Chapter 13

Inference About Comparing Two Populations

Comparing Two Populations

Previously we looked at techniques to estimate and test parameters for one population:

Population Mean μ

We will still consider these parameters when we are looking at ***two populations***, however our interest will now be:

➔ The ***difference*** between two means.

Difference between Two Means

Because we are comparing two population means, we use the statistic,

$$\bar{x}_1 - \bar{x}_2$$

which is an unbiased and consistent estimator of $\mu_1 - \mu_2$.

Sampling Distribution of $\bar{x}_1 - \bar{x}_2$

1. $\bar{x}_1 - \bar{x}_2$ is normally distributed if the original populations are normal –or– approximately normal if the populations are nonnormal and the sample sizes are large (n_1, $n_2 > 30$)

2. The expected value of $\bar{x}_1 - \bar{x}_2$ is $\mu_1 - \mu_2$

3. The variance of $\bar{x}_1 - \bar{x}_2$ is $\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$

and the standard error is: $\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$

Making Inferences About μ_1-μ_2

Since $\bar{x}_1 - \bar{x}_2$ is ***normally distributed*** if the original populations are normal –or– ***approximately normal*** if the populations are nonnormal and the sample sizes are large, then:

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

is a standard normal (or approximately normal) random variable.

We could use this to build the test statistic and the confidence interval estimator for $\mu_1 - \mu_2$.

Making Inferences About μ_1-μ_2

…except that, in practice, the z statistic is rarely used since the population variances are unknown.

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \quad ??$$

Instead we use a **t**-statistic.

We consider two cases for the unknown population variances: when we believe they are ***equal*** and conversely when they are ***not equal***.

Which test to use?

Which test statistic do we use? Equal variance or unequal variance?

Whenever there is insufficient evidence that the variances are unequal, it is preferable to perform the

equal variances t-test.

This is so, because for any two given samples:

The number of degrees of freedom for the **equal** variances case $\geq$ The number of degrees of freedom for the **unequal** variances case

Larger numbers of degrees of freedom have the same effect as having larger sample sizes

Testing the Population Variances

Testing the Population Variances

H_0: $\sigma_1^2 / \sigma_2^2 = 1$

H_1: $\sigma_1^2 / \sigma_2^2 \neq 1$

Test statistic: s_1^2 / s_2^2, which is F-distributed with degrees of freedom $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 2$.

The required condition is the same as that for the t-test of μ_1 - μ_2 , which is both populations are normally distributed.

Example 13.1

Millions of investors buy mutual funds choosing from thousands of possibilities.

Some funds can be purchased directly from banks or other financial institutions while others must be purchased through brokers, who charge a fee for this service.

This raises the question, can investors do better by buying mutual funds directly than by purchasing mutual funds through brokers.

Example 13.1

To help answer this question a group of researchers randomly sampled the annual returns from mutual funds that can be acquired directly and mutual funds that are bought through brokers and recorded the net annual returns, which are the returns on investment after deducting all relevant fees. Xm13-01

Can we conclude at the 5% significance level that directly-purchased mutual funds outperform mutual funds bought through brokers?

Example 13.1

IDENTIFY

To answer the question we need to compare the population of returns from direct and the returns from broker- bought mutual funds.

The data are obviously interval (we've recorded real numbers).

This problem objective - data type combination tells us that the parameter to be tested is the difference between two means μ_1- $\mu_{2.}$

Example 13.1

IDENTIFY

The hypothesis to be tested is that the mean net annual return from directly-purchased mutual funds (μ_1) is larger than the mean of broker-purchased funds (μ_2). Hence the alternative hypothesis is

