Notes on Liouville Field Theory and 2D String Theory

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This is a note on some aspects of Liouville field theory and non-critical string theory. The main references are [1–3].

1 Liouville field theory from 2D quantum gravity

We consider the Polyakov path integral of 2D gravity coupled to some scalar field $X^\mu, \quad \mu = 1, ..., D$, or a typical bosonic string theory:

$$Z = \int [dX][dg] \exp\left(-S_P - \mu_0 \int_\Sigma d^2\sigma\sqrt{g}\right). \tag{1}$$

Here Σ is the world-sheet and g is the dynamical metric on this world-sheet. S_P is the Polyakov action:

$$S_P = \frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{g}g^{ab}\partial_a X\cdot\partial_b X. \tag{2}$$

This theory has two world-sheet symmetries: diffeomorphism symmetry and Weyl invariance. In textbook string theory, we view both of these symmetries as gauge redundancies. We can fix the metric to unit form, and the integration over g is decomposed to integrals over diffeomorphisms v^a, Weyl mode φ, and the moduli t. After the change of variables, the measure is the Faddeev-Popov determinant which can be expressed with bc ghosts:

$$Z = \int [dt][d\varphi][dbdc][dX]\exp\left(-S_P - S_{gh}\right), \quad S_{gh} = \int d^2z\sqrt{g}(b\overline{\nabla}c + \tilde{b}\nabla\tilde{c}). \tag{3}$$

In non-critical string theory, we have non-zero Weyl anomalies, which can be deduced from the central charge of "matter" fields and ghost fields:

$$c_{tot} = c_X + c_{gh} = D - 26. \tag{4}$$

In $D \neq 26$ we do not view Weyl transformation as a gauge invariance. Rather, we view the Weyl mode as an independent dynamical degree of freedom. The path integral is now written as:

$$Z = \int [dt][d\varphi]_{\hat{g}e^\varphi}[dbdc]_{\hat{g}e^\varphi}[dX]_{\hat{g}e^\varphi}\exp\left(-S_P[X,\hat{g}] - S_{gh}[b,c,\hat{g}]\right). \tag{5}$$

One can calculate (or guess) the Weyl anomalies in these integration measures, and we get the famous Liouville action:

$$S_L[\phi, g] = \frac{1}{4\pi}\int d^2z\sqrt{g}\left(g^{ab}\partial_a\phi\partial_b\phi + \phi QR + 4\pi\mu e^{2b\phi}\right). \tag{6}$$

The partition function is now:

$$Z = \int [dt][d\varphi]_{\hat{g}}[dbdc]_{\hat{g}}[dX]_{\hat{g}} \exp\left(-S_P[X,\hat{g}] - S_{gh}[b,c,\hat{g}] - S_L[\phi,\hat{g}]\right). \tag{7}$$

The final answer of this partition function should be independent of the gauge choice. So if we do a local rescaling $\hat{g} \to e^{\sigma}\hat{g}$ of the fixed metric, and the Liouville field ϕ is also shifted by σ, the result should be the same. For this to be a symmetry, the total central charge of the theory should vanish:

$$c_{tot} = c_X + c_{gh} + c_L = 26 - D + c_L = 0. \tag{8}$$

We use this condition and match the Weyl anomaly to determine Q. Under infinitesimal change of metric δg_{ab}, the change in a typical expectation value is:

$$\delta\langle ...\rangle = -\frac{1}{4\pi}\int d^2\sigma\sqrt{g}\delta g_{ab}\langle T^{ab}...\rangle + \text{contact terms}. \tag{9}$$

For a Weyl transformation, $\delta g_{ab} = 2g_{ab}\delta\omega$. We get:

$$\delta_W\langle ...\rangle = -\frac{1}{2\pi}\int d^2\sigma\sqrt{g}\delta\omega\langle T^a_a...\rangle + \text{contact terms}. \tag{10}$$

We know for a given theory with a energy-momentum tensor, the Weyl anomaly $T^a_a = -\frac{c}{12}R$. Let's do the calculation in conformal gauge, $g_{ab} = e^{2\omega}\delta_{ab}$, then the Ricci scalar $R = -2\nabla^2\omega = -2e^{-2\omega}\delta_{ab}\partial^a\partial^b\omega$. Then the partition function is the case where $... = 1$:

$$\delta_W Z[e^{2\omega}\delta] = \frac{c}{12\pi}Z[e^{2\omega}\delta]\int d^2\sigma\delta\omega\delta_{ab}\partial^a\partial^b\omega. \tag{11}$$

This is integrated to be:

$$Z[e^{2\omega}\delta] = Z[\delta]\exp\left(-\frac{c}{24\pi}\int d^2\sigma\omega\nabla^2\omega\right). \tag{12}$$

Now let's calculate the conformal anomaly of a linear dilaton theory:

$$S = \frac{1}{4\pi}\int d^2\sigma\sqrt{g}\left(g^{ab}\partial_a\phi\partial_b\phi + QR\phi\right). \tag{13}$$

This is very much alike the Liouville theory except for the absence of an exponential potential. We note that the exponential potential term does not contribute in the energy-momentum tensor. So we only need to worry about the linear dilaton part of the action to obtain its central charge.

Now let's see how to calculate correlation functions on the sphere in linear dilaton theory [1]. Similar to the bosonic string calculation, we consider first the generating functional:

$$\begin{aligned} Z[J] &= \left\langle \exp\left(i\int d^2\sigma J(\sigma)\phi(\sigma)\right)\right\rangle \\ &= \int [d\phi]\exp\left[-\frac{1}{4\pi}\int d^2\sigma\sqrt{g}(g^{ab}\partial_a\phi\partial_b\phi + QR\phi) + i\int d^2\sigma J(\sigma)\phi(\sigma)\right]. \end{aligned} \tag{14}$$

We expand the ϕ fields in the eigenbasis of Laplacian:

$$\phi(\sigma) = \sum_I \phi_I\Phi_I(\sigma), \quad \nabla^2\Phi_I = -\omega_I^2\Phi_I, \tag{15}$$

with normalization:

$$\int d^2\sigma\sqrt{g}\Phi_I(\sigma)\Phi_{I'}(\sigma) = \delta_{II'}. \tag{16}$$

We also define other components:

$$J_I = \int d^2\sigma J(\sigma)\Phi_I(\sigma), \quad R_I = \int d^2\sigma\sqrt{g}R(\sigma)\Phi_I(\sigma). \tag{17}$$

Then the generating functional can be written as:

$$Z[J] = \int \prod_I d\phi_I \exp\left(-\frac{\omega_I^2 x_I^2}{4\pi} + i\phi_I J_I + \frac{QR_I\phi_I}{4\pi}\right). \tag{18}$$

The integral is Gaussian except for constant modes ϕ_0 which gives a Dirac delta function:

$$\begin{aligned}&\int d\phi_0 \exp\left[i\phi_0\left(J_0 - i\frac{QR_0}{4\pi}\right)\right]\\ =&\int d\phi_0 \exp\left[i\phi_0\left(J_0 - 2iQ\Phi_0\right)\right] = 2\pi\delta(J_0 - 2iQ\Phi_0).\end{aligned} \tag{19}$$

Here we've used Gauss-Bonnet theorm: $\int d^2\sigma\sqrt{g} = 4\pi\chi$. And non-zero mode integral gives:

$$\begin{aligned}&\prod_{I\neq 0}\int d\phi_I \exp\left(-\frac{\omega_I^2 x_I^2}{4\pi} + i\phi_I J_I + \frac{QR_I\phi_I}{4\pi}\right)\\ =&\prod_{I\neq 0}\left(\frac{4\pi^2}{\omega_I^2}\right)^{\frac{1}{2}} \exp\left[-\frac{\pi}{\omega_I^2}\left(J_I - \frac{iQR_I}{4\pi}\right)^2\right].\end{aligned} \tag{20}$$

$$\begin{aligned}Z[J] =&2\pi\delta(J_0 - 2iQ\Phi_0)\left(\det\frac{-\nabla^2}{4\pi^2}\right)^{-1/2}\\ &\times\exp\left[-\sum_{I\neq 0}\frac{\pi}{\omega_I^2}\int d^2\sigma d^2\sigma'\left(J(\sigma) - \frac{iQR(\sigma)}{4\pi}\right)\Phi_I(\sigma)\Phi_I(\sigma')\left(J(\sigma') - \frac{iQR(\sigma')}{4\pi}\right)\right]\\ =&2\pi\delta(J_0 - 2iQ\Phi_0)\left(\det\frac{-\nabla^2}{4\pi^2}\right)^{-1/2}\\ &\times\exp\left[-\frac{1}{2}\int d^2\sigma d^2\sigma'\left(J(\sigma) - \frac{iQR(\sigma)}{4\pi}\right)G'(\sigma,\sigma')\left(J(\sigma') - \frac{iQR(\sigma')}{4\pi}\right)\right].\end{aligned} \tag{21}$$

Here we define Green's function:

$$G'(\sigma,\sigma') = \sum_{I\neq 0}\frac{2\pi}{\omega_I^2}\Phi_I(\sigma)\Phi_I(\sigma'), \tag{22}$$

which satisfies the following differential equation:

$$\nabla^2 G'(\sigma,\sigma') = -2\pi\delta(\sigma-\sigma')g^{-1/2} - 2\pi\Phi_0^2. \tag{23}$$

On the sphere, Green's function is solved to be:

$$G'(\sigma,\sigma') = -\frac{1}{2}\log|z_{12}|^2 + f(z_1,\bar{z}_1) + f(z_2,\bar{z}_2). \tag{24}$$

The two additional functions will drop out in the end. For vertex operator expectation value:

$$\left\langle \left[e^{ik_1\phi}(\sigma_1)\right]_r \cdots \left[e^{ik_n\phi}(\sigma_n)\right]_r \right\rangle, \tag{25}$$

we have $J(\sigma) = \sum_{i=1}^n k_i \delta(\sigma - \sigma_i)$, and $J_0 = \Phi_0 \int d^2\sigma J(\sigma) = \Phi_0 \sum_i k_i$. The generating functional is now:

$$\begin{aligned} Z[J] \sim& \delta(\sum k - 2iQ) \exp\left(-\sum_{i<j} k_i k_j G'(\sigma_i, \sigma_j) - \frac{1}{2}\sum_i k_i^2 G'_r(\sigma_i, \sigma_i)\right) \\ &\times \exp\left[\int d^2\sigma \sum_j \frac{iQR(\sigma)}{4\pi} G'(\sigma, \sigma_j) k_j\right] \\ &\times \exp\left[\frac{1}{2}\int d^2\sigma d^2\sigma' Q^2 \frac{R(\sigma)R(\sigma')}{16\pi^2} G'(\sigma, \sigma')\right]. \end{aligned} \tag{26}$$

Now we want to find the change of measure, so we set all momentum to zero, and the metric to $e^{2\omega}\delta$, $R = -2\nabla^2\omega$. The Weyl anomaly is:

$$\begin{aligned} Z[e^{2\omega}\delta] \sim& \exp\left[\frac{1}{2}\int d^2\sigma d^2\sigma' Q^2 \frac{\nabla^2\omega(\sigma)\nabla^2\omega(\sigma')}{4\pi^2} G'(\sigma, \sigma')\right] \\ =& \exp\left[\frac{1}{2}\int d^2\sigma d^2\sigma' Q^2 \frac{\omega(\sigma')\nabla^2\omega(\sigma)}{4\pi^2} \nabla^2 G'(\sigma, \sigma')\right] \\ =& \exp\left[-\frac{1}{4\pi}\int d^2\sigma d^2\sigma' \omega(\sigma')\nabla^2\omega(\sigma)\delta^2(\sigma - \sigma')\right] \\ =& \exp\left[-\frac{1}{4\pi}\int d^2\sigma \omega(\sigma)\nabla^2\omega(\sigma)\right]. \end{aligned} \tag{27}$$

Matching this with (12), we get the central charge (note that the free 2D scalar itself has central charge 1):

$$c_L = 1 + 6Q^2 = 25 - D, \quad Q = \sqrt{\frac{25 - D}{6}}. \tag{28}$$

Now let's try to determine b. This can be simply done by calculating the $Te^{2b\phi}$ OPE, where we require $e^{2b\phi}$ to be a $(1,1)$ tensor:

$$T = -\partial\phi\partial\phi + Q\partial^2\phi. \tag{29}$$

The OPE can be calculated:

$$T(z)e^{2b\phi}(w) = \frac{\Delta e^{2b\phi}(w)}{(z-w)^2} + \ldots = \frac{-b^2 + bQ}{(z-w)^2}, \quad \Delta = b(Q - b) = 1. \tag{30}$$

Then we have:

$$Q = b + \frac{1}{b}. \tag{31}$$

We can also solve for b:

$$b = \frac{Q}{2} - \frac{1}{2}\sqrt{Q^2 - 4}. \tag{32}$$

So if $Q^2 - 4 = \frac{1-D}{6} \geq 0$, $D \neq 1$, b and Q are all real. When $c_X = D = 1$, there is a phase transition, and the theory at this point is the so-called $c = 1$ string theory.

