Notes on Large N Gauge Theories

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This note intends to give a comprehensive review of Large N gauge theories, including Yang-Mills theory and matrix models.

1 Examples of large N field theories

The main reference of this section is [1].

1.1 Large N ϕ^4 model

1.2 Gross-Neveu model

1.3 $\mathbb{C}P^{N-1}$ model

1.4 Large N QCD and 't Hooft model

For $SU(N)$ chromodynamics, the dynamical variables include:

1. A set of Dirac fermions, ψ^a, which lies in the fundamental representation of $SU(N)$;
2. A set of gauge fields, $A^a_{\mu b}$, which lies in the adjoint representation.

Similar to the ordinary Yang-Mills, the Lagrangian writes:

$$\mathcal{L} = \frac{g^2}{N}\left[-\frac{1}{4}F^a_{\mu\nu b}F^{\mu\nu b}_a + \bar{\psi}_a(i\gamma^\mu\partial_\mu + \gamma^\mu A^a_{\mu b})\psi^b - m\bar{\psi}_a\psi^a\right]. \tag{1}$$

By rescaling the field variables, we can also put the coupling $g/\sqrt{N}$ to the cubic interaction term and g^2/N to the quartic term.

We stick to the original Lagrangian, and we count the power of N. We invoke the double-line, and the vacuum diagram gives $N^{F-E+V} = N^{2-2g}$, where g is the genus of the surface triangulated by the ribbon diagram, F is the number of faces, E is the number of edges, V is the number of vertices.

The leading connected vacuum diagrams are proportional to N^2, where only gluons are involved and the diagram looks like the triangulation of a sphere. Diagrams embedded into a

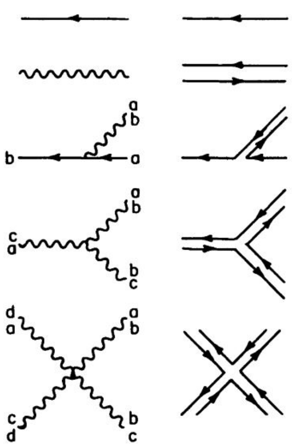

Figure 1: The double line representation of QCD Feynman graphs.

genus 0 surface are called planar diagrams. If we include quarks, effectively one of the faces is removed from a leading diagram, giving a single quark line as the boundary of the whole diagram. So the leading diagrams with quarks should only have one quark loop as the boudary of the diagram.

Exact solvability of 't Hooft model is to be added [2].

2 Large N phenomenology

2.1 Gluons and Mesons

We consider correlation functions of gluon single trace operators:

$$G = \mathrm{tr} F^m. \tag{2}$$

Intorducing sources, J_i, then in order to calculate $\langle G_1 ... G_n \rangle_c$, the generating functional writes:

$$Z[J_i] = \int \mathcal{D}A\mathcal{D}\psi \exp i(S + N\sum_i J_i G_i). \tag{3}$$

Since one external insertion cancels one N factor, we have:

$$\langle G_1 ... G_n \rangle_c \propto N^{2-n}. \tag{4}$$

For mesons, they are quark bilinears $B = \sqrt{N}\mathrm{tr}\bar{\psi}F^m\psi$. So similarly:

$$\langle B_1 ... B_n \rangle_c \propto N^{1-\frac{n}{2}}. \tag{5}$$

This simple counting has many interesting phenomenological implications. We list them as follows:

1. Single-trace operators and quark bilinears only create single-particle states by acting on the vacuum, which means that if you cut open a quark bilinear or glueball two point function,

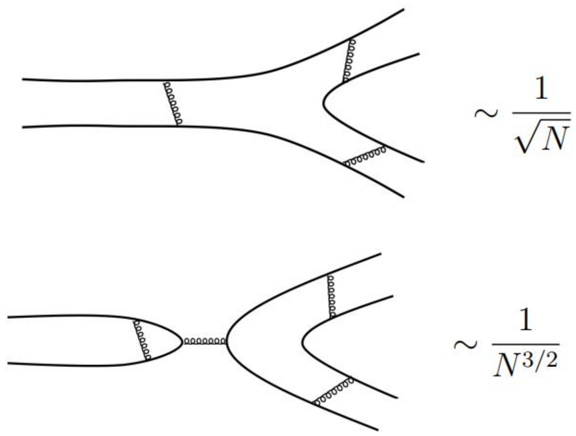

Figure 2: Meson decay diagrams. Here $B = \sqrt{N}\mathrm{tr}\bar{\psi}F^m\psi$. The first diagram is allowed and the particles produces should contain strange quarks, if the original meson is composed of two strange quarks. However, the latter diagram is prohibited by large N.