$$H_1: \mu_1 - \mu_2 > 0$$

and

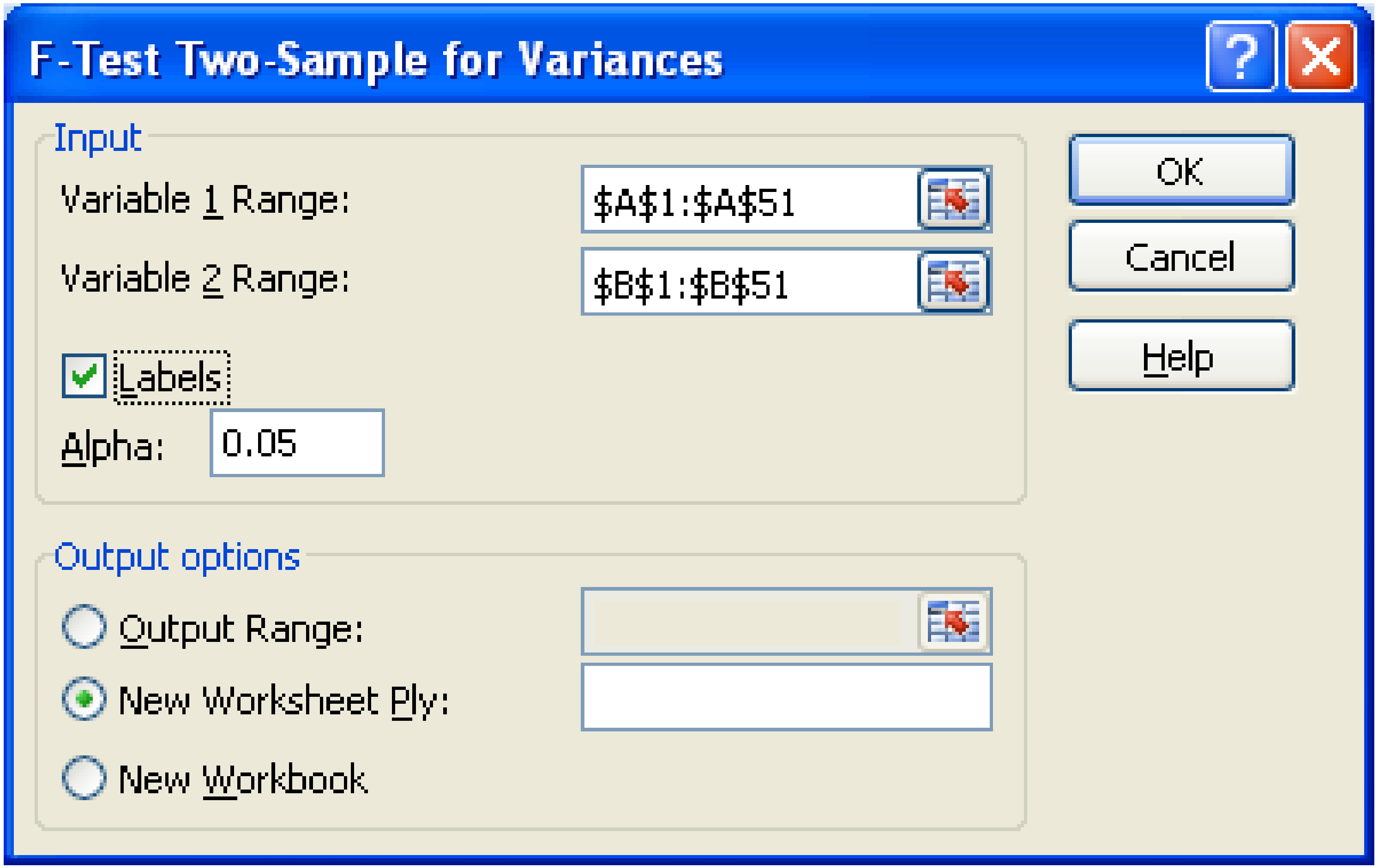
$$H_0: \mu_1 - \mu_2 = 0$$

To decide which of the t-tests of $\mu_1 - \mu_2$ to apply we conduct the F-test of σ_1^2 / σ_2^2 .

Example 13.1

IDENTIFY

Click Data, Data Analysis, and F-Test Two Sample for Variances



Example 13.1

IDENTIFY

	A	B	C
1	F-Test Two-Sample for Variances		
2			
3		*Direct*	*Broker*
4	Mean	6.63	3.72
5	Variance	37.49	43.34
6	Observations	50	50
7	df	49	49
8	F	0.86	
9	P(F<=f) one-tail	0.3068	
10	F Critical one-tail	0.6222	

The value of the test statistic is F = .86. Excel outputs the one-tail p-value. Because we're conducting a two-tail test, we double that value. Thus, the p-value of the test we're conducting is $2 \times .3068 = .6136$.

