Notes on Liouville Field Theory and 2D String Theory

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This is a note on some aspects of Liouville field theory and non-critical string theory. The main references are [1–3].

1 Liouville field theory from 2D quantum gravity

We consider the Polyakov path integral of 2D gravity coupled to some scalar field $X^\mu, \quad \mu = 1, ..., D$, or a typical bosonic string theory:

$$Z = \int [dX][dg] \exp\left(-S_P - \mu_0 \int_\Sigma d^2\sigma \sqrt{g}\right). \tag{1}$$

Here Σ is the world-sheet and g is the dynamical metric on this world-sheet. S_P is the Polyakov action:

$$S_P = \frac{1}{4\pi} \int_\Sigma d^2\sigma \sqrt{g} g^{ab} \partial_a X \cdot \partial_b X. \tag{2}$$

This theory has two world-sheet symmetries: diffeomorphism symmetry and Weyl invariance. In textbook string theory, we view both of these symmetries as gauge redundancies. We can fix the metric to unit form, and the integration over g is decomposed to integrals over diffeomorphisms v^a, Weyl mode φ, and the moduli t. After the change of variables, the measure is the Faddeev-Popov determinant which can be expressed with bc ghosts:

$$Z = \int [dt][d\varphi][dbdc][dX] \exp\left(-S_P - S_{gh}\right), \quad S_{gh} = \int d^2z \sqrt{g}(b\overline{\nabla}c + \tilde{b}\nabla\tilde{c}). \tag{3}$$

In non-critical string theory, we have non-zero Weyl anomalies, which can be deduced from the central charge of ”matter” fields and ghost fields:

$$c_{tot} = c_X + c_{gh} = D - 26. \tag{4}$$

In $D \neq 26$ we do not view Weyl transformation as a gauge invariance. Rather, we view the Weyl mode as an independent dynamical degree of freedom. The path integral is now written as:

$$Z = \int [dt][d\varphi]_{\hat{g}e^\varphi}[dbdc]_{\hat{g}e^\varphi}[dX]_{\hat{g}e^\varphi} \exp\left(-S_P[X, \hat{g}] - S_{gh}[b, c, \hat{g}]\right). \tag{5}$$

One can calculate (or guess) the Weyl anomalies in these integration measures, and we get the famous Liouville action:

$$S_L[\phi, g] = \frac{1}{4\pi} \int d^2z \sqrt{g} \left(g^{ab}\partial_a\phi\partial_b\phi + \phi QR + 4\pi\mu e^{2b\phi}\right). \tag{6}$$

The partition function is now:

$$Z = \int [dt][d\varphi]_{\hat{g}}[dbdc]_{\hat{g}}[dX]_{\hat{g}} \exp\left(-S_P[X,\hat{g}] - S_{gh}[b,c,\hat{g}] - S_L[\phi,\hat{g}]\right). \tag{7}$$

The final answer of this partition function should be independent of the gauge choice. So if we do a local rescaling $\hat{g} \to e^{\sigma}\hat{g}$ of the fixed metric, and the Liouville field ϕ is also shifted by σ, the result should be the same. For this to be a symmetry, the total central charge of the theory should vanish:

$$c_{tot} = c_X + c_{gh} + c_L = 26 - D + c_L = 0. \tag{8}$$

We use this condition and match the Weyl anomaly to determine Q. Under infinitesimal change of metric δg_{ab}, the change in a typical expectation value is:

$$\delta\langle ...\rangle = -\frac{1}{4\pi}\int d^2\sigma\sqrt{g}\delta g_{ab}\langle T^{ab}...\rangle + \text{contact terms}. \tag{9}$$

For a Weyl transformation, $\delta g_{ab} = 2g_{ab}\delta\omega$. We get:

$$\delta_W\langle ...\rangle = -\frac{1}{2\pi}\int d^2\sigma\sqrt{g}\delta\omega\langle T^a_a...\rangle + \text{contact terms}. \tag{10}$$

We know for a given theory with a energy-momentum tensor, the Weyl anomaly $T^a_a = -\frac{c}{12}R$. Let's do the calculation in conformal gauge, $g_{ab} = e^{2\omega}\delta_{ab}$, then the Ricci scalar $R = -2\nabla^2\omega = -2e^{-2\omega}\delta_{ab}\partial^a\partial^b\omega$. Then the partition function is the case where $... = 1$:

$$\delta_W Z[e^{2\omega}\delta] = \frac{c}{12\pi}Z[e^{2\omega}\delta]\int d^2\sigma\delta\omega\delta_{ab}\partial^a\partial^b\omega. \tag{11}$$

This is integrated to be:

$$Z[e^{2\omega}\delta] = Z[\delta]\exp\left(-\frac{c}{24\pi}\int d^2\sigma\omega\nabla^2\omega\right). \tag{12}$$

Now let's calculate the conformal anomaly of a linear dilaton theory:

$$S = \frac{1}{4\pi}\int d^2\sigma\sqrt{g}\left(g^{ab}\partial_a\phi\partial_b\phi + QR\phi\right). \tag{13}$$

This is very much alike the Liouville theory except for the absence of an exponential potential. We note that the exponential potential term $\int_\Sigma \sqrt{\hat{g}}e^{2b\phi}$ here is just the area $\int d^2\sigma\sqrt{g}$, which should be exactly invariant under the transformation we are interested in. So we only need to worry about the linear dilaton part of the action.

