CHAPTER 4
SENSITIVITY ANALYSIS AND THE SIMPLEX METHOD

1. Introduction

   . Sensitivity analysis in linear programming is concerned with examining how sensitive the optimal solution and the optimal objective function value are to changes in such parameters as the objective function coefficients (unit selling price, unit cost, etc.) and the right-hand-side values of the constraints (amount of resource available, minimum production requirement, etc.) without solving the new LP.

   . Some useful concepts in analyzing the output reports from Solver follow:

   (1) Constraints -

   (a) Binding (or tight): When a structural constraint is evaluated at the optimal solution, it is said to be binding if its left-hand-side (LHS) value is equal to the right-had-side (RHS) value.

   (b) Nonbinding (or loose): When a structural constraint is evaluated at the optimal solution, it is said to be nonbinding if its left-hand-side (LHS) value is not equal to the right-had-side (RHS) value.

   (c) Slack: When a structural constraint is evaluated at the optimal solution, the absolute difference between its LHS value and RHS value is called the slack.

   Remark: The slack for a binding constraint is 0 and that for a nonbinding constraint is positive.

   (d) Shadow price: The shadow price of the resource associated with a structural constraint is the effect on the objective function value as a result of the change (either increase or decrease) in the RHS value of the constraint by one unit.

   Remark: The shadow price of the resource associated with a binding structural constraint is nonzero, and it is 0 for a nonbinding structural constraint.

   (e) Allowable increase: This is the maximum amount by which the RHS value of a structural constraint may be increased without affecting the current shadow price.

   (f) Allowable decrease: This is the maximum amount by which the RHS value of a structural constraint may be decreased without affecting the current shadow price.

   (2) Objective function -

   (a) Reduced cost: In the objective function, the absolute value of the reduced cost of a decision variable is the minimum amount by which the coefficient of the variable must be improved in order for it to take on a positive value in the optimal solution.

   (b) Allowable increase: This is the maximum amount by which the coefficient of a decision variable may be increased without affecting the current optimal solution.

   (c) Allowable decrease: This is the maximum amount by which the coefficient of a decision variable may be decreased without affecting the current optimal solution.

   . The essence of sensitivity analysis in linear programming may be summarized below:

   (1) Constraints - The current shadow price of the resource associated with a structural constraint remains valid as long as the new RHS value falls within the range of [current RHS value - allowable decrease, current RHS value + allowable increase].

(2) Objective function - The current optimal solution to the LP remains valid as long as the new objective function coefficient falls within the range of [current objective coefficient - allowable decrease, current objective coefficient + allowable increase].

(3) If the new RHS value or the new objective coefficient falls outside the corresponding range, the new LP has to be solved to find the new optimal solution and the new objective function value.

2. Basic Sensitivity Analysis

- **Example 4.1:** A small furniture company in Richardson, TX, produces wooden chairs and tables for sale. Each chair requires 1 hour of labor time and 4 pounds of wood while each table requires 3 hours of labor time and 3 pounds of wood. The unit profit contributions of the chairs and tables are $4 and $5, respectively. There are 13 hours of labor time and 25 pounds of wood are available in the company.

  Let x be the number of chairs to be produced and y be the number of tables to be produced. A linear program for determining the product mix to maximize the total profit is shown below:

$$
\begin{array}{lrrcr}
\text{Maximize } Z = & 4x + & 5y & & \\
\text{subject to:} & x + & 3y & \leq & 13 \\
 & 4x + & 3y & \leq & 25 \\
 & x\ , & y & \geq & 0
\end{array}
$$

(1) Run Solver to solve the LP and present the Answer Report as well as the Sensitivity Report.

(2) Respond to the following questions based on the Answer Report:

(a) Summarize and interpret the optimal solution and the optimal objective function value.

(b) Is any of the constraints binding at optimality?

(c) How many hours of labor time are used at optimality? What is the slack?

(3) Respond to the following questions based on the Sensitivity Report:

(a) What will the total profit be if the labor time available is reduced by 1 hour due to routine training?

(b) Verify your answer to Part (a) above by reducing the number of hours of labor time available from 13 to 12 hours and running Solver to solve the new LP.

