# The Differential of a Function of Several Variables

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# 1 Double Integral

## 1.1 Introduction

*Volume Problem*: Find the volume $V$ of the solid $G$ enclosed between the surface $z = f(x, y)$ and the region $R$ in the $xy-$plane where $f(x, y)$ is continuous and non-negative on $R$.

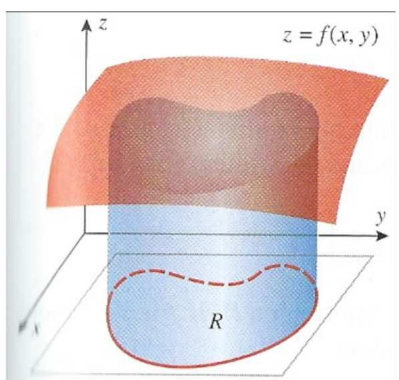



Figure 1: Volume.

*Mass Problem:* Find the mass $M$ of a lamina (a region $R$ in the $xy-$plane) whose density is a continuous nonnegative function $\rho(x, y)$

Let us consider the Volume Problem,

1. Divide the rectangle enclosing $R$ into subrectangles, and exclude all those rectangles that contain points outside $R$. Let $n$ be the number of all the rectangles inside $R$, and let $\Delta A_k = \Delta x_k \Delta y_k$ be the area of the $k-$th subrectangle.

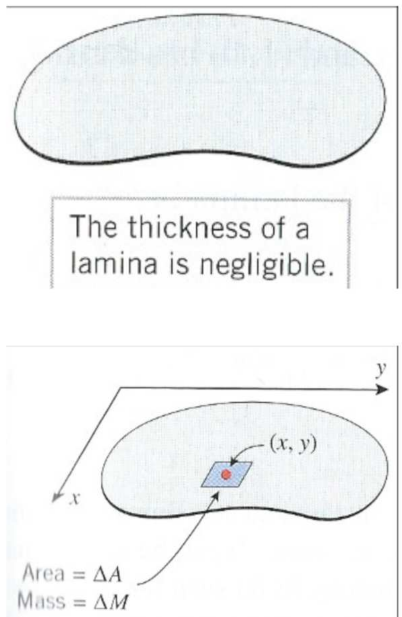



Figure 2: Mass.

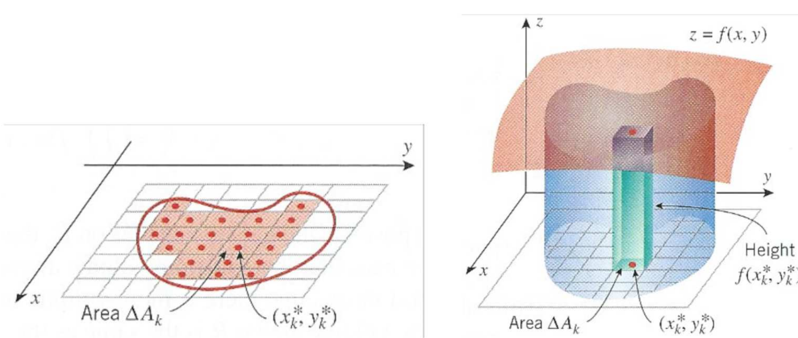



Figure 3: Volume.

2. Choose any point $(\xi_k, \eta_k)$ in the $k-$th subrectangle. The volume of a rectangular parallelepiped with base area $\Delta A_k$ and height $f(\xi_k, \eta_k)$ is $\Delta V_k = f(\xi_k, \eta_k)\Delta A_k$, thus,

$$V \approx \sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} f(\xi_k, \eta_k)\Delta A_k = \sum_{k=1}^{n} f(\xi_k, \eta_k)\Delta x_k \Delta y_k \tag{1}$$

   This sum is called the *Rimann sum*.

3. Take the sides of all the subrectangles to 0, and get

$$V = \lim_{\lambda(P)\to 0} \sum_{k=1}^{n} f(\xi_k, \eta_k)\Delta A_k = \iint_R f(x, y)\, \mathrm{d}A \tag{2}$$

   The last term is the notation for the limit of the Riemann sum, and it is called the *double integral* of $f(x, y)$ over $R$.

