Spectral-element method

1D wave equation

1D wave equation

Strong form: $\rho \partial_t^2 s = \partial_x (\mu \partial_x s)$

IC & BC: $\left\{ \begin{array}{rcl} s(x,0) & = & f(x) \\ s(L,t) & = & 0 \\ s(0,t) & = & 0 \end{array} \right.$ and $\left\{ \begin{array}{rcl} s(x,0) & = & f(x) \\ \partial_x s(L,t) & = & 0 \\ \partial_x s(0,t) & = & 0 \end{array} \right.$

Dirichlet boundary   Neumann boundary

1D wave equation

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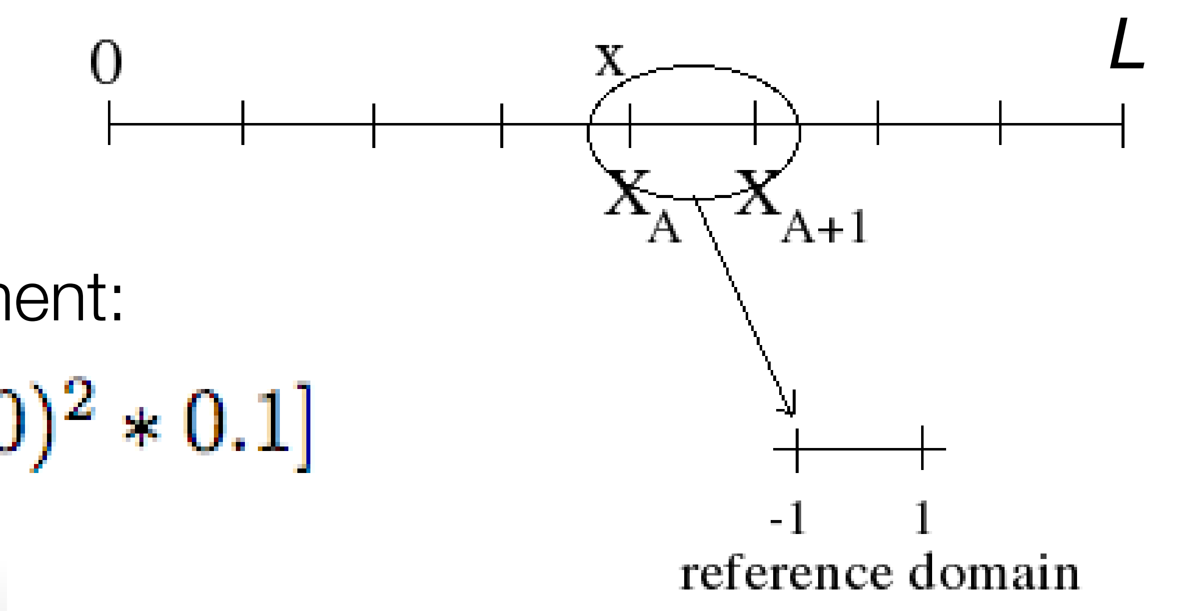
Dirichlet boundary

Neumann boundary



string with initial displacement:

$$f(x) = \exp[-(x-50)^2 * 0.1]$$

string properties:

$$\rho = 1 \; \& \; \mu = 1$$

Weak form

Weak form: $$\int_0^L \rho w \partial_t^2 s \mathrm{d}x = -\int_0^L \mu \partial_x w \partial_x s \mathrm{d}x + [w\mu\partial_x s]_0^L$$

displacement field (and test function) expanded on basis functions:

$$s(x(\xi), t) = \sum_{\alpha}^{N} s^{\alpha}(t) l_{\alpha}^{N}(\xi)$$

unknowns

Weak form

Weak form: $$\int_0^L \rho w \partial_t^2 s \mathrm{d}x = -\int_0^L \mu \partial_x w \partial_x s \mathrm{d}x + [w\mu\partial_x s]_0^L$$