(c) If additional wood is available, what is the maximum amount that you are willing to pay for one more pound of it? Why?

(d) Verify your answer to Part (c) above by changing the availability of the wood from 25 to 26 pounds and running Solver to solve the new LP.

(e) What the total profit is going to be if the amount of the wood available decreases from 25 to 15 pounds?

(f) Verify your answer to Part (e) above by changing the availability of the wood from 25 to 15 pounds and running Solver to solve the new LP.

[Solution] (1) Both the Answer Report and the Sensitivity Report are shown below:

Microsoft Excel 11.0 Answer Report
Worksheet: [LP.EX.xls]Sheet1
Report Created: 9/22/2007 6:34:00 PM

Target Cell (Max)

Cell	Name	Original Value	Final Value
$A$54	Z	0	31

Adjustable Cells

Cell	Name	Original Value	Final Value
$B$54	x	0	4
$C$54	y	0	3

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$55	LHS	13	$D$55<=$F$55	Binding	0
$D$56	LHS	25	$D$56<=$F$56	Binding	0

Microsoft Excel 12.0 Sensitivity Report
Worksheet: [EH573.1.EM.xls]Sheet1
Report Created: 1/28/2008 10:31:40 PM

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$15	x	4	0	4	2.666666667	2.333333333
$C$15	y	3	0	5	7	2

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$17	LHS	13	0.888888889	13	12	6.75
$D$18	LHS	25	0.777777778	25	27	12

(2) Based on the Answer Report:

(a) The optimal solution is $(x^*, y^*) = (4, 3)$ and the optimal objective function value is $Z^* = 31$. In other words, 4 chairs and 3 tables should be produced to maximize the total profit at $31.

(b) Both constraints are binding at optimality.

(c) Since the corresponding constraint (i.e., the first constraint) is binding and the slack is 0 hours, all of the 13 hours of labor time are used up at optimality. To see why this is the case, we plug $x^* = 4$ and $y^* = 3$ into the left-hand side of the first constraint ($x + 3y \leq 13$) to find that $x^* + 3y^* = 4 + 3(3) = 13$ hours of labor time are used to make 4 chairs along with 3 tables. As a consequence, no machine time is left idle, the constraint is binding, and hence the slack is 0.

(3) Based on the Sensitivity Report:

(a) Since the allowable decrease for the labor time is 6.75 hours and 1 hour is smaller than 6.75 hours, the current shadow price of the labor time, which is \$0.8889/hour, will remain unchanged under this new scenario. Consequently, the optimal total profit will decrease from \$31 to 31 - 0.8889 = \$30.1111 if the labor time availability is reduced by 1 hour due to routine training.

(b) Running Solver to solve the new LP with 12 hours (instead of 13 hours) of labor time being available, we see from the following Answer Report that the new optimal objective function value is $Z^* = 30.1111$. This is consistent with the prediction given in Part (a) above.

Microsoft Excel 12.0 Answer Report
Worksheet: [EH573.1.EM.xls]Sheet1
Report Created: 1/28/2008 10:45:10 PM

Target Cell (Max)

Cell	Name	Original Value	Final Value
\$A\$15	Z	0	30.11111111

Adjustable Cells

Cell	Name	Original Value	Final Value
\$B\$15	x	0	4.333333333
\$C\$15	y	0	2.555555556

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
\$D\$17	LHS	12	\$D\$17<=\$F\$17	Binding	0
\$D\$18	LHS	25	\$D\$18<=\$F\$18	Binding	0

(c) Since the shadow price of the wood is \$0.7778/pound, the allowable increase is 27 pounds, and 1 pound is smaller than 27 pounds, I will be willing to pay up to \$0.7778 to buy one more pound of it. This is because the additional wood will lead to an increase of the total profit by \$0.7778.

(d) Running Solver to solve the new LP with 26 pounds (instead of 25 pounds) of wood being available, we see from the following Answer Report that the new optimal objective function value is $Z^* = 31.7778$, which represents an increase in the total profit by 31.7778 - 31 = \$0.7778. Thus I will be willing to pay up to \$0.7778 to buy one more pound of the wood, which is consistent with the prediction given in Part (c) above.