## 1.2 The Darboux Criterion

Let us consider another criterion for Riemann integrability of a function, which is applicable only to real-valued function. Lower and Upper Darboux Sums. Let $f$ be a real-valued function on the interval $I$ and $P = I_i$ a partition of the interval $I$. We set

$$m_i = \inf_{x\in I_i} f(x), M_i = \sup_{x\in I_i} f(x). \tag{3}$$

The quantities

$$s(f, P) = \sum_i m_i |I_i|, S(f, P) = \sum_i M_i |I_i| \tag{4}$$

are called the *lower* and *upper sums* of the function $f$ over the interval $I$ corresponding to the partion $P$ of the interval.

The following relations hold between the Darboux sums a a function $f : I \to \mathbb{R}$:

1. $s(f, P) = \inf_\xi \sigma(f, P, \xi) \leq \sigma(f, P, \xi) \leq \sup_\xi \sigma(f, P, \xi) = S(f, P)$
2. If the partition $P'$ of the interval $I$ is obtained by refining intervals of the partion $P$, then $s(f, P) \leq s(f, P') \leq S(f, P') \leq S(f, P)$
3. The inequality $s(f, P_1) \leq S(f, P_2)$ holds for any pair of partition $P_1, P_2$ of the interval $I$.

## 1.3 Lower and Upper Integrals

The lower and upper Darboux integrals of the function $f : I \to \mathbb{R}$ over the interval $I$ are respectively

$$\underline{I} = \sup_P s(f, P), \overline{I} = \inf_P S(f, P) \tag{5}$$

where the supremum and infimum are taken over all partitions $P$ of the interval $I$.

**Theorem 1.1.** *For any bounded function:*

$$f : I \to \mathbb{R}, \lim_{\lambda(P)\to 0} s(f, P) = \underline{I}, \lim_{\lambda(P)\to 0} S(f, P) = \overline{I}$$

## 1.4 The Darboux Criterion for Integrability of a Real-valued Function

**Theorem 1.2** ((The Darboux Criterion)**.** *A real valued function $f : I \to \mathbb{R}$ defined on an interval $I \subset \mathbb{R}^{\ltimes}$ is integrable over that interval if and only if it is bounded on $I$ and its upper and lower Darboux integrals are equal.*

$$f \in \mathcal{R}(I) \equiv f \textit{ is bounded on } I, and \underline{I} = \overline{I}.$$

## 1.5 Double Integral Over Non-rectangular Regions

$$\int_a^b \int_c^d f(x, y)\, \mathrm{d}y\mathrm{d}x = \int_a^b \left[\int_c^d f(x, y)\right] \mathrm{d}x \tag{6}$$

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, \mathrm{d}y\mathrm{d}x = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y)\mathrm{d}y\right] \mathrm{d}x \tag{7}$$

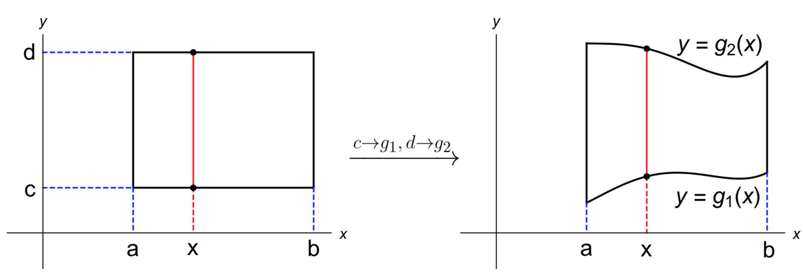



Figure 4: Double Integral.

