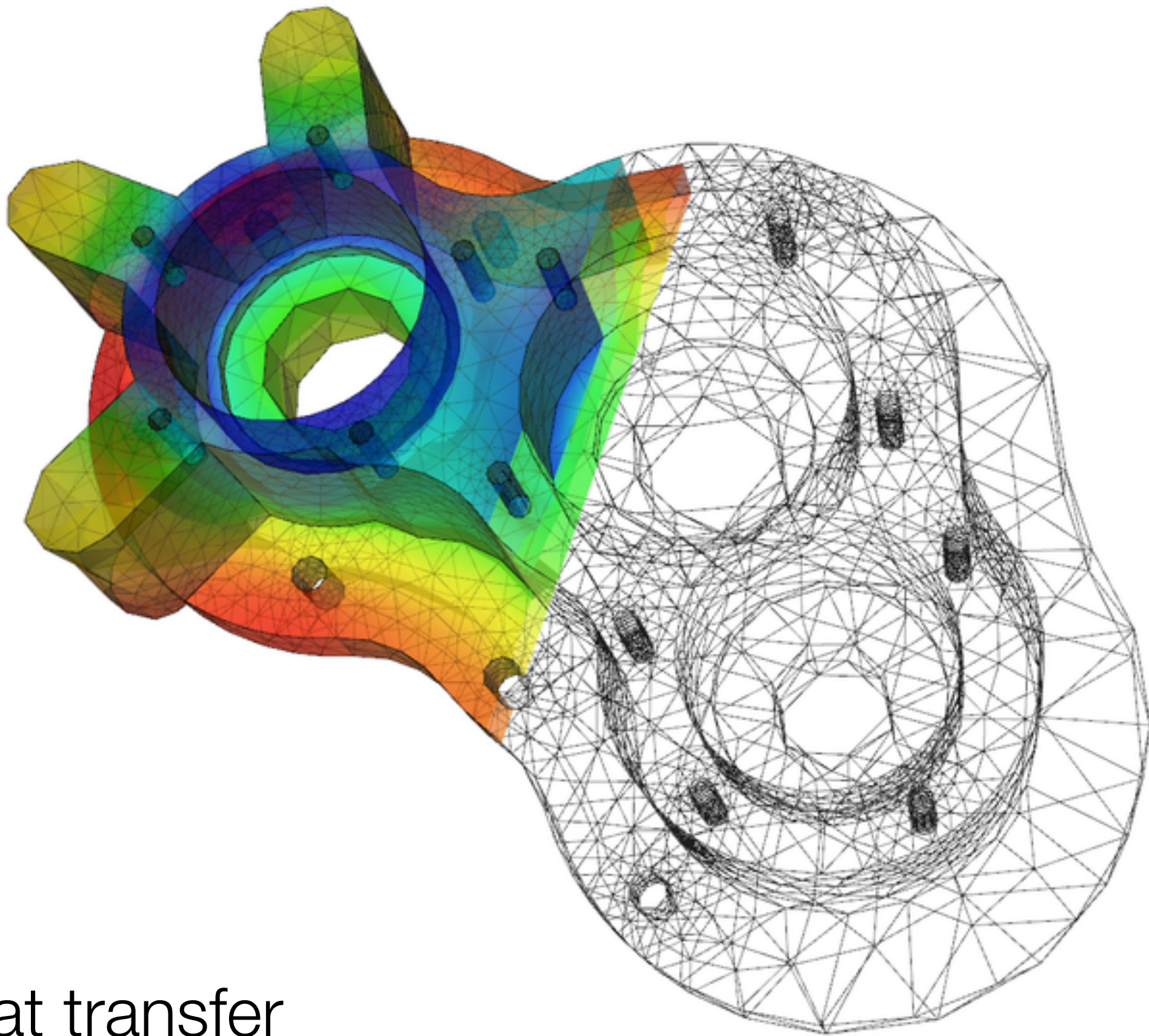


Finite-element methods



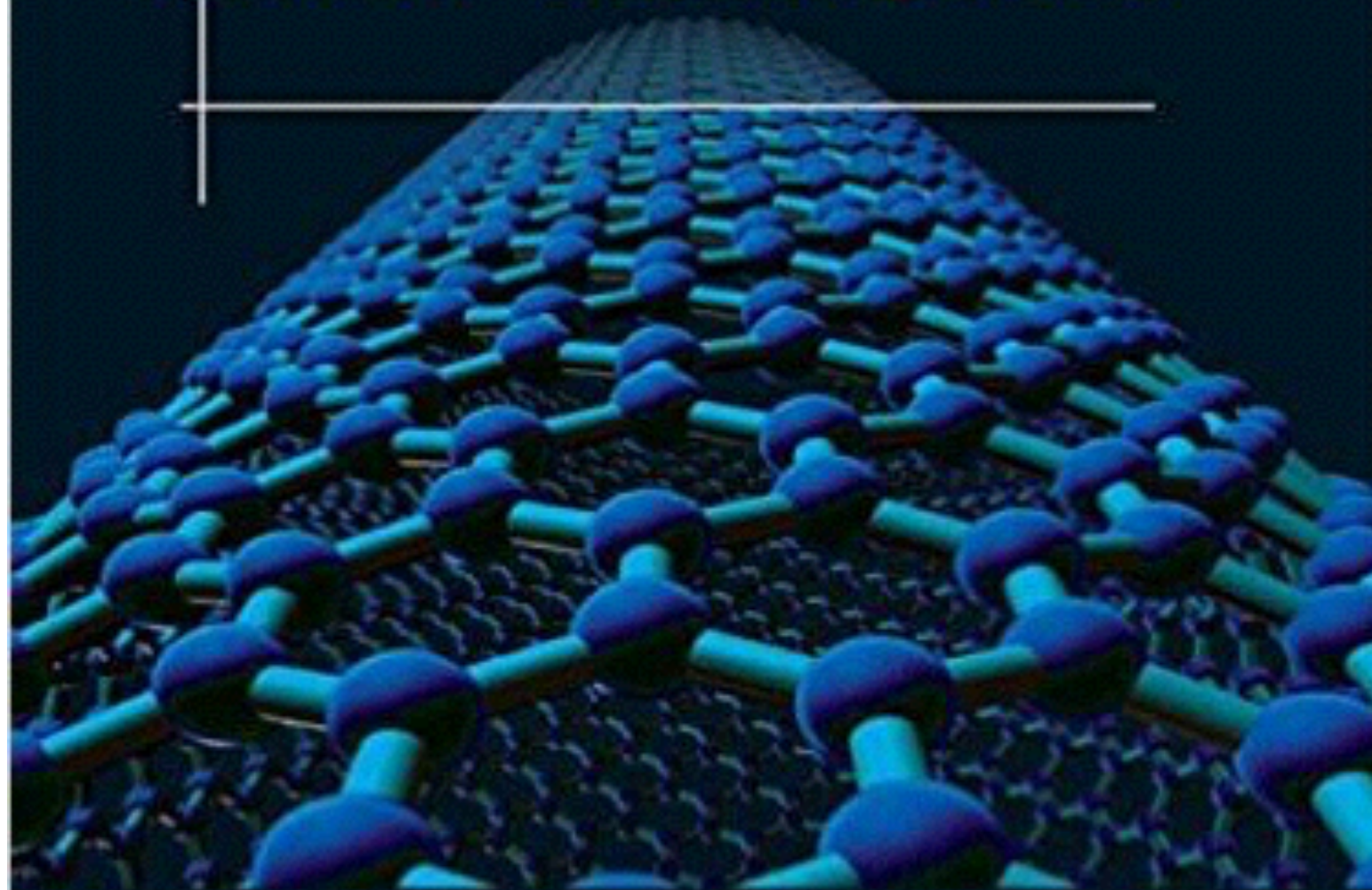


Heat transfer
civil engineering



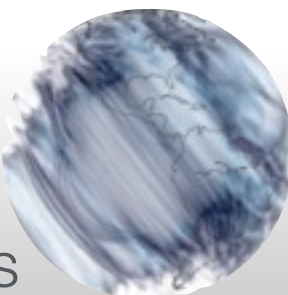
McGraw-Hill Nanoscience and Technology Series