Microsoft Excel 12.0 Answer Report
Worksheet: [EH573.1.MST.xls]Sheet1
Report Created: 1/28/2008 11:20:01 PM

Target Cell (Max)

Cell	Name	Original Value	Final Value
$A$15	Z	0	31.77777778

Adjustable Cells

Cell	Name	Original Value	Final Value
$B$15	x	0	4.333333333
$C$15	y	0	2.888888889

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$17	LHS	13	$D$17<=$F$17	Binding	0
$D$18	LHS	26	$D$18<=$F$18	Binding	0

(e) A decrease of 25 - 15 = 10 pounds of the wood is within the allowable decrease of 12 pounds. Therefore, the current shadow price of $0.7778/pound remains valid and the total profit will become $Z^* = 31 - 0.7778(10) = \$23.222$.

(f) Running Solver to solve the new LP with 15 pounds (instead of 25 pounds) of wood being available, we see from the following Answer Report that the new optimal objective function value is $Z^* = 23.222$. This is consistent with the prediction given in Part (e) above.

Microsoft Excel 12.0 Answer Report
Worksheet: [Exhibit 2.1.xls]Sheet1
Report Created: 2/1/2008 4:28:23 PM

Target Cell (Max)

Cell	Name	Original Value	Final Value
$A$15	Z	0	23.22222222

Adjustable Cells

Cell	Name	Original Value	Final Value
$B$15	x	0	0.666666667
$C$15	y	0	4.111111111

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$17	LHS	13	$D$17<=$F$17	Binding	0
$D$18	LHS	15	$D$18<=$F$18	Binding	0

- **Example 4.2:** Consider the Sensitivity Report generated by Solver for Example 4.1. What impact, if any, does each of the following changes have on the optimal solution and the optimal objective function value?

  (1) If the unit selling price of the table decreases from $5 to $4, what is the impact, if any, on the optimal solution and/or the optimal objective function value?

(2) Verify your answer to Part (1) above by changing the unit selling price of the table from \$5 to \$4 and running Solver to solve the new LP.

(3) If the unit selling price of the chair increases from \$4 to \$8, what is the impact, if any, on the optimal solution and/or the optimal objective function value?

(4) Verify your answer to Part (3) above by changing the unit selling price of the chair from \$4 to \$8 and running Solver to solve the new LP.

[Solution] The Sensitivity Report is reproduced below:

Microsoft Excel 12.0 Sensitivity Report
Worksheet: [EH573.1.EM.xls]Sheet1
Report Created: 1/28/2008 10:31:40 PM

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$B\$15	x	4	0	4	2.666666667	2.333333333
\$C\$15	y	3	0	5	7	2

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$D\$17	LHS	13	0.888888889	13	12	6.75
\$D\$18	LHS	25	0.777777778	25	27	12

(1) Observe that a decrease of \$1 from \$5 to \$4 for the unit selling price of the table (associated with the decision variable y) is smaller than the allowable decrease of \$2. As such, the current optimal solution of $(x^*, y^*) = (4, 3)$ remains optimal under the new scenario. However, the objective function value will become $4x^* + 4y^* = 4(4) + 4(3) = 28$.

(2) As indicated in the following Answer Report, the optimal solution is still $(x^*, y^*) = (4, 3)$ but the optimal objective function value becomes 28. Clearly, this is consistent with the prediction given in Part (1) above.

Microsoft Excel 12.0 Answer Report
Worksheet: [Exhibit 2.1.xls]Sheet1
Report Created: 2/1/2008 4:54:51 PM

Target Cell (Max)

Cell	Name	Original Value	Final Value
$A$15	Z	0	28

Adjustable Cells

Cell	Name	Original Value	Final Value
$B$15	x	0	4
$C$15	y	0	3

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$17	LHS	13	$D$17<=$F$17	Binding	0
$D$18	LHS	25	$D$18<=$F$18	Binding	0

(3) Note that an increase of $4 from $4 to $8 for the unit selling price of the chair (associated with the decision variable x) is greater than the allowable increase of $2.6667. As such, the current optimal solution of $(x^*, y^*) = (4, 3)$ is no longer optimal under this new scenario. Consequently, we need to solve the new LP to obtain the new optimal solution as well as the new optimal objective function value.