Now let's see how to calculate correlation functions on the sphere in linear dilaton theory [1]. Similar to the bosonic string calculation, we consider first the generating functional:

$$\begin{aligned} Z[J] &= \left\langle \exp\left(i\int d^2\sigma J(\sigma)\phi(\sigma)\right)\right\rangle \\ &= \int [d\phi]\exp\left[-\frac{1}{4\pi}\int d^2\sigma\sqrt{g}(g^{ab}\partial_a\phi\partial_b\phi + QR\phi) + i\int d^2\sigma J(\sigma)\phi(\sigma)\right]. \end{aligned} \tag{14}$$

We expand the ϕ fields in the eigenbasis of Laplacian:

$$\phi(\sigma) = \sum_I \phi_I\Phi_I(\sigma), \quad \nabla^2\Phi_I = -\omega_I^2\Phi_I, \tag{15}$$

with normalization:

$$\int d^2\sigma\sqrt{g}\Phi_I(\sigma)\Phi_{I'}(\sigma) = \delta_{II'}. \tag{16}$$

We also define other components:

$$J_I = \int d^2\sigma J(\sigma)\Phi_I(\sigma), \quad R_I = \int d^2\sigma\sqrt{g}R(\sigma)\Phi_I(\sigma). \tag{17}$$

Then the generating functional can be written as:

$$Z[J] = \int \prod_I d\phi_I \exp\left(-\frac{\omega_I^2 x_I^2}{4\pi} + i\phi_I J_I + \frac{QR_I\phi_I}{4\pi}\right). \tag{18}$$

The integral is Gaussian except for constant modes ϕ_0 which gives a Dirac delta function:

$$\begin{aligned}&\int d\phi_0 \exp\left[i\phi_0\left(J_0 - i\frac{QR_0}{4\pi}\right)\right] \\ =&\int d\phi_0 \exp\left[i\phi_0\left(J_0 - 2iQ\Phi_0\right)\right] = 2\pi\delta(J_0 - 2iQ\Phi_0).\end{aligned} \tag{19}$$

Here we've used Gauss-Bonnet theorm: $\int d^2\sigma\sqrt{g} = 4\pi\chi$. And non-zero mode integral gives:

$$\begin{aligned}&\prod_{I\neq 0}\int d\phi_I \exp\left(-\frac{\omega_I^2 x_I^2}{4\pi} + i\phi_I J_I + \frac{QR_I\phi_I}{4\pi}\right) \\ =&\prod_{I\neq 0}\left(\frac{4\pi^2}{\omega_I^2}\right)^{\frac{1}{2}} \exp\left[-\frac{\pi}{\omega_I^2}\left(J_I - \frac{iQR_I}{4\pi}\right)^2\right].\end{aligned} \tag{20}$$

$$\begin{aligned}Z[J] =&2\pi\delta(J_0 - 2iQ\Phi_0)\left(\det\frac{-\nabla^2}{4\pi^2}\right)^{-1/2} \\ &\times \exp\left[-\sum_{I\neq 0}\frac{\pi}{\omega_I^2}\int d^2\sigma d^2\sigma'\left(J(\sigma) - \frac{iQR(\sigma)}{4\pi}\right)\Phi_I(\sigma)\Phi_I(\sigma')\left(J(\sigma') - \frac{iQR(\sigma')}{4\pi}\right)\right] \\ =&2\pi\delta(J_0 - 2iQ\Phi_0)\left(\det\frac{-\nabla^2}{4\pi^2}\right)^{-1/2} \\ &\times \exp\left[-\frac{1}{2}\int d^2\sigma d^2\sigma'\left(J(\sigma) - \frac{iQR(\sigma)}{4\pi}\right)G'(\sigma,\sigma')\left(J(\sigma') - \frac{iQR(\sigma')}{4\pi}\right)\right].\end{aligned} \tag{21}$$

Here we define Green's function:

$$G'(\sigma,\sigma') = \sum_{I\neq 0}\frac{2\pi}{\omega_I^2}\Phi_I(\sigma)\Phi_I(\sigma'), \tag{22}$$

which satisfies the following differential equation:

$$\nabla^2 G'(\sigma,\sigma') = -2\pi\delta(\sigma-\sigma')g^{-1/2} - 2\pi\Phi_0^2. \tag{23}$$

On the sphere, Green's function is solved to be:

$$G'(\sigma,\sigma') = -\frac{1}{2}\log|z_{12}|^2 + f(z_1,\bar{z}_1) + f(z_2,\bar{z}_2). \tag{24}$$

The two additional functions will drop out in the end. For vertex operator expectation value:

$$\left\langle \left[e^{ik_1\phi}(\sigma_1)\right]_r \cdots \left[e^{ik_n\phi}(\sigma_n)\right]_r \right\rangle, \tag{25}$$

we have $J(\sigma) = \sum_{i=1}^n k_i\delta(\sigma - \sigma_i)$, and $J_0 = \Phi_0 \int d^2\sigma J(\sigma) = \Phi_0 \sum_i k_i$. The generating functional is now:

$$\begin{aligned} Z[J] \sim &\delta(\sum k - 2iQ) \exp\left(-\sum_{i<j} k_i k_j G'(\sigma_i, \sigma_j) - \frac{1}{2}\sum_i k_i^2 G'_r(\sigma_i, \sigma_i)\right) \\ &\times \exp\left[\int d^2\sigma \sum_j \frac{iQR(\sigma)}{4\pi} G'(\sigma, \sigma_j) k_j\right] \\ &\times \exp\left[\frac{1}{2}\int d^2\sigma d^2\sigma' Q^2 \frac{R(\sigma)R(\sigma')}{16\pi^2} G'(\sigma, \sigma')\right]. \end{aligned} \tag{26}$$