you should find the intermediate states are all singlet bound states of quarks and gluons, since confinement should happen for arbitrary N. If not, such diagrams can only appear at higher genus. Another intuition is that, since quantum fluctuation is greatly suppressed by large N, we should have a free, classical theory.

2. What follows from the last property is that, we should have infinite number of particles. For example, consider glueball propagator, we should sum over all tree-level propagators with poles from all possible massive particles:

$$\langle G(k)G(-k)\rangle_c = \sum_n \frac{|a_n|^2}{k^2 - M_n^2}. \tag{6}$$

However, asymptotic freedom implies that at large k, the correlation function should scale as the logarithm of k^2, yet if a finite number of single particle states are involved, this is impossible for there is no multi-particle branch cuts. Hence, in the $N \to \infty$ limit, Yang-Mills is a theory of an infinite number of free particles.

3. From the scaling of correlation functions, on can make observations of meson scattering amplitudes. Firstly, in the large N regime, meson scattering and decay is suppressed, as is shown in figure 2. Also, OZI rule becomes exact in this limit. The OZI rule states that for meson diagrams external quark legs cannot be divided into two sets disconnected by quark lines. Experimentally, $s\bar{s}$ cannot decay into mesons free of strange quarks.

4. The large N expansion also prohibits the observation of exotics, which are bound states of many quarks, since the 4-point connected quark bilinear correlation is suppressed by $1/N$.

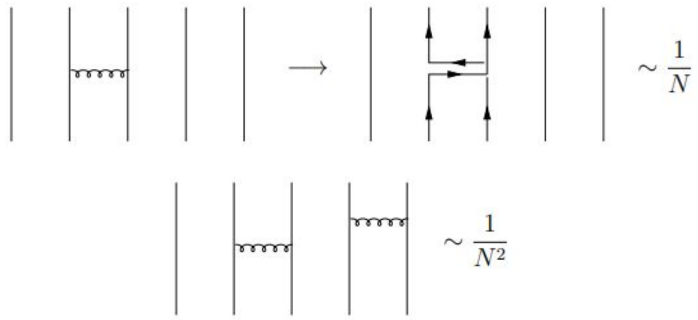

Figure 3: Baryon two-point functions with 1 and 2 gluon exchanges.

2.2 Baryons

For baryons the story is much more tricky. Baryons are quark bound states which involves an anti-symmetrized product of N quarks. In this subsection we introduce Witten's theory of baryons. [3].

To do that, we first consider rescaling the fields and move all the N dependence to vertices. The quark-gluon three point coupling is of order $1/\sqrt{N}$.

We first consider a baryon two-point function, as is shown in figure 3. If among all N quark lines there are two lines share an exchange of a gluon, the diagram scales as $N^2 \cdot \frac{1}{N} \sim N$. Now let's include the diagrams with four quarks that exchanged two gluons, we have the scaling $N^4 \left(\frac{1}{\sqrt{N}}\right)^4 \sim N^2$.

The problem emerges from the fact that the latter diagrams diverge even more severely than the lowest order diagrams. So it seems that a large N limit does not exist. However, we intend to argue that this limit exists. To do that, we first try to determine the N dependence of the baryon masses.

Baryon mass. By baryon masses, we mean:

$$\text{Baryon mass} \;=\; \text{quark masses} \;+\; \text{quark kinetic energy} \;+\; \text{quark-quark potential energy}. \quad (7)$$

The propagation amplitude for baryons is roughly $e^{-iM_B t}$, where M_B is the total baryon mass. We state that $M_B \sim N$. The consequence is that, for higher and higher powers of g^2, the amplitude should be more and more divergent in N. So the seemingly problem of the $1/N$ expansion comes from the N scaling behavior of M_B. To be exact, consider $M_B = Nf(g, M)$ where M is the bare quark mass, then if $f(g, M) = M(1 + g^2)$ the amplitude is:

$$e^{-itNM(1+g^2)} = e^{-iMt}(1 - iMtNg^2 - \frac{1}{2}M^2t^2N^2g^4 + ...). \quad (8)$$

Thus, the diagramatic method does not apply to the baryons. We use Hamiltonian and path integral formalism to analyze this question.

