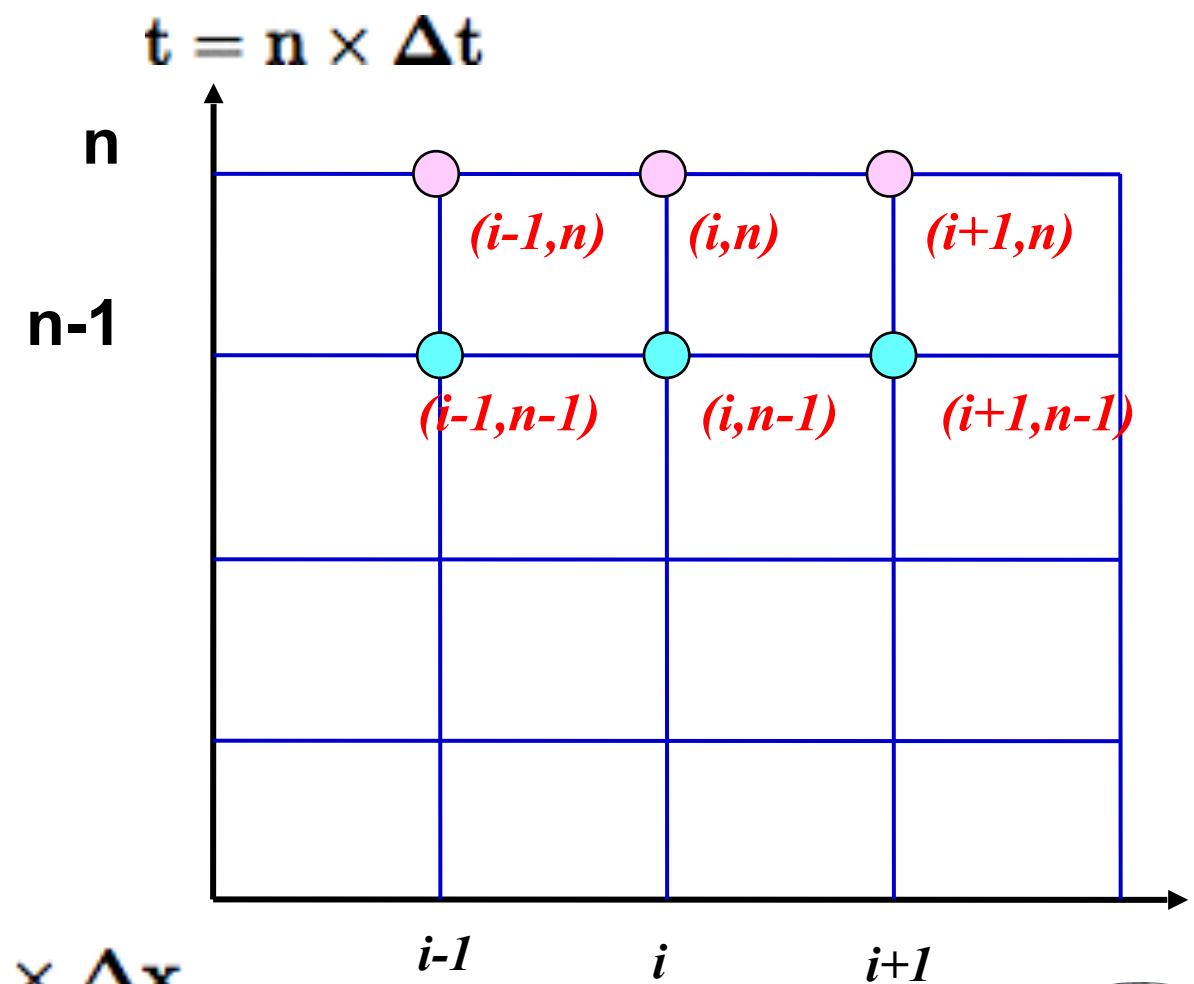
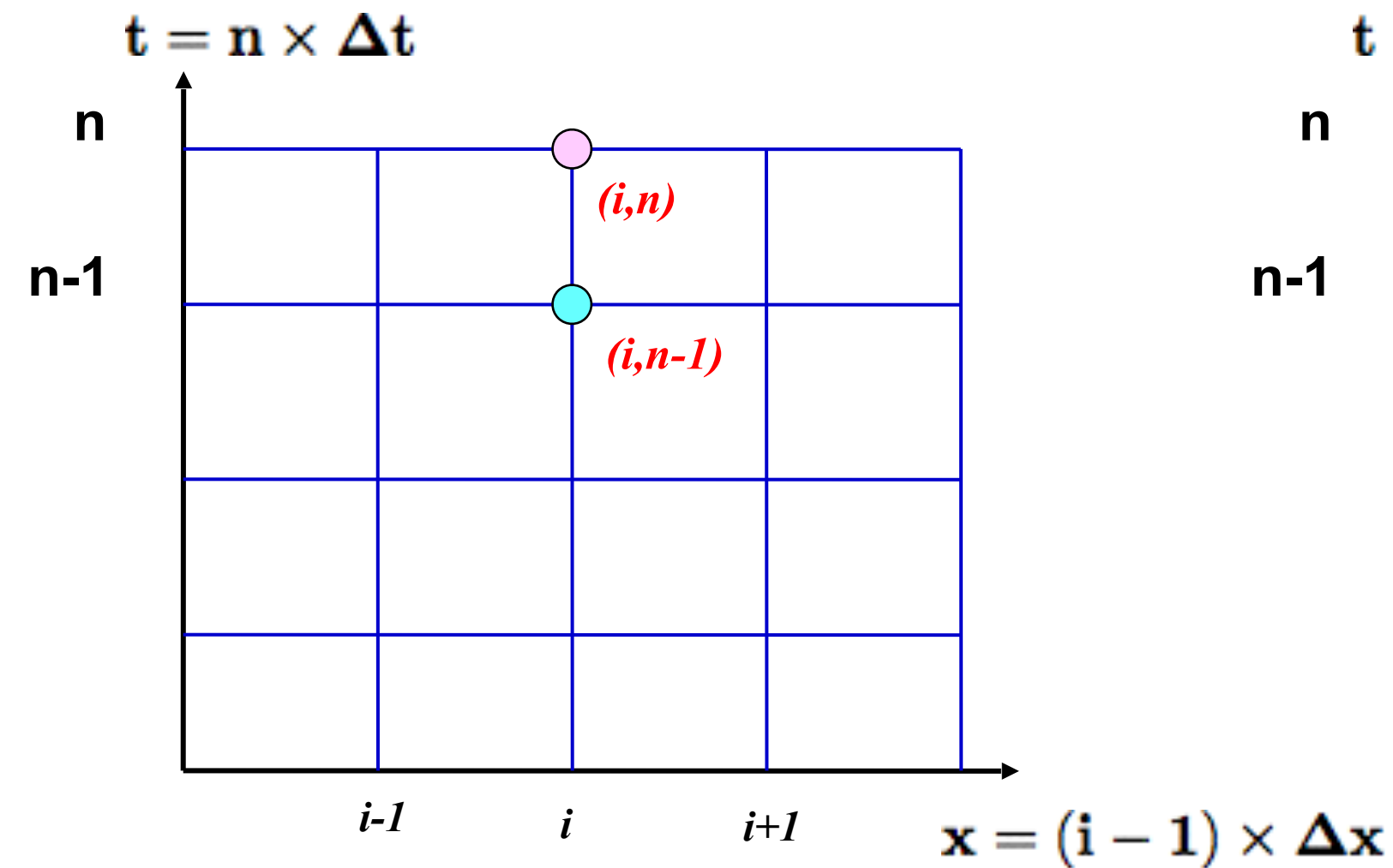
$$\partial_t T(x, t) \approx \frac{T(x, t) - T(x, t - \Delta t)}{\Delta t}$$

$$\equiv \frac{T_i^n - T_i^{n-1}}{\Delta t}$$

$$\partial_x^2 T(x, t) \approx \frac{1}{2} \frac{T(x + \Delta x, t) - 2T(x, t) + T(x - \Delta x, t)}{\Delta x^2}$$

$$+ \frac{1}{2} \frac{T(x + \Delta x, t - \Delta t) - 2T(x, t - \Delta t) + T(x - \Delta x, t - \Delta t)}{\Delta x^2}$$

$$\equiv \frac{1}{2} \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2} + \frac{1}{2} \frac{T_{i+1}^{n-1} - 2T_i^{n-1} + T_{i-1}^{n-1}}{\Delta x^2}$$



