CS 418: Interactive Computer Graphics

More Mathematics for Computer Graphics

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Slides adapted from
Professor John Hart's CS 418 Slides

Some Slides Adapted from
Angel and Shreiner: Interactive
Computer Graphics 7E © Addison-
Wesley 2015

3-D Affine Transformations

- An affine transformation is the sum of a linear transformation and a constant vector…
  - Technically, linear transformations preserve the origin
  - Translations map the origin to a new position

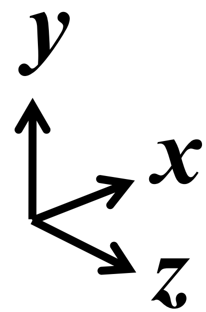
- General Form (with homogenous coordinates)

$$\begin{bmatrix} d & e & f & a \\ g & h & i & b \\ j & k & l & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} dx + ey + fz + a \\ gx + hy + iz + b \\ jx + ky + lz + c \\ 1 \end{bmatrix}$$

3-D Translation

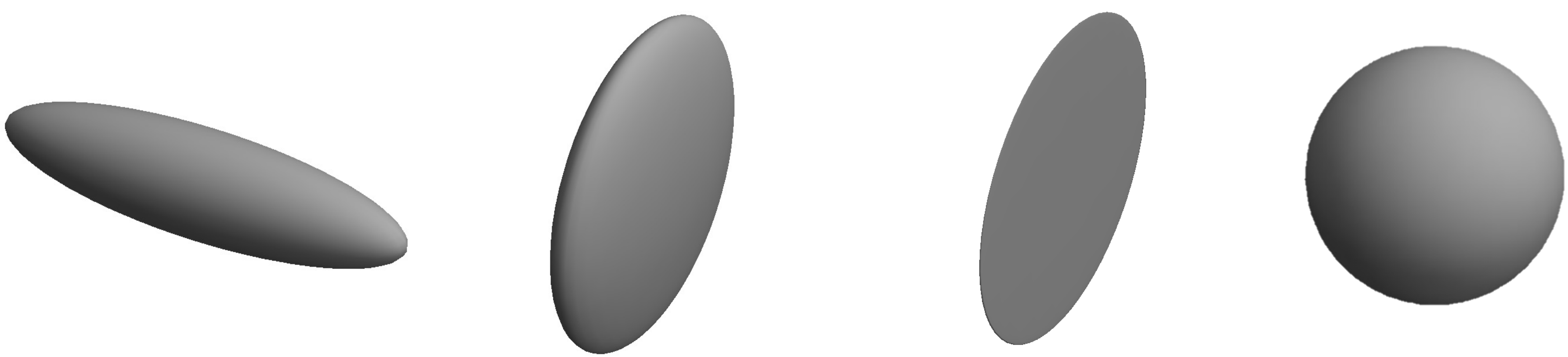
$$\begin{bmatrix} 1 & & & a \\ & 1 & & b \\ & & 1 & c \\ & & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + a \\ y + b \\ z + c \\ 1 \end{bmatrix}$$

Scale



$$\begin{bmatrix} a & & & \\ & b & & \\ & & c & \\ & & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} ax \\ by \\ cz \\ 1 \end{bmatrix}$$

Uniform Scale
$a = b = c = ¼$



Stretch
$a = b = 1$, $c = 4$

Squash
$a = b = 1$, $c = ¼$

Project
$a = b = 1$, $c = 0$

Invert
$a = b = 1$, $c = -1$

3-D Rotations

- About x-axis
  - rotates $y \rightarrow z$

$$\begin{bmatrix} 1 & & & \\ & \cos\theta & -\sin\theta & \\ & \sin\theta & \cos\theta & \\ & & & 1 \end{bmatrix}$$

- About y-axis
  - rotates $z \rightarrow x$

$$\begin{bmatrix} \cos\theta & & \sin\theta & \\ & 1 & & \\ -\sin\theta & & \cos\theta & \\ & & & 1 \end{bmatrix}$$

- About z-axis
  - rotates $x \rightarrow y$

$$\begin{bmatrix} \cos\theta & -\sin\theta & & \\ \sin\theta & \cos\theta & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

- Rotations do not commute!