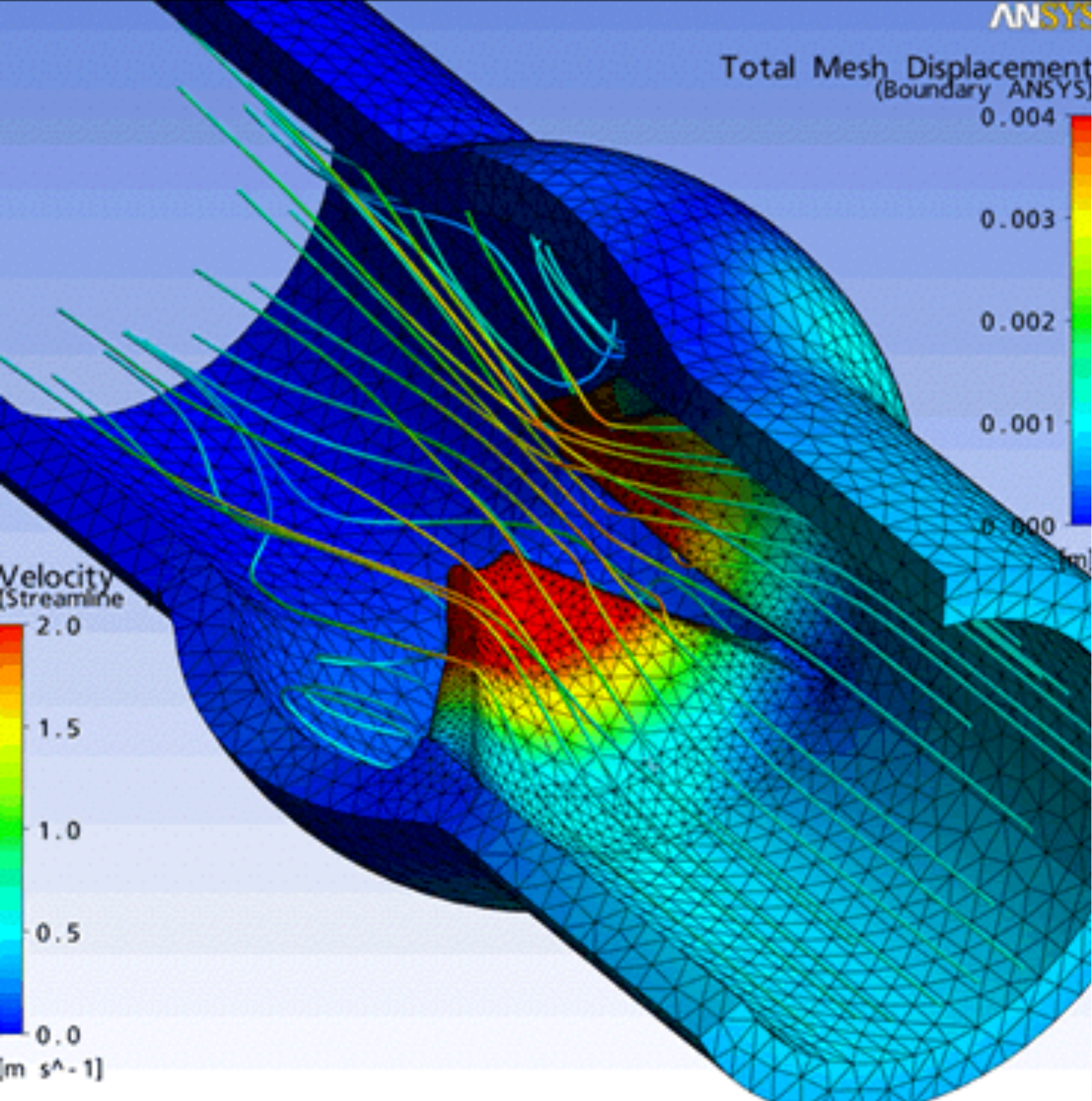
NANO / MICROSCALE HEAT TRANSFER



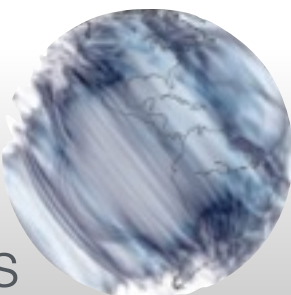
ZHUOMIN M. ZHANG

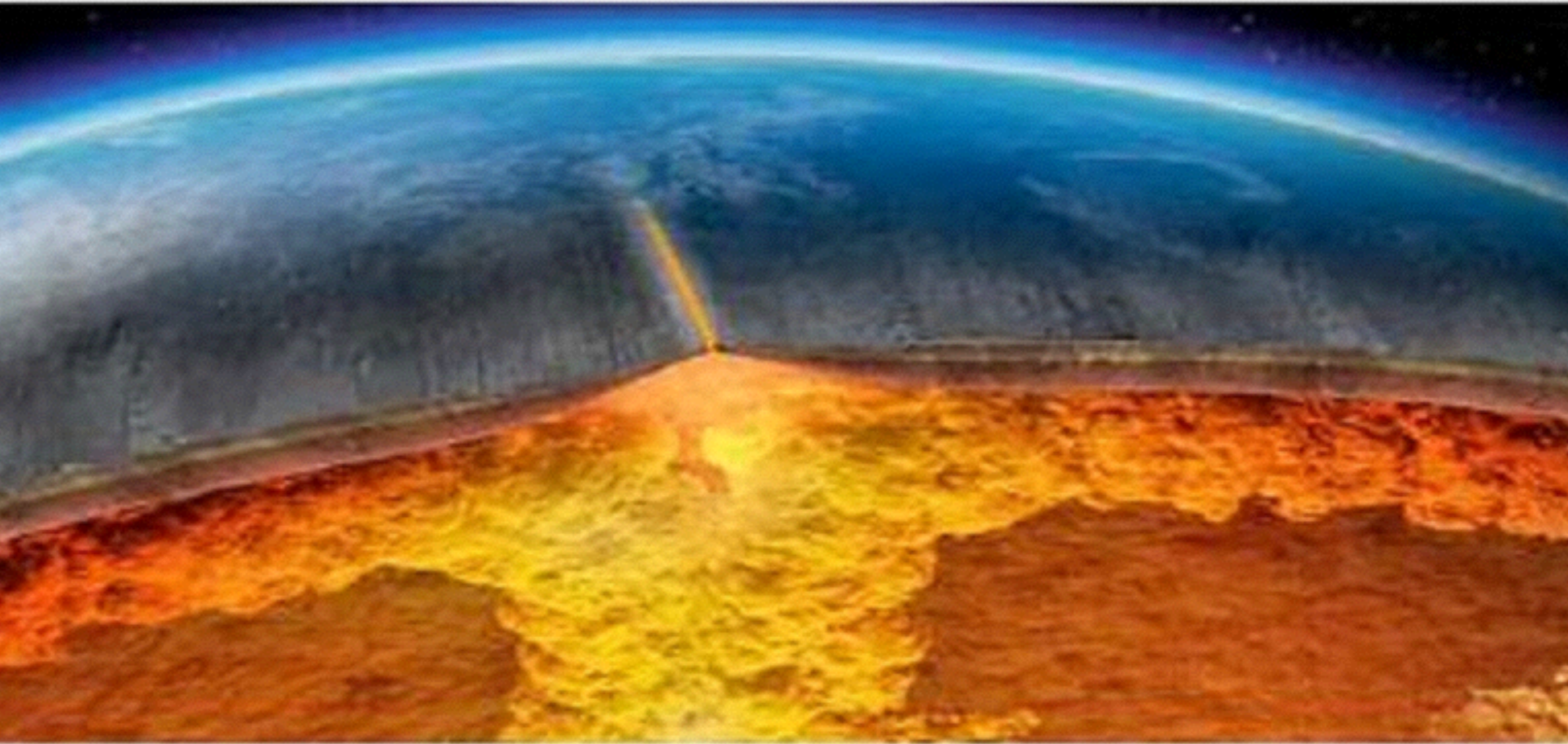
Heat transfer
nano-scales





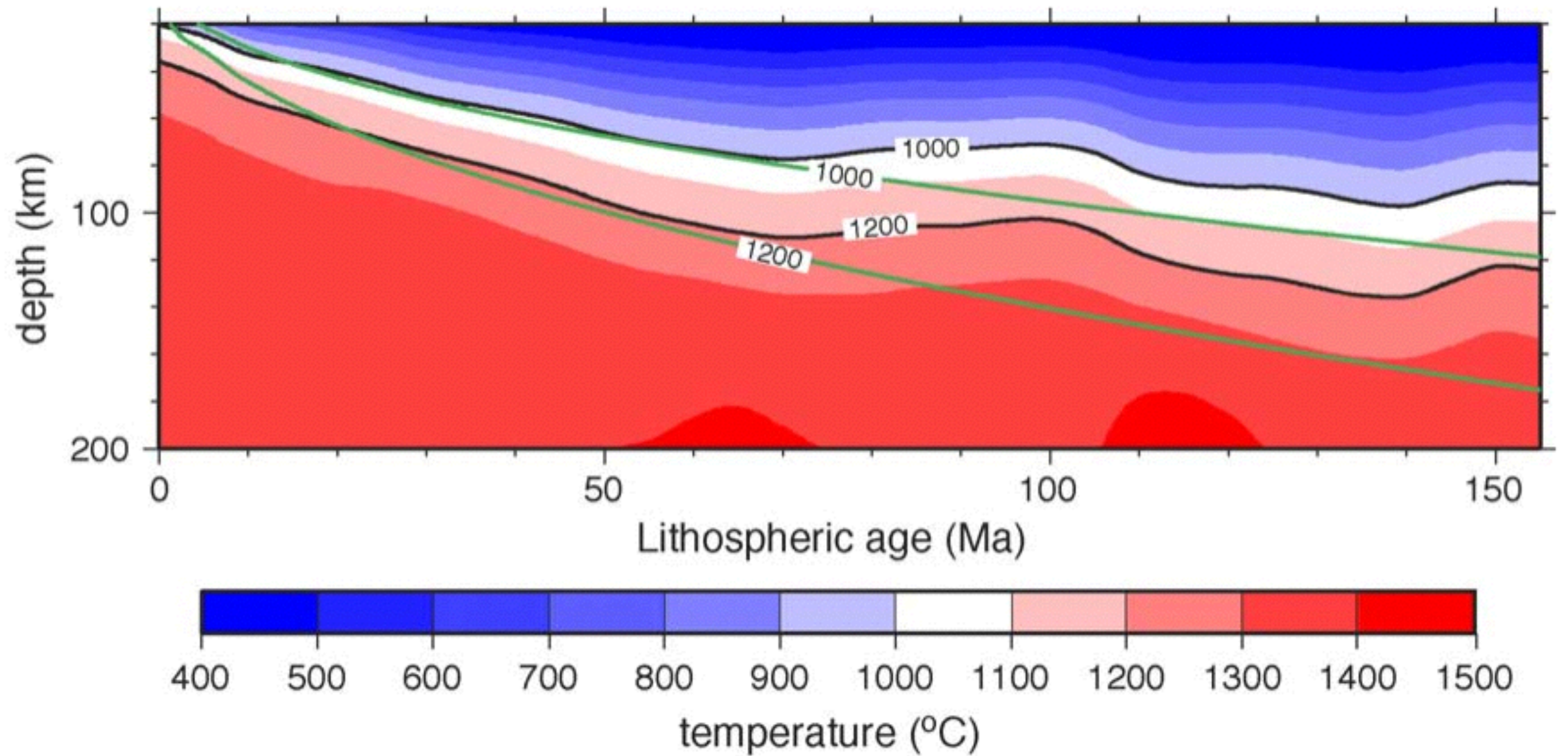
Heat transfer
biological tissue



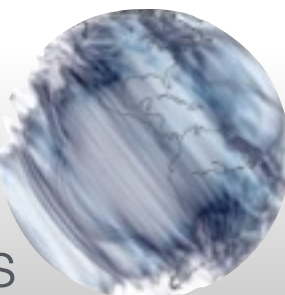