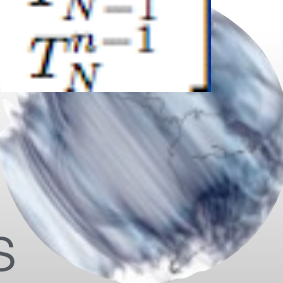
Crank-Nicolson scheme

$$\begin{aligned}\partial_t T(x, t) &\approx \frac{T(x, t) - T(x, t - \Delta t)}{\Delta t} && \equiv \frac{T_i^n - T_i^{n-1}}{\Delta t} \\ \partial_x^2 T(x, t) &\approx \frac{1}{2} \frac{T(x + \Delta x, t) - 2T(x, t) + T(x - \Delta x, t)}{\Delta x^2} && \equiv \frac{1}{2} \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2} + \frac{1}{2} \frac{T_{i+1}^{n-1} - 2T_i^{n-1} + T_{i-1}^{n-1}}{\Delta x^2} \\ &+ \frac{1}{2} \frac{T(x + \Delta x, t - \Delta t) - 2T(x, t - \Delta t) + T(x - \Delta x, t - \Delta t)}{\Delta x^2}\end{aligned}$$

$$\partial_t T(x, t) = D \partial_x^2 T(x, t)$$

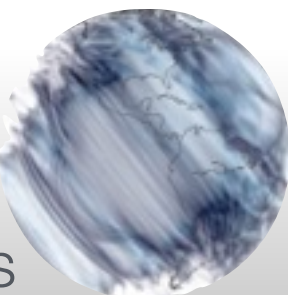
$$-D \frac{\Delta t}{2\Delta x^2} T_{i-1}^n + \left[1 + D \frac{\Delta t}{\Delta x^2} \right] T_i^n - D \frac{\Delta t}{2\Delta x^2} T_{i+1}^n = D \frac{\Delta t}{2\Delta x^2} T_{i-1}^{n-1} + \left[1 - D \frac{\Delta t}{\Delta x^2} \right] T_i^{n-1} + D \frac{\Delta t}{2\Delta x^2} T_{i+1}^{n-1}$$

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 \\ 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & a_{N-1} & b_{N-1} & c_{N-1} \\ 0 & 0 & 0 & 0 & a_N & b_N \end{bmatrix} \begin{bmatrix} T_1^n \\ T_2^n \\ T_3^n \\ \dots \\ T_{N-1}^n \\ T_N^n \end{bmatrix} = \begin{bmatrix} e_1 & f_1 & 0 & 0 & 0 & 0 \\ d_2 & e_2 & f_2 & 0 & 0 & 0 \\ 0 & d_3 & e_3 & f_3 & 0 & 0 \\ 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & d_{N-1} & e_{N-1} & f_{N-1} \\ 0 & 0 & 0 & 0 & d_N & e_N \end{bmatrix} \begin{bmatrix} T_1^{n-1} \\ T_2^{n-1} \\ T_3^{n-1} \\ \dots \\ T_{N-1}^{n-1} \\ T_N^{n-1} \end{bmatrix}$$



