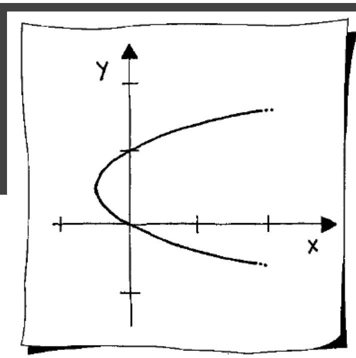
CS 418: Interactive Computer Graphics

Bezier Curves

Eric Shaffer

Geometric Modeling



We will finish the semester by briefly looking at some math for modeling

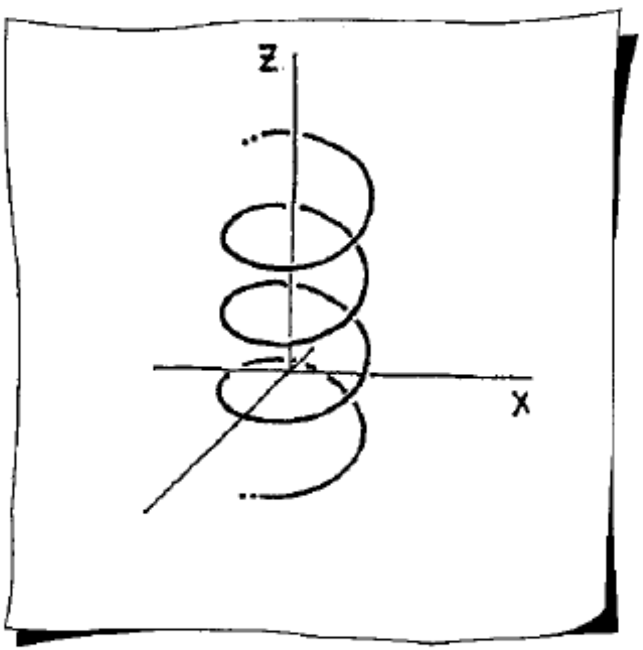
Geometric modeling is typically done by engineers and artists
- Assisted by computational tools (e.g. Maya or Blender or AutoCAD)
- The software provides a mathematical models of curves/surfaces

For rendering, ultimately everything will be turned into triangles

But modeling triangle-by-triangle would be too tedious

Also, using alternative representations can have other advantages
- More compact
- "Infinite resolution"
- Some tasks are easier
  - e.g. finding derivatives or deforming the geometry

Parametric Curves



Parametric curves defined in 3D:

$$\mathbf{x}(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} f(t) \\ g(t) \\ h(t) \end{bmatrix}$$

Simple example: a *helix*

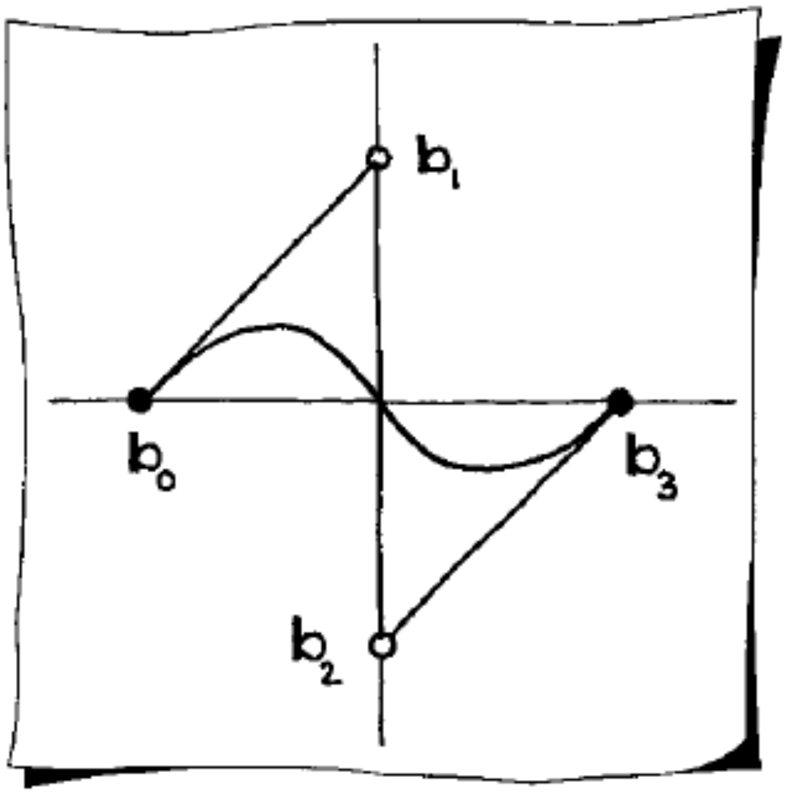
$$\mathbf{x}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix}$$

