作業

武國寧

1 解答題

設$a_n = \dfrac{1+(-1)^n}{n}, n = 1, 2, \cdots, a = 0.$

(1) 對下列ϵ分別求出極限定義中的N：

$$\epsilon_1 = 0.1, \epsilon_2 = 0.01, \epsilon_3 = 0.001$$

(2) 對$\epsilon_1, \epsilon_2, \epsilon_3$，可找到響應的$N$，這是否說明$a_n$趨於0？應該怎樣做才對？

(3) 對於任意給定的ϵ是否可以找到一個N?

2 證明題

按$\epsilon - N$定義證明：

(1) $\lim\limits_{n\to\infty} \dfrac{n}{n+1} = 1$

(2) $\lim\limits_{n\to\infty} \dfrac{3n^2+n}{2n^2-1} = \dfrac{3}{2}$

(3) $\lim\limits_{n\to\infty} \dfrac{n!}{n^n} = 0$

(4) $\lim\limits_{n\to\infty} \dfrac{n}{a^n} = 0 (a > 1)$

(5) $\lim\limits_{n\to\infty} \sqrt[n]{10} = 1$

3 證明題

證明：若$\lim\limits_{n\to\infty} a_n = a$，則對於任意的$k$，有$\lim\limits_{n\to\infty} a_{n+k} = a$

4 解答題

下面那些數列是有界數列、無界數列以及無窮大量：

(1) $\left\{[1+(-1)^n]\sqrt{n}\right\}$

(2) $\{\sin n\}$

(3) $\left\{\dfrac{n^2}{n-\sqrt{5}}\right\}$

(4) $\left\{2^{(-1)^n n}\right\}$

5 求下列極限

(1) $\displaystyle\lim_{n\to\infty}\frac{n^3+3n^2+1}{4n^3+2n+3}$

(2) $\displaystyle\lim_{n\to\infty}\frac{1+2n}{n^2}$

(3) $\displaystyle\lim_{n\to\infty}\left(\sqrt{n^2+n}-n\right)$

(4) $\displaystyle\lim_{n\to\infty}\left(\sqrt[n]{1}+\sqrt[n]{2}+\cdots+\sqrt[10]{n}\right)$

(5) $\displaystyle\lim_{n\to\infty}\frac{\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}}{\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^n}}$

(6) $\displaystyle\lim_{n\to\infty}\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}\right)$

(7) $\displaystyle\lim_{n\to\infty}\left(\sqrt{2}\sqrt[4]{2}\sqrt[8]{2}\cdots\sqrt[2^n]{2}\right)$

(8) $\displaystyle\lim_{n\to\infty}\left(\frac{1}{2}+\frac{3}{2^2}+\cdots+\frac{2n-1}{2^n}\right)$

(9) $\displaystyle\lim_{n\to\infty}\sqrt[n]{1-\frac{1}{n}}$

(10) $\displaystyle\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{n^2+2}+\cdots+\frac{1}{n^2+n}\right)$

6 證明下列極限存在並求其值

(1) 設$a_1 = \sqrt{2}, a_{n+1} = \sqrt{2a_n}, n = 1, 2, \cdots$

(2) 設$a_1 = \sqrt{c}(c > 0), a_{n+1} = \sqrt{c + a_n}, n = 1, 2, \cdots$

(3) 設$a_n = \dfrac{c^n}{n!}(c > 0), n = 1, 2, \cdots$

(4) 設$x_1 = a, x_2 = b, x_{n+2} = \dfrac{x_{n+1} + x_n}{2}(n = 1, 2, \cdots)$, 求$\lim\limits_{n\to\infty} x_n$

7 證明題

利用$\left\{\left(1 + \dfrac{1}{n}\right)^n\right\}$為單調遞增的結論，證明$\left\{\left(1 + \dfrac{1}{n+1}\right)^n\right\}$ 為單調遞增數列。

8 應用柯西收斂原理證明以下數列收斂

(1) $a_n = \dfrac{\sin 1}{2} + \dfrac{\sin 2}{2^2} + \cdots + \dfrac{\sin n}{2^n}$

(2) $a_n = 1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots + \dfrac{1}{n^2}$

9 證明題

證明：若單調數列$\{a_n\}$含有一個收斂子列，則$\{a_n\}$收斂。

10 證明題

證明：若$a_n > 0$,且$\lim\limits_{n\to\infty} \dfrac{a_n}{a_{n+1}} = l > 1$ 則$\lim\limits_{n\to\infty} a_n = 0$收斂。

