Higher-Order and Tuple-Based Massively-Parallel Prefix Sums

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Prefix Sum

- Given an array of values (integer or real values)

0	1	2	3	4	5	6	7
3	2	0	7	-6	1	-9	5

- Compute the array whose elements are the sum of all previous elements from the original array

0	1	2	3	4	5	6	7
3	5	5	12	6	7	-2	3

- A prefix scan is a generalization of the prefix sum where the operation doesn't have to be addition

Some Scan Operators

Operator	Identity element	Example
+	0	X + 0 = X
Min	Maximum Representative value	Min(X, ∞) = X
Max	Minimum Representative value	Max(X, -∞) = X
Multiply	1	X * 1 = X
Logical Or	FALSE	X || FALSE = X
Logical AND	TRUE	X && TRUE = X

Uses of Prefix Sums and Scans

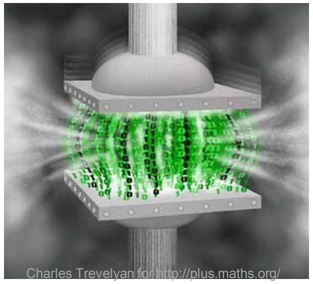
- Fundamental building block of parallel algorithms
  - Can be computed efficiently in parallel in log(*n*) steps
  - Help parallelize many seemingly serial algorithms
- Examples
  - Buffer allocation
  - Radix sort
  - Quicksort
  - String comparison
  - Lexical analysis
  - Run-length encoding
  - Histograms
  - Polynomial evaluation
  - Stream compaction
  - Data compression

Highlights

- GPU-friendly algorithm for prefix scans called **SAM**
- Novelties and features
  - Higher-order support that is communication optimal
  - Tuple-value support with constant workload per thread
  - Carry propagation scheme with O(1) auxiliary storage
  - Implemented in unified 100-statement CUDA kernel
- Results
  - Outperforms CUB by up to 2.9-fold on higher-order and by up to 2.6-fold on tuple-based prefix sums

Data Compression

- Data compression algorithms
  - **Data model** predicts next value in input sequence and emits difference between actual and predicted value
  - **Coder** maps frequently occurring values to produce shorter output than infrequent values



- Delta encoding
  - Widely used data model
  - Computes difference sequence (i.e., predicts current value to be the same as previous value in sequence)
  - Used in image compression, speech compression, etc.

Delta Coding

- Delta encoding is embarrassingly parallel
- Delta decoding appears to be sequential
  - Decoded prior value needed to decode current value
- Prefix sum decodes delta encoded values
  - Decoding can also be done in parallel

Input sequence	1, 2, 3, 4, 5, 2, 4, 6, 8, 10
Difference sequence (encoding)	1, 1, 1, 1, 1, -3, 2, 2, 2, 2
Prefix sum (decoding)	1, 2, 3, 4, 5, 2, 4, 6, 8, 10

Extensions of Delta Coding

- Higher orders
  - Higher-order predictions are often more accurate
    - First order $out_k = in_k - in_{k-1}$
    - Second order $out_k = in_k - 2 \cdot in_{k-1} + in_{k-2}$
    - Third order $out_k = in_k - 3 \cdot in_{k-1} + 3 \cdot in_{k-2} - in_{k-3}$
- Tuple values
  - Data frequently appear in tuples
    - Two-tuples $x_0, y_0, x_1, y_1, x_2, y_2, \ldots, x_{n-1}, y_{n-1}$
    - Three-tuples $x_0, y_0, z_0, x_1, y_1, z_1, \ldots, x_{n-1}, y_{n-1}, z_{n-1}$

Problem and Solution

- Conventional prefix sums are insufficient
  - Do not decode higher-order delta encodings
  - Do not decode tuple-based delta encodings
- Prior work
  - Requires inefficient workarounds to handle higher-order and tuple-based delta encodings
- SAM algorithm and implementation
  - Directly and efficiently supports these generalizations
  - Even supports combination of higher orders and tuples