For a general spinless primary field Φ_0. After couping to gravity, the operator should be "dressed" to consistently couple to gravity. The dressed operator is $\Phi = \Phi_0 e^{2\alpha\phi}$. To couple to gravity, we require that $\Delta(\Phi) = \Delta(\Phi_0) + \alpha(Q - \alpha) = 1$. This can be used to determine the gravitational dressing α.

1.1 c=1 string theory as a 2D critical string

The $c = 1$ string theory also has a critical string interpretation. To establish this point, let's first review string theory in general spacetime backgrounds and dimensions. In this case, an effective theory is that the string is coupled to external field excitations: tachyon field T, graviton $G_{\mu\nu}$, and dilaton Φ. Here we've assumed the Kalb-Ramond B field to be zero. The action can be written as:

$$S_\sigma = \frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{g}\left[g^{ab}G_{\mu\nu}(X)\partial_a X^\mu\partial_b X^\nu + R\Phi(X)\right]. \tag{33}$$

In the remaining of this section, we temporarily recover α' to illustrated some points more clearly. To the leading order in α' of the beta functions, we quote the results from eq.(3.7.14) in [4]:

$$\begin{aligned}\beta^G_{\mu\nu} =& \alpha'\mathbf{R}_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\Phi,\\ \beta^\Phi =& \frac{D-26}{6} - \frac{1}{2}\alpha'\nabla^2\Phi + \alpha'\nabla_\mu\Phi\nabla^\mu\Phi.\end{aligned} \tag{34}$$

A condition that the world-sheet theory is Weyl-invariant is:

$$\beta^G_{\mu\nu} = \beta^\Phi = 0. \tag{35}$$

This equality must be held at every order in α', so we need to find a background such that the beta functions all vanish. The simplest choice is that $D = 26$, the spacetime metric is flat, and the dilaton takes constant value. This returns to the critical string theory in $D = 26$ flat space.

A more non-trivial solution is:

$$G_{\mu\nu} = \eta_{\mu\nu}, \quad \Phi(X) = V_\mu X^\mu. \tag{36}$$

This theory is just a linear dilaton CFT coupled to the world-sheet gravity. According to our previous discussion, its central charge is $c = 1 + 6V_\mu V^\mu$. To make it free of Weyl anomaly, we can again get the condition:

$$V_\mu V^\mu = \frac{26-D}{6\alpha'}. \tag{37}$$

In $D \neq 26$, we can schematically take $26 - D$ of the spacetime dimensions compactified. We now take $D = 2$, where this theory can actually be solved. Since when $D < 26$, the gradient of Φ is spacelike, we take V^μ to have only one non-zero component at X^1 direction.

A small draw-back of this prescription is that the string coupling e^Φ becomes strong at large X^1. We now introduce again the tachyon field T. From tree-level string scattering amplitude, the effective action for tachyon field is [1, 4]:

$$S_T = -\frac{1}{2}\int d^D x\sqrt{-G}e^{-2\Phi}\left[G^{\mu\nu}\partial_\mu T\partial_\nu T - 4T^2\right]. \tag{38}$$

Inserting the background $\Phi = V_\mu X^\mu$, one can get the tachyon equation of motion:

$$-\partial^\mu\partial_\mu T(X) + 2V^\mu\partial_\mu T(X) - 4T(X) = 0. \tag{39}$$

The solutions are:

$$T(X) = \exp(q\cdot X), \quad (q-V)^2 = \frac{2-D}{6\alpha'}. \tag{40}$$

For $D = 2$, the solution is $q_0 = 0, q_1 = 2(\alpha')^{-1/2}$. Then $T(X) = \exp\Phi(X)$. Then we could see that X^1, or Φ, is now an interacting field theory with exponential potential and linear coupling to the Ricci curvature. This justifies our statement that the Liouville field is the spatial dimension of the 2D target space in $D = 2$ critical string theory. The exponential potential also prevents propagation to large X^1.

2 Semiclassical Liouville theory

Now we consider a semiclassical way to quantize the Liouville theory. Before all the discussion below we need to first define the semiclassical limit. If we rescale $\phi \to \frac{1}{2b}\phi$, then $\frac{1}{b}$ can be identified as the coupling constant of the theory. So $b \to 0$ is the semiclassical limit, where most of our calculation will be done in this section and the theory is most extensively studied. The most practical and obvious simplification is that we can do perturbative calculation with respect to μ here.

First, let's discuss the canonical quantization method on a cylinder which is mapped from a complex plane via the standard mapping:

$$z = e^{-iw}, \quad w = \sigma + i\tau = \sigma + t. \tag{41}$$

The Lagrangian is:

$$L = \frac{1}{4\pi}\partial^a \phi \partial_a \phi + \mu e^{2b\phi}. \tag{42}$$

The canonical momentum is:

$$\Pi = \frac{\partial L}{\partial \dot{\phi}} = \frac{1}{2\pi}\partial_t \phi. \tag{43}$$

We Fourier expand the canonical momentum and ϕ field:

$$\begin{aligned} \phi(\sigma, t) &= q + \sum_{n \neq 0} \frac{i}{n}[a_n(t)e^{-in\sigma} + b_n(t)e^{in\sigma}]; \\ \Pi(\sigma, t) &= p + \sum_{n \neq 0}[a_n(t)e^{-in\sigma} + b_n(t)e^{in\sigma}]. \end{aligned} \tag{44}$$

We impose the equal-time commutator:

$$[\phi(\sigma), \Pi(\sigma')] = i\delta(\sigma - \sigma'). \tag{45}$$

Using the mode expansion, this is re-written as:

$$[q, p] = i, \quad [a_n, a_m] = \frac{n}{2}\delta_{n,-m}, \quad [b_n, b_m] = \frac{n}{2}\delta_{n,-m}. \tag{46}$$

The Hamiltonian of the system on a cylinder is:

$$H = \frac{1}{2}\int_0^{2\pi} \frac{d\sigma}{2\pi}\left(T_{ww} + T_{\bar{w}\bar{w}}\right). \tag{47}$$

The new energy-momentum tensor can be obtained with the corresponding anomalous transformation rule:

$$T_{ww} = \left(\frac{\partial z}{\partial w}\right)^2 T_{zz} + \frac{c}{24} = -e^{-2iw}T_{zz} + \frac{1 + 6Q^2}{24}. \tag{48}$$

Since $z = \sigma + t$, $\bar{z} = \sigma - t$, we have:

$$\partial = \partial_z = \frac{1}{2}(\partial_\sigma + \partial_t), \quad \bar{\partial} = \partial_{\bar{z}} = \frac{1}{2}(\partial_\sigma - \partial_t). \tag{49}$$

So we have [2,3]:

$$
\begin{aligned}
T_{\pm\pm} =& \frac{1}{4}(\partial_\sigma\phi \pm 2\pi\Pi)^2 - \frac{Q}{4}(\partial_\sigma^2 + \partial_t^2 \pm 2\partial_t\partial_\sigma\phi) + \frac{1+6Q^2}{24} \\
=& \frac{1}{4}(2\pi\Pi \pm \partial_\sigma\phi)^2 - \frac{Q}{2}\partial_\sigma(2\pi\Pi \pm \phi) + Q\pi\mu b e^{2b\phi} + \frac{1+6Q^2}{24}.
\end{aligned} \tag{50}
$$

Using these set of operators, the Hamiltonian takes the form:

$$
H = \frac{1}{2}p^2 + 2\sum_{k>0}[a_{-k}a_k + b_{-k}b_k] + \mu\int_0^{2\pi} e^{2b\phi}. \tag{51}
$$

In the $b \to 0$ semiclassical limit, the interacting between the oscillating modes are suppressed. And if we solve the theory containing only the zero modes, we can act on the vacuum with the two set of operators to build the semiclassical Hilbert space:

$$
\mathcal{H} = \oplus_E \mathcal{F}_{E,Q} \otimes \overline{\mathcal{F}}_{E,Q}. \tag{52}
$$

Combining all the above discussion, the question is reduced to the quantum mechanics of the zero mode. The approximation throwing away the oscillation of the Liouville field is called the "mini-superspace" approximation. However, in the $c = 1$ string case, $b = 1$ and the theory is thus away from this regime. This means we cannot perturbatively solve this theory. We'll go to the $c = 1$ theory in later sections.

2.1 Solving the Liouville quantum mechanics

The quantum mechanical system has Hamiltonian:

$$
H_0 = \frac{1}{2}p^2 + 2\pi\mu e^{2bq} + \frac{Q^2}{4} = -\frac{1}{2}\partial_q^2 + 2\pi\mu e^{2bq} + \frac{Q^2}{4}. \tag{53}
$$

As $q \to -\infty$, the exponential interaction disappears. We label the state with a continuous parameter: momentum p. Then the eigenstates are just plane waves with eigenvalue Δ. More generally, the wave function can be written as:

$$
\psi_p(q) \sim e^{2ipq} + R(p)e^{-2ipq}, \quad p > 0, \quad \Delta(p, Q) = 2p^2 + \frac{Q^2}{4}. \tag{54}
$$

Here $R(p)$ is a reflecting coefficient.

For simplicity, we solve this in the "circumference of the universe" basis, $\ell = e^{bq}$. Then the eigenvalue equation goes to:

$$
\left[-b^2\left(l\frac{\partial}{\partial\ell}\right)^2 + 2\pi\mu\ell^2 + \frac{Q^2}{4}\right]\psi_p(l) = \left(2p^2 + \frac{Q^2}{4}\right)\psi_p(l). \tag{55}
$$

This equation is nothing but the Bessel equation, with appropriate boundary condition, the solution is:

$$
\psi_p(q) = \frac{1}{\pi}\sqrt{\frac{p}{b}\sinh\frac{2\pi p}{b}}K_{2ip/b}\left(\frac{\sqrt{2\pi\mu}\ell}{b}\right). \tag{56}
$$

Using an asymptotic expansion as $b \to 0$, we can get the reflection phase:

$$R(p) = -(\pi\mu b^{-2})^{-\frac{2ip}{b}} \frac{\Gamma(1+2ib^{-1}p)}{\Gamma(1-2ib^{-1}p)}. \tag{57}$$

We now try to understand the corresponding operator on the complex plane which can be mapped to this state.

For state with momentum p, the corresponding operator is conjectured to be $V_p(z,\bar{z}) = e^{2\alpha\phi} = e^{2ip\phi+Q\phi}$. Note that this is not a one-to-one correspondence. As an argument, we notice that the TV_p OPE can give its weight $\alpha(Q-\alpha) = \frac{Q^2}{4} + p^2$, and thus the energy:

$$H_0 = L_0 + \overline{L}_0 - \frac{c}{12} = 2p^2 - \frac{1}{12}. \tag{58}$$

This coincides with the asymptotic behavior of energy of the state with momentum p. So only states with $\alpha = \frac{Q}{2} + ip$ exist in the theory. However, let's consider the operator which measures the volume of the universe:

$$A = \int_\Sigma e^{2b\phi}. \tag{59}$$

The operator $e^{2b\phi}$ has an imaginary momentum, which corresponds to non-normalizable state. To further elaborate on this point, let's return to examine the semi-classical theory.