Example 13.1

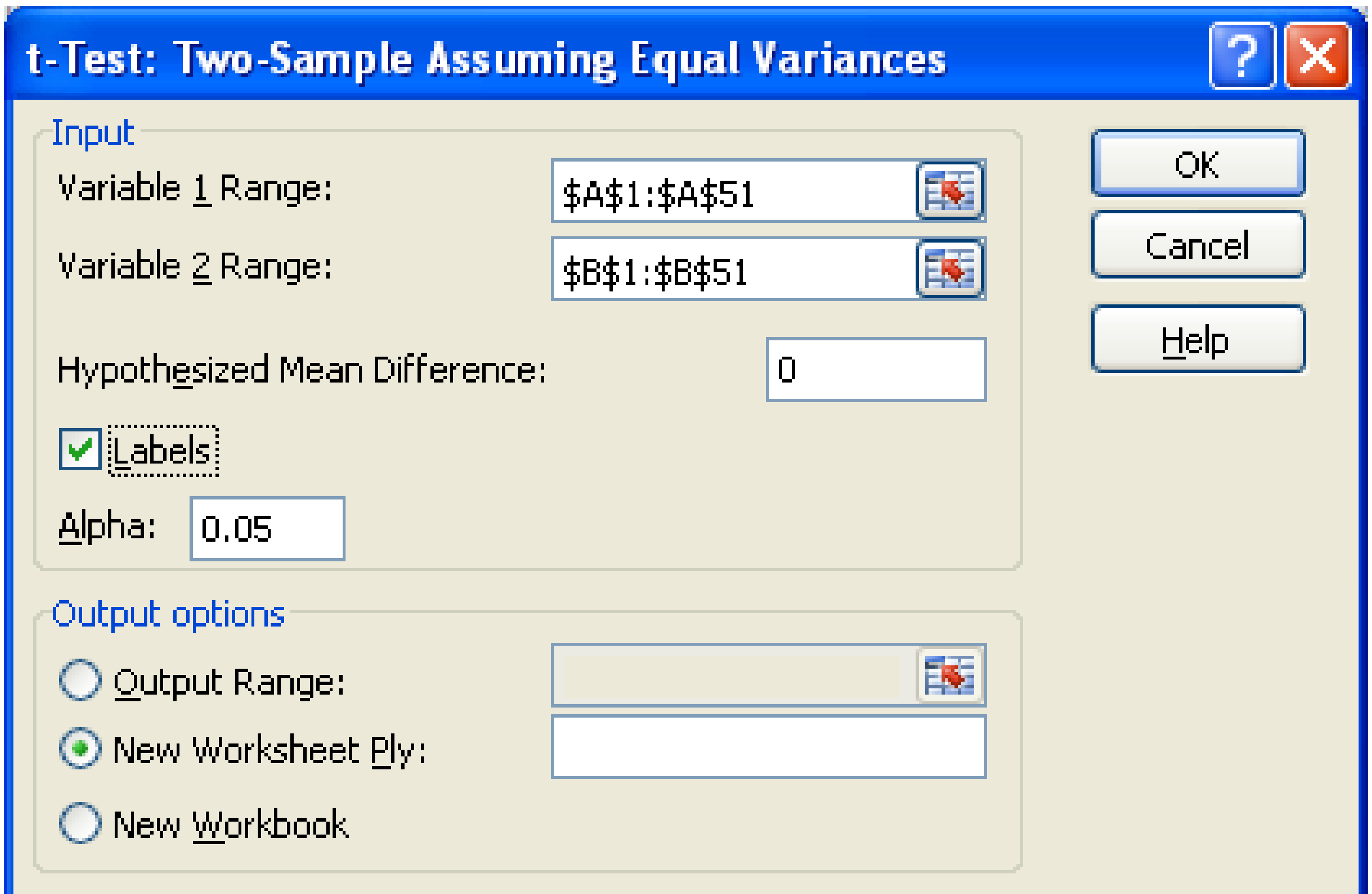
IDENTIFY

There is not enough evidence to infer that the population variances differ. It follows that we must apply the equal-variances t-test of $\mu_1 - \mu_2$

Example 13.1

COMPUTE

Click Data, Data Analysis, t-Test: Two-Sample Assuming Equal Variances



Example 13.1

COMPUTE

	A	B	C
1	t-Test: Two-Sample Assuming Equal Variances		
2			
3		*Direct*	*Broker*
4	Mean	6.63	3.72
5	Variance	37.49	43.34
6	Observations	50	50
7	Pooled Variance	40.41	
8	Hypothesized Mean Difference	0	
9	df	98	
10	t Stat	2.29	
11	P(T<=t) one-tail	0.0122	
12	t Critical one-tail	1.6606	
13	P(T<=t) two-tail	0.0243	
14	t Critical two-tail	1.9845	

Example 13.1

INTERPRET

The value of the test statistic is 2.29. The one-tail p-value is .0122.

