

Announcements

- HW 4 deadline extended to tomorrow, 6 pm
 - Can use at most two late days
- HW5 (the last HW) will be out soon
 - Due roughly at the end of this month
- Reminder: Midterm 2 will be December 2nd 9 am to 3rd 9 am
 - Only covers second half of the course: max flow, min cut, NP-hardness, approximation Algo, game theory
 - Takes around 3 hours, but you will have the entire day
 - Format is the same as midterm 1, open notes as well

CS6161: Design and Analysis of Algorithms (Fall 2020)

Introduction to Game Theory I



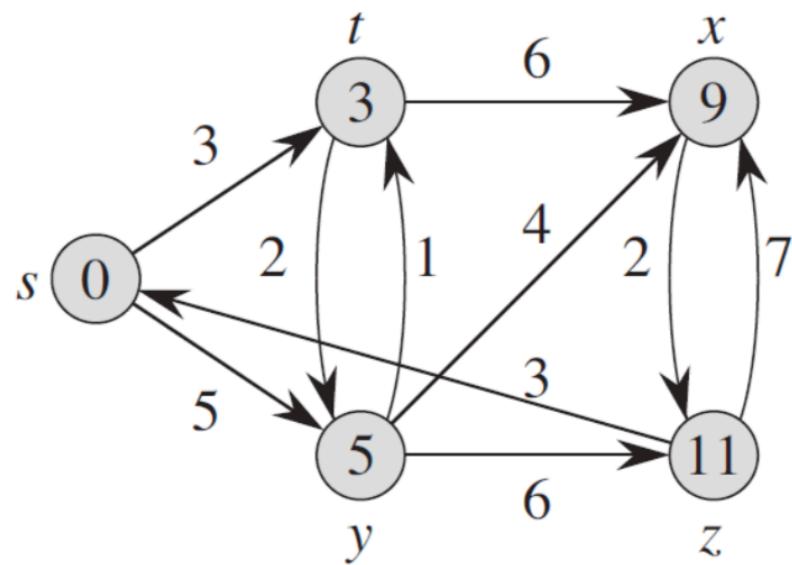
Instructor: Haifeng Xu

Outline

- Why Game Theory in Algorithm Design?
- Intro to Game Theory and Nash Equilibrium

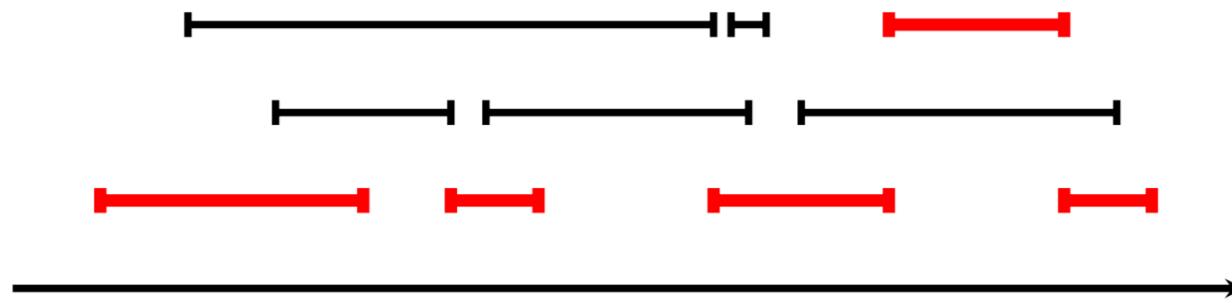
In all algorithm design problems we see so far, the designer is the ONLY agent and controls everything

- Shortest path – only the designer takes a path, no other routers and no congestion



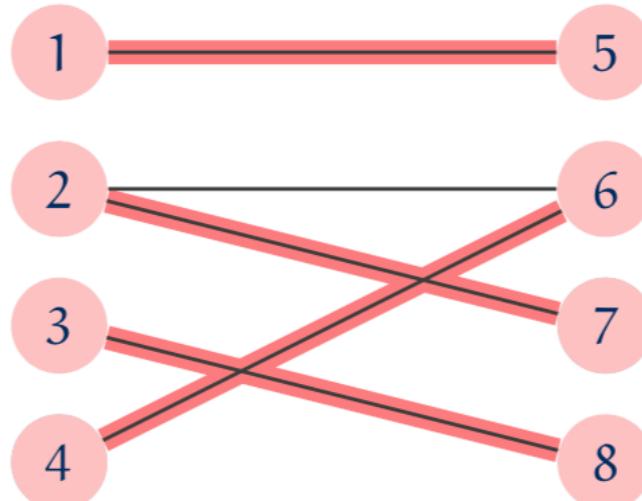
In all algorithm design problems we see so far, the designer is the **ONLY** agent and controls everything

