Probability Refresher and Cycle Analysis

A Quick Probability Refresher

- A random variable, X, can take on a number of different possible values
  - Example: the number of pigeons on the windowsill outside is a random variable with possible values 1,2,3,…
- Each time we observe (or sample) the random variable, it may take on a different value

A Quick Probability Refresher

- A random variable takes on each of these values with a specified probability
  - Example: $X = \{0, 1, 2, 3, 4\}$
  - $P[X=0] = .1$, $P[X=1] = .2$, $P[X=2] = .4$, $P[X=3] = .1$, $P[X=4] = .2$
- The sum of the probabilities of all values equals 1
  - $\Sigma_{\text{all values}}\, P[X=value] = 1$

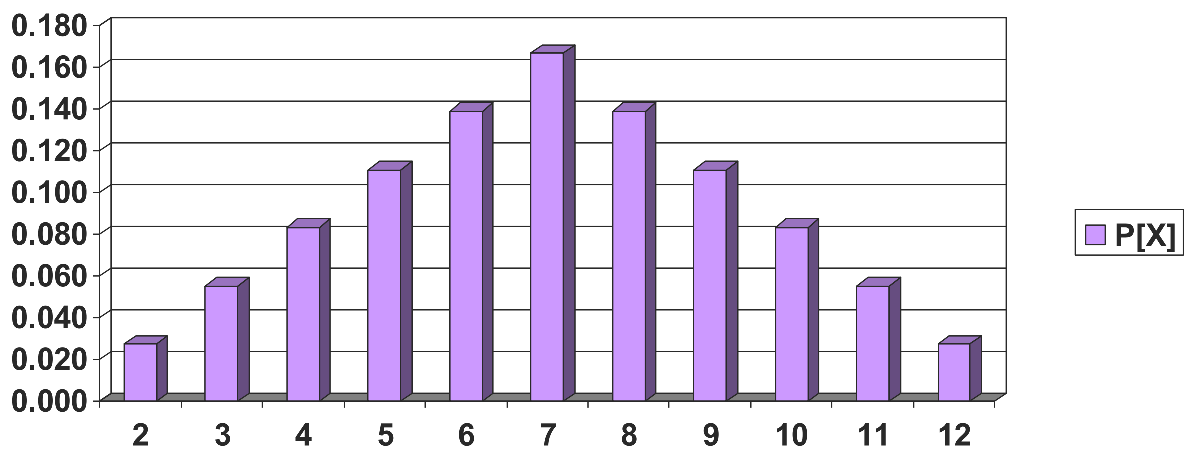
A Quick Probability Refresher

- Example
  - Suppose we throw two dice and the random variable, X, is the sum of the two dice
  - Possible values of X are $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
  - $P[X=2] = P[X=12] = 1/36$
  - $P[X=3] = P[X=11] = 2/36$
  - $P[X=4] = P[X=10] = 3/36$
  - $P[X=5] = P[X=9] = 4/36$
  - $P[X=6] = P[X=8] = 5/36$
  - $P[X=7] = 6/36$

Note: $\sum_{i=2}^{12} P[X=i] = 1$

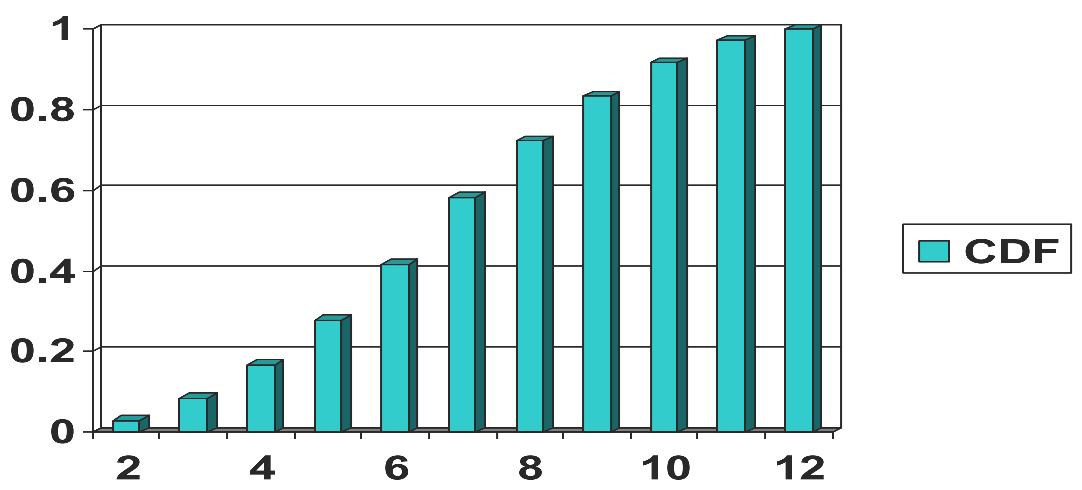
A Quick Probability Refresher

- A probability distribution function matches each possible value of a random variable with its associated probability