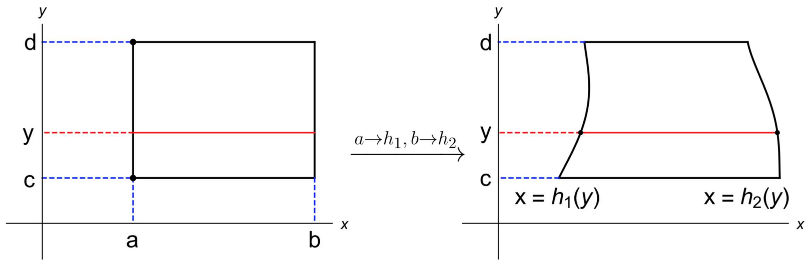



Figure 5: Double Integral.

$$\int_a^b \int_{h_1(y)}^{h_2(y)} f(x,y)\,\mathrm{d}x\mathrm{d}y = \int_a^b \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\mathrm{d}x\right] \mathrm{d}y \tag{8}$$

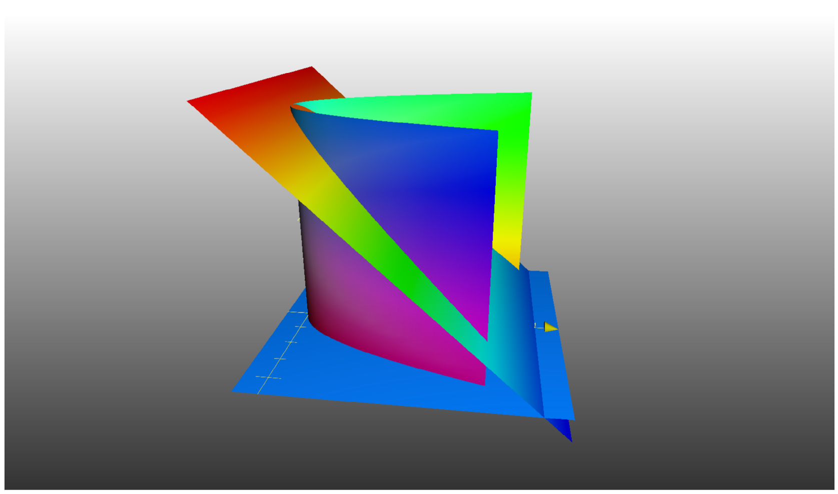
**Examples 1.** *1.* $\int_0^1 \int_1^2 \sin(2x - 3y)\mathrm{d}x\mathrm{d}y$

*2.* $\int_1^2 \int_0^1 \sin(2x - 3y)\mathrm{d}y\mathrm{d}x$

**Theorem 1.3.** *Let $D$ be the rectangle $[a,b] \times [c,d]$. If $f(x,y)$ is continuous on $D$ then*

$$\iint f(x,y)\,\mathrm{d}\sigma = \int_a^b \int_c^d f(x,y)\,\mathrm{d}y\mathrm{d}x = \int_c^d \int_a^b f(x,y)\,\mathrm{d}x\mathrm{d}y$$

**Examples 2.** *Find the volume $V$ of the solid enclosed by the surfaces $z = 0, y^2 = x, x + z = 1$*

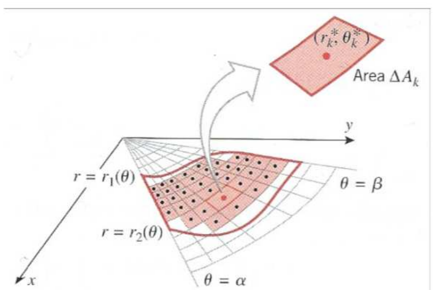


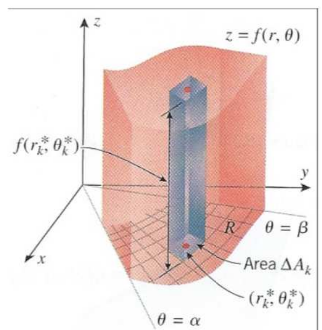
**Examples 3.** *Reverse the order of the integrals below:*

1. $\int_a^b \mathrm{d}x \int_a^x f(x,y)\mathrm{d}y$
2. $\int_0^1 \mathrm{d}y \int_0^{2y} f(x,y)\mathrm{d}x + \int_1^3 \mathrm{d}y \int_0^{3-y} f(x,y)\mathrm{d}x$

## 1.6 Double integrals in polar coordinates

Let us consider the volume problem in polar coordinates




$$\iint_D f(x,y)\,\mathrm{d}\sigma = \int_\alpha^\beta \mathrm{d}\theta \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta) r\,\mathrm{d}r$$