3-D Rotations

- About x-axis
  - rotates $y \to z$

$$\begin{bmatrix} 1 & & & \\ & \cos\theta & -\sin\theta & \\ & \sin\theta & \cos\theta & \\ & & & 1 \end{bmatrix}$$

- About y-axis
  - rotates $z \to x$

$$\begin{bmatrix} \cos\theta & & \sin\theta & \\ & 1 & & \\ -\sin\theta & & \cos\theta & \\ & & & 1 \end{bmatrix}$$

- About z-axis
  - rotates $x \to y$

$$\begin{bmatrix} \cos\theta & -\sin\theta & & \\ \sin\theta & \cos\theta & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

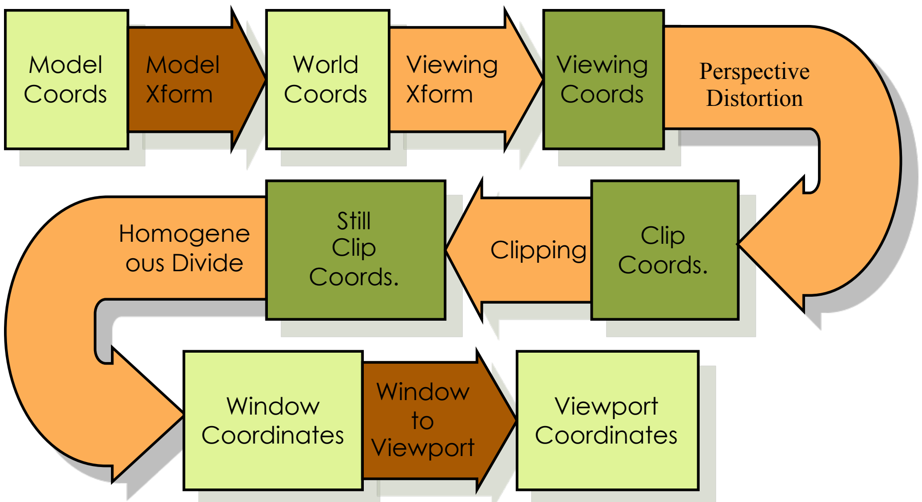
To understand rotation about y

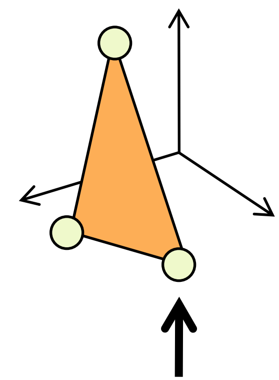
note that is like rotation about z with the axes mapped to each other

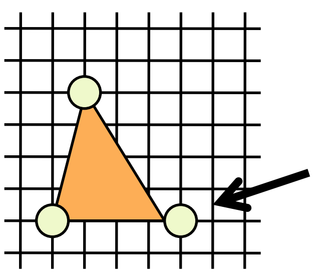
RotationY		RotationZ
Z	➔	X
X	➔	Y
Y	➔	Z

So entry in column X row Z in RotationY is the same as column Y row X in RotationZ

Graphics Pipeline

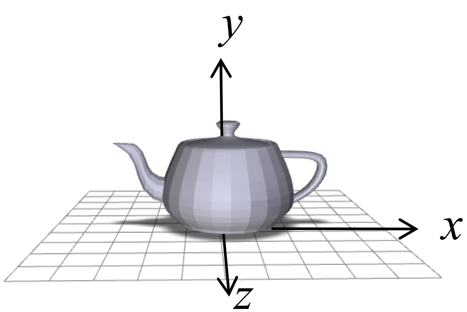






$$\begin{bmatrix} x_s \\ y_s \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \mathrm{W2V} \quad \end{bmatrix} \begin{bmatrix} \quad \mathrm{Persp} \quad \end{bmatrix} \begin{bmatrix} \quad \mathrm{View} \quad \end{bmatrix} \begin{bmatrix} \quad \mathrm{Model} \quad \end{bmatrix} \begin{bmatrix} x_m \\ y_m \\ z_m \\ 1 \end{bmatrix}$$

Transformation Order



$$\begin{bmatrix} x_s \\ y_s \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{M} \end{bmatrix} \begin{bmatrix} x_m \\ y_m \\ z_m \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_s \\ y_s \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathrm{R} \end{bmatrix} \begin{bmatrix} \mathrm{T} \end{bmatrix} \begin{bmatrix} x_m \\ y_m \\ z_m \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_s \\ y_s \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathrm{T} \end{bmatrix} \begin{bmatrix} \mathrm{R} \end{bmatrix} \begin{bmatrix} x_m \\ y_m \\ z_m \\ 1 \end{bmatrix}$$