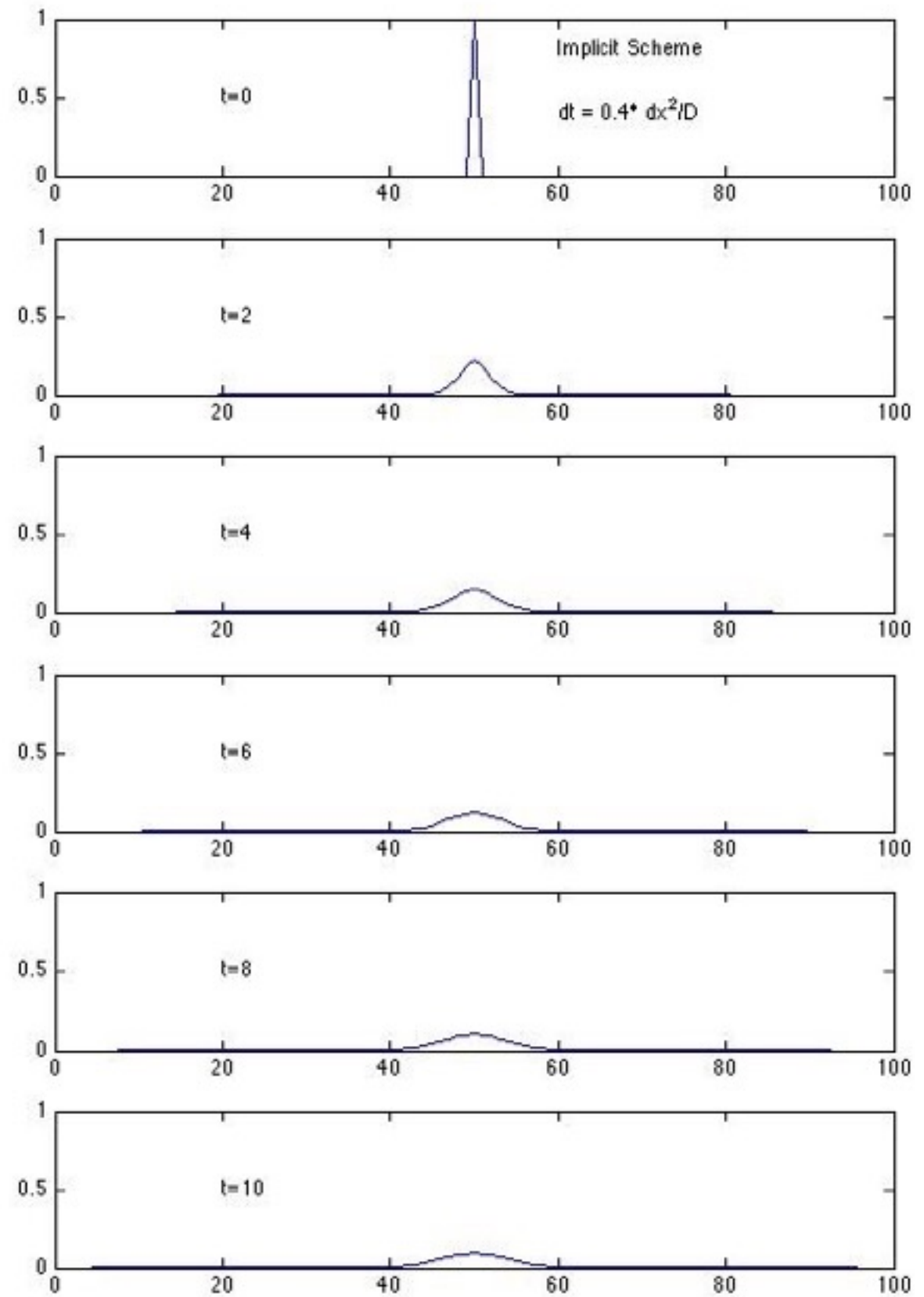
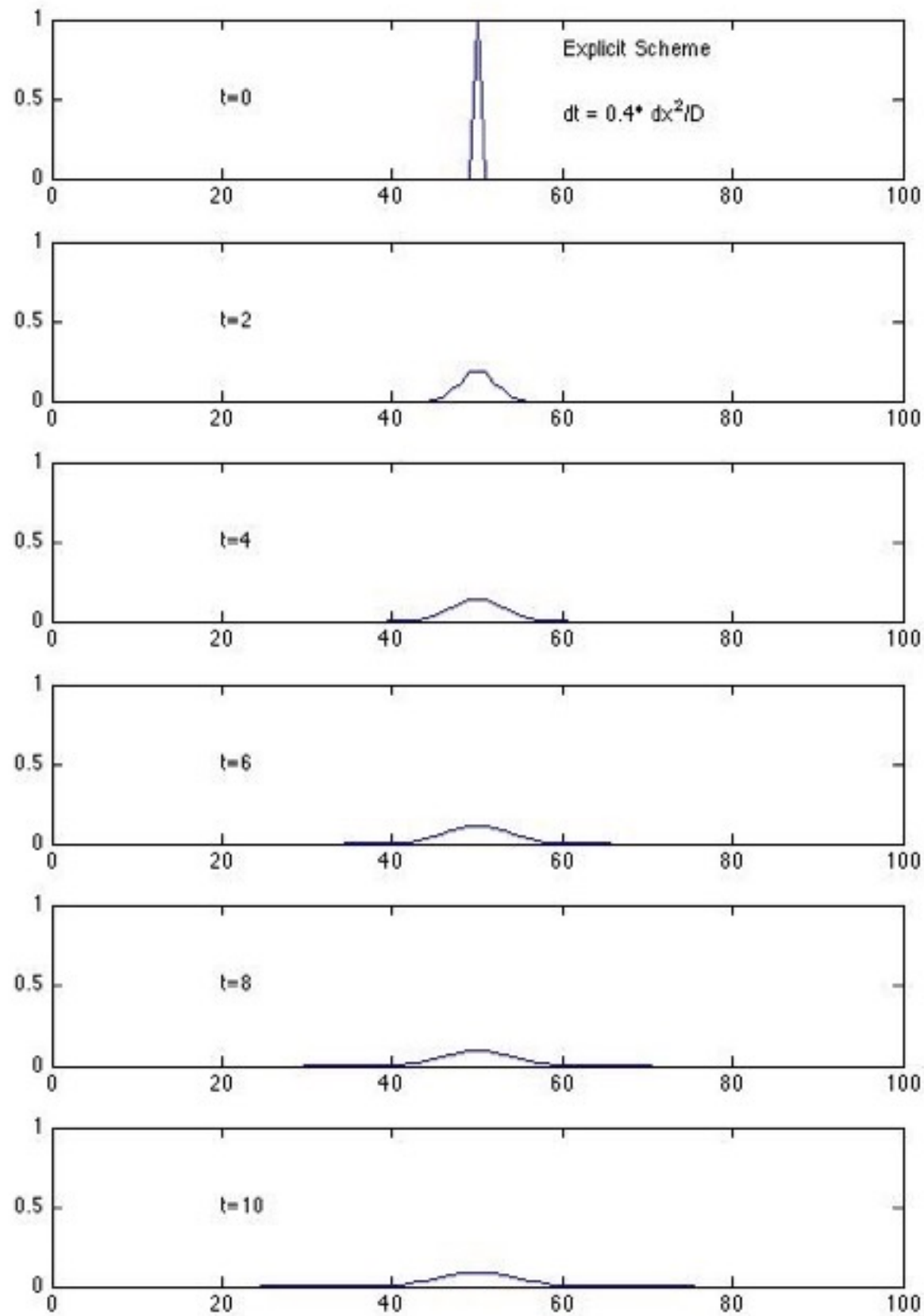
Explicit vs. Implicit schemes

