

Derivative

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1 Differentiable Functions

Definition 1.1. A function $f : E \rightarrow \mathbb{R}$ defined on a set E is differentiable at a point $a \in E$ that is a limit point of E if there exists a linear function $A(x - a)$ of the increment $x - a$ of the argument such that $f(x) - f(a)$ can be represented as

$$f(x) - f(a) = A(x - a) + o(x - a) \text{ as } x \rightarrow a, a \in E \quad (1)$$

In other words, a function is differentiable at a point a if the change in its values in a neighborhood of the point in question is linear up to a correction that is infinitesimal compared with the magnitude of the displacement $x - a$ for the point a .

Definition 1.2. The linear function $A(x - a)$ in Eq. 1 is called the differential of the function f at a .

The number A is unambiguously determined due to the uniqueness of the limit.

Definition 1.3. The number

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \quad (2)$$

is called the derivative of the function f at a .

Graphically, this definition says that the derivative of f at a is the slope of the tangent line to $y = f(x)$ at a , which is the limit as $x \rightarrow a$ of the slopes of the lines through $(x, f(x))$ and $(a, f(a))$.

We can also write

$$f'(a) = \lim_{\Delta x \rightarrow 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

Definition 1.4. A function $f : E \rightarrow \mathbb{R}$ defined on a set $E \subset \mathbb{R}$ is differentiable at a point $x \in E$ that is a limit point of E if

$$f(x+h) - f(x) = A(x)h + \alpha(x;h) \quad (3)$$

where $h \rightarrow A(x)h$ is a linear function in h and $\alpha(x;h) = o(h)$ as $h \rightarrow 0, x+h \in E$.

Definition 1.5. The function $h \rightarrow A(x)h$ of Definition 3, which is linear in h , is called the differential of the function $f : E \rightarrow \mathbb{R}$ at the point $x \in E$ and is denoted as $df(x)$ or $Df(x)$.

Thus, $df(x)(h) = A(x)h$.

From definitions 2 and 3 we have

$$\Delta f(x;h) - df(x)(h) = \alpha(x;h)$$

1.1 Some Examples

Examples 1. Let $f(x) = \sin x$. We shall show that $f'(x) = \cos x$.

Examples 2. We shall show that $\cos'(x) = -\sin x$.

Examples 3. If $f(t) = r \sin \omega t$, then $f'(t) = r\omega \cos \omega t$. If $f(t) = r \cos \omega t$, then $f'(t) = -r\omega \sin \omega t$.

Examples 4. The instantaneous velocity and instantaneous acceleration of a point mass. Suppose a point mass is moving in a plane and that in some given coordinate system its motion is described by differentiable function of time

$$x = x(t), y = y(t)$$

In particular, this motion is written as in the form

$$r(t) = (r \cos(\omega t + \alpha), r \sin(\omega t + \alpha))$$

Examples 5. The optic property of a parabolic mirror. Let us consider the parabola $y = \frac{1}{2p}x^2 (p > 0)$, and construct the tangent to it at the point $(x_0, y_0) = (x_0, \frac{1}{2p}x_0^2)$.

Examples 6.

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Examples 7. We shall show that

$$e^{x+h} - e^x = e^x h + o(h)$$

as $h \rightarrow 0$.

Examples 8. If $a > 0$, then $a^{x+h} - a^x = a^h (\ln a)h + o(h)$ as $h \rightarrow 0$.

2 The Basic Rules of Differentiation

2.1 Differentiation and the Arithmetic Operations

Theorem 2.1. If function $f : X \rightarrow \mathbb{R}$ and $g : X \rightarrow \mathbb{R}$ are differentiable at a point $x \in X$, then a) their sum is differentiable at x , and

$$(f + g)'(x) = (f' + g')(x),$$

b) their product is differentiable at x , and

$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x),$$

c) their quotient is differentiable at x if $g(x) \neq 0$, and

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

Corollary 2.2. The derivative of a linear combination of differentiable functions equals the same linear combination of the derivatives of these functions.

