

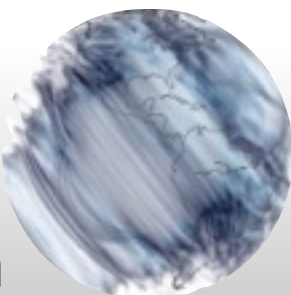
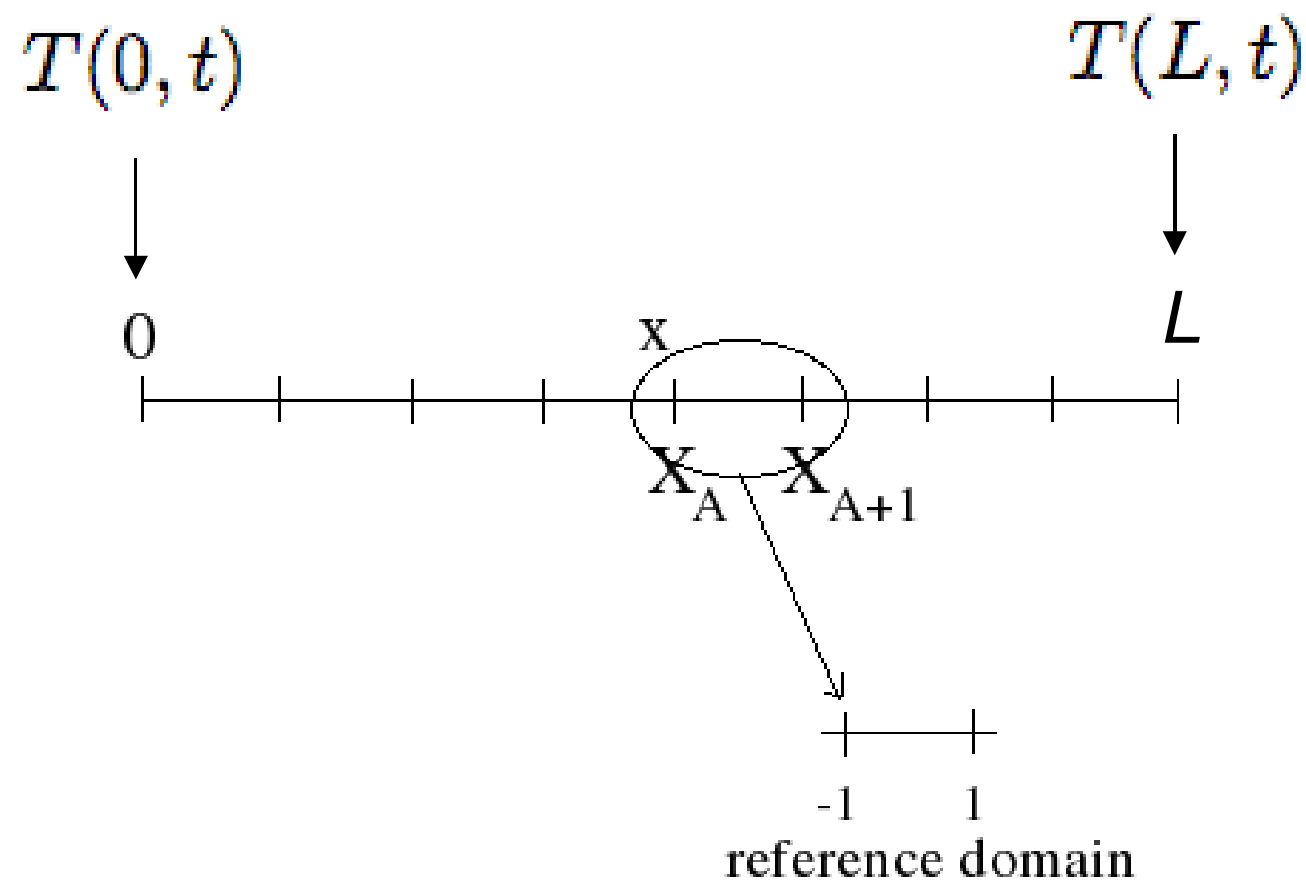
Spectral-element method



1D unsteady-state diffusion equation

Strong form: $\rho c_p \partial_t T - \partial_x (\kappa \partial_x T) = 0$

IC & BC:
$$\begin{cases} T(x, 0) = 0 \\ T(L, t) = 0 \\ T(0, t) = 10 \end{cases}$$