Now we want to find the change of measure, so we set all momentum to zero, and the metric to $e^{2\omega}\delta$, $R = -2\nabla^2\omega$. The Weyl anomaly is:

$$\begin{aligned} Z[e^{2\omega}\delta] \sim &\exp\left[\frac{1}{2}\int d^2\sigma d^2\sigma' Q^2 \frac{\nabla^2\omega(\sigma)\nabla^2\omega(\sigma')}{4\pi^2} G'(\sigma, \sigma')\right] \\ = &\exp\left[\frac{1}{2}\int d^2\sigma d^2\sigma' Q^2 \frac{\omega(\sigma')\nabla^2\omega(\sigma)}{4\pi^2} \nabla^2 G'(\sigma, \sigma')\right] \\ = &\exp\left[-\frac{1}{4\pi}\int d^2\sigma d^2\sigma' \omega(\sigma')\nabla^2\omega(\sigma)\delta^2(\sigma - \sigma')\right] \\ = &\exp\left[-\frac{1}{4\pi}\int d^2\sigma \omega(\sigma)\nabla^2\omega(\sigma)\right]. \end{aligned} \tag{27}$$

Matching this with (12), we get the central charge (note that the free 2D scalar itself has central charge 1):

$$c_L = 1 + 6Q^2 = 25 - D, \quad Q = \sqrt{\frac{25 - D}{6}}. \tag{28}$$

Now let's try to determine b. This can be simply done by calculating the $Te^{2b\phi}$ OPE, where we require $e^{2b\phi}$ to be a $(1, 1)$ tensor:

$$T = -\partial\phi\partial\phi + Q\partial^2\phi. \tag{29}$$

The OPE can be calculated:

$$T(z)e^{2b\phi}(w) = \frac{\Delta e^{2b\phi}(w)}{(z - w)^2} + ... = \frac{-b^2 + bQ}{(z - w)^2}, \quad \Delta = b(Q - b) = 1. \tag{30}$$

Then we have:

$$Q = b + \frac{1}{b}. \tag{31}$$

We can also solve for b:

$$b = \frac{Q}{2} - \frac{1}{2}\sqrt{Q^2 - 4}. \tag{32}$$

So if $Q^2 - 4 = \frac{1-D}{6} \geq 0$, $D \neq 1$, b and Q are all real. When $c_X = D = 1$, there is a phase transition, and the theory at this point is the so-called $c = 1$ string theory.

1.1 c=1 string theory as a 2D critical string

The $c = 1$ string theory also has a critical string interpretation. To establish this point, let's first review string theory in general spacetime backgrounds and dimensions. In this case, an effective theory is that the string is coupled to external field excitations: tachyon field T, graviton $G_{\mu\nu}$, and dilaton Φ. Here we've assumed the Kalb-Ramond B field to be zero. The action can be written as:

$$S_\sigma = \frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{g}\left[g^{ab}G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu + R\Phi(X)\right]. \tag{33}$$

In the remaining of this section, we temporarily recover α' to illustrated some points more clearly. To the leading order in α' of the beta functions, we quote the results from eq.(3.7.14) in [4]:

$$\begin{aligned}\beta^G_{\mu\nu} =& \alpha'\mathbf{R}_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\Phi,\\ \beta^\Phi =& \frac{D-26}{6} - \frac{1}{2}\alpha'\nabla^2\Phi + \alpha'\nabla_\mu\Phi\nabla^\mu\Phi.\end{aligned} \tag{34}$$

A condition that the world-sheet theory is Weyl-invariant is:

$$\beta^G_{\mu\nu} = \beta^\Phi = 0. \tag{35}$$

This equality must be held at every order in α', so we need to find a background such that the beta functions all vanish. The simplest choice is that $D = 26$, the spacetime metric is flat, and the dilaton takes constant value. This returns to the critical string theory in $D = 26$ flat space.

A more non-trivial solution is:

$$G_{\mu\nu} = \eta_{\mu\nu}, \quad \Phi(X) = V_\mu X^\mu. \tag{36}$$

This theory is just a linear dilaton CFT coupled to the world-sheet gravity. According to our previous discussion, its central charge is $c = 1 + 6V_\mu V^\mu$. To make it free of Weyl anomaly, we can again get the condition:

$$V_\mu V^\mu = \frac{26-D}{6\alpha'}. \tag{37}$$

In $D \neq 26$, we can schematically take $26 - D$ of the spacetime dimensions compactified. We now take $D = 2$, where this theory can actually be solved. Since when $D < 26$, the gradient of Φ is spacelike, we take V^μ to have only one non-zero component at X^1 direction.

