Notes about GANs

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Goals of this talk (adapted from ECE 547 lecture)

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- Discussing some issues related to GANs
- Simplified proofs
- Some research directions

Intuition

Classification:

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Which object is illustrated?

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- Car

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$$F(\boldsymbol{y}, x, \boldsymbol{w})$$

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$\boldsymbol{y}$: discrete output space

Parametric ($\boldsymbol{w}$) score function:

$$F(\boldsymbol{y}, x, \boldsymbol{w})$$

Model:

$$p_w(\boldsymbol{y}|x) = \frac{\exp F(\boldsymbol{y}, x, \boldsymbol{w})/\epsilon}{\sum_{\hat{\boldsymbol{y}}} \exp F(\hat{\boldsymbol{y}}, x, \boldsymbol{w})/\epsilon}$$

How about modeling a distribution $p(x)$ for the data?

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- Environment simulator (reinforcement learning, planning)
- Leveraging unlabeled data

Given data points x, how can we model $p(x)$?

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  - Use a variational auto-encoder

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Generative Adversarial Nets

Main idea:

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Advantage:

No summation over large probability spaces

Formulate the problem as a game between two players:

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Goal:

G tries to “trick” D by generating samples that are hard to distinguish

Pictorially:

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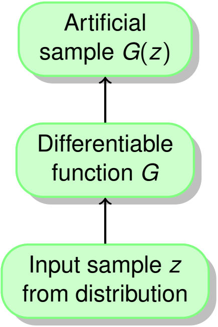

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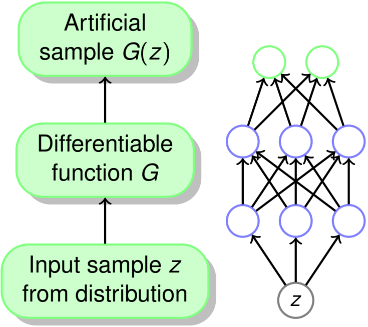

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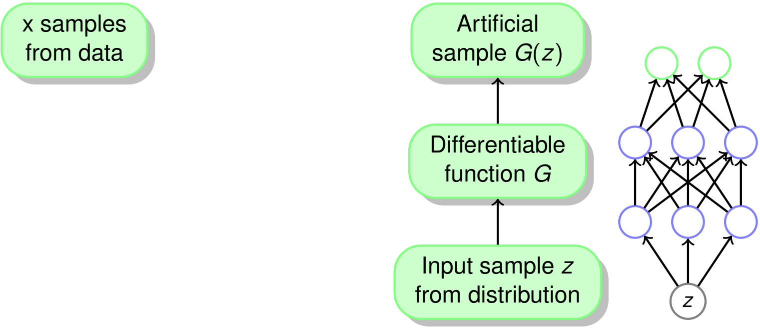

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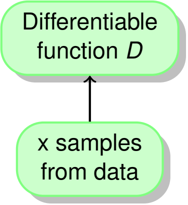