We observe that the p-value of the test is small (and the test statistic falls into the rejection region).

As a result we conclude that there is sufficient evidence to infer that on average directly-purchased mutual funds outperform broker-purchased mutual funds

Example 13.2

What happens to the family-run business when the boss's son or daughter takes over?

Does the business do better after the change if the new boss is the offspring of the owner or does the business do better when an outsider is made chief executive officer (CEO)?

In pursuit of an answer researchers randomly selected 140 firms between 1994 and 2002, 30% of which passed ownership to an offspring and 70% appointed an outsider as CEO.

Example 13.2

For each company the researchers calculated the operating income as a proportion of assets in the year before and the year after the new CEO took over.

The change (operating income after – operating income before) in this variable was recorded. Xm13-02

Do these data allow us to infer that the effect of making an offspring CEO is different from the effect of hiring an outsider as CEO?

Example 13.2

IDENTIFY

The problem objective is to compare two populations.

Population 1: Operating income of companies whose CEO is an offspring of the previous CEO

Population 2: Operating income of companies whose CEO is an outsider

The data type is interval (operating incomes).

Thus, the parameter to be tested is $\mu_1 - \mu_2$, where μ_1 = mean operating income for population 1 and μ_2 = mean operating income for population 2.

Example 13.2

IDENTIFY

We want to determine whether there is enough statistical evidence to infer that μ_1 is different from μ_2. That is, that $\mu_1 - \mu_2$ is not equal to 0. Thus,