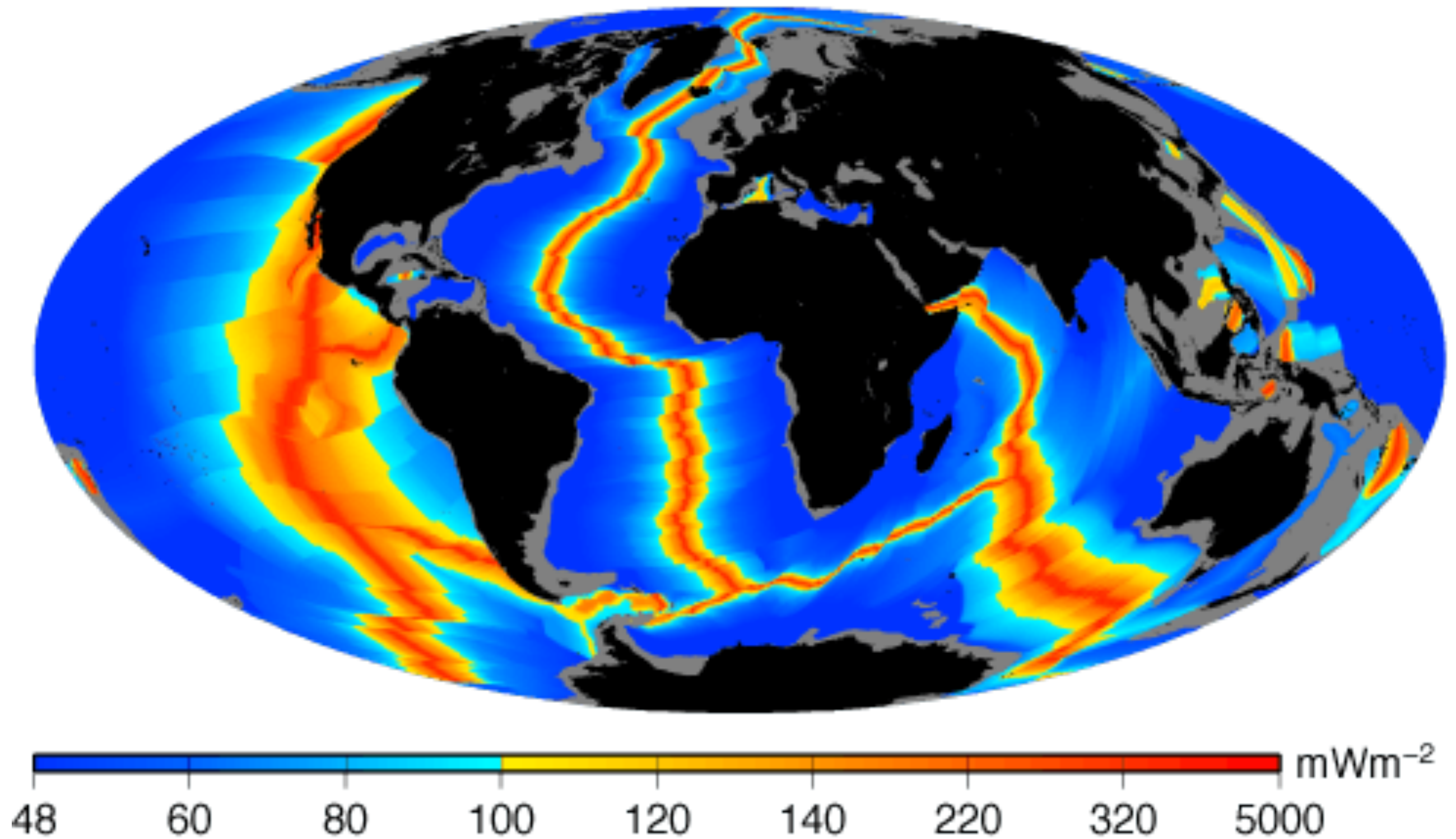
Heat transfer
half-space cooling





debate: half-space cooling of the Pacific plate
Ritzwoller et al. (EPSL, 2004)



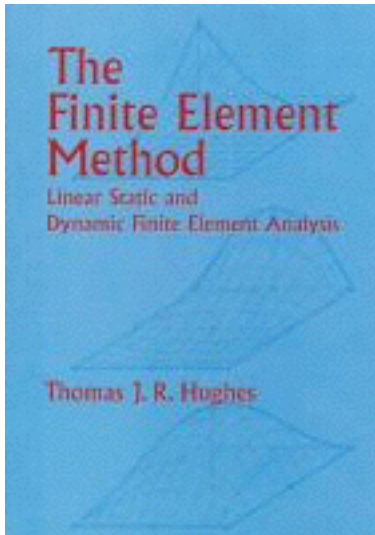


debate: geothermal heat transfer into the oceans
Labrosse (2009)



Finite-element literature

