
Chapter 4 函數的連續性

标签（空格分隔）： 連續 一致連續 初等函數

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§4.1 連續函數的定義

4.1.1 函數在一點連續

> **定義1.1** 設函數f在$U(x_0)$上有定義，若
>
> $$\lim_{x\to x_0} f(x)=f(x_0) \tag{1.1}$$
>
> 則稱函數f在x_0點連續。

- [x] 函數f在x_0點連續的邏輯語言為:

$$\forall \epsilon>0, \exists U(x_0), \forall x\in U(x_0) \Rightarrow |f(x)-f(x_0)|<\epsilon \tag{1.2}$$

- [x] 若記$\Delta y=f(x_0+\Delta x)-f(x_0)$, 函數$f(x)$在$x_0$點連續的充分必要條件為:

$$\lim_{\Delta x\to 0}\Delta y=0 \tag{1.3}$$

- [x] 函數$f(x)$在x_0點連續 $\lim\limits_{x\to x_0} f(x)=f(\lim\limits_{x\to x_0} x)=f(x_0)$.

> **定義1.2** 函數$f: E\to\mathbb{R}$為E上的連續函數，如果函數在E上的每一點連續。

例子1.1 如果$f(x)\equiv C, x\in E$,則$f(x)$在E上連續。
例子1.2 函數$f(x)=x$在$\mathbb{R}$上連續。
例子1.3 函數$f(x)=\sin x, f(x)=\cos x$ 在$\mathbb{R}$上連續。

例子1.4 函數$f(x) = a^x$在$\mathbb{R}$上連續。
例子1.5 函數$f(x) = \log_a x$在$\mathbb{R}$上連續。

4.1.2 間斷點及其分類

> **定義1.3** 若函數$f(x), x \in E$在$x_0 \in E$點不連續，則稱x_0為函數$f(x)$的一個間斷點。

- f在x_0無定義，或者$\lim\limits_{x \to x_0} f(x)$ 不存在， 則函數f在x_0點間斷。
- 若$\lim\limits_{x \to x_0} f(x) \neq f(x_0)$， 則函數$f$在$x_0$點間斷。
- 函數f在$x_0 \in E$點間斷的邏輯語言為：

$$\exists \epsilon_0 > 0, \forall \delta > 0, \exists \tilde{x} \in E \Rightarrow |f(\tilde{x}) - f(x_0)| \geq \epsilon_0 \tag{1.4}$$

例子1.6 函數$f(x) = \operatorname{sgn}(x)$在$x_0 = 0$點間斷。

> **定義1.3** 設$x_0 \in E$為函數$f(x) : E \to \mathbb{R}$的一個間斷點， 若存在一個連續函數$\tilde{f} : E \to \mathbb{R}$使得：$f|_{E\setminus a} = \tilde{f}|_{E\setminus a}$. 則稱$x_0$為函數$f(x)$的**可去間斷點**。
> * 如果x_0為函數$f(x)$的可去間斷點，則有$\lim\limits_{x \to x_0} f(x) = A$存在，但是$A \neq f(x_0)$.

例子1.7 討論函數

$$f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

在$x = 0$點的連續性。

> **定義1.4** 若函數$f(x)$在x_0點有：$\lim\limits_{x \to x_0^+} f(x) = A \neq \lim\limits_{x \to x_0^-} f(x) = B$，則稱$x_0$為函數$f(x)$的**跳躍間斷點**。

例子1.8 討論函數$y = \lfloor x \rfloor$的間斷點及其類型。

> **定義1.5** 可去間斷點和跳躍間斷點統稱為**第一類間斷點**， 不是第一類間斷點的間斷點稱為**第二類間斷點**。

例子1.9 討論函數$f(x) = \sin \dfrac{1}{x}$的間斷點及其類型。
例子1.10 討論***Dirichlet***函數

$$\mathcal{D}(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

的間斷點及其類型。
例子1.11 討論***Riemann***函數

$$\mathcal{R}(x) = \begin{cases} \frac{1}{n}, & x = \frac{m}{n} \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

的間斷點及其類型。

Homework

- **1.** 按照定義證明下列函數在其定義域上連續：
  (1). $f(x) = \dfrac{1}{x}$
  (2). $f(x) = |x|$

- **2.** 指出下列函數的間斷點並說明其類型：

  (1). $f(x) = x + \dfrac{1}{x}$
  (2). $f(x) = \dfrac{\sin x}{|x|}$
  (3). $f(x) = \lfloor |\cos x| \rfloor$
  (4). $f(x) = \operatorname{sgn}|x|$
  (5). $f(x) = \operatorname{sgn}(\cos x)$
  (6). $f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ -x, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

