曲线，曲面积分

武国宁*

中国石油大学-北京

2017年8月

1 第一类曲线曲面积分

1.1 第一类曲线积分

1.1.1 第一类曲线积分定义

Definition 1 设L是空间$\mathbb{R}^3$上可求长的连续曲线，其端点为A和B,函数$f(x,y,z)$在L上有界。令$A=P_0, B=P_n$，在L上顺次的插入分点$P_1,P_2,\cdots,P_{n-1}$. 分别在每个小弧段上任意取一点(ξ_i,η_i,ζ_i)， 并记第i个弧段$P_{i-1}P_i$的长度为$\Delta s_i(i=1,2,\cdots,n)$ 做和式

$$\sum_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)\Delta s_i$$

如果当所有小弧段的最长长度趋于零时，这个和的极限存在且唯一，与 分点和点的取法无关，则称该极限值为$f(x,y,z)$在曲线L上的第一类曲线 积分，记为：

$$\int_L f(x,y,z)\,\mathrm{d}s$$

即，

$$\int_L f(x,y,z)\,\mathrm{d}s = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)\Delta s_i$$

*电子邮件: wuguoning@163.com

其中$f(x,y,z)$ 称为被积函数，L称为积分路径。

1.1.2 第一类曲线积分性质

性质 1 *(线性)* 如果函数f,g在L上的第一类曲线积分存在，则对于任意常数$\alpha\in\mathbb{R},\beta\in\mathbb{R}$,有

$$\int_L (\alpha f+\beta g)\,\mathrm{d}s=\alpha\int_L f\,\mathrm{d}x+\beta\int_L g\,\mathrm{d}s.$$

性质 2 *(路径可加性)* 设曲线$L=L_1+L_2$，如果函数f在L上的第一类曲线积分存在，则 函数f在L_1和L_2上的积分也存在，反之亦然。且有

$$\int_L f\,\mathrm{d}s=\int_{L_1} f\,\mathrm{d}s+\int_{L_2} f\,\mathrm{d}s$$

Theorem 1.1 设函数$f(x,y,z)$在L上连续，则它在L上的第一类曲线积分存在，且有

$$\int_L f(x,y,z)\,\mathrm{d}s=\int_\alpha^\beta f(x(t),y(t),z(t))\sqrt{\dot{x(t)}^2+\dot{y(t)}^2+\dot{z(t)}^2}\,\mathrm{d}t$$

Example 1 计算$\int_L e^{\sqrt{x^2+y^2}}\,\mathrm{d}s$, 其中$L$ 为圆周 $x^2+y^2=a^2$,直线$y=x$及x轴在第一象限围城图形的边界。

Example 2 已知一条非均匀金属线L的方程为

$$x=e^t\cos t, y=e^t\sin t, z=e^t, 0\le t\le 1.$$

它在每一点的线密度与该点到原点的距离成反比，而且在点$(1,0,1)$处的线密度为*1*， 求它的质量。

1.2 曲面的面积

1.2.1 曲面面积的计算

设曲面Σ的方程为

$$x=x(u,v), y=y(u,v), z=z(u,v), (u,v)\in D$$

则

$$\boldsymbol{r}(u,v)=x(u,v)\boldsymbol{i}+y(u,v)\boldsymbol{j}+z(u,v)\boldsymbol{k},$$

相应的**Jacobi**矩阵

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

满秩，考虑D中的一个矩形微元σ,它的四个顶点为：

$$P_1(u_0,v_0),P_2(u_0+\Delta u,v_0),P_3(u_0+\Delta u,v_0+\Delta v),P_4(u_0,v_0+\Delta v)$$

它被影射为：

$$Q_1=\boldsymbol{r}(u_0,v_0),Q_2=\boldsymbol{r}(u_0+\Delta u,v_0),Q_3=\boldsymbol{r}(u_0+\Delta u,v_0+\Delta v),Q_4=\boldsymbol{r}(u_0+\Delta u,v_0+\Delta v).$$

那么有：

$$\overrightarrow{Q_1Q_2}=\boldsymbol{r}(u_0+\Delta u,v_0)-\boldsymbol{r}(u_0,v_0)=\boldsymbol{r}_u(u_0,v_0)\Delta u+\circ(\Delta u),$$

$$\overrightarrow{Q_1Q_4}=\boldsymbol{r}(u_0,v_0+\Delta v)-\boldsymbol{r}(u_0,v_0)=\boldsymbol{r}_v(u_0,v_0)\Delta v+\circ(\Delta v),$$

$$\Delta S\approx\|\boldsymbol{r}_u(u_0,v_0)\times\boldsymbol{r}_v(u_0,v_0)\|\Delta u\Delta v.$$

$$\mathrm{d}S=\|\boldsymbol{r}_u(u_0,v_0)\times\boldsymbol{r}_v(u_0,v_0)\|\Delta u\Delta v.$$

所以有，

$$\begin{aligned} S&=\iint_D\|\boldsymbol{r}_u(u_0,v_0)\times\boldsymbol{r}_v(u_0,v_0)\|\,\mathrm{d}u\mathrm{d}v\\ &=\iint_D\sqrt{EG-F^2}\,\mathrm{d}u\mathrm{d}v. \end{aligned}$$

其中，

$$\sqrt{EG-F^2}=\|\boldsymbol{r}_u\times\boldsymbol{r}_v\|=\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2+\left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2+\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2$$

现在考虑两种特殊情况：

1. 设曲面的方程为$z=f(x,y),(x,y)\in D$ 其中$f(x,y)$ 为连续可微函数， D 为

具有分段光滑边界的有界区域。这是有:

$$\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}.$$

这时有:

$$EG-F^2=\left(1+f_x^2\right)\left(1+f_y^2\right)-\left(f_xf_y\right)^2=1+f_x^2+f_y^2.$$

于是有,

$$S=\iint_D\sqrt{1+f_x^2(x,y)+f_y^2(x,y)}\,\mathrm{d}x\mathrm{d}y$$

2. 设曲面的方程为$H(x,y,z)=0$,其中$H(x,y,z)$是连续可微函数,且在Σ上$H_z(x,y,z)\neq 0$. 由隐函数存在条件有: $z=f(x,y),(x,y)\in D$ 从而有:

$$\begin{aligned}S&=\iint_D\sqrt{1+f_x^2+f_y^2}\,\mathrm{d}x\mathrm{d}y\\&=\iint_D\sqrt{1+\left(-\frac{H_x}{H_y}\right)^2+\left(-\frac{H_y}{H_z}\right)^2}\,\mathrm{d}x\mathrm{d}y\\&=\iint_D\frac{\|gradH\|}{|H_z|}\,\mathrm{d}x\mathrm{d}y\end{aligned}$$

Example 3 求抛物面$z=x^2+y^2$被平面$z=1$所截出的部分的面积。

Example 4 设Σ为球面$x^2+y^2+z^2=2Rz$包含在锥面$z^2=3\left(x^2+y^2\right)$内的部分,求它的面积。

1.3 第一类曲面积分

设空间中一曲面Σ上分布着质量,任意点(x,y,z)处的密度为$\rho(x,y,z)$,如何求 Σ的总质量。

Definition 2 设曲面Σ为有界光滑(或分片光滑)曲面,函数$z=f(x,y,z)$在曲面Σ 上有界。将曲面分成n片小的曲面$\Delta\Sigma_i,i=1,2,\cdots,n$,记$\Delta S_i$ 为第i块曲面的面积,在$\Delta\Sigma_i$上任取一点(ξ_i,η_i,ζ_i) ,作和

$$\sum_{i=1}^n f(\xi_i,\eta_i,\zeta_i)\Delta S_i$$

如果当分割的细度趋于零时，这个和的极限存在且唯一，则称此极限为$f(x,y,z)$在曲面Σ上的第一类曲面积分，记为$\iint_\Sigma f(x,y,z)\,\mathrm{d}S$.记为

$$\iint_\Sigma f(x,y,z)\,\mathrm{d}S = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)\Delta S_i,$$