A Brief Sampling of Other Useful Math

- Geometry:
  the study of shape and size in n-dimensional space
  - We are interested in objects that exist in two or three dimensions
- We will look at three basic geometric elements
  - Scalars
  - Vectors
  - Points
- Some useful operations
  - Dot product
  - Cross product
- Parametric form of equatsions

Scalars

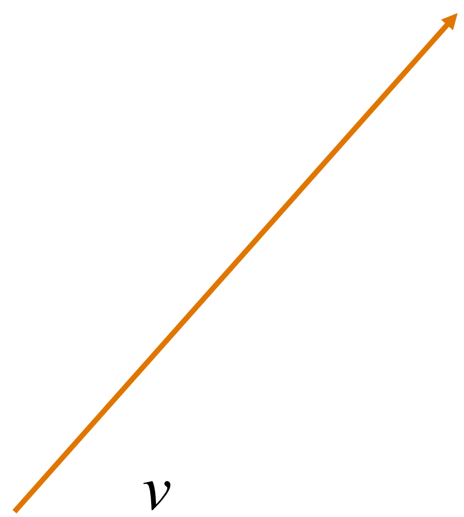
Scalars are quantities which express only a magnitude…
Think of them as just a single number like 3.14

- Scalars:
  - members of sets which
  - can be combined by two operations (addition and multiplication)
    - obey some fundamental laws (associativity, commutivity, inverses)
- Examples include the real numbers
  - under the ordinary rules with which we are familiar
- Scalars alone have no geometric properties

Vectors

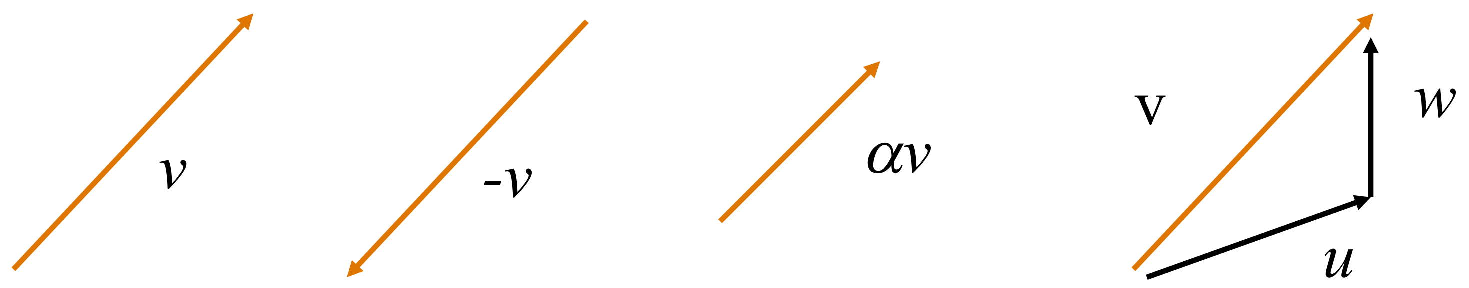
A vector is a quantity with two attributes
Direction
Magnitude

- Examples include
  - Force
  - Velocity
  - Directed line segments



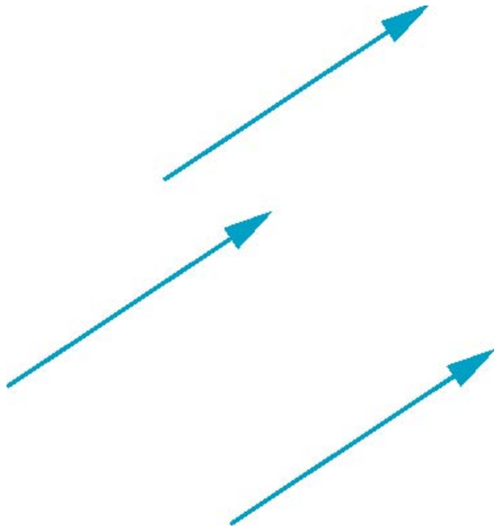
Vector Operations

- Every vector has an inverse
  - Same magnitude but points in opposite direction
- Every vector can be multiplied by a scalar
- There is a zero vector
  - Zero magnitude, undefined orientation
- The sum of any two vectors is a vector
  - Use head-to-tail axiom