- Shortest path – only the designer takes a path, no other routers and no congestion
- Job scheduling – all jobs honestly tell you (start, end) time, despite knowing that they may not be able to get completed



In all algorithm design problems we see so far, the designer is the **ONLY** agent and controls everything

- Shortest path – only the designer takes a path, no other routers and no congestion
- Job scheduling – all jobs honestly tell you (start, end) time, despite knowing that they may not be able to get completed
- Bipartite matching – a node is willing to be matched to any other node, no preferences



In all algorithm design problems we see so far, the designer is the ONLY agent and controls everything

- Shortest path – only the designer takes a path, no other routers and no congestion
- Job scheduling – all jobs honestly tell you (start, end) time, despite knowing that they may not be able to get completed
- Bipartite matching – a node is willing to be matched to any other node, no preferences
- Max flow – can optimize smuggling of drug, but there will be interdiction measures from security agencies
-

Real Algorithmic Problems Involve Multiple Agents

- Shortest path – need to avoid congestion caused by other drivers
- Job scheduling – you want your job to be completed by the server, and may adjust its (start, end) time if it helps
- Bipartite matching – you would want to be matched to a more preferred node, if that node also prefers you
- Max flow – can optimize smuggling of drug, but there will be interdiction measures from security agencies

Previous algorithmic techniques cannot handle such problems
with multiple strategic agents, a.k.a., games

Real Algorithmic Problems Involve Multiple Agents

- Shortest path – need to avoid congestion caused by other drivers
- Job scheduling – you want your job to be completed by the server, and may adjust its (start, end) time if it helps
- Bipartite matching – you would want to be matched to a more preferred node, if that node also prefers you
- Max flow – can optimize smuggling of drug, but there will be interdiction measures from security agencies

Game theory also studies optimization (i.e., algorithm design), but in settings with multiple agents, each maximizing their own objective

It turns out... things get much more complicated, **even counter-intuitive**, with multiple agents

- Even with two agents, each choosing among two actions

Example I: Prisoner's Dilemma

- Two members A,B of a criminal gang are arrested
- They are questioned in two separate rooms
 - ❖ No communications between them



	B	B stays silent	B betrays
A			
A stays silent	-1	0	
A betrays	0	-3	-2

Q: How should each prisoner act?

- Betray is the best action for each

Example I: Prisoner's Dilemma

- Two members A,B of a criminal gang are arrested
- They are questioned in two separate rooms
 - ❖ No communications between them



	B	B stays silent	B betrays
A	B stays silent	-1	0
A stays silent	-1	-3	
A betrays	0	-3	-2

Q: How should each prisoner act?

- Betray is the best action for each

Example I: Prisoner's Dilemma

- Two members A,B of a criminal gang are arrested
- They are questioned in two separate rooms
 - ❖ No communications between them

	B	B stays silent	B betrays
A			
A stays silent	-1	-1	0
A betrays	0	-3	-2

equilibrium

Q: How should each prisoner act?

- Betray is the best action for each
- But, (-1,-1) is a better outcome for both
- Why? What goes wrong?
 - Selfish behaviors lead to inefficient social outcome

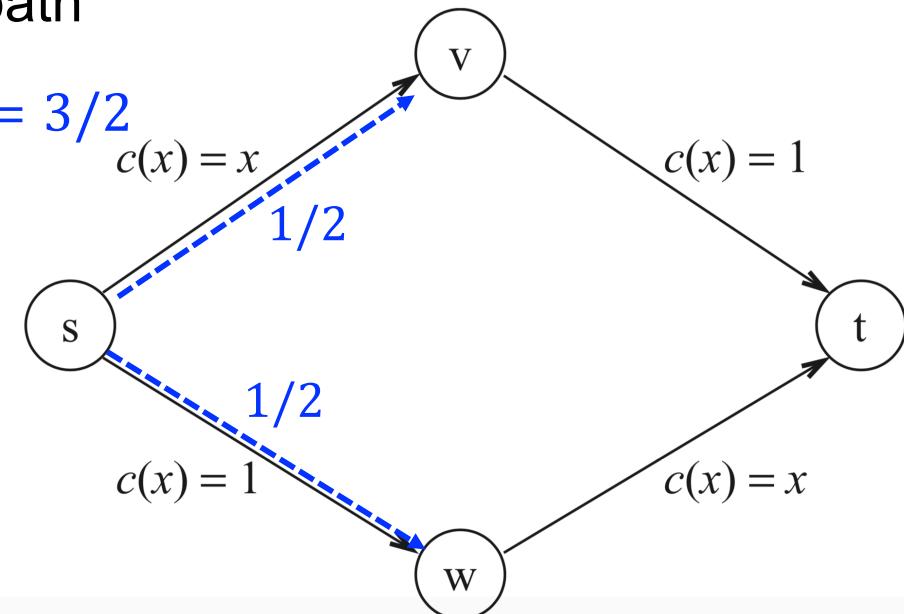
Example 2: Selfish Routing

- One unit flow from s to t which consists of (infinite) individuals, each controlling an infinitesimal small amount of flow
- Each individual wants to minimize his own travel cost

Q: What is the equilibrium status?