其中Σ为积分曲面，$f(x,y,z)$为被积函数。

1.3.1 第一类曲面积分的计算方法

设Σ的方程为:

$$x = x(u,v), y = y(u,v), z = z(u,v), (u,v)\in D.$$

$f(x,y,z)$在Σ上连续，则有:

$$\iint_\Sigma f(x,y,z)\,\mathrm{d}S = \iint_D f(x(u,v),y(u,v),z(u,v)\sqrt{EG-F^2}\,\mathrm{d}u\mathrm{d}v$$

特别的，当曲面Σ的方程为$z = z(x,y), (x,y)\in D$,则有

$$\iint_\Sigma f(x,y,z)\,\mathrm{d}S = \iint_D f(x,y,z(x,y)\sqrt{1+z_x^2(x,y)+z_y^2(x,y)}\,\mathrm{d}x\mathrm{d}y$$

Example 5 计算$I = \iint_\Sigma \sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}\,\mathrm{d}S$ 其中，Σ为椭球面$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1, a,b,c > 0$.

2 第二类曲线，曲面积分

2.1 第二类曲线积分

2.1.1 Vector Fields

Suppose a region in the plane or in the space is occupied by a moving fluid such as air or water. Imaging that the fluid is made up of a very large number of particles, and that any instant of time a particle has a velocity $\boldsymbol{v}$. If we take a picture of some particles at different position points at the same instant, we would expect to find that these velocities vary from position to position. We can think of a velocity vectors as being attached to each point of the fluid. Such a fluid exemplifies a **vector field.** Generally, a vector field on a domain in the plane or

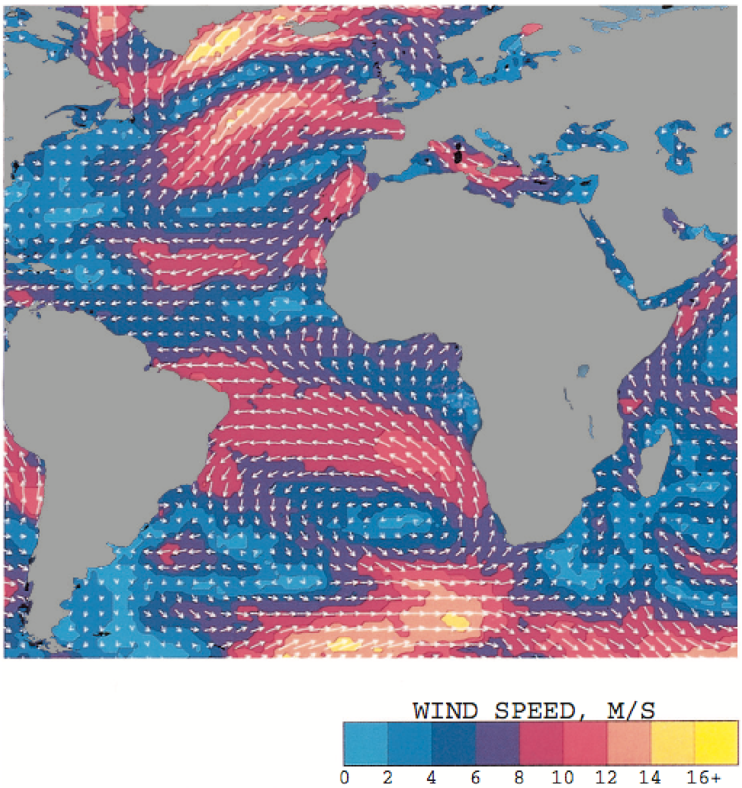

Figure 1: NASA's Seasat used radar to take 350,000 wind measurement over the world's oceans. The arrows show wind direction; their length and the color contouring indicate speed. Notice the heavy storm of Greenland.

in space is a function that assigns a vector to each point in the domain. A field of three-dimensional vectors might have a formula like

$$\boldsymbol{F}(x,y,z) = M(x,y,z)\boldsymbol{i} + N(x,y,z)\boldsymbol{j} + P(x,y,z)\boldsymbol{k}.$$

2.1.2 The Work of a Field

Let $\boldsymbol{F}(x)$ be a continuous force field acting in a domain G of the Euclidean space $\mathbb{R}^n$. The displacement of a test particle in the field is accompanied by work. We ask how we can compute the work done by the field in moving a unit test particle along a given trajectory, more precisely, a smooth path $\gamma : I(\gamma) \subset G$. It is know that in a constant field $\boldsymbol{F}$ the displacement by a vector $\boldsymbol{\xi}$ is associated with an amount of work $\langle \boldsymbol{F}, \boldsymbol{\xi} \rangle$

Suppose that the vector field

$$\boldsymbol{F(t)} = M(x(t), y(t), z(t)\boldsymbol{i} + N(x(t), y(t), z(t))\boldsymbol{j} + P(x(t), y(t), z(t))\boldsymbol{k}$$

represents a force throughout a region in space (it might be the force of gravity or an electromagnetic force of some kind), and $t \to \boldsymbol{r(t)}$ be a smooth mapping: $\gamma : I \to G$ defined on the closed interval $I = \{t \in \mathbb{R} | a \leq t \leq b\}$

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}, a \leq t \leq b,$$

is a smooth curve in the region.

We take a sufficiently fine partition of the closed interval $[a, b]$. Then on each interval $I_i = \{t \in I | t_{i-1} \leq t \leq t_i\}$ of the partition we have the equality

$$\Delta \boldsymbol{r}_i = \boldsymbol{r}_{i+1} - \boldsymbol{r}_i \approx \dot{\boldsymbol{r}}(t_i)\Delta t_i = (\dot{x}(t_i), \dot{y}(t_i), \dot{z}(t_i))\, \Delta t_i.$$

Since the field $\boldsymbol{F}(t)$ is continuous, it can be regarded a locally constant, and for that reason we can compute the work ΔA_i as

$$\Delta A_i \approx \langle \boldsymbol{F}(t_i), \dot{\boldsymbol{r}}(t_i)\Delta t_i \rangle.$$

$$A = \sum_i \Delta A_i \approx \sum_i \langle \boldsymbol{F}(t_i), \dot{\boldsymbol{r}}(t_i)\Delta t_i \rangle.$$

and so passing to the limit as the partition of the closed interval I is refined, we find that

$$A = \int_a^b \langle \boldsymbol{F}(t), \dot{\boldsymbol{r}}(t) \rangle \, \mathrm{d}t. \tag{1}$$

The expression$\langle \boldsymbol{F}(t), \dot{\boldsymbol{r}}(t) \rangle \mathrm{d}t$ is written as $\langle \boldsymbol{F}(t), \mathrm{d}\boldsymbol{r} \rangle$, then as assume the coordinates in $\mathbb{R}^3$ are Cartesian coordinates, we can give this expression the form

$$M\mathrm{d}x + N\mathrm{d}y + P\mathrm{d}z,$$

after which we can writes Eq. 1 as

$$A = \int_\gamma M\mathrm{d}x + N\mathrm{d}y + P\mathrm{d}z \tag{2}$$

or as

$$A = \int_\gamma \omega^1_{\boldsymbol{F}} \tag{3}$$

Formular 3 provides the precise meaning of the integrals of the work 1-form along the path γ.