Operations on Vectors

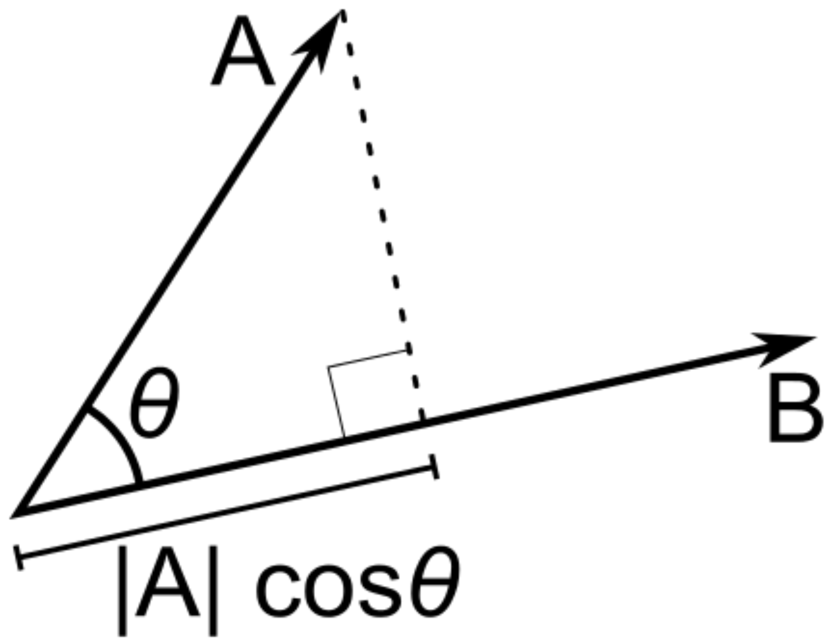
- Scalar-vector multiplication $u=\alpha v$
- Vector-vector addition: $w=u+v$
  - Allows expressions such as $v=u+2w-3r$
- Vectors lack position
- …need points to make things interesting



Dot Product of Two Vectors

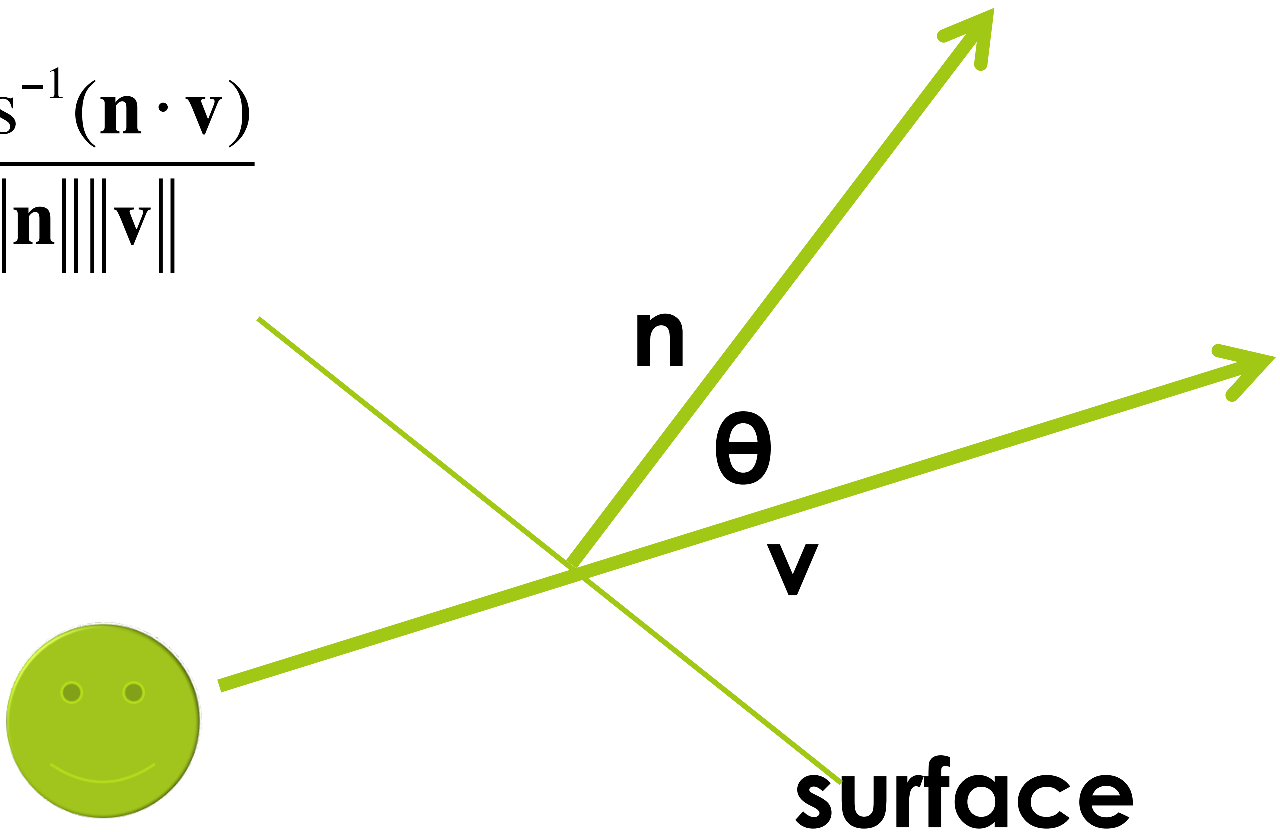
- Also known a ***scalar product***
- Also known ***as inner product*** in Euclidean Space
- In 3D we have :
  <a,b,c> • <d,e,f> = ad+be+cf
- Magnitude of a vector is $\|A\| = \sqrt{A \cdot A}$
- Important property of dot product:

$$A \cdot B = \|A\| \|B\| \cos\theta$$



Measuring the Angle Between Two Vectors

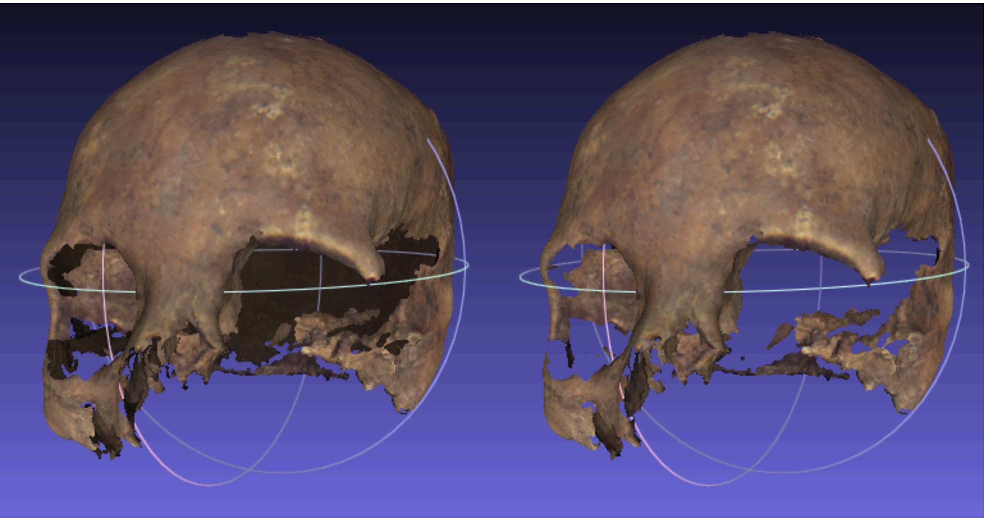
$$\theta = \frac{\cos^{-1}(\mathbf{n}\cdot\mathbf{v})}{\|\mathbf{n}\|\|\mathbf{v}\|}$$



Back Face Culling

- Decide whether the view vector runs from the surface to the eyepoint or **from the eyepoint to the surface**
  - For this test, we'll **use eyepoint to surface**.
- So, if $90 \leq \theta \leq 270$
  then dot product is negative and polygon faces viewer
- IF the dot product is positive ***then polygon does not face viewer***

Back Face Culling



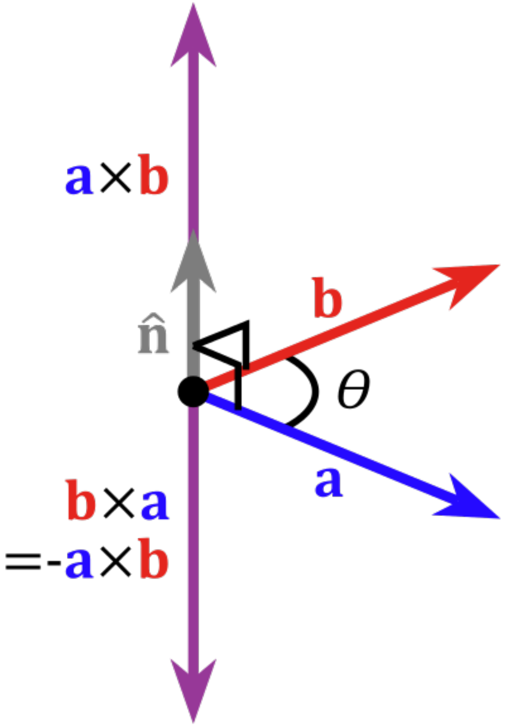
- Backface culling drops backfacing polygons from the pipeline.
- Why would backface culling be useful?
- What artifact do you see?
- Backface culling is not hidden surface removal

