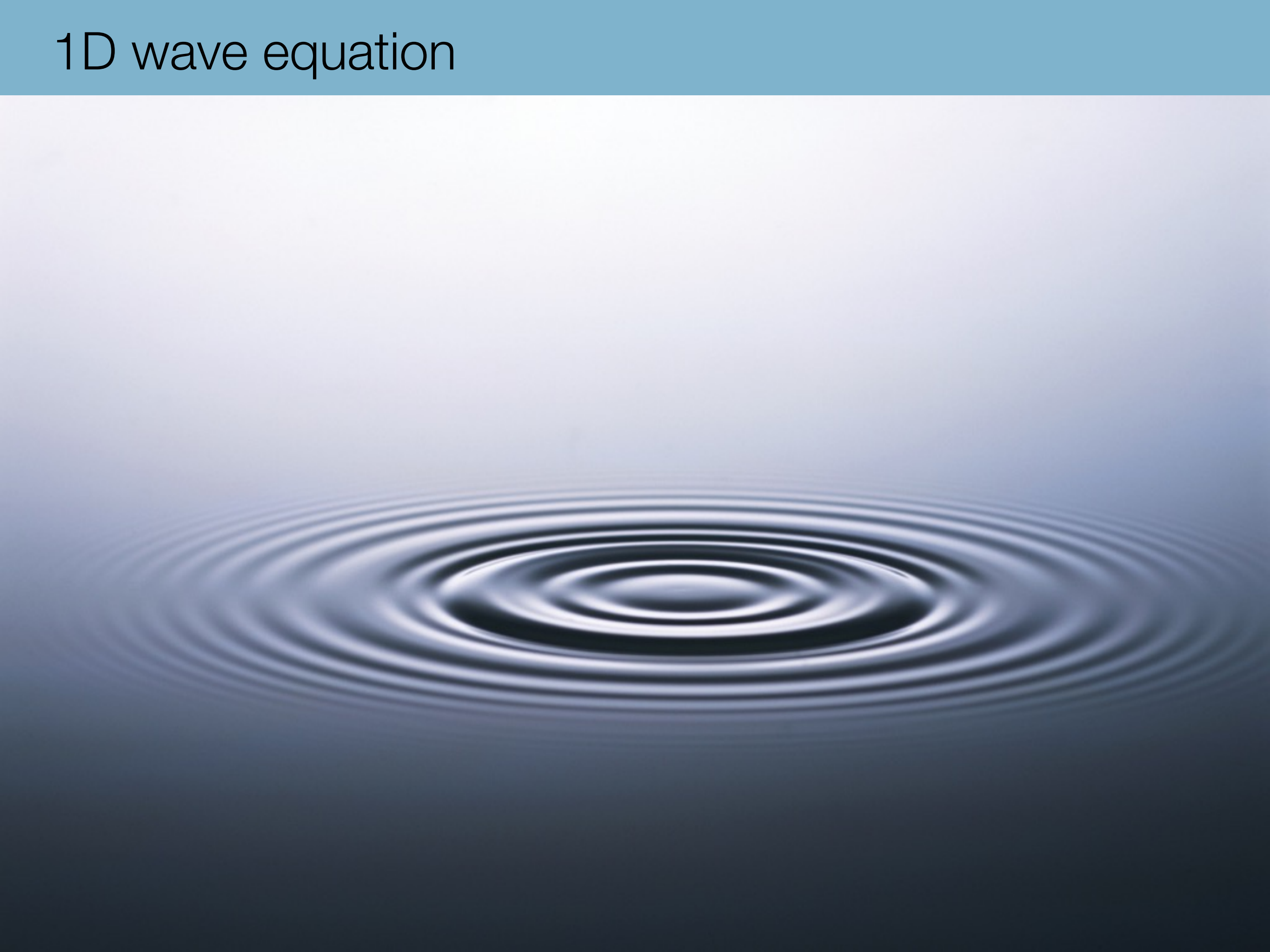


Spectral-element method



1D wave equation



1D wave equation

Strong form: $\rho \partial_t^2 s = \partial_x (\mu \partial_x s)$

IC & BC: $\begin{cases} s(x, 0) = f(x) \\ s(L, t) = 0 \\ s(0, t) = 0 \end{cases}$ and $\begin{cases} s(x, 0) = f(x) \\ \partial_x s(L, t) = 0 \\ \partial_x s(0, t) = 0 \end{cases}$

