Lagrange Multipler in Optimization

This section introduces *constrained optimization* of a function, generally a measure of *profit* or *utility.* We start with an unconstrained function and add a single *equality* constraint (limit).

A profit function $f(x, y)$ is given as:

$$f(x, y) = 40x + 110y - 16x^2 - 15xy - 73y^2$$

To *maximize* this function, we obtain the *partial derivatives* for each variable involved (x, y). These partial derivatives are

$$\begin{aligned} \frac{df}{dx} &= 40 - 15y - 32x \\ \frac{df}{dy} &= 110 - 146y - 15x \end{aligned}$$

Setting each of these equations to zero gives us the following set of simultaneous linear equations

$$\begin{aligned} 40 - 15y - 32x &= 0 \\ 110 - 146y - 15x &= 0 \end{aligned}$$

which is equivalent to

$$\begin{aligned} 15y + 32x &= 40 \\ 146y + 15x &= 110 \end{aligned}$$

Setting this up for solution as a *set of simultaneous linear equations* (2 equations in 2 unknowns) and using the original labels, we get

$$\begin{aligned} A &= \begin{pmatrix} 15 & 32 \\ 146 & 15 \end{pmatrix} \\ B &= \begin{pmatrix} 40 \\ 110 \end{pmatrix} \end{aligned}$$

Solving these equations in EXCEL gives us (approximately)

$$\begin{aligned} x &= 0.942208 \\ y &= 0.656622 \\ x + y &= 1.598831 \end{aligned}$$

which are the values of x and y which maximize the profit function for a value of 54.9584.

Thus far, the implicit assumption has been that there are neither limits to the amounts of x and y available individually nor in their relation to each other. Suppose, instead, that we have the following limit (*constraint*)

$$x + y = 1$$

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We deal with this by introducing a new (third) variable L called the *Lagrange multiplier* and use it to modify the original profit function to be

$$\begin{aligned} g(x, y, L) &= f(x, y) + L(1 - x - y) \\ &= 40x + 110y - 16x^2 - 15xy - 73y^2 + L(1 - x - y) \\ &= 40x + 110y - 16x^2 - 15xy - 73y^2 + L - Lx - Ly \end{aligned}$$

The resulting set of partial derivatives are

$$\begin{aligned} \frac{dg}{dx} &= 40 - 15y - 32x - L \\ \frac{dg}{dy} &= 110 - 146y - 15x - L \\ \frac{dg}{dL} &= -y - x + 1 \end{aligned}$$

Again, setting each of these equations to zero gives us a new set of simultaneous linear equations:

$$\begin{aligned} 40 - 15y - 32x - L &= 0 \\ 110 - 146y - 15x - L &= 0 \\ -y - x + 1 &= 0 \end{aligned}$$

which becomes

$$\begin{aligned} 15y + 32x + L &= 40 \\ 146y + 15x + L &= 110 \\ y + x &= 1 \end{aligned}$$

Setting this up for solution as a *set of simultaneous linear equations* (3 equations in 3 unknowns) and using the original labels, we get

$$A = \begin{pmatrix} 15 & 32 & 1 \\ 146 & 15 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 40 \\ 110 \\ 1 \end{pmatrix}$$

The solution for which (using EXCEL) is (approximately)

$$\begin{aligned} x &= 0.412162 \\ y &= 0.587838 \\ L &= 17.99324 \\ x+y &= 1 \end{aligned}$$

for a maximum profit of 49.57095. Note that the constraint is satisfied.