A small draw-back of this prescription is that the string coupling e^Φ becomes strong at large X^1. We now introduce again the tachyon field T. From tree-level string scattering amplitude, the effective action for tachyon field is [1, 4]:

$$S_T = -\frac{1}{2}\int d^D x\sqrt{-G}e^{-2\Phi}\left[G^{\mu\nu}\partial_\mu T\partial_\nu T - 4T^2\right]. \tag{38}$$

Inserting the background $\Phi = V_\mu X^\mu$, one can get the tachyon equation of motion:

$$-\partial^\mu\partial_\mu T(X) + 2V^\mu\partial_\mu T(X) - 4T(X) = 0. \tag{39}$$

The solutions are:

$$T(X) = \exp{(q\cdot X)}, \quad (q-V)^2 = \frac{2-D}{6\alpha'}. \tag{40}$$

For $D = 2$, the solution is $q_0 = 0, q_1 = 2(\alpha')^{-1/2}$. Then $T(X) = \exp\Phi(X)$. Then we could see that X^1, or Φ, is now an interacting field theory with exponential potential and linear coupling to the Ricci curvature. This justifies our statement that the Liouville field is the spatial dimension of the 2D target space in $D = 2$ critical string theory. Also, the exponential potential also prevents propagation to large X^1.

2 Semiclassical Liouville theory

Now we consider a semiclassical way to quantize the Liouville theory. First, let's discuss the canonical quantization method on a cylinder which is mapped from a complex plane via the standard mapping:

$$z = e^{-iw}, \quad w = \sigma + i\tau = \sigma + t. \tag{41}$$

The Lagrangian is:

$$L = \frac{1}{4\pi}\partial^a\phi\partial_a\phi + \mu e^{2b\phi}. \tag{42}$$

The canonical momentum is:

$$\Pi = \frac{\partial L}{\partial\dot{\phi}} = \frac{1}{2\pi}\partial_t\phi. \tag{43}$$

We Fourier expand the canonical momentum and ϕ field:

$$\begin{aligned}\phi(\sigma,t) &= q + \sum_{n\neq 0}\frac{i}{n}[a_n(t)e^{-in\sigma} + b_n(t)e^{in\sigma}];\\ \Pi(\sigma,t) &= p + \sum_{n\neq 0}[a_n(t)e^{-in\sigma} + b_n(t)e^{in\sigma}].\end{aligned} \tag{44}$$

We impose the equal-time commutator:

$$[\phi(\sigma), \Pi(\sigma')] = i\delta(\sigma - \sigma'). \tag{45}$$

Using the mode expansion, this is re-written as:

$$[q,p] = i, \quad [a_n, a_m] = \frac{n}{2}\delta_{n,-m}, \quad [b_n, b_m] = \frac{n}{2}\delta_{n,-m}. \tag{46}$$

The Hamiltonian of the system on a cylinder is:

$$H = \frac{1}{2}\int_0^{2\pi}\frac{d\sigma}{2\pi}\left(T_{ww} + T_{\bar{w}\bar{w}}\right). \tag{47}$$

The new energy-momentum tensor can be obtained with the corresponding anomalous transformation rule:

$$T_{ww} = \left(\frac{\partial z}{\partial w}\right)^2 T_{zz} + \frac{c}{24} = -e^{-2iw}T_{zz} + \frac{1+6Q^2}{24}. \tag{48}$$

Since $z = \sigma + t$, $\bar{z} = \sigma - t$, we have:

$$\partial = \partial_z = \frac{1}{2}(\partial_\sigma + \partial_t), \quad \bar{\partial} = \partial_{\bar{z}} = \frac{1}{2}(\partial_\sigma - \partial_t). \tag{49}$$

So we have [2,3]:

$$\begin{aligned}T_{\pm\pm} =& \frac{1}{4}(\partial_\sigma\phi \pm 2\pi\Pi)^2 - \frac{Q}{4}(\partial_\sigma^2 + \partial_t^2 \pm 2\partial_t\partial_\sigma\phi) + \frac{1+6Q^2}{24}\\ =& \frac{1}{4}(2\pi\Pi \pm \partial_\sigma\phi)^2 - \frac{Q}{2}\partial_\sigma(2\pi\Pi \pm \phi) + Q\pi\mu b e^{2b\phi} + \frac{1+6Q^2}{24}.\end{aligned} \tag{50}$$

Using these set of operators, the Hamiltonian takes the form:

$$H = \frac{1}{2}p^2 + 2\sum_{k>0}[a_{-k}a_k + b_{-k}b_k] + \mu\int_0^{2\pi} e^{2b\phi}. \tag{51}$$

So if we solve the theory containing only the zero modes, we can act on the vacuum with the two set of operators to build the Hilbert space:

$$\mathcal{H} = \oplus_E \mathcal{F}_{E,Q} \otimes \overline{\mathcal{F}}_{E,Q}. \tag{52}$$

2.1 Solving the Liouville quantum mechanics

The quantum mechanical system has Hamiltonian:

$$H_0 = \frac{1}{2}p^2 + 2\pi\mu e^{2bq} + \frac{Q^2}{4} = -\frac{1}{2}\partial_q^2 + 2\pi\mu e^{2bq} + \frac{Q^2}{4}. \tag{53}$$

As $q \to -\infty$, the exponential interaction disappears. We label the state with a continuous parameter: momentum p. Then the eigenstates are just plane waves with eigenvalue Δ. More generally, the wave function can be written as:

$$\psi_p(q) \sim e^{2ipq} + R(p)e^{-2ipq}, \quad p > 0, \quad \Delta(p, Q) = 2p^2 + \frac{Q^2}{4}. \tag{54}$$

Here $R(p)$ is a reflecting coefficient.