- Half unit flow through each path

- Total social cost = $\frac{3}{2} \left(\frac{1}{2} + \frac{1}{2} \right) = 3/2$

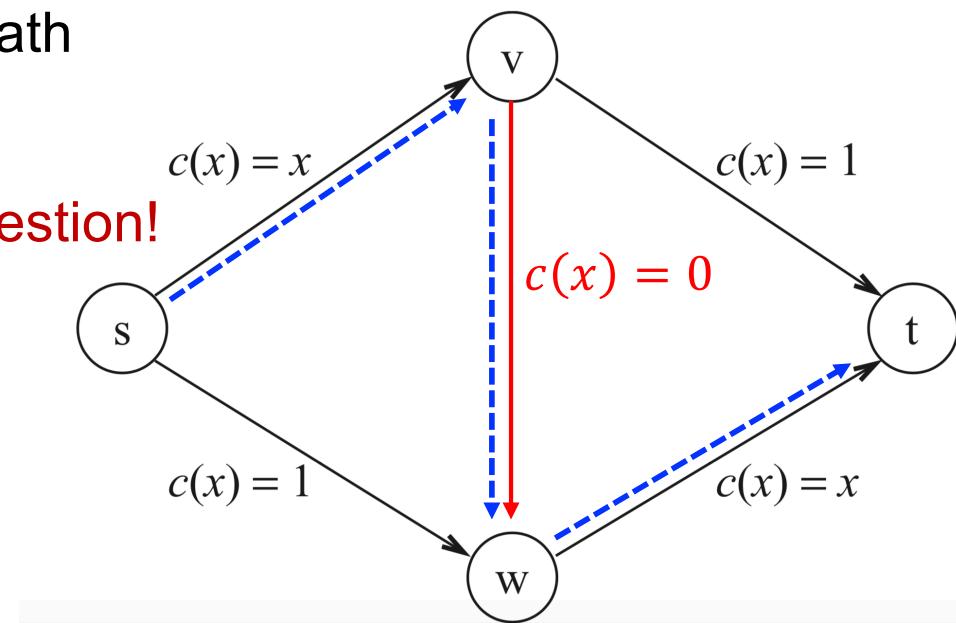


Example 2: Selfish Routing

- One unit flow from s to t which consists of (infinite) individuals, each controlling an infinitesimal small amount of flow
- Each individual wants to minimize his own travel cost

Q: If government builds a superior highway with 0 traveling cost, what is the new equilibrium status?

- Everyone takes the blue path
- Total social cost = 2
- More roads increase congestion!

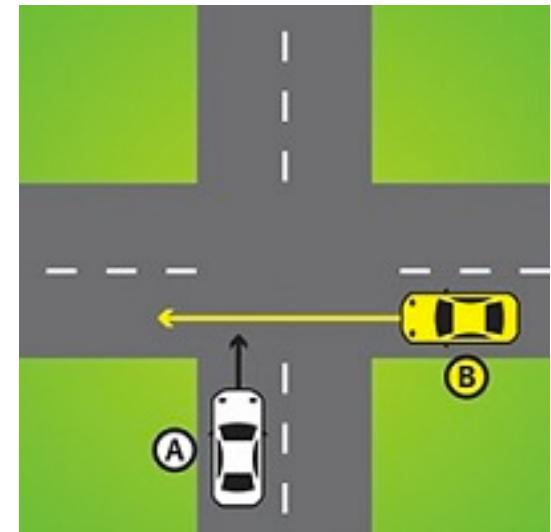


Example 3: Traffic Light Game

- Two cars heading to orthogonal directions

		B
		STOP
A		STOP
STOP		(-3, -2)
GO		(0, -2)

(-3, -2)	(-3, 0)
(0, -2)	(-100, -100)



Q: what are the equilibrium statuses?