3 Semiclassical states revisited

In this section we try to make sense of the non-normalizable states dual to operators in the form $e^{\alpha\phi}$. In ordinary CFT, the state dual to the primary operator $\mathcal{O}(z,\bar{z})$ is obtained by acting the operator valued at $z=0$ to the vacuum state $|0\rangle$. This gives you the mapping between the action of Virasoro generators on primary operators and the corresponding states:

$$L_0|\mathcal{O}\rangle = \bar{L}_0|\mathcal{O}\rangle = \Delta_\mathcal{O}|\mathcal{O}\rangle; \quad L_n|\mathcal{O}\rangle = \bar{L}_n|\mathcal{O}\rangle = 0, \quad n>0. \tag{60}$$

One can also get the descendant states by acting $L_n (n>0)$ on the primary state.

However, in Liouville field theory this naive approach is not feasible, owing to the fact that $|p=0\rangle$ is not actually in the Hilbert space. Alternatively, one could construct the state corresponding to $\mathcal{O}$ by doing a path integral over a disk with an $\mathcal{O}$ insertion in the center. By specifying the boundary condition on the boundary, one can obtain the wavefunction of this state in the basis of field configuration. The inner product is defined to be the path integral on the sphere obtained by gluing two disks. This is somewhat similar to the Hartle-Hawking construction of wavefunctions of the universe [5], as is shown in fig.1. Of course the path integral on the disk is divergent. We could cutoff the wave function $\Psi_\mathcal{O}(\phi)$ at some particular point ϕ_0. As the regulator is removed, we could either keep the norm of $\Psi_\mathcal{O}$ finite and have $\Psi_\mathcal{O}(\phi_0) \to 0$, or we hold $\Psi_\mathcal{O}(\phi_0)$ fixed and let the norm diverge. We will return to this point in section 4.

Having clarified the definition of these states, let's now discuss what classical geometry these states describe.

3.1 Classical Liouville solutions

For $Q = \frac{1}{b}$, the action (6) defines a classical conformal field theory, the corresponding Weyl transformation is:

$$\hat{g} \to e^\sigma \hat{g}, \quad \phi \to \phi - \sigma/2b. \tag{61}$$

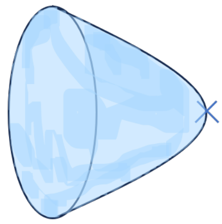

Figure 1: Illustration of Hartle-Hawking's way to prepare a state.

Then $R[\hat{g}] \to e^{-\sigma}(R[\hat{g}] - \nabla^2\sigma)$. The action transforms as:

$$\begin{aligned} S_L \to &\frac{1}{4\pi}\int d^2z\sqrt{g}e^{\sigma}\left[e^{-\sigma}g^{ab}\left(\partial_a\phi - \frac{1}{2b}\partial_a\sigma\right)\left(\partial_b\phi - \frac{1}{2b}\partial_b\sigma\right)\right. \\ &\left. + Q\, e^{-\sigma}(R - \nabla^2\sigma)(\phi - \sigma/2b) + 4\pi\mu e^{-\sigma}e^{2b\phi}\right] = S_L. \end{aligned} \tag{62}$$

So this defines a classical conformal field theory, and matches the reduction to $b \to 0$ of the quantum theory.

Also, note that now Weyl transformation acts non-linearly on ϕ, which means that ϕ can now be seen as a Goldstone boson emerging from the spontaneously broken Weyl symmetry by the choice of $\hat{g}$. The classical equation of motion is given by:

$$R[e^{2b\phi}\hat{g}] = -8\pi b^2\mu. \tag{63}$$

So the classical solution of Liouville theory describes a surface with constant negative curvature. A standard solution is the Poincare half plane:

$$ds^2 = e^{2b\phi}|dz|^2 = \frac{1}{4\pi b^2\mu}\frac{1}{(\mathrm{Im}z)^2}|dz|^2. \tag{64}$$

According to the uniformization theorem, every Riemann surface is conformally equivalent to:

1. CP^1, the Riemann sphere;

2. H, the Poincare half-plane;

3. H/Γ, where Γ is a discrete subgroup of the Mobius group $SL(2,\mathbb{R})$.

In general, for a Riemann surface X obtained by H/Γ, there is a projection map $\pi: \quad H \to X$, and an inverse, uniformization map:

$$f: \quad X \to H. \tag{65}$$

On the Riemann surface X, the metric takes a general form:

$$ds^2 \propto \frac{1}{\mu}\frac{\partial A\bar{\partial}B|dz|^2}{(1 - A(z)B(\overline{z}))^2}, \tag{66}$$

If we move the point z around a circle, the functions A, B will change under the conjugation of $SL(2,\mathbb{R})$. Based on the conjugacy class of the monodromy of these functions, the local geometry has three possibilities (also shown in fig.2):

1. Elliptic solutions: $A = z^a$, $B = \bar{z}^a$, $a \in \mathbb{R}$. In this situation the monodromy of A is $A \to e^{2ia\pi}A$. The metric takes the form:

$$ds^2 = e^{2b\phi}|dz|^2 \sim \frac{1}{\mu}\frac{a^2|dz|^2}{(z\bar{z})^{1-a}[1-(z\bar{z})^a]} \sim \frac{a^2}{\mu}\frac{d\sigma^2 + dt^2}{\sinh^2 at}. \tag{67}$$

This solution has a curvature singularity at $z = 0$

2. Parabolic solution: $A = i\log z$, $B = \frac{i}{\log\bar{z}}$. In this situation the monodromy of A is $A \to A - 2\pi$. The metric takes the form:

$$ds^2 \sim \frac{1}{\mu}\frac{|dz|^2}{z\bar{z}(\log z\bar{z})^2} \sim \frac{1}{\mu}\frac{d\sigma^2 + dt^2}{t^2}. \tag{68}$$

This geometry also has curvature singularity at $z = 0$.

3. Hyperbolic solution: $A = z^{im}$, $B = \bar{z}^{im}$, $m \in \mathbb{R}$. In this situation the monodromy of A is $A \to e^{-2\pi m}A$. The metric takes the form:

$$ds^2 \sim \frac{1}{\mu}\frac{m^2|dz|^2}{z\bar{z}\left[\sin\left(\frac{m}{2}\log(z\bar{z})\right)\right]^2} \sim \frac{1}{\mu}\frac{m^2(dt^2 + d\sigma^2)}{\sin^2(mt)}, \quad 0 < t < \frac{\pi}{m}. \tag{69}$$

For elliptic solutions, they indeed have imaginary energy, which are just what we discovered to be non-normalizable states. For hyperbolic solutions, they have real energy.

In addition, for the two singular solutions, the Liouville field ϕ also obeys the Liouville equation of motion with a source at $z = 0$, which corresponds to curvature singularity at the origin:

$$\frac{2}{\pi}\partial\bar{\partial}\phi - \frac{1}{4\pi}RQ - 2\mu b e^{2b\phi} + k\delta(z) = 0. \tag{70}$$

Motivated by the geometries illustrated above, the normalizable states were labelled "macroscopic states," and the non-normalizable states were labelled "microscopic states", according to the convention by Seiberg [3]. Semiclassically, the macroscopic states do not have a well-defined insertion point in the intrinsic geometry of the surface. The microscopic states, on the other hand, correspond semiclassically to the elliptic geometry, and thus to local operators — the operator insertion in this case is localized at the tip of the "funnel".

Since the asymptotic behavior of the energy eigenfunction should behave as e^{2ipq}, the $e^{-2ipq} = e^{-2(\alpha - \frac{Q}{2})q}$ should damp out as $q \to -\infty$. Thus we have the bound:

$$\mathrm{Re}\alpha \leq \frac{Q}{2}. \tag{71}$$

This is usually called the Seiberg bound.

4 Semiclassical amplitude

We calculate correlation functions in the form:

$$\left\langle \prod_{i=1}^{n} e^{2\alpha_i\phi(z_i)} \right\rangle = \int [d\phi] e^{-S[\phi]} \prod_{i=1}^{n} e^{2\alpha_i\phi(z_i)}. \tag{72}$$

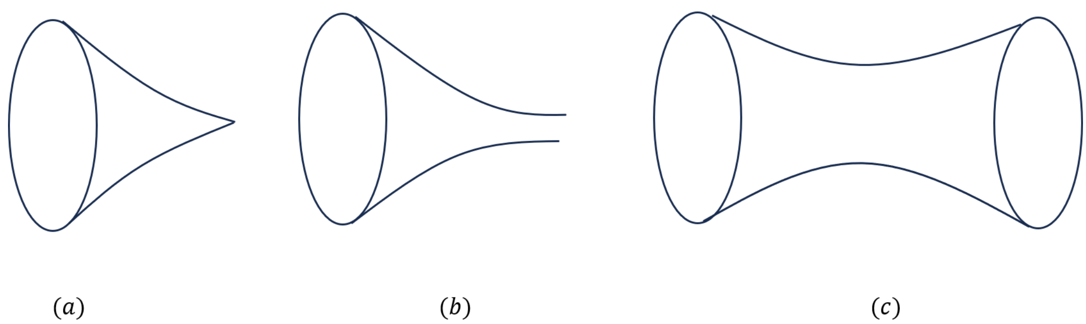

Figure 2: Three local geometries with constant negative curvature. (a): elliptic; (b): parabolic; (c): hyperbolic.

Semiclassically, this is evaluated using a saddle point approximation. The saddle point equation is:

$$\frac{2}{\pi}\partial\bar{\partial}\phi - \frac{1}{4\pi}RQ - 2\mu b e^{2b\phi} + \sum_i 2\alpha_i \delta(z - z_i) = 0. \tag{73}$$

Let's now shift the Liouville field $\phi \to \phi - \frac{\log \mu}{2b}$. This moves the μ dependence out of the Liouville action, and gives the μ scaling relation:

$$\left\langle \prod_{i=1}^{n} e^{2\alpha_i \phi(z_i)} \right\rangle \sim \mu^{\frac{(1-g)Q - \sum_i \alpha_i}{b}}. \tag{74}$$

This is called the KPZ scaling law. This Ward-Takahashi method can also be used to obtain the string susceptibility Γ_{str}, defined as $Z[A] \sim A^{(\Gamma_{\text{str}}-2)\chi/2-1}$. We consider the partition function with fixed area A:

$$Z[A] = \int [d\phi] e^{-S} \delta\Big(\int \sqrt{\hat{g}} e^{2b\phi} - A \Big). \tag{75}$$

We also shift $\phi \to \phi + \frac{\lambda}{2b}$, using again the Gauss-Bonnet theorem, we have:

$$\begin{gathered} \frac{1}{4\pi} \int \sqrt{\hat{g}} QR\phi \to \frac{1}{4\pi} \int \sqrt{\hat{g}} QR\phi + \frac{\lambda \chi Q}{2b} \\ \delta\left(\int \sqrt{\hat{g}} e^{2b\phi} - A \right) \to e^{-\lambda} \delta\left(\int \sqrt{\hat{g}} e^{2b\phi} - e^{-\lambda} A \right). \end{gathered} \tag{76}$$

Then we get:

$$Z[A] = \exp\left(-\frac{\lambda \chi Q}{2b} - \lambda \right) Z[e^{-\lambda} A]. \tag{77}$$

To separate the A dependence, we can choose $\lambda = \log A$, so that we get the scaling law:

$$Z[A] = A^{-\frac{\chi Q}{2b} - 1} Z[1]. \tag{78}$$

In the end we have the string susceptibility:

$$\Gamma_{\text{str}} = 2 - \frac{Q}{b}. \tag{79}$$

One could integrate the equation (73) over the surface Σ and use the Gauss-Bonnet theorem, we get:

$$-Q(2-2h)+2\sum_i \alpha_i - 2\mu b A(\Sigma)=0, \quad s \equiv \frac{1}{b}\left(Q(1-h)-\sum_i \alpha_i\right)<0. \tag{80}$$

It says that classically the KPZ exponent s is negative. After solving the saddle point value ϕ_{cl} one could just plug it into the path integral and obtain the result. However, $-S_{cl}-2\alpha_i\phi_{cl}(z_i)$ is now infinite. The divergence comes from the curvature source at each operator insertion. Near one of the insertion points z_i, the equation of motion (73) has a solution:

$$\phi_{cl}(z) \sim -\alpha_i \log|z-z_i|^2, \quad z \to z_i, \tag{81}$$

which diverges near $z = z_i$. This reflects the need for vertex operator renormalization. We regularize the action as follows:

$$\overline{S}_{cl} = \lim_{\epsilon\to 0}\left(\int_{\Sigma-\cup_i B(\epsilon,z_i)} \mathcal{L} - \sum_i \left(\Delta_i+\overline{\Delta}_i\right)\log\epsilon\right). \tag{82}$$

This also justifies that the divergence in the Hartle-Hawking construction introduced in the last section.