11 證明

給定$0 < a < b$令$x_1 = a, y_1 = b$。

(1) 若$x_{n+1} = \sqrt{x_n y_n}, y_{n+1} = \dfrac{x_n + y_n}{2}(n = 1, 2, \cdots)$, 證明$\{x_n\}, \{y_n\}$都收斂，且有$\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n$

(2) 若$x_{n+1} = \sqrt{x_n y_n}, y_{n+1} = \dfrac{2x_n y_n}{x_n + y_n}(n = 1, 2, \cdots)$, 證明$\{x_n\}, \{y_n\}$都收斂，且有$\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n$

12 解答題

利用$\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n = e$, 求下列極限：

1. $\lim\limits_{n\to\infty}\left(1-\dfrac{1}{n}\right)^n$
2. $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{n+1}$
3. $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{2n}\right)^n$
4. $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n^2}\right)^n$

13 按照定義證明下列極限

1. $\lim\limits_{x\to+\infty}\dfrac{6x+5}{x} = 6$
2. $\lim\limits_{x\to 2} x^2 - 6x + 10 = 2$
3. $\lim\limits_{x\to 2^-}\sqrt{4-x^2} = 0$
4. $\lim\limits_{x\to x_0}\cos x = \cos x_0$

14 討論函數在$x \to 0$的極限或左右極限

1. $f(x) = \dfrac{|x|}{x}$
2. $f(x) = \lfloor x \rfloor$
3. $f(x) = \begin{cases} 2^x, & x > 0 \\ 0, & x = 0 \\ 1 + x^2, & x < 0 \end{cases}$

15 求下列函數的極限

1. $\lim_{x \to \frac{\pi}{2}} \sin x - \cos x - x^2$
2. $\lim_{x \to 0} \frac{x^2 - 1}{2x^2 - x - 1}$
3. $\lim_{x \to 1} \frac{x^2 - 1}{2x^2 - x - 1}$
4. $\lim_{x \to 1} \frac{x^n - 1}{x^m - 1}$
5. $\lim_{x \to 4} \frac{\sqrt{1 + 2x} - 3}{\sqrt{x} - 2}$
6. $\lim_{x \to 0} \frac{\sqrt{a^2 + x} - a}{a}, a > 0$
7. $\lim_{x \to +\infty} \frac{(3x + 6)^{70}(8x - 5)^{20}}{(5x - 1)^{90}}$

16 利用夾逼原理求下列函數的極限

1. $\lim_{x \to -\infty} \frac{x - \cos x}{x}$
2. $\lim_{x \to -\infty} \frac{x \sin x}{x^2 - 4}$

17 證明題

設 $\lim_{x \to x_0} f(x) = A, \lim_{x \to x_0} g(x) = B$, 證明：

1. $\lim_{x \to x_0} [f(x) \pm g(x)] = A \pm B$
2. $\lim_{x \to x_0} [f(x)g(x)] = AB$
3. $\lim_{x \to x_0} \left[\frac{f(x)}{g(x)}\right] = \frac{A}{B}$

18 解答題

設

$$f(x)=\frac{a_0x^m+a_xx^{m-1}+\cdots+a_{m-1}x+a_m}{b_0x^n+b_1x^{n-1}+\cdots+b_{n-1}x+b_0}(a_0b_0\neq 0,m\leq n)$$

求$\lim\limits_{x\to\infty}f(x)$

19 證明題

設$f(x)>0$, $\lim\limits_{x\to x_0}f(x)=A$ 證明：$\lim\limits_{x\to x_0}\sqrt[n]{f(x)}=\sqrt[n]{A}, n\geq 2$

20 解答題

1. 敘述函數極限$\lim\limits_{x\to+\infty}f(x)$ 存在的海涅原理（歸結原則），並利用它證明$\lim\limits_{x\to+\infty}\cos x$不存在。
2. 敘述極限$\lim\limits_{x\to-\infty}f(x)$ 不存在的充分必要條件，並應用它證明極限$\lim\limits_{x\to-\infty}\sin x$不存在。