Corollary 2.3. If the functions f_1, \dots, f_n are differentiable at x , then

$$\begin{aligned} (f_1 f_2 \cdots f_n)'(x) &= f_1' f_2 \cdots f_n \\ &\quad + f_1 f_2' \cdots f_n + \cdots + f_1 f_2 \cdots f_n'. \end{aligned}$$

Corollary 2.4. It follows from the relation between the derivative and the differential that we have:

$$\begin{aligned} a) d(f + g)(x) &= df(x) + dg(x), \\ b) d(f \cdot g)(x) &= g(x)df(x) + f(x)dg(x), \\ c) d\left(\frac{f}{g}\right)(x) &= \frac{g(x)df(x) - f(x)dg(x)}{g^2(x)}. \end{aligned}$$

Examples 9. Find the derivative of $\tan x$ and $\cot x$.

2.2 Differentiation of a Composite Function (chain rule)

Theorem 2.5. If the function: $f : X \rightarrow Y \subset \mathbb{R}$ is differentiable at a point $x \in X$ and the function $g : Y \rightarrow \mathbb{R}$ is differentiable at the point $y = f(x) \in Y$, then the composite function $g \circ f : X \rightarrow \mathbb{R}$ is differentiable at x , and the differential $d(g \circ f)(x) : T\mathbb{R} \rightarrow T\mathbb{R}_{g(f(x))}$ of their composition equals the composition $dg(y) \circ df(x)$ of their differentials.

Proof. The conditions for differentiability of the function f and g have the form.

$$\begin{aligned}f(x+h) - f(x) &= f'(x)h + o(h), h \rightarrow 0, x+h \in X \\g(y+t) - g(y) &= g'(y)t + o(t), t \rightarrow 0, y+t \in Y\end{aligned}$$

□

Corollary 2.6. The derivative $(g \circ f)'(x)$ of the composition of differentiable real-valued functions equals the product $g'(f(x)) \cdot f'(x)$ of the derivatives of these functions computed at the corresponding points.

$$\frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta y} \cdot \frac{\Delta y}{\Delta x}$$

Examples 10. Let us show that for $\alpha \in \mathbb{R}$ we have $\frac{dx^\alpha}{dx} = \alpha x^{\alpha-1}$ in the domain $x > 0$, that is, $dx^\alpha = \alpha x^{\alpha-1}dx$

Examples 11. The derivative of the logarithm of the absolute value of a differentiable function is often called its logarithmic derivative.

$$d(\ln |f|)(x) = \frac{f'(x)}{f(x)}dx = \frac{df(x)}{f(x)}.$$

Examples 12. The absolute and relative errors in the value of a differentiable function caused by errors in the data for the argument.

$$f(x+h) - f(x) = f'(x)h + \alpha(x;h),$$

$$\frac{|f'(x)h|}{|f(x)|} = \frac{|df(x)h|}{|f(x)|}$$

2.3 Differentiation of an Inverse Function

Theorem 2.7. Let the function $f : X \rightarrow Y$ and $f^{-1} : Y \rightarrow X$ be mutually inverse and continuous at points x_0 and $f(x_0) = y_0 \in Y$ respectively. If f is differentiable at x_0 and $f'(x_0) \neq 0$, then f^{-1} is also differentiable at the point y_0 , and

$$(f^{-1})'(y_0) = (f'(x_0))^{-1}.$$

Remark. If we knew in advance that the function f^{-1} was differentiable at y_0 , we would find immediately by the identity $(f^{-1} \circ f)(x) = x$ and the theorem on differentiation of a composite function that $(f^{-1})' \cdot f'(x_0) = 1$.

Remark. The condition $f'(x_0) \neq 0$ is obviously equivalent to the statement that the mapping $h \rightarrow f'(x_0)h$ realized by the differential $df(x_0) : T\mathbb{R}(x_0) \rightarrow T\mathbb{R}(y_0)$ is invertible mapping $[df(x_0)]^{-1} : T\mathbb{R}(y_0) \rightarrow T\mathbb{R}(x_0)$ given by the formula $\tau \rightarrow (f'(x_0))^{-1} \tau$.

