Topic 1: Set

1. Floor and ceiling
   - $\lfloor\rfloor$: floor
   - $\lceil\rceil$: ceiling
   - $\lfloor -X \rfloor = -\lceil X \rceil$
   - $\lfloor X + t \rfloor = \lfloor X \rfloor + t$ for t ∈ Z
2. Divisibility, prime, gcd and lcm
   - $m \mid n$: m divides n (m is less)
   - $n \mid 0$ is true, and $0 \mid n$ is false, except n = 0
   - prime: $n > 1$ and $1 \mid n$ and $n \mid n$ only
   - relatively prime: $gcd(m, n) = 1$
   - gcd: greatest common divisor
   - lcm: least common multiple
   - gcd(m, n) * lcm(m, n) = $|m| * |n|$
   - Euclid's gcd algorithm: for m ¿ n, gcd(m, n) = gcd(m-n, n)
3. Set notation and construction
   - a set is a set of elements
   - Notation 1: S = {e1, e1, e1 ... }
   - Notation 2: S = {e: description of e}
   - symmetric difference 1: $A \oplus B = (A \cup B) \setminus (A \cap B)$
   - symmetric difference 2: $A \oplus B = (A \setminus B) \cup (B \setminus A)$
   - Subset: ⊆, Proper subset: ⋐
   - Power set: Pow(X) = $\{A : A \subseteq X\}$
   - Cardinality: $|X|$
   - Always: $|Pow(X)| = 2^{|X|}$
   - Set of Numbers: P ⊂ N ⊂ Z ⊂ Q ⊂ R
4. Laws of Sets Operations
   - Commutativity
   - Associativity
   - Distribution
   - Idempotence
   - Identity

- Double Complementation
- De morgan Laws: $(A \cup B))^C = A^C \cap B^C$, $(A \cap B))^C = A^C \cup B^C$

5. Cartesian product

- (a, b): ordered pair
- $A \times B = \{(a, b)|a \in A, b \in B\}$

6. Formal language

- Σ: alphabet – a finite, none empty set
- λ: a empty word
- Σ^k: set of all words of length k
- Σ^*: set of all words
- Σ^+: set of all none empty words

Topic 2: Function Matrix and Relation

1. Function Definition

- notation 1: $f : S \to T$
- notation 2: $f : x \mapsto y$
- notation 3: f(x) = y
- every input has an one and only one output
- Image: Im(f) = $\{f(x), x \in Dom(f)\}$
- $Im(f) \subset Codom(f)$
- Composition: $g \circ f = g(f(x))$ where Im(f) ⊂ Dom(g)
- Identity: $f \circ Id = Id \circ f = f$

2. Function inverse

- surjective(onto): every output has a related input

$$Im(f) = Codom(f)$$

- injective(one-to-one): every input has an unique output

$$x \neq y \implies f(x) \neq f(y)$$
$$f(x) = f(y) \implies x = y$$

- bijective

$$surjective and injective$$

- inverse

$$f^{-1} : y \to x$$

- $f : D \to C, S_D \subseteq D, S_C \subseteq C$, then:
  $f(S_D) \subseteq C$ is the image, and $f^{\Leftarrow}(S_C) \subseteq D$ is the inverse image
  if $f^{-1}(S_C) = f^{\Leftarrow}(S_C)$

3. Matrix

- M_{mn} m is row and n is column

$$\begin{bmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \dots \\ m_{m1} & m_{m2} & \dots & m_{mn} \end{bmatrix}$$

- Transpose M^T
  a matrix is called symmetric if $M^T = M$
- Sum
- product (first row second column)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11} * b_{11} + a_{12} * b_{21} + a_{13} * b_{31} & a_{11} * b_{12} + a_{12} * b_{22} + a_{13} * b_{32} \\ a_{21} * b_{21} + a_{22} * b_{21} + a_{13} * b_{31} & a_{21} * b_{12} + a_{22} * b_{22} + a_{23} * b_{32} \end{bmatrix}$$