A Quick Probability Refresher

- The cumulative distribution function of a random variable, X, is defined by
  - CDF: $P[X \le x] = \Sigma_{all\ y=x}\ P[x=y]$



A Quick Probability Refresher

- Expected Value
  - Can be thought of a "long term average" of observing the random variable a large number of times

$$E[X] = \bar{x} = \sum_{\text{All possible values of } x} Value * P[X = value]$$

- Example: dice - *E[X]*

$= 2*1/36 + 3*2/36 + 4*3/36 + 5*4/36 + 6*5/36 + 7*6/36 + 8*5/36 + 9*4/36 + 10*3/36 + 11*2/36 + 12*1/36$

A Quick Probability Refresher

- Average vs. Expected Value
  - Short term average
    - Suppose a random variable X is sampled N times
    - Let n_i = # of $X = i$ was observed
    - Average of samples
      - $= (n_0{*}0 + n_1{*}1 + n_2{*}2 + n_3{*}3 + \ldots)/N$
      - $= n_0/N{*}0 + n_1/N{*}1 + n_2/N{*}2 + n_3/N{*}3 + \ldots$
    - As $N \rightarrow \infty$, the ratio n_i/N becomes p_i
  - Thus, $E[X]$
    - $= \lim_{N\rightarrow\infty} [n_0/N{*}0 + n_1/N{*}1 + n_2/N{*}2 + n_3/N{*}3 + \ldots]$
    - $= p_0{*}0 + p_1{*}1 + p_2{*}2 + p_3{*}3 + \ldots$
    - $= \Sigma^{\infty}_{i=0}\, i * p_i$

A Quick Probability Refresher

- Continuous Random Variables
  - In many cases, a random variable takes a value drawn from a continuous interval
    - Ex: processing time for a packet may be any real value [0, ∞)
  - The distribution of possible values a continuous random variable can take is given by a probability density function, *F(x)*
  - $P(a \le x \le b) = \int_a^b F(x)dx = \Sigma^b_{i=a} P(x = i)$
  - $E[x] = \int_{-\infty}^{\infty} xF(x)dx = \Sigma_i * P(x = i)$

Probability Example

- Basic probability notions
  - Two useful rules
    - Probabilities of all possible events sum to 1
    - Probability of independent events
      - Product of probabilities of events
      - e.g., probability of two coins coming up heads $= 1/2 \times 1/2 = 1/4$
  - Calculating averages/expected values
    - Function f
    - Multiply f by probability for each possible event
    - Sum over all events

Probability Example - Problem

- Given a bag with *N* balls
  - *1 blue* ball
  - *N - 1 white* balls
- Algorithm
  - pick a ball
    - if *blue*, you win
    - else return to bag
  - repeat *N* times
- Question
  - What is your chance of winning for large *N*?

Probability Example - Solution

- Can write as a sum
  - Chance of finding *blue* on first try = $1/N$
  - On second try = $[(N-1)/N] * (1/N)$
  - Etc.
- Instead, write
  - *1 - (chance of losing)*
  - Parenthesized term
    - Product of N factors
    - Each factor = $(N-1)/N$
  - $1 - [(N - 1)/N]^N$

Probability Example - Solution

- For $N = 2$,
  - $1/2$ first is *white*
  - $1/2$ second is *white*
  - $1/4$ both are *white*
  - $3/4$ chance to win $= 1 - (1/2)^2$
- For $N=3$,
  - $2/3$ first is *white*
  - $2/3$ second is *white*
  - $2/3$ third is *white*
  - $8/27$ all three are *white*
  - $19/27$ chance to win $= 1 - (2/3)^3$ $(< 3/4)$

Probability Example - Solution

- *N=4* probability of win = 68%
- *N=5* probability of win = 67%
- *N=8* probability of win = 66%
- large *N*? *0*?

