The Differential of a Function of Several Variables

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1 Double Integral

1.1 Introduction

Volume Problem: Find the volume V of the solid G enclosed between the surface $z = f(x, y)$ and the region R in the $xy-$plane where $f(x, y)$ is continuous and non-negative on R.

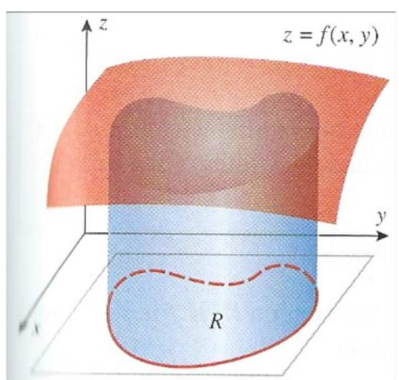


Figure 1: Volume.

Mass Problem: Find the mass M of a lamina (a region R in the $xy-$plane) whose density is a continuous nonnegative function $\rho(x, y)$

Let us consider the Volume Problem,

1. Divide the rectangle enclosing R into subrectangles, and exclude all those rectangles that contain points outside R. Let n be the number of all the rectangles inside R, and let $\Delta A_k = \Delta x_k \Delta y_k$ be the area of the $k-$th subrectangle.

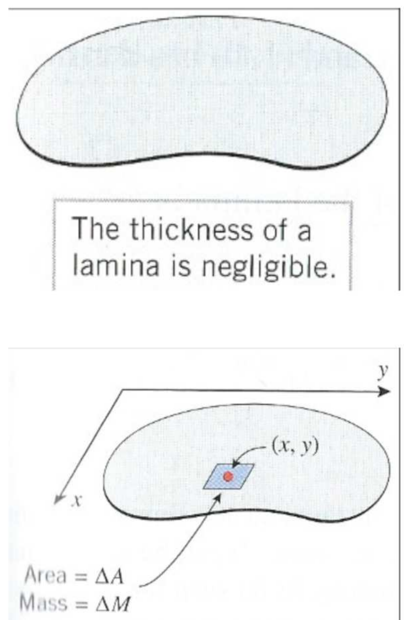


Figure 2: Mass.

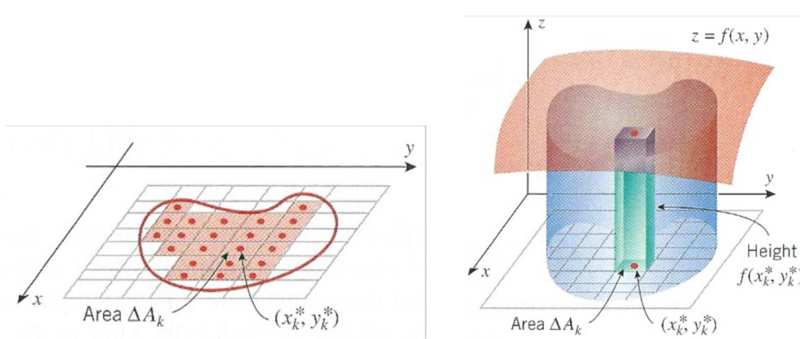


Figure 3: Volume.

2. Choose any point (ξ_k, η_k) in the $k-$th subrectangle. The volume of a rectangular parallelepiped with base area ΔA_k and height $f(\xi_k, \eta_k)$ is $\Delta V_k = f(\xi_k, \eta_k)\Delta A_k$, thus,

$$V \approx \sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} f(\xi_k, \eta_k)\Delta A_k = \sum_{k=1}^{n} f(\xi_k, \eta_k)\Delta x_k \Delta y_k \tag{1}$$

   This sum is called the *Rimann sum*.

3. Take the sides of all the subrectangles to 0, and get

$$V = \lim_{\lambda(P)\to 0} \sum_{k=1}^{n} f(\xi_k, \eta_k)\Delta A_k = \iint_R f(x, y)\,\mathrm{d}A \tag{2}$$

   The last term is the notation for the limit of the Riemann sum, and it is called the *double integral* of $f(x, y)$ over R.