When the KPZ exponent $s = Q(1-h)-\sum_i \alpha_i \geq 0$, there is no classical solutions. In this case, we can use semi-classical method by considering the fixed-area correlation function:

$$\left\langle \prod_{i=1}^n e^{2\alpha_i\phi(z_i)}\right\rangle_A = \int [d\phi] e^{-S[\phi]} e^{\mu A}\prod_{i=1}^n e^{2\alpha_i\phi(z_i)}\delta\left(\int \sqrt{\hat{g}}e^{2b\phi}-A\right). \tag{83}$$

The scaling law of fix-area correlation can be obtained similarly using Ward-Takahashi method, it scales with A as:

$$\left\langle \prod_{i=1}^n e^{2\alpha_i\phi(z_i)}\right\rangle_A \sim A^{-s-1}. \tag{84}$$

One could do a Laplace transformation to get the scaling of the fixed-cosmological constant correlation function:

$$\int_0^\infty \frac{dA}{A}A^{-s}e^{-\mu A} = \mu^{-s}\Gamma(-s). \tag{85}$$

This matches the KPZ scaling law. For $s<0$ the integral converges. However, if $s=0$, the integral diverges logarithmically while for $s>0$ it diverges following the power law, at small area limit as a UV divergence in terms of the length measured by $g = e^{2b\phi}\hat{g}$. We could do a UV cutoff in the integration over g. This gives the result:

$$\left\langle \prod_{i=1}^n e^{2\alpha_i\phi(z_i)}\right\rangle = P(\mu) + C\mu^s \left\langle \prod_{i=1}^n e^{2\alpha_i\phi(z_i)}\right\rangle_{A=1}, \tag{86}$$

where $P(\mu)$ is a polynomial to the degree $[s]$.

We also comment here on the semi-classical meaning of the Seiberg bound. The local geometry around a local operator insertion will take the elliptic form:

$$e^{2b\phi} = \frac{1}{\pi\mu b^2}\frac{a^2}{|z|^{2-2a}(1-|z|^{2a})^2}. \tag{87}$$

The corresponding momentum $p = \lim_{t\to-\infty} \partial_t \phi = i\left(\alpha - \frac{Q}{2}\right)$. This gives us $\alpha = \frac{1-a}{2b}$. Since πa is the deficit angle for a classical geometry, we require $a \geq 0$. This gives the semiclassical bound for one operator insertion: $\frac{1}{2b} \geq \alpha$. The WKB approximation is good when $b \to 0$, and $Q \sim \frac{1}{b}$. Then we've retained the Seiberg bound: $\alpha < \frac{Q}{2}$.

4.1 Examples

In the end of this section let's calculate some examples. First we can consider one-loop order the partition function of $c = 1$ Liouville with a 1-dimensional, compactified target space. We take the radius of the target circle to be R.

In this case, the power of μ vanishes since $h = 1$. For more general cases, perturbative with respect to μ is simply not feasible since the power of μ is fix by WT identity. However, for the torus case we can deal with the Liouville field as if they were free. To see this more explicitly, we first integrate over the zero mode of $\phi = \phi_0 + \bar{\phi}$. Then the non-zero mode path integral becomes simply free:

$$\int d\phi_0 e^{2\sum_i \alpha_i \phi_0} e^{-Q\chi\phi_0 - 4\pi\mu e^{2b\phi_0}\int\sqrt{\tilde{g}}e^{2b\bar{\phi}}} = \frac{1}{2b}\Gamma(-s)B^s, \tag{88}$$

where $B = 4\pi\mu \int \sqrt{\tilde{g}} e^{2b\bar{\phi}}$. It is proposed that when $s \geq 0$, the integral over the oscillating mode $\bar{\phi}$ can be done using free field techniques where there is no classical field configuration. Then the result can be analytically continued back to the $s < 0$ regime. We'll come back to this point in section 7.

Back to this problem, the only contribution from the zero-mode is given by the Liouville volume: $V_\phi = \int d\phi e^{-\mu e^{2b\phi}} = -\frac{1}{2b}\log\mu$.

The remaining integral over the oscillating modes of ϕ is the same as free fields, as is hypothesized above. We omit the calculation here. The fixed-area partition function is:

$$\langle 1 \rangle_A \sim \frac{1}{4\pi A b\sqrt{2\tau_2}|\eta(q)|^2}. \tag{89}$$

When doing the Laplace to fixed-cosmological constant result, one could easily see that there is a logarithmical divergence.

There are also some easy examples which can be obtained purely from the covariance of $SL(2,\mathbb{C})$. Similar to the results in ordinary conformal field theory, the structure of 2-point and 3-point function is fixed by Mobius invariance:

$$e^{2\alpha\phi} \to |\beta z + \delta|^{-4\Delta_\alpha} e^{2\alpha\phi}, \quad \text{under } z \to (\alpha z + \gamma)/(\beta z + \delta). \tag{90}$$

The structure of the two and three-point functions are thus as follows:

$$\begin{aligned} \left\langle e^{2\alpha_1\phi(z_1)} e^{2\alpha_2\phi(z_2)} \right\rangle &= |z_{12}|^{-4\Delta_1} \left(N(\alpha_1)\delta_{\alpha_2, Q-\alpha_1} + B(\alpha_1)\delta_{\alpha_2,\alpha_1} \right); \\ \left\langle e^{2\alpha_1\phi(z_1)} e^{2\alpha_2\phi(z_2)} e^{2\alpha_3\phi(z_3)} \right\rangle &= \frac{C[\alpha_1,\alpha_2,\alpha_3]}{|z_{12}^{\Delta_{123}} z_{13}^{\Delta_{132}} z_{23}^{\Delta_{241}}|}. \end{aligned} \tag{91}$$

Here $\Delta_{ijk} = \Delta_i + \Delta_j - \Delta_k$.

5 c=1 matrix quantum mechanics

In this section we try to solve the $c = 1$ string theory with matrix quantum mechanics. We first go through some basic techniques in random matrix theory, and we dive into the relation between $c = 1$ matrix quantum mechanics and its corresponding world-sheet theory.

5.1 General matrix models

A general matrix model is described by a partition function in the form of a matrix integral:

$$Z = \int [dM] \exp\left(-N \operatorname{Tr} V(M)\right), \tag{92}$$

where V is taken as polynomial. This can be seen as a string theory embedded into a zero dimensional target space. Through a ribbon graph analysis, the free energy $F = \log Z$ arrange itself into a genus expansion. For example, we take the potential to be $V(M) = \frac{1}{2}M^2 + \frac{\kappa}{3!}M^3$, then the free energy takes the form:

$$\log Z = \sum_{h=0} N^{2-2h} Z_h(\kappa). \tag{93}$$

If we naively take the $N \to \infty$ limit, we might conclude that only planar graphs will survive in this limit. However, we can make this model more interesting by taking the double-scaling limit. It is known, and remains to be shown below, that the partition function at a certain genus has a scaling behavior near a critical couping κ_c which is independent of the genus:

$$Z_h(\kappa) \sim (\kappa_c - \kappa)^{(2-\Gamma_{\rm str})\chi/2}. \tag{94}$$

This scaling behavior is somewhat similar to that of Liouville theory. In the double-scaling limit, we simultaneously take $\kappa \to \kappa_c$ and $N \to \infty$, while keeping $\mu_r = N(\kappa_c - \kappa)^{(2-\Gamma_{\rm str})/2}$ fixed. Then the free energy goes to a power series of μ_r:

$$\log Z = \sum_h \mu_r^{2-2h} f_g. \tag{95}$$

To solve this model, there are two major methods, and the one we'll use frequently is the orthogonal polynomial formalism.

First, we can view the matrix model as a $U(N)$ gauge theory, based on the invariance under the conjugation of $U \in U(N)$. This allows us to rewrite the measure:

$$[dM] \to \prod_i^N d\lambda_i dU \Delta_{\rm FP}, \tag{96}$$

where dU is the Haar measure on the $U(N)$ group. The Jacobian determinant FP can be obtained by Faddeev-Popov method [6]. Since the action is independent of U, the integral over $U(N)$ decouples. The gauge-fixed result is as follows:

$$Z = \int \prod_{i=1}^N d\lambda_i \Delta^2(\lambda) \exp\left(-N \sum_i V(\lambda_i)\right). \tag{97}$$

Here the Faddeev-Popov determinant is the Vandermonde determinant: $\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j) = \det(\lambda_i^{j-1})$.

Now let's introduce orthogonal polynomials, defined as follows:

$$\int_{-\infty}^{\infty} d\lambda e^{-V(\lambda)} P_n(\lambda) P_m(\lambda) = h_n \delta_{nm}. \tag{98}$$

We normalize the polynomials to be $P_n(\lambda) = \lambda^n + ...$, then the Vandermonde determinant can be expressed in terms of these polynomials:

$$\Delta(\lambda) = \det(\lambda_i^{j-1}) = \det(P_{j-1}(\lambda_i)). \tag{99}$$

This equality can be seen from the fact that any polynomial to the degree n can be obtained by the linear combination of different columns in the Vandermonde determinant, which leaves the determinant unchanged. Then we have:

$$\begin{aligned} Z =& \int \prod_{i=1}^{N} d\lambda_i e^{-V(\lambda_i)} \sum_{\pi\pi'} (-1)^{\pi}(-1)^{\pi'} \prod_k \prod_l P_{\pi(k)-1}(\lambda_k) P_{\pi'(l)-1}(\lambda_l) \\ =& N! \prod_{i=0}^{N-1} h_i = N! h_0 \prod_{k=1}^{N-1} f_k^{N-k}, \quad f_k = h_k/h_{k-1}. \end{aligned} \tag{100}$$

In the large N limit, k/N can be seen as a continuous variable $\xi = k/N \in [0,1]$. f_k is now a function of ξ. The partition function is now:

$$\log Z \sim N \log N - N + \log h_0 + N^2 \int_0^1 d\xi (1-\xi) \log f(\xi), \quad \frac{1}{N^2} \log Z \sim \int_0^1 d\xi (1-\xi) \log f(\xi). \tag{101}$$

Note that we haven't done any approximation so far so this expression is for all-genus partition function.

Now let's assume that the potential $V(\lambda)$ is even, and note that the following recursion relation holds:

$$\lambda P_n(\lambda) = P_{n+1} + r_n P_{n-1}. \tag{102}$$

The right hand side does not contain P_n because $\int d\lambda d^{-V(\lambda)} P_n(\lambda) P_n(\lambda) = 0$ for even $V(\lambda)$. Let's now consider:

$$\int d\lambda e^{-V} P_n(\lambda) \lambda P_{n-1}(\lambda) = r_n h_{n-1} = h_n, \quad r_n = f_n = h_n/h_{n-1}. \tag{103}$$

So in this particular case $f_n = r_n$. Similarly:

$$\int d\lambda e^{-V} P_n'(\lambda) P_{n-1}(\lambda) = \int d\lambda e^{-V} n P_{n-1}(\lambda) P_{n-1}(\lambda) = n h_{n-1}, \tag{104}$$

also, by integrating by parts, we have:

$$\int d\lambda e^{-V} P_n'(\lambda) P_{n-1}(\lambda) = -\int d\lambda P_n(\lambda) \frac{d}{d\lambda} \left(e^{-V} P_{n-1}(\lambda) \right) = \int d\lambda e^{-V} P_n(\lambda) V' P_{n-1}(\lambda). \tag{105}$$

Now we have all the tools to solve the partition function. For concreteness let's consider a particular potential:

$$\begin{aligned} V(\lambda) =& \frac{N}{2g} \left(\lambda^2 + \lambda^4 + b\lambda^6 \right) \\ gV' =& N \left(\lambda + 2\lambda^3 + 3b\lambda^5 \right). \end{aligned} \tag{106}$$

Now using (104), (105), we get:

$$gn/N = r_n + 2r_n(r_{n-1} + r_n + r_{n-1}) + 3b(rrr \text{ terms}). \tag{107}$$

There are 10 rrr terms, since $\int d\lambda e^V \lambda^5 P_n P_{n-1}$ is like going 3 stairs up and 2 stairs down, resulting in 10 possibilities, each gives a factor h_{n-1}.