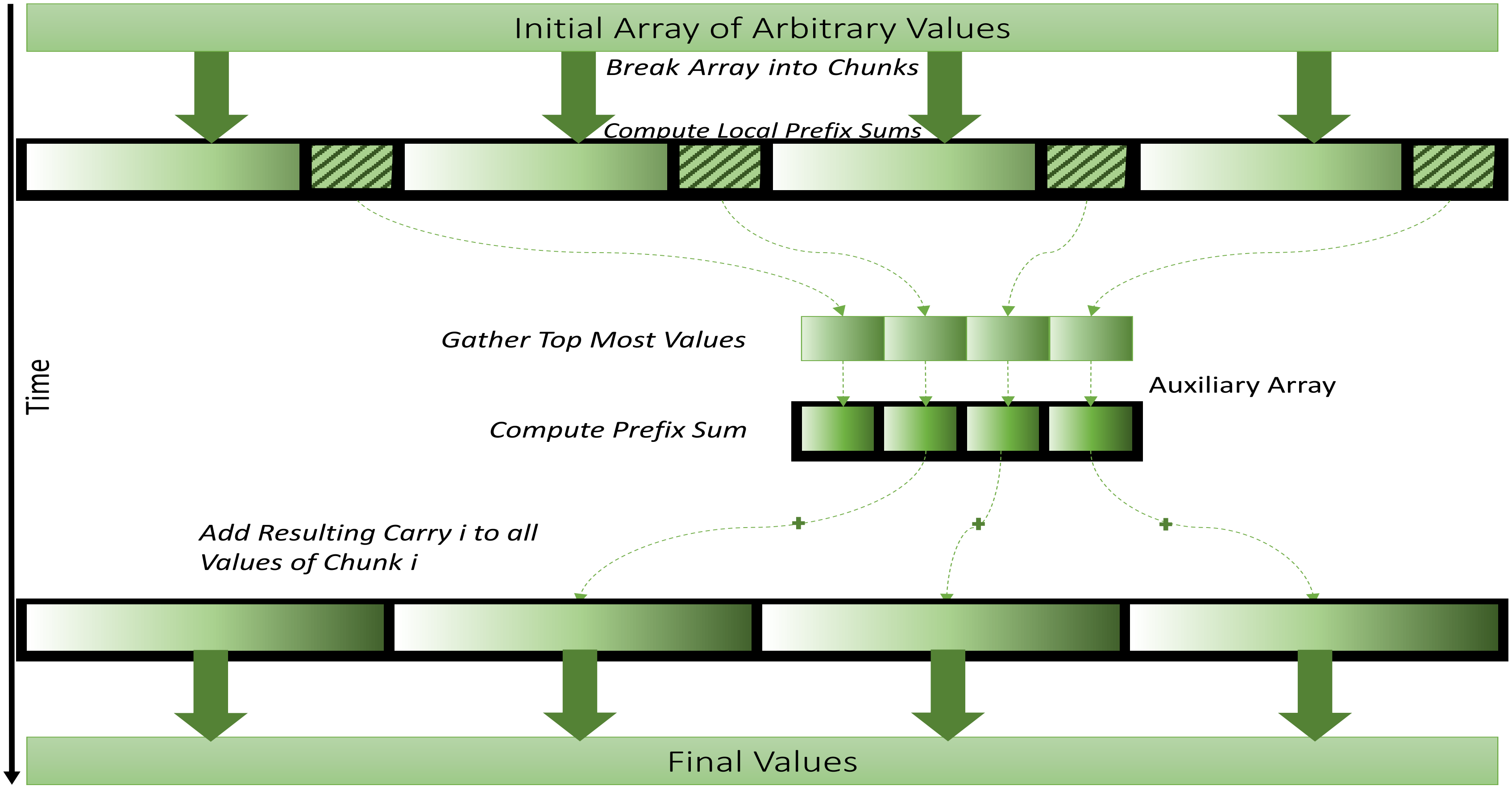
Work Efficiency of Prefix Sums

- Sequential prefix sum requires only a single pass
  - $2n$ data movement through memory
  - Linear $O(n)$ complexity

```
out[0] = 0
for i from 1 to n do
    out[i] = out[i - 1] + in[i - 1]
```

- Parallel algorithm should have same complexity
  - $O(n)$ applications of the sum operator