(4) As indicated in the following Answer Report, the optimal solution turns out to be $(x^*, y^*) = (6.25, 0)$ and the new maximum objective function value is $Z^* = 50$. Obviously, this is consistent with the prediction given in Part (3) above.

Microsoft Excel 12.0 Answer Report
Worksheet: [Exhibit 2.1.xls]Sheet1
Report Created: 2/1/2008 5:00:43 PM

Target Cell (Max)

Cell	Name	Original Value	Final Value
$A$15	Z	0	50

Adjustable Cells

Cell	Name	Original Value	Final Value
$B$15	x	0	6.25
$C$15	y	0	0

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$17	LHS	6.25	$D$17<=$F$17	Not Binding	6.75
$D$18	LHS	25	$D$18<=$F$18	Binding	0

. **Example 4.3:** Lone Star Company in Little Rock, AR, is developing a new paint that must have a brilliance rating of at least 15 degrees, a hue rating of at least 32 degrees, and a clarity rating of at least 21 degrees. The two ingredients to be mixed to produce the new product are Alpha and Beta, which cost \$7 per pound and \$9 per pound, respectively. Each pound of Alpha contributes to 3 degrees of brilliance, 8 degrees of hue, and 7 degrees of clarity whereas each pound of Beta contributes 5 degrees of brilliance, 8 degrees of hue, and 3 degrees of clarity. Relevant information has been summarized in the following table:

Criterion	Alpha	Beta	Minimum requirement
Brilliance	3	5	15
Hue	8	8	32
Clarity	7	3	21
Cost	7	9	

Currently, only 1.2 pounds of Beta are available. Let A be the amount (in pounds) of Alpha and B be the amount (in pounds) of Beta to be mixed to develop the new paint. A linear program for determining the quantity of each ingredient to be use to minimize the total cost is presented below:

$$
\begin{array}{llrcr}
\text{Minimize } Z = & 7A + 9B & & & \\
\text{subject to:} & 3A + 5B & \geq & 15 \\
& 8A + 8B & \geq & 32 \\
& 7A + 3B & \geq & 21 \\
& B & \leq & 1.2 \\
& A\ ,\ B & \geq & 0
\end{array}
$$

(1) Run Solver to solve the LP, summarize the optimal solution as well as the optimal objective function value, and interpret them. Be sure to show both the Answer Report and the Sensitivity Report.

(2) Based on the Answer Report, how many constraints are nonbinding at optimality? What are the respective slacks? What do they mean in this problem and where do they come from?

(3) If a local supplier would like to provide additional Beta at a cost of \$2 per pound, should the offer be accepted based on the Sensitivity Report? Why or why not?

{Solution] (1) Both the Answer Report and the Sensitivity Report are displayed below:

Microsoft Excel 12.0 Answer Report
Worksheet: [EH573.1.MST.xls]Sheet2
Report Created: 2/1/2008 12:13:10 PM

Target Cell (Min)

Cell	Name	Original Value	Final Value
$A$15	Z	0	31.8

Adjustable Cells

Cell	Name	Original Value	Final Value
$B$15	A	0	3
$C$15	B	0	1.2

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$17	LHS	15	$D$17>=$F$17	Binding	0
$D$18	LHS	33.6	$D$18>=$F$18	Not Binding	1.6
$D$19	LHS	24.6	$D$19>=$F$19	Not Binding	3.6
$D$20	LHS	1.2	$D$20<=$F$20	Binding	0

Microsoft Excel 12.0 Sensitivity Report
Worksheet: [EH573.1.MST.xls]Sheet2
Report Created: 2/1/2008 12:13:11 PM

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$15	A	3	0	7	1E+30	1.6
$C$15	B	1.2	0	9	2.666666667	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$17	LHS	15	2.333333333	15	1E+30	0.6
$D$18	LHS	33.6	0	32	1.6	1E+30
$D$19	LHS	24.6	0	21	3.6	1E+30
$D$20	LHS	1.2	-2.666666667	1.2	0.3	1.2

It is seen from the Answer Report that the optimal solution is $(A^*, B^*) = (3, 1.2)$ and the optimal objective function value is 31.8. In other words, 3 pounds of Alpha and 1.2 pounds of Beta should be mixed to make the new paint so that the total cost is minimized at $31.8.