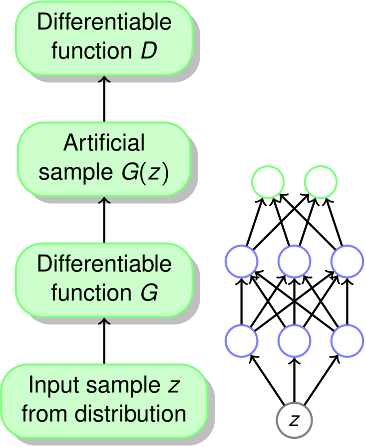

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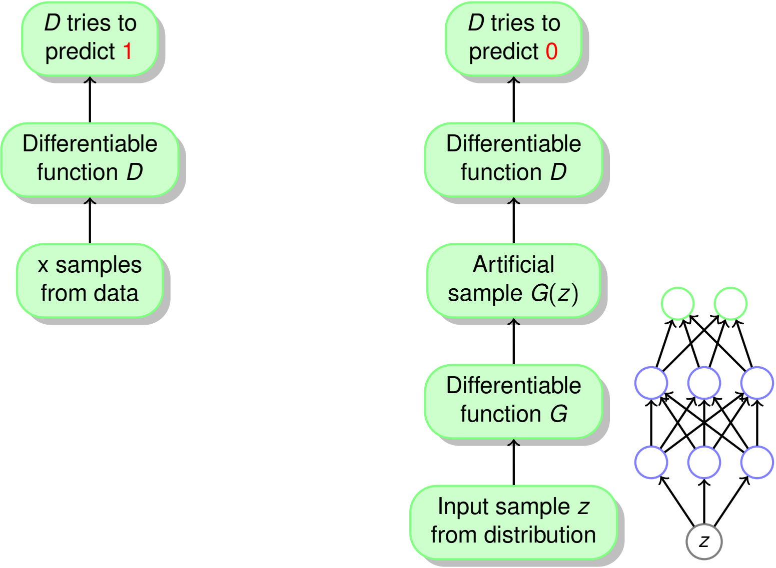

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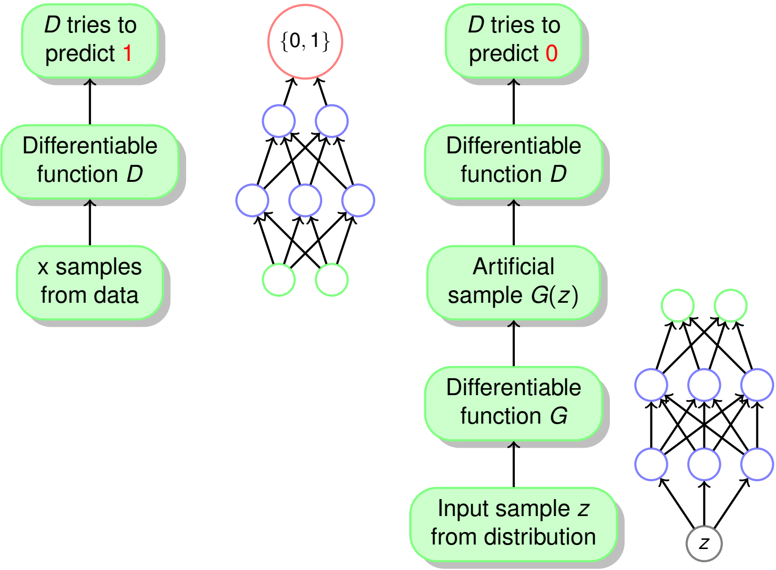

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In practice:

Heuristics make this optimization more stable.

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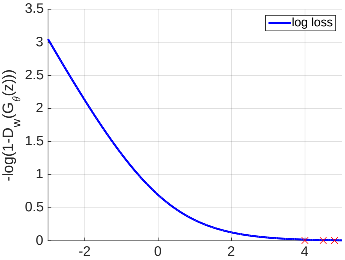

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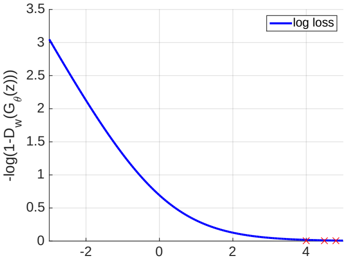

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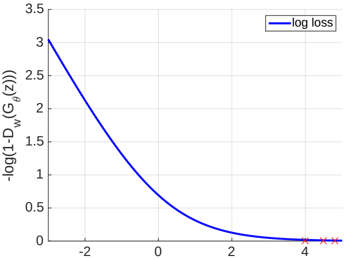

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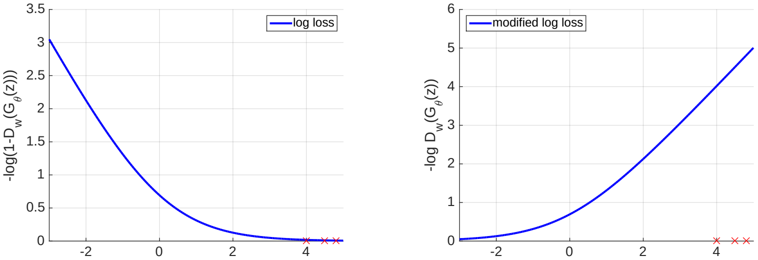