1.2 The Darboux Criterion

Let us consider another criterion for Riemann integrability of a function, which is applicable only to real-valued function. Lower and Upper Darboux Sums. Let f be a real-valued function on the interval I and $P = I_i$ a partition of the interval I. We set

$$m_i = \inf_{x\in I_i} f(x), M_i = \sup_{x\in I_i} f(x). \tag{3}$$

The quantities

$$s(f, P) = \sum_i m_i|I_i|, S(f, P) = \sum_i M_i|I_i| \tag{4}$$

are called the *lower* and *upper sums* of the function f over the interval I corresponding to the partion P of the interval.

The following relations hold between the Darboux sums a a function $f : I \to \mathbb{R}$:

1. $s(f, P) = \inf_\xi \sigma(f, P, \xi) \leq \sigma(f, P, \xi) \leq \sup_\xi \sigma(f, P, \xi) = S(f, P)$

2. If the partition P' of the interval I is obtained by refining intervals of the partion P, then $s(f, P) \leq s(f, P') \leq S(f, P') \leq S(f, P)$

3. The inequality $s(f, P_1) \leq S(f, P_2)$ holds for any pair of partition P_1, P_2 of the interval I.

1.3 Lower and Upper Integrals

The lower and upper Darboux integrals of the function $f : I \to \mathbb{R}$ over the interval I are respectively

$$\underline{I} = \sup_P s(f, P), \overline{I} = \inf_P S(f, P) \tag{5}$$

where the supremum and infimum are taken over all partitions P of the interval I.

Theorem 1.1. *For any bounded function:*

$$f : I \to \mathbb{R}, \lim_{\lambda(P)\to 0} s(f, P) = \underline{I}, \lim_{\lambda(P)\to 0} S(f, P) = \overline{I}$$

1.4 The Darboux Criterion for Integrability of a Real-valued Function

Theorem 1.2 ((The Darboux Criterion)**.** *A real valued function $f : I \to \mathbb{R}$ defined on an interval $I \subset \mathbb{R}^\ltimes$ is integrable over that interval if and only if it is bounded on I and its upper and lower Darboux integrals are equal.*

$$f \in \mathcal{R}(I) \equiv f \textit{ is bounded on } I, \textit{and} \underline{I} = \overline{I}.$$

1.5 Double Integral Over Non-rectangular Regions

$$\int_a^b \int_c^d f(x, y)\,\mathrm{d}y\mathrm{d}x = \int_a^b \left[\int_c^d f(x, y)\right] \mathrm{d}x \tag{6}$$

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,\mathrm{d}y\mathrm{d}x = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y)\mathrm{d}y\right] \mathrm{d}x \tag{7}$$

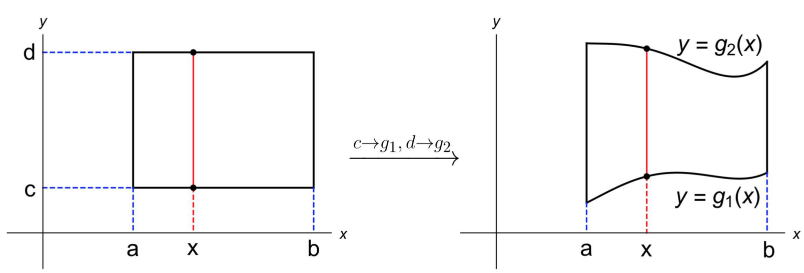


Figure 4: Double Integral.

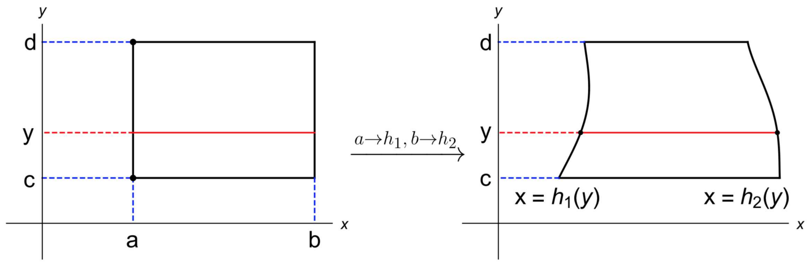


Figure 5: Double Integral.

$$\int_a^b \int_{h_1(y)}^{h_2(y)} f(x,y)\,\mathrm{d}x\mathrm{d}y = \int_a^b \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\mathrm{d}x\right]\mathrm{d}y \tag{8}$$