Results



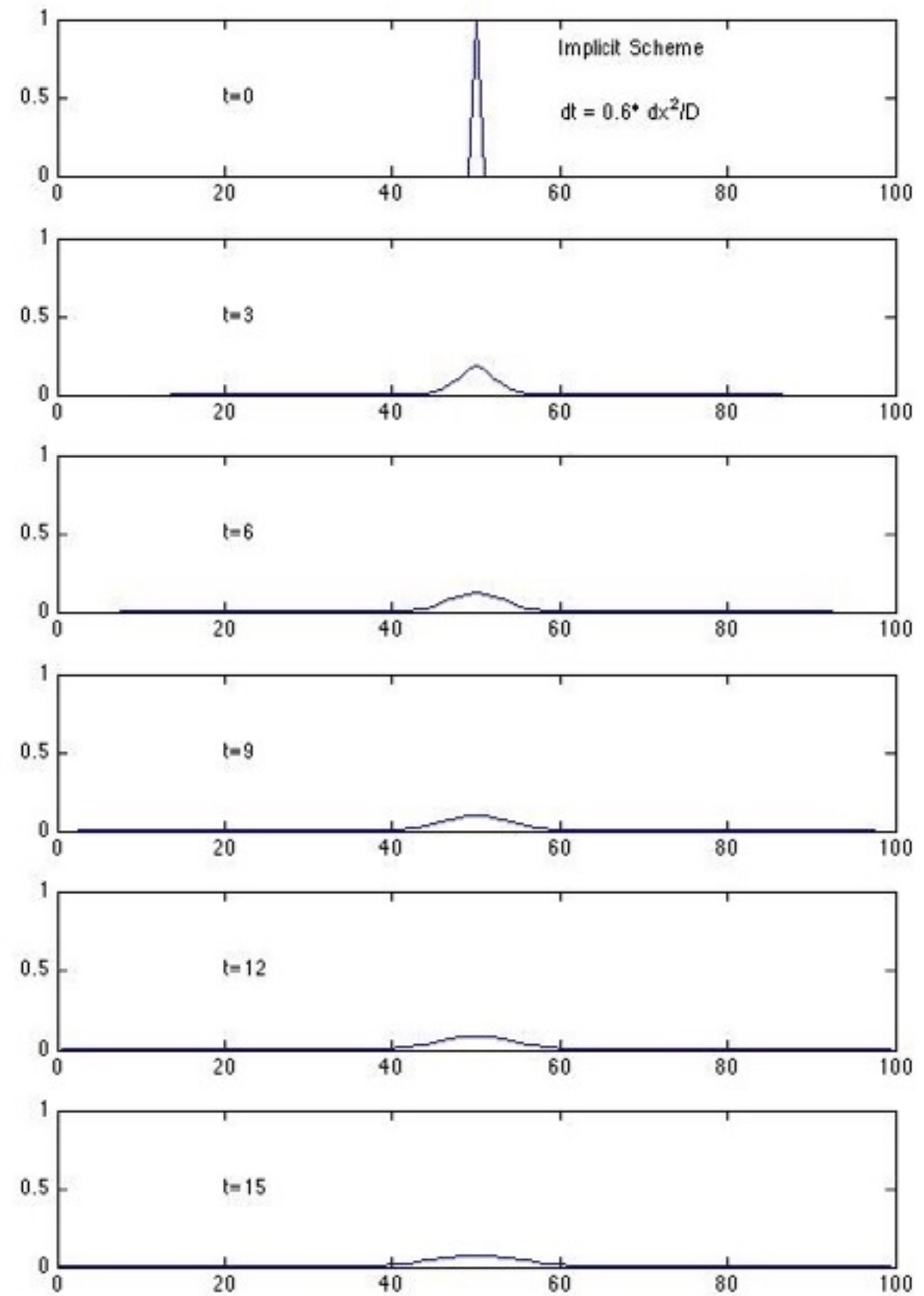
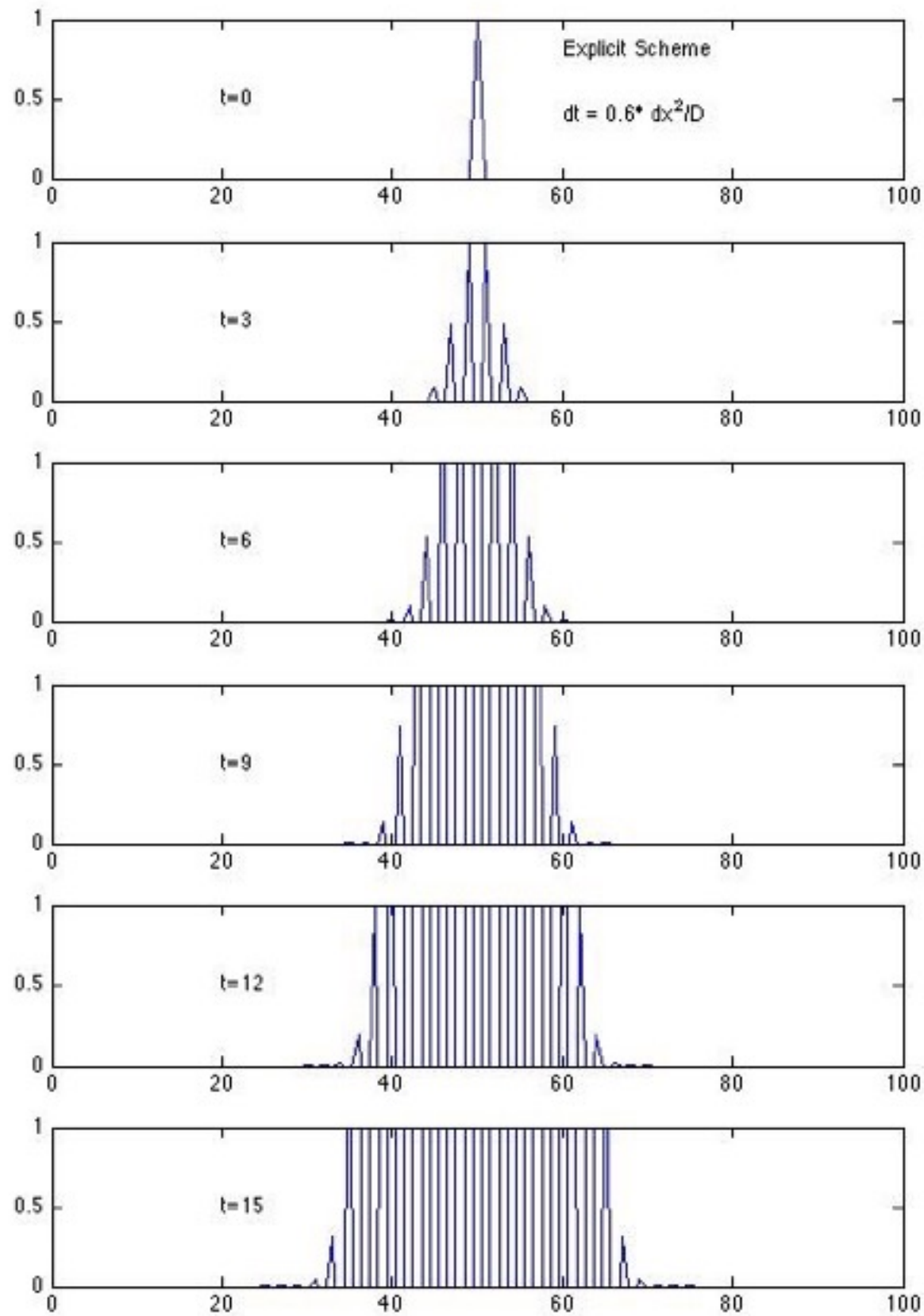
Explicit vs. Implicit schemes