For simplicity, we solve this in the "circumference of the universe" basis, $l = e^{bq}$. Then the eigenvalue equation goes to:

$$\left[-b^2\left(l\frac{\partial}{\partial l}\right)^2 + 2\pi\mu l^2 + \frac{Q^2}{4}\right]\psi_p(l) = \left(2p^2 + \frac{Q^2}{4}\right)\psi_p(l). \tag{55}$$

This equation is nothing but the Bessel equation, with appropriate boundary condition, the solution is:

$$\psi_p(q) = \frac{1}{\pi}\sqrt{\frac{p}{b}\sinh\frac{2\pi p}{b}}K_{2ip/b}\left(\frac{\sqrt{2\pi\mu}l}{b}\right). \tag{56}$$

Using an asymptotic expansion as $b \to 0$, we can get the reflection phase:

$$R(p) = -(\pi\mu b^{-2})^{-\frac{2ip}{b}}\frac{\Gamma(1 + 2ib^{-1}p)}{\Gamma(1 - 2ib^{-1}p)}. \tag{57}$$

We now try to understand the corresponding operator on the complex plane which can be mapped to this state.

For state with momentum p, the corresponding operator is conjectured to be $V_p(z, \bar{z}) = e^{2\alpha\phi} = e^{2ip\phi + Q\phi}$. Note that this is not a one-to-one correspondence. As an argument, we notice that the TV_p OPE can give its weight $\alpha(Q - \alpha) = \frac{Q^2}{4} + p^2$, and thus the energy:

$$H_0 = L_0 + \overline{L}_0 - \frac{c}{12} = 2p^2 - \frac{1}{12}. \tag{58}$$

This coincides with the asymptotic behavior of energy of the state with momentum p. So only states with $\alpha = \frac{Q}{2} + ip$ exist in the theory. However, let's consider the operator which measures the volume of the universe:

$$A = \int_\Sigma e^{2b\phi}. \tag{59}$$

The operator $e^{2b\phi}$ has an imaginary momentum, which corresponds to non-normalizable state. To further elaborate on this point, let's return to examine the semi-classical theory.

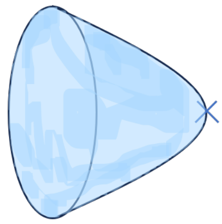

Figure 1: Illustration of Hartle-Hawking's way to prepare a state.

3 Semiclassical states revisited

In this section we try to make sense of the non-normalizable states dual to operators in the form $e^{\alpha\phi}$. In ordinary CFT, the state dual to the primary operator $\mathcal{O}(z,\bar{z})$ is obtained by acting the operator valued at $z=0$ to the vacuum state $|0\rangle$. This gives you the mapping between the action of Virasoro generators on primary operators and the corresponding states:

$$L_0|\mathcal{O}\rangle = \bar{L}_0|\mathcal{O}\rangle = \Delta_{\mathcal{O}}|\mathcal{O}\rangle; \quad L_n|\mathcal{O}\rangle = \bar{L}_n|\mathcal{O}\rangle = 0, \quad n>0. \tag{60}$$

One can also get the descendant states by acting $L_n(n>0)$ on the primary state.

However, in Liouville field theory this naive approach is not feasible, owing to the fact that $|p=0\rangle$ is not actually in the Hilbert space. Alternatively, one could construct the state corresponding to $\mathcal{O}$ by doing a path integral over a disk with an $\mathcal{O}$ insertion in the center. By specifying the boundary condition on the boundary, one can obtain the wavefunction of this state in the basis of field configuration. The inner product is defined to be the path integral on the sphere obtained by gluing two disks. This is somewhat similar to the Hartle-Hawking construction of wavefunctions of the universe, as is shown in 1. Of course the path integral on the disk is divergent. We could cutoff the wave function $\Psi_{\mathcal{O}}(\phi)$ at some particular point ϕ_0. As the regulator is removed, we could either keep the norm of $\Psi_{\mathcal{O}}$ finite and have $\Psi_{\mathcal{O}}(\phi_0)\to 0$, or we hold $\Psi_{\mathcal{O}}(\phi_0)$ fixed and let the norm diverge. We will return to this point in section 4.

Having clarified the definition of these states, let's now discuss what classical geometry these states describe.

3.1 Classical Liouville solutions

For $Q=\frac{1}{b}$, the action (6) defines a classical conformal field theory, the corresponding Weyl transformation is:

$$\hat{g}\to e^{\sigma}\hat{g}, \quad \phi\to\phi-\sigma/2b. \tag{61}$$

Then $R[\hat{g}]\to e^{-\sigma}(R[\hat{g}]-\nabla^2\sigma)$. The action transforms as:

$$\begin{aligned} S_L \to & \frac{1}{4\pi}\int d^2z\sqrt{g}e^{\sigma}\left[e^{-\sigma}g^{ab}\left(\partial_a\phi-\frac{1}{2b}\partial_a\sigma\right)\left(\partial_b\phi-\frac{1}{2b}\partial_b\sigma\right)\right. \\ & \left. +Q\,e^{-\sigma}(R-\nabla^2\sigma)(\phi-\sigma/2b)+4\pi\mu e^{-\sigma}e^{2b\phi}\right]=S_L. \end{aligned} \tag{62}$$

So this defines a classical conformal field theory. But now how to understand eq.(31)? If we rescale $\phi\to\frac{1}{2b}\phi$, then $\frac{1}{b}$ can be identified as the coupling constant of the theory. So $b\to 0$ is the classical limit, where most of our calculation is done.