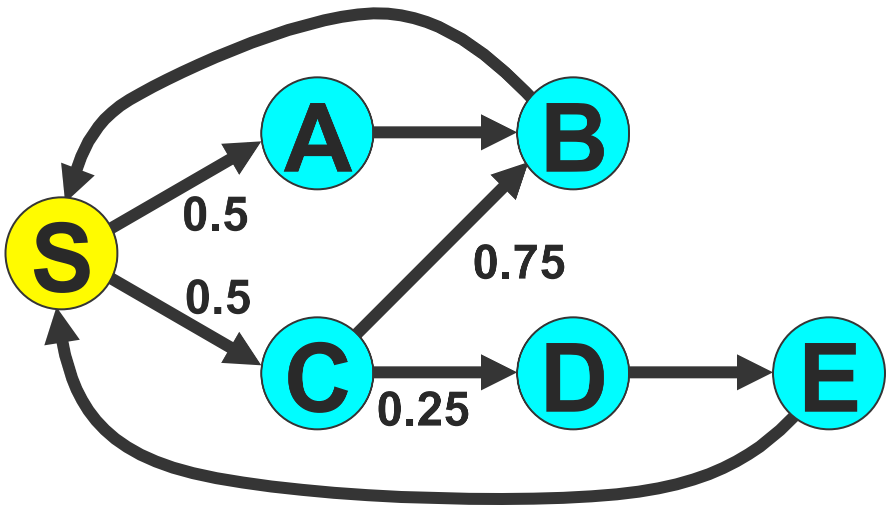
$$\lim_{N\to\infty}\left(\frac{N-1}{N}\right)^{N}$$

Fun Example

- Flip a coin repeatedly.
  - Two heads in a row scores 1 point.
  - Scoring pairs may not overlap
    - (e.g., three heads in a row does not score 2 points).
- On average, how many points do you score per flip?
- Would you play this game in Las Vegas for
  - $1 per flip and $5 per point?
  - $1 per flip and $7 per point?

A Different Example

- What fraction of time (on average) is spent in state E?

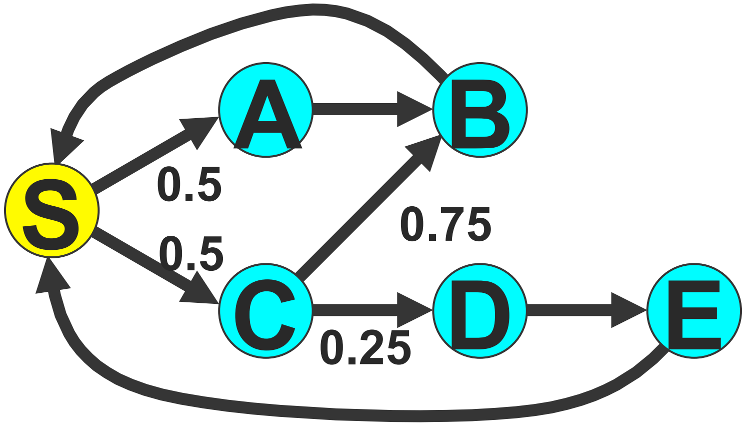


Cycle Analysis

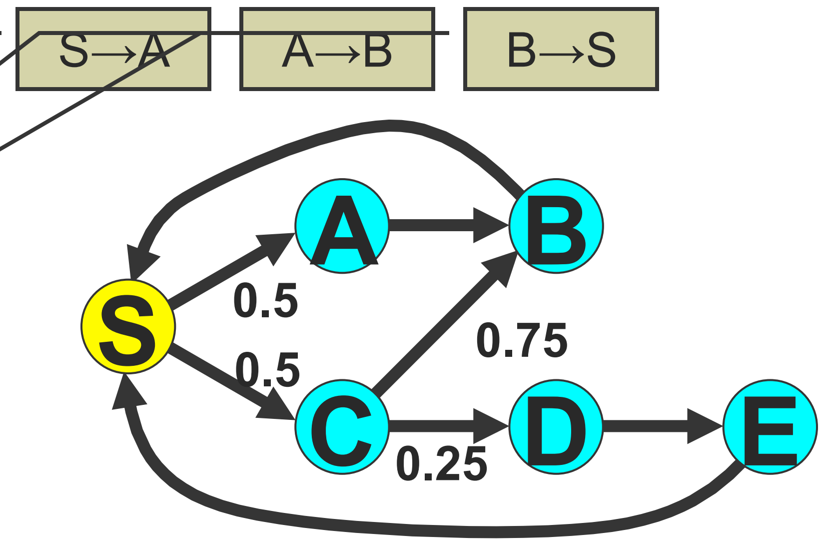
- Start with a discrete Markov process
  - Transitions happen periodically (every Δt)
  - Probabilities independent of past/future behavior
- Form all possible cyclic sequences (cycles)
  - Pick a "start" state
  - List all cycles from that state
  - Calculate probability per cycle
  - Calculate average cycle length
- Can calculate expected values of cycle-dependent properties with average length and cycle probabilities