The expression of Equation 2 can also be written as

$$A = \int_\gamma \boldsymbol{F} \cdot \boldsymbol{T}\,\mathrm{d}s \tag{4}$$

where $\boldsymbol{T} = (\cos\alpha, \cos\beta, \cos\gamma)$ is the unit tangent vector.

Example 6 *Consider the force field* $\boldsymbol{F} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ *defined at all points of the plane* $\mathbb{R}^2$ *except the origin. Let us compute the work of this field along the curve* γ_1 *defined as* $x = \cos t, y = \sin t, 0 \leq t \leq 2\pi$*, and along the curve defined by* $x = 2 + \cos t, y = \sin t, 0 \leq t \leq 2\pi$

Example 7 *Let* $\boldsymbol{r}$ *be the radius vector of a point* $(x, y, z) \in \mathbb{R}^3$ *and* $r = |\boldsymbol{r}|$*. Suppose a force field* $\boldsymbol{F} = f(r)\boldsymbol{r}$ *is defined everywhere in* $\mathbb{R}^3$ *except at the origin. This is so-called central force field. Let up find the work of* $\boldsymbol{F}$ *on a path:* $\gamma : [0,1] \to \mathbb{R}^3 \setminus 0$

$$\begin{aligned}
\int_\gamma f(r)(x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z) &= \frac{1}{2}\int_\gamma f(r)\mathrm{d}(x^2+y^2+z^2) = \frac{1}{2}\int_0^1 f(r(t))\,\mathrm{d}r^2(t) \\
&= \frac{1}{2}\int_0^1 f\left(\sqrt{u(t)}\right)\,\mathrm{d}u(t) = \frac{1}{2}\int_{r_0^2}^{r_1^2} f\left(\sqrt{u}\right)\,\mathrm{d}u \\
&= \Phi(r_0, r_1).
\end{aligned}$$

In particular, for the gravitational field $\frac{1}{r^3}\boldsymbol{r}$ *of a unit point mass located at the origin, we obtain*

$$\Phi(r_0, r_1) = \frac{1}{2}\int_{r_0^2}^{r_1^2} \frac{1}{u^{\frac{3}{2}}}\,\mathrm{d}u = \frac{1}{r_0} - \frac{1}{r_1}$$

Example 8 *Find the work done by a variable force over a space curve, where the force is* $\boldsymbol{F} = (y - x^2)\boldsymbol{i} + (z - y^2)\boldsymbol{j} + (x - z^2)\boldsymbol{k}$ *over the curve* $\boldsymbol{r}(t) = t\boldsymbol{i} + t^2\boldsymbol{j} + t^3\boldsymbol{k}$*, from* $(0, 0, 0)$ *to* $(1, 1, 1)$.

Example 9 *Find flow along a helix: A fluid's velocity field is* $\boldsymbol{F} = x\boldsymbol{i} + z\boldsymbol{j} + k\boldsymbol{k}$*. Find the flow along the helix* $\boldsymbol{r}(t) = \cos t\boldsymbol{i} + \sin t\boldsymbol{j} + t\boldsymbol{k}$

2.1.3 Flux Across a Plane Curve

To find the rate at which a fluid is entering or leaving a region enclosed by a smooth curve C in the xy-plane, we calculate the line integral over C of $\boldsymbol{F} \cdot \boldsymbol{n}$, the scalar component of the fluid's velocity field in the direction of the curve's outward-pointing normal vector. The value of this integral is the flux [1] of $\boldsymbol{F}$ across C.

Definition 3 *If C is a smooth curve in the domain of a continuous vector field* $\boldsymbol{F} = M(x, y)\boldsymbol{i} + N(x, y)\boldsymbol{j}$ *in the plane and if* $\boldsymbol{n}$ *is the outward-pointing unit normal vector on C, the flux of* $\boldsymbol{F}$ *across C is :*

$$A = \int_C \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}s \tag{5}$$

To evaluate the integral of Equation 5, we begin with a smooth parameterization

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}, a \leq t \leq b.$$

If the motion is counterclockwise, then

$$\boldsymbol{n} = \boldsymbol{T} \times \boldsymbol{k},$$

and if the motion is clockwise, then

$$\boldsymbol{n} = -\boldsymbol{T} \times \boldsymbol{k},$$

where $\boldsymbol{T} = \dfrac{\dot{\boldsymbol{r}}(t)}{\|\boldsymbol{r}(t)\|}$. So for counterclockwise motion, the calculation of Equation 5 is:

[1] Flux is a Latin word for flow, but many flux calculation involve no motion at all.

$$
\begin{aligned}
A &= \oint_C \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}s = \int_a^b \boldsymbol{F} \cdot \frac{\dot{\boldsymbol{r}}(t) \times \boldsymbol{k}}{\|\dot{\boldsymbol{r}}(t)\|} \|\dot{\boldsymbol{r}}(t)\| \, \mathrm{d}t \\
&= \int_a^b \boldsymbol{F} \cdot \dot{\boldsymbol{r}}(t) \times \boldsymbol{k} \, \mathrm{d}t = \oint_C M \mathrm{d}y - N \mathrm{d}x
\end{aligned} \tag{6}
$$

Example 10 *Fine the flux of $\boldsymbol{F} = (x - y)\boldsymbol{i} + x\boldsymbol{j}$ across the circle $x^2 + y^2 = 1$ in the xy-plane.*

(**Method I**) Parametrization the circle: $x = \cos t, y = \sin t, 0 \le t \le 2\pi$

$$
\begin{aligned}
A &= \oint \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}s = \int_0^{2\pi} (\cos t - \sin t, \cos t) \cdot (\cos t, \sin t) \ \mathrm{d}t \\
&= \int_0^{2\pi} \cos^2 t \, \mathrm{d}t = \pi
\end{aligned}
$$

(**Method II**)

$$
A = \oint_C M \mathrm{d}y - N \mathrm{d}x = \int_0^{2\pi} \cos t - \sin t \, \mathrm{d} \sin t - \cos t \, \mathrm{d} \cos t = \pi
$$

3 Surface Area and Surface Integrals

We know how to integrate a function over a flat region in a plane, but what if the function is defined over a curved surface? To evaluate one of these so-called surface integrals, we rewrite it as double integral over a region in a coordinate plane beneath the surface.

3.1 Surface Area

Figure 3 shows a surface S lying above its "shadow" region R in a plane beneath it. The surface is defined by the equation $f(x, y, z) = c$. If the surface is smooth (∇f is continuous and never vanishes on S). We can define and calculate it area as a double integral over R.

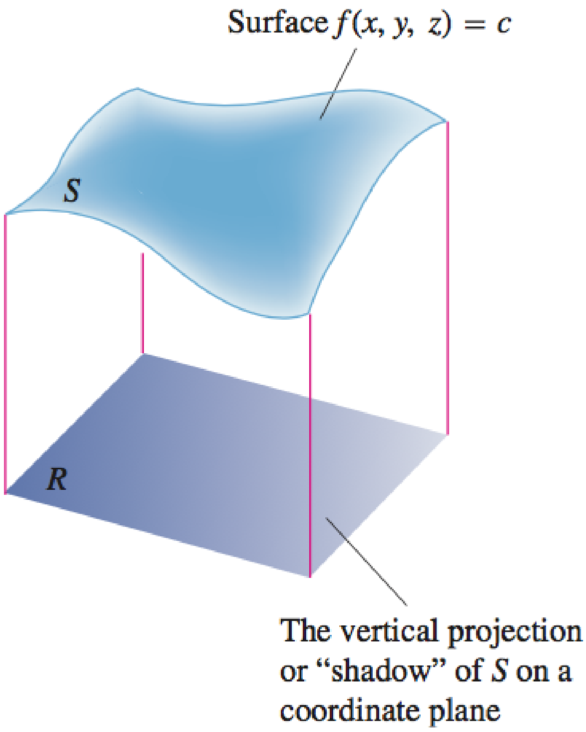

Figure 2: As we soon see, the integral of a function $g(x, y, z)$ over a surface S in space can be calculated by evaluating a related double integral over the vertical projection or "shadow" of S on a coordinate plane.