Bezier Curves

Bezier Curves

- Type of polynomial curve
- Curve is defined by a modeler (artist) by specifying control points
- Can be defined to generate a polynomial of any degree
  - Cubics are most common
  - Higher degree curve requires more control points
- Can be joined together to form piecewise polynomial curves
- Can form the basis of Bezier patches which define a surface
- Named after Pierre Bezier
  - French Mechanical Engineer worked for Renault
  - Lived 1910-1999

Cubic Bezier Curves



$$\mathbf{x}(t) = \begin{bmatrix} -(1-t)^3 + t^3 \\ 3(1-t)^2 t - 3(1-t)t^2 \end{bmatrix}$$

Shape?
Rewrite as a combination of points

$$\mathbf{x}(t) = (1-t)^3 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 3(1-t)^2 t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3(1-t)t^2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + t^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Four points form a polygon
– Resembles curve for $t \in [0, 1]$

Cubic Bezier Curves

Define a cubic Bézier curve by

$$\mathbf{x}(t) = (1-t)^3\mathbf{b}_0 + 3(1-t)^2 t\mathbf{b}_1 + 3(1-t)t^2\mathbf{b}_2 + t^3\mathbf{b}_3$$

2D or 3D points $\mathbf{b}_i$ are the Bézier control points
Control points form the Bézier polygon of the curve
Also written as

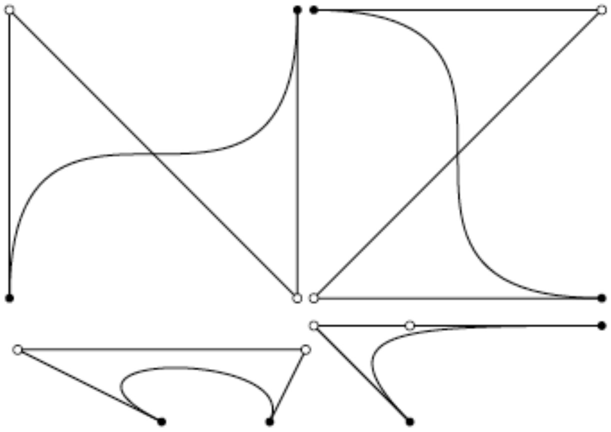
$$\mathbf{x}(t) = B_0^3(t)\mathbf{b}_0 + B_1^3(t)\mathbf{b}_1 + B_2^3(t)\mathbf{b}_2 + B_3^3(t)\mathbf{b}_3$$

B_i^3 are called the cubic Bernstein polynomials
The $\mathbf{b}_i$ are called the coefficients of the polynomial $\mathbf{x}(t)$

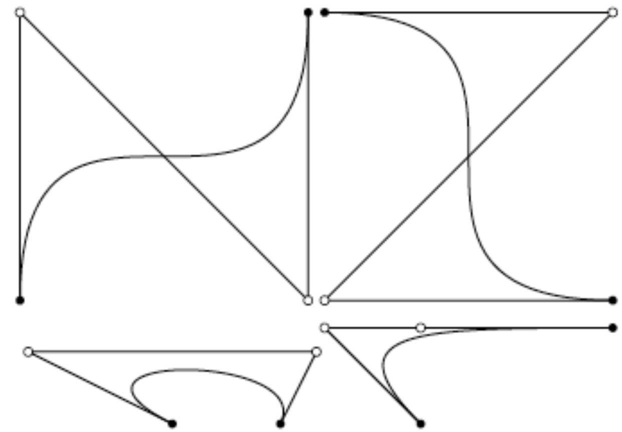
Cubic Bezier Curves

Important Properties of Bezier Curves

- Endpoint Interpolation
- Symmetry
- Invariance under affine transformations
- Convex hull property
- Linear precision



Cubic Bezier Curves

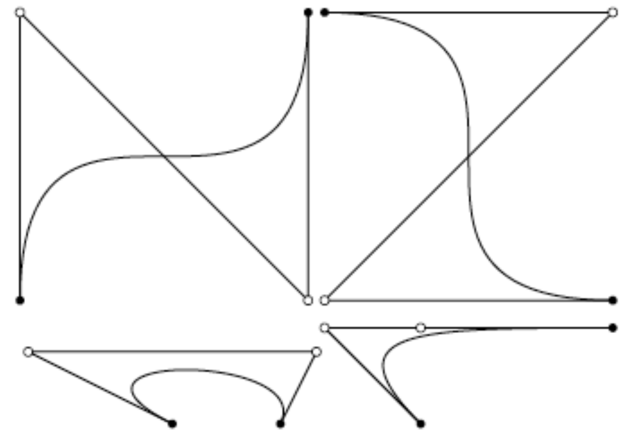


Important Properties of Bezier Curves

- Endpoint Interpolation
  The curve will pass through the first and last control points:
  $x(0.0) = b_0$
  $x(1.0) = b_3$