**Examples 4.**

$$\iint_D \frac{\mathrm{d}\sigma}{\sqrt{1-x^2-y^2}}$$

$D : \{(x,y)|\, x^2+y^2 \le 1\}$

**Examples 5.** *Compute the volume of the Figure(1.6), which the surfaces are*

$$x^2+y^2+z^2 \le R^2, x^2+y^2 = Rx$$

**Examples 6.**

$$\iint_D e^{-(x^2+y^2)}\,\mathrm{d}\sigma.$$

$D : x^2+y^2 \le R^2$

**Examples 7.** *Compute the volume*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1$$

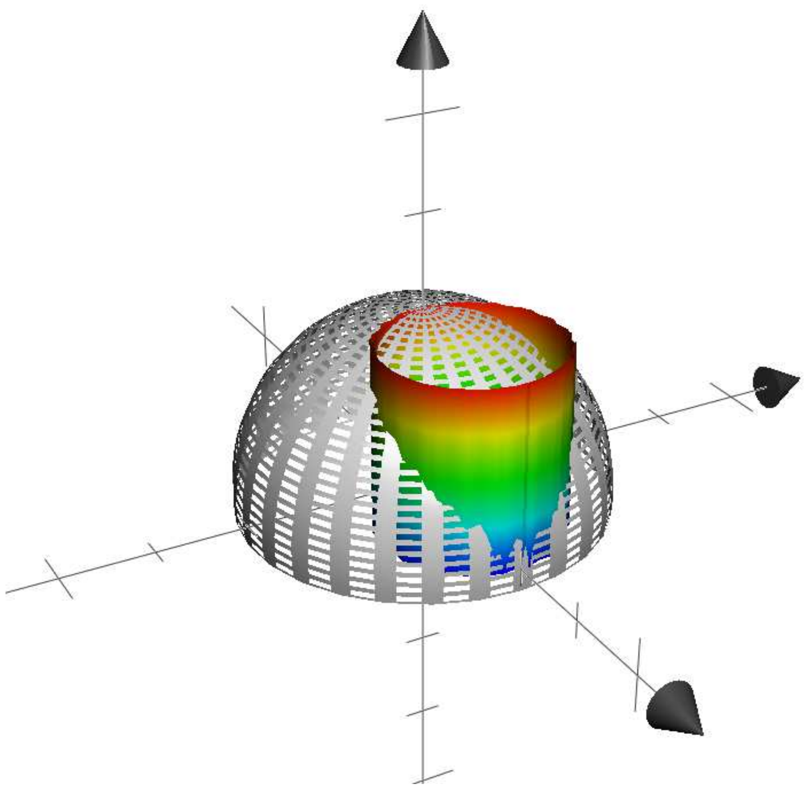


Figure 6: Volume.

**Examples 8.**

$$\iint_D e^{\frac{x-y}{x+y}}\,\mathrm{d}\sigma.$$

$D : x + y \leq 1, x \geq 0, y \geq 0.$

**Examples 9.**

$$\int_{-\infty}^{+\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$$

## 1.7 Triple Integrals

Find the mass $M$ of a solid $G$ whose density (the mass per unit volume) is a continuous nonnegative function $\rho(x, y, z)$.

### 1.7.1 Evaluating triple integrals over rectangular boxex

$$\iiint_G f(x,y,z)\,\mathrm{d}v = \int_a^b \mathrm{d}x \int_c^d \mathrm{d}y \int_k^l f(x,y,z)\,\mathrm{d}z$$

$$\iiint_G f(x,y,z)\,\mathrm{d}v = \iint_D \mathrm{d}\sigma \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,\mathrm{d}z$$

$$\iiint_G f(x,y,z)\,\mathrm{d}v = \int_a^b \mathrm{d}x \iint_D f(x,y,z)\,\mathrm{d}\sigma$$

**Examples 10.**

$$\iiint_V \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{x^2+y^2}$$

$V : x = 1, x = 2, z = 0, y = x, z = y$

**Examples 11.**

$$\iiint_V (x^2 + y^2 + z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$V : z = \sqrt{x^2 + y^2}, z = 1$