$$\text{factor} = 0.4 \rightarrow dt = 0.4 * dt^{**2} / D$$



Explicit vs. Implicit schemes

$$\text{factor} = 0.6 \rightarrow dt = 0.6 * dt^{**2} / D$$

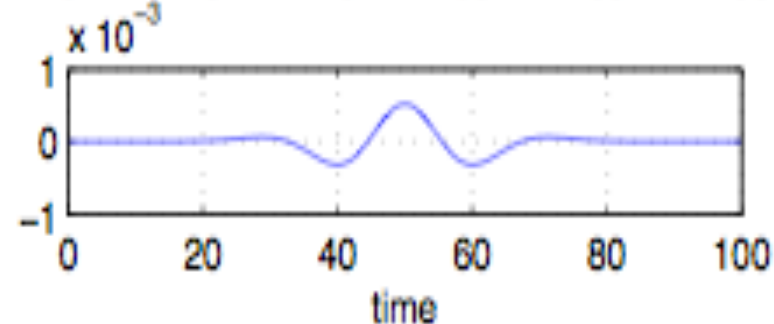
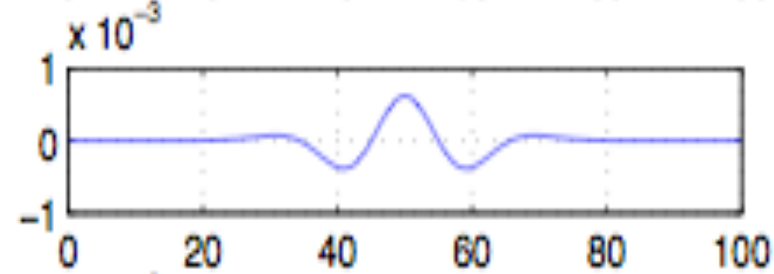
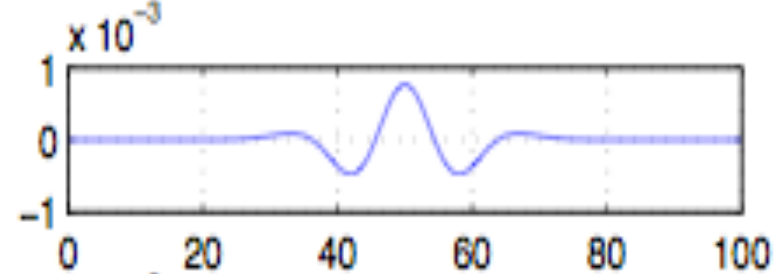
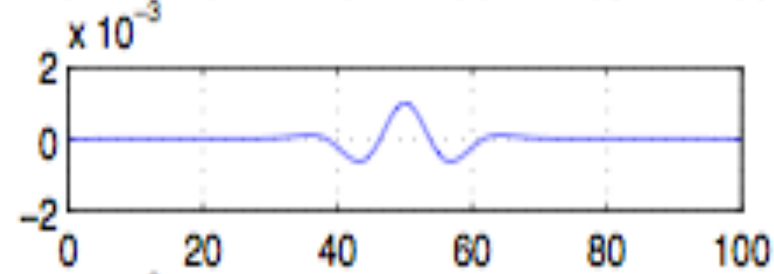
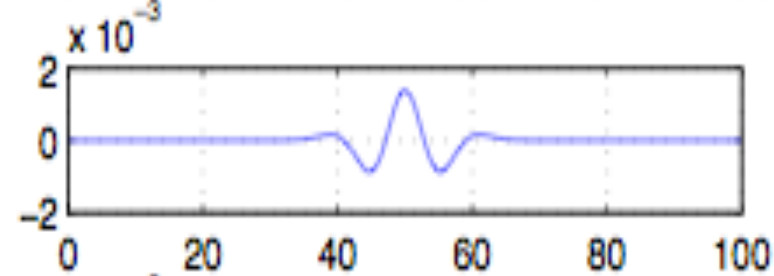
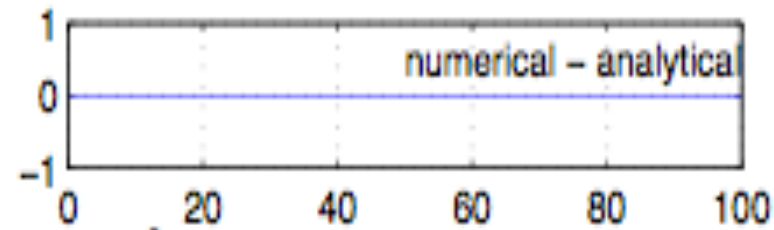
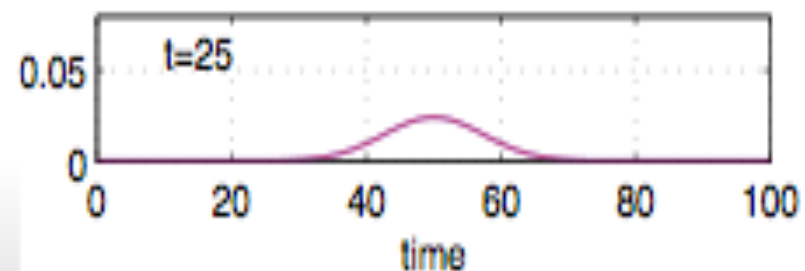
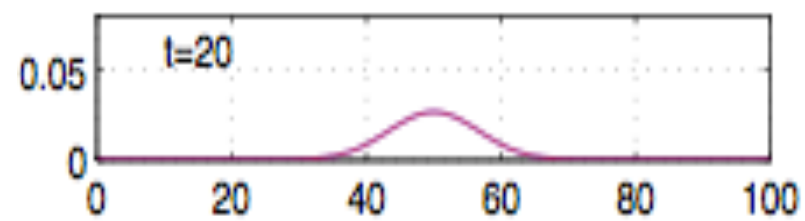
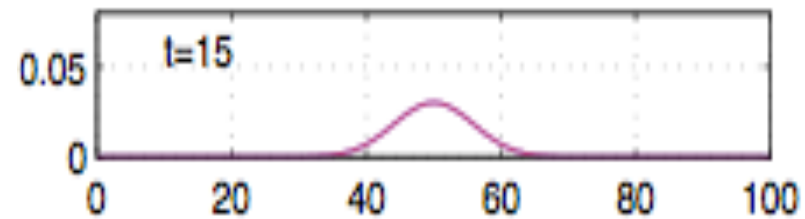
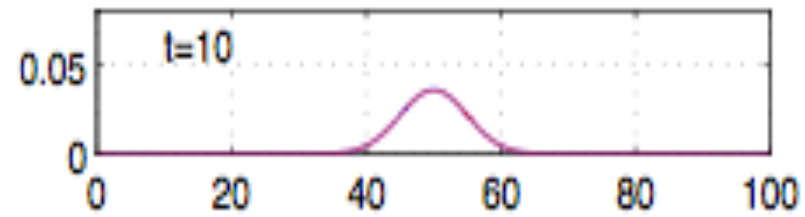
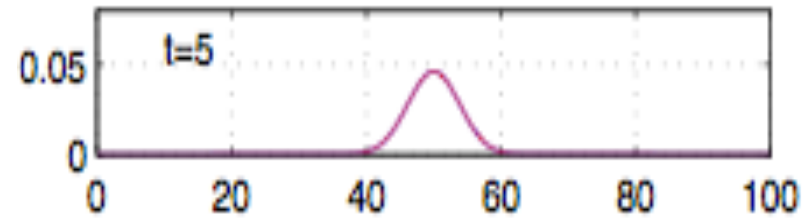
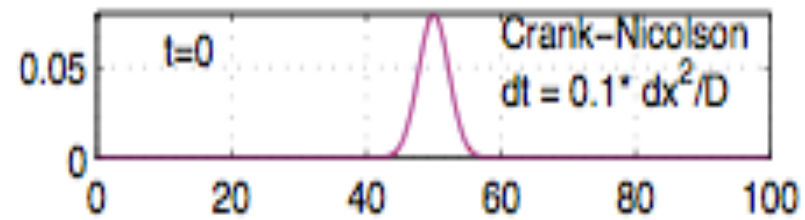