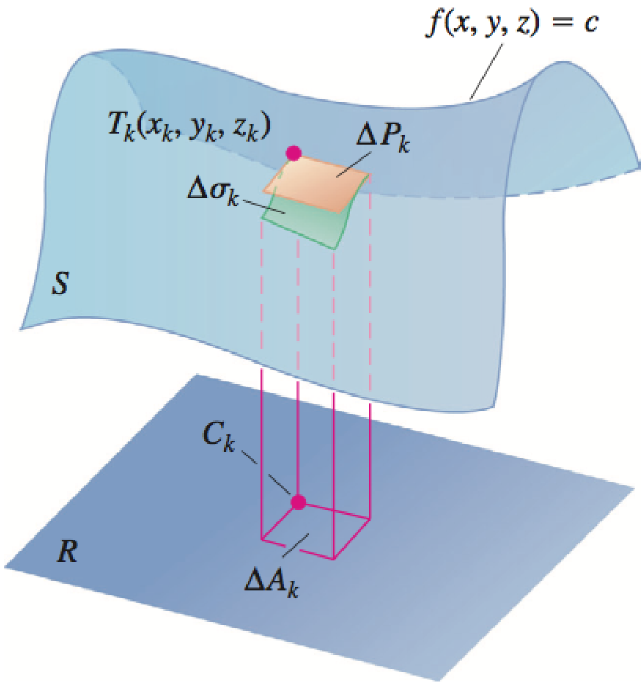

Figure 3: A surface S and its vertical projection onto a plane beneath it. You can think of R as the shadow of S on the plane. The tangent plane ΔP_k approximates the surface patch $\Delta\sigma_k$ above ΔA_k.

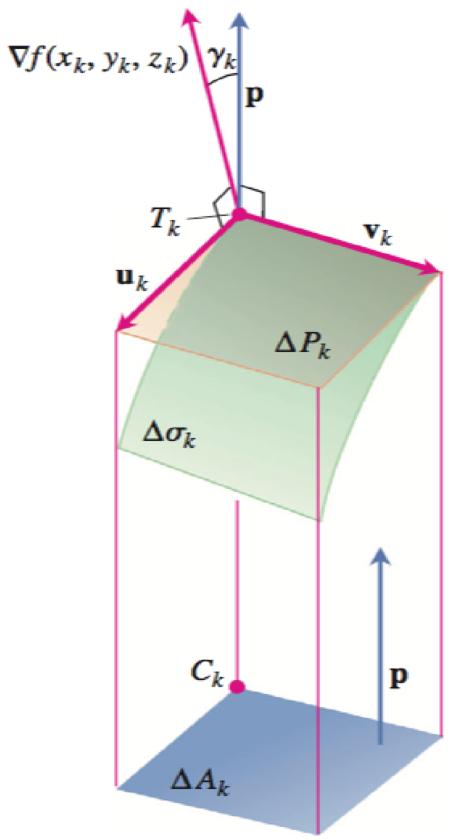

Figure 4: Magnified view from the preceding figure. The vector $\boldsymbol{u}_k \times \boldsymbol{v}_k$ is parallel to the vector ∇f because both vectors are normal to the plane of ΔP_k.

The first step in defining the area of S is to partition the region R into small rectangles ΔA_k of the kind we would use if we were defining an integral over R. Directly above each ΔA_k lies a patch of surface $\Delta\sigma_k$ that we may approximate by a parallelogram ΔP_k in the tangent plane to S at a point $T_k(x_k, y_k, z_k)$ in $\Delta\sigma_k$.

Figure 4 give a magnified view of $\Delta\sigma_k$ and ΔP_k, showing the gradient vector ∇f_k at T_k and a unit vector $\boldsymbol{p}$ that is normal to R. The figure also shows the angle γ_k between ∇f_k and $\boldsymbol{p}$. In our case, this translates into the statement

$$|(\boldsymbol{u}_k \times \boldsymbol{v}_k) \cdot \boldsymbol{p}| = \Delta A_k$$

or

$$\Delta P_k\, |\cos\gamma_k| = \Delta A_k$$

or

$$\Delta P_k = \frac{\Delta A_k}{|\cos\gamma_k|}$$

We will have $\cos\gamma_k \neq 0$ is ∇f is not parallel to the ground plane and $\nabla f \cdot \boldsymbol{p} \neq 0$.

Since the patches ΔP_k approximates the surface patches $\Delta\sigma_k$ that fit together to make S, the sum

$$\sum \Delta P_k = \sum \frac{\Delta A_k}{|\cos\gamma_k|} \tag{7}$$

If we refined the partition of R. In fact, the sums on the right-hand side of the equation 7 are approximating sums for the double integral.

$$\iint_R \frac{1}{|\cos\gamma|}\, \mathrm{d}A. \tag{8}$$

We therefore define the area of S to be the value of this integral whenever it exists. For any surface $f(x, y, z) = c$, we have $|\nabla f \cdot \boldsymbol{p}| = |\nabla f|\, |\boldsymbol{p}|\, |\cos\gamma|$, so

$$\frac{1}{|\cos\gamma|} = \frac{|\nabla f|}{|\nabla f \cdot \boldsymbol{p}|}.$$

The area of the surface $f(x, y, z) = c$ over a closed and bounded plane region R is

$$S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \boldsymbol{p}|}\, \mathrm{d}A. \tag{9}$$

Example 11 *Find the area of the surface cut from the bottom of the paraboloid*

$x^2 + y^2 - z = 0$ *by the plane* $z = 4$.

We have

$$f(x,y,z) = x^2 + y^2 - z, \nabla f = 2x\boldsymbol{i} + 2y\boldsymbol{j} - \boldsymbol{k},$$
$$|\nabla f \cdot \boldsymbol{p}| = |\nabla f \cdot \boldsymbol{k}| = 1.$$
$$S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \boldsymbol{p}|}\,\mathrm{d}x\mathrm{d}y = \iint_{x^2+y^2\le 4} \sqrt{4x^2 + 4y^2 + 1}\,\mathrm{d}x\mathrm{d}y.$$

Example 12 *Find the area of the cap cut from the hemisphere* $x^2 + y^2 + z^2 = 2$, *by the cylinder* $x^2 + y^2 \le 1$ *in the xy-plane.*

3.2 Surface Integrals

Suppose, for example, that we have an electrical charge distributed over a surface $f(x,y,z) = c$ like the one shown if Figure 3 and that the function $g(x,y,z)$ gives the charge per unit area (charge density) at each point on S. The we may calculate the total charge on S as an integral of below:

$$\text{Totalcharge} \approx \sum g(x_k, y_k, z_k)\Delta P_k = \sum g(x_k, y_k, z_k)\frac{\Delta A_k}{|\cos\gamma|}$$

If f, the function defining the surface S, and its first derivatives are continuous, and if g is continuous over S, then the sums on the right-hand side of the last equation approach the limit

$$\iint_R g(x,y,z)\frac{\mathrm{d}A}{|\cos\gamma|} = \iint_R g(x,y,z)\frac{|\nabla f|}{|\nabla f \cdot \boldsymbol{p}|}\,\mathrm{d}A. \tag{10}$$