$$\rightarrow \quad \mathbf{M}\partial_t^2\mathbf{s} = \mathbf{K}\mathbf{s}$$

mass matrix

stiffness matrix

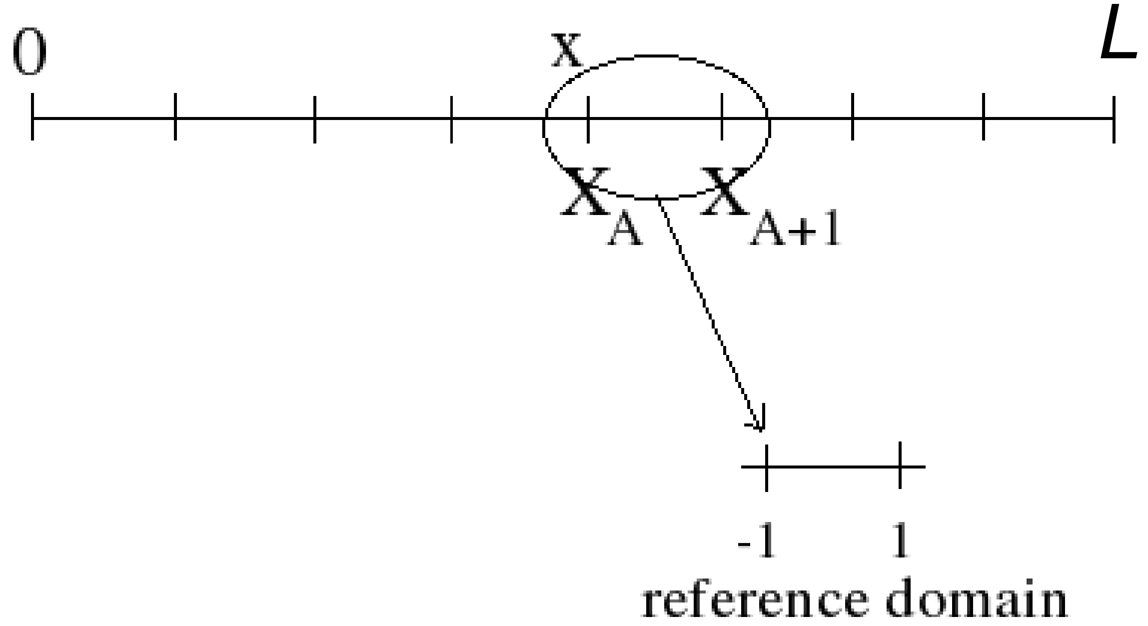
zero

force vector

Reference domain

Definition of the reference domain:

Consider the mapping $\xi : [X_A, X_{A+1}] \to [\xi_1, \xi_2]$, such that



$$\begin{cases} \xi(X_A) & = \xi_1 = -1 \\ \xi(X_{A+1}) & = \xi_2 = 1 \end{cases}$$

$$x(\xi) = \sum_{a=1}^{2} X_a N_a(\xi)$$

with shape functions being degree-1 Lagrange polynomials

$$N_a(\xi) = \frac{1}{2}(1 + \xi_a \xi) \quad \text{a=1,2}$$

Jacobian: $J = \dfrac{\partial x}{\partial \xi}$

Reference domain

Interpolation:

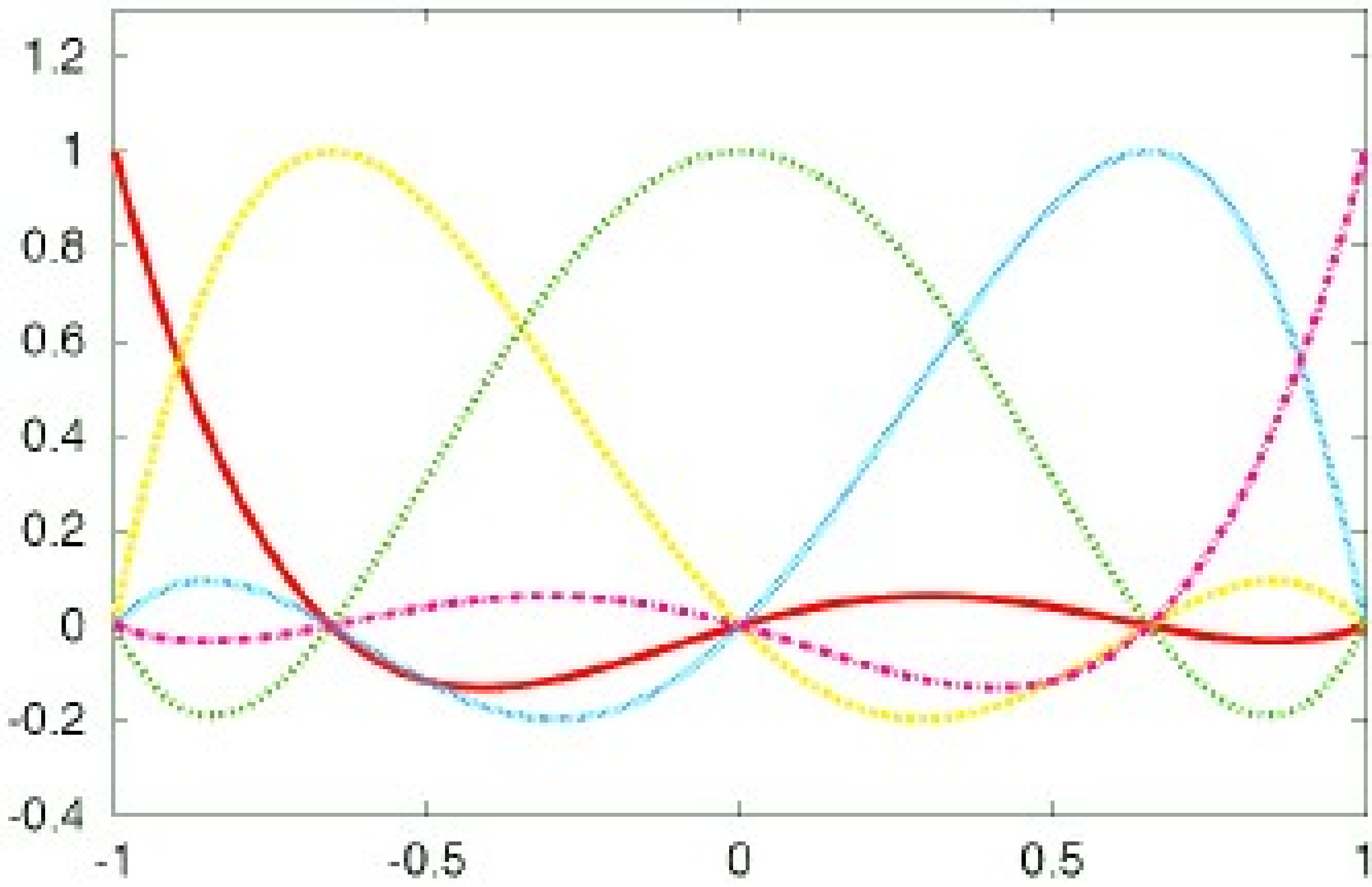
$$s(x(\xi),t) = \sum_{\alpha}^{N} s^{\alpha}(t) l_{\alpha}^{N}(\xi)$$

Gauss-Lobatto-Legendre quadrature integration rule:

$$\begin{aligned}\int_{\Omega_e} s(x,t)\mathrm{d}x &= \int_{-1}^{1} s(x(\xi),t) J^{\alpha} \mathrm{d}\xi \\ &\sim \sum_{\alpha}^{N} \omega_{\alpha} s^{\alpha}(t) J^{\alpha}\end{aligned}$$

Basis functions

Lagrange polynomials:

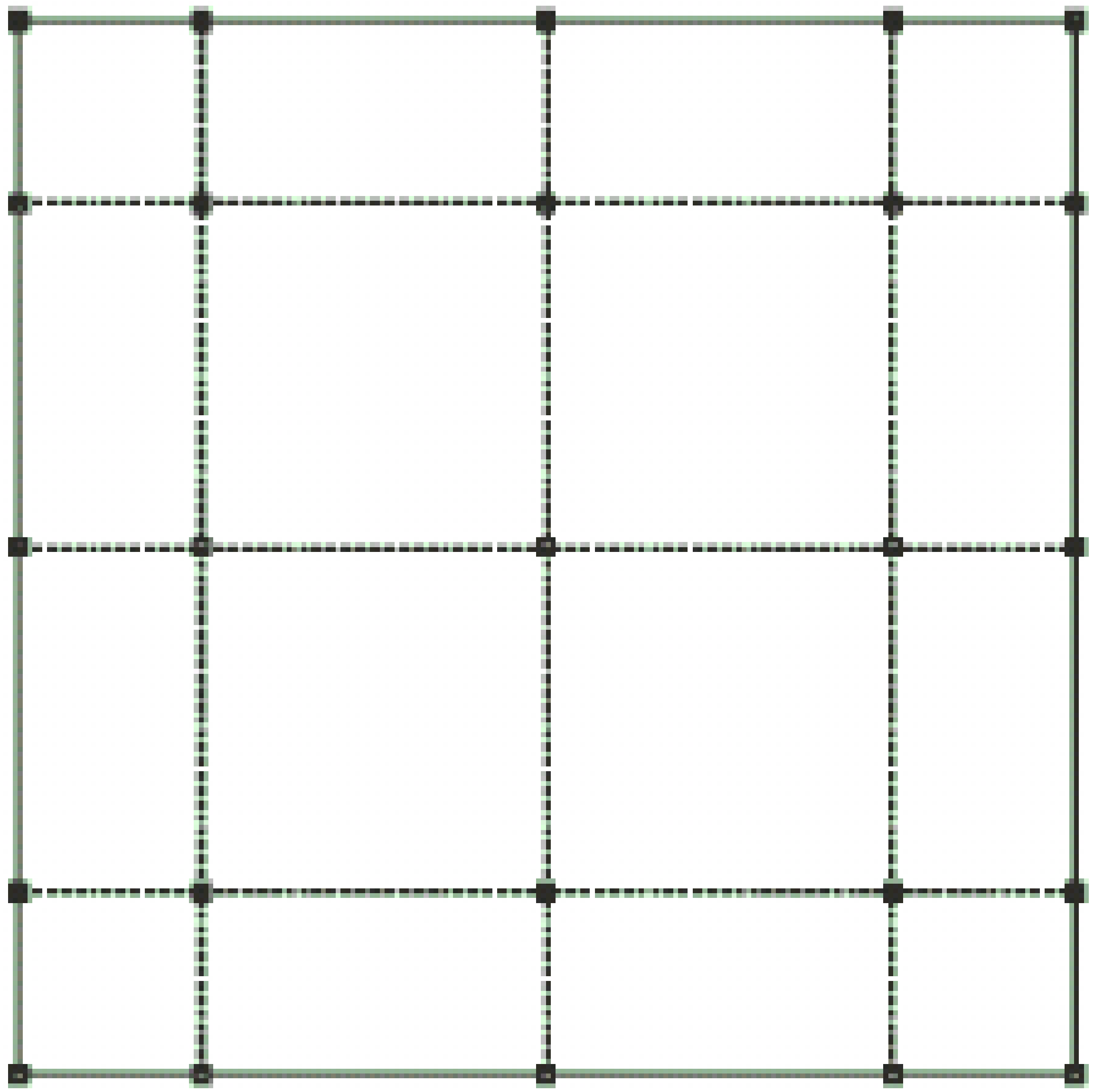


degree-4 polynomials

Lagrange polynomials property: $l_{\alpha}^{N}(\xi_{\beta}) = \delta_{\alpha\beta}$

Basis functions

Gauss-Lobatto-Legendre points:



degree-4 GLL points
(2D quad example)