- **3.** 延托下列函數，使其在$\mathbb{R}$上連續：
  (1). $f(x) = \dfrac{x^3 - 8}{x - 2}$
  (2). $f(x) = \dfrac{1 - \cos x}{x^2}$
  (3). $f(x) = x \cos \dfrac{1}{x}$

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§ 4.2 連續函數的性質

4.2.1 連續函數的局部性質

> **定理2.1** 設$f : E \to \mathbb{R}$在$x_0 \in E$點連續，則下列結論成立：
> (1). 函數$f : E \to \mathbb{R}$在x_0的某鄰域$U_\delta(x_0)$上有界。
> (2). 如果$f(x_0) \neq 0$，則存在x_0的某鄰域$U_\delta(x_0)$，對於$\forall x \in U_\delta(x_0)$，有$f(x)$與$f(x_0)$同號。
> (3). 如果函數$g : U_E(x_)) \to \mathbb{R}$定義在$x_0$的某鄰域上，且在$x_0$點連續，則下列函數在$x_0$點連續：
>
> > a) $(f + g)(x) = f(x) + g(x)$
> > b) $(f \cdot g)(x) = f(x) \cdot g(x)$
> > c) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} (g(x_0) \neq 0)$
>
> (4). 如果函數$g : Y \to \mathbb{R}$在$b \in Y$點連續，且f滿足：$f : E \to Y, f(x_0) = b$，f在x_0點連續，則復合函數$g \circ f$在x_0點連續。

Example 2.1 求$\lim_{x\to 1}\sin(1-x^2)$

example 2.2 求極限：(1) $\lim_{x\to 0}\sqrt{2-\frac{\sin x}{x}}$; (2) $\lim_{x\to\infty}\sqrt{2-\frac{\sin x}{x}}$

Example 2.3 An algebraic polynomail $P(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n$ is a continuous function on $\mathbb{R}$.

Example 2.4 A rational function $R(x)=\dfrac{P(x)}{Q(x)}$ (a quotient polynomial) is continuous wherever it is defined, that is, $Q(x)\neq 0$.

4.2.2 Global Properties of Continuous Functions

> **Theorem2 (The Bolzano-Cauchy intermediate-value theorem)**
> If a function that is continuous on a closed interval assumes values with different signs at the endpoints of the interval, then there is a point in the interval where it assume the value 0
>
> > In logical symbols, this theorem has the following expression.
> >
> > $$f\in C[a,b]\wedge f(a)\cdot f(b)<0\Rightarrow \exists c\in[a,b], f\text{©}=0.$$

Remarks to Theorem2

Remarks 1 The proof of the theorem provides a very simple algorithm for find a root of the equation $f(x)=0$ on an interval at whose endpoints a continuous function $f(x)$ has values with opposite signs.

```
   # bisection method for zeros finding
   #!/usr/bin/env python
   # -*- coding: UTF-8 -*-

class ZeroFinding(object):
    def __init__(self, fun):
        '''
        fun: the input function.
        root: the roots of the input function.
        val: the function value at the root.
        '''
        self.fun = fun
        self.root = 0.0
        self.val = 0.0

    def __str__(self):
        '''
        print the root and val
        '''
        return 'root\t:%f\nval\t:%f\n' % (self.root, self.val)

    def bisection(self, a, b):
        import os, sys
        import numpy as np
        '''
```

```
        bisection method for zeros finding
        '''
        EPSILON = 0.000001
        if self.fun(a)*self.fun(b) > 0:
            print('The values at the end points assume not opposite signs')
            sys.exit(1)
        while abs(b-a)>EPSILON:
            c = (a + b) / 2.0
            if abs(self.fun(c)) < EPSILON:
                self.root = c
                self.val = self.fun(c)
                return
            elif self.fun(c)*self.fun(a) > 0:
                a = c
            else:
                b = c

        self.root = c
        self.val = self.fun(c)


def f(x):
    '''
    define a test function
    '''
    return x*x*x-1.0


if __name__ == '__main__':
    obj = ZeroFinding(f)
    obj.bisection(-1.0, 2.0)
    print(obj)
```