The Surface Area Differential and the Differential Form for Surface Integrals

$$\boldsymbol{d}\sigma = \frac{|\nabla f|}{|\nabla f \cdot \boldsymbol{p}|}$$

$$\iint_S g\,\mathrm{d}\sigma$$

Example 13 *Integrate* $g(x,y,z) = xyz$ *over the surface of the cube cut from the first octant by the planes* $x = 1, y = 1, z = 1$ *(Figure 5)*

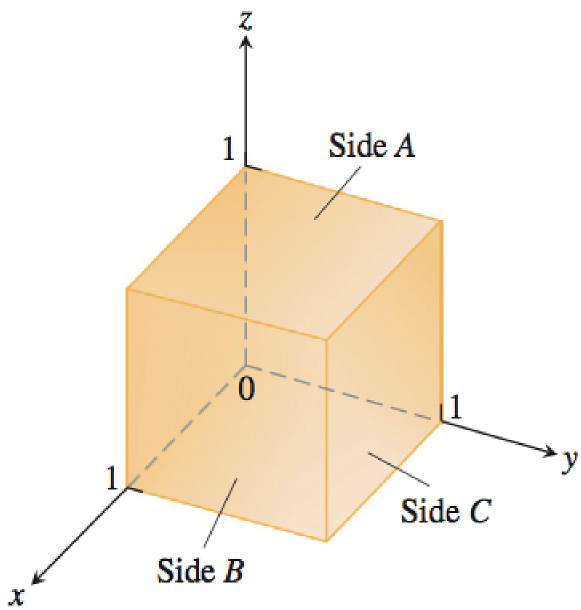

Figure 5: The cube in example above.

3.3 Orientation

We call a smooth surface S **orientable** or **two-sided** if it is possible to define a field $\boldsymbol{n}$ of unit normal vectors on S that varies continuously with position. Once $\boldsymbol{n}$ has been chosen, we say that we have **oriented** the surface, and we call the surface together with its normal field an **oriented surface**. The vector $\boldsymbol{n}$ at any point is called the **positive direction** at that point.

3.4 Surface Integral for Flux

Suppose that $\boldsymbol{F}$ is a continuous vector field defined over an oriented surface S and that $\boldsymbol{n}$ is the chosen unit normal field on the surface. We call the integral of $\boldsymbol{F} \cdot \boldsymbol{n}$ over S the flux of $\boldsymbol{F}$ across S in the positive direction.

Definition 4 *The **flux** of a three-dimensional vector field $\boldsymbol{F}$ across an oriented surface S in the direction of $\boldsymbol{n}$ is*

$$Flux = \iint_S \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}\sigma$$

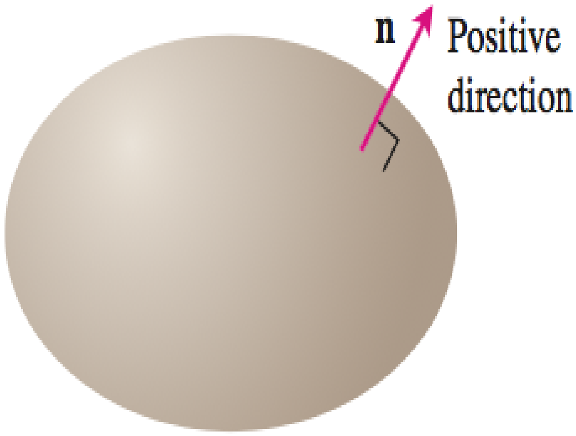

Figure 6: Smooth closed surface in space is orientable. The outward unit normal vector defines the positive direction at each point.

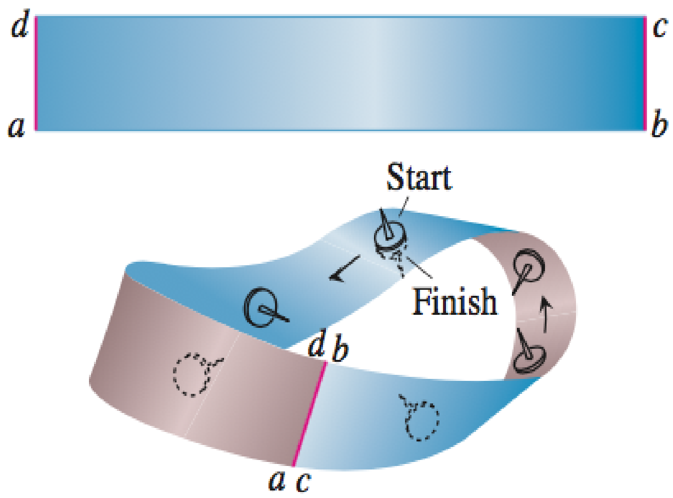

Figure 7: The Mobius band is a non-orientable or one-side surface.

The definition is analogous to the flux of a two-dimensional field $\boldsymbol{F}$ across a plane curve C. In the plane, the flux is

$$\int_C \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}s,$$

the integral of the scalar component of $\boldsymbol{F}$ normal to the curve.

If $\boldsymbol{F}$ is the velocity field of a three-dimensional fluid flow, the flux of $\boldsymbol{F}$ across surface S is the net rate at which fluid is crossing S in the chosen positive direction. If S is part of a surface $g(x, y, z) = c$, then $\boldsymbol{n}$ may be taken to be one of the two fields

$$\boldsymbol{n} = \pm \frac{\nabla g}{|\nabla g|} \tag{11}$$

depending on which one gives the preferred direction. The corresponding flux is

$$\begin{aligned} \text{Flux} &= \iint_S \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}\sigma \\ &= \iint_S \left(\boldsymbol{F} \cdot \frac{\pm \nabla g}{|\nabla g|} \right) \frac{|\nabla g|}{|\nabla g \cdot \boldsymbol{p}|} \, \mathrm{d}A \qquad (12) \\ &= \iint_S \boldsymbol{F} \cdot \frac{\nabla g}{|\nabla g \cdot \boldsymbol{p}|} \, \mathrm{d}A \end{aligned}$$

Example 14 *Find the flux of $\boldsymbol{F} = yz\boldsymbol{j} + z^2\boldsymbol{k}$ outward through the surface S cut from the cylinder $y^2 + z^2 = 1, z \geq 0$ by the plane $x = 0$ and $x = 1$. See Figure 6.*

Solution:

The outward normal field on S may be calculated from the gradient of $g(x, y, z) = y^2 + z^2$ to be

$$\boldsymbol{n} = +\frac{\nabla g}{|\nabla g|} = \frac{2y\boldsymbol{j} + 2z\boldsymbol{k}}{\sqrt{4y^2 + 4z^2}} = \frac{2y\boldsymbol{j} + 2z\boldsymbol{k}}{2} = y\boldsymbol{j} + z\boldsymbol{k}.$$

With $\boldsymbol{p} = \boldsymbol{k}$, we also have

$$\mathrm{d}\sigma = \frac{|\nabla g|}{|\nabla g \cdot \boldsymbol{k}|} \mathrm{d}A = \frac{2}{|2z|} \mathrm{d}A = \frac{1}{z} \mathrm{d}A.$$

$$\boldsymbol{F} \cdot \boldsymbol{n} = \left(yz\boldsymbol{j} + z^2\boldsymbol{k} \right) \cdot (y\boldsymbol{j} + z\boldsymbol{k}) = z\left(y^2 + z^2\right) = z.$$

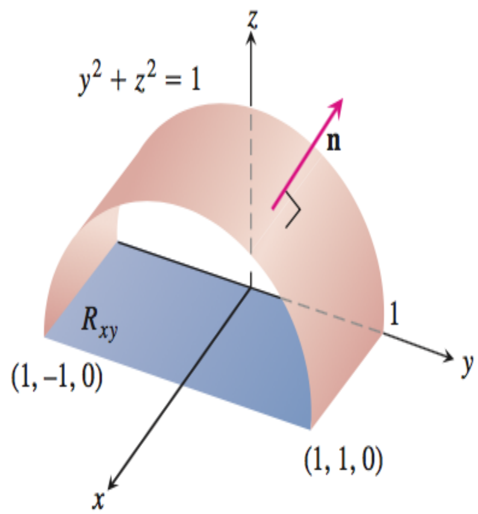

Figure 8: Calculating the flux of a vector field outward through this surface. The area of the shadow region R_{xy} is 2.