First we consider baryons made from very heavy quarks, that is, their dynamics are non-relativistic. We can write down a Shrodinger equation with the Hamiltonian:

$$H = NM + \sum_i \left(-\frac{\nabla_i^2}{2M}\right) - \frac{g^2}{N}\sum_{i<j}\frac{1}{|x_i - x_j|}. \tag{9}$$

Here we take M to be the single, bare quark mass. The quark-quark interaction is of order $1/N$. Here perturbation theory with respect to the Coulumb potential term is not legitimate, for there are $\sim N^2$ terms in the sum over quark pairs.

Instead, our approximation is the so-called Hatree-Fock approximation. Here in the large N limit, the quark-quark interaction is negligible, however, the total potential experienced by one of the quarks is of order one.

To find the groud state of baryons, each quark should be in the groud state of the average potential. We use the ansatz:

$$\psi(x_1, \ldots, x_N) = \prod_{i=1}^{N}\phi(x_i), \tag{10}$$

for many-body wave function.

We use the variational principle to determine the wave function. We define ϵ as the average energy per quark, we calculate $\langle\psi|H - N\epsilon|\psi\rangle$:

$$NM + N\int d^3x\frac{|\nabla\phi(x)|^2}{2M} + \frac{N^2}{2}\left(-\frac{g^2}{N}\right)\int d^3x d^3y\frac{|\phi(x)|^2|\phi(y)|^2}{|x-y|} - N\epsilon\int d^3x|\phi(x)|^2. \tag{11}$$

We vary the energy expectation with respect to ϕ. This gives us the Shrodinger equation:

$$-\frac{\nabla^2}{2M}\phi(x) - g^2\phi(x)\int\frac{d^3y|\phi(y)|^2}{|x-y|} = \epsilon\phi(x). \tag{12}$$

We can convert this to a differential equation and solve it numerically. However, we intend not to do that here, but to point out two take-away message:

1. The baryon masses are of order N.
2. The size of baryons is really independent of N. By size, we mean the spatial profile of the wave function, which is obtained from a N independent equation.

Another important point to address is that the Hatree-Fork approximation ansatz becomes exact in the $N \to \infty$ limit. The reason is that, the large N limit corresponds to a weakly coupled regime. Though the total potential experienced by one quark is of order one, it is the sum of many small terms. In this situation, the fluctuation will be largely suppressed. For example, consider a random variable $A \sim 1$, and it's the sum of N variables $A_i \sim 1/N, \quad i = 1, ..., N$. if the fluctuation is small and independent, we have the variance:

$$\langle A^2\rangle - \langle A\rangle^2 = \langle(A_1 + \delta_1 + ... + A_N + \delta_N)^2\rangle - (A_1 + ... + A_N)^2 \sim N \times \delta^2. \tag{13}$$

This is a very small value, where we have assumed that the one-point expectation of fluctuation vanishes. So it is legitimate to say that quarks experiences averaged, classical potential. A more

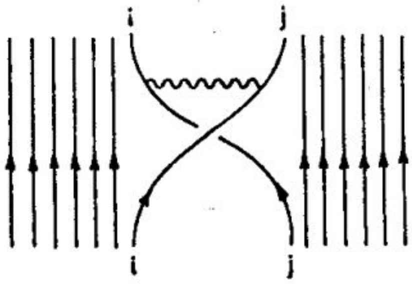

Figure 4: Baryon scattering diagrams with a pair of quarks exchanged and connected by a gluon line.

concrete way to see that is by noticing that the ansatz wave function is the eigenstate of a modified Hamiltonian:

$$\hat{H} = NM + \sum_i \left(-\frac{\nabla_i^2}{2M}\right) + \sum_i V(x_i), \quad V(x) = -g^2 \int \frac{dy|\phi(y)|^2}{|x-y|}. \tag{14}$$

We treat the difference between these two Hamiltonians as a perturbation, and the time-independent perturbation theory will give corrections to the baryon masses negligible in the large N limit. Hence, we argue in the large N limit, the Hatree-Fock wave function is exact.

