The Differential of a Function of Several Variables

Guoning Wu

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1 Double Integral

1.1 Introduction

Volume Problem: Find the volume V of the solid G enclosed between the surface $z = f(x, y)$ and the region R in the $xy-$plane where $f(x, y)$ is continuous and non-negative on R.

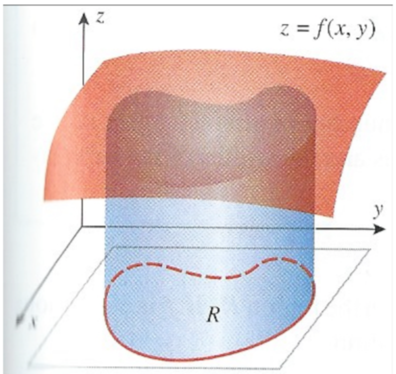
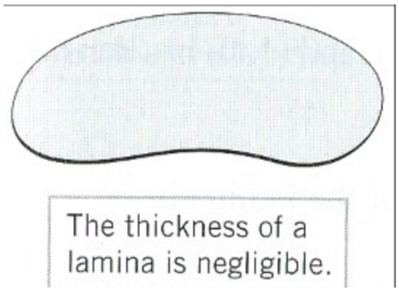


Figure 1: Volume.

Mass Problem: Find the mass M of a lamina (a region R in the $xy-$plane) whose density is a continuous nonnegative function $\rho(x, y)$

Let us consider the Volume Problem,

1. Divide the rectangle enclosing R into subrectangles, and exclude all those rectangles that contain points outside R. Let n be the number of all the rectangles inside R, and let $\Delta A_k = \Delta x_k \Delta y_k$ be the area of the $k-$th subrectangle.



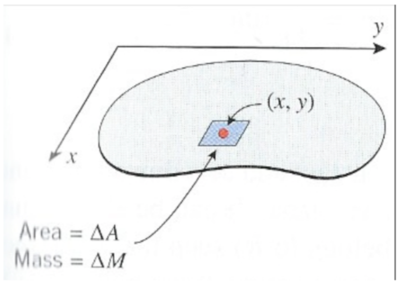
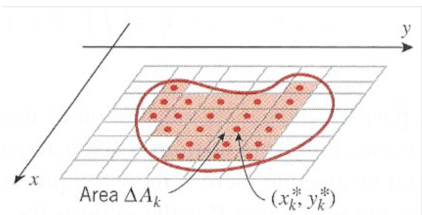


Figure 2: Mass.



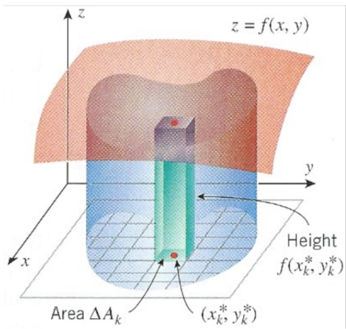


Figure 3: Volume.

2. Choose any point (ξ_k, η_k) in the $k-$th subrectangle. The volume of a rectangular parallelepiped with base area ΔA_k and height $f(\xi_k, \eta_k)$ is $\Delta V_k = f(\xi_k, \eta_k)\Delta A_k$, thus,

$$V \approx \sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} f(\xi_k, \eta_k)\Delta A_k = \sum_{k=1}^{n} f(\xi_k, \eta_k)\Delta x_k \Delta y_k \tag{1}$$

   This sum is called the *Rimann sum*.

3. Take the sides of all the subrectangles to 0, and get

$$V = \lim_{\lambda(P)\to 0} \sum_{k=1}^{n} f(\xi_k, \eta_k)\Delta A_k = \iint_R f(x, y)\,\mathrm{d}A \tag{2}$$

   The last term is the notation for the limit of the Riemann sum, and it is called the *double integral* of $f(x, y)$ over R.

1.2 The Darboux Criterion

Let us consider another criterion for Riemann integrability of a function, which is applicable only to real-valued function. Lower and Upper Darboux Sums. Let f be a real-valued function on the interval I and $P = I_i$ a partition of the interval I. We set

$$m_i = \inf_{x\in I_i} f(x), M_i = \sup_{x\in I_i} f(x). \tag{3}$$

The quantities

$$s(f, P) = \sum_i m_i|I_i|, S(f, P) = \sum_i M_i|I_i| \tag{4}$$

are called the *lower* and *upper sums* of the function f over the interval I corresponding to the partion P of the interval.