Taking the large N limit, we get:

$$\begin{aligned} g\xi =& W(r) \\ =& g_c + \frac{1}{2} W''|_{r=r_c} (r(\xi) - r_c)^2 + ... \end{aligned} \tag{108}$$

Here $W(r) = r + 6r^2 + 30br^2$, and r_c is the critical point of $W(r)$, g_c is its critical value. Then we have:

$$r - r_c \sim (g_c - g\xi)^{1/2}. \tag{109}$$

Then the partition function gives:

$$\frac{1}{N^2} Z = \int_0^1 d\xi (1-\xi) \log(r_c + (g_c - g\xi)^{1/2}) \sim (g_c - g)^{5/2}. \tag{110}$$

This is the genus zero result, and we now have the susceptibility $\Gamma = -\frac{1}{2}$.

5.2 Integrable hierarchy

Let's now go back to (107) and try to obtain the all genus partition function, i.e.:

$$\begin{aligned} g\xi =& W(r) + 2r(\xi)(r(\xi+\epsilon) + r(\xi-\epsilon) - 2r(\xi)) \\ =& g_c + \frac{1}{2} W''|_{r=r_c} (r - r_c)^2 + 2r(\xi)(r(\xi+\epsilon) + r(\xi-\epsilon) - 2r(\xi)) + ... \end{aligned} \tag{111}$$

As is shown, the double-scaling limit is to take simultaneously $g \to g_c$ and $N \to \infty$ in a particular way. Since $g - g_c$ has length dimension $+2$, we define $g - g_c = \kappa^{-4/5} a^2$ with a as a parameter with the dimension of length(note that the scaling is $N(g-g_c)^{1-\Gamma_{\text{str}}/2} \sim 1$). The limit is taken by taking $a \to 0$ and scaling a with respect to N as:

$$a \sim N^{-5/2} \sim \epsilon^{5/2}. \tag{112}$$

The integral (101) is dominated by $\xi = 1$, so we change the variable from ξ to z, the relation is $g_c - g\xi = a^2 z$. Our ansatz for r is this regime is $r = r_c + au(z)$. Then the relation is now:

$$z = u^2 - \frac{1}{3} u''', \tag{113}$$

after proper rescaling. The solution to this equation characterizes all genus partition function for pure gravity.

For potentials of which the first non-zero critical point appearing at the mth derivative, the susceptibility is $\Gamma_{\text{str}} = \frac{1}{m}$. The scaling ansatz is now $r = r_c + a^{-2\Gamma_{\text{str}}} u(z)$, $1/N = a^{2-\Gamma_{\text{str}}}$, $g - g_c = \kappa^{2/(\Gamma_{\text{str}}-2)} a^2$. This corresponds to different matter coupled to gravity. The differential equation resulting from this scaling behavior and (5.2) is the mth member of the KdV hierarchy of differential equations.

We now want to make sense of the emergence of KdV hierarchy in matrix models. Let's first rescale the orthogonal polynomials for simplicity:

$$\Pi_n = P_n / \sqrt{h_n}, \quad d\lambda \int e^{-V} \Pi_n \Pi_m = \delta_{nm}. \tag{114}$$

Define an operator $Q_{nm} = \sqrt{r_m}\delta_{m,n+1} + \sqrt{r_n}\delta_{m+1,n}$, such that:

$$\begin{aligned}\lambda\Pi_n = \sqrt{\frac{h_{n+1}}{h_n}}\Pi_{n+1} + r_n\sqrt{\frac{h_n}{h_{n-1}}}\Pi_{n-1} \\ = Q_{nm}\Pi_m.\end{aligned} \tag{115}$$

So the rescaled orthogonal polynomial is an eigenvector of the operator Q. We plug in the ansatz $r = r_c + a^{2/m}u(z)$:

$$Q \to (r_c + a^{2/m}u(z))e^{\epsilon\partial_\xi} + (r_c + a^{2/m}u(z))e^{-\epsilon\partial_\xi}. \tag{116}$$

The term leading in a is:

$$Q = 2r_c^{1/2} + \frac{a^{2/m}}{\sqrt{r_c}}(u + r_c\kappa^2\partial_z^2). \tag{117}$$

Another operator can be defined canonically:

$$A_{nm}\Pi_m = \frac{d}{d\lambda}\Pi_n. \tag{118}$$

The exact form of this operator can be obtained from:

$$0 = \int d\lambda \frac{d}{d\lambda}\left(\Pi_n\Pi_m e^{-V}\right), \quad A + A^T = V'(Q). \tag{119}$$

Here we've differentiated term by term and used the fact that $\int e^{-V}\Pi_n\Pi_m\lambda^k = (Q^k)_{nm}$.

The matrix A does not have any symmetric or anti-symmetric property, so it is most convenient to convert to a matrix P:

$$P \equiv A - \frac{1}{2}V'(Q) = \frac{1}{2}(A - A^T). \tag{120}$$

So a commutation relation is satisfied:

$$[P.Q] = 1. \tag{121}$$

If $V = \sum_{k=0}^{\ell} a_k\lambda^{2k}$, the matrix elements of P are non-zero for $|m-n| \leq 2\ell - 1$. In the continuum, $W' = ...W^{(2\ell-1)} = 0$ can tune P to be a $2\ell - 1$ order differential operator. The differential equations are in fact results of this commutator.

5.3 c=1 matrix quantum mechanics

We refer to [2,7,8].

We first try to heuristically make sense of the matrix quantum mechanics approach. Generally a string theory embedded in 1D can be written as:

$$Z = \int [dX][dg_{ab}] \exp\left(-\frac{1}{4\pi}\int \sqrt{g}g^{ab}\partial_a X\partial_b X - \frac{1}{4\pi}\int \Phi R - \lambda\int\sqrt{g}\right). \tag{122}$$

On a world-sheet, the Einstein-Hilbert term gives the genus of the surface, and the cosmological constant term gives the area. We discretize the surface, such that the bosons X live on a 2-dimensional lattice, then the cosmological term is proportional to the total number of vertices on the lattice. Then the discretized partition function is:

$$Z \sim \sum_h g_0^{2-2h}\sum_\Lambda \kappa^V \prod_{i=1}^{V}\int dX_i \prod_{\langle ij\rangle} G(X_i, X_j), \tag{123}$$

where V is the number of vertices and Λ labels different triangulations of genus h. $G(X_i, X_j) = \exp\left(\frac{1}{2}(X_i - X_j)^2\right)$, but this is actually unimportant, since the coupled matter field may vary.

Now let's consider a matrix quantum mechanics:

$$Z = \int [d\Phi] \exp\left[-\int_{-L/2}^{L/2} dx \left(\frac{1}{2}(\partial_x \Phi)^2 + U(\Phi)\right)\right]. \tag{124}$$

If we take $U(\Phi) = -\frac{1}{2}\Phi^2 + \frac{\kappa}{3!}\Phi^3$, the Feynman diagram expansion also takes the form (123), where the i, j indices now labels the interaction vertex. The only difference is the propagator, which could be verified to be $G(x_i, x_j) = \exp(-|x_i - x_j|)$.

Now let's try to solve this matrix quantum mechanics. First let's obtain its Hamiltonian. Generally the Hamiltonian is written as [8]:

$$H = \text{Tr}\left(\frac{1}{2}P^2 + U(\Phi)\right), \tag{125}$$

where P is the momentum conjugate to Φ, and should be written as $P_{ij} = -i\frac{\partial}{\partial \Phi_{ji}}$.

The integration variable can be change to a diagonal matrix Λ and two unitaries to diagonalize the matrix:

$$\Phi = \Omega^{-1}\Lambda\Omega, \quad \Omega \in U(N). \tag{126}$$

In terms of the two variables Λ and Ω, there is a $S_N \ltimes U(1)^N$ symmetry. The action is the following:

$$\begin{aligned} U(1)^N: &\quad \Lambda \to \Lambda, \quad \Omega \to T^{-1}\Omega, \quad T = \text{diag}\{e^{i\theta_1}, ..., e^{i\theta_N}\}; \\ S_N: &\quad \Lambda \to W_{ij}^{-1}\Lambda W_{ij}, \quad \Omega \to W_{ij}^{-1}\Omega, \end{aligned} \tag{127}$$

where W_{ij} is the permutation matrix. In terms of $\Lambda = \text{diag}\{\lambda_1, ..., \lambda_N\}$ and Ω, the action of the Hamiltonian (125) on the state Ψ in the Hilbert space is:

$$H = -\frac{1}{2}\sum_{i,j}\frac{\partial}{\partial \Phi_{ji}}\frac{\partial}{\partial \Phi_{ij}} + \sum_i V(\lambda_i). \tag{128}$$

Since $\Phi_{ij} = (\Omega)^\dagger_{ik}\lambda_k\Omega_{kj}$, then:

$$\frac{\partial}{\partial \Phi_{ij}} = \sum_k \frac{\partial \lambda_k}{\partial \Phi_{ij}}\frac{\partial}{\partial \lambda_k} + \sum_{m,n}\frac{\partial \Omega_{mn}}{\partial \Phi_{ij}}\frac{\partial}{\partial \Omega_{mn}}. \tag{129}$$

To further simplify, note that the i-th column is the eigenvector of Φ, we get:

$$\Omega_{ik}\Phi_{kj} = \lambda_i\Omega_{ij}, \quad \Phi_{ij}(\Omega^\dagger)_{jk} = \lambda_k(\Omega^\dagger)_{ik}. \tag{130}$$

We vary the first equation:

$$\delta\Omega_{ik}\Phi_{kj} + \Omega_{ik}\delta\Phi_{kj} = \delta\lambda_i\Omega_{ij} + \lambda_i\delta\Omega_{ij}. \tag{131}$$

Act with $(\Omega^\dagger)_{jl}$, we get:

$$\delta\Omega_{ik}\lambda_l(\Omega^\dagger)_{kl} + \Omega_{ik}\delta\Phi_{kj}(\Omega^\dagger)_{jl} = \delta\lambda_i\delta_{il} + \lambda_i\delta\Omega_{ij}(\Omega^\dagger)_{jl}. \tag{132}$$

Then we get:

$$\frac{\partial \lambda_i}{\partial \Phi_{mn}} = (\Omega^\dagger)_{ni}\Omega_{im}, \quad \frac{\partial \Omega_{ij}}{\partial \Phi_{mn}} = \sum_{i\neq k}\frac{\Omega_{im}(\Omega^\dagger)_{nk}}{\lambda_i - \lambda_k}\Omega_{kj}. \tag{133}$$

Having obtained this relation, we can get the kinetic term as follows:

$$H = \sum_i \left(-\frac{\partial^2}{\partial \lambda_i^2} + V(\lambda_i) \right) + \frac{1}{2} \sum_{i,j} \left(-\frac{1}{\lambda_i - \lambda_j} \frac{\partial}{\partial \lambda_i} + \frac{R_{ij} R_{ji}}{(\lambda_i - \lambda_j)^2} \right). \tag{134}$$

Here $R_{ij} = \Omega_{jm} \frac{\partial}{\partial_{im}}$ is the $U(N)$ generator. We can remove the terms linear in $\frac{1}{\lambda_i - \lambda_j}$ with a linear transformation $H \to H' = \Delta H \Delta^{-1}$, so that now H' acts on the wavefunction $\Psi'(\Lambda, \Omega) = \Delta \Psi(\Lambda, \Omega)$. The Hamiltonian is now:

$$H = \sum_i \left(-\frac{\partial^2}{\partial \lambda_i^2} + V(\lambda_i) \right) + \frac{1}{2} \sum_{i,j} \frac{R_{ij} R_{ji}}{(\lambda_i - \lambda_j)^2}. \tag{135}$$

Returning to the partition function, the path integral calculates the following amplitude:

$$Z = \lim_{L \to \infty} \langle f | e^{-HL} | i \rangle. \tag{136}$$

Thus, only the ground state contribution remains. So $Z = \lim_{L \to \infty} e^{-E_0 L}$. The question is now converted to solving the lowest energy level, of which the eigenfunction lies in the singlet sector of $U(N)$ representation. Thus, we can discard the angular term in the Hamiltonian. In the singlet sector, the wavefunction only depends on the energy eigenvalues, and should be anti-symmetric after swapping two eigenvalues. This effectively is a system describing N non-interacting fermions.