books



Thomas J.R. Hughes.

The Finite Element Method : linear static and dynamic finite element analysis,
Mineola, N.Y. : Dover, 2000.



O. C. Zienkiewicz, R. L. Taylor and J. Z. Zhu.

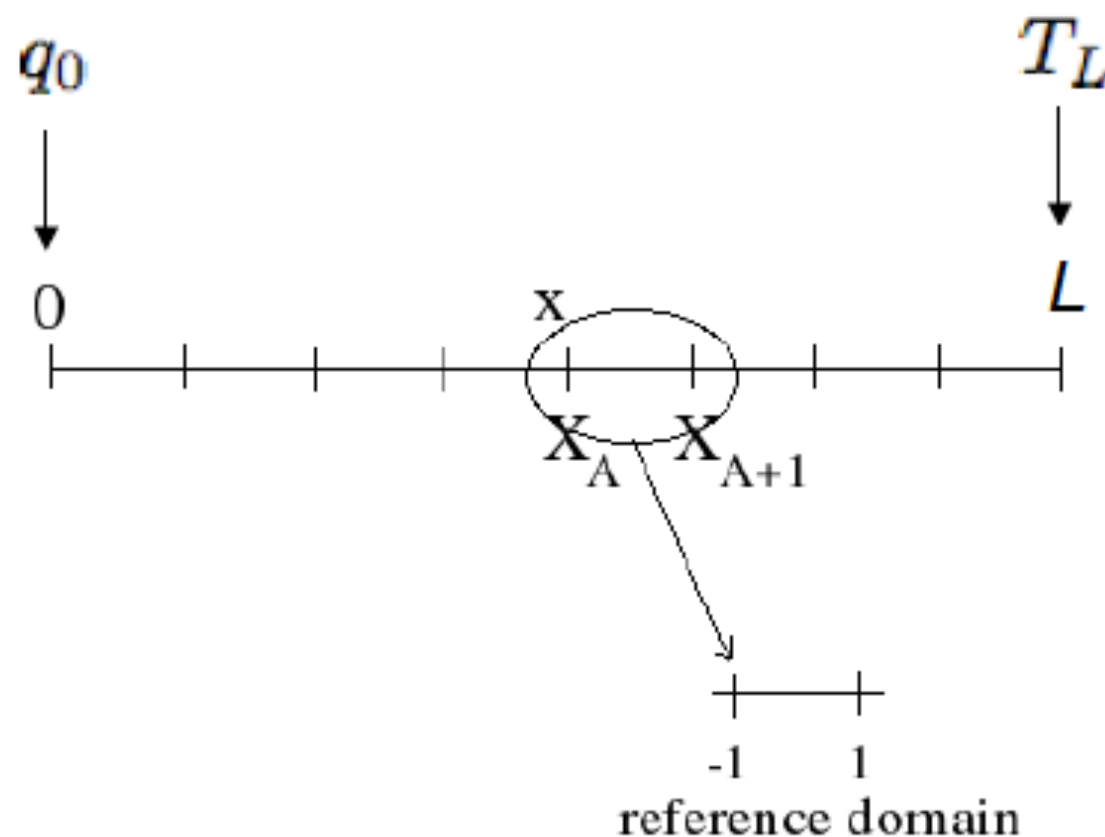
The Finite Element Method : its basis and fundamentals,
ISBN: 978-1-85617-633-0.