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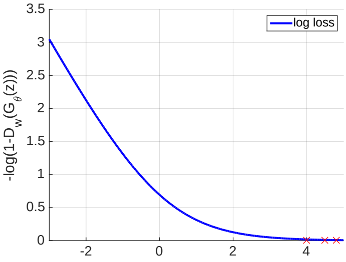
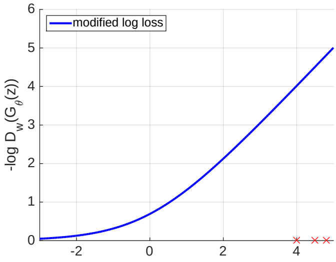
- Issue: no joint cost function for D and G





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- Convex in w

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- Convex in w
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- Good empirical results
- Additional insights

Primal:

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Dual:

Primal:

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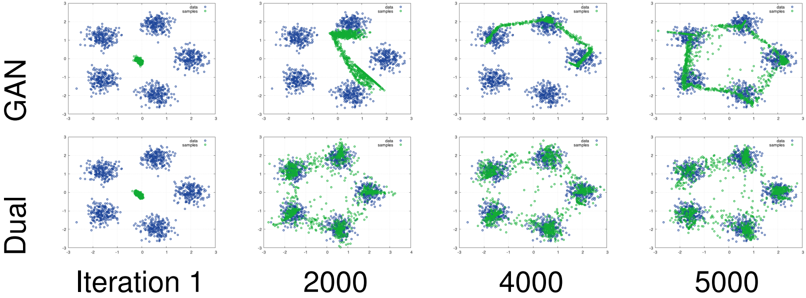
Dual:

$$\begin{array}{ll} \max\limits_{\theta,\lambda_x,\lambda_z} & \frac{-1}{2C}\left\|\sum\limits_{\boldsymbol{x}}(1-\lambda_x)\boldsymbol{x} - \sum\limits_{\boldsymbol{z}}\lambda_z G_\theta(\boldsymbol{z})\right\|_2^2 + \sum\limits_{\boldsymbol{x}} H(\lambda_x) + \sum\limits_{\boldsymbol{z}} H(\lambda_z) \\ \text{s.t.} & \begin{array}{ll} \forall \boldsymbol{x} & 0 \le \lambda_x \le 1 \\ \forall \boldsymbol{z} & 0 \le \lambda_z \le 1 \end{array} \end{array}$$

Some results on toy data:



Plenty of additional impressive tricks.

Analysis of generative adversarial nets:

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What is the optimal discriminator assuming arbitrary capacity?

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Euler-Lagrange formalism:

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S(D) = \int_x L(x, D, \dot{D}) dx
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Consequently:

What is the optimal discriminator assuming arbitrary capacity?

$$\frac{\partial L(x, D, \dot{D})}{\partial D} = -\frac{p_{\text{data}}}{D} + \frac{p_G}{1-D} = 0$$

Consequently:

$$D^*(x) = \frac{p_{\text{data}}(x)}{p_{\text{data}}(x) + p_G(x)}$$

Given the optimal discriminator, what is the optimal generator?

$$-\int_x p_{\text{data}}(x)\log D^*(x) + p_G(x)\log(1 - D^*(x))dx$$

$$=$$

$$=$$

Given the optimal discriminator, what is the optimal generator?

$$
\begin{aligned}
& -\int_x p_{\text{data}}(x) \log D^*(x) + p_G(x) \log(1 - D^*(x)) dx \\
= \quad & -\int_x p_{\text{data}}(x) \log \frac{p_{\text{data}}(x)}{p_{\text{data}}(x) + p_G(x)} + p_G(x) \log \frac{p_G(x)}{p_{\text{data}}(x) + p_G(x)} dx \\
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& -\int_x p_{\text{data}}(x)\log D^*(x) + p_G(x)\log(1-D^*(x))dx \\
= & -\int_x p_{\text{data}}(x)\log\frac{p_{\text{data}}(x)}{p_{\text{data}}(x)+p_G(x)} + p_G(x)\log\frac{p_G(x)}{p_{\text{data}}(x)+p_G(x)}dx \\
=
\end{aligned}
$$

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Given the optimal discriminator, what is the optimal generator?