4. Relation Property

- a relation from S to T is a subset of $R \subseteq S \times T$
- x is related to y, denote xRy or R(x, y), or x,y $\in$ R, can be True or False
- Reflexive: $\forall x \implies xRx$
- Anti-reflexive: $\forall x \implies x \not R\, x$
- a relation can not be both reflexive and anti-reflexive
- Symmetric: $\forall x, y, xRy \implies yRx$
- Anti-Symmetric: $\forall x, y, xRy \wedge yRx \implies x = y$
- a relation can be both symmetric and anti-reflexive
- Transitive: $\forall x, y, z, xRy \wedge yRz \implies xRz$

5. Equivalence relation and Order relations

- Equivalence Relation: reflexive, symmetric and Transitive
- Partial Order$\preceq$: reflexive, antisymmetric, transitive
- lub: least upper bound, $x \in S \wedge x \succeq a, \forall a \in A$

- glb: greatest lower bound, $x \in S \wedge x \preceq a, \forall a \in A$
- Lattice: a poset where lub and glb exist for every pair of elements, then they exist for every finite subset
- Total Order$\leq$: Partial Order, Linearity: arrange every elements in a line $\forall a, b, a \leq b \vee b \leq a$
- Well order:Total Order, every subset has a least element
- Lexicographic order
- lenlex order

Topic 3: Graph theory

1. Definition: a collection of vertices and edges
   - terminology:
   - incident: edge is incident to vertices
   - adjacent: vertex is adjacent to its neighbour vertices
   - isolated
   - types of graph:
   - Undirected graph: edge = {v1, v2}
   - directed graph: edge = (v1, v2)
   - v(G) = $|V|$, e(G) = $|E|$
2. Degree
   - degree: number of edges attached to the vertex
   - Regular graph: all degree are Equivalence
   - $\Sigma deg(v) = 2 \times e(G)$
   - the degree is always even
   - there is an even number of vertices of odd degree
   - $\Sigma outdeg(v) = \Sigma indeg(v) = e(G)$
3. path
   - simple path(edge): $e_i \neq e_j$
   - close path: $v_0 = v_n$
   - acyclic path(vertex): $v_i \neq v_j$
   - cycle: acyclic path and close path
   - acyclic graph: graph contains no cycle
   - Edge traversal:

- Euler path: path containing every edge exactly once
- iff either it has exactly two vertices of odd degree
- Euler circuit: closed Euler path
- iff all deg(v) is even
- Vertex traversal:
- hamiltonian path: visit every vertex exactly once
- hamiltonian circuit: closed hamiltonian path

4. Connect graph

   - connected graph:
   - if there is an x-y path $\forall x, y \in V$, denote with k(G)
   - strongly connected graph(directed):
   - each pair of vertices joined by a directed path in both directions
   - complete graph K_n:
   - every vertex connected to each other, $\frac{n(n-1)}{2}$ edges.
   - complete bipartitie graph $K_{m,n}$:
   - all vertices form difference parts are connected, vertices from the same part are disconnected.
   -

5. Tree

   - acyclic, connected
   - acyclic, $|V_G| = |E_G| + 1$
   - one simple path between any two vertices
   - connect, becomes disconnected if any single edge is removed
   - acyclic, has a cycle is any single edge is added

6. Graph Isomorphisms

   - $\phi : V_G \to V_H$ is bijection
   - $(x, y) \in E_G$ iff $(\phi(x), \phi(y)) \in E_G$

7. Colouring, Cliques

   - Chromatic number $\chi(G) : theminimumcolour$
   - $\chi(Tree) = 2$
   - $\chi(cyclewithevenvertices) = 2, \chi(cyclewithoddvertices) = 3$
   - Clique number $\kappa(G)$: the largest complete subgraph
   - Planar graph: without intersection
   - a graph is nonplanar then it must contain a subdivision of K_5 or $K_{3,3}$

Topic 4: Logic

1. proposition Logic

   - statement: a declarative sentence that can be True or False
   - Well-formed formula(wff)
   - Connectives: $\neg p, p \wedge q, p \vee q, p \implies q$ is a wff.
   - A implies B