GLL points are the n+1 roots of $(1-\xi^2)P_n'(\xi) = 0$

P_n : Legendre polynomial of degree n

Mass matrix

Local (element) resolution: Mass matrix

$$
\begin{aligned}
\int_{\Omega_e} w\rho\partial_t^2 s\,\mathrm{d}x &= \int_{-1}^{1} \rho(x(\xi))w(x(\xi))\partial_t^2 s(x(\xi),t)J\mathrm{d}\xi \\
&\sim \sum_{\gamma=0}^{N}\omega_\gamma\rho^\gamma J^\gamma \sum_{\beta}^{N} w^\beta l_\beta^N(\xi_\gamma)\sum_{\alpha}^{N} \partial_t^2 s^\alpha l_\alpha^N(\xi_\gamma) \\
&= \sum_{\gamma=0}^{N} \boxed{\omega_\gamma\rho^\gamma J^\gamma w^\gamma}\,\partial_t^2 s^\gamma
\end{aligned}
$$

diagonal matrix

Stiffness matrix

Local (element) resolution: Stiffness matrix

$$\int_{\Omega_e} \mu \partial_x w \partial_x s \mathrm{d}x = \int_{-1}^{1} \mu(x(\xi))[\partial_x w(x(\xi))][\partial_x s(x(\xi),t)] J \mathrm{d}\xi$$

$$\sim \sum_{\alpha=0}^{N} \boxed{\omega_\alpha \mu^\alpha [\sum_\beta^N w^\beta l'^N_\beta(\xi_\alpha)\partial_x \xi(\xi_\alpha)][\sum_\gamma^N s^\gamma l'^N_\gamma(\xi_\alpha)\partial_x \xi(\xi_\alpha)] J^\alpha}$$

=> $M_\alpha \partial_t^2 s^\alpha(t) = \sum_\gamma^N K_{\alpha,\gamma} s^\gamma(t)$

Matricial form:

$$\mathbf{M}\partial_t^2 \mathbf{s} = \mathbf{K}\mathbf{s}$$

Boundaries

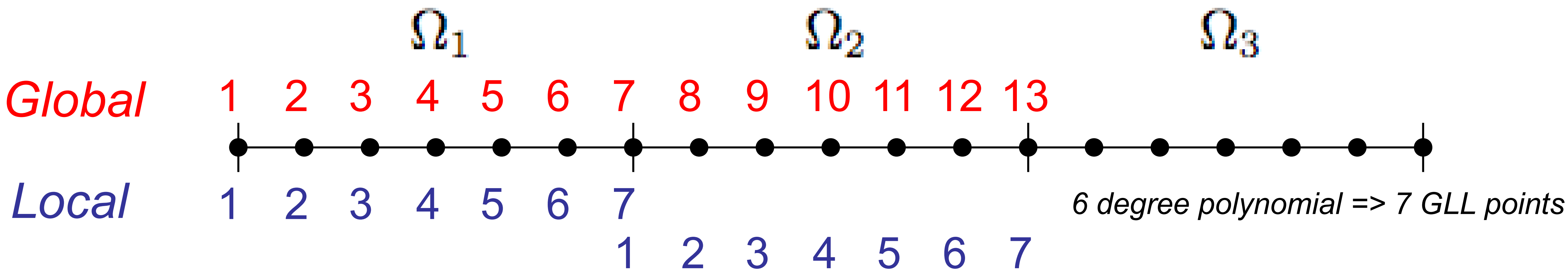
Boundary conditions:

$$w\mu\partial_x s = \omega^N \mu^N \sum_{\alpha=0}^{N} s^\alpha l_\alpha'^N(\xi_N)\partial_x \xi(\xi_N) \quad \text{for } x = 0$$

$$w\mu\partial_x s = \omega^0 \mu^0 \sum_{\alpha=0}^{N} s^\alpha l_\alpha'^N(\xi_0)\partial_x \xi(\xi_0) \quad \text{for } x = L$$

Assembly

Local to Global:



You need an array which links Local *(where the calculation is done)* to Global *(where you want to know the results which are marched in time):*

$$\mathrm{iglob}(\mathrm{ielmn}, i) = j$$

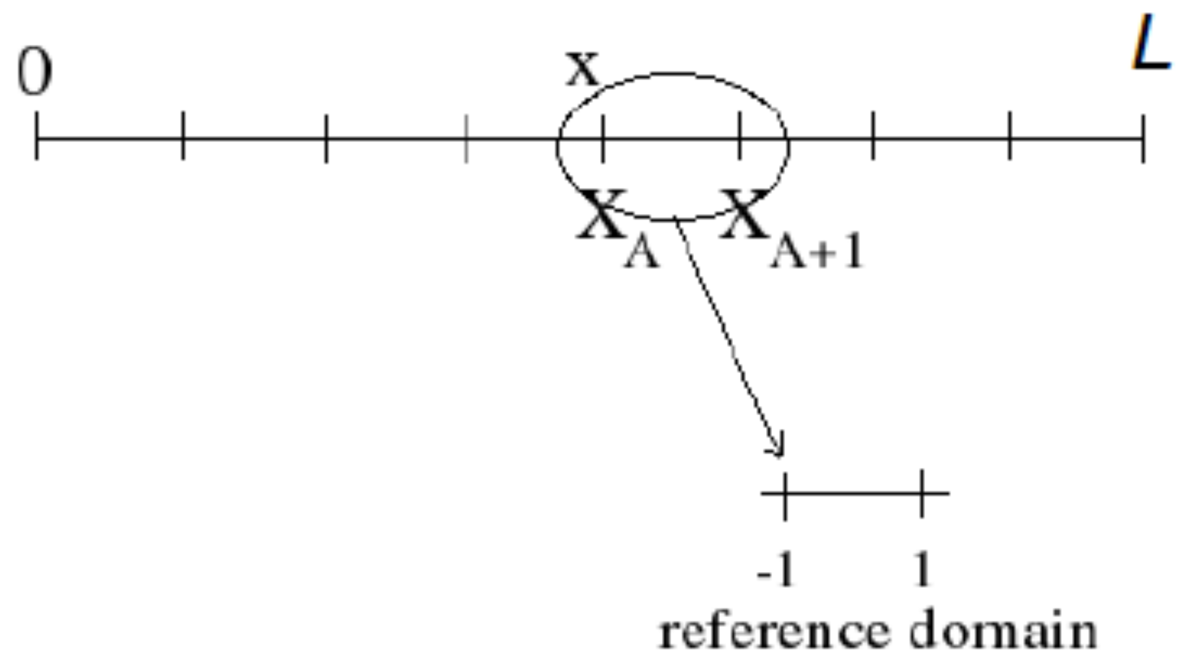
The element number (2)

The local number (4)

The global number (10)

Assembly

Assembling: back to global level



$M_{local}(\text{ielmn}, i) = \omega_\alpha \rho^\alpha J^\alpha$
$M_{global}(:) = 0$
!loop over the elements
do ielmn=1,Nel
!loop over the GLL points
    do i=1,NGLL
!get the global index
    j=iglob(ielmn,i) $M_{global}(j) = M_{global}(j) + M_{local}(\text{ielmn}, i)$
    enddo
enddo

Time stepping

Time scheme: Newmark algorithm

- Predictor:

$$
\begin{aligned}
d_{n+1} &= d_n + \Delta t v_n + \frac{1}{2}\Delta t^2 a_n \\
v_{n+1} &= v_n + \frac{1}{2}\Delta t a_n \\
a_{n+1} &= 0 \quad \text{(initialization at the beginning of each time step)}
\end{aligned}
$$

- Solve:

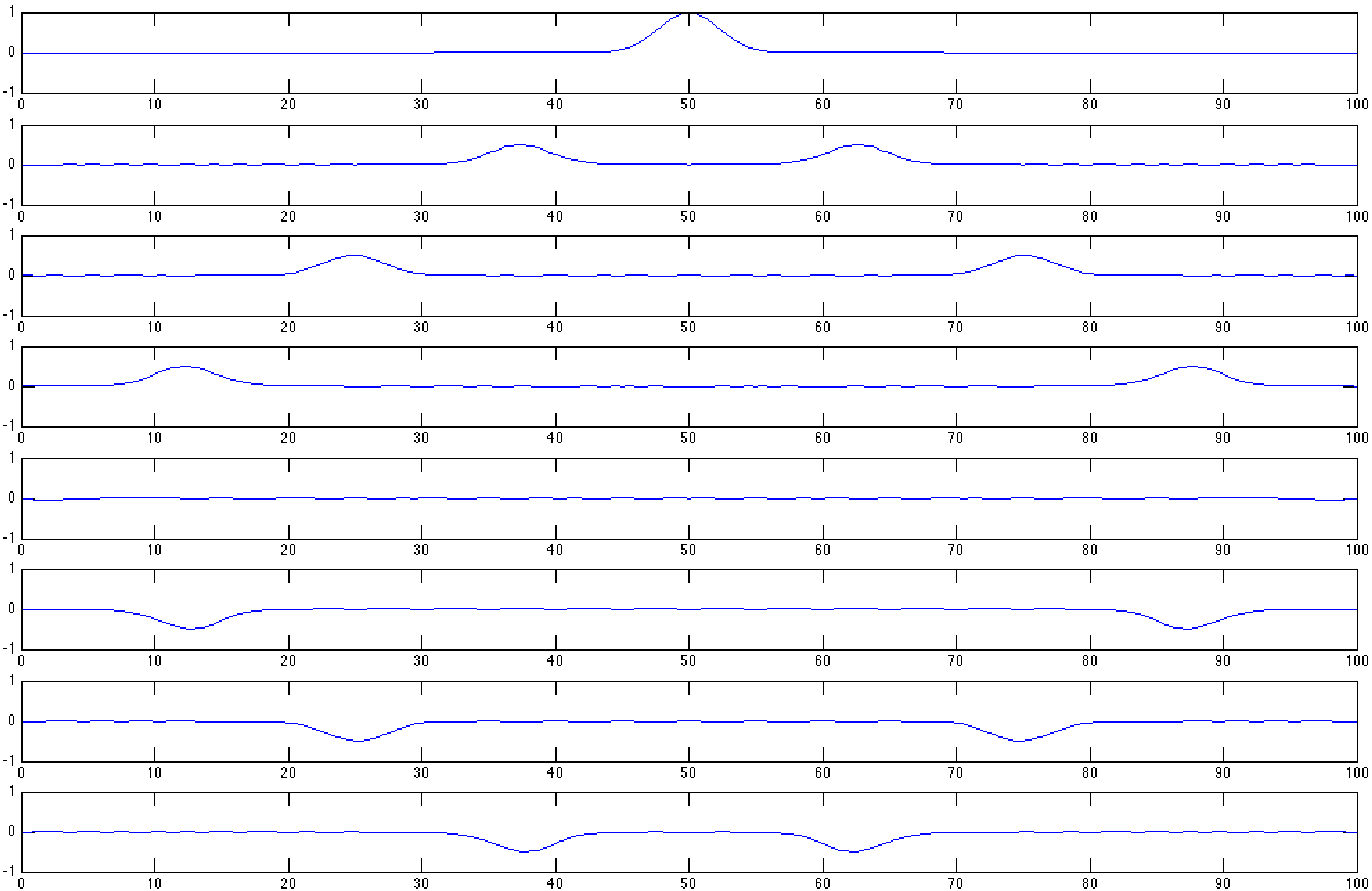
$$
\begin{aligned}
F_{n+1} &= K d_{n+1} \\
M \Delta a &= F_{n+1}
\end{aligned}
$$

- Corrector:

$$
\begin{aligned}
a_{n+1} &= a_{n+1} + \Delta a \\
v_{n+1} &= v_{n+1} + \frac{1}{2} a_{n+1} \\
d_{n+1} &= d_{n+1}
\end{aligned}
$$

SEM - 1D unsteady-state diffusion equation

Results: *(a) Dirichlet boundary*



SEM - 1D unsteady-state diffusion equation

Results: *(b) Neumann boundary*