Topic 1: Set

1. Floor and ceiling
   - $\lfloor\rfloor$: floor
   - $\lceil\rceil$: ceiling
   - $\lfloor -X \rfloor = -\lceil X \rceil$
   - $\lfloor X + t \rfloor = \lfloor X \rfloor + t$ for t ∈ Z
2. Divisibility, prime, gcd and lcm
   - $m \mid n$: m divides n (m is less)
   - $n \mid 0$ is true, and $0 \mid n$ is false, except n = 0
   - prime: $n > 1$ and $1 \mid n$ and $n \mid n$ only
   - relatively prime: $gcd(m, n) = 1$
   - gcd: greatest common divisor
   - lcm: least common multiple
   - gcd(m, n) * lcm(m, n) = $|m| * |n|$
   - Euclid's gcd algorithm: for $m > n$, gcd(m, n) = gcd(m-n, n)
3. Set notation and construction
   - symmetric difference:
   - $A \oplus B = (A \cup B) \setminus (A \cap B)$
   - $A \oplus B = (A \setminus B) \cup (B \setminus A)$
   - Subset: $\subseteq$, Proper subset: $\Subset$
   - Power set: Pow(X) = $\{A : A \subseteq X\}$
   - Cardinality: $|X|$
   - Always: $|Pow(X)| = 2^{|X|}$
   - Set of Numbers: P ⊂ N ⊂ Z ⊂ Q ⊂ R
   - De morgan Laws:
   - $(A \cup B))^C = A^C \cap B^C$
   - $(A \cap B))^C = A^C \cup B^C$
   - Cartesian product:
   - $A \times B = \{(a, b) | a \in A, b \in B\}$
4. Formal language
   - Σ: alphabet – a finite, none empty set
   - λ: a empty word
   - Σ^k: set of words of length k
   - Σ^*: set of all words
   - Σ^+: set of none empty words

Topic 2: Function Matrix and Relation

1. Function Definition
   - f(x)=y, $f : S \to T$, $f : x \mapsto y$
   - every input has an one and only one output
   - Image: Im(f) = $\{f(x), x \in Dom(f)\}$
   - Composition: $g \circ f = g(f(x))$ where Im(f) ⊂ Dom(g)
   - Identity: $f \circ Id = Id \circ f = f$
2. Function inverse
   - surjective(onto): every output has a related input
     $Im(f) = Codom(f)$
   - injective(one-to-one): every input has an unique output
     $x \neq y \implies f(x) \neq f(y)$
     $f(x) = f(y) \implies x = y$
   - bijective: surjective and injective
   - inverse function:
     $f^{-1} : y \to x$
   - inverse image:
     $f^{\Leftarrow}(B) = \{s \in C : f(s) \in B\} \in C$
   - $f^{-1}(B) = f^{\to}(B)$
3. Matrix
   - Transpose: M^T
   - symmetric: $M^T = M$
   - Product:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{21} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

4. Relation
   - a relation from S to T is a subset of $R \subseteq S \times T$, can be True or False
   - denote: xRy, R(x, y), x,y ∈ R
   - Reflexive: $\forall x \implies xRx$
   - Anti-reflexive: $\forall x \implies x \not R x$
   - Symmetric: $\forall x, y, xRy \implies yRx$
   - Anti-Symmetric: $\forall x, y, xRy \wedge yRx \implies x = y$
   - Transitive: $\forall x, y, z, xRy \wedge yRz \implies xRz$
   - a relation CAN NOT be both reflexive and anti-reflexive
   - a relation CAN be both symmetric and anti-reflexive
5. Equivalence relation and Order relations
   - Equivalence Relation: reflexive, symmetric and Transitive
   - Partial Order $\preceq$:
     reflexive, antisymmetric, transitive
     lub(least upper bound), $x \in S \wedge x \succeq a, \forall a \in A$
     glb(greatest lower bound), $x \in S \wedge x \preceq a, \forall a \in A$
     Lattice: a poset where lub and glb exist for every pair of elements, then they exist for every finite subset
   - Total Order $\leq$:
     a partial order with Linearity
     arrange every elements in a line
     $\forall a, b, a \leq b \vee b \leq a$
   - Well Order:
     a total order with every subset has a least element