Dirichlet boundary

Neumann boundary



1D wave equation

Strong form: $\rho \partial_t^2 s = \partial_x (\mu \partial_x s)$

IC & BC: $\begin{cases} s(x, 0) = f(x) \\ s(L, t) = 0 \\ s(0, t) = 0 \end{cases}$ and $\begin{cases} s(x, 0) = f(x) \\ \partial_x s(L, t) = 0 \\ \partial_x s(0, t) = 0 \end{cases}$

Dirichlet boundary

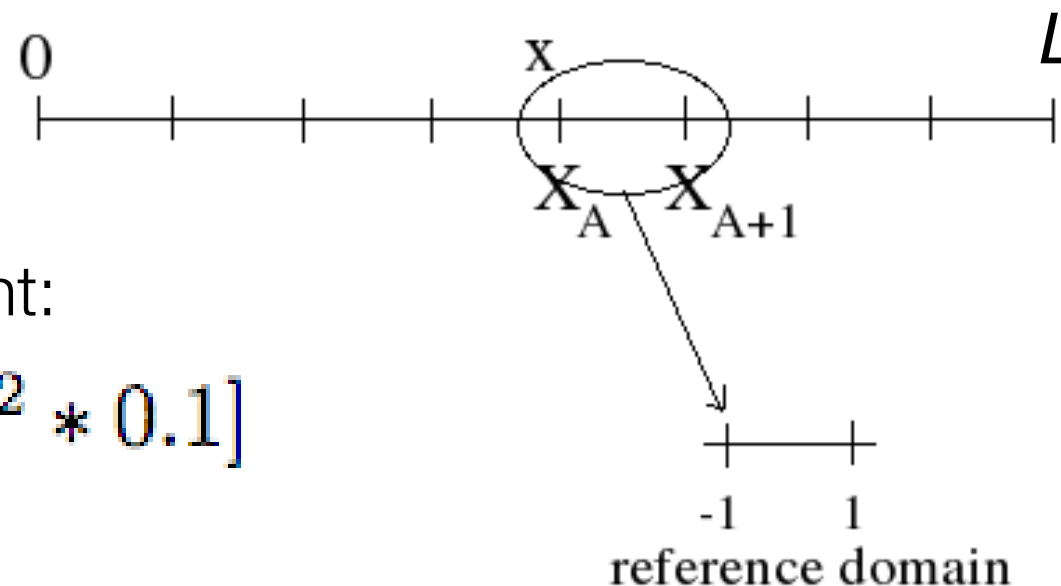
Neumann boundary

string with initial displacement:

$$f(x) = \exp[-(x - 50)^2 * 0.1]$$

string properties:

$$\rho = 1 \ \& \ \mu = 1$$



Weak form

Weak form:
$$\int_0^L \rho w \partial_t^2 s dx = - \int_0^L \mu \partial_x w \partial_x s dx + [w \mu \partial_x s]_0^L$$

displacement field (and test function) expanded on basis functions:

$$s(x(\xi), t) = \sum_{\alpha}^N s^{\alpha}(t) l_{\alpha}^N(\xi)$$

unknowns



Weak form

Weak form: $\int_0^L \rho w \partial_t^2 s dx = - \int_0^L \mu \partial_x w \partial_x s dx + [w \mu \partial_x s]_0^L$