- Symmetry
  Specifying contol points in the order
  b_0,b_1,b_2,b_3
  generates the same curve as the order:
  b_3,b_2,b_1,b_0

Cubic Bezier Curves



Important Properties of Bezier Curves

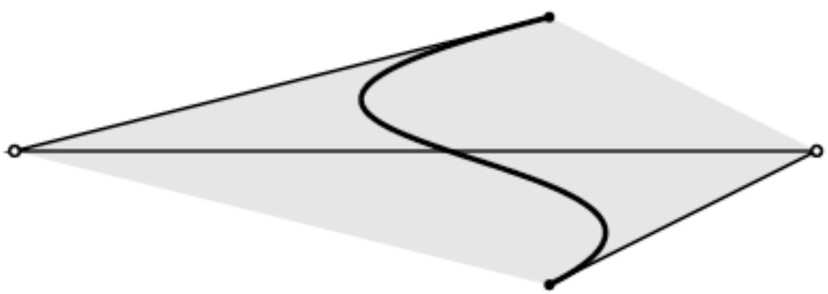
- Invariance under affine transformations

  Transforming the control polygon similarly transforms the curve

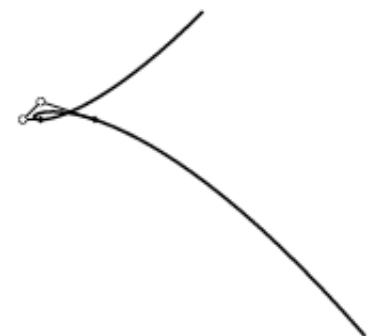
- Linear Precision

  If b_1 and b_2 are evenly spaced on a straight line, the cubic Bezier curve will be the linear interpolant between b_0 and b_3

Convex Hull Property



The convex hull property



A Bézier curve for $t \in [-1, 2]$

Extrapolation: t outside $[0, 1]$
– Curve not within convex hull (in general)
– Unpredictable behavior

Derivatives

Differentiate each component with respect $t \Rightarrow$ the tangent vector

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = -3(1-t)^2\mathbf{b}_0 + [3(1-t)^2 - 6(1-t)t]\mathbf{b}_1 + [6(1-t)t - 3t^2]\mathbf{b}_2 + 3t^2\mathbf{b}_3$$

Group like terms

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = 3[\mathbf{b}_1 - \mathbf{b}_0](1-t)^2 + 6[\mathbf{b}_2 - \mathbf{b}_1](1-t)t + 3[\mathbf{b}_3 - \mathbf{b}_2]t^2$$

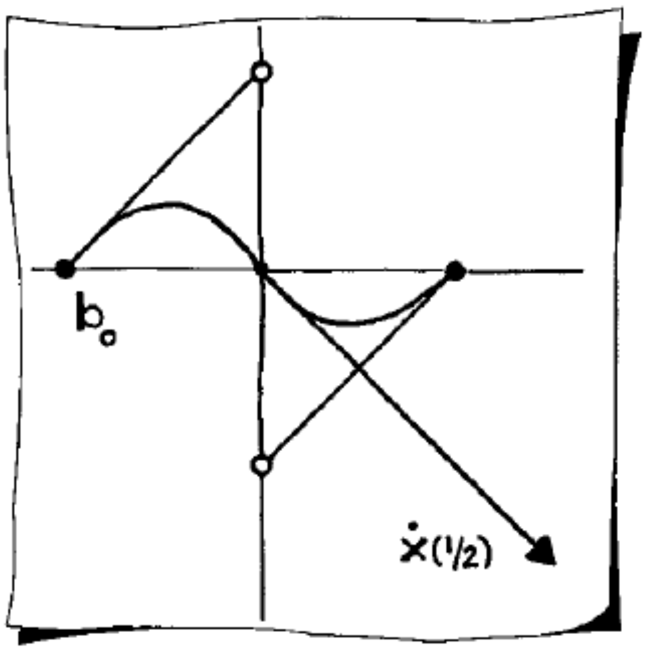
Abbreviated as

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = 3\Delta\mathbf{b}_0(1-t)^2 + 6\Delta\mathbf{b}_1(1-t)t + 3\Delta\mathbf{b}_2 t^2$$

where $\Delta\mathbf{b}_i$ is known as the forward difference

Shorten notation: $\dot{\mathbf{x}}(t) \equiv \mathrm{d}\mathbf{x}(t)/\mathrm{d}t$