Crank-Nicolson scheme vs. analytic solution

Results

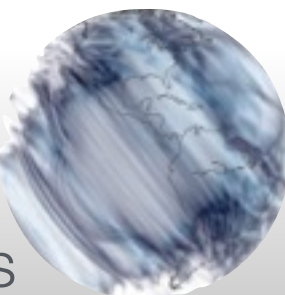


Crank-Nicolson scheme vs. analytic solution

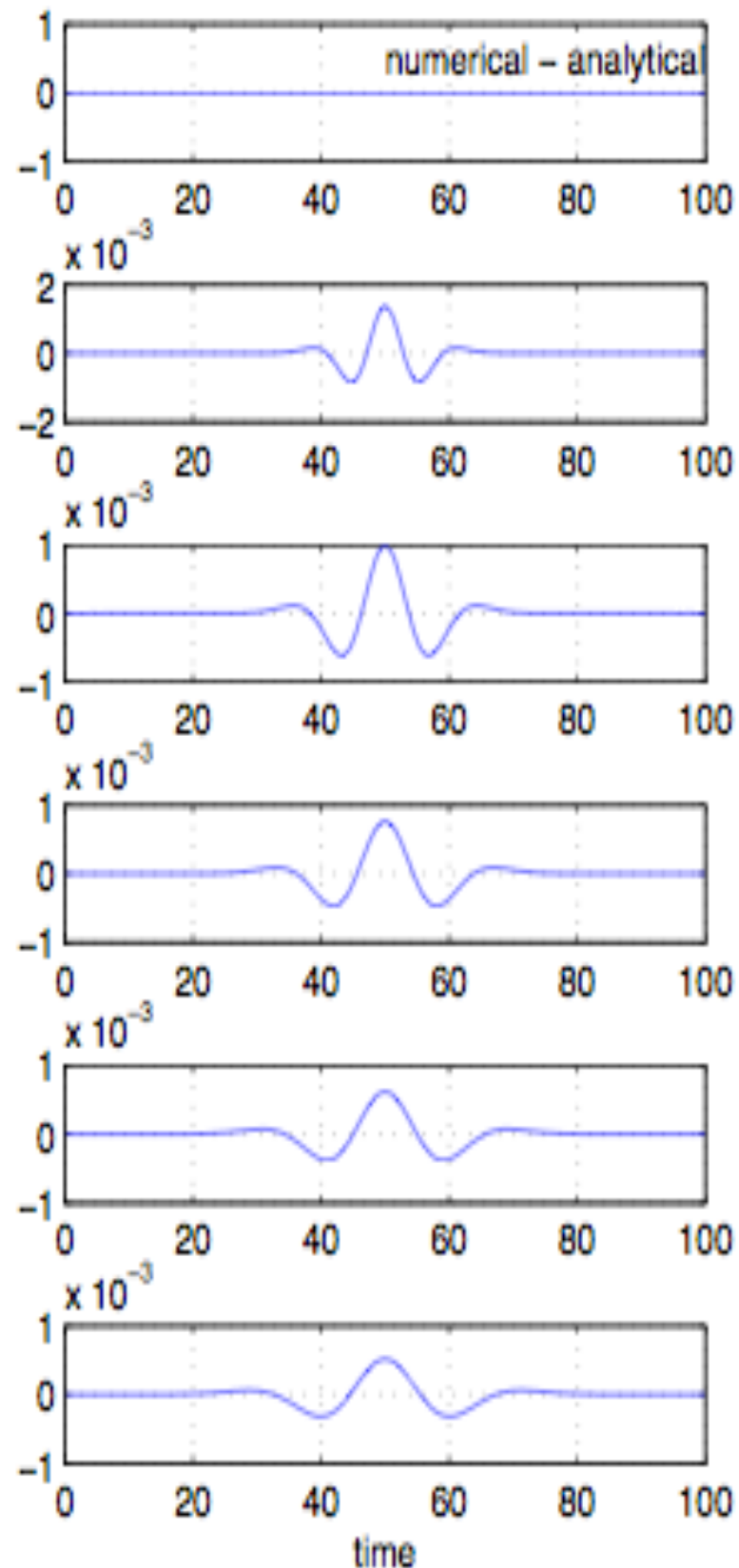
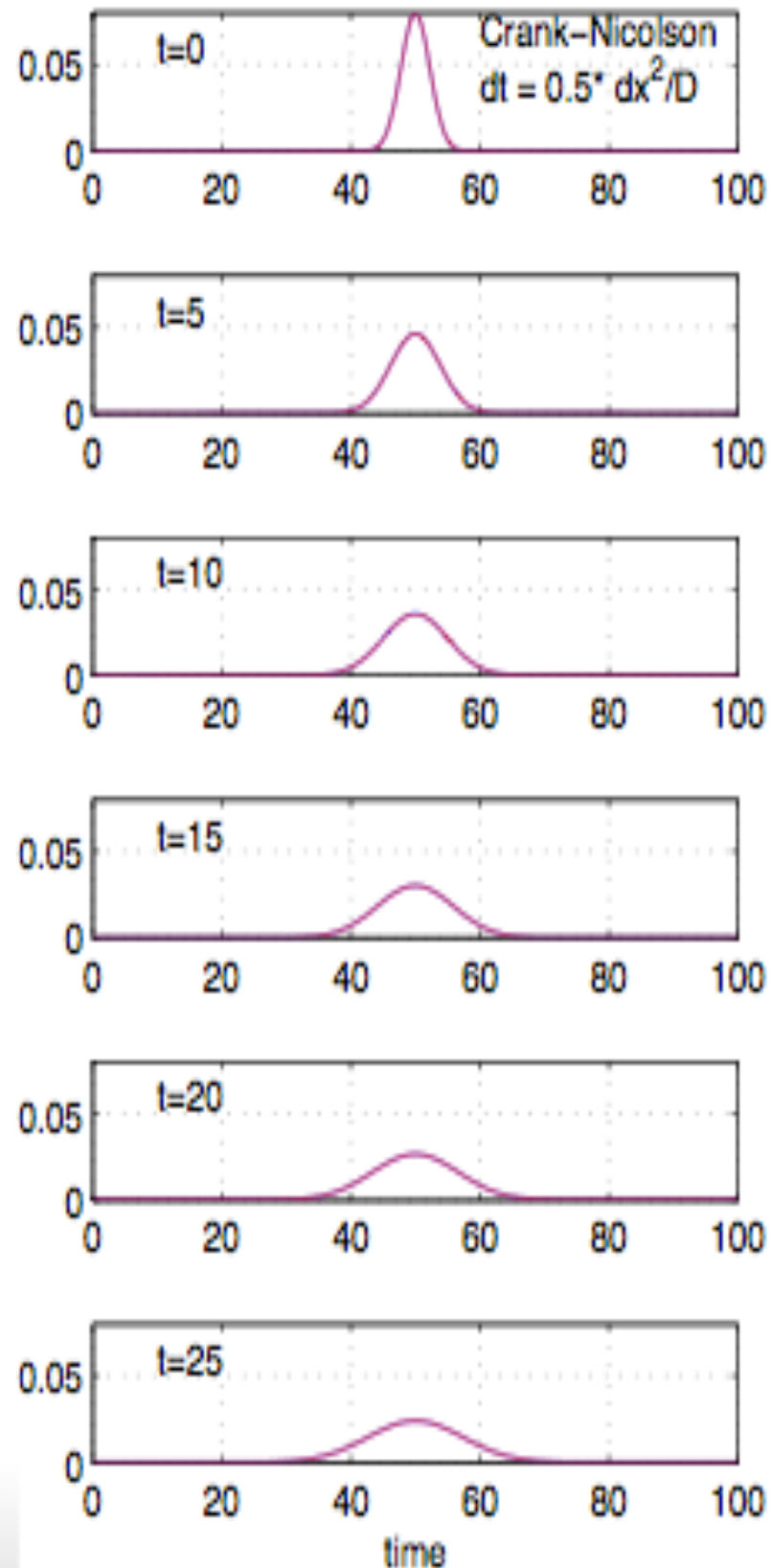