1D unsteady-state diffusion equation

Strong form: $\rho c_p \partial_t T - \partial_x (\kappa \partial_x T) = f$

Boundary conditions:
$$\begin{cases} T(L, t) &= T_L \\ -\kappa \partial_x T(0, t) &= q_0 \\ T(x, 0) &= T_0(x) \end{cases}$$



FEM - 1D unsteady-state diffusion equation

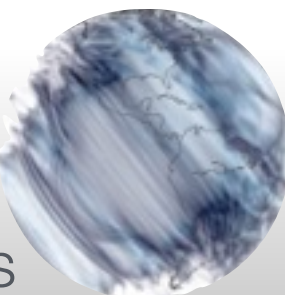
Weak form:
$$\int_0^L \rho c_p w \partial_t T dx = - \int_0^L \kappa \partial_x w \partial_x T dx + q_0 w(0) + \int_0^L w f dx$$

Test function and temperature field expanded on some basis functions:

$$w(x) = \sum_{A=1}^N c_A N_A(x)$$

$$T(x) = \sum_{A=1}^{N_{el}} d_A N_A(x) + T_1 N_{n+1}(x)$$

unknown



FEM - 1D unsteady-state diffusion equation

Weak form: $\int_0^L \rho c_p w \partial_t T dx = - \int_0^L \kappa \partial_x w \partial_x T dx + q_0 w(0) + \int_0^L w f dx$

We are solving: $\mathbf{M}\dot{\mathbf{d}} + \mathbf{K}\mathbf{d} = \mathbf{F}$

capacity matrix

stiffness matrix

force vector



FEM - 1D unsteady-state diffusion equation

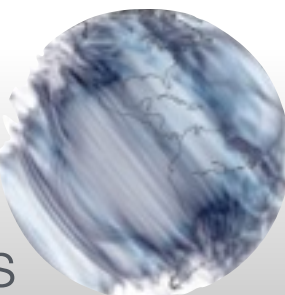
Global level:

$$\begin{aligned} M_{AB} &= (N_A, \rho c_p N_B) \\ &= \int_0^L \rho c_p N_A N_B dx \end{aligned}$$

$$K_{AB} = a(N_A, N_B) = \int_0^L \kappa \partial_x N_A \partial_x N_B dx$$

and

$$\begin{aligned} F_A &= (N_A, f) + N_A(0)q_0 - a(N_A, N_{n+1})T_L \\ &= \int_0^L N_A f dx + N_A(0)q_0 - \int_0^L \partial_x N_A \partial_x N_{n+1} dx \end{aligned}$$



FEM - 1D unsteady-state diffusion equation

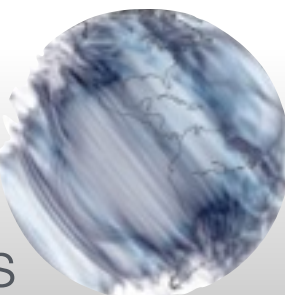
Global level:

$$M_{AB} = \int_0^L \rho c_p N_A N_B dx$$
$$K_{AB} = \int_0^L \kappa \partial_x N_A \partial_x N_B dx$$
$$F_A = \int_0^L N_A f dx + N_A(0) q_0 - \int_0^L \partial_x N_A \partial_x N_{n+1} dx$$

→ $\mathbf{M}\dot{\mathbf{d}} + \mathbf{K}\mathbf{d} = \mathbf{F}$ where $\dot{\mathbf{d}} = \partial_t \mathbf{d}$

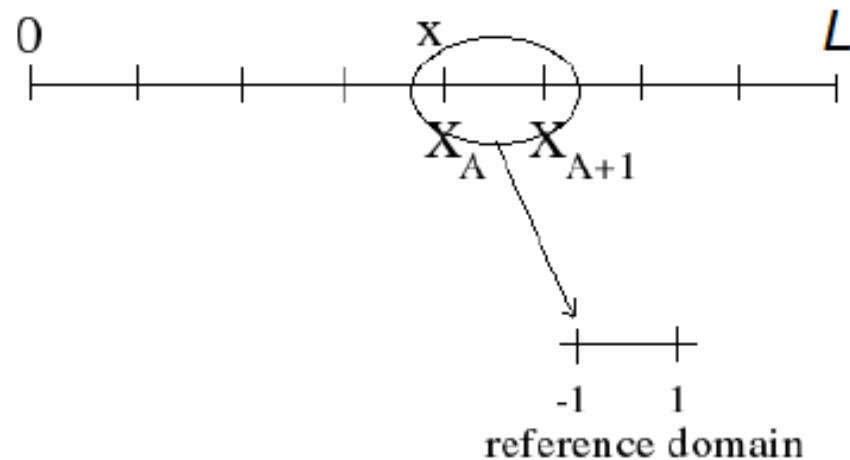
mapping using Jacobian:

$$\int_0^1 g(x) dx = \sum_{\Omega_e} \int_{\Omega_e} g(x) dx = \sum_{\Omega_e} \int_{-1}^1 g(x(\xi)) J d\xi$$



FEM - 1D unsteady-state diffusion equation

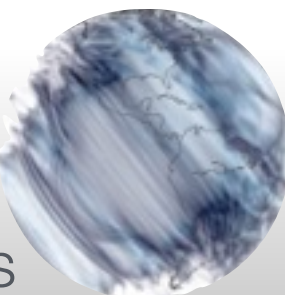
Local level: Consider the mapping $\xi : [X_A, X_{A+1}] \rightarrow [\xi_1, \xi_2]$, such that



$$\begin{cases} \xi(X_A) &= \xi_1 = -1 \\ \xi(X_{A+1}) &= \xi_2 = 1 \end{cases}$$

=> Linear shape functions:

$$N_a(\xi) = \frac{1}{2}(1 + \xi_a \xi) \quad a=1,2$$



FEM - 1D unsteady-state diffusion equation

Local (element) resolution:

$$N_a(\xi) = \frac{1}{2}(1 + \xi_a \xi) \quad a=1,2$$

Capacity matrix: $m_{ab}^e = (N_a, \rho c_p N_b) = \int_{\Omega_e} \rho c_p N_a N_b dx$

↓ *Change of variables (reference domain)*

$$m_{ab}^e = \frac{h_e}{2} \int_{-1}^1 \rho c_p N_a N_b d\xi$$

↓ *Matricial form*

$$m^e = \frac{\rho c_p h_e}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$



FEM - 1D unsteady-state diffusion equation

Local (element) resolution:

$$N_a(\xi) = \frac{1}{2}(1 + \xi_a \xi) \quad a=1,2$$

Stiffness matrix: $k_{ab}^e = a(N_a, \kappa N_b) = \int_{\Omega_e} \kappa \partial_x N_a \partial_x N_b dx$