(2) According to the Answer Report, the second and the third constraints are nonbinding at optimality with respective slacks of 1.6 units and 3.6 units. The implication is that the new paint to be produced contains 1.6 degrees higher than the minimum requirement of 32 degrees of hue and 3.6 degrees higher than the minimum requirement of 21 degrees of clarity.

Notice that $A^* = 3$ pounds of Alpha and $B^* = 1.2$ pounds of Beta are to be mixed. The total rating of hue in the new paint is $8A^* + 8B^* = 8(3) + 8(1.2) = 33.6$ degrees and the minimum requirement is 32 degrees, hence $33.6 - 32 = 1.6$ degrees higher than necessary. Likewise, the total rating of clarity contained in the new paint is $7A^* + 3B^* = 7(3) +$

3(1.2) = 24.6 degrees and the minimum requirement is 21 degrees, hence 24.6 - 21 = 3.6 degrees higher than necessary.

(3) We see from the bottom row of the Sensitivity Report that the shadow price of the Beta is \$-2.6667/pound, which means that the total cost will go down by \$2.6667 for each additional pound of the Beta used so long as the change is within the allowable limit. As \$2.6667 > \$2, the offer should be accepted but only up to 0.3 pounds of additional Beta should be purchased. This is because the allowable increase for the shadow price to remain \$-2.6667 is 0.3.

3. Duality Theory

. Every linear programming model has two forms. The first or original form is called the primal, and the second form is derived from the primal and is called the dual.

. Since both the primal and the dual are associated with the same problem, their respective solutions are closely related to each other. Specifically, the optimal solution values in the primal are equal to the shadow prices in the dual, and the shadow prices in the primal are equal to the optimal solution values in the dual. In addition, the optimal objective function value for the primal is the same as that for the dual.

. The importance of duality in linear programming lies in the following facts:

(1) The solution to the dual provides a basis for understanding some of the fundamental economic relationships that exist in the primal. This is particularly useful for managers to make sound business decisions.

(2) It allows some primal LP problems with more constraints than decision variables to be transformed into equivalent dual LP problems with less constraints than decision variables, which are much easier to solve. (In general, the amount of time required to solve an LP is in proportion to the number of constraints but depends little on the number of decision variables.)

(3) LPs involving many decision variables and only two constraints may be transformed into an equivalent dual LP involving only two decision variables (and many constraints) that may be solved graphically when no computer is available.

. Primal/dual transformation - The rules described in the exhibit on the last page of this chapter may be used to help you with the transformation between the primal and the dual of an LP.

Remark: A decision variable in an LP is said to be unrestricted in sign (UIS) if it is allowed to take on a positive value, 0, or a negative value.

. **Example 4.4:** Consider the following linear program, which was formulated for the product mix problem in Example 2.15. Derive the dual linear program.

$$
\begin{array}{llrrl}
\text{Maximize } Z = & 4x + & 5y & & \\
\text{subject to:} & x + & 3y & \le & 13 \\
 & 4x + & 3y & \le & 25 \\
 & x\ , & y & \ge & 0
\end{array}
$$

[Solution]

$$
\begin{array}{llrrl}
\text{Minimize } Z = & 13u + & 25v & & \\
\text{subject to:} & u + & 4v & \ge & 4 \\
 & 3u + & 3v & \ge & 5 \\
 & u\ , & v & \ge & 0
\end{array}
$$

. **Example 4.5:** Derive the dual of the following LP:

$$
\begin{array}{lrrrcr}
\text{Minimize } Z = & 3x_1 - & 4x_2 + & 5x_3 & & \\
\text{subject to:} & 2x_1 + & 3x_2 + & 3x_3 & \geq & 27 \\
 & x_1 - & & x_3 & \geq & 0 \\
 & 4x_1 + & 2x_2 + & 4x_3 & \leq & 55 \\
 & & x_2 & & \leq & 4 \\
 & x_1\ , & x_2\ , & x_3 & \geq & 0
\end{array}
$$