The zeros of the function: $f(x) = x^3 - 1$ is:

```
root      :1.000000
val       :0.000001
```

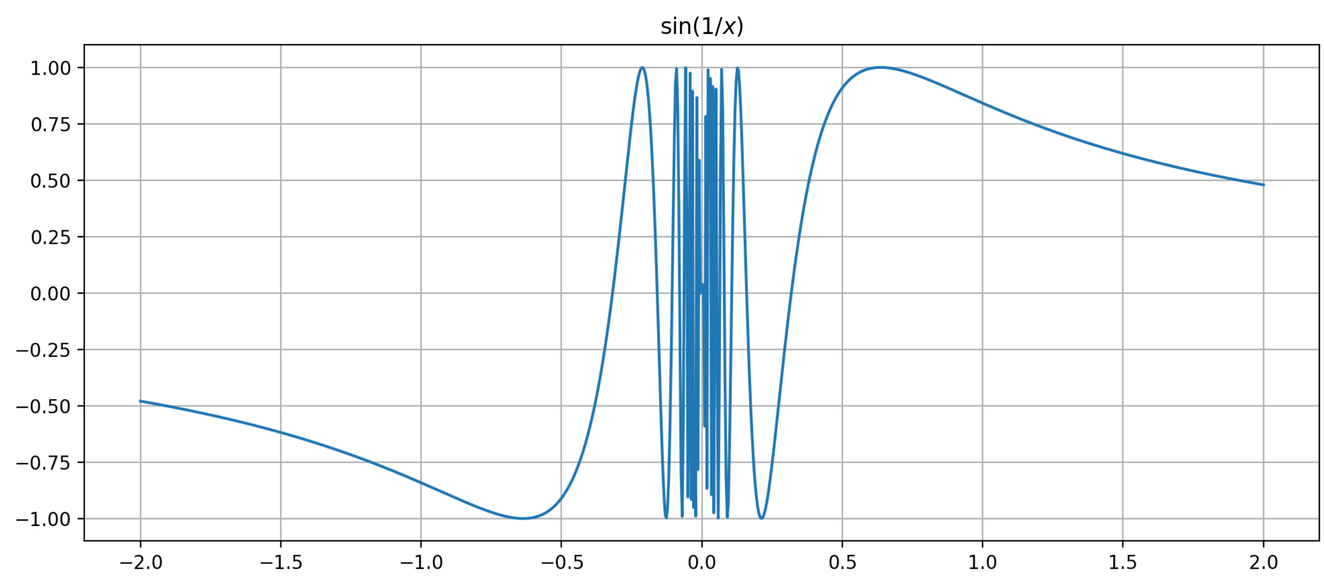
Remark2 The theorem asserts that it is impossible to pass continuously from positive to negative values without assuming the value zero along the way.

> **Corollary to Theorem 2** If the function ϕ is continuous on an open interval and assumes values $\phi(a) = A, \phi(b) = B$ at points a, b, then for any number C between A and B, there is a point ξ between a, b at which $\phi(\xi) = C$.

> **Theorem 3(The Weierstrass maximum-value theorem)** A function that is continuous on a closed interval is bounded on that interval. Moreover there is a point in the interval where the function assume its **maximum** vale and a point where it assumes its **minimal** value.

> **Definition1** A function $f : E \to \mathbb{R}$ is uniformly continuous on a set $E \subset \mathbb{R}$ if for every $\epsilon > 0$ there exist a $\delta > 0$ such that $|f(x_1) - f(x_2)| < \epsilon$ for all points $x_1, x_2 \in E$ such that $|x_1 - x_2| < \delta$.
> **logicial language:** $f : E \to \mathbb{R}$ is uniformly continuous :=
>
> $$\forall \epsilon > 0, \exists \delta > 0, \forall x_1, x_2 \in E, |x_1 - x_2| < \epsilon \Rightarrow |f(x_1) - f(x_2)| < \epsilon.$$
>
> **Remark 1** Generally speaking, uniformly continuity implies pointwise continuity, pointwise continuity doesn't implies uniformly continuity.

Example2.4 The function $f(x) = \sin\left(\frac{1}{x}\right)$ is continuous at the open interval $(0, 1)$. However, at each neighborhood of 0, the function assumes both $1, -1$, so the function is not uniformly continuous at the interval $(0, 1)$.



