Matrix Algebra

Definitions

Scalar single numerical value

Matrix rectangular array of values with r rows and c columns; *e.g.* the size of matrix A is 2 by 3 (or 2 x 3)

$$A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

Square Matrix has $r = c$; *e.g.*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Symmetric Matrix a *square* matrix A is *symmetric iff* $A_{i,j} = A_{j,i}$; *e.g.*

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

where $A_{1,2} = A_{2,1} = b$

Dense Matrix a matrix with no (or few) zero entries

Sparse Matrix a matrix with mostly zero entries

Matrix Diagonal the collection of values from a *square* matrix where the row and column numbers are equal ($A_{i,i}$)

Identity Matrix a *square, sparse* matrix I where the *matrix diagonal* is 1; more explicitly, where $I_{i,j} = 0$ when $i \neq j$ and $I_{j,j} = 1$; *e.g.*

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Operators and Properties

Assume the following

$$
\begin{aligned}
z &= scalar \\
A &= \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \\
B &= \begin{pmatrix} g & h \\ i & j \\ k & l \end{pmatrix} \\
C &= \begin{pmatrix} m & n & o \\ p & q & r \end{pmatrix}
\end{aligned}
$$

Matrices *conform* when the *magnitudes* of their respective rows and columns allow a given operation. Common notation has *scalar* quantities as lower case letters and *matrix* quantities as upper case letters. We will be using single upper-case letters to designate matrices and single lower-case letters to designate scalars.

Addition operator is "+"; operation conforms for matrices when both have the same number of rows and columns; operation always conforms for matrix/scalar mix; operation is *commutative* (order independent) for both conformal matrix-matrix and matrix-scalar situations *e.g.*

$$\begin{aligned} A + B &= B + A \\ &= \begin{pmatrix} (a+g) & (b+h) \\ (c+i) & (d+j) \\ (e+k) & (f+l) \end{pmatrix} \\ z + A &= A + z \\ &= \begin{pmatrix} (a+z) & (b+z) \\ (c+z) & (d+z) \\ (e+z) & (f+z) \end{pmatrix} \end{aligned}$$

Subtraction operator is "-"; operation conforms for matrices when both have the same number of rows and columns; operation always conforms for matrix/scalar mix; operation is *not commutative* (order independent) for both conformal matrix-matrix and matrix-scalar situations *e.g.*

$$\begin{aligned} A - B &= \begin{pmatrix} (a-g) & (b-h) \\ (c-i) & (d-j) \\ (e-k) & (f-l) \end{pmatrix} \\ &\neq B - A \\ A - z &= \begin{pmatrix} (a-z) & (b-z) \\ (c-z) & (d-z) \\ (e-z) & (f-z) \end{pmatrix} \\ &\neq z - A \end{aligned}$$

Multiplication (matrix-scalar) operator is shown as an elevated dot; operation always conforms and is *commutative*; *e.g.*

$$\begin{aligned} z \cdot A &= A \cdot z \\ &= \begin{pmatrix} (a \cdot z) & (b \cdot z) \\ (c \cdot z) & (d \cdot z) \\ (e \cdot z) & (f \cdot z) \end{pmatrix} \\ I \cdot z &= z \cdot I \end{aligned}$$

Dot Product the sum of the element by element multiplication of 2 row and/or column segments of a matrix; the row(s) and/or column(s) must be of the same length

Multiplication (matrix-matrix) operator is shown as an elevated dot; operation conforms when first matrix column count and second matrix row count are equal; operation is *not commutative e.g.*; given an s by t matrix and an t by u matrix, the result will be a s by u matrix formed by *dot products* of each row from the first listed matrix and each column from the second listed matrix; *e.g.*

$$\begin{aligned} A \cdot C &= \begin{pmatrix} (b \cdot p + a \cdot m) & (b \cdot q + a \cdot n) & (b \cdot r + a \cdot o) \\ (d \cdot p + c \cdot m) & (d \cdot q + c \cdot n) & (d \cdot r + c \cdot o) \\ (f \cdot p + e \cdot m) & (f \cdot q + e \cdot n) & (f \cdot r + e \cdot o) \end{pmatrix} \\ &\neq C \cdot A \end{aligned}$$

And $I \cdot A = A \cdot I$ *only* when I and A are *conformal*.

Inverse *division* does exist as either an operator or an operation; *inverse* is used with *matrix multiplication* to accomplish the same effect; applicable only for square matrix; when *determinant* is 0 no unique inverse exists; operation satisfies the relationship

$$A^{-1} \cdot A = I$$

Transpose converts an r by c matrix into a c by r matrix; each column of the original matrix becomes a row in the resulting matrix; operator can be either a *superscript* T or a *prime* ('), so

$$\begin{aligned} A &= \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \\ A^T &= \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} \\ A^T &= A' \end{aligned}$$

Example

A set of n simultaneous linear equations in n unknowns is specified by a matrix A which is n by n of *coefficient values* for the equations, a matrix X which is n by 1 representing the *unknown* quatities being solved for, and a matrix B which is n by 1 and represents the *right hand side* (RHS) of the following equation

$$A \cdot X = B$$

The solution can be found through the following steps

$$\begin{aligned} A \cdot X &= B \\ A^{-1} \cdot A \cdot X &= A^{-1} \cdot B \\ I \cdot X &= A^{-1} \cdot B \\ X &= A^{-1} \cdot B \end{aligned}$$

So, suppose we have a *system of linear equations* which we need to solve. As an example, we will use

$$\begin{aligned} 3x + 4y &= 29 \\ 2x + 5y &= 31 \end{aligned}$$

Following the notation already used in this example, we would set up a matrix **A** and a matrix **B** in EXCEL. We want to solve for a matrix **X**. These matrices look like

$$\begin{aligned} A &= \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} \\ B &= \begin{pmatrix} 29 \\ 31 \end{pmatrix} \end{aligned}$$

And we have out solution

$$X = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

where, from the original problem, $x = 3$ and $y = 5$.

See the example spreadsheet for how to perform these operations in *EXCEL*.