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n} \,\mathrm{d}\sigma = \iint_S z \frac{1}{z} \,\mathrm{d}A = \iint_{R_{xy}} \mathrm{d}A = 2.$$

3.5 Parametrized Surfaces

We have defined curves in the plane in three different ways:

Explicit form: $y = f(x)$,

Implicit form: $F(x, y) = 0$,

Parametric vector form: $\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j}$.

We have analogous definition of surface in space:

Explicit form: $z = f(x, y)$,

Implicit form: $F(x, y, z) = 0$.

There is also a parametric form that gives the position of a point on the surface as a vector function of two variables. The present section extends the investigation of surface area and surface integrals to surface described parametrically.

3.5.1 Parametrizations of Surfaces

Let

$$\boldsymbol{r}(u,v) = f(u,v)\boldsymbol{i} + g(u,v)\boldsymbol{j} + h(u,v)\boldsymbol{k} \tag{13}$$

be a continuous vector function that is defined on a region R in the uv-plane.

Example 15 *Find a parametrization of the cone*

$$z = \sqrt{x^2 + y^2}, 0 \leq z \leq 1.$$

Example 16 *Find a parametrization of the sphere*

$$x^2 + y^2 + z^2 = a^2$$

Example 17 *Find a parametrization of the cylinder*

$$x^2 + (y-3)^2 = 9, 0 \leq z \leq 5.$$

3.5.2 Surface Area

For details see **section 1.2.1**.

3.5.3 Surface Integral

For detail see **section 1.3**

3.5.4 Flux

Example 18 *Find the flux of* $\boldsymbol{F} = yz\boldsymbol{i} + x\boldsymbol{j} - z^2\boldsymbol{k}$ *outward through the parabolic cylinder* $y = x^2$, $0 \leq x \leq x, 0 \leq z \leq 4$.

4 Path Independence, Potential Functions, and Conservative Fields

In gravitational and electric fields, the amount of work it takes to move a mass or a charge from one point to another depends only on the object's initial and final positions and not on the path taken in between.

4.1 Path Independence

If A and B are two points in an open region D in space, the work $\int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$ done in moving a particle from A to B by a field $\boldsymbol{F}$ defined on D usually depends on the path taken. For some special field, however, the integral's value is the same for all paths from A to B.

Definition 5 *Let $\boldsymbol{F}$ be a field defined on an open region D in space, and suppose that for any two points A and B the work $\int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$ done in moving from A to B is the same over all paths from A to B. Then the integral $\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$ is path independent in D and the field $\boldsymbol{F}$ is conservative on D.*

Under differentiability conditions normally met in practice, a field $\boldsymbol{F}$ is conservative if and only if it is the gradient of a scalar function, that is, if and only if $\boldsymbol{F} = \nabla f$ for some f. The function f is called the potential function.

Definition 6 *If $\boldsymbol{F}$ is a field defined on D and $\boldsymbol{F} = \nabla f$ for some scalar function f on D, then f is called a* ***potential function for $\boldsymbol{F}$***.

4.2 Line Integrals in Conservative Fields

Theorem 4.1 The Fundamental Theorem of Line Integral
Let $\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j} + P\boldsymbol{k}$ be a vector field whose component are continuous throughout an open connected D in space. Then there exists a differentiable function f such that

$$\boldsymbol{F} = \nabla f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}$$

if and only if for all points A and B in D the value of $\int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$ is independent of the path joining A to B in D.

If the integral is independent of the path from A to B, its value is

$$\int_A^B \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = f(B) - f(A).$$

Example 19 *Find the work done by the conservative field*

$$\boldsymbol{F} = yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k}$$

along any smooth curve C joining the point $A(-1, 3, 9)$ to $B(1, 6, -4)$.

Theorem 4.2 Closed-Loop Property of Conservative Fields *The following statements are equivalent.*

1. *$\int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = 0$ around every closed loop in D.*
2. *The field $\boldsymbol{F}$ is conservative on D.*

4.3 Find Potentials for Conservative Fields

Theorem 4.3 *Suppose that the domain of $\boldsymbol{F}$ is connected and simply connected. Let Let $\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j} + P\boldsymbol{k}$ be a vector field whose component have continuous first partial derivatives. Then, $\boldsymbol{F}$ is conservative if and only if*

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y},$$

Example 20 *Show that $\boldsymbol{F} = (e^x \cos y + yz)\boldsymbol{i} + (xz - e^x \sin y)\boldsymbol{j} + (yz + z)\boldsymbol{k}$ is conservative and find a potential function for it.*

5 Green's Theorem in the Plane

In this section we consider how to evaluate the integral if it is not associated with a conservative vector field, but is a flow or flux integral across a closed curve in the xy-plane.

5.1 Divergence

We need new ideas for Green's theorem. The first is the idea of the divergence of a vector field at a point, sometimes called the flux density of the vector field by physicists and engineers.

Suppose that $\boldsymbol{F}(x, y) = M(x, y)\boldsymbol{i} + N(x, y)\boldsymbol{j}$ is the velocity field of a fluid flow in the plane and that the first partial derivatives of M and N are continuous at each point of a region R. Let (x, y) be a point in R and let A be a small rectangle with one corner at (x, y) that, along with its interior, lies entirely in R. The sides of the rectangle parallel to the coordinate axes, have lengths of Δx and Δy. The rate at which fluid leaves the rectangle across the bottom edge is approximately, see Figure 9

$$\boldsymbol{F}(x, y) \cdot -\boldsymbol{j}\Delta x = -N(x, y)\Delta x$$

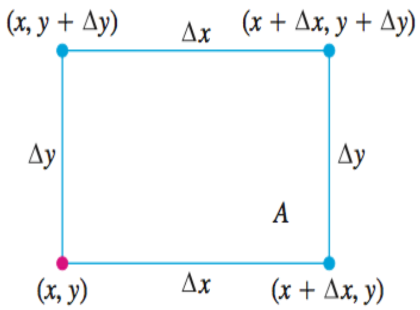

Figure 9: The rectangle for defining the divergence (flux density) of a vector field at point (x, y).

Exit Rates:

1. Top: $\boldsymbol{F}(x, y + \Delta y) \cdot \boldsymbol{j}\Delta x = N(x, y + \Delta y)\Delta x$
2. Bottom:$\boldsymbol{F}(x, y) \cdot -\boldsymbol{j}\Delta x = -N(x, y)\Delta x$
3. Right: $\boldsymbol{F}(x + \Delta x, y) \cdot \boldsymbol{i}\Delta y = M(x + \Delta x, y)\Delta y$
4. Left: $\boldsymbol{F}(x, y) \cdot -\boldsymbol{i}\Delta y = -M(x, y)\Delta y$.