**Examples 12.**

$$\iiint_V \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\mathrm{d}v$$

$V : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$

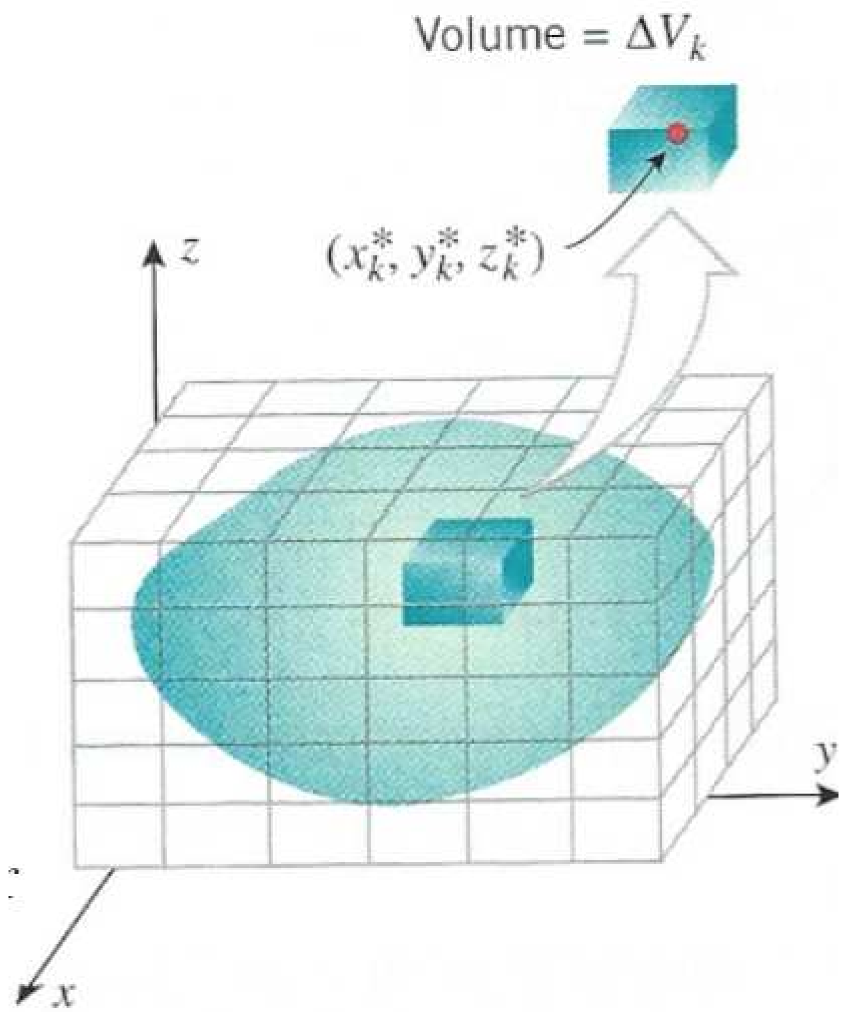



Figure 7: Mass.

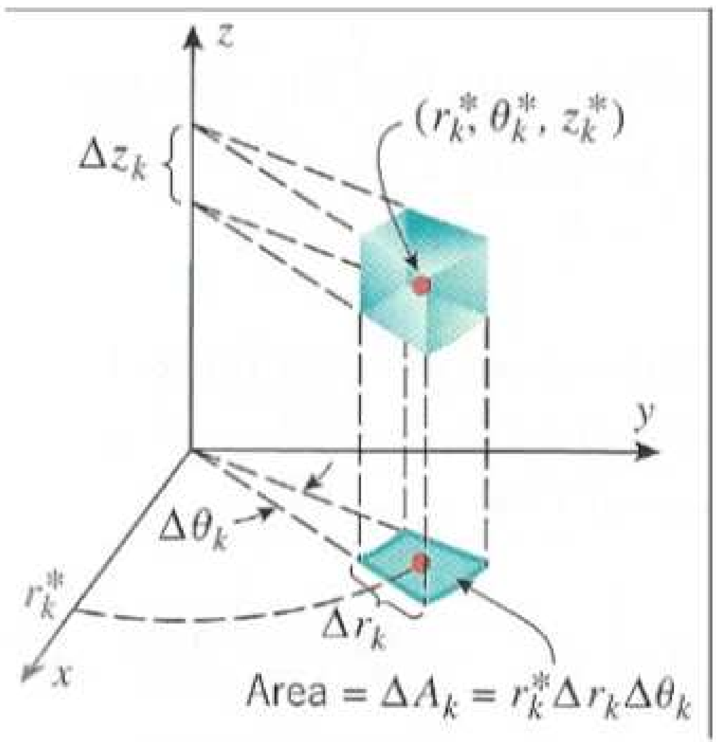



Figure 8: Cylindrical.

