APPENDIX

D Basic Definitions from Linear Systems Theory

D.1 LINEAR TIME INVARIANT (LTI) SYSTEMS

A discrete linear time-invariant system is characterized uniquely by its *impulse response sequence*, $h(n)$. This is the output of the system when its input is excited by the impulse sequence, $\delta(n)$, that is,

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 1 \end{cases} \tag{D.1}$$

When its input is excited by a sequence $x(n)$, its output sequence is given by the *convolution* of $x(n)$ with $h(n)$, defined as

$$y(n) = \sum_{k=-\infty}^{+\infty} h(k)x(n-k) = \sum_{k=-\infty}^{+\infty} x(k)h(n-k) \equiv h(n) * x(n) \tag{D.2}$$

For continuous time systems the convolution becomes an integral, that is,

$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)\,d\tau \\ &= \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)\,d\tau \equiv x(t) * y(t) \end{aligned} \tag{D.3}$$

where $h(t)$ is the impulse response of the system, that is, the output when its input is excited by the Dirac delta function $\delta(t)$, defined by

$$\delta(t) = 0, \quad \text{for } t \neq 0, \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(t)dt = 1 \tag{D.4}$$

Linear time-invariant systems can be:

- *Causal*: Their impulse response is zero for $n < 0$. Otherwise, they are known as *noncausal*. Observe that only causal systems can be realized in real time. This is because for noncausal systems, the output at time n would require knowledge of future samples $x(n+1), x(n+2), \ldots$, which in practice is not possible.

- *Finite impulse response (FIR)*: The corresponding impulse response is of finite extent. If this is not the case, the systems are known as *infinite impulse response* (IIR) systems. For a causal FIR system the input–output relation becomes

$$y(n) = \sum_{k=0}^{L-1} h(k)x(n-k) \tag{D.5}$$

where L is the length of the impulse response. When a system is FIR but noncausal, it can become causal by delaying its output. Take for example the system with impulse response $0, \ldots, 0, h(-2), h(-1), h(0), h(1), h(2), 0, \ldots$. Then,

$$\begin{aligned} y(n-2) &= h(-2)x(n) + h(-1)x(n-1) + h(0)x(n-2) \\ &\quad + h(1)x(n-3) + h(2)x(n-4) \end{aligned} \tag{D.6}$$

That is, at time "n" the output corresponds to the delayed time "$n-2$." The delay is equal to the maximum negative index of nonzero impulse coefficient.

D.2 TRANSFER FUNCTION

The z-transform of the impulse response, defined as

$$H(z) = \sum_{n=-\infty}^{+\infty} h(n)z^{-n} \tag{D.7}$$

is known as the *transfer function* of the system. The free parameter z is a complex variable. The definition in (D.7) is meaningful, provided that the series converges. For most of the sequences of our interest this is true for some region in the complex plane. It can easily be shown that for causal and FIR systems the region of convergence is of the form

$$|z| > |R|, \quad \text{for some } |R| < 1 \tag{D.8}$$

that is, it is the exterior of a circle in the complex plane, centered at the origin, and it contains the unit circle ($|z| = 1$). Let $X(z)$ and $Y(z)$ be the z-transforms of the input and output sequences of a linear time-invariant system. Then (D.2) is shown to be equivalent to

$$Y(z) = H(z)X(z) \tag{D.9}$$

If the unit circle is in the region of convergence of the respective z-transforms (for example, for causal FIR systems), then for $z = \exp(-j\omega)$ we obtain the equivalent Fourier transform and

$$Y(\omega) = H(\omega)X(\omega) \tag{D.10}$$

If the impulse response of a linear time-invariant system is delayed by r samples, for example, to make it causal in case it is noncausal, the transfer function of the delayed system is given by $z^{-r}H(z)$.

D.3 SERIAL AND PARALLEL CONNECTION

Consider two LTI systems with responses $h_1(n)$ and $h_2(n)$, respectively. Figure D.1a shows the two systems connected in serial and Figure D.1b in parallel. The overall impulse responses are easily shown to be

$$\text{Serial} \quad h(n) = h_1(n) * h_2(n) \tag{D.11}$$

$$\text{Parallel} \quad h(n) = h_1(n) + h_2(n) \tag{D.12}$$

D.4 TWO-DIMENSIONAL GENERALIZATIONS

A two-dimensional linear time-invariant system is also characterized by its two-dimensional impulse response sequence $H(m, n)$, which in the case of images is known as a *point spread function*. On filtering an input image array $X(m, n)$ by $H(m, n)$ the resulting image array is given by the two-dimensional convolution

$$\begin{aligned} Y(m, n) &= \sum_k \sum_l H(m-k, n-l)X(k, l) \equiv H(m, n) * *X(m, n) \\ &= \sum_k \sum_l H(k, l)X(m-k, n-l) \end{aligned} \tag{D.13}$$

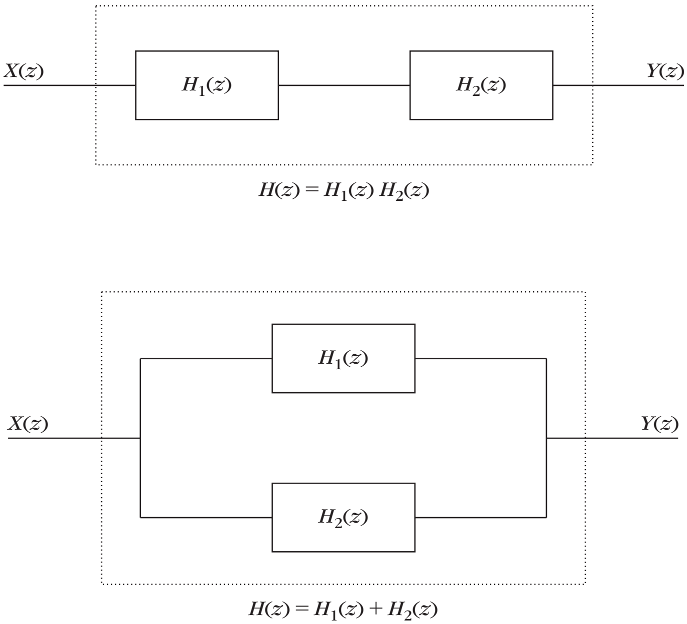

FIGURE D.1

Serial and parallel connections of LTI systems.