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$$
p_G(x) = p_{\text{data}}(x)
$$

Recall:

GANs optimize Jensen-Shannon divergence

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How about other divergences/distances?

Wasserstein GAN:

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Other ways to measure distance between two distributions:

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Wasserstein distance:

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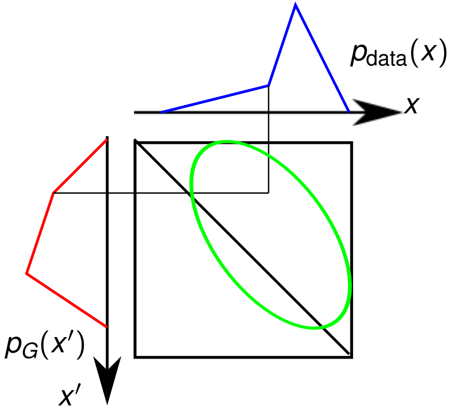
Pictorially:



How to formulate GANs using Wasserstein distance?

$$\min_{p_G} W(p_{\text{data}}, p_G) = \min_{p_G} \min_{p_J(x,x')\in\Pi(p_{\text{data}},p_G)} \mathbb{E}_{p_J}[\|x - x'\|]$$

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Results:

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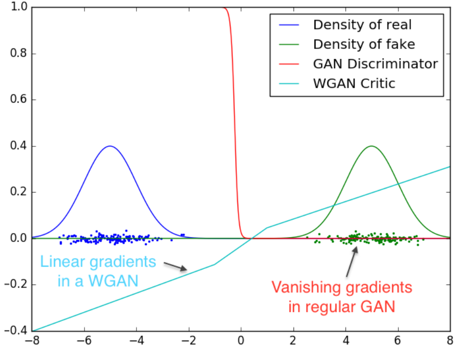

1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
−8
−6
−4
−2
0
2
4
6
8
Density of real
Density of fake
GAN Discriminator
WGAN Critic
Linear gradients in a WGAN
Vanishing gradients in regular GAN

Wasserstein GAN objective:

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Linear GAN dual:

$$\begin{array}{ll} \max\limits_{\theta,\lambda_x,\lambda_z} & \frac{-1}{2C}\left\|\sum_{\boldsymbol{x}}(1-\lambda_x)\boldsymbol{x} - \sum_{\boldsymbol{z}}\lambda_z G_\theta(\boldsymbol{z})\right\|_2^2 + \sum_{\boldsymbol{x}} H(\lambda_x) + \sum_{\boldsymbol{z}} H(\lambda_z) \\ \text{s.t.} & \begin{array}{ll} \forall \boldsymbol{x} & 0 \leq \lambda_x \leq 1 \\ \forall \boldsymbol{z} & 0 \leq \lambda_z \leq 1 \end{array}, \end{array}$$

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Maximum mean discrepancy: [Gretton et al.]

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Maximum mean discrepancy: [Gretton et al.]

$$MMD(x, G_\theta(z)) := \max_f \left(\mathbb{E}_x[f(x)] - \mathbb{E}_z[G_\theta(z)]\right)$$

Other ways to optimize the Wasserstein distance:

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Sliced Wasserstein distance: [Rabin et al. (ECCV 2010); Texture Mixing]

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$$\tilde{W}(X, X')^2 = \int_{w \in \Omega} \min_{\sigma_w} \sum_i |(X_i - X'_{\sigma_w(i)})^T w|^2 dw$$

with permutation: σ_w

Discussion