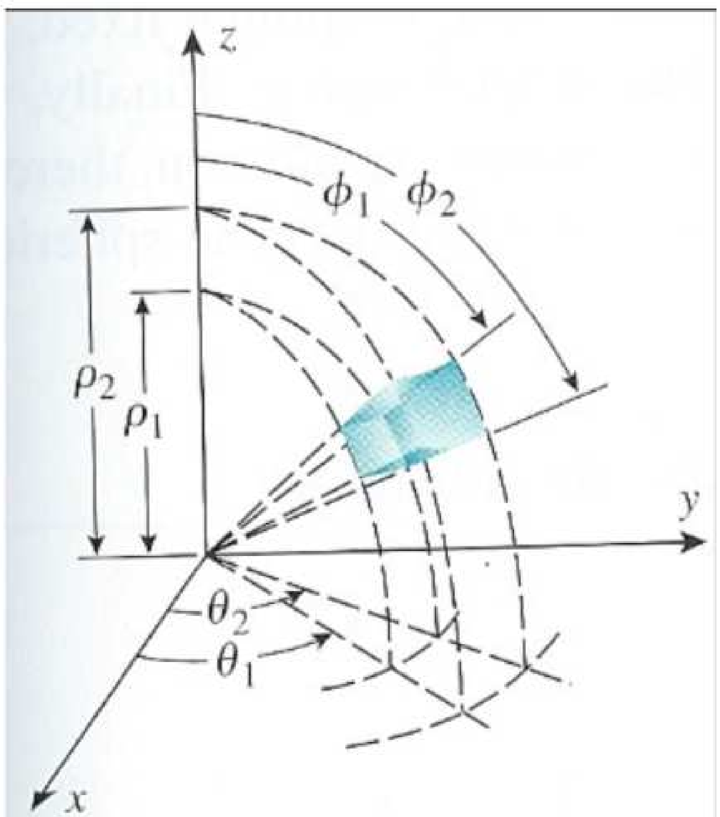



Figure 9: Spherical.

### 1.7.2 Triple integrals in cylindrical and spherical coordinates

$$\iiint_V f(x,y,z)\,\mathrm{d}v = \iiint_{V'} f(r\cos\theta, r\sin\theta, z) r\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}z.$$

$$\iiint_V f(x,y,z)\,\mathrm{d}v = \iiint_{V'} f(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi) r^2\sin\phi\,\mathrm{d}r\mathrm{d}\phi\mathrm{d}\theta.$$

**Examples 13.**

$$\iiint_\omega \left(x^2+y^2\right)\,\mathrm{d}v,$$

*$\omega$ is the volume bounded by surfaces $z = x^2+y^2, z = h$*

**Examples 14.**

$$\iiint_\omega ze^{-(x^2+y^2+z^2)}\,\mathrm{d}v,$$

*$\omega$ is the volume bounded by surfaces $z = \sqrt{x^2+y^2}, x^2+y^2+z^2 = 1$*