Derivatives



Example

$$\mathbf{x}(t) = (1-t)^3 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 3(1-t)^2 t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3(1-t)t^2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + t^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\dot{\mathbf{x}}(t) = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (1-t)^2 + 6 \begin{bmatrix} 0 \\ -2 \end{bmatrix} (1-t)t + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} t^2$$

$$\dot{\mathbf{x}}(0.5) = \begin{bmatrix} 1.5 \\ -1.5 \end{bmatrix}$$

Piecing Together Curves

Tangent vectors at the curve's endpoints:

$$\dot{\mathbf{x}}(0) = 3\Delta\mathbf{b}_0 \qquad \dot{\mathbf{x}}(1) = 3\Delta\mathbf{b}_2$$

$\Rightarrow$ control polygon is tangent to the curve at the endpoints
– property helps with piecing together several Bézier curves

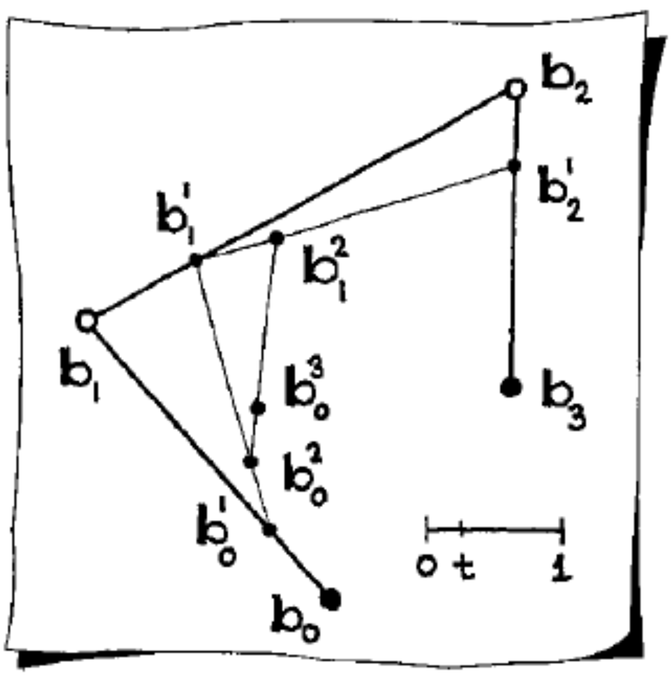
The de Casteljau Algorithm

The de Casteljau algorithm is probably the most important algorithm of all of CAGD

Paul de Faget de Casteljau invented it in 1959

The de Casteljau algorithm is a recursive algorithm that constructs the point $\mathbf{x}(t)$ on a Bézier curve

The de Casteljau Algorithm



Given: $\mathbf{b}_0, \ldots, \mathbf{b}_3$

and a parameter value t

Find: $\mathbf{x}(t)$

Compute:

$$\mathbf{b}_0^1 = (1-t)\mathbf{b}_0 + t\mathbf{b}_1$$
$$\mathbf{b}_1^1 = (1-t)\mathbf{b}_1 + t\mathbf{b}_2$$
$$\mathbf{b}_2^1 = (1-t)\mathbf{b}_2 + t\mathbf{b}_3$$

$$\mathbf{b}_0^2 = (1-t)\mathbf{b}_0^1 + t\mathbf{b}_1^1$$
$$\mathbf{b}_1^2 = (1-t)\mathbf{b}_1^1 + t\mathbf{b}_2^1$$

$$\mathbf{x}(t) = \mathbf{b}_0^3 = (1-t)\mathbf{b}_0^2 + t\mathbf{b}_1^2$$

Simply repeated linear interpolation!

The de Casteljau Algorithm

A convenient schematic tool for describing the algorithm
– Arrange the involved points in a triangular diagram

$$\begin{array}{llll}
\mathbf{b}_0 & & & \\
\mathbf{b}_1 & \mathbf{b}_0^1 & & \\
\mathbf{b}_2 & \mathbf{b}_1^1 & \mathbf{b}_0^2 & \\
\mathbf{b}_3 & \mathbf{b}_2^1 & \mathbf{b}_1^2 & \mathbf{b}_0^3
\end{array}$$

In the implementation of the de Casteljau algorithm:
– Not necessary to use a 2D array to simulate the triangular diagram
– A 1D array of control points is sufficient
For example $\mathbf{b}_0^1$ is calculated and loaded into $\mathbf{b}_0$
(Must save original control polygon)