$$H_1: \mu_1 - \mu_2 \neq 0$$

and

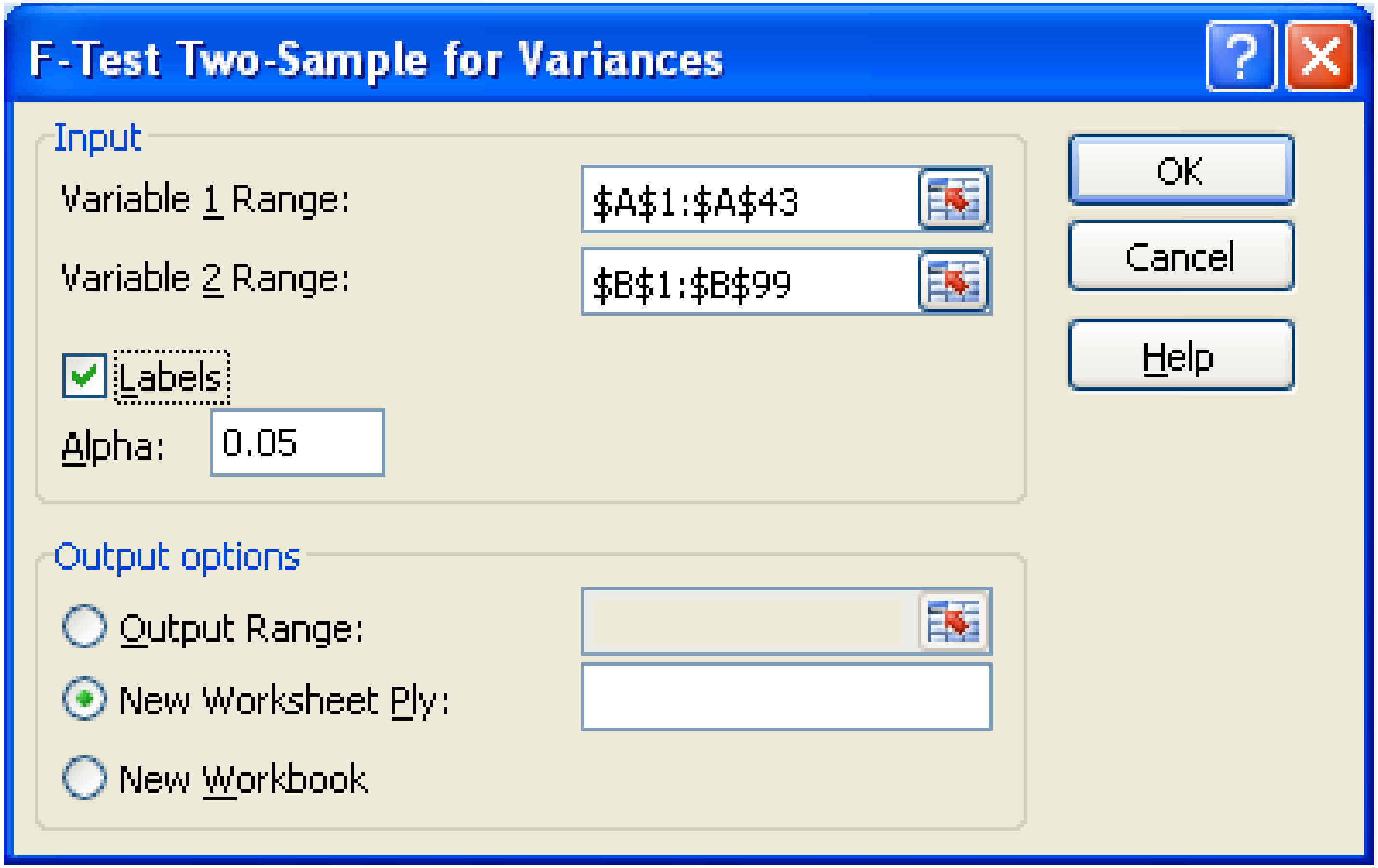
$$H_0: \mu_1 - \mu_2 = 0$$

We need to determine whether to use the equal-variances or unequal-variances t –test of $\mu_1 - \mu_2$.

Example 13.2

IDENTIFY

Click Data, Data Analysis, and F-Test Two Sample for Variances



Example 13.2

IDENTIFY

	A	B	C
1	F-Test Two-Sample for Variances		
2			
3		*Offspring*	*Outsider*
4	Mean	-0.10	1.24
5	Variance	3.79	8.03
6	Observations	42	98
7	df	41	97
8	F	0.47	
9	P(F<=f) one-tail	0.0040	
10	F Critical one-tail	0.6314	

The value of the test statistic is $F = .47$. The p-value of the test we're conducting is $2 \times .0040 = .0080$.