Hierarchical Parallel Prefix Sum



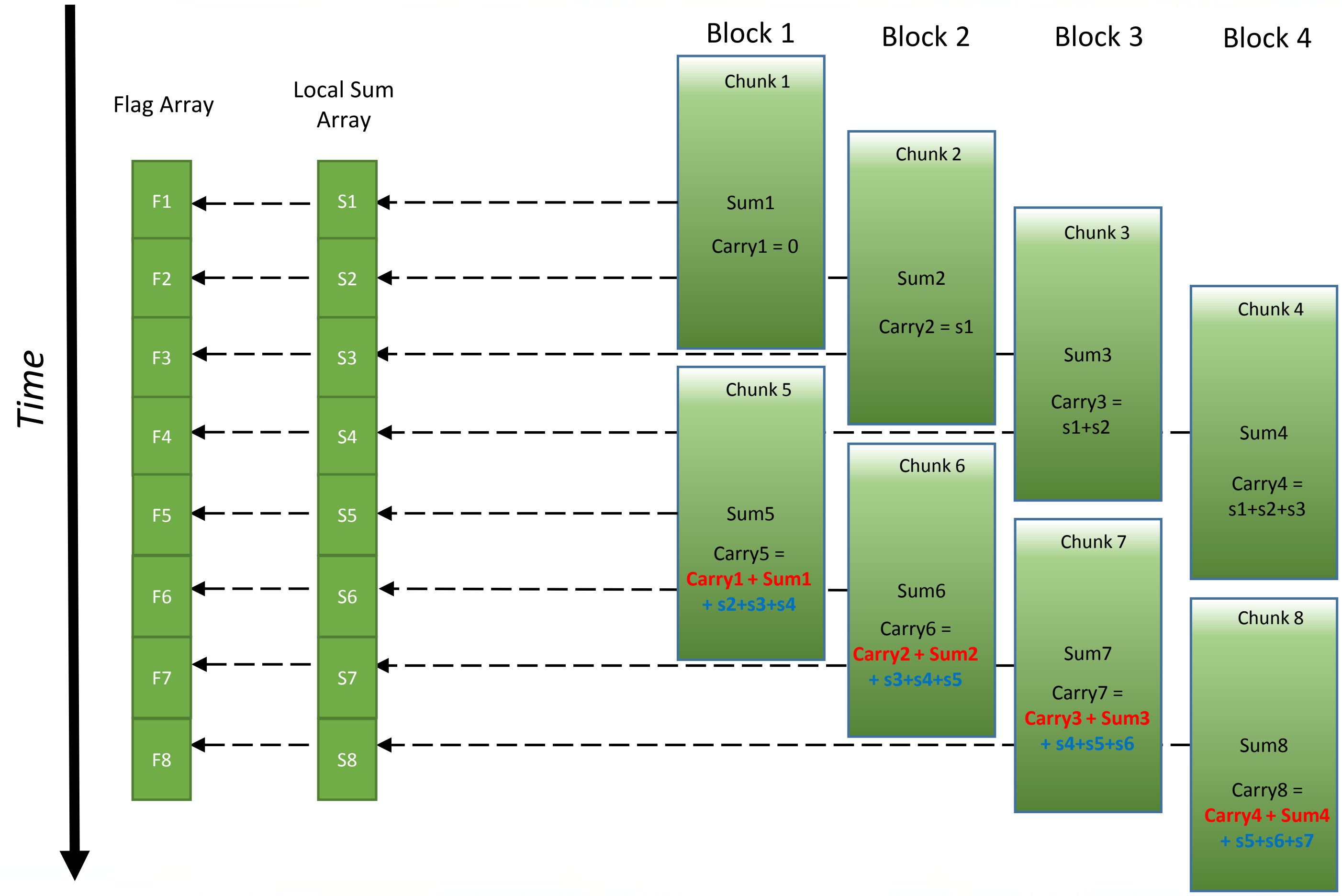
Standard Prefix-Sum Implementation

- Based on 3-phase approach
- Reads and writes every element twice
  - $4n$ main-memory accesses
- Auxiliary array is stored in global memory
  - Calculation is performed across blocks
- High-performance implementations
  - Allocate and process several values per thread
- Thrust and CUDPP use this hierarchical approach

SAM Base Implementation

- Intra-block prefix sums
  - Computes prefix sum of each chunk conventionally
  - Writes local sum of each chunk to auxiliary array
  - Writes ready flag to second auxiliary array
- Inter-block prefix sums
  - Reads local sums of all prior chunks
  - Adds up local sums to calculate carry
  - Updates all values in chunk using carry
  - Writes final result to global memory

Pipelined Processing of Chunks



Carry Propagation Scheme

- Persistent-block-based implementation
  - Same block processes every k^{th} chunk
  - Carries require only O(1) computation per chunk
- Circular-buffer-based implementation
  - Only 3*k* elements needed at any point in time
  - Local sums and ready flags require O(1) storage
- Redundant computations for latency hiding
  - Write-followed-by-independent-reads pattern
  - Multiple values processed per thread (fewer chunks)

Higher-order Prefix Sums

Higher-order Prefix Sums

- Higher-order difference sequences can be computed by repeatedly applying first order

Input values	1, 2, 3, 4, 5, 2, 4, 6, 8, 10
First order	1, 1, 1, 1, 1, -3, 2, 2, 2, 2
Second order	1, 0, 0, 0, 0, -4, 5, 0, 0, 0

- Prefix sum is the inverse of order-1 differencing
  - *K* prefix sums will decode an order-*k* sequence
- No direct solution for computing higher orders
  - Must use iterative approach
  - Other codes' memory accesses proportional to order

Higher-order Prefix Sums (cont.)

