Notes on Bosonic String Amplitudes

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This is a short note on the basics of bosonic strings and the calculation of their amplitudes.

1 The world-sheet action and string spectrum

String theory describes an 1-dimensional extended object. The 1-dimensional string sweeps a 2-dimensional surface Σ. The string itself is parametrized by the coordinate σ. Along with the proper time τ of the trajectory, they parametrize the world-sheet Σ. The world-sheet Σ is embedded in the D-dimensional spacetime, with induced metric $h_{ab} = G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$. A simple and natural guess of the action is just the area of the world-sheet with tension $T = \frac{1}{2\pi\alpha'}$:

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int_\Sigma d\tau d\sigma \sqrt{-\det h_{ab}}. \tag{1}$$

This action is called Nambu-Goto action and has the following symmetry:

1. Spacetime Poincare symmetry: $X'^\mu(\tau,\sigma) = \Lambda^\mu_\nu X^\nu(\tau,\sigma) + a^\mu$;
2. Diffeomorphism: $\tau' = \tau'(\tau,\sigma), \sigma' = \sigma'(\tau,\sigma)$.

A more convenient action for quantization is the Polyakov action, which involves a dynamical metric γ_{ab}:

$$S_P = -\frac{1}{4\pi\alpha'}\int_\Sigma d\tau d\sigma \sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu. \tag{2}$$

Here we have taken the spacetime metric to be flat. This action classically agrees with the Nambu-Goto action, obtained by varying the metric:

$$\delta_\gamma S_P = -\frac{1}{4\pi\alpha'}\int_\Sigma d\tau d\sigma \sqrt{-\gamma}\delta\gamma^{ab}(h_{ab} - \frac{1}{2}h_{cd}\gamma_{ab}\gamma^{cd}). \tag{3}$$

The solution implies $h_{ab}\sqrt{-h} = \gamma_{ab}\sqrt{-\gamma}$. This can be used to eliminate γ in the action, and gives the NG action.

The Polyakov action has the following symmetries:

1. Spacetime Poincare symmetry;

2. Diffeomorphism:

$$X'^{\mu}(\tau',\sigma') = X^{\mu}(\tau,\sigma); \quad \frac{\partial\sigma'^{c}}{\partial\sigma^{a}}\frac{\partial\sigma'^{d}}{\partial\sigma^{b}}\gamma'_{cd}(\tau',\sigma') = \gamma_{ab}(\tau,\sigma). \tag{4}$$

;

3. Weyl transformation:

$$X'^{\mu}(\tau,\sigma) = X^{\mu}(\tau,\sigma); \quad \gamma'_{ab}(\tau,\sigma) = e^{2\omega(\tau,\sigma)}\gamma_{ab}(\tau,\sigma). \tag{5}$$

We view the latter two types of invariance as world-sheet gauge symmetries.

1.1 String spectrum