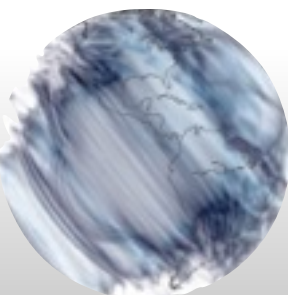
Weak form

Weak form:
$$\int_0^L \rho c_p w \partial_t T dx = - \int_0^L \kappa \partial_x w \partial_x T dx + [w \kappa \partial_x T]_0^L$$

Temperature field (and test function) expanded on basis functions:

$$T(x(\xi), t) = \sum_{\alpha}^N T^{\alpha}(t) l_{\alpha}^N(\xi)$$

unknowns



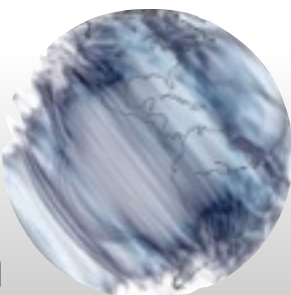
Weak form

Weak form: $\int_0^L \rho c_p w \partial_t T dx = - \int_0^L \kappa \partial_x w \partial_x T dx + [w \kappa \partial_x T]_0^L$

$\Rightarrow \mathbf{M} \dot{\mathbf{d}} + \mathbf{K} \mathbf{d} = \mathbf{F}$ where $\dot{\mathbf{d}} = \partial_t \mathbf{d}$

capacity matrix stiffness matrix force vector

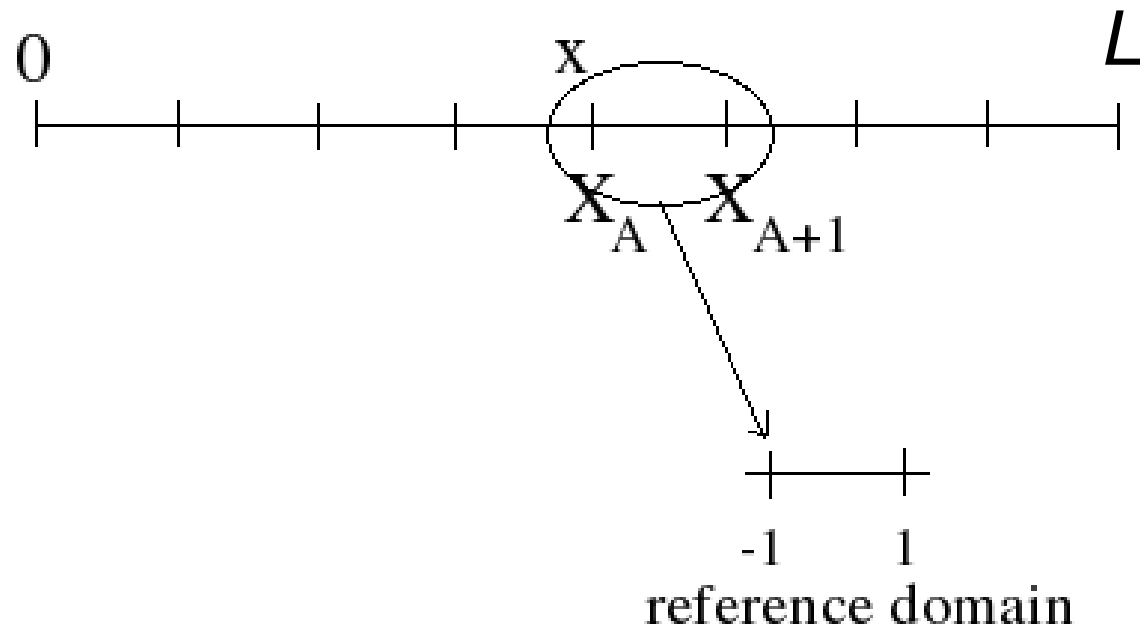
zero



Reference domain

Definition of the reference domain:

Consider the mapping $\xi : [X_A, X_{A+1}] \rightarrow [\xi_1, \xi_2]$, such that



$$\begin{cases} \xi(X_A) &= \xi_1 = -1 \\ \xi(X_{A+1}) &= \xi_2 = 1 \end{cases}$$

$$x(\xi) = \sum_{a=1}^2 X_a N_a(\xi)$$

with shape functions being degree-1 Lagrange polynomials

$$N_a(\xi) = \frac{1}{2}(1 + \xi_a \xi) \quad a=1,2$$

$$\text{Jacobian: } J = \frac{\partial x}{\partial \xi}$$

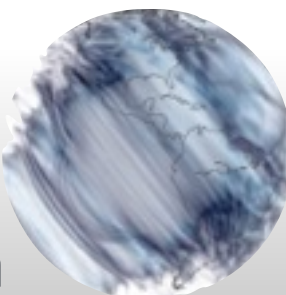
Reference domain

Interpolation:

$$T(x(\xi), t) = \sum_{\alpha}^N T^{\alpha}(t) l_{\alpha}^N(\xi)$$

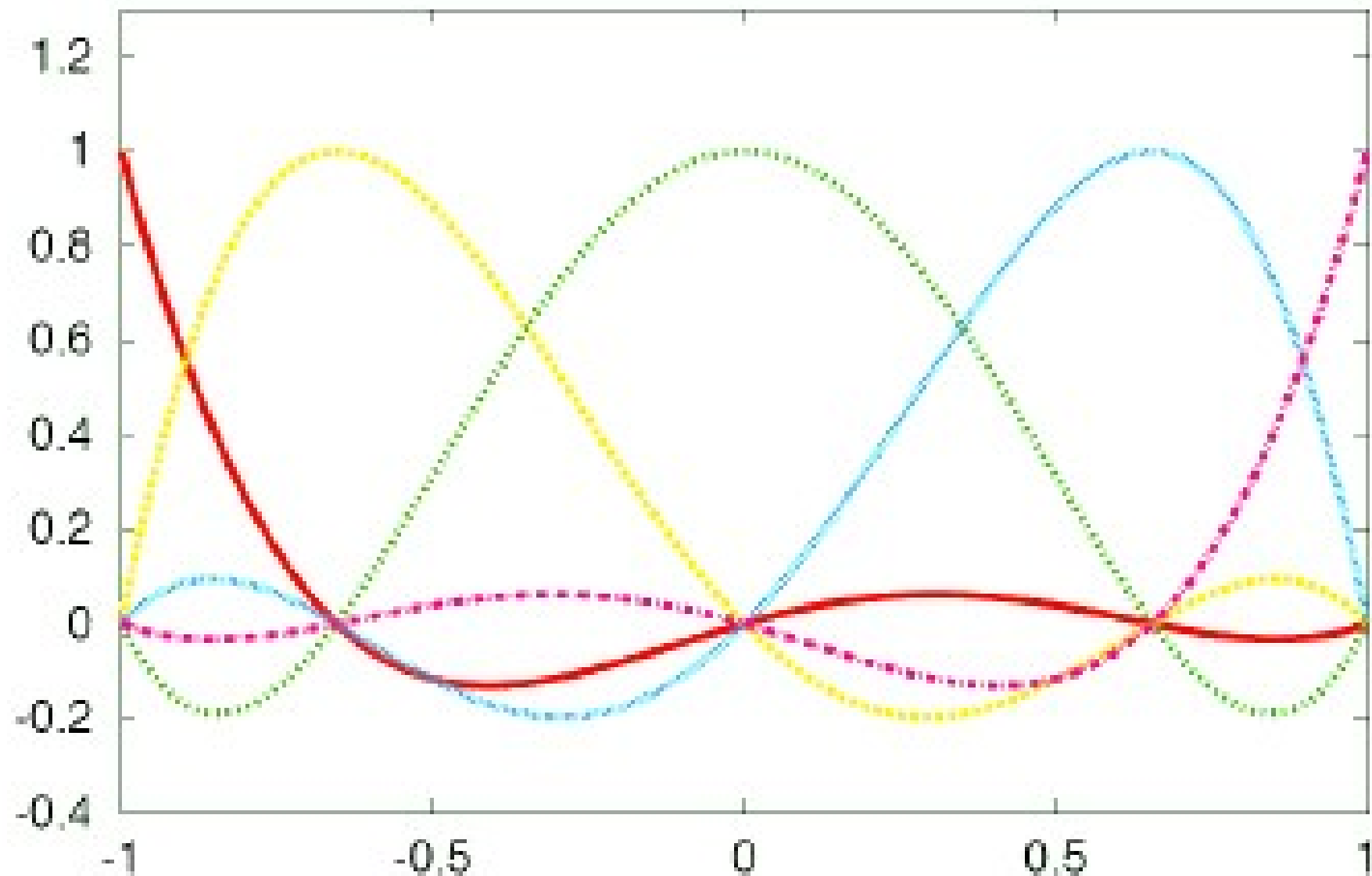
Gauss-Lobatto-Legendre quadrature integration rule:

$$\begin{aligned} \int_{\Omega_e} T(x, t) dx &= \int_{-1}^1 T(x(\xi), t) J^{\alpha} d\xi \\ &\sim \sum_{\alpha}^N \omega_{\alpha} T^{\alpha}(t) J^{\alpha} \end{aligned}$$



Basis functions

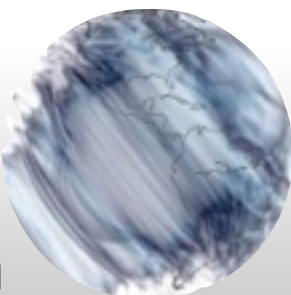
Lagrange polynomials:



degree-4
polynomials

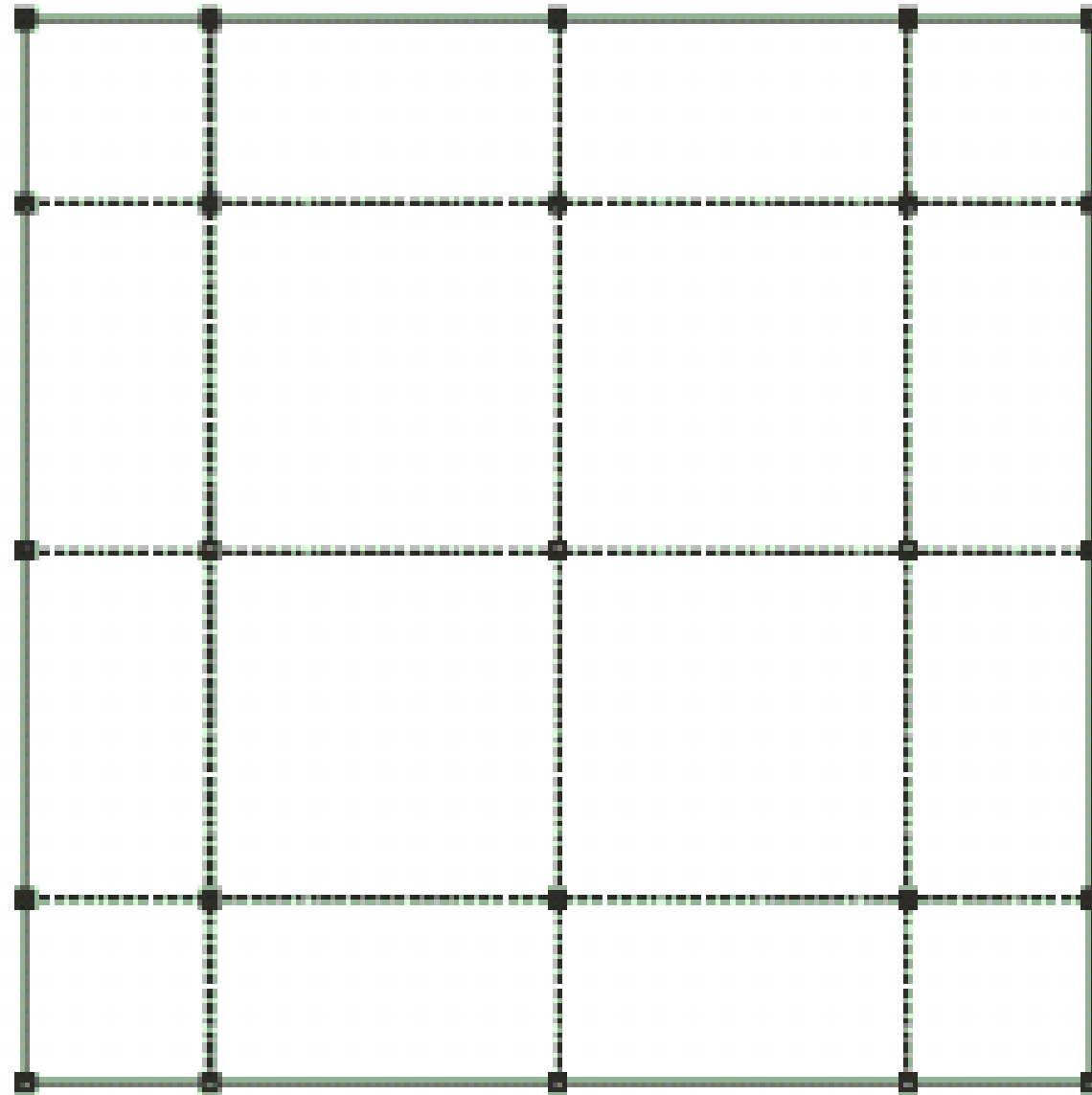
Lagrange polynomials property: $l_{\alpha}^N(\xi_{\beta}) = \delta_{\alpha\beta}$