↓ *Change of variables (reference domain)*

$$k_{ab}^e = \frac{2}{h_e} \int_{-1}^1 \kappa \partial_\xi N_a \partial_\xi N_b d\xi$$

↓ *Matricial form*

$$k^e = \frac{\kappa}{h_e} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$



FEM - 1D unsteady-state diffusion equation

Local (element) resolution:

$$N_a(\xi) = \frac{1}{2}(1 + \xi_a \xi) \quad a=1,2$$

Force vector: $f_a^e = \int_{\Omega_e} N_A f dx + \begin{cases} \delta_{a1} q_0 & \text{for } e = 1 \\ -k_{a2}^e T_L & \text{for } e = N_{el} \\ 0 & \text{else} \end{cases}$

↓ *Change of variables (reference domain)*

$$f_a^e = \frac{h_e}{2} \int_{-1}^1 N_a f d\xi + \text{boundary terms}$$

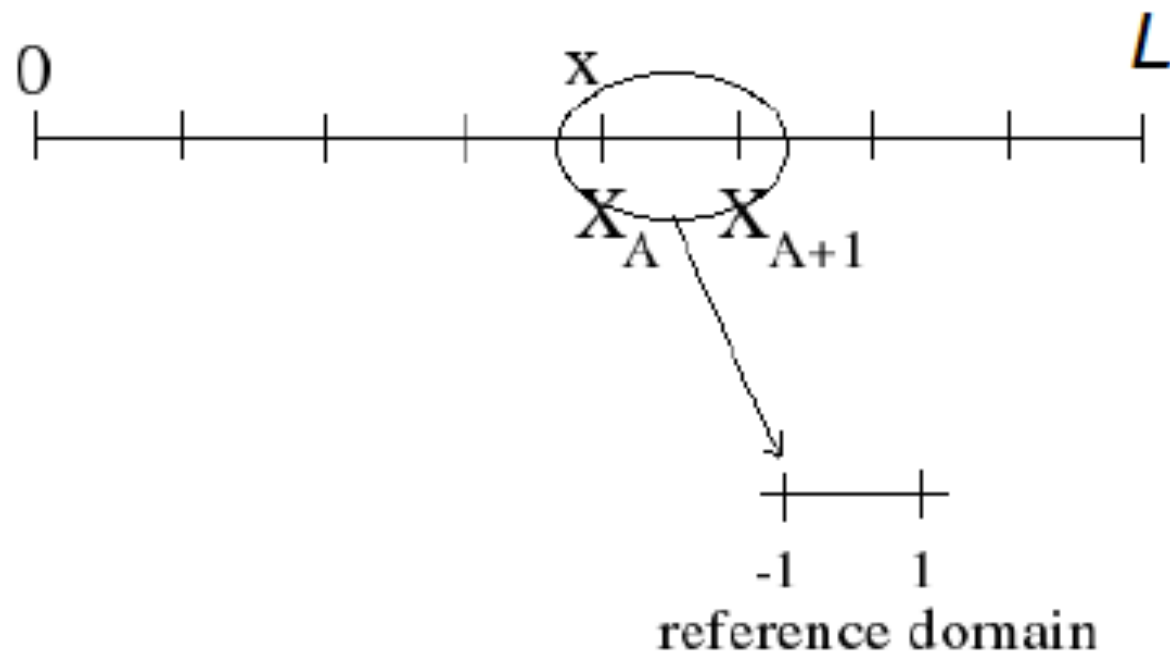
↓ *Matricial form*

$$f^e = \frac{h_e}{6} \begin{pmatrix} 2f_1 + f_2 \\ f_1 + 2f_2 \end{pmatrix} + \text{boundary terms}$$



FEM - 1D unsteady-state diffusion equation

Assembling: back to global level



$iglob(i,ielem) =$

$$\begin{cases} ielem & \text{if } i=1 \\ ielem+1 & \text{if } i=2 \end{cases}$$

code example:

```
do ielem = 1, nelem
  do i = 1,2
    u(iglob(i,ielem)) = u(iglob(i,ielem)) + u_local(i,ielem)
  end
end
```



FEM - 1D unsteady-state diffusion equation

Time scheme: Predictor-Corrector algorithm

- Predictor:

$$\begin{aligned}d_{n+1} &= d_n + (1 - \alpha)\Delta t \dot{d}_n \\ \dot{d}_{n+1} &= 0 \quad (\text{initialization at the beginning of each time step})\end{aligned}$$

- Solve:

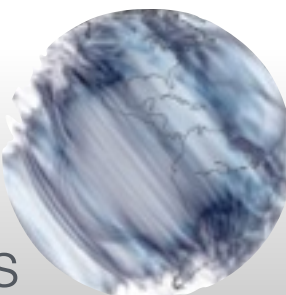
$$\begin{aligned}rhs &= F - M\dot{d}_{n+1} - Kd_{n+1} \\ \delta\dot{d}_{n+1} &= M^{-1}rhs\end{aligned}$$

- Corrector:

$$\begin{aligned}d_{n+1} &= d_{n+1} + \alpha\Delta t \dot{d}_{n+1} \\ \dot{d}_{n+1} &= \dot{d}_{n+1} + \delta\dot{d}_{n+1}\end{aligned}$$

where Δt is the time step.

$$\begin{cases} \alpha = 0 & \text{forward differences} \\ \alpha = 1/2 & \text{midpoint rule} \\ \alpha = 1 & \text{backward differences} \end{cases}$$



FEM - 1D unsteady-state diffusion equation

FEM solution: *initial, simple harmonic function*

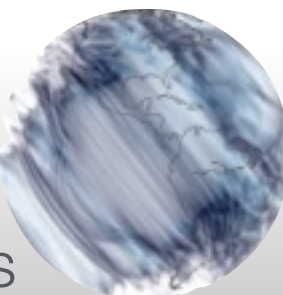
$$\rho c_p \partial_t T - \partial_x (\kappa \partial_x T) = f = 0$$

Boundary conditions:

$$\begin{cases} T(L, t) &= T_L = 1 \\ -\kappa \partial_x T(0, t) &= q_0 = 0 \\ T(x, 0) &= T_0(x) = 1 + \cos(x) \quad \text{in } [0, L] \text{ and } L = \frac{\pi}{2} \end{cases}$$