Cross Product of Two Vectors

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

$$\mathbf{b} = \langle b_1, b_2, b_3 \rangle$$

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - b_2 a_3, a_3 b_1 - b_3 a_1, a_1 b_2 - b_1 a_2 \rangle$$

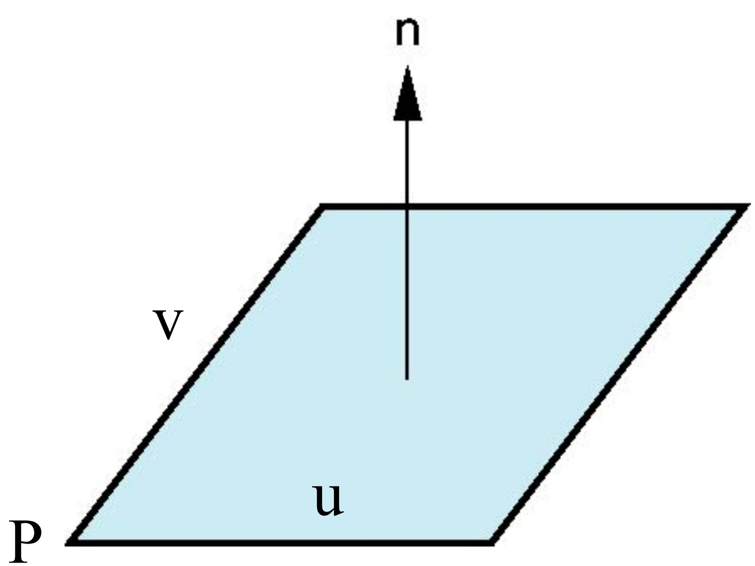


Important Property:

The cross product yields a vector orthogonal to the original two vectors

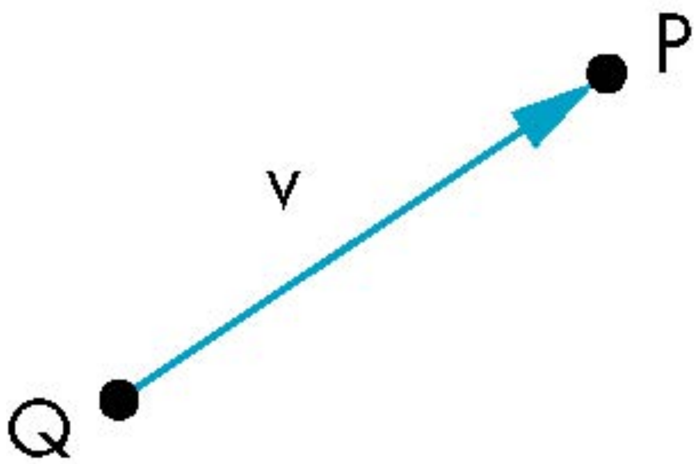
Normals

- In three dimensional spaces, every plane has a vector orthogonal to it called the **normal vector**



Points

- Location in space
- Operations allowed between points and vectors
  - Point-point subtraction yields a vector
  - Equivalent to point-vector addition



$$v = P - Q$$

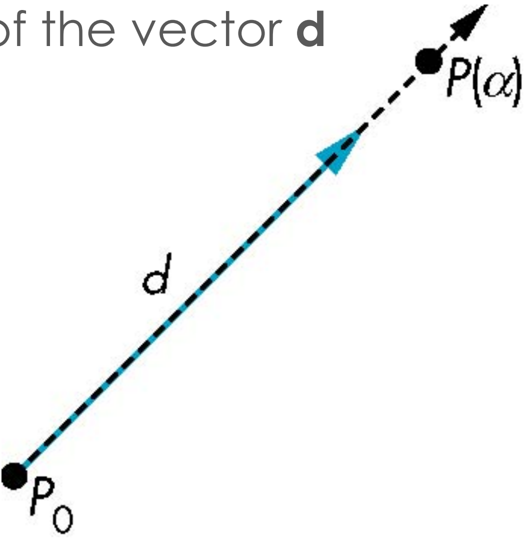
$$P = v + Q$$

Affine Spaces

- A vector space with points
- Operations
  - Vector-vector addition
  - Scalar-vector multiplication
  - Point-vector addition
  - Scalar-scalar operations
- For any point define
  - $1 \bullet P = P$
  - $0 \bullet P = \mathbf{0}$ (zero vector)