Example 13.2

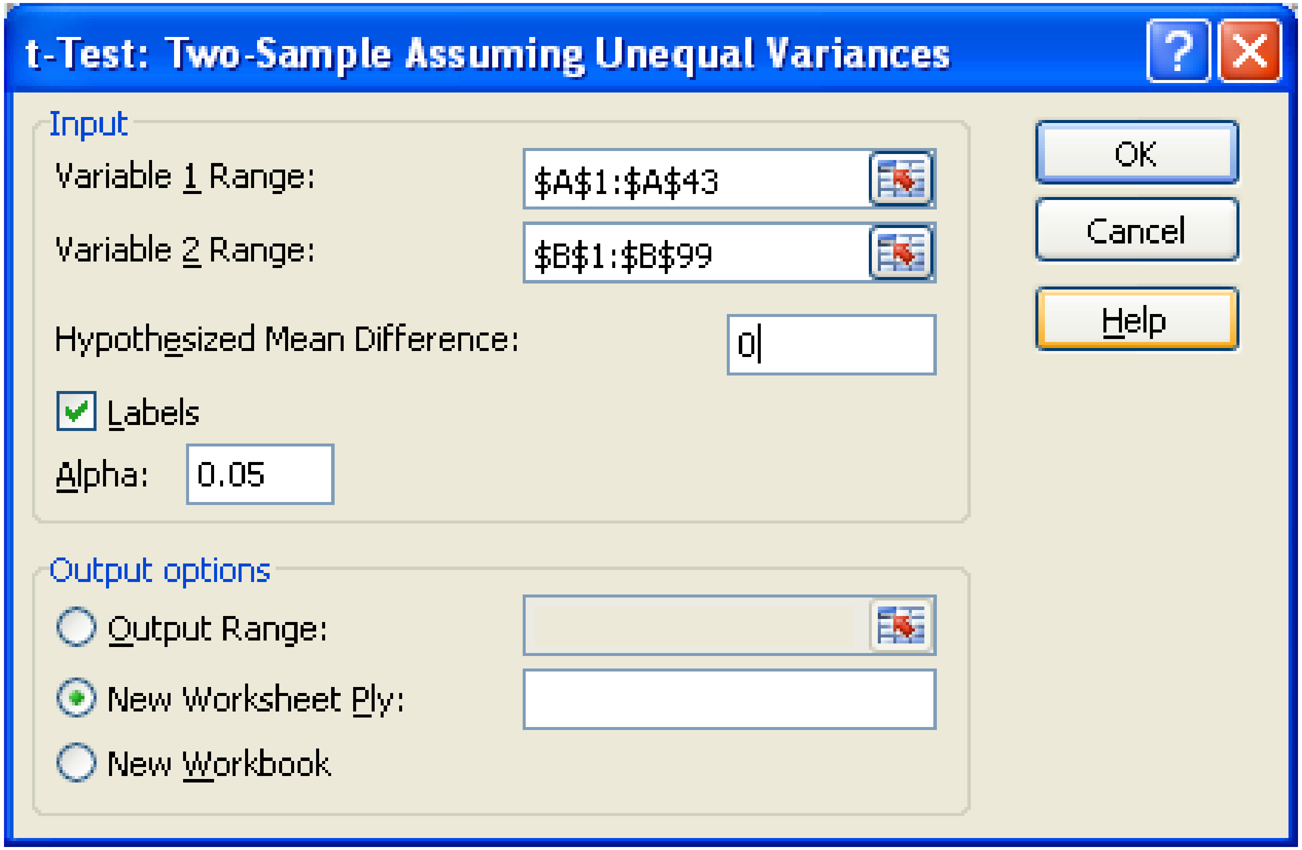
IDENTIFY

Thus, the correct technique is the unequal-variances t-test of $\mu_1 - \mu_{2.}$

Example 13.2

COMPUTE

Click Data, Data Analysis, t-Test: Two-Sample Assuming Unequal Variances



Example 13.2...

COMPUTE

	A	B	C
1	t-Test: Two-Sample Assuming Unequal Variances		
2			
3		*Offspring*	*Outsider*
4	Mean	-0.10	1.24
5	Variance	3.79	8.03
6	Observations	42	98
7	Hypothesized Mean Difference	0	
8	df	111	
9	t Stat	-3.22	
10	P(T<=t) one-tail	0.0008	
11	t Critical one-tail	1.6587	
12	P(T<=t) two-tail	0.0017	
13	t Critical two-tail	1.9816	

Example 13.2…

INTERPRET

The t-statistic is – 3.22 and its p-value is .0017. Accordingly, we conclude that there is sufficient evidence to infer that the mean times differ.

Checking the Required Condition

Both the equal-variances and unequal-variances techniques require that the populations be normally distributed. As before, we can check to see whether the requirement is satisfied by drawing the histograms of the data.

Example 13.3

Despite some controversy, scientists generally agree that high-fiber cereals reduce the likelihood of various forms of cancer.

However, one scientist claims that people who eat high-fiber cereal for breakfast will consume, on average, fewer calories for lunch than people who don't eat high-fiber cereal for breakfast.

Example 13.3

If this is true, high-fiber cereal manufacturers will be able to claim another advantage of eating their product--potential weight reduction for dieters.

As a preliminary test of the claim, 150 people were randomly selected and asked what they regularly eat for breakfast and lunch.

Example 13.3

Each person was identified as either a consumer or a nonconsumer of high-fiber cereal, and the number of calories consumed at lunch was measured and recorded. Xm13-03

Can the scientist conclude at the 5% significance level that his belief is correct?

Example 13.3

$$H_0 : (\mu_1 - \mu_2) = 0$$

$$H_1 : (\mu_1 - \mu_2) < 0$$

	A	B	C
1	t-Test: Two-Sample Assuming Unequal Variances		
2			
3		*Consumers*	*Nonconsumers*
4	Mean	604.02	633.23
5	Variance	4103	10670
6	Observations	43	107
7	Hypothesized Mean Difference	0	
8	df	123	
9	t Stat	-2.09	
10	P(T<=t) one-tail	0.0193	
11	t Critical one-tail	1.6573	
12	P(T<=t) two-tail	0.0386	
13	t Critical two-tail	1.9794	

Example 13.3

The value of the test statistic is −2.09.