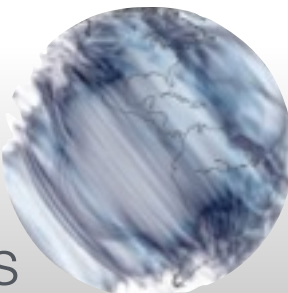
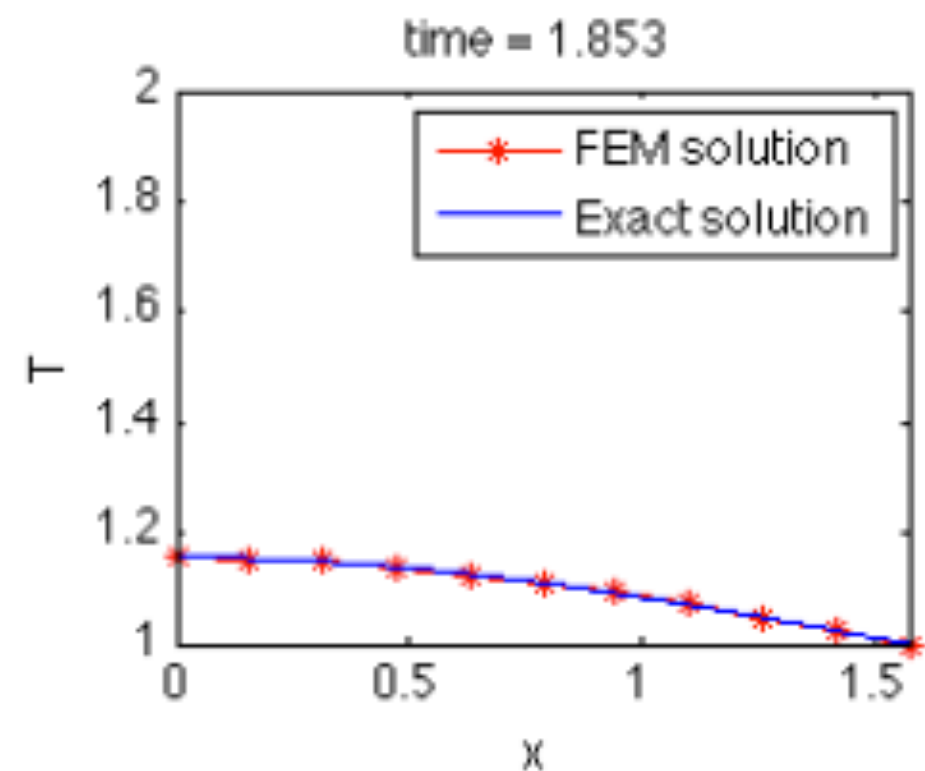
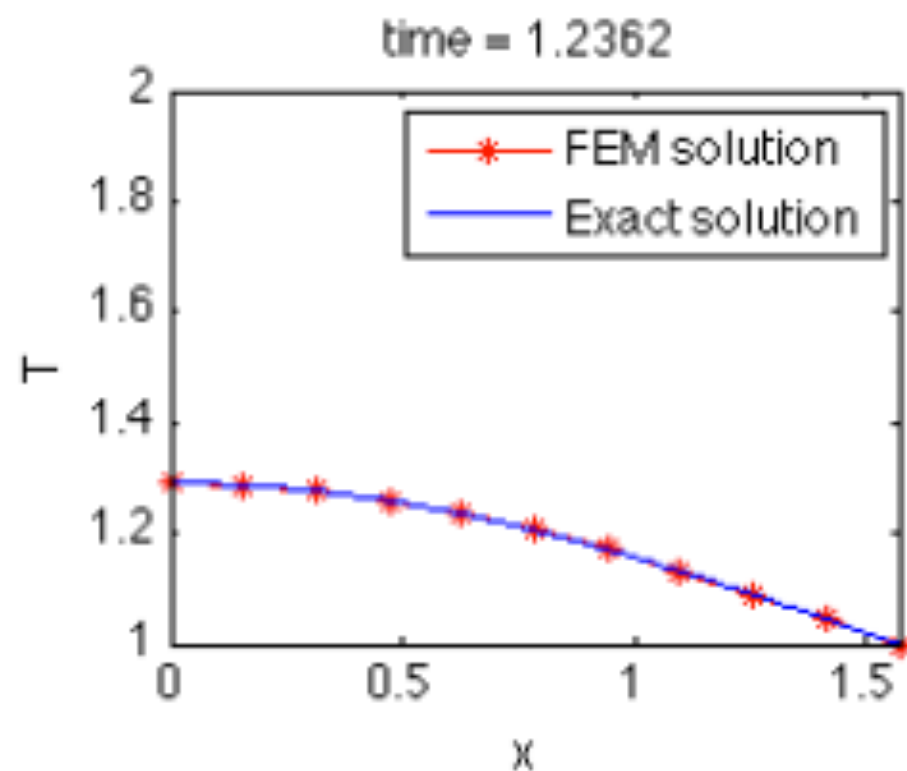
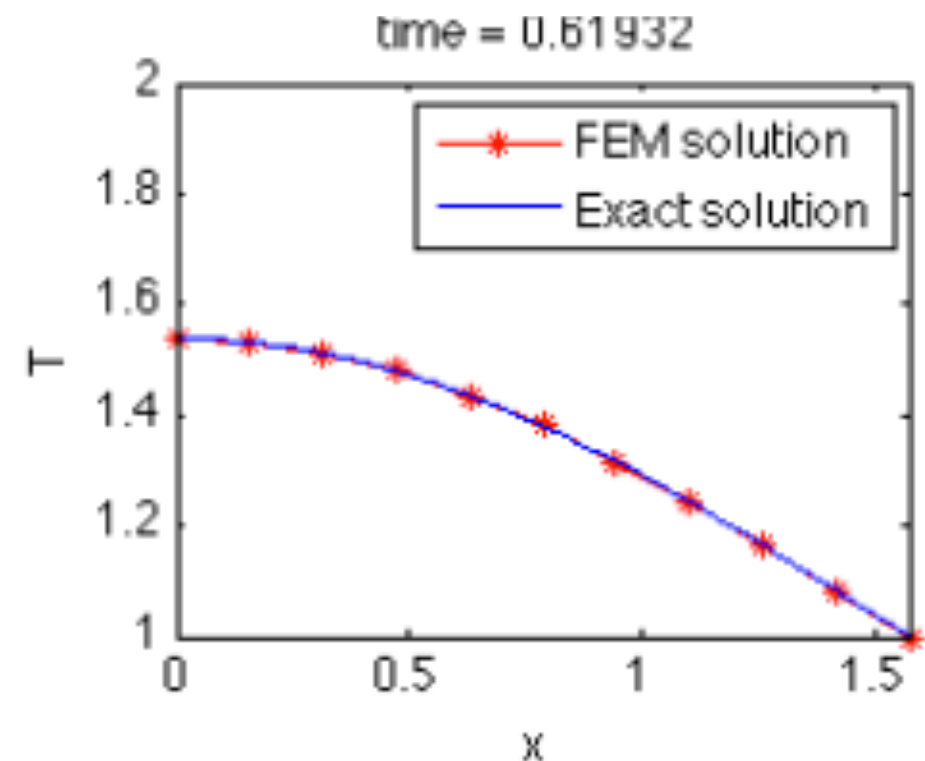
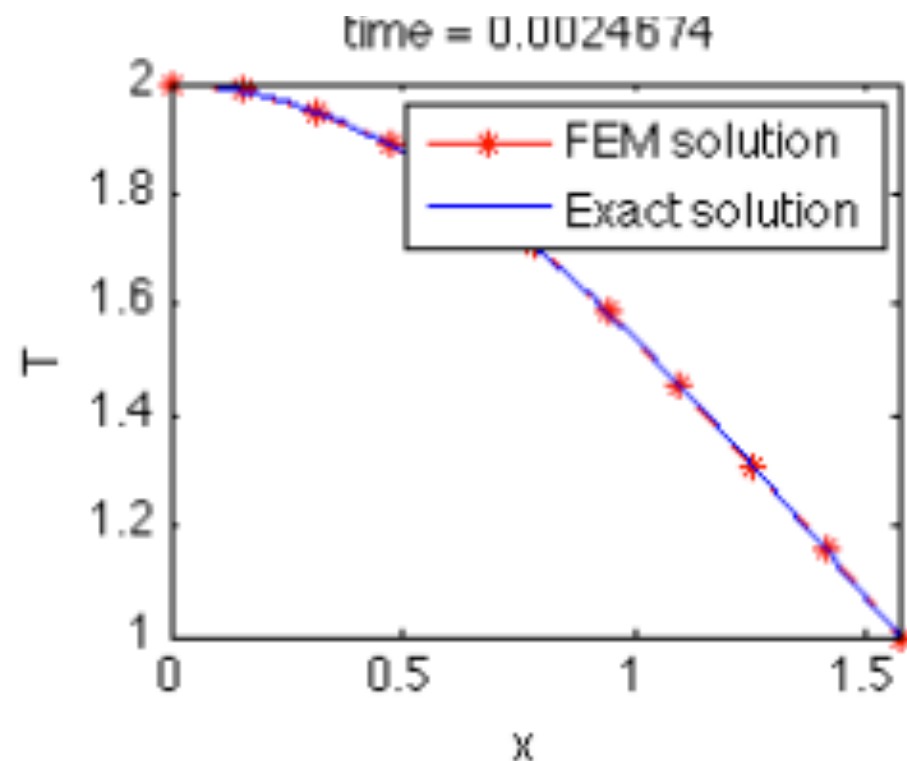
has exact solution:

$$T(x, t) = 1 + e^{-t} \cos(x)$$



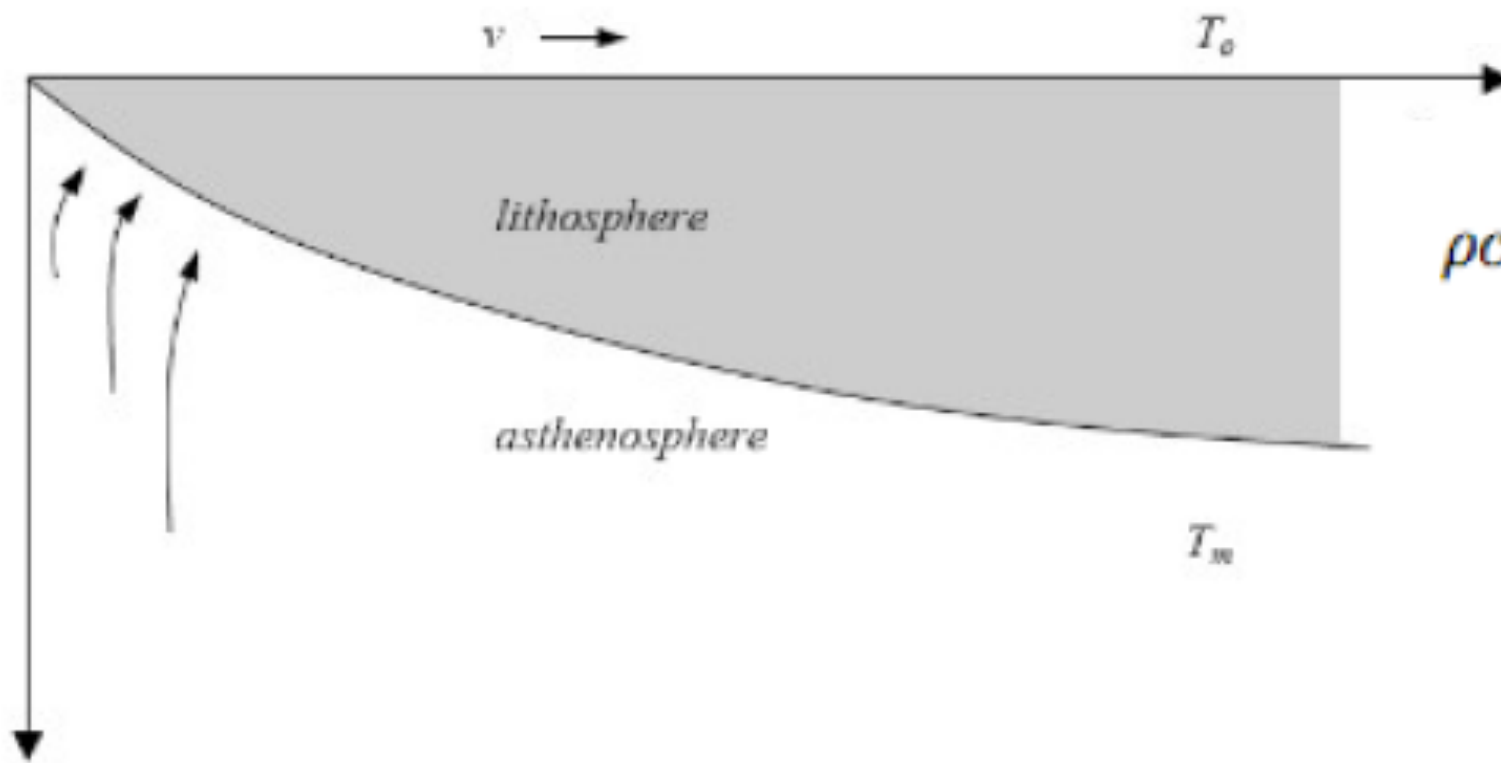
FEM - 1D unsteady-state diffusion equation

FEM solution: *simple harmonic function*



FEM - 1D unsteady-state diffusion equation

FEM solution: *half-space cooling*



$$\rho c_p \partial_t \theta = \partial_x (\kappa \partial_x \theta) \quad \text{in } 0 < x < \infty,$$

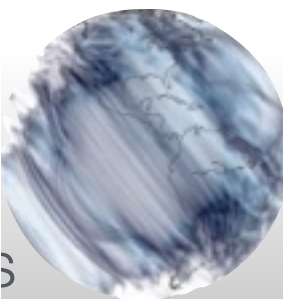
where $\theta = \frac{T - T_m}{T_0 - T_m}$
non-dimensional variable

Boundary conditions:

$$\begin{aligned} T(0, t) &= T_0 \quad (\text{surface temperature}) \\ T(x \rightarrow \infty, t) &\rightarrow T_m \\ T(x, 0) &= T_m \quad (\text{initial temperature}) \end{aligned}$$

has exact solution:

$$\theta = \operatorname{erfc} \frac{x}{2\sqrt{\frac{\kappa}{\rho c_p} t}}$$



FEM - 1D unsteady-state diffusion equation

FEM solution: *half-space cooling*