Now the Hamiltonian acting on the states in the singlet sector as follows:

$$H = -\sum_i \left(\frac{d^2}{d\lambda_i^2} + U(\lambda_i) \right). \tag{137}$$

For concreteness, let's consider a specific potential:

$$V(\lambda) = -\frac{\lambda^2}{2} + g\frac{\lambda^4}{4!}. \tag{138}$$

Due to Pauli's exclusion principle, in the ground state the fermions pile up to the fermi level $-\mu$. μ depends on g and N. The large N limit we're interested in is to take $N \to \infty$ while holding μ fixed, as is shown in fig.3. This amounts to zooming in to the region near $\lambda = 0$, and $g \to 0$ in this limit. We introduce the density of states as:

$$\rho(\epsilon) = \sum_n \delta(\epsilon - \epsilon_n) \tag{139}$$

Then the total particle number and energy are:

$$N = \int \rho(\epsilon) \theta(-\mu - \epsilon) d\epsilon, \quad E = \int \epsilon \rho(\epsilon) \theta(-\mu - \epsilon) d\epsilon. \tag{140}$$

Then:

$$\rho(-\mu) = -\frac{\partial N}{\partial \mu}, \quad \frac{\partial E}{\partial N} = -\mu. \tag{141}$$

In this way, the knowledge of $\rho(\epsilon)$ is enough to calculate the energy as a function of N. Using the resolvent, we have:

$$\rho(\epsilon) = \frac{1}{\pi} \mathrm{Im}\,\mathrm{Tr} \left[\frac{1}{\frac{1}{2}p^2 - \frac{1}{2}\lambda^2 + \mu - i\epsilon} \right]. \tag{142}$$

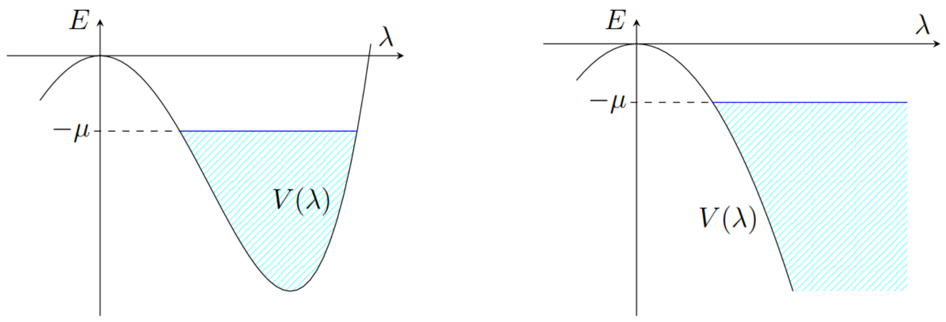

Figure 3: Semiclassical description of the free fermion Hamiltonian before (left) and after (right) the double-scaling limit is taken. This figure is replicated from [8].

The harmonic oscillator propagator is well-known:

$$\left\langle x \left| \frac{1}{\frac{1}{2}p^2 - \frac{1}{2}\lambda^2 + \mu - i\epsilon} \right| x \right\rangle = \int_0^\infty dT e^{-\mu T} \sqrt{\frac{\omega}{2\pi \sinh \omega T}} \exp\left[-\omega(2x^2\cosh\omega T - 2x^2)/2\sinh\omega T\right]. \tag{143}$$

In this case, $\omega \to -i$. We rotate the integration contour such that effectively $T \to iT$. Integrating over x, we have:

$$\frac{\partial\rho}{\partial\mu} = \frac{1}{\pi}\mathrm{Im}\int_0^\infty dT e^{-i\mu T}\frac{T/2}{\sinh(T/2)}. \tag{144}$$

One could do an asymptotic expansion in μ^{-1} and do the integration. Then the energy and chemical potential μ can be obtained. For details (which are very non-trivial but long and tedious), one could refer to [1]. As a parallel but important case, one could consider the target space X is now compact with radius R. Now the partition function writes:

$$Z = \int [d\Phi]\exp\left[-\int_0^{2\pi R} dx\left(\frac{1}{2}(\partial_x\Phi)^2 + U(\Phi)\right)\right] = \text{ Tr }\left[e^{-2\pi R H}\right]. \tag{145}$$

This amounts to doing the calculation in finite temperature. The particle number is written down using Fermi-Dirac distribution:

$$N = \int_{-\infty}^{\infty}\rho(\epsilon)d\epsilon\frac{1}{1+e^{2\pi R(\epsilon+\mu)}}. \tag{146}$$

In the thermodynamic limit, the free energy can be written as:

$$\frac{\partial \log Z}{\partial N} = -\mu. \tag{147}$$

The following discussion is parallel. The non-trivial property is that the result is manifestly invariant under T-duality:

$$R = \to \frac{1}{R}, \quad \mu_r \to R\mu_r. \tag{148}$$

Here μ_r is introduced to renormalize the expansion in the double-scaling limit with respect to N. At $R = 1$ the matrix quantum mechanics is self-dual under T-duality.

6 Tachyon scattering amplitudes

In this section we discuss the scattering amplitude of tachyons. There are various ways to do it. From the matrix quantum mechanics side, this calculation can be done directly or through the collective field approach. In the world-sheet theory, the amplitude can be calculated through string perturbation theory. We present all these calculations in this section. It is good to know that although we can reproduce some of the matrix model results from the world-sheet theory, the $c = 1$ string theory side of this duality is not yet understood non-perturbatively as is done in matrix models. There are some interesting developments working on this direction in recent years, but in this note we just stick to some examples.

6.1 World-sheet scattering amplitudes

We first present a world-sheet calculation here which is probably more related to the discussion in the earlier few chapters. For $c = 1$, as is stated before, the semiclassical approximation and mini-superspace approximation does not work well at this point since now different Fourier modes couple to each other. However, as $\phi \to -\infty$, a subset of states are still described by (54) since the interaction between zero modes and oscillating modes is suppressed in this limit. However, the reflection phase is different and is given by [9]:

$$R(p) = -\left(\pi\mu\frac{\Gamma(b^2)}{\Gamma(1-b^2)}\right)^{-\frac{2iP}{b}} \frac{\Gamma\left(\frac{2iP}{b}\right)\Gamma\left(2iPb\right)}{\Gamma\left(-\frac{2iP}{b}\right)\Gamma\left(-2iPb\right)}. \tag{149}$$

This state is created by the following vertex operator:

$$V_P \to e^{(Q+2iP)} + R(P)e^{(Q-2iP)}. \tag{150}$$

The two-point function is normalized as:

$$\langle V_{P_1}(z_1,\bar{z}_1)V_{P_2}(z_2,\bar{z}_2)\rangle = \frac{\pi\delta(P_1-P_2)}{|z_{12}|^{4\Delta(P_1)}}. \tag{151}$$

In this normalization, the structural constant is given by the DOZZ formula [9,10], which we will discuss in section 7:

$$C(P_1,P_2,P_3) = \frac{1}{\Upsilon_1(1+i(P_1+P_2+P_3))}\left[\frac{2P_1\Upsilon_1(1+2iP_1)}{\Upsilon_1(1+i(P_2+P_3-P_1))} \times (2 \text{ permutations})\right]. \tag{152}$$

Here Υ_1 is a special case of the Barnes double Gamma function.

For a CFT, the data is fully encoded in the spectrum of Virasoro primaries and structure constants. In principle we can solve any correlation function on arbitrary Riemann surfaces. For these data to be self-consistent, they should satisfy a series of conditions, e.g., crossing symmetry, modular invariance. The 4-point function of Liouville field theory $\langle V_{P_1}(z_1,\bar{z}_1)V_{P_2}(z_2,\bar{z}_2)V_{P_3}(z_3,\bar{z}_3)V_{P_2}(z_3,\bar{z}_3)\rangle$ is not known in a closed form, but can be expanded in terms of conformal blocks. Calculating the OPE between $V_{P_1}(z_1,\bar{z}_1)V_{P_2}(z_2,\bar{z}_2)$ gives:

$$\langle V_{P_1}(z,\bar{z})V_{P_2}(0)V_{P_3}(1)V_{P_2}(\infty)\rangle = \int dP' C(P_1,P_2,P')C(P',P_3,P_4)F^{(P')}_{P_1P_2P_3P_4}(z,\bar{z}). \tag{153}$$

The crossing ratio is $\frac{z_{12}z_{34}}{z_{32}z_{14}} = z$. Note that the operator defined at infinity is:

$$V_{P_4}(\infty) = \lim_{z\to\infty} z^{2\Delta}\bar{z}^{2\overline{\Delta}}V_{P_4}(z,\bar{z}). \tag{154}$$

The t-channel is obtained by the OPE $V_{P_1}V_{P_3}$:

$$\langle V_{P_1}(z,\bar{z})V_{P_2}(0)V_{P_3}(1)V_{P_2}(\infty)\rangle = \int dP' C(P_1,P_3,P')C(P',P_2,P_4)F^{(P')}_{P_1P_3P_2P_4}(1-z,1-\bar{z}). \quad (155)$$

Similarly for the u-channel:

$$\langle V_{P_1}(z,\bar{z})V_{P_2}(0)V_{P_3}(1)V_{P_2}(\infty)\rangle = (z\bar{z})^{-2-2P_1}\int dP' C(P_1,P_4,P')C(P',P_3,P_2)F^{(P')}_{P_1P_3P_2P_4}(1/z,1/\bar{z}). \quad (156)$$

Thus, the crossing symmetry relates $z, 1-z, 1/z$ and do a S_3 operation on the external weights. This maps the region $D=\{|z-1|<1, \mathrm{Re}z<\frac{1}{2}\}$ to the whole complex plane.

To be finished.

6.2 Direct calculation in matrix models

The direct calculation in matrix quantum mechanics is highly technical. Before delving into these calculation, let's first clarify some concepts and draw a general road map.

First, the gravitationally dressed tachyon vertex operator is $V_q = e^{iqX}e^{2\alpha\phi}$. When $D = c_X = 1$, we have $Q^2 = \frac{25-D}{6} = 4$. The dressed operator has conformal weight:

$$\Delta = \frac{q^2}{4} + \alpha(Q-\alpha). \quad (157)$$

To consistently couple to gravity, $\Delta = 1$, so the dressed operator should take the form:

$$e^{iqX}e^{(2\pm q)\phi}. \quad (158)$$

When $q>0$, it takes on the positive sign and vice versa. We define $q>0$ as the incoming wave and $a<0$ as the outgoing wave, so we can combine them into $V_q = e^{iqX}e^{(2-|q|)\phi}$.

We now first discuss the situation where one incoming wave with momentum q_{N+1} is scattered into N outgoing wave with momenta $q_1, ..., q_N$.

We claim that the vertex operator V_q corresponds to insertions of macroscopic loop operators $W(\ell) = \mathrm{Tr}\, e^{\ell\Phi}$ in matrix model. The mapping is by shrinking the circumference L of the loop to zero, which is more explicitly written as:

$$\hat{W}(q,\ell) = \int dx e^{iqx}\,\mathrm{Tr}\, e^{\ell\Phi(x)}, \quad V_q = \lim_{\ell\to 0}\ell^{-|q|}\hat{W}(q,\ell). \quad (159)$$

Let's try to make sense of this correspondence. First, the macroscopic loop operator $W = e^{\ell\Phi(x)}$ can be expanded into local operators:

$$W(\ell) = \sum_{n=0}^{\infty}\sigma_j \ell^{j+1/2}. \quad (160)$$

For Gaussian potential and $b=1$, σ_j behaves like $e^{2\alpha_j\phi}$, with $\alpha_j = \frac{1}{2}(2-j)$. Comparing this operator to the dressed vertex operator, we expect the remaining operator in the $\ell\to 0$ limit should have $j=|q|$. So the correspondence can be argued.