The one-tail p-value is .0193.

We conclude that there is sufficient evidence to infer that consumers of high-fiber cereal do eat fewer calories at lunch than do nonconsumers.

Matched Pairs Experiment…

Previously when comparing two populations, we examined independent samples.

If, however, an observation in one sample is ***matched*** with an observation in a second sample, this is called a ***matched pairs experiment***.

To help understand this concept, let's consider Example 13.4

Example 13.4

In the last few years a number of web-based companies that offer job placement services have been created.

The manager of one such company wanted to investigate the job offers recent MBAs were obtaining.

In particular, she wanted to know whether finance majors were being offered higher salaries than marketing majors.

Example 13.4

In a preliminary study she randomly sampled 50 recently graduated MBAs half of whom majored in finance and half in marketing.

From each she obtained the highest salary (including benefits) offer (Xm13-04).

Can we infer that finance majors obtain higher salary offers than do marketing majors among MBAs?

Example 13.4

IDENTIFY

The parameter is the difference between two means (where μ_1 = mean highest salary offer to finance majors and μ_2 = mean highest salary offer to marketing majors).

Because we want to determine whether finance majors are offered higher salaries, the alternative hypothesis will specify that is greater than.

Calculation of the F-test of two variances indicates the use the equal-variances test statistic.

Example 13.4

IDENTIFY

The hypotheses are

$$H_0: (\mu_1 - \mu_2) = 0$$

$$H_1: (\mu_1 - \mu_2) > 0$$

The Excel output is:

Example 13.4

COMPUTE

	A	B	C
1	t-Test: Two-Sample Assuming Equal Variances		
2			
3		*Finance*	*Marketing*
4	Mean	65,624	60,423
5	Variance	360,433,294	262,228,559
6	Observations	25	25
7	Pooled Variance	311,330,926	
8	Hypothesized Mean Difference	0	
9	df	48	
10	t Stat	1.04	
11	P(T<=t) one-tail	0.1513	
12	t Critical one-tail	1.6772	
13	P(T<=t) two-tail	0.3026	
14	t Critical two-tail	2.0106	

Example 13.4

INTERPRET

The value of the test statistic ($t = 1.04$) and its p-value (.1513) indicate that there is very little evidence to support the hypothesis that finance majors attract higher salary offers than marketing majors.

Example 13.4

INTERPRET

We have some evidence to support the alternative hypothesis, but not enough.

Note that the difference in sample means is

$$\bar{x}_1 - \bar{x}_2 = (65{,}624 - 60{,}423) = 5{,}201$$

Example 13.5

Suppose now that we redo the experiment in the following way.

We examine the transcripts of finance and marketing MBA majors.

We randomly sample a finance and a marketing major whose grade point average (GPA) falls between 3.92 and 4 (based on a maximum of 4).

We then randomly sample a finance and a marketing major whose GPA is between 3.84 and 3.92.

Example 13.5

We continue this process until the 25th pair of finance and marketing majors are selected whose GPA fell between 2.0 and 2.08.

(The minimum GPA required for graduation is 2.0.)

As we did in Example 13.4, we recorded the highest salary offer . Xm13-05

Can we conclude from these data that finance majors draw larger salary offers than do marketing majors?

Example 13.5

IDENTIFY

The experiment described in Example 13.4 is one in which the samples are independent. That is, there is no relationship between the observations in one sample and the observations in the second sample. However, in this example the experiment was designed in such a way that each observation in one sample is matched with an observation in the other sample. The matching is conducted by selecting finance and marketing majors with similar GPAs. Thus, it is logical to compare the salary offers for finance and marketing majors in each group. This type of experiment is called **matched pairs**.