Examples 13. We shall show that $\arcsin' y = \frac{1}{1-y^2}$ for $|y| < 1$.

Examples 14. $\operatorname{arccot}' y = -\frac{1}{1+y^2}$, $\arctan' y = \frac{1}{1+y^2}$

Examples 15. The hyperbolic and inverse hyperbolic functions and their derivatives. The function

$$\begin{aligned}\sinh x &= \frac{1}{2} (e^x - e^{-x}) \\ \cosh x &= \frac{1}{2} (e^x + e^{-x})\end{aligned}$$

are respectively the hyperbolic sine and hyperbolic cosine of x . These functions, which for the time being have been introduced purely formally, arise just as naturally in many problems as the circular functions $\sin x$ and $\cos x$.

We remark that

$$\begin{aligned}\sinh(-x) &= -\sinh x \\ \cosh(-x) &= \cosh x\end{aligned}$$

Moreover, the following basic identity is obvious

$$\cosh^2 x - \sinh^2 x = 1$$

The graphs of the functions $y = \sinh x$ and $y = \cosh x$ are shown in Fig 2. The inverse of the hyperbolic sine is

$$x = \ln(y + \sqrt{1+y^2})$$

Thus,

$$\sinh^{-1} y = \ln(y + \sqrt{1+y^2})$$

Similarly, using the monotonicity of the function $y = \cosh x$ on its definition, we have

$$\begin{aligned}\cosh^{-1}(y) &= \ln(y + \sqrt{y^2 - 1}) \\ \cosh_+^{-1}(y) &= \ln(y - \sqrt{y^2 - 1})\end{aligned}$$

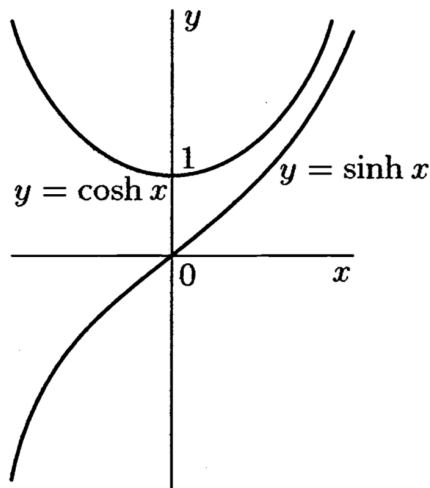


Figure 1: Hyperbolic functions.

From the definitions given above, we find

$$\begin{aligned}\sinh' x &= \cosh x, \\ \cosh' x &= \sinh x,\end{aligned}$$

and by the theorem on the derivative of an inverse function, we find

$$\begin{aligned}(\sinh^{-1} y)' &= \frac{1}{\sinh' x} = \frac{1}{\cosh' x} = \frac{1}{\sqrt{1+y^2}} \\ (\cosh^{-1} y)' &= \frac{1}{\cosh' x} = \frac{1}{\cosh' x} = \frac{1}{-\sqrt{\cosh^2 x - 1}} = -\frac{1}{\sqrt{y^2 - 1}}, y > 1 \\ (\cosh_+^{-1} y)' &= \frac{1}{\cosh' x} = \frac{1}{-\sqrt{\cosh^2 x - 1}} = -\frac{1}{\sqrt{y^2 - 1}}, y > 1\end{aligned}$$

Like $\tan x$ and $\cot x$ one can consider the functions

$$\tanh x = \frac{\sinh x}{\cosh x}, \text{ and } \coth x = \frac{\cosh x}{\sinh x}$$

called the hyperbolic tangent and hyperbolic cotangent respectively, and also the functions inverse to them, the area tangent

$$\tanh^{-1} y = \frac{1}{2} \ln \frac{1+y}{1-y}, |y| < 1, \coth^{-1} y = \frac{1}{2} \ln \frac{y+1}{y-1}, |y| > 1,$$