- SAM is more efficient
  - Internally iterates only the computation phase
  - Does not read and write data in each iteration
  - Requires only 2*n* main-memory accesses for any order
- SAM's higher-order implementation
  - Employs multiple sum arrays, one per order
    - Each sum array is an O(1) circular buffer
  - Needs O(1) non-Boolean ready 'flags'
    - Uses counts to indicate iteration of current local sum

Tuple-based Prefix Sums

Tuple-based Prefix Sums

- Data may be tuple based $x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}$
- Other codes have to handle tuples as follows
  - Reordering elements, compute, undo reordering
    - Slow due to reordering and may require extra memory

$$x_0, x_1, \ldots, x_{n-1} \mid y_0, y_1, \ldots, y_{n-1}$$

$$\Sigma_0^0 x_i, \Sigma_0^1 x_i, \ldots, \Sigma_0^{n-1} x_i \mid \Sigma_0^0 y_i, \Sigma_0^1 y_i, \ldots, \Sigma_0^{n-1} y_i$$

$$\Sigma_0^0 x_i, \Sigma_0^0 y_i, \Sigma_0 1 x_i, \Sigma_0^1 y_i, \ldots, \Sigma_0^{n-1} x_i, \Sigma_0^{n-1} y_i$$

  - Defining a tuple data type as well as the plus operator
    - Slow for large tuples due to excessive register pressure

Tuple-based Prefix Sums (cont.)

- SAM is more efficient
  - No reordering
  - No special data types or overloaded operators
  - Always same amount of data per thread
- SAM's tuple implementation
  - Employs multiple sum arrays, one per tuple element
    - Each sum array is an O(1) circular buffer
    - Uses modulo operations to determine which array to use
  - Still employs single O(1) Boolean flag array