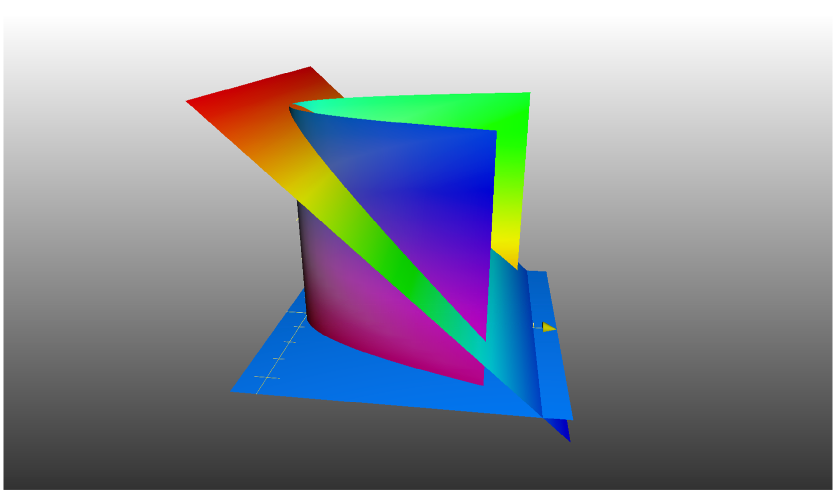
Examples 1. *1.* $\int_0^1 \int_1^2 \sin(2x-3y)\mathrm{d}x\mathrm{d}y$

2. $\int_1^2 \int_0^1 \sin(2x-3y)\mathrm{d}y\mathrm{d}x$

Theorem 1.3. *Let D be the rectangle $[a,b]\times[c,d]$. If $f(x,y)$ is continuous on D then*

$$\iint f(x,y)\,\mathrm{d}\sigma = \int_a^b \int_c^d f(x,y)\,\mathrm{d}y\mathrm{d}x = \int_c^d \int_a^b f(x,y)\,\mathrm{d}x\mathrm{d}y$$

Examples 2. *Find the volume V of the solid enclosed by the surfaces $z = 0, y^2 = x, x+z = 1$*



Examples 3. *Reverse the order of the integrals below:*

1. $\displaystyle\int_a^b \mathrm{d}x \int_a^x f(x,y)\mathrm{d}y$
2. $\displaystyle\int_0^1 \mathrm{d}y \int_0^{2y} f(x,y)\mathrm{d}x + \int_1^3 \mathrm{d}y \int_0^{3-y} f(x,y)\mathrm{d}x$

1.6 Double integrals in polar coordinates

2 作業

1. 根據二重積分的性質，比較下列積分的大小
   (a) $\iint_D (x+y)^2 \,\mathrm{d}x\mathrm{d}y, \iint_D (x+y)^3 \,\mathrm{d}x\mathrm{d}y$，其中$D$為$x$軸，$y$軸與直線$x+y=1$ 所圍成的區域。
   (b) $\iint_D \ln(x+y) \,\mathrm{d}x\mathrm{d}y, \iint_D [\ln(x+y)]^2 \,\mathrm{d}x\mathrm{d}y$，其中$D=[3,5]\times[0,1]$。
2. 計算下列重積分
   (a) $\iint_D x^3+3x^2y+y^3 \,\mathrm{d}x\mathrm{d}y$，其中$D=[0,1]\times[0,1]$。
   (b) $\iint_D xye^{x^2+y^2} \,\mathrm{d}x\mathrm{d}y$，其中$D=[a,b]\times[c,d]$。
   (c) $\iint_D xy^2 \,\mathrm{d}x\mathrm{d}y$，其中$D$由拋物線$y^2=2px$ 和直線$x=\frac{p}{2}(p>0)$所圍成的區域。
   (d) $\iint_D e^{x+y} \,\mathrm{d}x\mathrm{d}y$，其中$D$為$\{(x,y)|\,|x|+|y|\le 1\}$所圍成的區域。
   (e) $\iint_D x \,\mathrm{d}x\mathrm{d}y$，其中$D$為拱線的一拱$x=a(t-\sin t), y=a(1-\cos t)(0\le t\le 2\pi)$所圍成的區域。