SEM - 1D unsteady-state diffusion equation



Basis functions

Gauss-Lobatto-Legendre points:

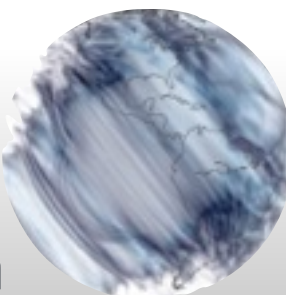


degree-4 GLL points
(2D quad example)

GLL points are the $n+1$ roots of $(1 - \xi^2)P'_n(\xi) = 0$

P_n : Legendre polynomial of degree n

SEM - 1D unsteady-state diffusion equation

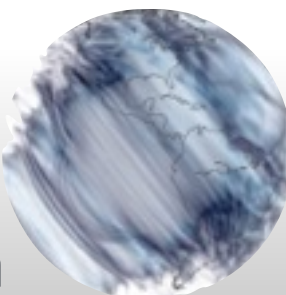


Mass matrix

Local (element) resolution: Capacity/mass matrix

$$\begin{aligned}\int_{\Omega_e} w \rho c_p \partial_t T dx &= \int_{-1}^1 \rho(x(\xi)) c_p(x(\xi)) w(x(\xi)) \partial_t T(x(\xi), t) J d\xi \\ &\sim \sum_{\alpha=0}^N \omega_{\alpha} \rho^{\alpha} c_p^{\alpha} J^{\alpha} \sum_{\beta} w^{\beta} l_{\beta}^N(\xi_{\alpha}) \sum_{\gamma} \partial_t T^{\gamma} l_{\gamma}^N(\xi_{\alpha}) \\ &= \sum_{\alpha=0}^N \boxed{\omega_{\alpha} \rho^{\alpha} c_p^{\alpha} J^{\alpha} w^{\alpha}} \partial_t T^{\alpha}\end{aligned}$$

diagonal matrix



Stiffness matrix

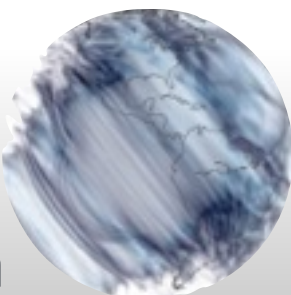
Local (element) resolution: Stiffness matrix

$$\begin{aligned}\int_{\Omega_e} \kappa \partial_x w \partial_x T dx &= \int_{-1}^1 \kappa(x(\xi)) [\partial_x w(x(\xi))] [\partial_x T(x(\xi), t)] J d\xi \\ &\sim \sum_{\alpha=0}^N \omega_{\alpha} \kappa^{\alpha} \left[\sum_{\beta}^N w^{\beta} l'_{\beta}{}^N(\xi_{\alpha}) \partial_x \xi(\xi_{\alpha}) \right] \left[\sum_{\gamma}^N T^{\gamma} l'_{\gamma}{}^N(\xi_{\alpha}) \partial_x \xi(\xi_{\alpha}) \right] J^{\alpha}\end{aligned}$$

$$\Rightarrow M_{\alpha} \partial_t T^{\alpha}(t) = \sum_{\gamma}^N K_{\alpha,\gamma} T^{\gamma}(t)$$

Matricial form:

$$\mathbf{M} \partial_t \mathbf{T} = \mathbf{K} \mathbf{T}$$

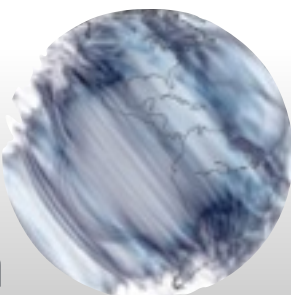


Boundaries

Boundary conditions:

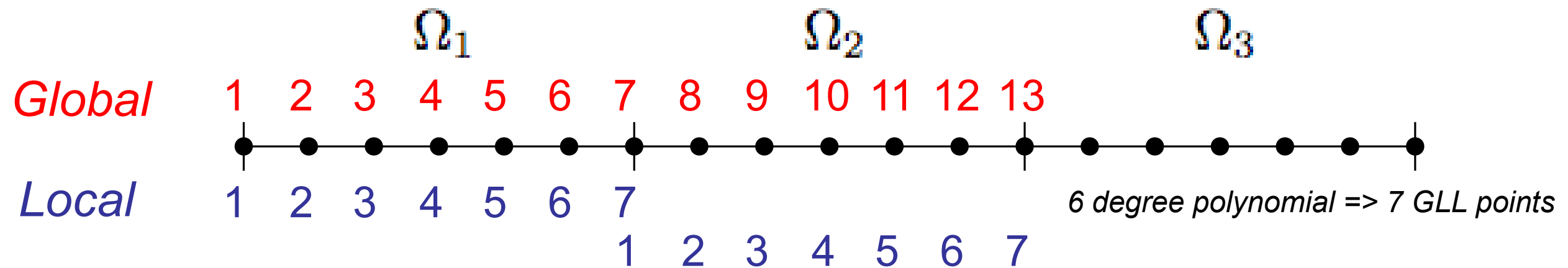
$$w\kappa\partial_x T = \omega^N \kappa^N \sum_{\alpha=0}^N T^\alpha l'_\alpha{}^N(\xi_N) \partial_x \xi(\xi_N) \quad \text{for } x = 0$$

$$w\kappa\partial_x T = \omega^0 \kappa^0 \sum_{\alpha=0}^N T^\alpha l'_\alpha{}^N(\xi_0) \partial_x \xi(\xi_0) \quad \text{for } x = L$$

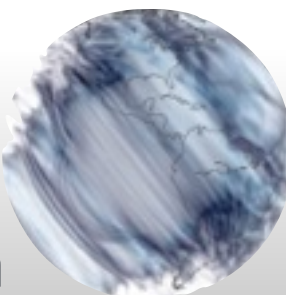
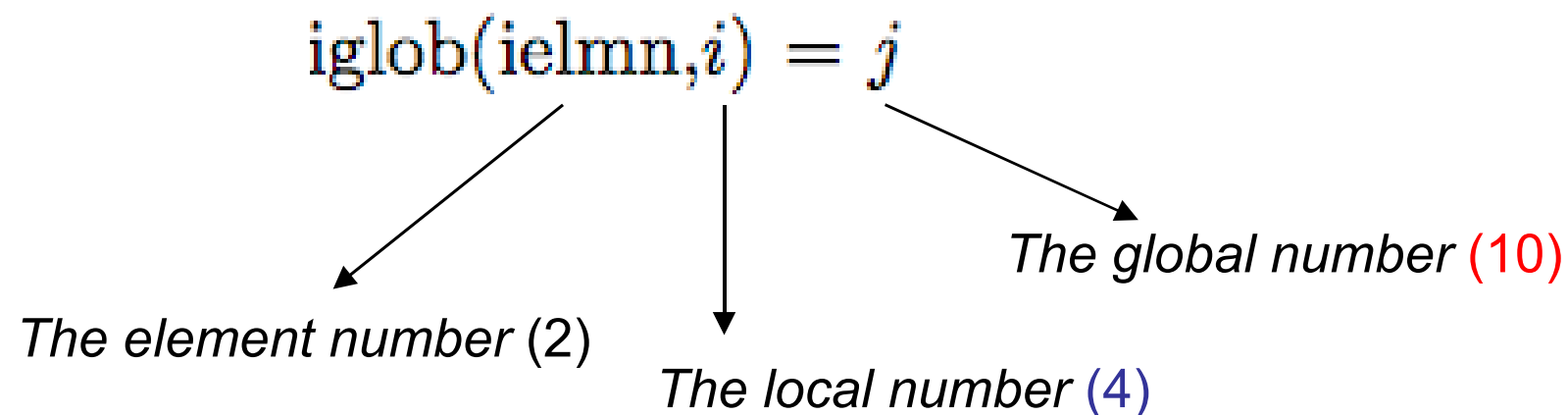