Combining opposite pairs gives:

$$\textbf{Flux across rectangle boundary} \approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x \Delta y.$$

We now divide by $\Delta x \Delta y$ to estimate the total flux per unit area of flux density for the rectangle:

$$\text{divF} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

Definition 7 *The divergence (flux density) of a vector field $\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j}$ at the point (x, y) is*

$$\text{div}\boldsymbol{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

Example 21 *Find the divergence of $\boldsymbol{F}(x, y) = (x^2 - y)\boldsymbol{i} + (xy - y^2)\boldsymbol{j}$.*

5.2 Spin Around an Axis: The k-Component of Curl

The second idea we need for Green's theorem has to do with measuring how a paddle wheel spins at a point in a fluid flowing in a plane region. This idea gives some sense of how the fluid is circulating around axes located at different points and perpendicular to the region. Physicists sometimes refer to this as the circulation density of a vector field $\boldsymbol{F}$ at a point. To obtain it, we return to the velocity field.

$$\boldsymbol{F}(x, y) = M(x, y)\boldsymbol{i} + N(x, y)\boldsymbol{j}$$

and the rectangle A. The rectangle is redrawn here as Figure 10.

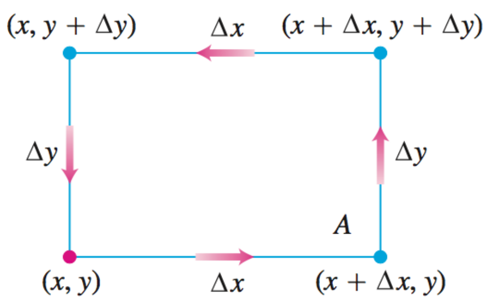

Figure 10: The rectangle for defining the curl (circulation density) of a vector field at point (x, y).

The counterclockwise circulation of the velocity $\boldsymbol{F}$ around the boundary of A is the sum of flow rates along the sides.

1. Top: $\boldsymbol{F}(x, y+\Delta y) \cdot -\boldsymbol{i}\Delta x = -M(x, y+\Delta y)\Delta x$
2. Bottom:$\boldsymbol{F}(x, y) \cdot \boldsymbol{i}\Delta x = M(x, y)\Delta x$
3. Right: $\boldsymbol{F}(x+\Delta x, y) \cdot \boldsymbol{j}\Delta y = N(x+\Delta x, y)\Delta y$
4. Left: $\boldsymbol{F}(x, y) \cdot -\boldsymbol{j}\Delta y = -N(x, y)\Delta y.$

We add opposite pairs to get:

$$\textbf{Circulation along the boundary} \approx \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x \Delta y.$$

Definition 8 *The k-component of the curl (circulation density) of a vector field* $\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j}$ *at the point* (x, y) *is the scalar*

$$\mathrm{curl}\boldsymbol{F} \cdot \boldsymbol{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

Example 22 *Find the k-component of the curl for the vector field*

$$\boldsymbol{F}(x, y) = (x^2 - y)\boldsymbol{i} + (xy - y^2)\boldsymbol{j}$$

5.3 Two Forms for Green's Theorem

In one form, Green's Theorem says that under suitable conditions the outward flux of a vector field across a simple closed curve in the plane equals the double integral of the divergence of the field over the region enclosed by the curve.

Theorem 5.1 Flux-Divergence or Normal Form
The outward flux of a field $\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j}$ *across a simple closed curve* C *equals the double integral of div* $\boldsymbol{F}$ *over the region* R *enclosed by* C.

$$\oint_C \boldsymbol{F} \cdot \boldsymbol{n}\, \mathrm{d}s = \oint_C M\, \mathrm{d}y - N\, \mathrm{d}x = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) \mathrm{d}x\mathrm{d}y \qquad (14)$$

In another form, Green's Theorem says that the counterclockwise circulation of a vector around a simple closed curve is the double integral of the k-component of the curl of the field over the region enclosed by the curve.

Theorem 5.2 (Circulation-Curl or Tangential Form) *The counterclockwise circulation of a field $\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j}$ around a simple closed curve C in the plane equals the double integral of $(\mathrm{curl}\boldsymbol{F}) \cdot \boldsymbol{k}$ over the region R enclosed by C.*

$$\oint_C \boldsymbol{F} \cdot \boldsymbol{T}\,\mathrm{d}s = \oint_C M\,\mathrm{d}x + N\,\mathrm{d}y = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathrm{d}x\mathrm{d}y. \tag{15}$$

Example 23 *Verify both forms of Green's Theorem for the field*

$$\boldsymbol{F}(x,y) = (x-y)\boldsymbol{i} + x\boldsymbol{j}$$

and the region R bounded by the unit circle

$$C : \boldsymbol{r}(t) = \cos t\boldsymbol{i} + \sin t\boldsymbol{j}, 0 \le t \le 2\pi.$$

Example 24 *Evaluate the integral*

$$\oint_C xy\,\mathrm{d}y - y^2\,\mathrm{d}x$$

where C is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$.

Example 25 *Calculate the outward flux of the field $\boldsymbol{F}(x,y) = x\boldsymbol{i} + y^2\boldsymbol{j}$ across the square bounded by the lines $x = \pm 1$ and $y = \pm 1$.*

Example 26 *Verify the circulation form of Green's Theorem on the annular ring $R : h^2 \le x^2 + y^2 \le 1, 0 < h < 1$, if*

$$M = \frac{-y}{x^2+y^2}, N = \frac{x}{x^2+y^2}$$

Here R is not simply connected.

6 The Divergence Theorem and a Unified Theory

The divergence form of Green's Theorem in the plane states that the net outward flux of the field across a simple closed curve can be calculated by integrating the divergence of the field over the region enclosed by the curve. The corresponding theorem in three dimensions, called the Divergence Theorem, states that the net outward flux of a vector field across a closed surface in space can be calculated by integrating the divergence of the field over the region enclosed by the surface.

6.1 Divergence in Three dimension

The divergence of a vector field $\boldsymbol{F} = M(x,y,z)\boldsymbol{i} + N(x,y,z)\boldsymbol{j} + P(x,y,z)\boldsymbol{k}$ is the scalar function

$$\mathrm{div}\boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

Example 27 *Find the divergence of* $\boldsymbol{F} = 2xz\boldsymbol{i} - xy\boldsymbol{j} - z\boldsymbol{k}$.

6.2 The Gauss-Ostrogradskii Formula

Theorem 6.1 *Let* $\mathbb{R}^3$ *be three-dimensional space with a fixed coordinates system* x, y, z *and* $\overline{D}$ *a compact domain in* $\mathbb{R}^3$ *bounded by piecewise-smooth surface. Let* P, Q, R *be smooth functions in the closed domain* $\overline{D}$*. Then the following relation holds:*

$$\iiint_{\overline{D}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) \mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$= \iint_{\partial\overline{D}} P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy.$$

If we denote $\boldsymbol{F} = (P, Q, R)$, the Gauss-Ostrogradskii theorem can be written as

$$\iiint_{\overline{D}} \nabla \cdot \boldsymbol{F}\, dV = \iint_{\partial\overline{D}} \boldsymbol{F} \cdot \boldsymbol{n} d\sigma.$$

Example 28 *Find the flux of* $\boldsymbol{F} = xy\boldsymbol{i} + yz\boldsymbol{j} + xz\boldsymbol{k}$ *outward through the surface of the cube cut from the first octant by the plane* $x = 1, y = 1, z = 1$.