Also, note that now Weyl transformation acts non-linearly on ϕ, which means that ϕ can now be seen as a Goldstone boson emerging from the spontaneously broken Weyl symmetry by the choice of $\hat{g}$. The classical equation of motion is given by:

$$R[e^{2b\phi}\hat{g}] = -8\pi b^2\mu. \tag{63}$$

So the classical solution of Liouville theory describes a surface with constant negative curvature. A standard solution is the Poincare half plane:

$$ds^2 = e^{2b\phi}|dz|^2 = \frac{1}{4\pi b^2\mu}\frac{1}{(\mathrm{Im}z)^2}|dz|^2. \tag{64}$$

According to the uniformization theorem, every Riemann surface is conformally equivalent to:

1. CP^1, the Riemann sphere;
2. H, the Poincare half-plane;
3. H/Γ, where Γ is a discrete subgroup of the Mobius group $SL(2,\mathbb{R})$.

In general, for a Riemann surface X obtained by H/Γ, there is a projection map $\pi: \quad H \to X$, and an inverse, uniformization map:

$$f: \quad X \to H. \tag{65}$$

On the Riemann surface X, the metric takes a general form:

$$ds^2 \propto \frac{1}{\mu}\frac{\partial A\bar{\partial}B|dz|^2}{(1-A(z)B(\bar{z}))^2}, \tag{66}$$

If we move the point z around a circle, the functions A, B will change under the conjugation of $SL(2,\mathbb{R})$. Based on the conjugacy class of the monodromy of these functions, the local geometry has three possibilities (also shown in fig.2):

1. Elliptic solutions: $A = z^a$, $B = \bar{z}^a$, $a \in \mathbb{R}$. In this situation the monodromy of A is $A \to e^{2ia\pi}A$. The metric takes the form:

$$ds^2 = e^{2b\phi}|dz|^2 \sim \frac{1}{\mu}\frac{a^2|dz|^2}{(z\bar{z})^{1-a}[1-(z\bar{z})^a]} \sim \frac{a^2}{\mu}\frac{d\sigma^2+dt^2}{\sinh^2 at}. \tag{67}$$

   This solution has a curvature singularity at $z = 0$

2. Parabolic solution: $A = i\log z$, $B = \frac{i}{\log \bar{z}}$. In this situation the monodromy of A is $A \to A - 2\pi$. The metric takes the form:

$$ds^2 \sim \frac{1}{\mu}\frac{|dz|^2}{z\bar{z}(\log z\bar{z})^2} \sim \frac{1}{\mu}\frac{d\sigma^2+dt^2}{t^2}. \tag{68}$$

   This geometry also has curvature singularity at $z = 0$.

3. Hyperbolic solution: $A = z^{im}$, $B = \bar{z}^{im}$, $m \in \mathbb{R}$. In this situation the monodromy of A is $A \to e^{-2\pi m}A$. The metric takes the form:

$$ds^2 \sim \frac{1}{\mu}\frac{m^2|dz|^2}{z\bar{z}\left[\sin\left(\frac{m}{2}\log(z\bar{z})\right)\right]^2} \sim \frac{1}{\mu}\frac{m^2(dt^2+d\sigma^2)}{\sin^2(mt)}, \quad 0 < t < \frac{\pi}{m}. \tag{69}$$

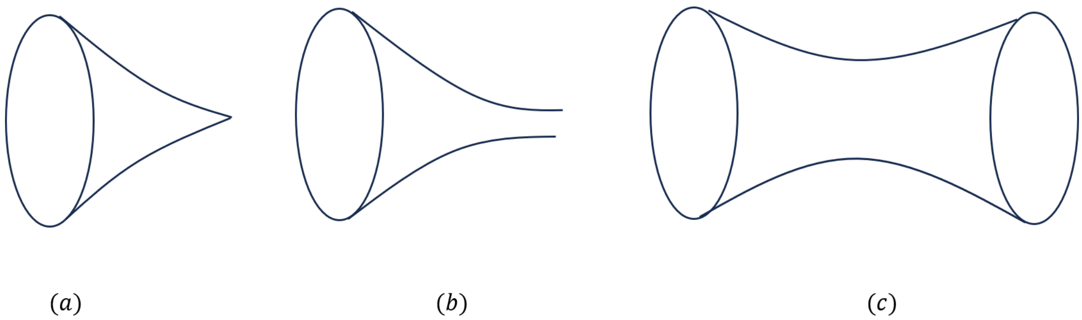

Figure 2: Three local geometries with constant negative curvature. (a): elliptic; (b): parabolic; (c): hyperbolic.

For elliptic solutions, they indeed have imaginary energy, which are just what we discovered to be non-normalizable states. For hyperbolic solutions, they have real energy.

In addition, for the two singular solutions, the Liouville field ϕ also obeys the Liouville equation of motion with a source at $z = 0$, which corresponds to curvature singularity at the origin:

$$\frac{2}{\pi}\partial\bar{\partial}\phi - \frac{1}{4\pi}RQ - 2\mu b e^{2b\phi} + k\delta(z) = 0. \tag{70}$$

Motivated by the geometries illustrated above, the normalizable states were labelled "macroscopic states," and the non-normalizable states were labelled "microscopic states", according to the convention by Seiberg [3]. Semiclassically, the macroscopic states do not have a well-defined insertion point in the intrinsic geometry of the surface. The microscopic states, on the other hand, correspond semiclassically to the elliptic geometry, and thus to local operators — the operator insertion in this case is localized at the tip of the "funnel".