So the problem is converted to obtaining the correlation function of loop operators in the double-scaled matrix quantum mechanics. Before that, we discuss this technique in zero-dimensional matrix model.

In section 5.1, we showed that the matrix integral can be reduced to an integral involving only eigenvalues with Vandermonde determinant in the measure. We interpret this determinant as a Slater determinant of an N-fermion quantum mechanical system. We introduce the second-quantized fermion operator:

$$\Psi(\lambda) = \sum_n a_n \psi_n(\lambda), \tag{161}$$

where the orthonormal wave functions can be constructed from the orthogonal polynomials:

$$\psi_n(\lambda) = \frac{1}{\sqrt{h_n}} P_n(\lambda) e^{-\frac{1}{2}NV(\lambda)}. \tag{162}$$

The creation operator a_n and annihilation operator $a_n^\dagger$ constitutes the ordinary algebra:

$$\{a_n, a_m^\dagger\} = \delta_{n,m}. \tag{163}$$

The ground state of the system is the fermi sea, which satisfies:

$$\begin{aligned} a_n|N\rangle &= 0, \quad n \geq N; \\ a_n^\dagger|N\rangle &= 0, \quad n < N. \end{aligned} \tag{164}$$

We also define the second-quantized eigenvalue operator:

$$\hat{\lambda}^n = \int d\lambda \lambda^n \Psi^\dagger(\lambda)\Psi(\lambda), \tag{165}$$

such that the normal observables can be obtained via:

$$\left\langle \prod_{i=1} \operatorname{Tr} \Phi^{n_i} \right\rangle = \langle N| \prod_i \hat{\lambda}^{n_i} |N\rangle. \tag{166}$$

We can verify this result for instance:

$$\begin{aligned} \langle \operatorname{Tr} \Phi^n \rangle =& \frac{\int \prod_i d\lambda_i (\det P_{j-1}(\lambda_i))^2 \left(\sum_k \lambda_k^n\right) \prod_m e^{-NV(\lambda)}}{N! \prod_i h_i} \\ =& \frac{N}{N! \prod_i h_i} (N-1)! \sum_{j=0}^{N-1} \int d\lambda (P_j(\lambda))^2 \lambda^n e^{-NV(\lambda)} \\ =& \sum_j \int d\lambda (P_j(\lambda))^2 \lambda^n e^{-NV(\lambda)} \\ =& \sum_j d\lambda \int_\lambda \psi_j(\lambda)\psi_j(\lambda)\lambda^n \\ =& \int d\lambda \left\langle N|\Psi^\dagger \lambda^n \Psi|N \right\rangle = \langle N| \prod_i \hat{\lambda}^{n_i} |N\rangle. \end{aligned} \tag{167}$$

In the second line above, we've used the orthogonality. The N in front comes from the λ summation and $(N-1)!$ is the combinatorics of permuting the other $N-1$ eigenvalues.

We can now define the macroscopic loop operator $W(\ell)$ here as follows:

$$W(\ell) = e^{\ell\hat{\lambda}}. \tag{168}$$

By Wick's theorem, all the correlation functions in principle can be expressed in terms of fermion two-point functions:

$$K_N(\lambda_1,\lambda_2) = \left\langle N|\Psi^\dagger(\lambda_1)\Psi(\lambda_2)|N\right\rangle. \tag{169}$$

In particular, $\rho(\lambda) = K_N(\lambda,\lambda)$.

Now we have to study the double-scaling limit of K_N. orthogonal polynomials have the following recursion relation:

$$\lambda\psi_n = \sqrt{r_{n+1}}\psi_{n+1} + \sqrt{r_n}\psi_{n-1}. \tag{170}$$

Then:

$$\begin{aligned} K_N(\lambda_1,\lambda_2) =& \sum_{n=0}^{N-1}\psi_n(\lambda_1)\psi_n(\lambda_2) \\ =& \sqrt{r_{N+1}}\frac{\psi_{N+1}(\lambda_1)\psi_N(\lambda_2) - \psi_{N+1}(\lambda_2)\psi_N(\lambda_1)}{\lambda_1-\lambda_2}. \end{aligned} \tag{171}$$

So we need to investigate the double-scaling behavior of the wavefuncions themselves. In the continuum limit, for an mth critical potential, $Na^{2+1/m} = \kappa^{-1}$. Near the edge point of eigenvalue distribution, $\lambda = \lambda_c + a^{2/m}\tilde{\lambda}$, $n/N = 1 - a^2(z-\mu)$. Then $z > \mu$. Here μ is introduced to ensure the fermi sea condition. The filling condition now becomes $z > \mu$. Also the Green's function has a well-defined limit:

$$K(\lambda_c + a^{2/m}\tilde{\lambda}_1, \lambda_c + a^{2/m}\tilde{\lambda}_2) \to a^{-2/m}K_{\text{cont.}}(\tilde{\lambda}_1,\tilde{\lambda}_2), \text{as } a\to 0. \tag{172}$$

Here:

$$K_{\text{cont.}}(\lambda_1,\lambda_2) = \int_\mu^\infty dz\psi(z,\lambda_1)\psi(z,\lambda_2). \tag{173}$$

This could be deduced from the continuum limit of Christoffel-Darboux formula (171):

$$\begin{aligned} K(\lambda_1,\lambda_2) \propto& \frac{\psi_{N+1}(\lambda_1)\psi_N(\lambda_2) - \psi_{N+1}(\lambda_2)\psi_N(\lambda_1)}{\lambda_1-\lambda_2} \\ \to& \frac{\psi'(\mu,\lambda_1)\psi(\mu,\lambda_2) - \psi'(\mu,\lambda_2)\psi(\mu,\lambda_1)}{\lambda_1-\lambda_2} \\ =& a^{-2/m}\frac{\psi'(\mu,\tilde{\lambda}_1)\psi(\mu,\tilde{\lambda}_2) - \psi'(\mu,\tilde{\lambda}_2)\psi(\mu,\tilde{\lambda}_1)}{\tilde{\lambda}_1-\tilde{\lambda}_2} \end{aligned} \tag{174}$$

Here the prime denotes the derivative with respect to μ. To proceed, one may take the derivative of μ in the last line and integrate to obtain (173).

Having justified these concepts, we can now discuss the fermi field theory formulation. We now could define the fermion field operator:

$$\hat{\psi}(\lambda) = \int dz a(z)\psi(z,\lambda). \tag{175}$$

We also have the ground state condition:

$$a(z)|\mu\rangle = 0, \quad z<\mu; \quad a^\dagger(z)|\mu\rangle = 0, \quad z>\mu. \tag{176}$$

Also, the loop operator becomes $W(\ell) = \int d\lambda e^{\ell\lambda}\hat{\psi}^\dagger\hat{\psi}(\lambda)$.

6.2.1 Fermi field theory of c=1 matrix quantum mechanics

In the previous section we've introduced the fermi field theory for double scaled matrix integral. In this section we apply this method to $c = 1$ quantum mechanics.

First, let's remind ourselves how the double scaling limit is taken. For $c = 1$ matrix model, the partition function is given by the ground state energy:

$$\log Z = N \sum_{i=1}^{N} \epsilon_i. \tag{177}$$

The potential for individual fermion in the singlet sector is given by:

$$H_i = -\frac{d^2}{d\lambda_i^2} + U(\lambda_i). \tag{178}$$

To take the double scaling limit, we need to first calculate the genus h free energy $\mathcal{F}_h(g, U)$. Then we isolate the leading singular behavior as $g \to g_c$, then we can determine the scaling scheme and take the limit.

Let's first look at the genus 0 free energy with potential $U(\lambda) = -\frac{1}{2}\lambda^2 + \frac{g}{4!}\lambda^4$. The important part is just the maximum in the form of a quadratic potential. We'll see in the following.

In the large N limit, the density of states becomes a continuous function. In phase space, since here $1/N$ is in the place of $\hbar$, the states are represented by a point (λ, p), each with a volume $1/N$. Since we have N fermions, the phase space area is now $O(1)$. The total area is determined by the fermi level:

$$N = N \int \frac{dpd\lambda}{2\pi} \theta(\epsilon_F - \epsilon). \tag{179}$$

Then the fluid should have total area 1. This effectively is the saddle point equation and determines ϵ_F as a function of g. The total energy is:

$$E_0 = N \int \frac{dpd\lambda}{2\pi} \epsilon\theta(\epsilon_F - \epsilon), \quad \mathcal{F} = N^2 \int \frac{dpd\lambda}{2\pi} \epsilon\theta(\epsilon_F - \epsilon). \tag{180}$$

Then $\mathcal{F}_0 = \int \frac{dpd\lambda}{2\pi} \epsilon\theta(\epsilon_F - \epsilon)$. Now we want to see how the singularity arises in $\mathcal{F}_0$. As g decreases, the shape of the equal energy trajectories in phase space first continuously changes and shrinks. When $g \to g_c$, the area of the trajectory will approach 1. Which means at this point $\epsilon = 0$ trajectory coincides with the fermi surface. This happens when the tip of the potential coincides with the fermi level ϵ_F. So the singular behavior arises from near the origin. Since this potential is symmetric, the final result is twice the single sided result. In one side, the states distribute between two turning points λ_1 and λ_2 which is $[\sqrt{-2\epsilon_F}, \infty)$ in the double scaling limit. Then the singular part of the free energy is:

$$\begin{aligned}
\mathcal{F}_0 =& 2\int_{\lambda_1}^{\lambda_2} \frac{d\lambda}{2\pi} \left[\frac{1}{6}(p_+^3 - p_-^3) + \left(-\frac{1}{2}\lambda^2 + \frac{g}{4}\lambda^4\right)(p_+ - p_-)\right] \\
\sim& \frac{2}{\pi} \int_{\sqrt{-2\epsilon_F}}^{\infty} d\lambda \left(\frac{1}{3}(\epsilon + \frac{1}{2}\lambda^2) - \frac{1}{2}\lambda^2 + \frac{g}{4}\lambda^4\right) \sqrt{\lambda^2 + 2\epsilon_F} \\
\sim& \frac{1}{\pi}\left(\epsilon_F^2 + \frac{1}{4} g\epsilon_F^3\right) \log(-\epsilon_F).
\end{aligned} \tag{181}$$

A more straight forward way to see the support of singularity is that after the p integration $\sqrt{\lambda^2+2\epsilon_F}$ comes out. Here $p_\pm = \pm\sqrt{\frac{1}{2}\lambda^2+\epsilon_F}$ defines the upper and lower branches of the fermi surface. The leading nonanalytic behavior of $\mathcal{F}_0$ is:

$$\mathcal{F}_0 = \epsilon_F^2 \log(-\epsilon_F), \quad N^2\mathcal{F}_0 = \mu^2 \log\mu. \tag{182}$$

Here $\mu = -N\epsilon_F \sim O(1)$. This scaling behavior is independent of the quartic part of the potential but only the quadratic maximum. Also, the scaling variable which is held fixed is $\mu = -\epsilon_F(g)N$ instead of powers of $g - g_c$. The relation between ϵ_F and g can be obtained:

$$1 = 2\int_{\lambda_1}^{\lambda_2} \frac{d\lambda}{2\pi}(p_+ - p_-). \tag{183}$$

This is not any simple function of g.

Now let's begin to analyze the $c=1$ fermi field theory. The double scaled fermi field operator can be written down analogously:

$$\hat{\psi}(\lambda, x) = \int d\nu e^{i\nu x} a_\epsilon(\nu)\psi^\epsilon(\nu,\lambda). \tag{184}$$

Here $\epsilon = \pm$ represents the parity of the energy eigenfunction $\psi_\epsilon(\nu,\lambda)$ with respect to the Hamiltonian:

$$\left(-\frac{1}{2}\frac{d^2}{d\lambda^2} + U(\lambda)\right)\psi_\epsilon(\nu,\lambda) = \nu\psi_\epsilon(\nu,\lambda). \tag{185}$$

With quadratic potential, the solutions are Whittaker functions. The fermi sea vacuum is defined as usual.