Example 13.5

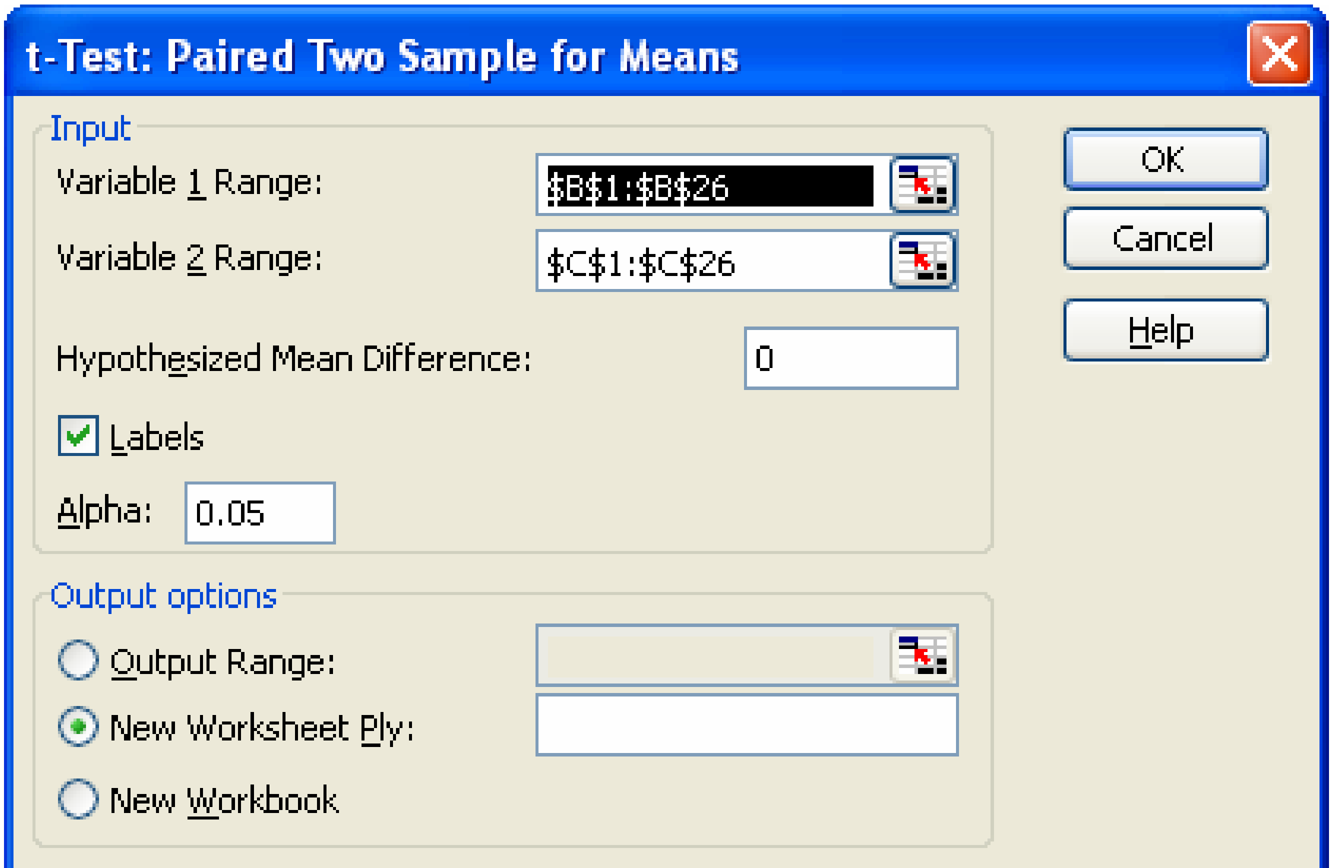
IDENTIFY

For each GPA group, we calculate the matched pair difference between the salary offers for finance and marketing majors.

Example 13.5

COMPUTE

Click Data, Data Analysis, t-Test: Paired Two- Sample for Means



Example 13.5

COMPUTE

	A	B	C
1	t-Test: Paired Two Sample for Means		
2			
3		*Finance*	*Marketing*
4	Mean	65,438	60,374
5	Variance	444,981,810	469,441,785
6	Observations	25	25
7	Pearson Correlation	0.9520	
8	Hypothesized Mean Difference	0	
9	df	24	
10	t Stat	3.81	
11	P(T<=t) one-tail	0.0004	
12	t Critical one-tail	1.7109	
13	P(T<=t) two-tail	0.0009	
14	t Critical two-tail	2.0639	

Example 13.5

INTERPRET

The p-value is .0004. There is overwhelming evidence that Finance majors do obtain higher starting salary offers than their peers in Marketing.