CS 519: Interactive Computer Graphics Fall 2015
In-class Worksheet 3

Vector Fields and Numerical Integration

1. Gradient

Suppose we have a 3D scalar field defined by $f(x, y, z) = x^2 + y^2 + z^2 - 1$
What is the ∇f?

$$\langle \partial f/\partial x, \partial f/\partial y, \partial f/\partial z \rangle =$$

$$\langle 2x, 2y, 2z \rangle$$

2. Divergence

Suppose we have a 3D vector field defined by
$v = \langle x^2yz^2, e^{2xy}, xy \rangle$
What is $\nabla \cdot v$?

$$\partial v_x/\partial x + \partial v_y/\partial y + \partial v_z/\partial z =$$

$$2xyz^2 + 2xe^{2xy} + 0$$

3. Curl

Suppose we have a 3D vector field defined by
$v = \langle x^2yz^2, x^2yz^2, xyz \rangle$
What is $\nabla \times v$?

$$\left\langle \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}, \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}, \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right\rangle$$

$$xz - 2x^2yz, 2x^2yz - yz, 2xyz^2 - x^2z^2$$

4. Laplacian

What is $\nabla^2 f$ using the function from question 1?

6

5. **Euler Integration**

A vector field in which describes a velocity can be thought of as an Ordinary Differential Equation (ODE). If we pick a seed point p, we can integrate numerically to recover a function that describes a position at any point in time. Since we are working numerically, the recovered function will be a piece-wise linear approximation of the actual function.

Euler integration is the simplest method for solving an ODE. Starting at some point *p*, We simply step using some step-size *h* in the direction of the velocity vector…and then repeat: $p_{i+1} = p_i + h\, v(p_i)$

Perform 2 steps of Euler integration on starting from (1,1) using the vector function $v = \langle 2x, 2y \rangle$ and a step-szie of $h = \frac{1}{2}$

p_0 (1,1)

$p_1 = (1,1) + \frac{1}{2}(2,2) = (2,2)$

$p_2 = (2,2) + \frac{1}{2}(4,4) = (4,4)$

6. **Runge-Kutta Integration**

RK integration uses a predictor-corrector formulation in which several Euler steps are averaged to find the actual step. Heun's method is the name for the RK method using 2 Euler steps (RK2):

$$p_{i+1} = p_i + \frac{h}{2} v(p_i) + \frac{h}{2} v(p_i + h\, v(p_i))$$

This significantly reduces error in the integration.

Perform one step of Heun's method with the same vector-field, starting point and step-size as in question5.

$p_0 = (1,1)$

$p_1 = (1,1) + \frac{1}{4}(2,2) + \frac{1}{4}(4,4)$

$= (2\frac{1}{2}, 2\frac{1}{2})$