Assembly

Local to Global:



You need an array which links Local (*where the calculation is done*) to Global (*where you want to know the results which are marched in time*):



Assembly

Assembling: back to global level

$$M_{local}(ielmn, i) = \omega_{\alpha} \rho^{\alpha} c_p^{\alpha} J^{\alpha}$$

$$M_{global}(:) = 0$$

!loop over the elements

do ielmn=1,Nel

!loop over the GLL points

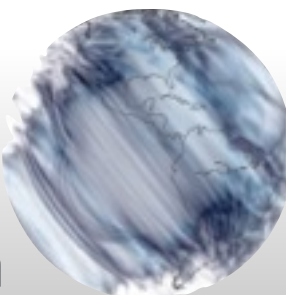
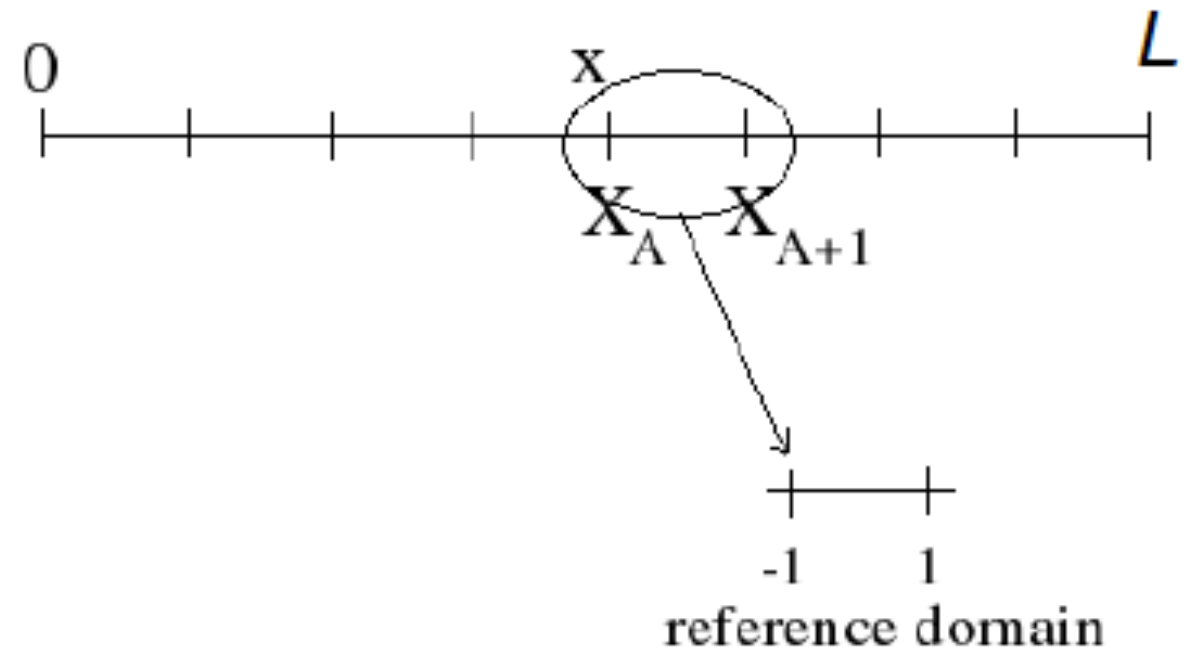
do i=1,NGLL

!get the global index

$$j = \text{iglob}(ielmn, i) \quad M_{global}(j) = M_{global}(j) + M_{local}(ielmn, i)$$

enddo

enddo



Time stepping

Time scheme: Predictor-Corrector algorithm

- Predictor:

$$T_{n+1} = T_n + \frac{1}{2}\Delta t \dot{T}_n$$

$$\dot{T}_{n+1} = 0 \quad (\text{initialization at the beginning of each time step})$$

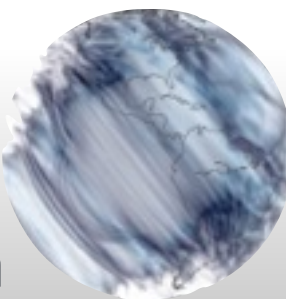
- Solve:

$$\begin{aligned} rhs &= -KT_{n+1} \\ \delta \dot{T}_{n+1} &= M^{-1} rhs \end{aligned}$$