Gaussian heat source
 $dx = 0.5$
 $dt = 0.1$

Red: analytical
Blue: numerical



Crank-Nicolson scheme vs. analytic solution

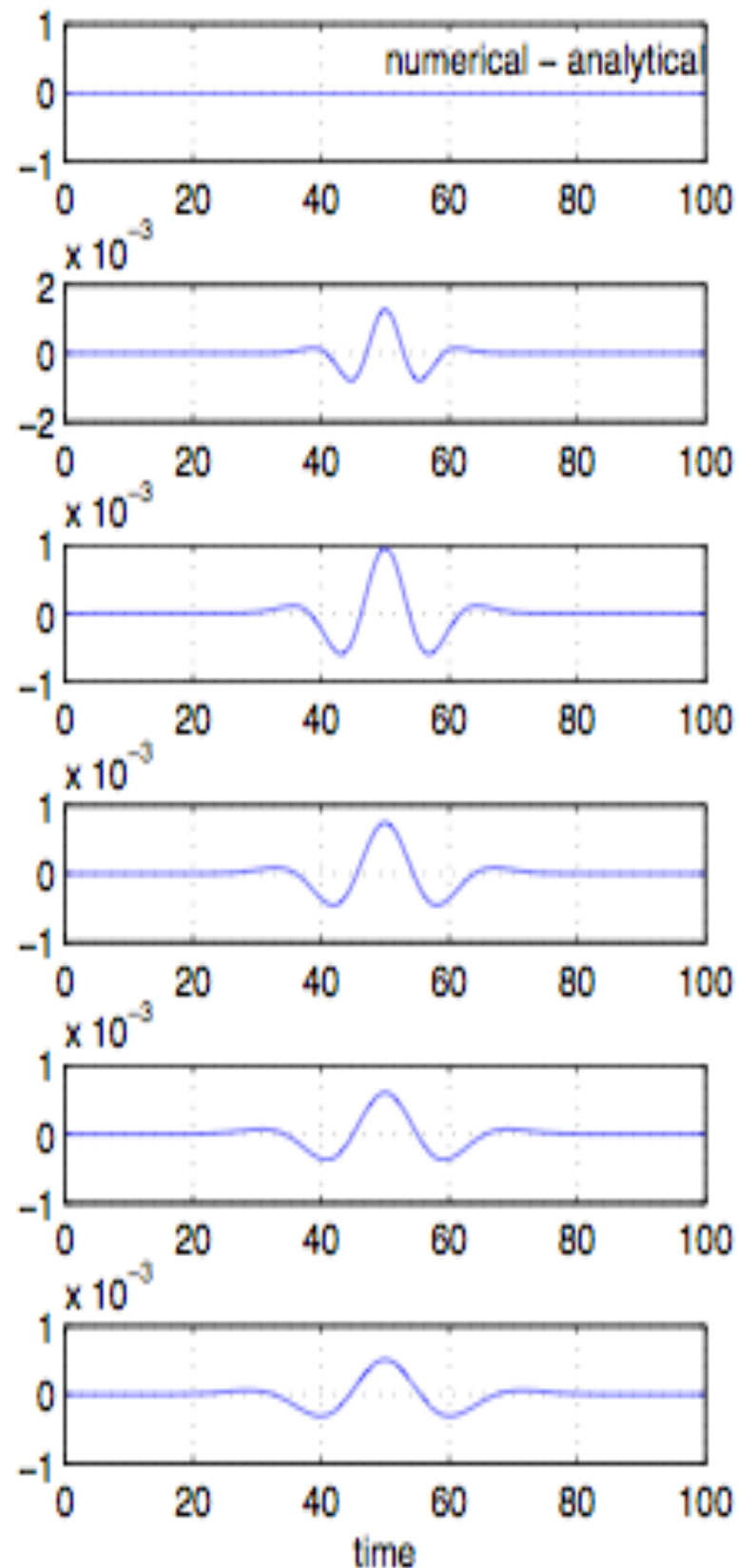
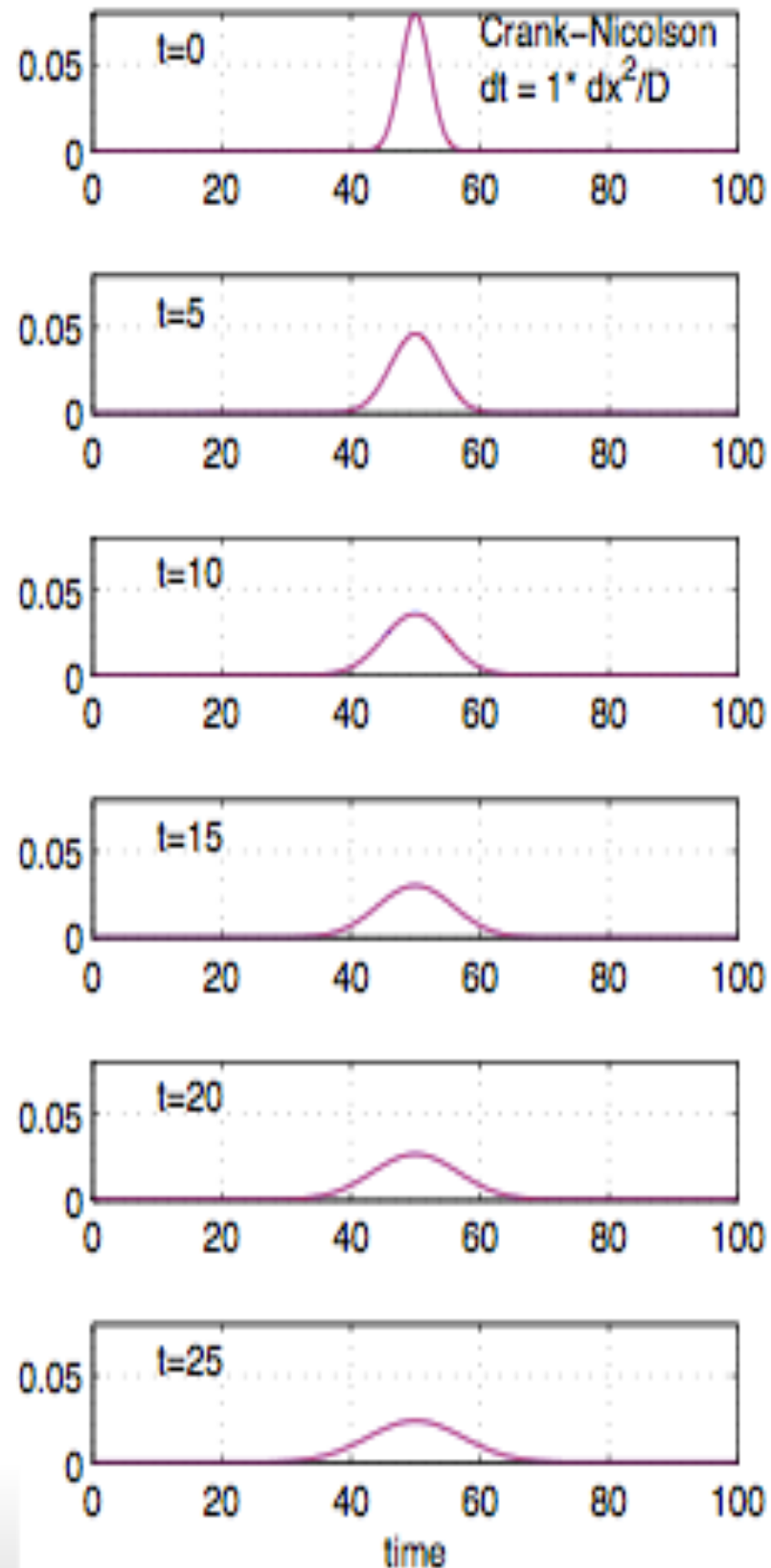


Gaussian heat source
 $dx = 0.5$
 $dt = 0.5$

Red: analytical
Blue: numerical



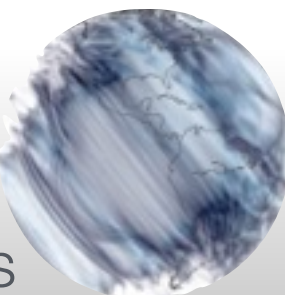
Crank-Nicolson scheme vs. analytic solution