**Examples 15.** *Compute the volume of the* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2} \le 1$

**Examples 16.** *Compute the volume of* $x_1^2+x_2^2+\cdots+x_n^2 \le R^2$

## 2 作業

1. 根據二重積分的性質，比較下列積分的大小
   (a) $\iint_D (x+y)^2 \, \mathrm{d}x\mathrm{d}y, \iint_D (x+y)^3 \, \mathrm{d}x\mathrm{d}y$，其中$D$為$x$軸，$y$軸與直線$x+y=1$ 所圍成的區域。
   (b) $\iint_D \ln(x+y) \, \mathrm{d}x\mathrm{d}y, \iint_D [\ln(x+y)]^2 \, \mathrm{d}x\mathrm{d}y$，其中$D=[3,5]\times[0,1]$。
2. 計算下列重積分
   (a) $\iint_D x^3+3x^2y+y^3 \, \mathrm{d}x\mathrm{d}y$，其中$D=[0,1]\times[0,1]$。
   (b) $\iint_D xye^{x^2+y^2} \, \mathrm{d}x\mathrm{d}y$，其中$D=[a,b]\times[c,d]$。
   (c) $\iint_D xy^2 \, \mathrm{d}x\mathrm{d}y$，其中$D$由拋物線$y^2=2px$ 和直線$x=\frac{p}{2}(p>0)$所圍成的區域。
   (d) $\iint_D e^{x+y} \, \mathrm{d}x\mathrm{d}y$，其中$D$為$\{(x,y)|\,|x|+|y|\le 1\}$所圍成的區域。
   (e) $\iint_D x \, \mathrm{d}x\mathrm{d}y$，其中$D$為拱線的一拱$x=a(t-\sin t), y=a(1-\cos t)(0\le t\le 2\pi)$所圍成的區域。
3. 採用極坐標變換計算下列積分
   (a) $\iint_D \sin\sqrt{x^2+y^2} \, \mathrm{d}\sigma, D=\left\{(x,y)|\,\pi^2\le x^2+y^2\le 4\pi^2\right\}$
   (b) $\iint_D (x+y) \, \mathrm{d}\sigma, D=\left\{(x,y)|\,x^2+y^2\le x+y\right\}$
   (c) $\iint_D |xy| \, \mathrm{d}\sigma, D=\left\{(x,y)|\,x^2+y^2\le a^2\right\}$
   (d) $\iint_D f'(x^2+y^2) \, \mathrm{d}\sigma, D=\left\{(x,y)|\,x^2+y^2\le R^2\right\}$
4. 做適當的座標變換計算下列積分
   (a) $\iint_D (x+y)\sin(x-y) \, \mathrm{d}\sigma, D=\{(x,y)|\,0\le x+y\le\pi, 0\le x-y\le\pi\}$
   (b) $\iint_D e^{\frac{x}{x+y}} \, \mathrm{d}\sigma, D=\{(x,y)|\,x+y\le 1, x\ge 0, y\ge 0\}$
5. 計算下列幾何體的體積
   (a) $V$是由$z=x^2+y^2, z=x+y$所圍成的立體。

(b) $V$是由$z^2 = \dfrac{x^2}{4} + \dfrac{y^2}{9}$和$2z^2 = \dfrac{x^2}{4} + \dfrac{y^2}{9}$所圍成的立體。

6. 選取適當的座標變換計算下列積分

   (a) $\iiint_\Omega (x^2+y^2+z^2)\,\mathrm{d}v$ 其中$\Omega$為球$x^2+y^2+z^2 \le 1$

   (b) $\iiint_\Omega \sqrt{1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}}\,\mathrm{d}v$ 其中$\Omega$為球$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2} \le 1$

   (c) $\iiint_\Omega \dfrac{z\ln(1+x^2+y^2+z^2)}{1+x^2+y^2+z^2}\,\mathrm{d}v$ 其中$\Omega$為球$x^2+y^2+z^2 \le 1, z \ge 0$

   (d) $\iiint_\Omega (x+y-z)(x-y+z)(y+z-x)\,\mathrm{d}v$ 其中$\Omega$為$0 \le x+y-z \le 1, 0 \le x-y+z \le 1, 0 \le y+z-x \le 1$

7. 計算下列積分

   (a) $\int_\Omega \sqrt{x_1+x_2+\cdots+x_n}\,\mathrm{d}v$ 其中$\Omega$為$x_1+x_2+\cdots+x_n \le 1, x_i \ge 0 (i=1,2,\cdots,n)$

   (b) $\int_\Omega \sqrt{x_1^2+x_2^2+\cdots+x_n^2}\,\mathrm{d}v$ 其中$\Omega$為$x_1^2+x_2^2+\cdots+x_n^2 \le 1$