Parametric Form of a Line

- Consider all points of the form
  - $P(a)=P_0 + a\,\mathbf{d}$
  - Set of all points that pass through P_0 in the direction of the vector **d**



Parametric Form of a Line

- This form is known as the parametric form of the line
  - More robust and general than other forms
  - Extends to curves and surfaces

- Two-dimensional forms
  - Explicit: $y = mx + h$
  - Implicit: $ax + by + c = 0$
  - Parametric:

    $x(a) = (1-a)x_0 + a)x_1$

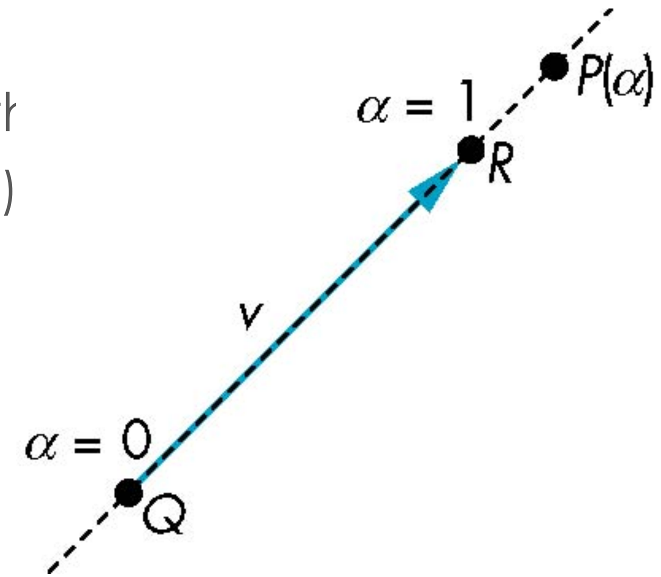
    $y(a) = (1-a)y_0 + ay_1$

Rays and Line Segments

- If a >= 0,
  then P(a) is the *ray* leaving P_0 in the direction **d**

- If we use two points to define v, th
  P(a) = Q + a (R-Q)=Q+av =aR + (1-a)

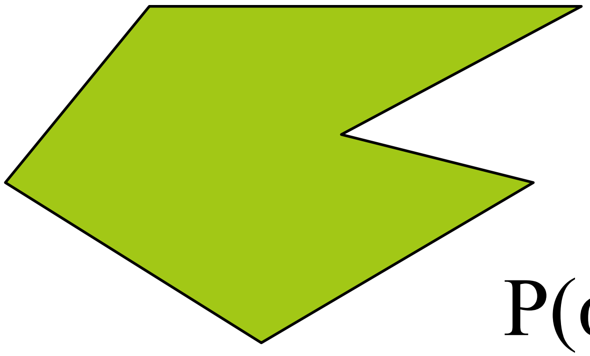
  For 0<=a<=1 we get
  all the points on the *line segment* joining R and Q



Curves and Surfaces

- Curves are one parameter entities of the form P(a) where the function is nonlinear
- Surfaces are formed from two-parameter functions P(a, b)
  - Linear functions give planes and polygons

$P(\alpha)$



$P(\alpha, \beta)$

Plane Equations

Scalar equation of a plane is $ax + by + cz = d$

Vector equation of a plane is $\vec{\mathbf{n}} \cdot (p - p_0) = 0$
Where $\vec{\mathbf{n}}$ is the normal and p_0 a point defining the plane.
Any point *p* for which the equation holds is on the plane.