A	B	A $\implies$ B
F	F	T
F	T	T
T	F	F
T	T	T

   - A unless B: $\neg B \implies A$
   - A just in case B: $A \iff B$

A	B	A $\implies$ B
F	F	T
F	T	F
T	F	F
T	T	T

   - $A \iff B = (A \implies B) \vee (B \implies A)$

2. Logic Equivalence

   - $\phi \equiv \varphi$: have the same truth value
   - Excluded middle contradiction
   - Identity
   - Idempotence
   - Double Negation
   - Commutativity
   - Associativity
   - Distribution
   - De Morgan's Law: $\neg(p \wedge q) \equiv \neg p \vee q$, $\neg(p \vee q) \equiv \neg p \wedge q$
   - Implication

3. formula

   - Satisfiable: it can ber true for some assignment of truth value of its basic propositions
   - Validity Tautology: $\models \phi$ if it is true for all propositions

4. argument

- Validity, Entailment: an argument is valid if conclusion is true when all premises are true
- $\phi_1, \phi_2 \dots \phi_n \models \phi$

5. Theorem: $\phi \equiv \varphi$ iff $\models (\phi \iff \varphi)$

6. Proof Method

- Contrapositive: $A \implies B \iff (\neg B \implies \neg A)$
- Contradiction: $A \iff \neg A \implies (B \wedge \neg B)$
- case
- substitution

7. Boolean Function

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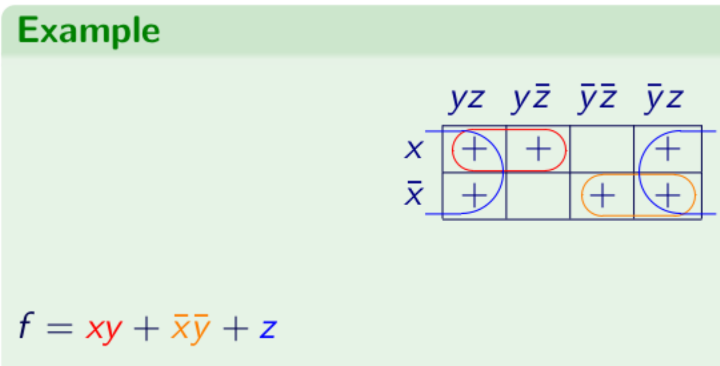
$\wedge$	and	.	conjuction
$\vee$	or	+	disjunction
$\neg$	not	$\overline{p}$	negation

- CNF: $\prod C = C_1.C_2.C_3 \dots C_n$
- DNF(prefered): $\sum C = C_1 + C_2 + C_3 \dots C_n$
- absortion: $x + xy = x$
- combining the opposites: $xy + x\overline{y} = x$
- Demorgan's law
- double Negation
- 

Topic 5: Induction and Recursion

1. Mathematical Induction

- Base case: first thing is true
- Incuctive Hypothesis: Assume sth is true for k, prove that k+1 is true

- conclusion
- Strong induction: $P(m) \land P(m+1) \land \cdots \land P(k) \implies P(k+1)$
- F-B induction: $P(k) \implies p(k+1)$ for some k, $P(k) \implies p(k-1)$ for other k.