By the rules for differentiation we have

$$\begin{aligned}\tanh' x &= \frac{1}{\cosh^2 x} \\ \coth' x &= -\frac{1}{\sinh x}\end{aligned}$$

By the theorem on the derivative of an inverse function

3 Table of Derivatives of the Basic Elementary Functions

4 Higher-order Derivative

If a function $f : E \rightarrow \mathbb{R}$ is differentiable at every point $x \in E$, then a new function $f' : E \rightarrow \mathbb{R}$ arises, whose value at a point $x \in E$ equals the derivative $f'(x)$ of the function f at that point.

The function $f' : E \rightarrow \mathbb{R}$ may itself have a derivative $(f'(x))' : E \rightarrow \mathbb{R}$ on E , called the second derivative of the original function f and denoted by one of the following two symbols:

$$f''(x), \frac{d^2 f(x)}{dx^2}$$

and if we wish to indicate explicitly the variable of differentiation in the first case, we also write, for example, $f''_{xx}(x)$

Definition 4.1. By induction, if the derivative $f^{(n-1)}(x)$ of order $n - 1$ of f has been defined, then the derivative of order n is defined by the formula:

$$f^{(n)}(x) = \frac{d}{dx} f^{(n-1)}(x)$$

The following notations are conventional for the derivative of order n :

$$(f^{(n)}(x))(x)$$

The set of functions $f : E \rightarrow \mathbb{R}$ having continuous derivatives up to order n inclusive will be denoted as $C^{(n)}(E, \mathbb{R})$, and by the simpler symbol $C^{(n)}(E)$.

Function $f(x)$	Derivative $f'(x)$	Restrictions on domain of $x \in \mathbb{R}$
1. C (const)	0	
2. x^α	$\alpha x^{\alpha-1}$	$x > 0$ for $\alpha \in \mathbb{R}$ $x \in \mathbb{R}$ for $\alpha \in \mathbb{N}$
3. a^x	$a^x \ln a$	$x \in \mathbb{R}$ ($a > 0, a \neq 1$)
4. $\log_a x $	$\frac{1}{x \ln a}$	$x \in \mathbb{R} \setminus 0$ ($a > 0, a \neq 1$)
5. $\sin x$	$\cos x$	
6. $\cos x$	$-\sin x$	
7. $\tan x$	$\frac{1}{\cos^2 x}$	$x \neq \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$
8. $\cot x$	$-\frac{1}{\sin^2 x}$	$x \neq \pi k, k \in \mathbb{Z}$
9. $\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$ x < 1$
10. $\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$	$ x < 1$
11. $\arctan x$	$\frac{1}{1+x^2}$	
12. $\operatorname{arccot} x$	$-\frac{1}{1+x^2}$	
13. $\sinh x$	$\cosh x$	
14. $\cosh x$	$\sinh x$	
15. $\tanh x$	$\frac{1}{\cosh^2 x}$	
16. $\coth x$	$-\frac{1}{\sinh^2 x}$	$x \neq 0$
17. $\operatorname{arsinh} x = \ln(x + \sqrt{1+x^2})$	$\frac{1}{\sqrt{1+x^2}}$	
18. $\operatorname{arcosh} x = \ln(x \pm \sqrt{x^2-1})$	$\pm \frac{1}{\sqrt{x^2-1}}$	$ x > 1$
19. $\operatorname{artanh} x = \frac{1}{2} \ln \frac{1+x}{1-x}$	$\frac{1}{1-x^2}$	$ x < 1$
20. $\operatorname{arcoth} x = \frac{1}{2} \ln \frac{x+1}{x-1}$	$\frac{1}{x^2-1}$	$ x > 1$

Figure 2: Table of Derivatives of the Basic Elementary Functions.

Examples 16.