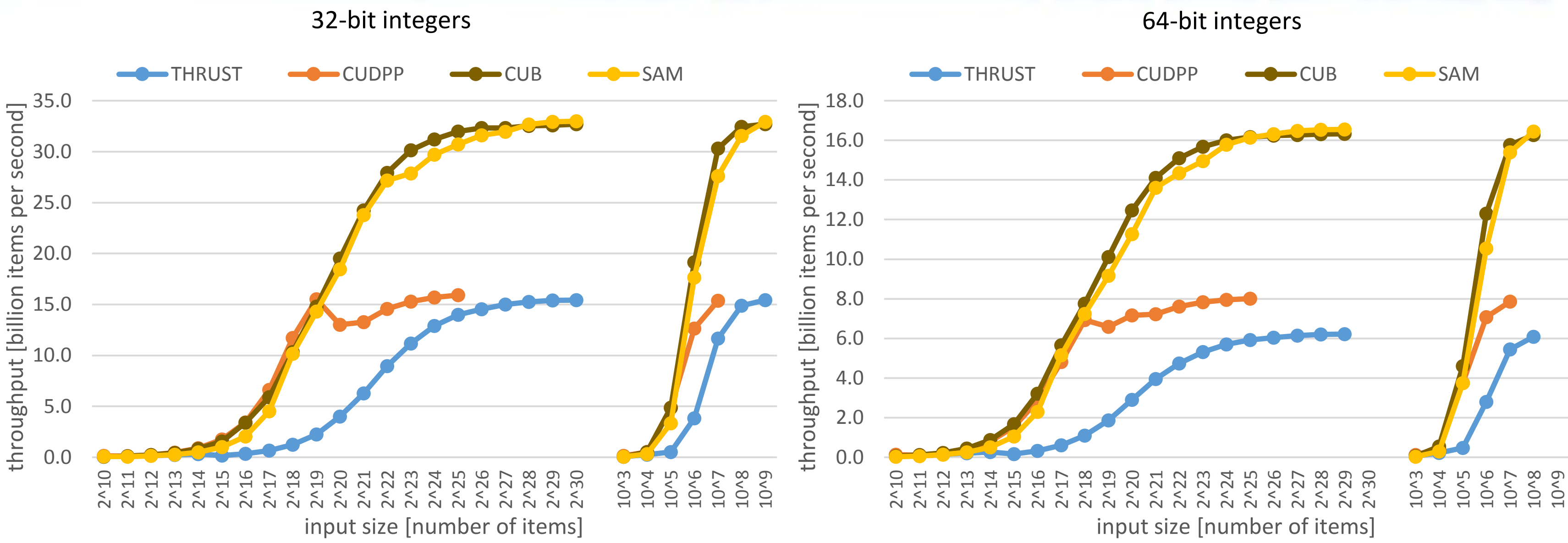
Experimental Methodology

- Evaluate following prefix sum implementations
  - Thrust library (from CUDA Toolkit 7.5)
    - *4n* memory accesses
  - CUDPP library 2.2
    - *4n* memory accesses
  - CUB library 1.5.1
    - *2n* memory accesses
  - SAM 1.1
    - *2n* memory accesses

GPU	GeForce Titan X	Tesla K40c
Architecture	Maxwell	Kepler
PE	3072	2880
Multiprocessors	24	15
Persistent Blocks	48	30
Global Memory	12 GB	12 GB
Peak Bandwidth	336 GB/s	288 GB/s

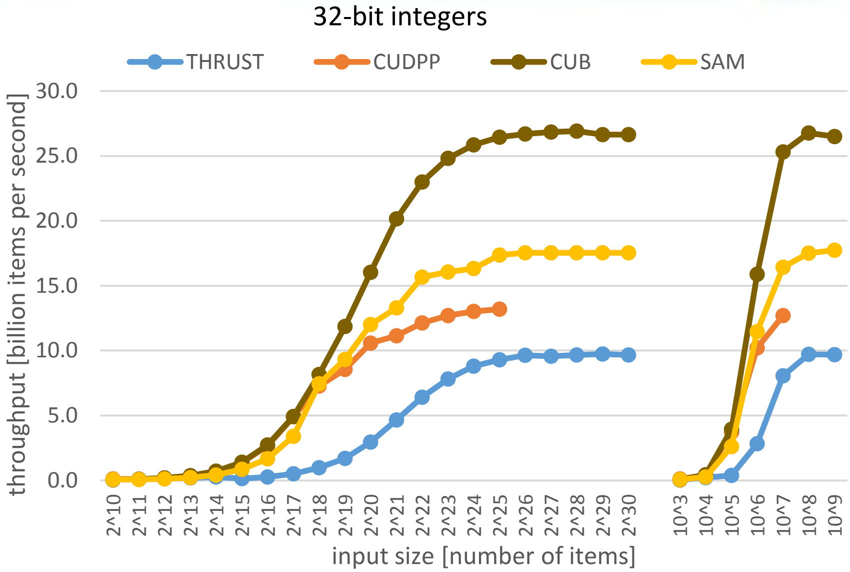
Performance Evaluation

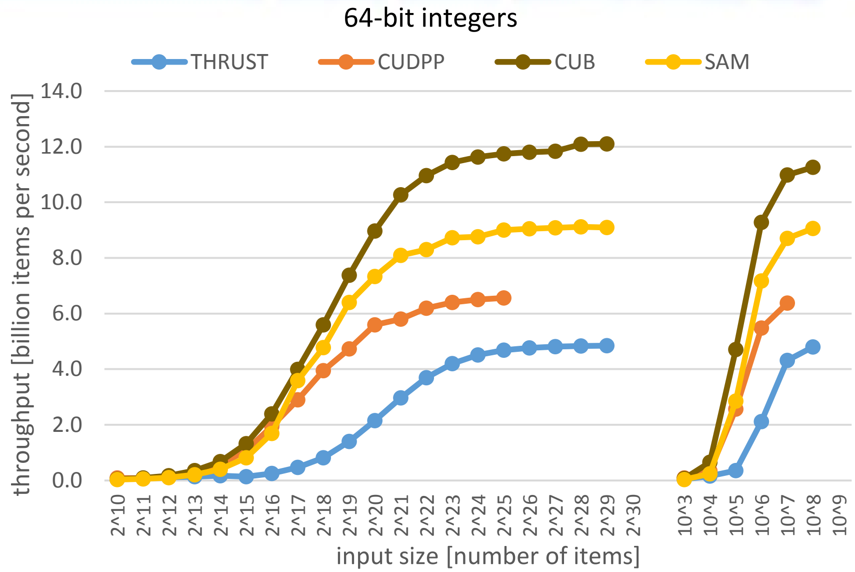
Prefix Sum Throughputs (Titan X)