[Solution]

$$
\begin{array}{lrrrrcr}
\text{Maximize } Z = & 27y_1 + & & 55y_3 + & 4y_4 & & \\
\text{subject to:} & 2y_1 + & y_2 + & 4y_3 & & \leq & 3 \\
 & 3y_1 + & & 2y_3 + & y_4 & \leq & -4 \\
 & 3y_1 - & y_2 + & 4y_3 & & \leq & 5 \\
 & y_1\ , & y_2 & & & \geq & 0 \\
 & & & y_3\ , & y_4 & \leq & 0
\end{array}
$$

- **Example 4.6:** Formulate the dual of the linear program presented below:

$$
\begin{array}{lrrcr}
\text{Maximize } Z = & 3x_1 + & 5x_2 & & \\
\text{subject to:} & 2x_1 + & 3x_2 & \leq & 8 \\
 & 5x_1 + & 4x_2 & = & 9 \\
 & x_1 + & x_2 & \leq & 3 \\
 & x_1\ , & x_2 & \geq & 0
\end{array}
$$

[Solution]

$$
\begin{array}{lrrrcr}
\text{Minimize } Z = & 8y_1 + & 9y_2 + & 3y_3 & & \\
\text{subject to:} & 2y_1 + & 5y_2 + & y_3 & \geq & 3 \\
 & 3y_1 + & 4y_2 + & y_3 & \geq & 5 \\
 & y_1\ , & & y_3 & \geq & 0 \\
 & & y_2 & & & \text{UIS}
\end{array}
$$

- **Example 4.7:** Derive the dual of the following LP:

$$
\begin{array}{lrrrcr}
\text{Minimize } Z = & 24x_1 + & 28x_2 + & 18x_3 & & \\
\text{subject to:} & 4x_1 + & x_2 + & 2x_3 & \geq & 3 \\
 & 2x_1 + & 7x_2 + & 3x_3 & \geq & 7 \\
 & x_1\ , & & x_3 & \geq & 0 \\
 & & x_2 & & & \text{UIS}
\end{array}
$$

[Solution]

$$
\begin{array}{lrrcr}
\text{Maximize } Z = & 3u + & 7v & & \\
\text{subject to:} & 4u + & 2v & \leq & 24 \\
 & u + & 7v & = & 28 \\
 & 2u + & 3v & \leq & 18 \\
 & u\ , & v & \geq & 0
\end{array}
$$

- **Example 4.8:** Formulate the dual of the LP model presented below:

$$
\begin{array}{lrrcr}
\text{Maximize} & 2x_1 + & x_2 & & \\
\text{subject to:} & x_1 + & x_2 & = & 2 \\
 & 2x_1 - & x_2 & \geq & 3 \\
 & x_1 - & x_2 & \leq & 1 \\
 & x_1 & & \geq & 0 \\
 & & x_2 & & \text{UIS}
\end{array}
$$

[Solution]

$$\begin{array}{llrrrcr}
\text{Minimize } Z = & 2y1 + & 3y2 + & y3 & & \\
\text{subject to:} & y1 + & 2y2 + & y3 & \geq & 2 \\
 & y1 - & y2 - & y3 & = & 1 \\
 & y1 & & & & \text{UIS} \\
 & & y2 & & \leq & 0 \\
 & & & y3 & \geq & 0
\end{array}$$

- **Example 4.9:** Derive the dual of the following linear program:

$$\begin{array}{lrrrcr}
\text{Maximize } Z = & 3x1 + & 4x2 - & 2x3 & & \\
\text{subject to:} & 4x1 - & 12x2 + & 3x3 & \leq & 12 \\
 & -2x1 + & 3x2 + & x3 & \leq & 6 \\
 & -5x1 + & x2 - & 6x3 & \geq & -40 \\
 & 3x1 + & 4x2 - & 2x3 & = & 10 \\
 & x1 & & & \geq & 0 \\
 & & x2 & & \leq & 0 \\
 & & & x3 & & \text{UIS}
\end{array}$$