Scattering of baryons. First we should determine how strong is the baryon-baryon interaction. A typical and leading contribution to scattering amplitudes is dipicted in figure 4. Now we investigate how this diagram scales with N. We have a N for each choice of quarks in the two baryons, and one $1/N$ from the gluon propagator. Thus, this diagram scale as N. It is also good to note that if no gluon line is involved, the exchanged quark should carry the same quantum number, which will contribute only one N in the quark choosing procedure. Also since we do not put gluon line, the diagram also scales as N.

Here we meet the same problem: the baryon-baryon force is growing in proportion to N. This dilemma can also be resolved by realizing that the baryon mass is of order N. The N scaling behavior guarantees that the interaction will not vanish compared to the kinetic energy as $N \to \infty$. Now we try to write down an ansatz for many-body wave functions. Since we are now considering $2N$ quarks with only N colors, we should put N quarks in one state ϕ_1 and the rest of the quarks in another orthogonal state ϕ_2. In each group the wave function is anti-symmetrized in the colors of quarks, and we should anti-symmetrize them with respect to the choice of which N of the $2N$ quarks are put into which group. To be exact, the wave function is:

$$\psi(x_1, ..., x_{2N}) = \sum_P (-1)^P \prod_{i=1}^N \phi_1(x_i, t) \prod_{j=1}^N \phi_2(x_j, t). \tag{15}$$

Note that for many-body dynamics, the ansatz we use is time-dependent.

Also by varying the groud state energy, we have the Schrodinger equation:

$$i\frac{\partial}{\partial t}\phi_1(x,t) = -\frac{\nabla^2}{2M}\phi_1(x,t) - g^2\phi_1(x,t)\int \frac{d^3y|\phi(y,t)|^2}{|x-y|} - g^2\phi_2(x,t)\int \frac{d^3y\phi_2^*\phi_1(y,t)}{|x-y|}. \tag{16}$$

The last term describes the baryon-baryon interaction.

Now let's consider the meson-baryon scattering. First, since we only have to choose once from N quarks, the interaction is of order 1. For baryons, this is negligible compared to the kinetic energy, but the interaction is of the same order as the meson mass. So the mesons are scattered by baryons. The wave function ansatz is:

$$\psi(x_1, ..., x_N, x', y, t) = \sum_P \prod_{i=1}^{N} \phi(x_i, t) u(x', y, t), \tag{17}$$

where $u(x', y, t)$ is the wave function of a quark and an anti-quark.

To lowest order, the equation of motion of ϕ is not influenced by u, but the equation for u is:

$$\begin{aligned} i\frac{\partial}{\partial t}u(x,y,t) = &-\frac{\nabla_x^2}{2M}u(x,y,t) - \frac{\nabla_y^2}{2M}u(x,y,t) - g^2\frac{u(x,y,t)}{|x-y|} \\ &- g^2\phi(x)\int \frac{dz\phi^*(z,t)u(z,y,t)}{|x-z|} - g^2\phi(x)\int \frac{dz\phi^*(z,t)u(z,y,t)}{|z-y|}. \end{aligned} \tag{18}$$

Excited states. So far we only focused on the baryon ground states, now let's look at excited baryon states.

We have seen that the ground state baryon can be described by a mean field approximation, namely the single quark wave function is determined by the averaged potential it experiences. The averaged potential is:

$$V(x) = -g^2 \int \frac{dy|\phi(y)|^2}{|x-y|}. \tag{19}$$

So it's easy to guess that the excited baryon states are given by exciting a few quarks so that they are in the low-lying excited states of the averaged potential. If the excitation energies of this equation are ϵ_k, then the excited baryons will have masses:

$$M_B^* = M_B + \sum n_k \epsilon_k, \tag{20}$$

where n_k is the number of quarks in the kth excited state. So the level spacings of the first few excited states are independent of N.

Now we consider another limit, which is similar to the so-called double-scaled limit. That is, when taking N to infinity, we also take the number of excited quarks to infinity. To be exact, we take pN quarks in an excited state and $(1-p)N$ quarks in the ground state.

