Polya's enumeration theorem. 计数原理.

1.1. Intro.

Counting is a crucial in math competitions. You may encountered these problems in AMC.

2007-12B-16.

Each face of a regular tetrahedron is painted either red, white, or blue. Two colorings are considered indistinguishable if two congruent tetrahedra with those colorings can be rotated so that their appearances are identical. How many distinguishable colorings are possible?
(A) 15 (B) 18 (C) 27 (D) 54 (E) 81

If we calculate it in a brutal approach, it's time-consuming and we often miss or overcount some cases. Essentially, we want to identify coloring that are "equivalent" under rotations. (i.e. they can be transformed from one to another by rotation) and calculate equivalent ones.

With the notion of orbit of group, we can easily do this.

1.2. G-action.

Def. 1. A Group is a set G equipped with a binary operation $*$ s.t:

① Closeness : $\forall g, h \in G$, $g*h \in G$

② Associative : $\forall a, b, c \in G$, $(a*b)*c = a*(b*c)$

③ Indentity : $\exists e \in G$ s.t. $\forall g \in G$, $g*e = e*g = g$

④ Inverse : $\forall g \in G$, $\underbrace{\exists!}_{\text{unique}}\, g^{-1} \in G$ s.t. $g*g^{-1} = g^{-1}*g = e$

Rmk. You can view it as an algebraic structure, like vector space.

Def. 2: A group action of group G on set X is a map $G \curvearrowright X$ s.t:

$$G \curvearrowright X : \begin{aligned} G \times X &\to X \\ (g, x) &\mapsto gx \end{aligned}$$

And $\forall x \in X$, $(e, x) \mapsto x$

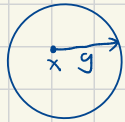
$$(g_1 g_2)x = g_1(g_2 x)$$

Rmk. You can view it as we impose group structure on a set.

Def.3. An orbit of $x \in X$ under group action is a set.

$$\mathrm{Orb}(x) = \{gx \mid \forall g \in G\}.$$

Rmk.



It's all element x can reach under an action.

Rmk. We can see some correspondence.

Orb(x)		Same coloring
All elements x can reach, under group action	$\Longleftrightarrow$	All coloring a coloring can reach by rotation

, and if we want the later ones to be identified as a same coloring, we can correspondedly identify elements in an orbit as the same.

Theorem: Consider finite G and $G \curvearrowright X$, let r be the number of orbit,

$X_g = \{x \in X \mid gx = x\}$. then

$$r = \frac{1}{|G|} \sum |X_g|.$$

This is called Burnside Lemma, the proof is deferred due to its prior knowledge

1.3. Apply it.

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Solve: We want to identify 1). G and 2) X. | r is the number of dist. coloring

$|G| = 4 \times 3 = 12$. (fix one side and rotate 3 sides, respectively)

$|X| = 3^4 = 81$. $\sum |X_g| = \underbrace{81}_{|X_e|} + \underbrace{3 \times 3 \times (4+4)}_{3\times 3\times(4+4)} + \underbrace{3 \times 3 \times 3}_{3\times3\times3} = 180$

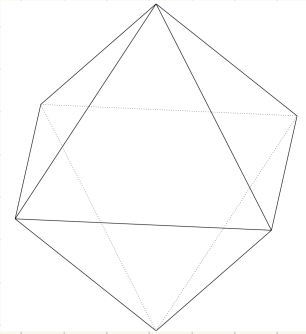
$\therefore r = \frac{1}{|G|} \cdot \sum |X_g| = \frac{180}{12} = 15$

$3 \times 3 \times (\underset{\circlearrowleft 120°}{4} + \underset{\circlearrowleft 240°}{4})$

$3 \times 3 \times 3$ 3 axes

2000-12-25

Eight congruent equilateral triangles, each of a different color, are used to construct a regular octahedron. How many distinguishable ways are there to construct the octahedron? (Two colored octahedrons are distinguishable if neither can be rotated to look just like the other.)



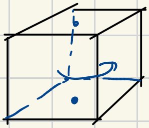
Solve: $|G| \cong |S_4| = 4!$

$|X| = 8! \Rightarrow \sum |X_g| = 8!$ (only identity map preserve it, because all are different color)

$\therefore r = \frac{1}{|G|} \sum |X_g| = \frac{8!}{4!} = \boxed{1680}$

How many different ways are there to color a cube with 6 colors

Solve: $|G| = 4 \times 6 = 24$ (fixed one side, rotate remaining 4 sides respectively)



$|X| = 6!$, $\sum |X| = 6!$ (all color are different, so only identical map preserve coloring)

$r = \frac{1}{|G|} \sum |X_g| = \frac{1}{24} \cdot 6! = \boxed{30}$

Rmk. 遇到全不同色: $|X| = \sum |X_g|$

否则分不同 G 中的 symmetry 来考虑.