$f(x)$	$f'(x)$	$f''(x)$	\dots	$f^{(n)}(x)$
a^x	$a^x \ln a$	$a^x \ln^2 a$	\dots	$a^x \ln^n a$
e^x	e^x	e^x	\dots	e^x
$\sin x$	$\cos x$	$-\sin x$	\dots	$\sin(n + n\pi/2)$
$\cos x$	$-\sin x$	$-\cos x$	\dots	$\cos(n + n\pi/2)$
$(1+x)^\alpha$	$\alpha(1+x)^{\alpha-1}$	$\alpha(\alpha-1)(1+x)^{\alpha-2}$	\dots	$\alpha(\alpha-1)\dots(\alpha-n+1)(1+x)^{\alpha-n}$
x^α	$\alpha x^{\alpha-1}$	$\alpha(\alpha-1)x^{\alpha-2}$	\dots	$\alpha(\alpha-1)\dots(\alpha-n+1)x^{\alpha-n}$
$\log_a x $	$\frac{1}{\ln a} x^{-1}$	$\frac{-1}{\ln a} x^{-2}$	\dots	$\frac{(-1)^{n-1}(n-1)!}{\ln a} x^{-n}$
$\ln x $	x^{-1}	$-x^{-2}$	\dots	$(-1)^{n-1}(n-1)!x^{-n}$

Examples 17 (Leibniz's formula). Let $u(x)$ and $v(x)$ be functions having derivatives up to order n inclusive on a common set E . The following formula of Leibniz holds for the n th derivative of their product:

$$(uv)^{(n)} = \sum_{m=0}^n \binom{n}{m} u^{(n-m)} v^{(m)} \quad (4)$$

Examples 18. If $P_n(x) = c_0 + c_1x + \dots + c_nx^n$, then

$$\begin{aligned}
 P_n(0) &= c_0 \\
 P'_n(x) &= c_1 + 2c_2x + \dots + nc_nx^{n-1} \Rightarrow P'_n(0) = c_1 \\
 P''_n(x) &= 2c_2 + 3 \cdot 2c_3x + \dots + n(n-1)c_nx^{n-2} \Rightarrow P''_n(0) = 2!c_2 \\
 P_n^{(3)}(x) &= 3 \cdot 2c_3 + \dots + n(n-1)(n-2)c_nx^{n-3} \Rightarrow P_n^{(3)}(0) = 3!c_3 \\
 &\vdots \\
 P_n^{(n)}(x) &= n(n-1)(n-2)\dots 2c_n \Rightarrow P_n^{(n)}(0) = n!c_n \\
 P_n^{(k)} &= 0 \quad \text{for } k > n.
 \end{aligned}$$

Thus, the polynomial $P_n(x)$ can be written as

$$P_n(x) = P_n^{(0)} + \frac{1}{1!}P_n^{(1)}(0)x + \frac{1}{2!}P_n^{(2)}(0)x^2 + \dots + \frac{1}{n!}P_n^{(n)}(0)x^n.$$

Examples 19. Using Leibniz's formula and the fact that all the derivatives of a polynomial of order higher than the degree of the polynomial are zero, find the n th derivative of the following functions:

$$x^2 \sin x, x^2 \sinh x, x^2 \ln x, x^2 \sin x \cos x, x^2 e^x.$$

Examples 20. Let $f(x) = \arctan x$, find the values $f^{(n)}(0)$ ($n = 1, 2, \dots$),

Examples 21. Let f be a differentiable function on \mathbb{R} . Show that

1. if f is an even function, then f' is an odd function,
2. if f is an odd function, then f' is an even function,
3. f' is odd $\Leftrightarrow f$ is even.