21 計算題，求下列極限

1. $\lim\limits_{x\to 0}\frac{\sin 2x}{x}$
2. $\lim\limits_{x\to 0}\frac{\sin x^3}{(\sin x)^2}$
3. $\lim\limits_{x\to\frac{\pi}{2}}\frac{\cos x}{x-\frac{\pi}{2}}$
4. $\lim\limits_{x\to 0}\frac{\tan x}{x}$
5. $\lim\limits_{x\to 0}\frac{\tan^{-1}x}{x}$
6. $\lim\limits_{x\to 0}\frac{\tan x-\sin x}{x^3}$
7. $\lim\limits_{x\to 0}\frac{\sin^2 x-\sin^2 a}{x-a}$

8. $\lim_{x \to 0} \dfrac{\sin 4x}{\sqrt{x+1}-1}$

9. $\lim_{x \to 0} \dfrac{\sqrt{1-\cos x^2}}{1-\cos x}$

22 計算題，求下列極限

1. $\lim_{x \to \infty} \left(1 - \dfrac{2}{x}\right)^{-x}$

2. $\lim_{x \to 0} (1+ax)^{\frac{1}{x}}, a \in \mathbb{R}$

3. $\lim_{x \to 0} (1+\tan x)^{\cot x}$

4. $\lim_{x \to +\infty} \left(\dfrac{3x+2}{3x-1}\right)^{2x-1}$

5. $\lim_{x \to +\infty} \left(1 + \dfrac{\alpha}{x}\right)^{\beta x}, \alpha, \beta \in \mathbb{R}$

23 證明題

證明：$\lim_{x \to 0} \left\{ \lim_{n \to \infty} \cos x \cos \dfrac{x}{2} \cos \dfrac{x}{2^2} \cdots \cos \dfrac{x}{2^n} \right\} = 1$

24 證明下列各式

1. $2x - x^2 = O(x)(x \to 0)$

2. $x \sin \sqrt{x} = O(x^{\frac{3}{2}})(x \to 0^+)$

3. $\sqrt{1+x} - 1 = o(1)(x \to 0)$

4. $(1+x)^n = 1 + nx + o(x)(x \to 0, n \in \mathbb{Z})$

5. $2x^3 + x^2 = O(x^3)(x \to \infty)$

6. $o(g(x)) \pm o(g(x)) = o(g(x))(x \to x_0)$

7. $o(g_1(x)) \cdot o(g_2(x)) = o(g_1(x)g_2(x))(x \to x_0)$

25 是確定α的值，使下列函數與x^α當$x \to 0$時為同階無窮小量

1. $\sin 2x - 2\sin x$
2. $\dfrac{1}{1+x} - (1-x)$
3. $\sqrt{1+\tan x} - \sqrt{1-\sin x}$
4. $\sqrt[5]{3x^2 - 4x^3}$

26 是確定α的值，使下列函數與x^α當$x \to \infty$時為同階無窮大量

1. $\sqrt{x^2 + x^5}$
2. $x + x^2(2 + \sin x)$
3. $(1+x)(1+x^2)\cdots(1+x^n)$

27 求下列曲線的漸近線

1. $y = \dfrac{1}{x}$
2. $y = \tan^{-1} x$
3. $y = \dfrac{3x^3+4}{x^2-2x}$

28 確定下列a與α,使下列各無窮小量或無窮大量等價於ax^α

1. $u(x) = x^5 - 3x^4 + 2x^3 (x \to 0, x \to \infty)$
2. $u(x) = \dfrac{x^5 + 2x^2}{3x^4 - x^3} (x \to 0, x \to \infty)$
3. $u(x) = \sqrt{x^3} + \sqrt[3]{x^2} (x \to 0+, x \to +\infty)$

4. $u(x) = \sqrt{x + \sqrt{x + \sqrt{x}}}(x \to 0+, x \to +\infty)$
5. $u(x) = \sqrt{1+3x} - \sqrt[3]{1+2x}(x \to 0, x \to +\infty)$
6. $u(x) = \sqrt{x^2+1} - x(x \to +\infty)$
7. $u(x) = \sqrt{x^3+x} - x^{\frac{3}{2}}(x \to 0+)$
8. $u(x) = \sqrt{1 + x\sqrt{x}} - e^{2x}(x \to 0+)$
9. $u(x) = \ln\cos x - \tan^{-1} x^2(x \to 0)$
10. $u(x) = \sqrt{1+\tan x} - \sqrt{1-\sin x}(x \to 0)$