- SAM and CUB outperform the other approaches (2*n* vs. 4*n*)
- For 64-bit values, throughputs are about half (but same GB/s)
- SAM matches cudaMemcpy throughput at high end (264 GB/s)
  - Surprising since SAM was designed for higher orders and tuples

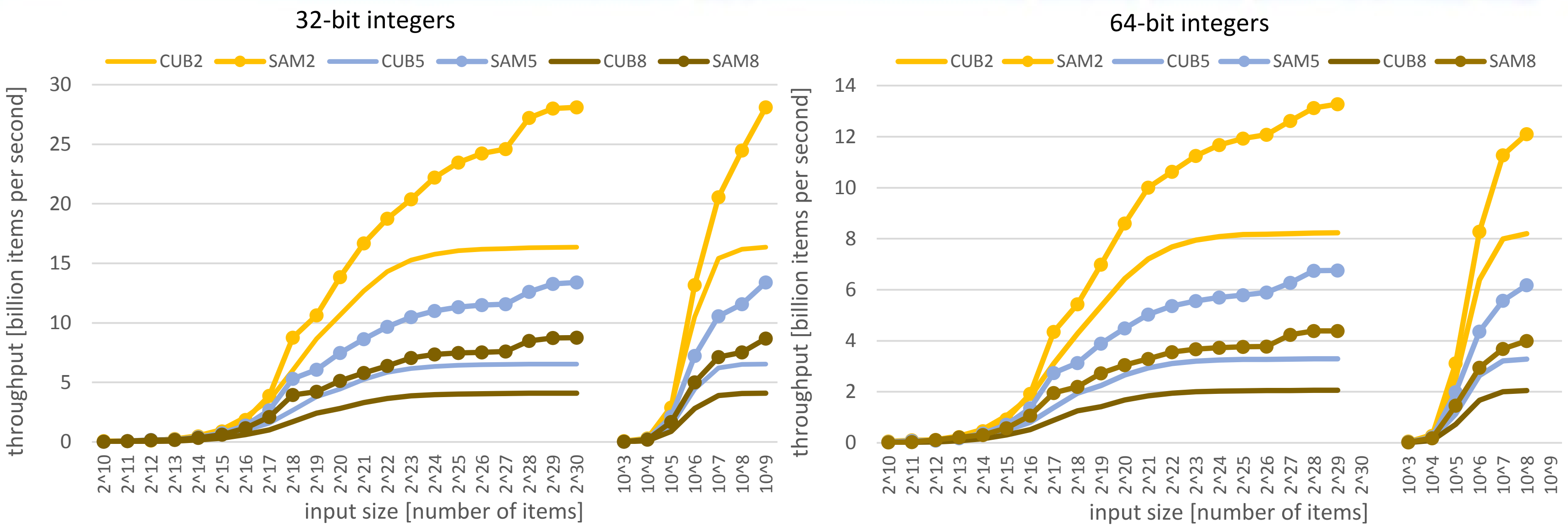
Prefix Sum Throughputs (K40)





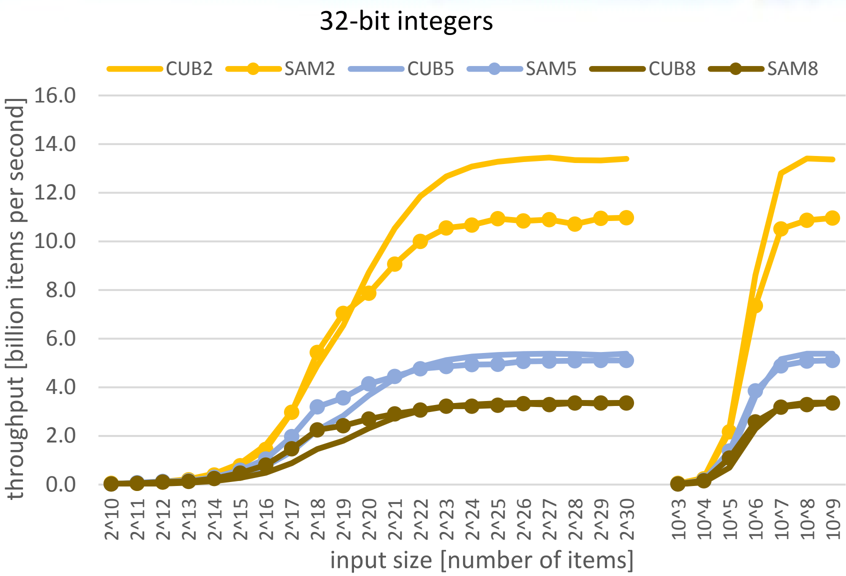
- K40 throughputs are lower for all algorithms
- SAM is faster than Thrust/CUDPP on medium and large inputs
- CUB outperforms SAM by 50% on large inputs on 32-bits ints
  - SAM's implementation is not a particularly good fit for this older GPU