5 作業

5.1 解答題

設

$$f(x) = \begin{cases} x^2, & x \geq 3 \\ ax + b, & x < 3 \end{cases}$$

試確定 a, b 的值，使 f 在 $x = 3$ 處可導。

5.2 解答題

求下列曲線在指定點處的切線，法線方程。

(1) $y = \frac{x^2}{4}, P(2, 1)$

(2) $y = \cos x, P(0, 1)$

5.3 解答題

求下列函數的導數

(1) $f(x) = |x|^3$

(2) $f(x) = \begin{cases} x + 1, & x \geq 0 \\ 1, & x < 0 \end{cases}$

5.4 解答題

設函數

$$f(x) = \begin{cases} x^\alpha \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

試問：

(1) α 為何值時，函數在 $x = 0$ 點連續；

(2) α 為何值時，函數在 $x = 0$ 點可導。

5.5 求下列函數的導數

(1) $y = 3x^2 + 2$

(2) $y = \frac{1 - x^2}{1 + x + x^2}$

(3) $y = x^n + nx$

(4) $y = \frac{x}{m} + \frac{m}{x} + 2\sqrt{x} + \frac{2}{\sqrt{x}}$

(5) $y = x^3 \log_3 x$

(6) $y = e^x \cos x$

(7) $y = (x^2 + 1)(3x - 1)(1 - x^3)$

(8) $y = \frac{\tan x}{x}$

(9) $y = \frac{x}{1 - \cos x}$

(10) $y = \frac{1 + \ln x}{1 - \ln x}$

(11) $y = x\sqrt{1 - x^2}$

(12) $y = (x^2 - 1)^3$

(13) $y = \left(\frac{1 + x^2}{1 - x} \right)^3$

(14) $y = \ln(\ln x)$

(15) $y = \ln(\sin x)$

(16) $y = \ln(x + \sqrt{1 + x^2})$

(17) $y = \ln \left(\frac{\sqrt{1 + x} - \sqrt{1 - x}}{\sqrt{1 + x} + \sqrt{1 - x}} \right)$

(18) $y = (\sin x + \cos x)^3$

(19) $y = (\sin x^2)^3$

(20) $y = \arcsin(\sin^2 x)$

$$(21) \quad y = x^{x^x}$$

$$(22) \quad y = \sqrt{x + \sqrt{x + \sqrt{x}}}$$

$$(23) \quad y = \sin(\sin(\sin x))$$

$$(24) \quad y = \sin\left(\frac{x}{\sin\left(\frac{x}{\sin x}\right)}\right)$$

$$(25) \quad y = (x - a_1)^{a_1}(x - a_2)^{a_2} \cdots (x - a_n)^{a_n}$$

5.6 求下列函數在指定點的高階導數

$$(1) \quad f(x) = 3x^3 + 4x^2 - 5x - 9, \text{ 求 } f'''(1), f^{(4)}(x)(1)$$

$$(2) \quad f(x) = \frac{x}{\sqrt{1+x^2}}, \text{ 求 } f''(0), f''(1), f''(-1)$$

5.7 求下列函數的高階導數

$$(1) \quad f(x) = x \ln x, \text{ 求 } f''(x)$$

$$(2) \quad f(x) = e^{-x^2}, \text{ 求 } f'''(x)$$

$$(3) \quad f(x) = \ln(1+x), \text{ 求 } f^{(5)}(x)$$

$$(4) \quad f(x) = x^3 e^x, \text{ 求 } f^{(10)}(x)$$

5.8 解答題

設 f 為二階可導函數，求下列函數的二階導數

$$(1) \quad f(\ln x)$$

$$(2) \quad f(x^n)$$

$$(3) \quad f(f(x))$$

5.9 解答題

求下列函數的 n 階導數

$$(1) \ y = \ln x$$

$$(2) \ y = a^x (a > 0, a \neq 1)$$

$$(3) \ y = \frac{1}{x(1-x)}$$

$$(4) \ y = \frac{\ln x}{x}$$

$$(5) \ y = \frac{x^n}{1-x}$$

5.10 解答題

求下列參數方程所確定的函數的二階導數

$$(1) \ \begin{cases} x = a \cos^3 t \\ y = a \sin^3 t \end{cases}$$

$$(2) \ \begin{cases} x = e^t \cos t \\ y = e^t \sin t \end{cases}$$