The de Casteljau Algorithm

Example

$$\mathbf{x}(t) = (1-t)^3 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 3(1-t)^2 t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3(1-t)t^2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + t^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Evaluate at $t = 0.5$

$$\begin{array}{ccccc}
\begin{bmatrix} -1.0 \\ 0.0 \end{bmatrix} & & & & \\
\begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix} & \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix} & & & \\
\begin{bmatrix} 0.0 \\ -1.0 \end{bmatrix} & \begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix} & \begin{bmatrix} -0.25 \\ 0.25 \end{bmatrix} & & \\
\begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix} & \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} & \begin{bmatrix} 0.25 \\ -0.25 \end{bmatrix} & \begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix} & = \mathbf{x}(0.5)
\end{array}$$

The Matrix Form and Monomials

A cubic Bézier curve:

$$\mathbf{b}(t) = B_0^3(t)\mathbf{b}_0 + B_1^3(t)\mathbf{b}_1 + B_2^3(t)\mathbf{b}_2 + B_3^3(t)\mathbf{b}_3$$

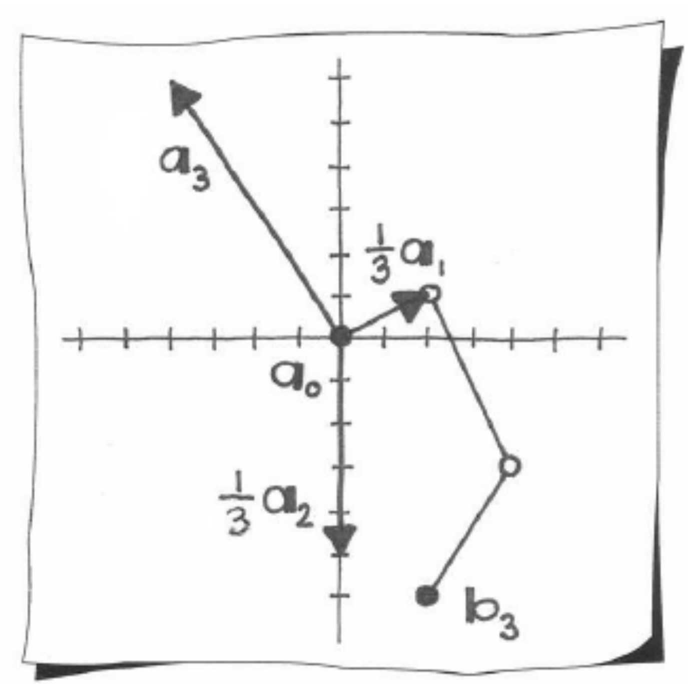
Rewritten in matrix form:

$$\mathbf{b}(t) = \begin{bmatrix} \mathbf{b}_0 & \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} \begin{bmatrix} B_0^3(t) \\ B_1^3(t) \\ B_2^3(t) \\ B_3^3(t) \end{bmatrix}$$

A more concise formulation using matrices:

$$\mathbf{b}(t) = \begin{bmatrix} \mathbf{b}_0 & \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} \begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$

The Matrix Form and Monomials



Monomial polynomials are the most familiar type
– Cubic case: $1, t, t^2, t^3$
Can reformulate a Bézier curve

$$\begin{aligned}\mathbf{b}(t) = \mathbf{b}_0 &+ 3t(\mathbf{b}_1 - \mathbf{b}_0)\\ &+ 3t^2(\mathbf{b}_2 - 2\mathbf{b}_1 + \mathbf{b}_0)\\ &+ t^3(\mathbf{b}_3 - 3\mathbf{b}_2 + 3\mathbf{b}_1 - \mathbf{b}_0)\end{aligned}$$

$$= \mathbf{a}_0 + \mathbf{a}_1 t + \mathbf{a}_2 t^2 + \mathbf{a}_3 t^3$$

Geometric interpretation of $\mathbf{a}_i$ and $\mathbf{b}_i$ different

The Matrix Form and Monomials

The monomial coefficients $\mathbf{a}_i$ are defined as

$$\begin{bmatrix}\mathbf{a}_0 & \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix} = \begin{bmatrix}\mathbf{b}_0 & \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3\end{bmatrix} \begin{bmatrix}1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1\end{bmatrix}$$

Inverse process:

$$\begin{bmatrix}\mathbf{b}_0 & \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3\end{bmatrix} = \begin{bmatrix}\mathbf{a}_0 & \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix} \begin{bmatrix}1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1\end{bmatrix}^{-1}$$

The square matrix in this equation is nonsingular

$\Rightarrow$ Any cubic curve can be written in Bézier or monomial form