2. $\sum n = frac(a_1 + a_n)n2$

3. $\sum n^2 = fracn(n+1)(2n+1)6$

4. $\prod n = fraca_1(1 - q^n)1 - q$

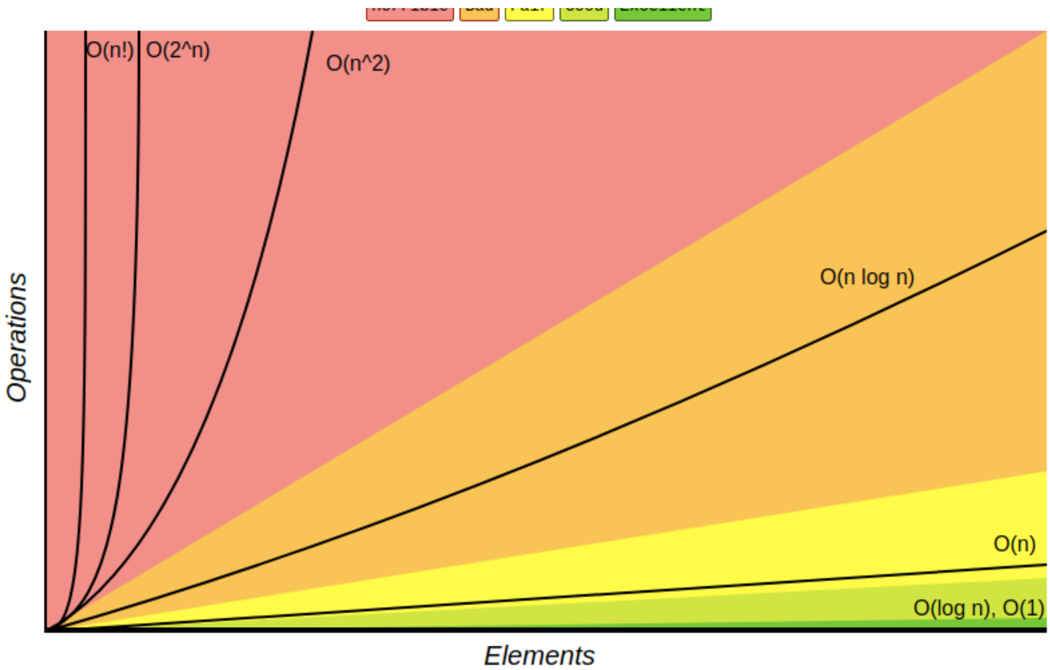
5. Recursive

   - Basis: some initial terms are specified.
   - Recursive Process: later terms states as functional expressions of earlier terms.
   - Correctness: if the computation of any later term can be reduced to the initial values give in basis.

Topic 6: Programs Analysis

1. big O

   - O(f): all function that are asymptotically less than f, also be called upper bound
   - which means that $\exists n_0$ for $n > n_0, g < f$
   - $\Omega(f)$: lower bound
   - $\Theta(f)$: tight bound
   - complexity in terms of input size, N
   - drop constants, machine-independent
   - worst-case
   - $O(1) < O(logn) < O(n) < O(nlogn) < O(n^2) < O(2^n) < O(n!)$

- 

2. Master Theorem

   - $T(n) = d^a T(f\frac{n}{d}) + \Theta(n^b)$
   - $O(n^a), a > b$
   - $O(n^a logn), a = b$
   - $O(n^b), a < b$

Topic 7: Counting and Probability

1. Counting

   - Union rule: for disjoint base sets $|S_1 \cup \ldots S_n| = \sum |S_n|$
   - Product rule: $S_1 \times \cdots \times S_n = \prod S_n$
   - permutation:
   - select r items from a size n set, without repetition, order matters
   - $\Pi(n, r) = \frac{n!}{(n-r)!}$
   - combination:
   - select r items from a size n set, without repetition, order doesn't matter
   - $\binom{n}{k=\frac{n!}{(n-r)!r!}}$

2. Probability

- Uniform probability distribution: each sample has a same probability in sample space Ω
- Event: a collection of outcomes from Ω
- in a uniform distribution, $P(E) = \frac{|E|}{|\Omega|}$
- Two sets: $|A \cup B| = |A| + |B| - |A \cap B|$
- Three sets: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

3. Conditional Probability

- Conditional Probability: $P(B|A) = \frac{P(A \cap B)}{P(A)}$ for $P(A) \neq 0$
- Total Probability: $P(B) = P(A \cap B) + P(\not{A} \cap B)$
- Bayes' Theorem: $P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(\not{A})P(B|\not{A})}$

4. Independent Event, Mutually exclusive

- A and B are independent($A \perp B$) iff:
- $P(A \cap B) = P(A)P(B)$
- $P(A|B) = P(A)$ for $P(A) \neq 0$
- $P(B|A) = P(B)$ for $P(B) \neq 0$
- $A \perp B \iff A^C \perp B \iff A \perp B^C \iff A^C \perp B^C$
- A and B are mutually exclusive iff $P(A \cap B) = \Phi$

5. Expectation

- Definition: $E(X) = \sum_{k \in Z} P(X = k)k$
- linearity of expected value:
- $E(X + Y) = E(X) + E(Y)$
- $E(cX) = cE(X)$
- Standard deviation: ϱ
- variance: ϱ^2
- $\varrho^2 = E((X - E(X))^2) = E(X^2) - E(X)^2$