It is useful to write out explictly the negation of property of uniform continuity for a function.

> $f : E \to \mathbb{R}$ is not uniformly continuous on E if and only if
>
> $$\exists \epsilon_0 > 0, \forall \delta > 0, \exists x_1, x_2 \in E, |x_1 - x_2| < \delta, |f(x_1) - f(x_2)| \geq \epsilon_0$$

Example 2.5 If $f : E \to \mathbb{R}$ is unbounded on every neighborhood of a fixed point $x_0 \in E$, then f is not uniformly continuous on E.

Example 2.6 The function $f(x) = x^2$ is continuous on $\mathbb{R}$ but not uniformly continuous on $\mathbb{R}$.

Example 2.7 The function $f(x) = \sin(x^2)$ is continous on $\mathbb{R}$ but not uniformly continuous on it.

> **Theorem 4(The Cantor-Heine theorem on uniform continuity)** A function that is continuous on a closed interval is uniformly continuous on that interval.

> **Proposition 1** A continuous mapping $f : E \to \mathbb{R}$ of a closed interval $E = [a, b]$ into $\mathbb{R}$ is injective if and only if f is strictly monotonic on $[a, b]$.

> **Proposition 2** Each strictly monotonic function $f : X \to \mathbb{R}$ defined on a numerical set $X \subset \mathbb{R}$ has an inverse $f^{-1} : Y \to \mathbb{R}$ defined on the set $Y = f(X)$ of values f, and has the same kind of monotonicity on Y that f has on X.

> **Proposition 3** The discontinuities of a function $f : E \to \mathbb{R}$ that is monotonic on the set $E \subset \mathbb{R}$ can be only discontinuities of **fist kind**.
>
> > **Corollary 1** If a is a point of discontinuity of a monotonic function $f : E \to \mathbb{R}$, then at least one of the limits:
> >
> > $$\lim_{x \to a^+} f(x) = f(a+0), \lim_{x \to a^-} f(x) = f(a-0)$$
> >
> > exists.
> >
> > **Corollary 2** The set of points of discontinuities of a monotonic function is at most countable.

> **Proposition 4(A criterion for continuity of a monotonic function)** A monotonic function $f : E \to \mathbb{R}$ defined on a closed interval $E = [a, b]$ is continuous if and only if its set of values $f(E)$ is the closed interval with endpoints $f(a)$ and $f(b)$.
>
> **Theorem 5(The inverse function theorem)** A function $f : X \to \mathbb{R}$ that is strictly monotonic on a set $X \subset \mathbb{R}$ has an inverse $f^{-1} : Y \to \mathbb{R}$ of values f. The function $f^{-1} : Y \to \mathbb{R}$ is monotonic and has the same type of monotonicity on Y that f has on X.
> If in addition X is a closed interval $[a, b]$ and f is continuous on X, then the set $Y = f(X)$ is the closed interval with endpoints $f(a)$ and $f(b)$ and the function $f^{-1} : Y \to \mathbb{R}$ is continuous on it.

Homework

- **1.** 設f, g在區間I上連續，記

  $$F(x) = \max\{f(x), g(x)\}, G(x) = \min\{f(x), g(x)\}.$$

  證明F, G在區間I上連續。

- **2.** 設f為$\mathbb{R}$上的連續函數，常數$c > 0$,記

$$F(x)=\begin{cases} -c, & 若 f(x)<-c \\ f(x)\ , & 若 |f(x)|\le c \\ c, & 若 f(x)>c \end{cases}$$

證明$F(x)$在$\mathbb{R}$上連續。

- **3.**若對於任何充分小的$\epsilon>0, f$在$[a+\epsilon, b-\epsilon]$上連續，能否推出f在(a,b)內連續?
- **4.**求極限
  (1) $\lim\limits_{x\to\frac{\pi}{4}}(\pi-x)\tan x$;
  (2) $\lim\limits_{x\to 1^+}\dfrac{x\sqrt{1+2x}-\sqrt{x^2-1}}{x+1}$
- **5.** 證明：任何一個實係數奇次方程至少有一個實根。
- **6.** 試用一致連續的定義證明：若f, g在區間I上一致連續，則$f+g$也在I上一致連續。
- **7.** 證明：$f(x)=x^2$在$[a,b]$上一致連續，但在$\mathbb{R}$上不一致連續。

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