

The Differential of a Function of Several Variables

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1 Double Integral

1.1 Introduction

Volume Problem: Find the volume V of the solid G enclosed between the surface $z = f(x, y)$ and the region R in the xy -plane where $f(x, y)$ is continuous and non-negative on R .

Mass Problem: Find the mass M of a lamina (a region R in the xy -plane) whose density is a continuous nonnegative function $\rho(x, y)$

Let us consider the Volume Problem,

1. Divide the rectangle enclosing R into subrectangles, and exclude all those rectangles that contain points outside R . Let n be the number of all the rectangles inside R , and let $\Delta A_k = \Delta x_k \Delta y_k$ be the area of the k -th subrectangle.
2. Choose any point (ξ_k, η_k) in the k -th subrectangle. The volume of a rectangular parallelepiped with base area ΔA_k and height $f(\xi_k, \eta_k)$ is $\Delta V_k = f(\xi_k, \eta_k) \Delta A_k$, thus,

$$V \approx \sum_{k=1}^n \Delta V_k = \sum_{k=1}^n f(\xi_k, \eta_k) \Delta A_k = \sum_{k=1}^n f(\xi_k, \eta_k) \Delta x_k \Delta y_k \quad (1)$$

This sum is called the *Rimann sum*.

3. Take the sides of all the subrectangles to 0, and get

$$V = \lim_{\lambda(P) \rightarrow 0} \sum_{k=1}^n f(\xi_k, \eta_k) \Delta A_k = \iint_R f(x, y) \, dA \quad (2)$$

The last term is the notation for the limit of the Riemann sum, and it is called the *double integral* of $f(x, y)$ over R .

1.2 The Darboux Criterion

Let us consider another criterion for Riemann integrability of a function, which is applicable only to real-valued function. Lower and Upper Darboux Sums. Let f be a real-valued function on the interval I and $P = I_i$ a partition of the interval I . We set

$$m_i = \inf_{x \in I_i} f(x), M_i = \sup_{x \in I_i} f(x). \quad (3)$$

The quantities

$$s(f, P) = \sum_i m_i |I_i|, S(f, P) = \sum_i M_i |I_i| \quad (4)$$

are called the *lower* and *upper sums* of the function f over the interval I corresponding to the partition P of the interval.

The following relations hold between the Darboux sums of a function $f : I \rightarrow \mathbb{R}$:

1. $s(f, P) = \inf_{\xi} \sigma(f, P, \xi) \leq \sigma(f, P, \xi) \leq \sup_{\xi} \sigma(f, P, \xi) = S(f, P)$
2. If the partition P' of the interval I is obtained by refining intervals of the partition P , then $s(f, P) \leq s(f, P') \leq S(f, P') \leq S(f, P)$
3. The inequality $s(f, P_1) \leq S(f, P_2)$ holds for any pair of partitions P_1, P_2 of the interval I .

1.3 Lower and Upper Integrals

The lower and upper Darboux integrals of the function $f : I \rightarrow \mathbb{R}$ over the interval I are respectively

$$\underline{I} = \sup_P s(f, P), \bar{I} = \inf_P S(f, P) \quad (5)$$

where the supremum and infimum are taken over all partitions P of the interval I .

Theorem 1.1. *For any bounded function:*

$$f : I \rightarrow \mathbb{R}, \lim_{\lambda(P) \rightarrow 0} s(f, P) = \underline{I}, \lim_{\lambda(P) \rightarrow 0} S(f, P) = \bar{I}$$

1.4 The Darboux Criterion for Integrability of a Real-valued Function

Theorem 1.2 ((The Darboux Criterion). *A real valued function $f : I \rightarrow \mathbb{R}$ defined on an interval $I \subset \mathbb{R}^\times$ is integrable over that interval if and only if it is bounded on I and its upper and lower Darboux integrals are equal.*

$$f \in \mathcal{R}(I) \equiv f \text{ is bounded on } I, \text{ and } \underline{I} = \bar{I}.$$

1.5 Double Integral Over Non-rectangular Regions

$$\int_a^b \int_c^d f(x, y) \, dy \, dx = \int_a^b \left[\int_c^d f(x, y) \, dy \right] \, dx \quad (6)$$

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \right] \, dx \quad (7)$$

$$\int_a^b \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy = \int_a^b \left[\int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \right] \, dy \quad (8)$$

Examples 1. 1. $\int_0^1 \int_1^2 \sin(2x - 3y) \, dx \, dy$

2. $\int_1^2 \int_0^1 \sin(2x - 3y) \, dy \, dx$

Theorem 1.3. *Let D be the rectangle $[a, b] \times [c, d]$. If $f(x, y)$ is continuous on D then*

$$\iint f(x, y) \, d\sigma = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

Examples 2. *Find the volume V of the solid enclosed by the surfaces $z = 0, y^2 = x, x + z = 1$*

Examples 3. *Reverse the order of the integrals below:*

1. $\int_a^b dx \int_a^x f(x, y) \, dy$

2. $\int_0^1 dy \int_0^{2y} f(x, y) \, dx + \int_1^3 dy \int_0^{3-y} f(x, y) \, dx$

1.6 Double integrals in polar coordinates

2 作業

1. 根據二重積分的性質，比較下列積分的大小

(a) $\iint_D (x+y)^2 dx dy, \iint_D (x+y)^3 dx dy$ ，其中 D 為 x 軸， y 軸與直線 $x+y=1$ 所圍成的區域。

(b) $\iint_D \ln(x+y) dx dy, \iint_D [\ln(x+y)]^2 dx dy$ ，其中 $D = [3, 5] \times [0, 1]$ 。

2. 計算下列重積分

(a) $\iint_D x^3 + 3x^2y + y^3 dx dy$ ，其中 $D = [0, 1] \times [0, 1]$ 。

(b) $\iint_D xy e^{x^2+y^2} dx dy$ ，其中 $D = [a, b] \times [c, d]$ 。

(c) $\iint_D xy^2 dx dy$ ，其中 D 由拋物線 $y^2 = 2px$ 和直線 $x = \frac{p}{2} (p > 0)$ 所圍成的區域。

(d) $\iint_D e^{x+y} dx dy$ ，其中 D 為 $\{(x, y) \mid |x| + |y| \leq 1\}$ 所圍成的區域。

(e) $\iint_D x dx dy$ ，其中 D 為拱線的一拱 $x = a(t - \sin t), y = a(1 - \cos t) (0 \leq t \leq 2\pi)$ 所圍成的區域。