[Solution]

$$\begin{array}{lrrrrcr}
\text{Minimize} & 12y1 + & 6y2 - & 40y3 + & 10y4 & & \\
\text{subject to:} & 4y1 - & 2y2 - & 5y3 + & 3y4 & \geq & 3 \\
 & -12y1 + & 3y2 + & y3 + & 4y4 & \leq & 4 \\
 & 3y1 + & y2 - & 6y3 - & 2y4 & = & -2 \\
 & y1 \;, & y2 & & & \geq & 0 \\
 & & & y3 & & \leq & 0 \\
 & & & & y4 & & \text{UIS}
\end{array}$$

- Solving special linear programs - A linear programming model involving decision variables that either must take on negative values or are unrestricted in sign may be reformulated as one where the decision variables are all nonnegative according to the following rules:

(1) A variable (x) that must take on a negative value - Replace each x with -xp in the LP, where xp is nonnegative.

(2) A variable (y) that is unrestricted in sign - Replace each y with yp - yn in the LP, where yp and yn are both nonnegative.

The modified LP is then solved by running Solver in the normal fashion. However, the optimal solution obtained needs to be converted to find the optimal solution to the original problem.

- **Example 4.10:** Solve the following linear program by running Solver:

$$\begin{array}{lrrrrcr}
\text{Maximize } Z = & 12x1 - & x2 - & 5x3 - & 20x4 & & \\
\text{subject to:} & -2x1 + & x2 + & x3 + & x4 & \geq & 25 \\
 & -3x1 + & & x3 + & 4x4 & \geq & 30 \\
 & x1 & & & & \leq & 0 \\
 & & x2 & & & & \text{UIS} \\
 & & & x3 \;, & x4 & \geq & 0
\end{array}$$

[Solution] Replacing x1 with -x1n and replace x2 with x2p - x2n, where x1n, x2p, and x2n are all nonnegative, the original LP becomes:

$$\begin{array}{lrrrrrcr}
\text{Maximize } Z = & -12x1p - & x2p + & x2n - & 5x3 - & 20x4 & & \\
\text{subject to:} & 2x1p + & x2p - & x2n + & x3 + & x4 & \geq & 25 \\
 & 3x1p + & & & x3 + & 4x4 & \geq & 30 \\
 & x1p \;, & x2p \;, & x2n \;, & x3 \;, & x4 & \geq & 0
\end{array}$$

Running Solver to solve the new LP, we find that the optimal solution is ($x1p^*$, $x2p^*$, $x2n^*$, $x3^*$, $x4^*$) = (10, 5, 0, 0, 0) and the optimal objective function value is $Z^* = -125$. It follows that the optimal solution to the original LP is ($x1^*$, $x2^*$, $x3^*$, $x4^*$) = ($-x1p^*$, $x2p^* - x2n^*$, $x3^*$, $x4^*$) = (-10, 5 - 0, 0, 0) = (-10, 5, 0, 0) with an optimal objective function value of $Z^* = -125$.

. Relationships between primal and dual - As pointed out earlier, the key relationships between the primal and the dual of a linear program are:

(1) The optimal solution values in the primal are equal to the shadow prices in the dual,

(2) The shadow prices in the primal are equal to the optimal solution values in the dual, and

(3) The optimal objective function value for the primal is the same as that for the dual.

(4) The dual of the dual is the primal.

Maximization problem		Minimization problem
Number of constraints	1	Number of variables
(≤) constraint	2	Nonnegative variable
(≥) constraint	3	Nonpositive variable
(=) constraint	4	Unrestricted variable
Number of variables	5	Number of constraints
Nonnegative variable	6	(≥) constraint
Nonpositive variable	7	(≤) constraint
Unrestricted variable	8	(=) constraint
Objective function coefficient for j^{th} variable	9	RHS constant for j^{th} constraint
RHS constant for i^{th} constraint	10	Objective function coefficient for i^{th} variable
Technological coefficient in constraint i for variable j (a_{ij})	11	Technological coefficient in constraint j for variable i (a_{ji})