- Corrector:

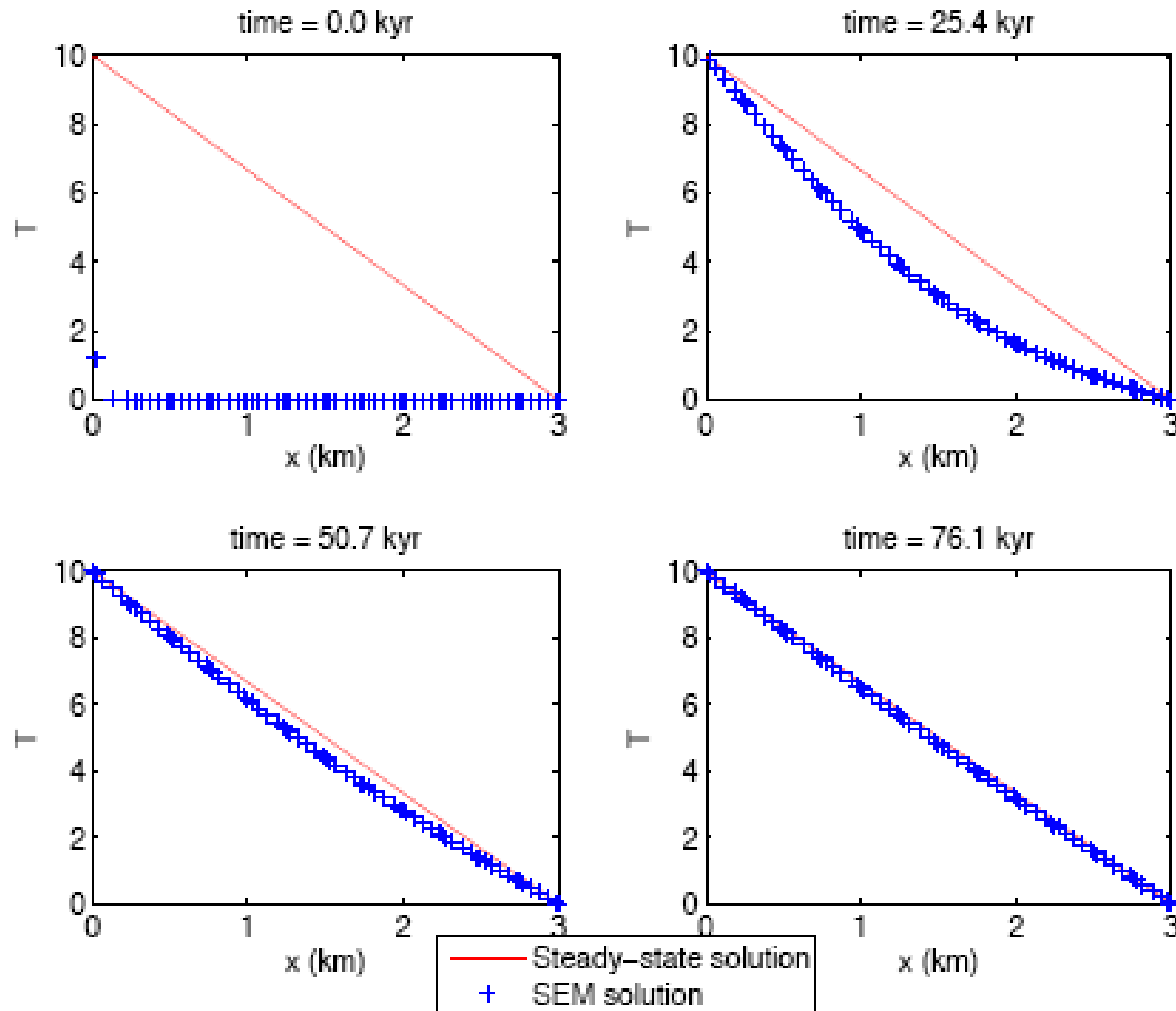
$$T_{n+1} = T_{n+1} + \frac{1}{2}\Delta t \delta \dot{T}_{n+1}$$

$$\dot{T}_{n+1} = \dot{T}_{n+1} + \delta \dot{T}_{n+1}$$



SEM - 1D unsteady-state diffusion equation

Results: *even element spacing*



SEM - 1D unsteady-state diffusion equation

Results: *variable conductivity*