As is done above, we still use an ansatz for many-body wave function. We take the single-body excited wave function to be $\phi_1(x)$ and the ground state to be $\phi_0(x)$, then the full wave function is:

$$\psi(x_1, ..., x_N) \sum_P (-1)^P \prod_{i=1}^{pN} \phi_1(x_i) \prod_{j=1}^{(1-p)N} \phi_0(x_j). \tag{21}$$

After that we can follow the footsteps above. However, we wish to have a more elegant way to solve the highly excited baryon states. Let us first return to the one baryon problem, where the wave function and Schrodinger equation are written in (10) and (12), respectively. However, given any initial conditions, we can obtain a set of solutions to time dependent Schrodinger equation, but these solutions do not necessarily describe the excited eigenstate.

Here we do this semiclassically by invoking the DHN formula, that we look for solutions that are periodic in time with period T. We quantize the theory by requiring that the action $\int_0^T \langle\psi|H - i\partial_t|\psi\rangle = 2\pi n$. This gives the energy eigenstates.

Baryons made from light quarks. In the above discussion we assumed the quark mass to be large, so that we only consider non-relativistic dynamics. Here we do this for two dimensional quantum chromodynamics. In 2-dimension spacetime, we can move one degree of freedom for gauge field by gauge fixing. We use path integral methods to study this case.

The action of two dimensional QCD with an arbitrary quark mass is as follows:

$$I = \int d^2x \left[\bar{\psi}_a(i\gamma_\mu\partial^\mu - m)\psi^a - F^a_{b\mu\nu}F^{b\mu\nu}_a - \frac{g}{\sqrt{N}}\bar{\psi}_a\gamma_\mu A^\mu\psi^a\right]. \tag{22}$$

Here we also choose the axial gauge, $A_1 = 0$, and integrate out the remaining component. This reduces to a linear potential, $\frac{g^2}{N}|x-y|$.

We introduce an auxiliary field σ as we did in Gross-Neveu model and $\mathbb{C}P^{N-1}$ model, by adding in the action an extra term:

$$I \to I - (\sigma + \bar{\psi}\psi)^2. \tag{23}$$

The equation of motion is not modified, since if we put σ on shell the action simply reduces to the original one. if we obtain a 4-fermi interaction, now the interaction becomes in the form:

$$\mathcal{L}_{int} = -\sigma^2 - 2\sigma\bar{\psi}\psi. \tag{24}$$

Here since in the effective action the 4-fermi interaction is non-local, we modify this method. Roughly, the on-shell solution of $\sigma(x,y,t)$ should be:

$$\sigma(x,y,t) = \bar{\psi}_i(x,t)\psi^i(y,t). \tag{25}$$

The modified action is:

$$\begin{aligned} I = &\int dxdt\bar{\psi}_a(i\gamma_\mu\partial^\mu - m)\psi^a + \int dxdydt\sigma(x,y,t)\sigma^*(x,y,t) + \frac{g}{\sqrt{N}} \\ &(\int dxdydt\bar{\psi}_a(x,t)\psi^a(y,t)\sigma(x,y,t)\sqrt{|x-y|} + \text{h.c.}). \end{aligned} \tag{26}$$

For baryon two-point function, we calculate:

$$\langle J(x)J^\dagger(0)\rangle = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\sigma\mathcal{D}\sigma^* e^{iI}\psi_1(x)...\psi_N(x)\psi_1^\dagger(0)...\psi_N^\dagger(0). \tag{27}$$

After integrating out the Dirac fermions, we obtain a effective action for σ:

$$\begin{aligned} &\int d\sigma d\sigma^* \exp N\left(\operatorname{Tr}\log(i\gamma_\mu\partial^\mu - g\sigma) + i\int dxdydt\sigma^*\sigma + \log S(x,0;g\sigma)\right) \\ =&\int d\sigma d\sigma^* \exp N\Gamma(g,\sigma). \end{aligned} \tag{28}$$

where $S(x, 0; g\sigma)$ is a quark propagator from 0 to x in the backgroud of field σ. Here we've rescaled $\sigma/\sqrt{N} \to \sigma$.