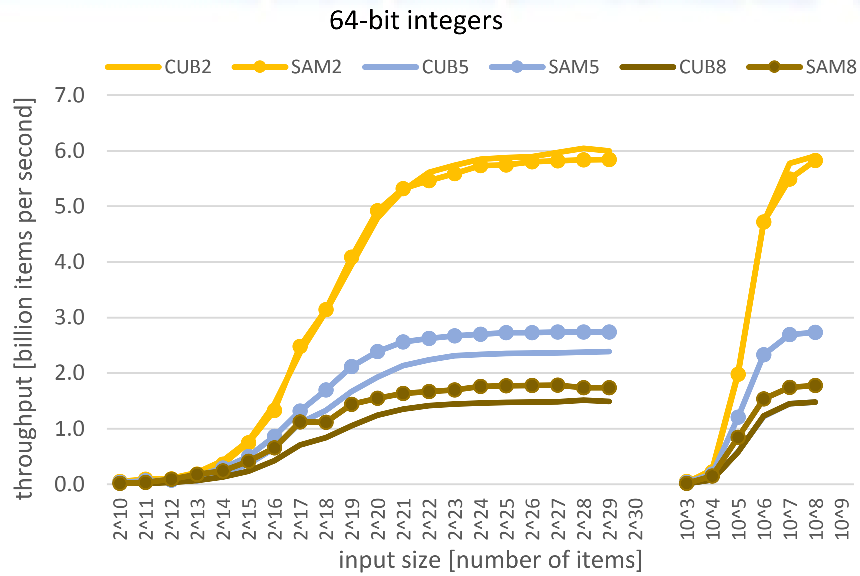
Higher-order Throughputs (Titan X)



- Throughputs decrease as order increases due to more iterations
- SAM's performance advantage increases with higher orders
  - Always executes 2*n* global memory accesses
  - Outperforms CUB by 52% on order 2, 78% on order 5, and 87% on order 8

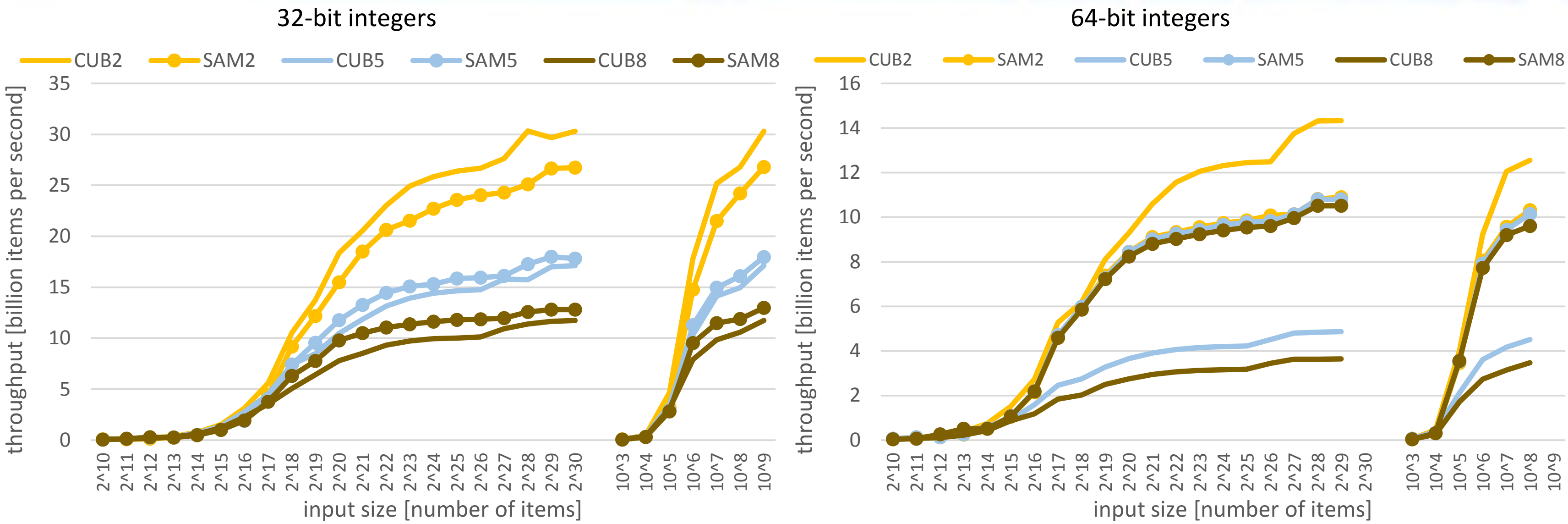
Higher-order Throughputs (K40)