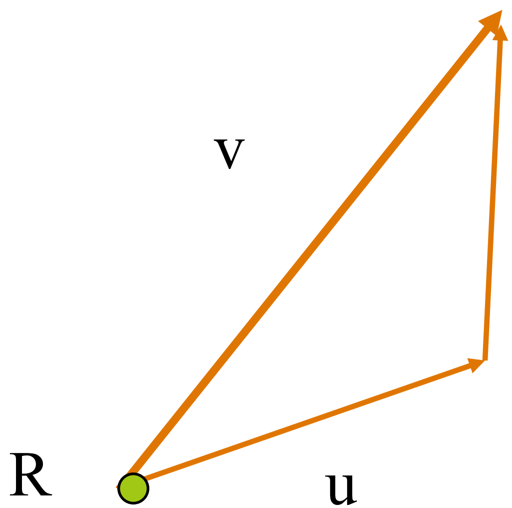
Given three points p_0, $p_{1,}$ and p_2 you can generate the plane equation:

$$\vec{\mathbf{n}} = (p_1 - p_0) \times (p_2 - p_0)$$

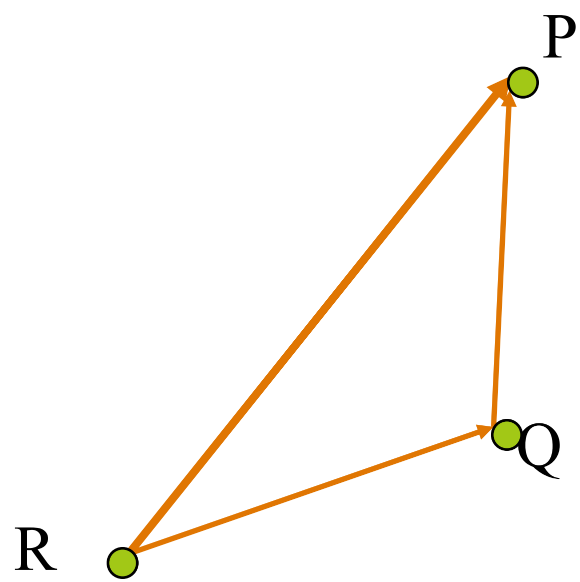
…and then use the vector form above

Planes

- Can also be defined parametrically using 2 vectors



$$P(\alpha,\beta)=R+\alpha u+\beta v$$



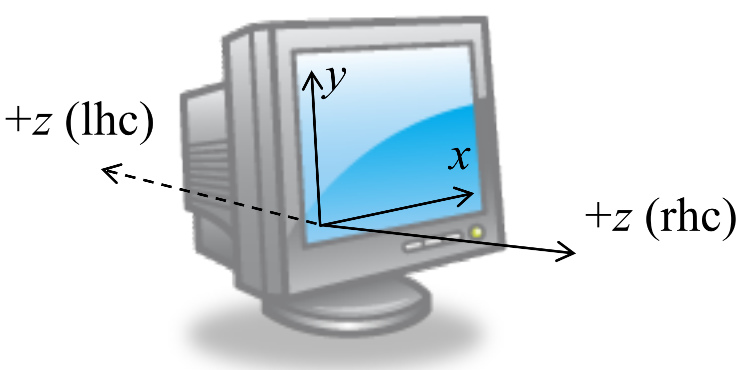
$$P(\alpha,\beta)=R+\alpha(Q-R)+\beta(P-Q)$$

3-D Coordinates

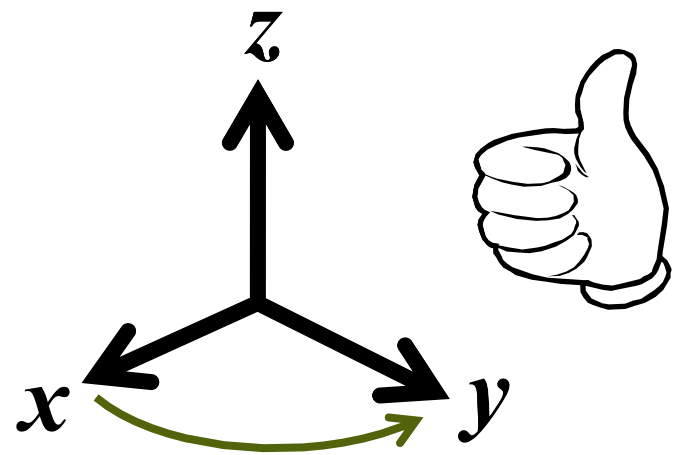
WebGL Clip Space coordinate system is a right-handed system

- Points represented by 4-vectors
- Need to decide orientation of coordinate axes

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Right Handed Coord. Sys.



Left Handed Coord. Sys.