Since the asymptotic behavior of the energy eigenfunction should behave as e^{2ipq}, the $e^{-2ipq} = e^{-2(\alpha-\frac{Q}{2})q}$ should damp out as $q \to -\infty$. Thus we have the bound:

$$\mathrm{Re}\alpha \leq \frac{Q}{2}. \tag{71}$$

This is usually called the Seiberg bound.

4 Semiclassical amplitude

We calculate correlation functions in the form:

$$\left\langle \prod_{i=1}^{n} e^{2\alpha_i\phi(z_i)} \right\rangle = \int [d\phi] e^{-S[\phi]} \prod_{i=1}^{n} e^{2\alpha_i\phi(z_i)}. \tag{72}$$

Semiclassically, this is evaluated using a saddle point approximation. The saddle point equation is:

$$\frac{2}{\pi}\partial\bar{\partial}\phi - \frac{1}{4\pi}RQ - 2\mu b e^{2b\phi} + \sum_i 2\alpha_i\delta(z - z_i) = 0. \tag{73}$$

Let's now shift the Liouville field $\phi \to \phi - \frac{\log\mu}{2b}$. This moves the μ dependence out of the Liouville action, and gives the μ scaling relation:

$$\left\langle \prod_{i=1}^{n} e^{2\alpha_i\phi(z_i)} \right\rangle \sim \mu^{\frac{(1-g)Q - \sum_i \alpha_i}{b}}. \tag{74}$$

This is called the KPZ scaling law. This Ward-Takahashi method can also be used to obtain the string susceptibility Γ_{str}, defined as $Z[A] \sim A^{(\Gamma_{\text{str}}-2)\chi/2-1}$. We consider the partition function with fixed area A:

$$Z[A] = \int [d\phi] e^{-S} \delta\Big(\int \sqrt{\hat{g}} e^{2b\phi} - A\Big). \tag{75}$$

We also shift $\phi \to \phi + \frac{\lambda}{2b}$, using again the Gauss-Bonnet theorem, we have:

$$\begin{gathered} \frac{1}{4\pi}\int \sqrt{\hat{g}} Q R \phi \to \frac{1}{4\pi}\int \sqrt{\hat{g}} Q R \phi + \frac{\lambda \chi Q}{2b} \\ \delta\left(\int \sqrt{\hat{g}} e^{2b\phi} - A\right) \to e^{-\lambda} \delta\left(\int \sqrt{\hat{g}} e^{2b\phi} - e^{-\lambda} A\right). \end{gathered} \tag{76}$$

Then we get:

$$Z[A] = \exp\left(-\frac{\lambda \chi Q}{2b} - \lambda\right) Z[e^{-\lambda} A]. \tag{77}$$

Then we can choose $\lambda = \log A$, so that we get the scaling law:

$$Z[A] = A^{-\frac{\chi Q}{2b} - 1} Z[1]. \tag{78}$$

Then we have the string susceptibility:

$$\Gamma_{\text{str}} = 2 - \frac{Q}{b}. \tag{79}$$

One could integrate the equation (73) over the surface Σ and use the Gauss-Bonnet theorem, we get:

$$-Q(2-2h) + 2\sum_i \alpha_i - 2\mu b A(\Sigma) = 0, \quad s \equiv \frac{1}{b}\left(Q(1-h) - \sum_i \alpha_i\right) < 0. \tag{80}$$

It says that classically the KPZ exponent s is negative. After solving the saddle point value ϕ_{cl} one could just plug it into the path integral and obtain the result. However, now $-S_{cl} - 2\alpha_i \phi_{cl}(z_i)$ is now infinite. The divergence comes from the curvature source at each operator insertion. Near one of the insertion points z_i, the equation of motion (73) has a solution:

$$\phi_{cl}(z) \sim -\alpha_i \log |z - z_i|^2, \quad z \to z_i, \tag{81}$$

which diverges near $z = z_i$. This reflects the need for vertex operator renormalization. We regularize the action as follows:

$$\overline{S}_{cl} = \lim_{\epsilon \to 0}\left(\int_{\Sigma - \cup_i B(\epsilon, z_i)} \mathcal{L} - \sum_i \left(\Delta_i + \overline{\Delta}_i\right) \log \epsilon\right). \tag{82}$$

This also justifies that the divergence in the Hartle-Hawking construction introduced in the last section.