So we can calculate the loop correlation functions in this field theory, of which the second quantized form is:

$$\hat{W}(\ell, q) = \int dx d\lambda e^{iqx}\hat{\psi}^\dagger\hat{\psi}(\lambda, x)e^{\lambda\ell}. \tag{186}$$

So in principle we calculate the correlation function:

$$G(x_1,\lambda_1,...,x_n,\lambda_n) = \left\langle \mu|\hat{\psi}^\dagger\hat{\psi}(x_1,\lambda_1)...\hat{\psi}^\dagger\hat{\psi}(x_n,\lambda_n)|\mu\right\rangle. \tag{187}$$

Since the double scaled model is Gaussian, the correlation function can be obtained through Wick contraction provided the resolvent for the inverse harmonic oscillator. In the end we can obtain an series expansion in terms of μ, and the scaling variable $\mu = \kappa^{-1}$ is in the place of the string coupling. The result is:

$$\langle\mu|\prod_i W(\ell_i, q_i)|\mu\rangle = \prod_i \ell_i^{|q_i|}\sum_h \kappa^{-2+2h+n}\mathcal{A}_{h,n}(q_1,...,q_n), \tag{188}$$

where:

$$\mathcal{A}_{h,n}(q_1,...,q_n) = \langle\tilde{V}_{q_1}...\tilde{V}_{q_n}\rangle_{h,n}. \quad \tilde{V}_q = \Gamma(q)c\tilde{c}e^{iqX}e^{(2-|q|)\phi}. \tag{189}$$

Here $\Gamma(q)$ is a normalization obtained by comparing the first few tree level results with the Liouville calculation. There is a subtlety arising from the two disconnected support of λ so that we cannot Laplace transform with respect to λ. A solution is to evaluate the Fourier transform of the density correlation $G(x_1,\lambda_1,...,x_n,\lambda_n)$. More explicitly:

$$M(x_1,z_1,...,x_n,z_n) = \int_{-\infty}^{\infty}\prod_{i=1}^n d\lambda_i e^{i\lambda_i z_i}G(x_1,\lambda_1,...,x_n,\lambda_n). \tag{190}$$

The answer will split into two pieces. In the first piece we can continue $z \to i\ell$ and in the second piece we can continue $z \to -i\ell$. This effectively gives the contribution from the two "worlds" of eigenvalues.

However, the actual computation is very complicated. In the following we'll introduce a diagrammatic rule to calculate these amplitudes.

6.3 Collective field theory approach

In the last subsection we explained how the dynamics of $c = 1$ matrix model can be formulated in non-relativistic fermi field theory by calculating the correlation functions of second quantized macroscopic loops. In this section we still adopt the fermion story but choose an alternative approach by studying the collective fluctuation of fermi sea.

In the ground state, the fermi surface is given by $p_\pm^{(0)} = \pm\sqrt{\lambda^2 + 2\epsilon_F}$. We parameterize the fluctuation with a collective field $\eta(\lambda)$. Then:

$$\rho = \frac{1}{2\pi}\left(p_+ - p_-\right) = \rho^{(0)}(\lambda) + \frac{1}{\sqrt{\pi}}\partial_\lambda \eta(\lambda). \tag{191}$$

Here $\rho(\lambda) = \frac{1}{\pi}\sqrt{\lambda^2 + 2\epsilon_F}$ is the ground state of fermion density. Similarly we could define the fermion momentum density:

$$\Pi_p(\lambda) = \sum_{i=1}^{N} p_i \delta(\lambda - \lambda_i) = \frac{p_+^2 - p_-^2}{4\pi}. \tag{192}$$

They satisfy the classical commutation relation:

$$\{\rho(\lambda), \Pi_p(\lambda')\} = \rho(\lambda')\frac{d}{d\lambda'}\delta(\lambda - \lambda'). \tag{193}$$

From this we can deduce that the canonical momentum with respect to η is:

$$\Pi(\lambda) = -\frac{1}{2\sqrt{2}}\left(p_+ + p_-\right), \tag{194}$$

such that $\{\eta(\lambda), \Pi(\lambda')\} = \delta(\lambda - \lambda')$. Then the Hamiltonian can be written as the following:

$$\begin{aligned}
H &= \sum_i \left(\frac{1}{2}p_i^2 - \frac{1}{2}\lambda_i^2 - \epsilon_F\right) \\
&= \int \frac{dp}{2\pi}\int_{\sqrt{-2\epsilon_F}}^{\infty} d\lambda \left(\frac{p^2}{2} - \frac{\lambda^2}{2} - \epsilon_F\right) \\
&= \int_{\sqrt{-2\epsilon_F}}^{\infty} d\lambda \left(\frac{p_+^3 - p_-^3}{6} - \frac{\lambda^2}{2}(p_+ - p_-) - \epsilon_F(p_+ - p_-)\right) \\
&= \int_{\sqrt{-2\epsilon_F}}^{\infty} d\lambda \left(\frac{1}{2}\sqrt{\lambda^2 + 2\epsilon_F}(\Pi(\lambda)^2 + (\partial_\lambda\eta)^2) + \frac{\sqrt{\pi}}{2}\Pi(\lambda)^2\partial_\lambda\eta + \frac{\sqrt{\pi}}{6}(\partial_\lambda\eta)^3\right) \\
&= \int_0^\infty d\tau \left[\frac{1}{2}(\Pi^2 + (\partial_\tau\eta)^2) + \frac{\sqrt{\pi}}{12\mu\sinh^2\tau}\left(2\Pi^2\partial_\tau\eta + (\partial_\tau\eta)^3\right)\right].
\end{aligned} \tag{195}$$

This is a theory of a massless boson with position dependent coupling. Here we've introduced another variable $\lambda = \sqrt{-2\epsilon_F}\cosh\tau$.

Now we're going to obtain the tree-level amplitude using the underlying W_∞-symmetry. For an inverse harmonic oscillator, $H = \frac{1}{2}p^2 - \frac{1}{2}\lambda^2$. The equation of motion is simple:

$$\dot{\lambda} = p, \quad \dot{p} = \lambda. \tag{196}$$

From this we can identify two conserved quantities:

$$v = (-\lambda - p)e^{-t}, \quad w = (\lambda - p)e^{t}, \tag{197}$$

and so as the following series of quantities:

$$\mathcal{C}_{mn} = \int \frac{dpd\lambda}{2\pi} v^m w^n. \tag{198}$$

Generally, the shape of the fermi surface is parameterized as follows:

$$\begin{aligned} \lambda =&(1 + a(\sigma)) \cosh(\sigma - t); \\ p =&(1 + a(\sigma)) \sinh(\sigma - t). \end{aligned} \tag{199}$$

Here the time evolution is obtained from the Hamiltonian equation of motion (note that now p, λ is not independent):

$$\partial_t p = \{H, p\} = \lambda - p\partial_\lambda p. \tag{200}$$

Suppose the upper(lower) branch of the fermi surface is parameterized by $\sigma_+(\sigma_-)$, solving the first equation of (199) gives $\sigma_\pm(t, \lambda)$. The no fluctuation solution is $\bar{\sigma}_\pm = t \pm \tau$. Suppose the fluctuation is $\delta\sigma_\pm = \sigma_\pm - \bar{\sigma}_\pm$. As $t \to -\infty (t \to \infty)$, the asymptotic behavior when $\tau \to \infty$ is:

$$\delta\sigma_\pm = \mp \log\left(1 + a(t \pm \tau + \delta\sigma_\pm(t \pm \tau))\right). \tag{201}$$

In this sense $\delta\sigma_\pm$ is now a function of one variable $t \pm \tau$. Also, the fermi momentum $p_\pm$ can be expanded:

$$p_\pm = \pm\sqrt{\lambda^2 - (1 + a(\sigma_\mp))^2} \sim \pm\lambda \mp \frac{1}{2\lambda}\left(1 + \psi_\mp(t \mp \tau)\right), \tag{202}$$

where $\psi_\pm = -1 + \left((1 + a(\sigma_\pm))^2\right)^2$. Note that p_- correspond to incoming wave ψ_+. Now let's return to the evaluation of v, w. As $t \to -\infty$:

$$\begin{aligned} v =&(-p - \lambda)e^{-t} = -(1 + a(t + \tau + \delta\sigma_+(t + \tau)))e^{\tau - t + \delta\sigma_+} \to e^{\tau - t}\sqrt{\pi}(\Pi - \partial_\tau\eta) \equiv e^{\tau - t}\epsilon_+; \\ w =&(p - \lambda)e^{t} \to 2e^{-\tau + t}. \end{aligned} \tag{203}$$

Correspondingly, when $t \to \infty$, $v \to 2e^{-\tau - t}$, $w \to e^{\tau + t}\sqrt{\pi}(-\Pi - \partial_\tau\eta) \equiv e^{\tau + t}\epsilon_-$. Since we have the W_∞ conserved quantity $\mathcal{C}_{mn}$, matching the incoming and outgoing wave gives:

$$\begin{aligned} \mathcal{C}_{mn} =&\frac{2^n}{2\pi(m+1)} \int_{-\infty}^{\infty} dt e^{(n-m)(t-\tau)} \left(\epsilon_+^{m+1} - \mu^{m+1}\right) \\ =&\frac{2^m}{2\pi(n+1)} \int_{-\infty}^{\infty} dt e^{(n-m)(t+\tau)} \left(\epsilon_-^{n+1} - \mu^{n+1}\right). \end{aligned} \tag{204}$$

The quantized fluctuation can be written as:

$$S(\tau, t) = \int \frac{dk}{2\pi k\sqrt{2}} \left(a_k e^{-i|k|t + ik\tau} + a_k^\dagger e^{i|k|t - ik\tau}\right). \tag{205}$$

Plug this into the conservation relation and set $m = 0, n = ik$, We can obtain the relation between the in-state creation operator $a_k^\dagger$ and the out-state creation operators $a_{k_i}^\dagger$:

$$\begin{aligned} a_k^\dagger = &\left(\frac{1}{2}\mu\right)^{-ik} \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{i}{\sqrt{\pi}\mu}\right)^{n-1} \frac{\Gamma(1-ik)}{\Gamma(2-n-ik)} \\ &\times \int_{-\infty}^{0} dk_1 dk_2 ... dk_n (a_{k_1}^\dagger - a_{k_1})...(a_{k_n}^\dagger - a_{k_n}) \delta(\pm|k_1|... \pm |k_n| - k). \quad k > 0, k_i < 0 \end{aligned} \tag{206}$$

Here $\pm|k_i|$ depends whether the corresponding operator in the product after expansion is $a_{k_i}^\dagger$ or a_{k_i}. These operators satisfy the commutation relation:

$$[a_k, a_{k'}^\dagger] = 2\pi|k|\delta(k - k'). \tag{207}$$

The tree level S-matrix is defined as:

$$\langle k_1...k_n; out|k_1'...k_m'; in\rangle, \quad |k_1'...k_m'; in\rangle = a_{k_1'}^\dagger ... a_{k_m'}^\dagger |0\rangle. \tag{208}$$

Using the relation between the incoming operators and outgoing operators and commutation relation, one can obtain these amplitudes.

6.4 Diagrammatics for closed string scattering

In this subsection we introduce a diagrammatic rule which allows us to calculate $c = 1$ matrix model S-matrix non-perturbatively in an elegant way. The calculation is originally done in [11] and we mainly follow [8].

Outline: fermionization $\rightarrow$ particle/hole scattering $\rightarrow$ rebozonization.

Reflecting amplitudes: R. Nonperturbatively $R < 1$ by an amount of $e^{2\pi E}$ because of the tunneling effects between the two disconnected patch of fermi seas.

To be added.

7 DOZZ formula

In this section we mainly refer to [12]. The original papers are [9, 10]. **To be added**

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