Example 29 *Find the net outward flux of the field*

$$\boldsymbol{F} = \frac{x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}}{\rho^3}, \rho = \sqrt{x^2 + y^2 + z^2}$$

across the boundary of the region: $D : 0 < a^2 \leq x^2 + y^2 + z^2 \leq b^2$

6.3 Gauss's Law: One of the Four Great Laws of Electromagnetic Theory

The outward flux of $\boldsymbol{E}$ across any sphere centered at the origin is $\frac{q}{\epsilon_0}$, and the result is not confined to spheres. In electro- magnetic theory, the electric field created by a point charge q located at the origin is

$$\boldsymbol{E}(x, y, z) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\boldsymbol{r}|^2}\frac{r}{|r|} = \frac{q}{4\pi\epsilon_0}\frac{x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}}{\rho^3}$$

Gauss's law:

$$\iint_S \boldsymbol{E} \cdot \boldsymbol{n} d\sigma = \frac{q}{\epsilon_0}$$

7 Stokes' Theorem

In three dimensional, the circulation around a point P in a plane is described with a vector. This vector is normal to the plane of the circulation and points in the direction that gives it a right-hand relation to the circulation line. The length of the vector gives the rate of the fluid's rotation, which usually varies as the circulation plane is tilted about P. It turns out that the vector of greatest circulation in a flow with velocity field $\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j} + P\boldsymbol{k}$ is the curl vector

$$\text{curl}\boldsymbol{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\boldsymbol{k}. \tag{16}$$

If we denote

$$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}. \tag{17}$$

The curl of $\boldsymbol{F}$ is $\nabla \times \boldsymbol{F}$:

$$\begin{aligned}
\nabla \times \boldsymbol{F} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} \\
&= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\boldsymbol{k} \\
&= \mathrm{curl}\boldsymbol{F}.
\end{aligned}$$

Example 30 *Find the curl of* $\boldsymbol{F} = (x^2 - y)\,\boldsymbol{i} + 4z\boldsymbol{j} + x^2\boldsymbol{k}$.

Example 31 *For a 1-form*

$$\omega = P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z$$

defined in a domain D *in* $\mathbb{R}^3$ *we obtian*

$$\mathrm{d}\omega = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathrm{d}z \wedge \mathrm{d}x + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y$$

Theorem 7.1 *Let* S *be an oriented piecewise-smooth compact two dimensional surface with boundary* ∂S *embedded in a domain* $G \subset \mathbb{R}^3$, *in which a smooth 1-form* $\omega = P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z$ *is defined. Then the following relation holds:*

$$\int_{\partial S} P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z = \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathrm{d}y\wedge\mathrm{d}z + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathrm{d}z\wedge\mathrm{d}x + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\wedge\mathrm{d}y$$

where the orientation of the boundary ∂S *is chosen consisting with the orientation of the surface* S.

In other notation, this means that

$$\int_S \mathrm{d}\omega = \int_{\partial S} \omega.$$

Let us show the proof of the Stokes theorem. To avoid expressions that are really too cumbersome, we shall write out only the first, main part of its two

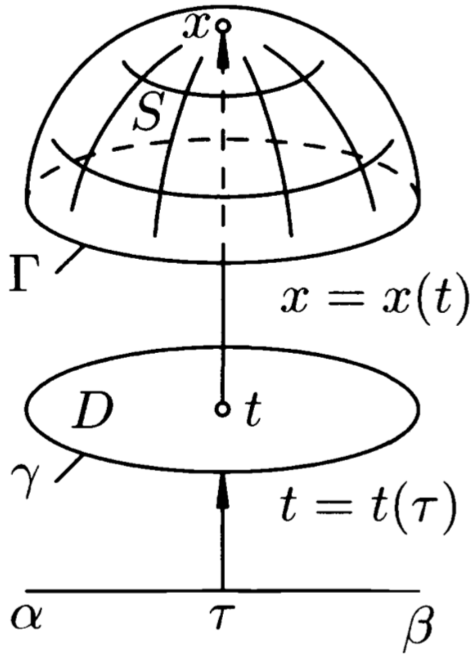

Figure 11: Parameters of a surface for Stokes.

expressions, and with some simplifications even in that. To be specific, let us introduce the notation x^1, x^2, x^3 for the coordinates of a point $x \in \mathbb{R}^3$ and verify only that

$$\int_{\partial S} P(x)\mathrm{d}x^1 = \iint_S \frac{\partial P}{\partial x^2}\mathrm{d}x^2 \wedge \mathrm{d}x^1 + \frac{\partial P}{\partial x^3}\mathrm{d}x^3 \wedge \mathrm{d}x^1$$

For simplicity we shall assume that S can be obtained by a smooth mapping $x = x(t)$ of domain D in the plane $\mathbb{R}^2$ of the variables t^1, t^2 and bounded by a smooth curve $\gamma = \partial D$ parametrized via a mapping $t = t(\tau)$ by the points of the closed interval $\alpha < \tau < \beta$ (See Figure 11). Then the boundary $\Gamma = \partial S$ of the surface S can be written as $x = x(t(\tau))$, where τ ranges over the closed inteval $[\alpha, \beta]$. Using the definiton of the integral over a curve, Green's formula for a plane domain D, and the definiton of the integral over a parametrized surface, we find successively.

$$
\begin{aligned}
\int_\Gamma P(x)\,\mathrm{d}x^1 &= \int_\alpha^\beta P\left(x(t(\tau))\right)\left(\frac{\partial x^1}{\partial t^1}\frac{\mathrm{d}t^1}{\mathrm{d}\tau}+\frac{\partial x^1}{\partial t^2}\frac{\mathrm{d}t^2}{\mathrm{d}\tau}\right)\mathrm{d}\tau \\
&= \int_\gamma P\left(x(t)\right)\left(\frac{\partial x^1}{\partial t^1}\mathrm{d}t^1+\frac{\partial x^1}{\partial t^2}\mathrm{d}t^2\right) \\
&= \iint_D \left[\frac{\partial}{\partial t^1}\left(P\frac{\partial x^1}{\partial t^2}\right)-\frac{\partial}{\partial t^2}\left(P\frac{\partial x^1}{\partial t^1}\right)\right]\mathrm{d}t^1\wedge\mathrm{d}t^2 \\
&= \iint_D \left(\frac{\partial P}{\partial t^1}\frac{\partial x^1}{\partial t^2}-\frac{\partial P}{\partial t^2}\frac{\partial x^1}{\partial t^1}\right)\mathrm{d}t^1\wedge\mathrm{d}t^2 \\
&= \iint_D \sum_{i=1}^3\left(\frac{\partial P}{\partial x^i}\frac{\partial x^i}{\partial t^1}\frac{\partial x^1}{\partial t^2}-\frac{\partial P}{\partial x^i}\frac{\partial x^i}{\partial t^2}\frac{\partial x^1}{\partial t^1}\right)\mathrm{d}t^1\wedge\mathrm{d}t^2 \\
&= \iint_D \left(\frac{\partial P}{\partial x^2}\frac{\partial x^2}{\partial t^1}+\frac{\partial P}{\partial x^3}\frac{\partial x^3}{\partial t^1}\right)\frac{\partial x^1}{\partial t^2} \\
&- \left(\frac{\partial P}{\partial x^2}\frac{\partial x^2}{\partial t^2}+\frac{\partial P}{\partial x^3}\frac{\partial x^3}{\partial t^2}\right)\frac{\partial x^1}{\partial t^1}\,\mathrm{d}t^1\wedge\mathrm{d}t^2 \\
&= \iint_D \left(\frac{\partial P}{\partial x^2}\left|\frac{\partial(x^2,x^1)}{\partial(t^1,t^2)}\right|+\frac{\partial P}{\partial x^3}\left|\frac{\partial(x^3,x^1)}{\partial(t^1,t^2)}\right|\right)\mathrm{d}t^1\wedge\mathrm{d}t^2 \\
&= \iint_S \left(\frac{\partial P}{\partial x^2}\mathrm{d}x^2\wedge\mathrm{d}x^1+\frac{\partial P}{\partial x^3}\,\mathrm{d}x^3\wedge\mathrm{d}x^1\right)
\end{aligned}
$$