When the KPZ exponent $s = Q(1-h) - \sum_i \alpha_i \geq 0$, there is no classical solutions. In this case, we can use semi-classical method by considering the fixed-area correlation function:

$$\left\langle \prod_{i=1}^{n} e^{2\alpha_i \phi(z_i)} \right\rangle_A = \int [d\phi] e^{-S[\phi]} e^{\mu A} \prod_{i=1}^{n} e^{2\alpha_i \phi(z_i)} \delta\left(\int \sqrt{\hat{g}} e^{2b\phi} - A\right). \tag{83}$$

The scaling law of fix-area correlation can be obtained similarly using Ward-Takahashi method, it scales with A as:

$$\left\langle \prod_{i=1}^{n} e^{2\alpha_i \phi(z_i)} \right\rangle_A \sim A^{-s-1}. \tag{84}$$

One could do a Laplace transformation to get the scaling of the fixed-cosmological constant correlation function:

$$\int_0^\infty \frac{dA}{A} A^{-s} e^{-\mu A} = \mu^{-s}\Gamma(-s). \tag{85}$$

This matches the KPZ scaling law. For $s < 0$ the integral converges. However, if $s = 0$, the integral diverges logarithmically while for $s > 0$ it diverges following the power law, at small area limit as a UV divergence in terms of the length measured by $g = e^{2b\phi}\hat{g}$. We could do a UV cutoff in the integration over g. This gives the result:

$$\left\langle \prod_{i=1}^{n} e^{2\alpha_i \phi(z_i)} \right\rangle = P(\mu) + C\mu^s \left\langle \prod_{i=1}^{n} e^{2\alpha_i \phi(z_i)} \right\rangle_{A=1}, \tag{86}$$

where $P(\mu)$ is a polynomial to the degree $[s]$.

We also comment here on the semi-classical meaning of the Seiberg bound. The local geometry around a local operator insertion will take the elliptic form:

$$e^{2b\phi} = \frac{1}{\pi\mu b^2} \frac{a^2}{|z|^{2-2a}(1-|z|^{2a})^2}. \tag{87}$$

The corresponding momentum $p = \lim_{t\to-\infty} \partial_t \phi = i\left(\alpha - \frac{Q}{2}\right)$. This gives us $\alpha = \frac{1-a}{2b}$. Since πa is the deficit angle for a classical geometry, we require $a \geq 0$. This gives the semiclassical bound for one operator insertion: $\frac{1}{2b} \geq \alpha$. The WKB approximation is good when $b \to 0$, and $Q \sim \frac{1}{b}$. Then we've retained the Seiberg bound: $\alpha < \frac{Q}{2}$.

4.1 Examples

In the end of this section let's calculate some examples. First we can consider one-loop order the partition function of $c = 1$ Liouville with a 1-dimensional, compactified target space. We take the radius of the target circle to be R.

In this case, the power of μ vanishes since $h = 1$. For more general cases, perturbative with respect to μ is simply not feasible since the power of μ is fix by WT identity. However, for the torus case we can deal with the Liouville field as if they were free. To see this more explicitly, we first integrate over the zero mode of $\phi = \phi_0 + \bar{\phi}$. Then the non-zero mode path integral becomes simply free:

$$\int d\phi_0 e^{2\sum_i \alpha_i \phi_0} e^{-Q\chi\phi_0 - 4\pi\mu e^{2b\phi_0}\int\sqrt{\hat{g}}e^{2b\bar{\phi}}} = \frac{1}{2b}\Gamma(-s)B^s, \tag{88}$$

where $B = 4\pi\mu \int \sqrt{\hat{g}} e^{2b\bar{\phi}}$. It is proposed that when $s \geq 0$, the integral over the oscillating mode $\bar{\phi}$ can be done using free field techniques where there is no classical field configuration. Then the result can be analytically continued back to the $s < 0$ regime. We'll come back to this point in section 7.

Back to this problem, the only contribution from the zero-mode is given by the Liouville volume: $V_\phi = \int d\phi e^{-\mu e^{2b\phi}} = -\frac{1}{2b}\log\mu$.

The remaining integral over the oscillating modes of ϕ is the same as free fields, as is hypothesized above. We omit the calculation here. The fixed-area partition function is:

$$\langle 1 \rangle_A \sim \frac{1}{4\pi A b \sqrt{2\tau_2} |\eta(q)|^2}. \tag{89}$$

When doing the Laplace to fixed-cosmological constant result, one could easily see that there is a logarithmical divergence.

There are also some easy examples which can be obtained purely from the covariance of $SL(2,\mathbb{C})$. Similar to the results in ordinary conformal field theory, the structure of 2-point and 3-point function is fixed by Mobius invariance:

$$e^{2\alpha\phi} \to |\beta z + \delta|^{-4\Delta_\alpha} e^{2\alpha\phi}, \quad \text{under } z \to (\alpha z + \gamma)/(\beta z + \delta). \tag{90}$$

The structure of the two and three-point functions are thus as follows:

$$\begin{aligned} \left\langle e^{2\alpha_1\phi(z_1)} e^{2\alpha_2\phi(z_2)} \right\rangle &= |z_{12}|^{-4\Delta_1} \left(N(\alpha_1)\delta_{\alpha_2, Q-\alpha_1} + B(\alpha_1)\delta_{\alpha_2,\alpha_1} \right); \\ \left\langle e^{2\alpha_1\phi(z_1)} e^{2\alpha_2\phi(z_2)} e^{2\alpha_3\phi(z_3)} \right\rangle &= \frac{C[\alpha_1, \alpha_2, \alpha_3]}{|z_{12}^{\Delta_{123}} z_{13}^{\Delta_{132}} z_{23}^{\Delta_{241}}|}. \end{aligned} \tag{91}$$

Here $\Delta_{ijk} = \Delta_i + \Delta_j - \Delta_k$.

5 c=1 matrix quantum mechanics

6 The DOZZ formula

7 Teschner's proposal

In this section we mainly refer to [5].

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