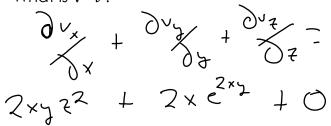
Vector Fields and Numerical Integration

1. Gradient

2. Divergence

Suppose we have a 3D vector field defined by $v = \langle x^2yz^2, e^{2xy}, xy \rangle$ What is $\nabla \cdot v$?



3. Curl

Suppose we have a 3D vector field defined by $v = \langle x^2yz^2, x^2yz^2, xyz \rangle$ $\left(\frac{\partial J_2}{\partial y} - \frac{\partial J_2}{\partial z}\right) \frac{\partial J_2}{\partial z} - \frac{\partial J_2}{\partial x} = \frac{\partial J_2}{\partial x} + \frac{\partial J_2}{\partial x} = \frac{\partial J_2}{\partial x}$

4. Laplacian

What is $\nabla^2 f$ using the function from question 1?



5. Euler Integration

A vector field in which describes a velocity can be thought of as an Ordinary Differential Equation (ODE). If we pick a seed point p, we can integrate numerically to recover a function that describes a position at any point in time. Since we are working numerically, the recovered function will be a piece-wise linear approximation of the actual function.

Euler integration is the simplest method for solving an ODE. Starting at some point p, We simply step using some step-size h in the direction of the velocity vector...and then repeat: $p_{i+1} = p_i + h v(p_i)$

Perform 2 steps of Euler integration on starting from (1,1) using the vector function $v = \langle 2x, 2y \rangle$ and a step-szie of $h = \frac{1}{2}$

$$P_{0}(1,1)$$
 $P_{1}=(1,1)+\frac{1}{2}(2,2)=(2,2)$
 $P_{2}=(2,2)+\frac{1}{2}(4,4)=(4,4)$

6. Runge-Kutta Integration

RK integration uses a predictor-corrector formulation in which several Euler steps are averaged to find the actual step. Heun's method is the name for the RK method using 2 Euler steps (RK2):

$$p_{i+1} = p_i + \frac{h}{2}v(p_i) + \frac{h}{2}v(p_i + h v(p_i))$$

This significantly reduces error in the integration.

Perform one step of Heun's method with the same vector-field, starting point and step-size as in question5.

$$P_{0} = (1,1)$$
 $P_{1} = (1,1) + \frac{1}{4}(2,2) + \frac{1}{4}(4,4)$
 $= (2,2) + \frac{1}{4}(4,4)$