Example

cycle **probability**



Example



cycle **probability**

ABS

CBS

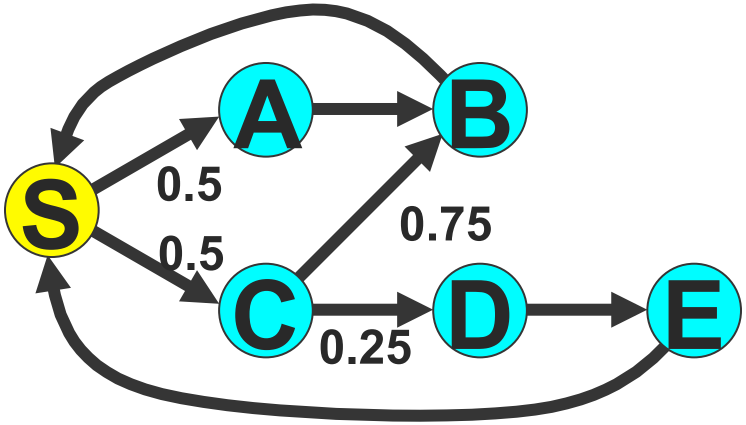
CDES

average cycle length

ABS CBS CDES

Example

Amount of time spent in E when in cycle CDES



- average fraction of time spent in E
  = 1•0.125 periods/cycle

Probability of cycle CDES

- dividing by average length…
  = 0.125 / 3.125 = 0.04

Fun Example

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- Would you play this game in Las Vegas for
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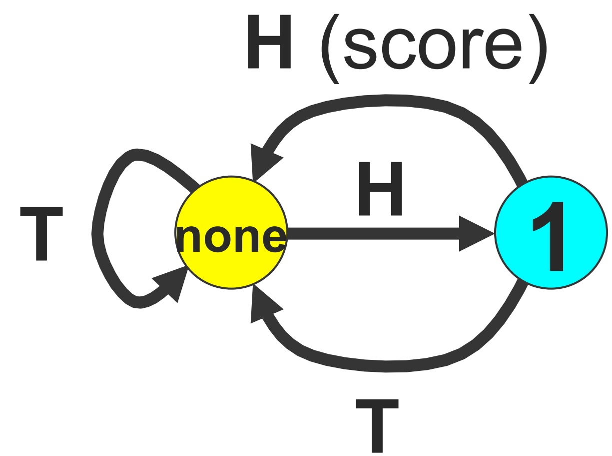
  - \$1 per flip and \$7 per point?

Fun Example

cycle	probability
T	1/2
HT	1/4
HH	1/4



average cycle length

average score per cycle

average score per flip

Fun Example

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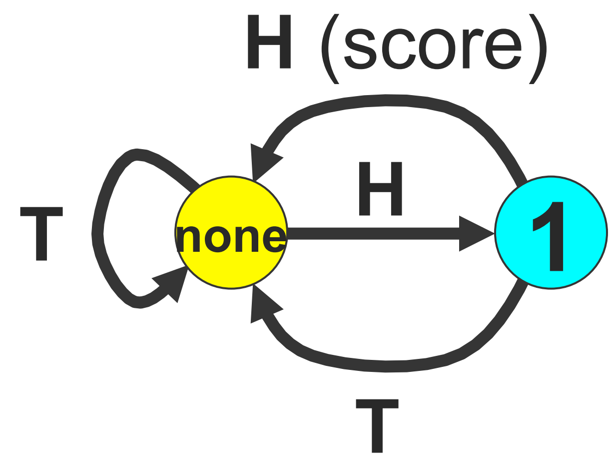

average cycle length = 1/2 + 1/2 + 1/2 = 3/2 flips
average score per cycle = 1/4 points
average score per flip = (1/4) / (3/2) = 1/6 pts/flip

(Good luck getting $7 per point in Vegas!)