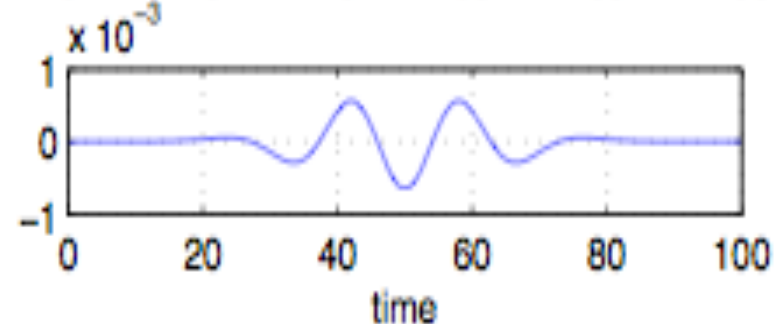
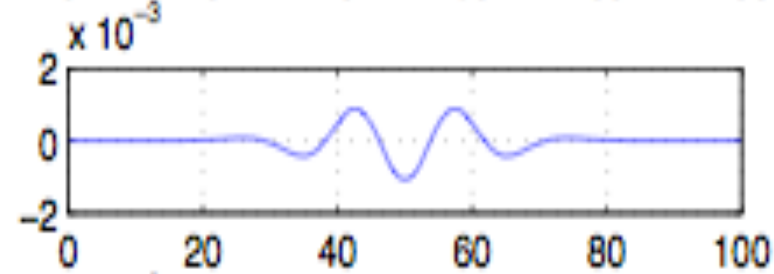
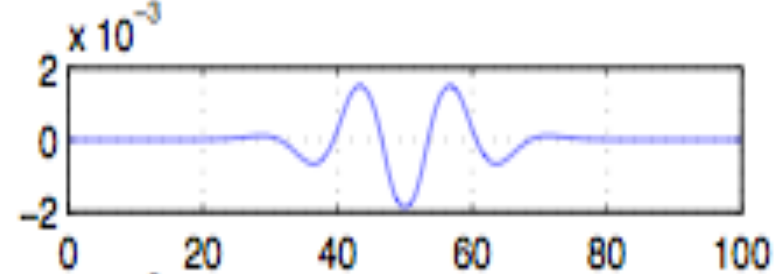
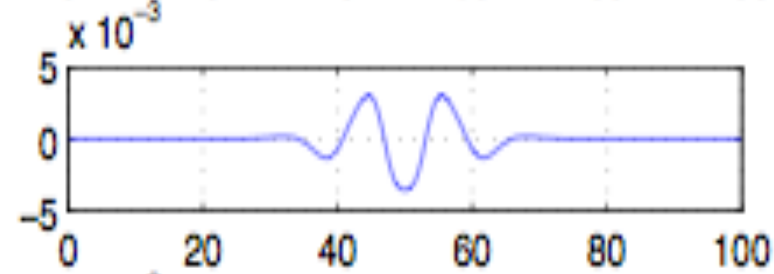
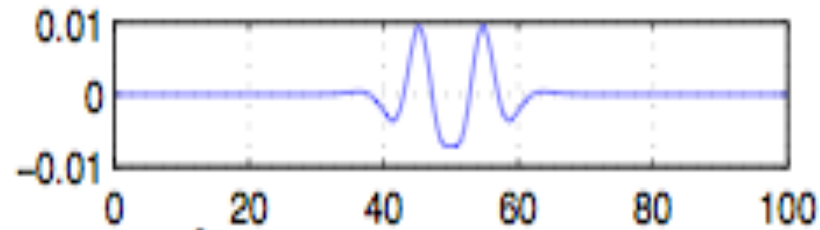
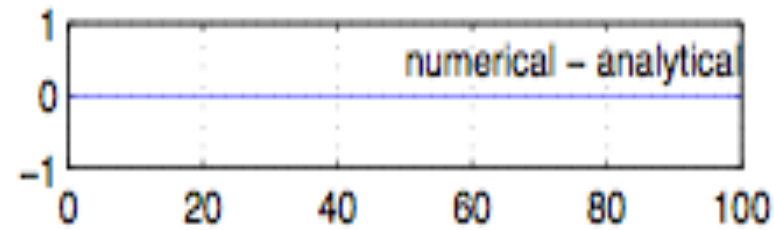
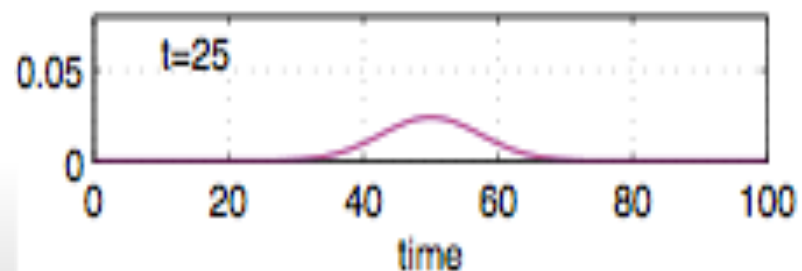
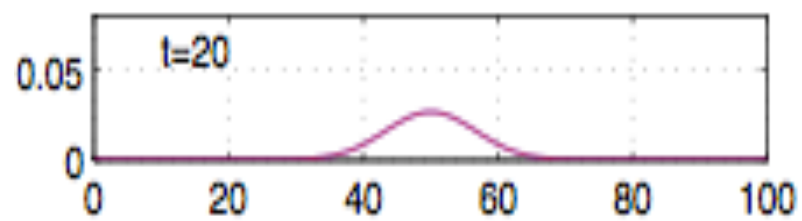
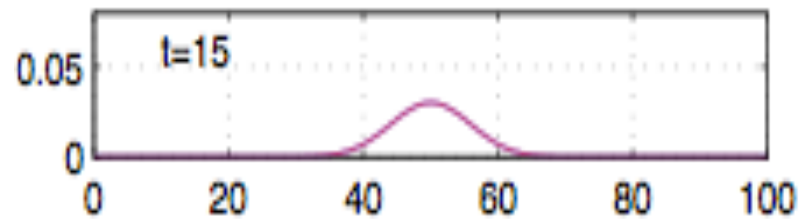
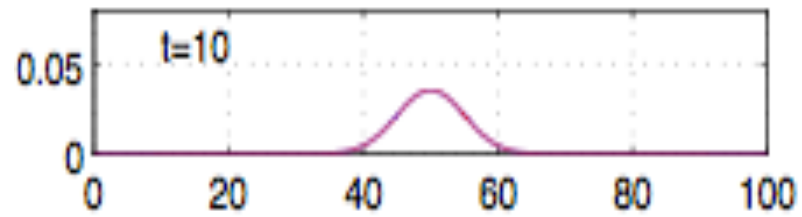
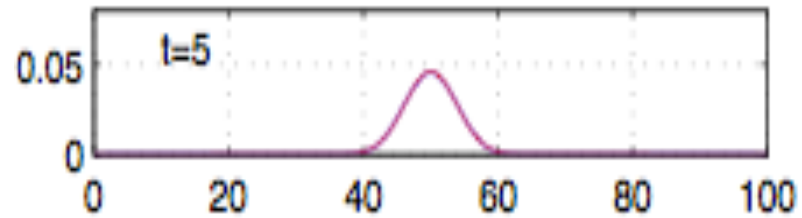
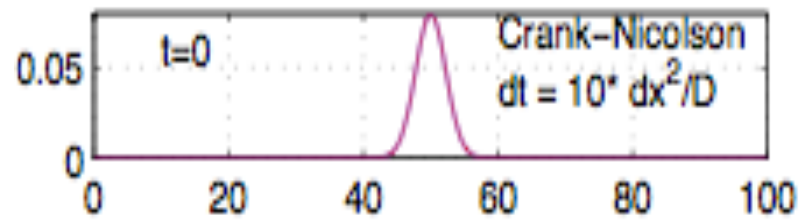
Gaussian heat source
 $dx = 0.5$
 $dt = 1$

where
explicit scheme
becomes
unstable

Red: analytical
Blue: numerical



Crank-Nicolson scheme vs. analytic solution



Gaussian heat source
 $dx = 0.5$
 $dt = 10$

where
explicit scheme
becomes
unstable

Red: analytical
Blue: numerical