$$\rightarrow \mathbf{M} \partial_t^2 \mathbf{s} = \mathbf{K} \mathbf{s}$$

mass matrix

stiffness matrix

force vector

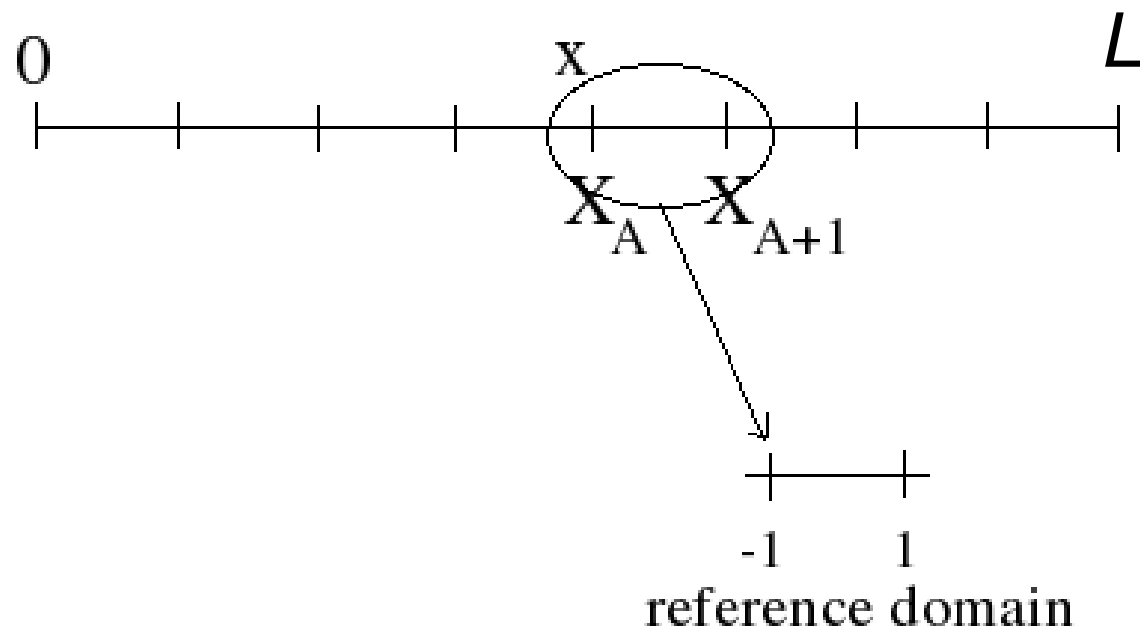
zero



Reference domain

Definition of the reference domain:

Consider the mapping $\xi : [X_A, X_{A+1}] \rightarrow [\xi_1, \xi_2]$, such that



$$\begin{cases} \xi(X_A) &= \xi_1 = -1 \\ \xi(X_{A+1}) &= \xi_2 = 1 \end{cases}$$

$$x(\xi) = \sum_{a=1}^2 X_a N_a(\xi)$$

with shape functions being degree-1 Lagrange polynomials

$$N_a(\xi) = \frac{1}{2}(1 + \xi_a \xi) \quad a=1,2$$

$$\text{Jacobian: } J = \frac{\partial x}{\partial \xi}$$

Reference domain

Interpolation:

$$s(x(\xi), t) = \sum_{\alpha}^N s^{\alpha}(t) l_{\alpha}^N(\xi)$$

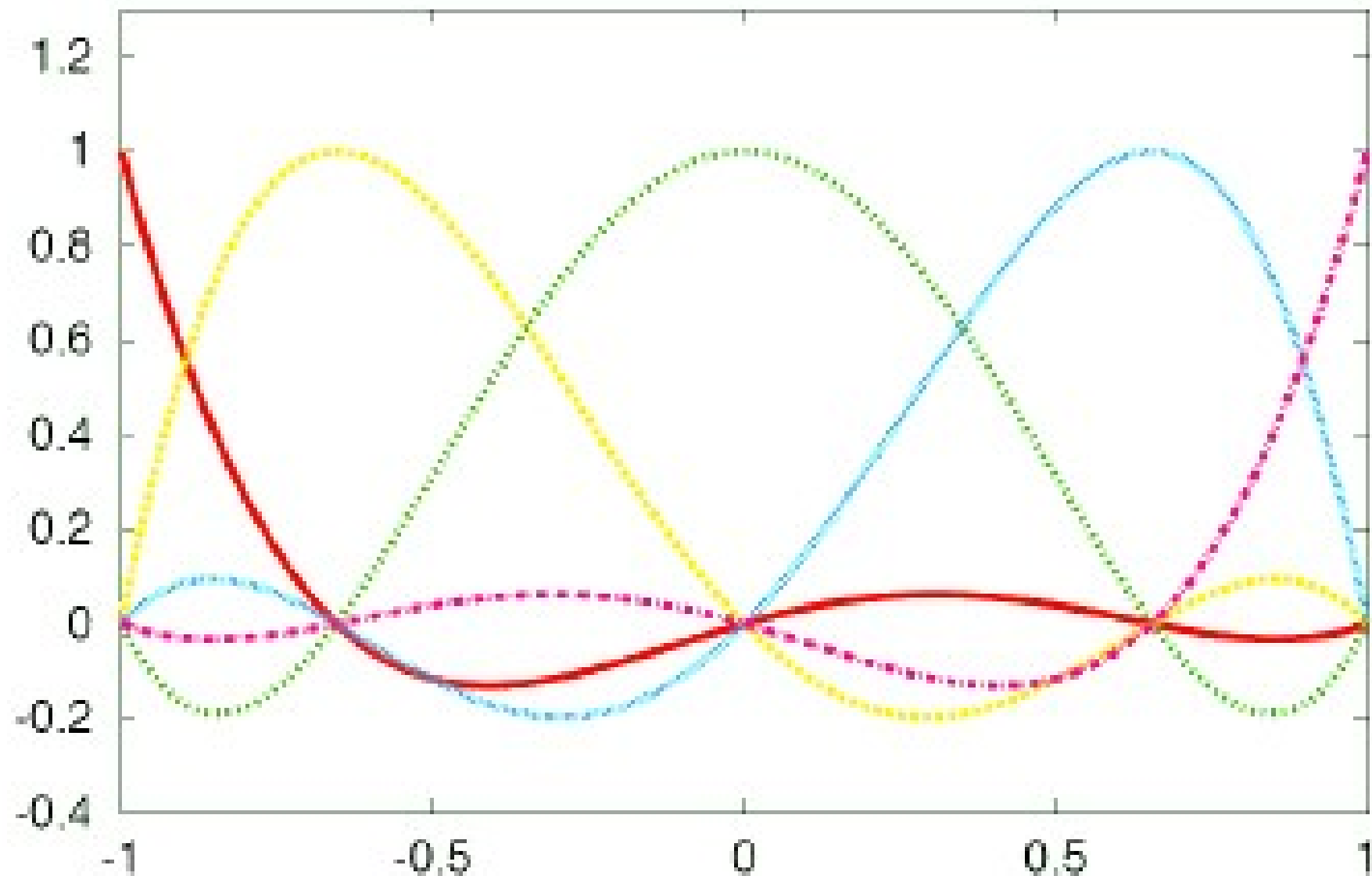
Gauss-Lobatto-Legendre quadrature integration rule:

$$\begin{aligned} \int_{\Omega_e} s(x, t) dx &= \int_{-1}^1 s(x(\xi), t) J^{\alpha} d\xi \\ &\sim \sum_{\alpha}^N \omega_{\alpha} s^{\alpha}(t) J^{\alpha} \end{aligned}$$



Basis functions

Lagrange polynomials:



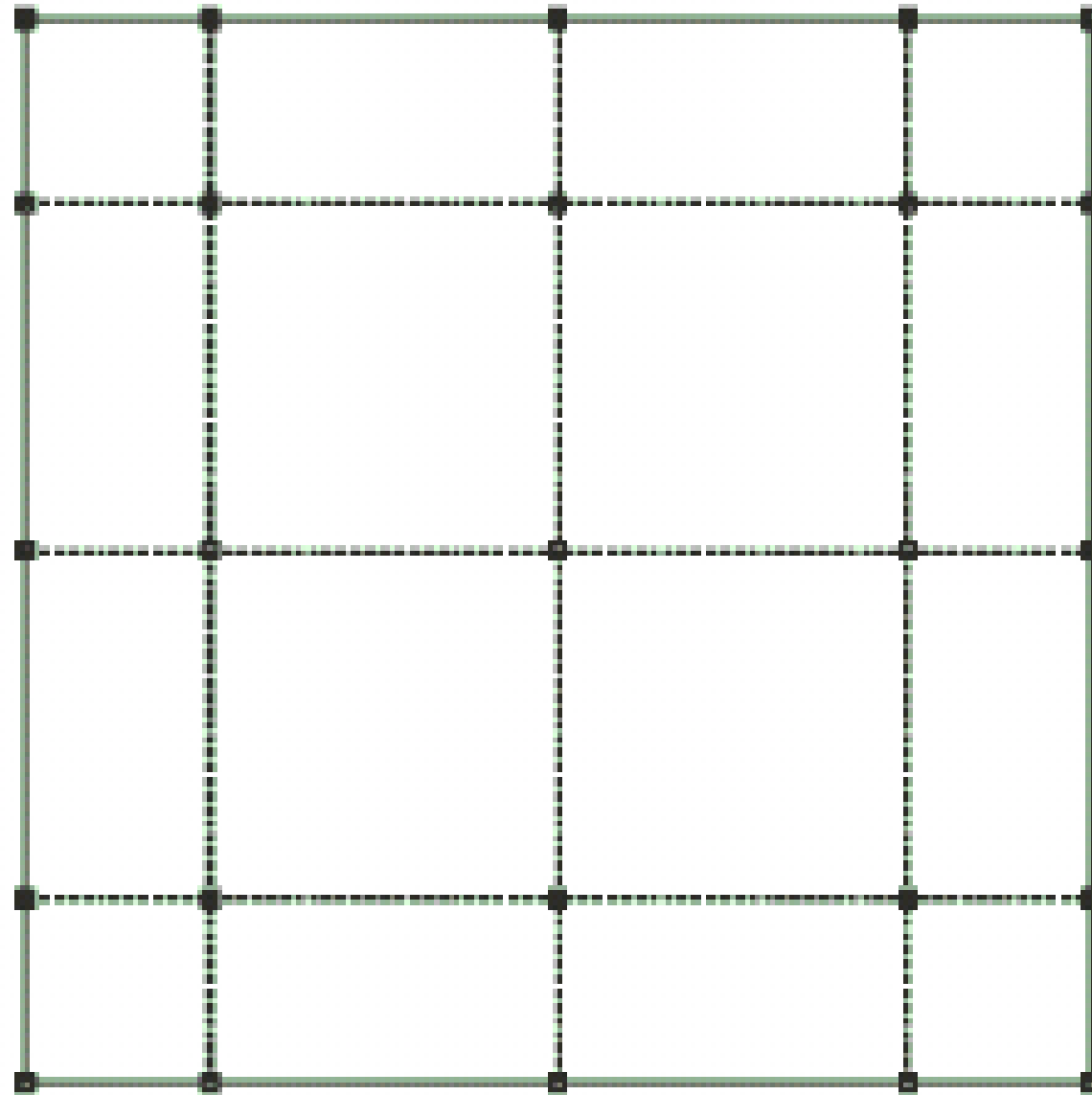
degree-4
polynomials

Lagrange polynomials property: $l_{\alpha}^N(\xi_{\beta}) = \delta_{\alpha\beta}$



Basis functions

Gauss-Lobatto-Legendre points:



degree-4 GLL points
(2D quad example)

GLL points are the $n+1$ roots of $(1 - \xi^2)P'_n(\xi) = 0$

P_n : Legendre polynomial of degree n



Mass matrix

Local (element) resolution: Mass matrix

$$\begin{aligned}\int_{\Omega_e} w \rho \partial_t^2 s dx &= \int_{-1}^1 \rho(x(\xi)) w(x(\xi)) \partial_t^2 s(x(\xi), t) J d\xi \\ &\sim \sum_{\gamma=0}^N \omega_\gamma \rho^\gamma J^\gamma \sum_{\beta}^N w^\beta l_\beta^N(\xi_\gamma) \sum_{\alpha}^N \partial_t^2 s^\alpha l_\alpha^N(\xi_\gamma) \\ &= \sum_{\gamma=0}^N \boxed{\omega_\gamma \rho^\gamma J^\gamma w^\gamma} \partial_t^2 s^\gamma\end{aligned}$$

diagonal matrix



Stiffness matrix

Local (element) resolution: Stiffness matrix

$$\begin{aligned}\int_{\Omega_e} \mu \partial_x w \partial_x s dx &= \int_{-1}^1 \mu(x(\xi)) [\partial_x w(x(\xi))] [\partial_x s(x(\xi), t)] J d\xi \\ &\sim \sum_{\alpha=0}^N \omega_{\alpha} \mu^{\alpha} \left[\sum_{\beta}^N w^{\beta} l'_{\beta}{}^N(\xi_{\alpha}) \partial_x \xi(\xi_{\alpha}) \right] \left[\sum_{\gamma}^N s^{\gamma} l'_{\gamma}{}^N(\xi_{\alpha}) \partial_x \xi(\xi_{\alpha}) \right] J^{\alpha}\end{aligned}$$

$$\Rightarrow M_{\alpha} \partial_t^2 s^{\alpha}(t) = \sum_{\gamma}^N K_{\alpha, \gamma} s^{\gamma}(t)$$