The following relations hold between the Darboux sums a a function $f : I \to \mathbb{R}$:

1. $s(f, P) = \inf_\xi \sigma(f, P, \xi) \le \sigma(f, P, \xi) \le \sup_\xi \sigma(f, P, \xi) = S(f, P)$

2. If the partition P' of the interval I is obtained by refining intervals of the partion P, then $s(f, P) \le s(f, P') \le S(f, P') \le S(f, P)$

3. The inequality $s(f, P_1) \le S(f, P_2)$ holds for any pair of partition P_1, P_2 of the interval I.

1.3 Lower and Upper Integrals

The **lower** and **upper** Darboux integrals of the function $f : I \to \mathbb{R}$ over the interval I are respectively

$$\underline{I} = \sup_P s(f, P), \overline{I} = \inf_P S(f, P) \tag{5}$$

where the supremum and infimum are taken over all partitions P of the interval I.

Theorem 1.1. *For any bounded function:*

$$f : I \to \mathbb{R}, \lim_{\lambda(P)\to 0} s(f, P) = \underline{I}, \lim_{\lambda(P)\to 0} S(f, P) = \overline{I}$$

1.4 The Darboux Criterion for Integrability of a Real-valued Function

Theorem 1.2 ((The Darboux Criterion). *A real valued function $f : I \to \mathbb{R}$ defined on an interval $I \subset \mathbb{R}^{\ltimes}$ is integrable over that interval if and only if it is bounded on I and its upper and lower Darboux integrals are equal.*

$$f \in \mathcal{R}(I) \equiv f \textit{ is bounded on } I, and \underline{I} = \overline{I}.$$

1.4.1 Double Integral Over Non-rectangular Regions

$$\int_a^b \int_c^d f(x, y)\, \mathrm{d}y\mathrm{d}x = \int_a^b \left[\int_c^d f(x, y)\right] \mathrm{d}x \tag{6}$$

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, \mathrm{d}y\mathrm{d}x = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y)\mathrm{d}y\right] \mathrm{d}x \tag{7}$$

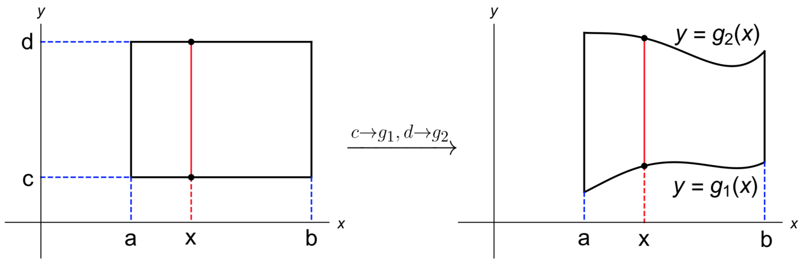


Figure 4: Double Integral.

$$\int_a^b \int_{h_1(y)}^{h_2(y)} f(x, y)\, \mathrm{d}x\mathrm{d}y = \int_a^b \left[\int_{h_1(y)}^{h_2(y)} f(x, y)\mathrm{d}x\right] \mathrm{d}y \tag{8}$$

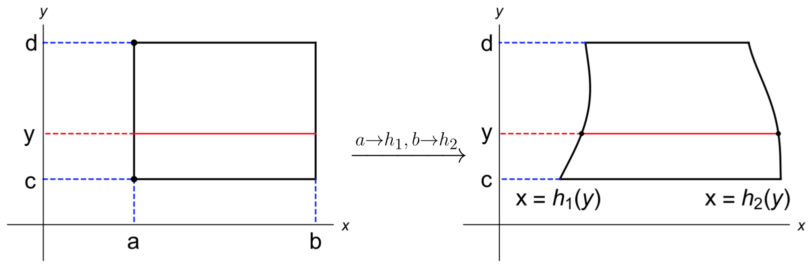


Figure 5: Double Integral.