OPRE 6398 Prescriptive Analytics
Solutions to Homework 6

2. Given $\lambda = 5$, $\mu = 60/6 = 10$, and $\rho = 5/10 = 0.5$.
   The probability that there are n cars in the center is $P_n = 0.5^n (1 - 0.5) = 0.5^{n+1}$

   So the average number of cars that are parked on the street at any point in time is

   $P_5(5 - 4) + P_6(6 - 4) + P_7(7 - 4) + \ldots = 0.5^6(1) + 0.5^7(2) + 0.5^7(3) + \ldots = 0.5^6(1 + 2(0.5) + 3(0.5^2) + \ldots) = 0.5^6((1 + 0.5 + 0.5^2 + \ldots) + (0.5 + 0.5^2 + 0.5^3 + \ldots) + (0.5^2 + 0.5^3 + 0.5^4 + \ldots) + \ldots) =$
   $0.5^6(2 + 0.5(2) + 0.52(2) + \ldots) = 0.56(2)(1 + 0.5 + 0.5^2 + \ldots) = (0.56)(2)(2) = 0.0625$

   which may also be interpreted as the probability that a car is parked on the street at any point in time. Since the total number of cars arriving at the facility per week is 5(48) = 240, 0.0625(240) = 15 of them are parked on the street. As such, the weekly fines to be paid by the center are 15(0.2)(25) = $75.

3. Note that $\lambda = 20$ and $\mu = 60/2 = 30$.

   (1) Since $\rho = 20/30 = 2/3$, the probability that a customer will spend an average of 3 to 6 minutes waiting for service is $P(3/60 < t' < 6/60) = P(0.05 < t' < 0.1) = P(t' > 0.05) - P(t' > 0.1) = (2/3)e^{0.05(20-30)} - (2/3)e^{0.1(20-30)} \approx 0.4044 - 0.2453 = 0.1591$

   (2) Note that $\lambda = 60/2.5 = 24$ and 5 minutes = 5/60 = 1/12 hours. Since $W = 1/(\mu - \lambda) = 1/(\mu - 24) \le 1/12$, we have $\mu - 24 \ge 12$ or $\mu \ge 36$. Therefore, Linda must serve at least 36 customers per hour to ensure that a noontime customer will not spend more than 5 minutes in the yogurt shop.

4. Note that $\lambda = 17$ and $\mu = 60/3 = 20$.

   (1) The average number of planes stacking in the air and waiting for permission to land is $L_q = (17)^2/[20(20 - 17)] \approx 4.82$ airplanes.

   (2) Since $W_q = 17/[20(20 - 17)] = 17/60$ hours = 17 minutes, the average cost of fuel burned by an airplane waiting to land is 20(10)(17) = $3,400.

   (3) Since $\rho = 17/20 = 0.85$, the chance of finding less than three planes in the system is $P_{<3} = 1 - P_{>2} = 1 - (0.85)^{2+1} \approx 0.3859$.

   (4) The utilization of the runway is $\rho = 0.85$.

5. (1) Currently, for each secretary, $\lambda = 3$ and $\mu = 4$. It takes an average of $W = 1/(4 - 3) = 1$ hour to get a letter typed. If the two secretaries are pooled, then $k = 2$, $\lambda = 6$, $\mu = 4$, and $\lambda/k\mu = 6/[2(4)] = 0.75$. We then have

   $P_0 \approx \dfrac{1}{\sum_{n=0}^{2-1} \dfrac{(\frac{6}{4})^n}{n!} + \dfrac{(\frac{6}{4})^2}{(2-1)!} \cdot \dfrac{4}{2(4)-6}} \approx 0.1429$. It follows that $L_q = \dfrac{(\frac{6}{4})^2 (6)(4)}{(2-1)![2(4)-6]^2} (0.1429) \approx 1.9292$ and $W_q = 1.9292/6 \approx 0.3215$. Thus, it now takes an average of $W = 0.3215 + 1/4 = 0.5715$ hours to get a letter typed. Since $0.5715 < 1$, I suggest that the two secretaries be pooled.

(2) The implication is that while people tend to have their own private services rather than to centralize services in an organization, building empires in the case of service facilities can be very costly.