Topic 3: Graph theory

1. Definition: a collection of vertices and edges
   - Undirected graph: edge = {v1, v2}
   - directed graph: edge = (v1, v2)
   - denote: v(G) = $|V|$, e(G) = $|E|$
2. Degree
   - Degree: number of edges attached to the vertex
   - Regular graph: all degree are Equivalence
   - $\Sigma deg(v) = 2 \times e(G)$
     the degree is always even
     there is an even number of vertices of odd degree
   - $\Sigma outdeg(v) = \Sigma indeg(v) = e(G)$
3. Path
   - simple path(edge): $e_i \neq e_j$
   - close path: $v_0 = v_n$
   - acyclic path(vertex): $v_i \neq v_j$
   - cycle: acyclic path and close path
   - acyclic graph: graph contains no cycle
   - Edge traversal:
     Euler path: path containing every edge exactly once
     iff either it has exactly two vertices of odd degree
     Euler circuit: closed Euler path
     iff all deg(v) is even
   - Vertex traversal:
     hamiltonian path: visit every vertex exactly once
     hamiltonian circuit: closed hamiltonian path
4. Connect graph
   - connected graph:
     if there is an x-y path $\forall x, y \in V$, denote with k(G)
   - strongly connected graph(directed): each pair of vertices joined by a directed path in both directions
   - complete graph K_n:
     every vertex connected to each other, $\frac{n(n-1)}{2}$ edges.
   - complete bipartitie graph $K_{m,n}$:
     all vertices form difference parts are connected, vertices from the same part are disconnected.
5. Tree definitions:
   - acyclic, connected
   - acyclic, $|V_G| = |E_G| + 1$
   - one simple path between any two vertices
   - connect, becomes disconnected if any single edge is removed
   - acyclic, has a cycle is any single edge is added
6. Graph Isomorphisms
   - $\phi : V_G \to V_H$ is bijection

- $(x, y) \in E_G$ iff $(\phi(x), \phi(y)) \in E_G$

7. Colouring and Cliques
   - Chromatic number
     $\chi(G)$: the minimum colour
   - $\chi(Tree) = 2$
     χ (cycle with even vertices) $= 2$
     χ (cycle with odd vertices) $= 3$
   - Clique number
     $\kappa(G)$: the largest complete subgraph
   - Planar graph: without intersection
   - a graph is nonplanar then it must contain a subdivision of K_5 or $K_{3,3}$

Topic 4: Logic

1. proposition Logic
   - statement: a declarative sentence that can be True or False
   - Well-formed formula(wff)
   - Connectives: $\neg p, p \wedge q, p \vee q, p \implies q$ is a wff.
   - A implies B: $A \implies B$

A	B	A $\implies$ B
F	F	T
F	T	T
T	F	F
T	T	T

   - A unless B: $\neg B \implies A$
   - A just in case B: $A \iff B$

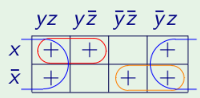
A	B	A $\implies$ B
F	F	T
F	T	F
T	F	F
T	T	T

     $A \iff B = (A \implies B) \vee (B \implies A)$

2. Logic Equivalence
   - $\phi \equiv \varphi$: have the same truth value
   - De Morgan's Law:
     $\neg(p \wedge q) \equiv \neg p \vee q$
     $\neg(p \vee q) \equiv \neg p \wedge q$
3. formula
   - Satisfiable: it can ber true for some assignment of truth value of its basic propositions
   - Validity Tautology: $\models \phi$ if it is true for all propositions
4. argument
   - Validity, Entailment: an argument is valid if conclusion is true when all premises are true
   - $\phi_1, \phi_2 \dots \phi_n \models \phi$
5. Theorem: $\phi \equiv \varphi$ iff $\models (\phi \iff \varphi)$
6. Proof Method
   - Contrapositive: $A \implies B \iff (\neg B \implies \neg A)$
   - Contradiction: $A \iff \neg A \implies (B \wedge \neg B)$
   - case
   - substitution
7. Boolean Function
   -