The importance is that here N is in the place of $1/\hbar$. So the large N limit should be suppressing all the quantum fluctuations away from the classical configuration. So expanding near the classical solution σ_{cl}, we have the heuristic result:

$$\frac{\exp iN\Gamma(g, \sigma_{cl})}{\det\left(\frac{\delta\Gamma}{\delta\sigma(x)\delta\sigma(y)}\right)}. \tag{29}$$

In the 't Hooft model, the effective action and the one-loop determinant can be obtained quite easily by Schwinger-Keldysh equations. Here the bi-local field σ has the quantum number of a meson, hence the procedure should be identical to what 't Hooft has done. However, if we include the baryon, this one-loop determinant is hard to obtain. What we can say is that the propagation of this effective meson is modified by the presence of the baryon.

Baryons as the monopoles of QCD. Now let's move our focus to the mass and size of baryons. The take-home message tells us that:

$$M_B \sim N, \quad \text{Size of baryons is } N \text{ independent.} \tag{30}$$

This seems like a typical feature of a soliton, like a 't Hooft-Polyakov monopole where the size independent of coupling and energy which scales as $1/g^2$. Here the effective coupling of large N QCD is $g_s \sim \frac{1}{N}$. In AdS/CFT, this is referred to as the coupling dictionary between large N CFTs and strings.

Back to where we were, this suggests baryons are a rather special kind of soliton. Indeed, soliton configuration is a classical solution, and the large N limit here corresponds to a classical limit of QCD. In QCD string context, they correspond to D-branes. Pictorially, the baryon is a vertex on which N QCD flux tubes can end.

3 Large N matrix models and quantum gravity

From the QCD discussion, we've seen that in the $N \to \infty$ limit, the theory goes to classic. We do not have quark loops, quarks in baryons sit in their mean-field ground state.

In this section, we turn to a simpler gauge theory: matrix models. For large N matrix models, all gauge invariant observables are given by their classical value. The field configuration here is called the **master fields**. Also, it is conjectured that QCD has its string dual [4], and a 2D non-critical string dual of random matrices is explored a long time ago [5] [6]. More recently, a duality between 2D quantum Jackiw-Teitelboim gravity and matrix integrals was established by Saad, Stanford and Shenker [7]. In the quantum chaos context, random matrix theory characterize appropriate observables in chaotic quantum systems, like the level statistics and the random time evolution [8]. You might refer to [9] for a comprehensive introduction to random matrices.

3.1 Motivation: worldsheet theory of quantum gravity

We consider some matter scalar fields X are coupled to a two dimensional surface Σ with metric g, with a cosmological constant term:

$$Z = \int \frac{\mathcal{D}g\mathcal{D}X}{\mathbf{Vol}(Diff)} \exp\left[-S_M(X,g) - \frac{\mu_0}{8\pi}\int d^2\xi\sqrt{g}\right]. \tag{31}$$

For a matter field coupled to two dimensinal gravity, we now argue that the action must be conformal. This can be seen by noticing that in two dimensions, the trace of the stress tensor must be zero. For the free bosonic string, the action will be $S_M = \frac{1}{8\pi}\int d^2\xi\sqrt{g}g^{ab}G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$. The X fields now are the embedding of Σ into a D-dimensional spacetime.

This path integral is not defined to be satisfactory. We justify this integral by specifying its measure. The measures are determined by:

$$\begin{aligned} \mathcal{D}_g\delta X e^{-||\delta X||_g^2} &= 1; \\ \mathcal{D}_g\delta g e^{-\frac{1}{2}||\delta g||_g^2} &= 1. \end{aligned} \tag{32}$$

These two measures are invariant under diffeomorphisms but not necessarily under conformal transformations $g \to e^\sigma g$:

$$\mathcal{D}_{e^\sigma g}X = e^{\frac{D}{48\pi}S_L(\sigma)}\mathcal{D}_g X, \tag{33}$$

where $S_L(\sigma)$ is the **Liouville action**:

$$S_L(\sigma) = \int d^2\xi\sqrt{g}\left(\frac{1}{2}g^{ab}\partial_a\sigma\partial_b\sigma + R\sigma + \mu e^\sigma\right). \tag{34}$$

3.2 Master fields

We refer to [10] in this subsection.

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