Matricial form:

$$\mathbf{M} \partial_t^2 \mathbf{s} = \mathbf{K} \mathbf{s}$$



Boundaries

Boundary conditions:

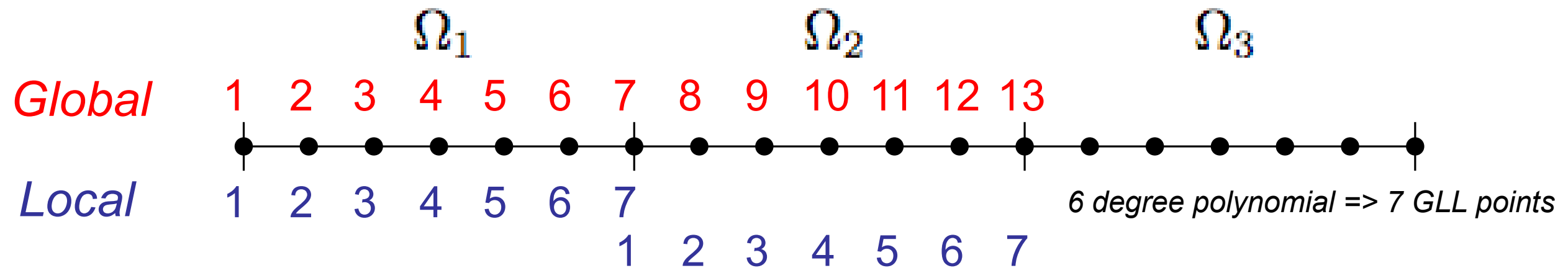
$$w\mu\partial_x s = \omega^N \mu^N \sum_{\alpha=0}^N s^\alpha l'_\alpha{}^N(\xi_N) \partial_x \xi(\xi_N) \quad \text{for } x = 0$$

$$w\mu\partial_x s = \omega^0 \mu^0 \sum_{\alpha=0}^N s^\alpha l'_\alpha{}^N(\xi_0) \partial_x \xi(\xi_0) \quad \text{for } x = L$$

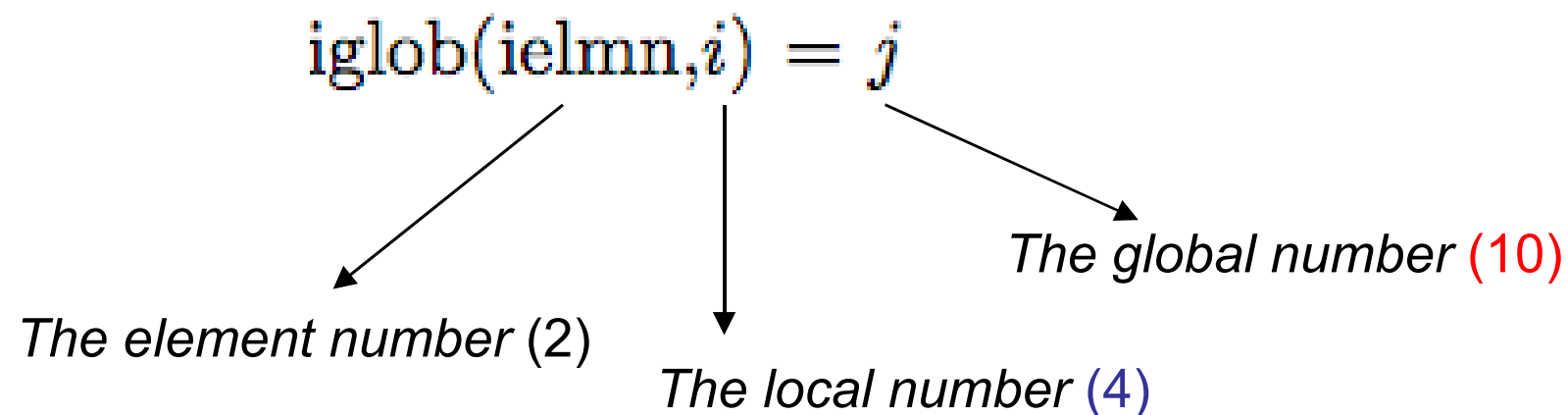


Assembly

Local to Global:



You need an array which links Local (*where the calculation is done*) to Global (*where you want to know the results which are marched in time*):



Assembly

Assembling: back to global level

$$M_{local}(ielmn, i) = \omega_{\alpha} \rho^{\alpha} J^{\alpha}$$

$$M_{global}(:) = 0$$

!loop over the elements

do ielmn=1,Nel

!loop over the GLL points

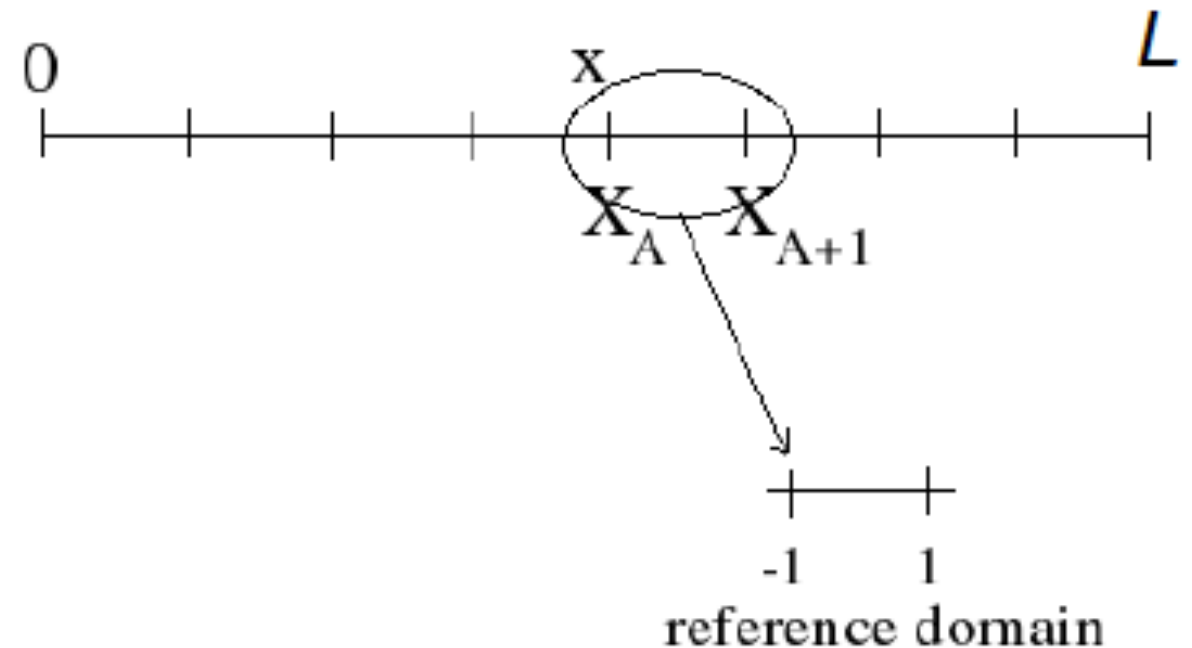
do i=1,NGLL

!get the global index

$$j = \text{iglob}(ielmn, i) \quad M_{global}(j) = M_{global}(j) + M_{local}(ielmn, i)$$

enddo

enddo



Time stepping

Time scheme: Newmark algorithm

- Predictor:

$$d_{n+1} = d_n + \Delta t v_n + \frac{1}{2} \Delta t^2 a_n$$

$$v_{n+1} = v_n + \frac{1}{2} \Delta t a_n$$

$$a_{n+1} = 0 \quad (\text{initialization at the beginning of each time step})$$

- Solve:

$$F_{n+1} = K d_{n+1}$$

$$M \Delta a = F_{n+1}$$

- Corrector:

$$a_{n+1} = a_{n+1} + \Delta a$$

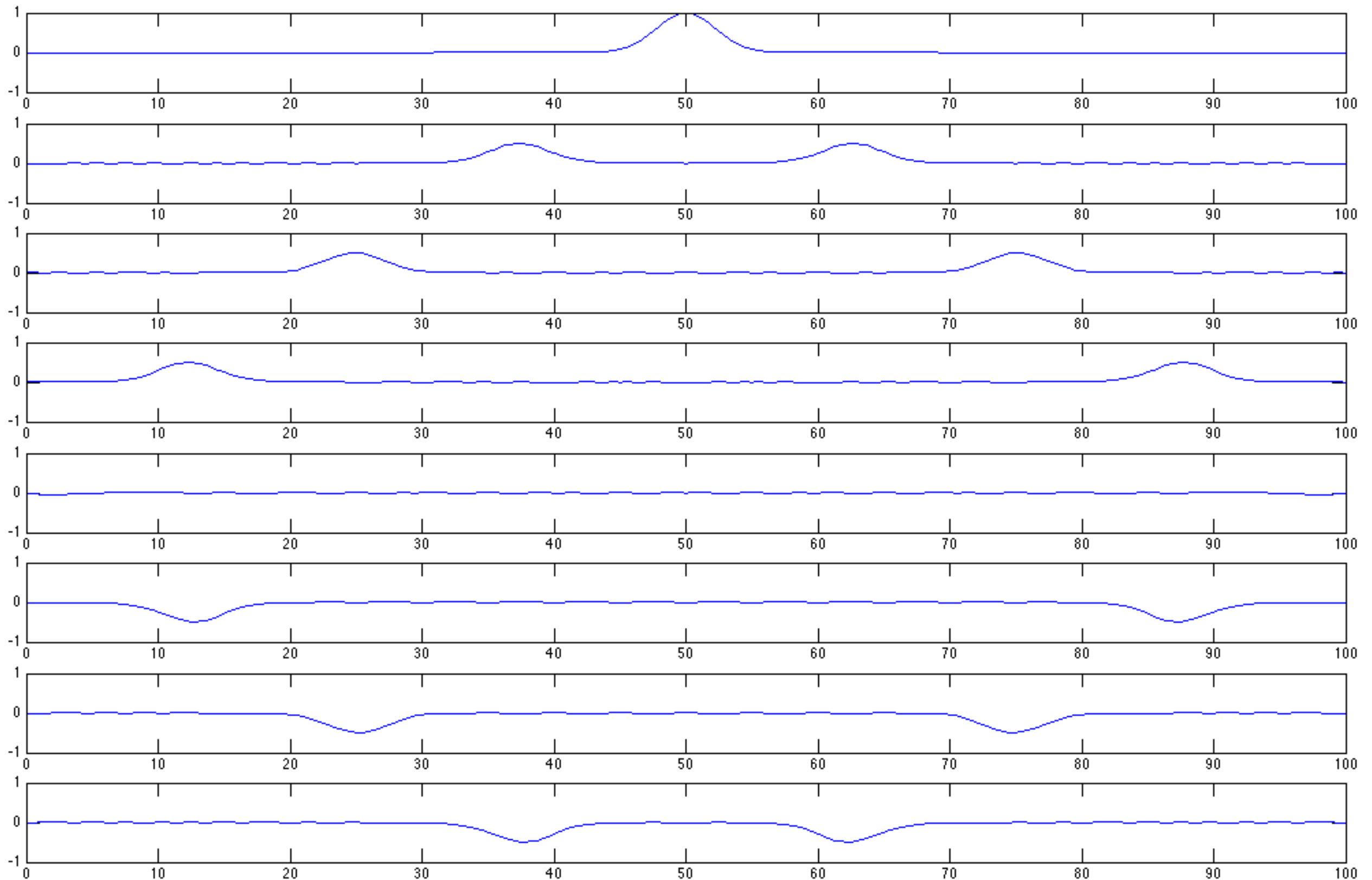
$$v_{n+1} = v_{n+1} + \frac{1}{2} \Delta a$$

$$d_{n+1} = d_{n+1}$$



SEM - 1D unsteady-state diffusion equation

Results: (a) *Dirichlet boundary*



SEM - 1D unsteady-state diffusion equation

Results: (b) *Neumann boundary*