$\wedge$	and	.	conjuction
$\vee$	or	+	disjunction
$\neg$	not	$\overline{p}$	negation

   - CNF: $\prod C = C_1.C_2.C_3 \dots C_n$
   - DNF(prefered): $\sum C = C_1 + C_2 + C_3 \dots C_n$
   - absorption: $x + xy = x$
   - combining the opposites: $xy + x\overline{y} = x$
   - Demorgan's law
   - double Negation
   - **Example**



     $f = xy + \bar{x}\bar{y} + z$

Topic 5: Induction and Recursion

1. Mathematical Induction
   - Base case: first thing is true
   - Incuctive Hypothesis: Assume sth is true for k, prove that k+1 is true
   - conclusion
   - Strong induction: $P(m) \wedge P(m+1) \wedge \dots \wedge P(k) \implies P(k+1)$
   - F-B induction: $P(k) \implies p(k+1)$ for some k, $P(k) \implies p(k-1)$ for other k.
2. $\sum n = \frac{(a_1 + a_n)n}{2}$
3. $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\prod n = \frac{a_1(1-q^n)}{1-q}$
5. Recursive
   - Basis: some initial terms are specified.
   - Recursive Process: later terms states as functional expressions of earlier terms.
   - Correctness: if the computation of any later term can be reduced to the initial values give in basis.

Topic 6: Programs Analysis

1. Big O: complexity in terms of input size
   - O(f): also be called upper bound all function that are asymptotically less than f, which means that $\exists n_0$ for $n > n_0, g < f$
   - $\Omega(f)$: lower bound
   - $\Theta(f)$: tight bound
   - $O(1) < O(logn) < O(n) < O(nlogn) < O(n^2) < O(2^n) < O(n!)$
2. Master Theorem
   - $T(n) = d^a T(f\frac{n}{d}) + \Theta(n^b)$
   - $O(n^a), a > b$
   - $O(n^a logn), a = b$
   - $O(n^b), a < b$

Topic 7: Counting and Probability

1. Counting
   - Union rule: for disjoint base sets $|S_1 \cup \dots S_n| = \sum |S_n|$
   - Product rule: $S_1 \times \dots \times S_n = \prod S_n$
   - permutation:
     select r items from a size n set, without repetition, order matters
     $$\Pi(n, r) = \frac{n!}{(n-r)!}$$
   - combination:
     select r items from a size n set, without repetition, order doesn't matter
     $$\binom{n}{k} = \frac{n!}{(n-r)!r!}$$
   - k balls into n box:
     $\binom{n+k-1}{k}$
2. Probability
   - Uniform probability distribution: each sample has a same probability in sample space Ω
     in a uniform distribution, $P(E) = \frac{|E|}{|\Omega|}$
   - Event: a collection of outcomes from Ω
   - Two sets: $|A \cup B| = |A| + |B| - |A \cap B|$
   - Three sets: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
3. Conditional Probability and Independence
   - Conditional Probability:
     $P(B|A) = \frac{P(A \cap B)}{P(A)}$ for $P(A) \neq 0$
   - A and B are independent($A \perp B$) iff:
     $P(A \cap B) = P(A)P(B)$
     $P(A|B) = P(A)$ for $P(A) \neq 0$
     $P(B|A) = P(B)$ for $P(B) \neq 0$
     $A \perp B \iff A^C \perp B \iff A \perp B^C \iff A^C \perp B^C$
   - A and B are mutually exclusive iff $P(A \cap B) = \Phi$
4. Expectation
   - Definition:
     $E(X) = \sum_{k \in Z} P(X = k)k$
   - linearity of expected value:
     $E(X + Y) = E(X) + E(Y)$
     $E(cX) = cE(X)$
   - Standard deviation: ϱ
   - variance: ϱ^2
     $= E((X - E(X))^2)$
     $= E(X^2) - E(X)^2$