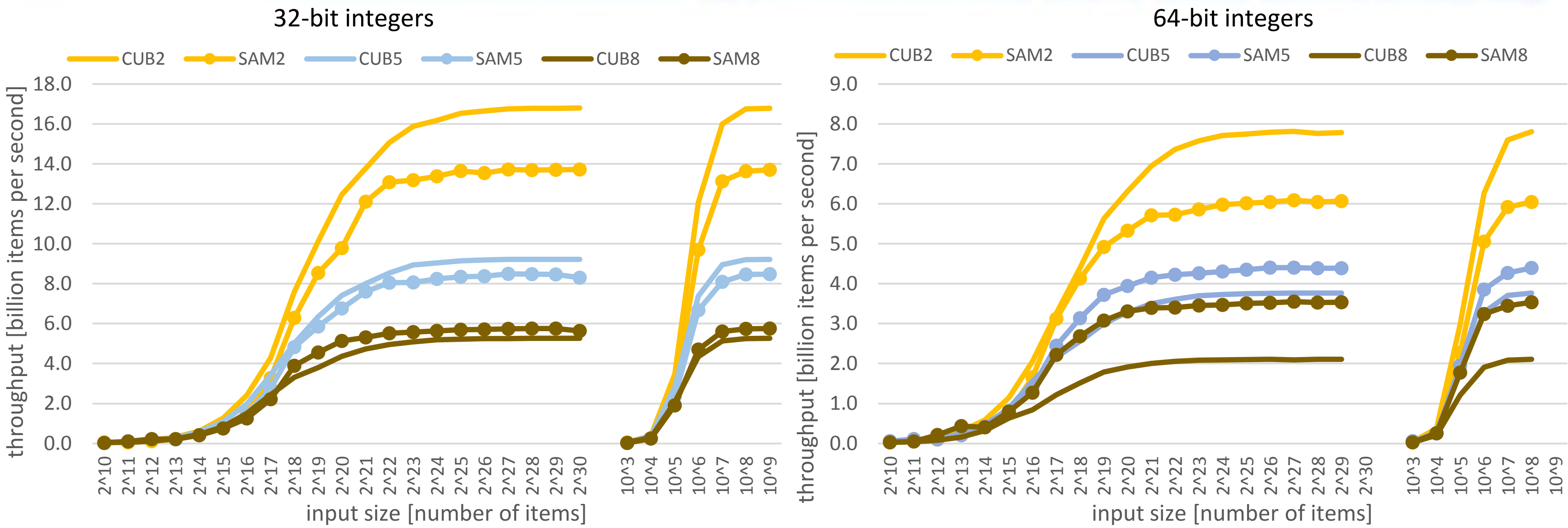
- CUB outperforms SAM on orders 2 and 5, but not on order 8
  - Again, SAM's relative performance increases with higher orders
- SAM's relative performance over CUB is higher on 64-bit values
  - Baseline advantage of CUB over SAM is smaller for 64-bit values

Tuple-based Throughputs (Titan X)



- Throughputs decrease with larger tuple sizes due to extra work
- SAM's performance advantage increases with larger tuple sizes
  - Larger tuples increase register pressure in CUB but not in SAM
  - SAM is 17% slower on 2-tuples but 20% faster on 5-tuples and 34% faster on 8-tuples

Tuple-based Throughputs (K40)



- SAM outperforms CUB on 8-tuples (and larger tuples)
  - Again, SAM's relative performance increases with larger tuple sizes
- The benefit of SAM over CUB is higher with 64-bit values
  - SAM already outperforms CUB on 5-tuples

Summary

- SAM directly supports generalized prefix scans
  - Higher-order prefix scans
  - Tuple-based prefix scans
- SAM performance on Maxwell and Kepler GPUs
  - Reaches cudaMemcpy throughput on large inputs
  - Outperforms all alternatives by up to 2.9x on order-eight and by up to 2.6x on eight-tuple prefix sums
  - SAM's relative performance increases with higher orders and larger tuple sizes

Question?

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http://cs.txstate.edu/~burtscher/research/SAM/

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