Answer: (STOP, GO) and (GO, STOP)

Key Characteristics of These Games

- Each agent wants to maximize her own payoff
- An agent's payoff depends on other agents' actions
- The interaction stabilizes at a state where no agent can increase his payoff via **unilateral deviation**
 - Remark: this does not mean they achieve globally best possible utility (e.g., prisoner's dilemma)

Strategic Games Are Ubiquitous

➤ Pricing

<input type="checkbox"/> Spirit Airlines (2) \$438	6:30am - 8:15am United	2h 45m (Nonstop) BOS - ORD Very Good Flight (8.1/10) Details & baggage fees ▾	5 left at \$236 roundtrip	Select
<input type="checkbox"/> Morning (5:00am - 11:59am)				
<input type="checkbox"/> Afternoon (12:00pm - 5:59pm)	9:23am - 11:27am American Airlines	3h 4m (Nonstop) BOS - ORD Very Good Flight (8.3/10) Details & baggage fees ▾	\$236 roundtrip	Select
<input type="checkbox"/> Evening (6:00pm - 11:59pm)				
<input type="checkbox"/> Early Morning (12:00am - 4:59am)	7:01am - 9:10am American Airlines	3h 9m (Nonstop) BOS - ORD Very Good Flight (8.3/10) Details & baggage fees ▾	\$236 roundtrip	Select
<input type="checkbox"/> Morning (5:00am - 11:59am)				
<input type="checkbox"/> Afternoon (12:00pm - 5:59pm)	5:30am - 8:50am Delta	4h 20m (1 stop) BOS - 42m in DTW - ORD Satisfactory Flight (6.4/10) Details & baggage fees ▾	1 left at \$246 roundtrip	Select
<input type="checkbox"/> Evening (6:00pm - 11:59pm)				

[Run this search to ad](#)

Strategic Games Are Ubiquitous

- Pricing
- Sponsored search
 - Drives 90%+ of Google's revenue

Google search results for "where to buy cruise vacation". The results page shows a header with the Google logo, a search bar containing the query, and navigation links for All, Shopping, Images, News, Videos, More, Settings, and Tools. Below the header, it says "About 103,000,000 results (0.63 seconds)".

The first result is a sponsored ad for Carnival Cruise Line, highlighted with a red box. It features the text "Cruises | Caribbean Vacations | Carnival Cruise Line" and "Ad www.carnival.com/". It includes two promotional offers: "\$1.03" for "2-5 Day Cruises" and "\$1.03" for "6-9 Day Cruises". Below these offers is the text "Make Your Vacation Dreams A Reality With A Carnival® Cruise. Book Online Today! Signature Dining." and "Set Sail On These Quick Getaways That Fit Any Calendar, Anytime.".

The second result is another sponsored ad for Carnival Cruise Line, showing a 3D model of a cruise ship and the text "3-D Cruise Ship Centerpiece \$6.65 Zoom Party".

The third result is an ad for Expedia Cruises, showing the text "Expedia Cruises | Cruise Vacations" and "Ad www.expedia.com/Cruises". It includes the text "Find the Perfect Cruise at the Best Price on Expedia, the #1 Travel Site To Book Cruises. Last Minute Cruise Deals. Best Price Guaranteed. 4,000 Cruises Worldwide. Luxury Cruises Available. Destinations: Caribbean, Bahamas, Alaska, Mexico, Europe, Bermuda, Hawaii, Canada/New England." and a large price of "\$1.02".

The fourth result is an ad for VacationToGo.com, showing the text "2019 Cruises 82% Off | Compare All Cruise Lines | VacationsToGo.com" and "Ad www.vacationstogo.com/". It includes the text "Book today for best price and selection on 2019 cruises. Save up to 82% on every Ship. Last-Minute Cruise Deals · Age 55+ Discounts · Caribbean up to 82% Off - Huge Carnival Deals" and a large price of "\$0.60".

The fifth result is an ad for Kayak.com, showing the text "KAYAK® Cruise Search | Find the Cheapest Cruise Deals | kayak.com" and "Ad www.kayak.com/vacations-go/last". It includes the text "Rating for kayak.com: 2.9 - 605 reviews" and a large price of "\$0.21".

On the right side of the results, there is a link "See cruise vac..." and a button "Sponsored ⓘ". At the bottom right, there is a link "More on Google" with an arrow icon.

Strategic Games Are Ubiquitous

- Pricing
- Sponsored search
 - Drives 90%+ of Google's revenue
- FCC's Allocation of spectrum to radio frequency users

The screenshot shows the top navigation bar of the FCC website. On the left is the FCC logo and name. To the right are two search/filter options: "Browse by CATEGORY" and "Browse by BUREAUS & OFFICES". Further right is a search bar with a magnifying glass icon. Below the header is a blue navigation bar with links: About the FCC, Proceedings & Actions, Licensing & Databases, Reports & Research, News & Events, and For Consumers.

Home / Economics and Analytics /

Auctions

Proceedings & Actions

Proceedings and Actions Overview

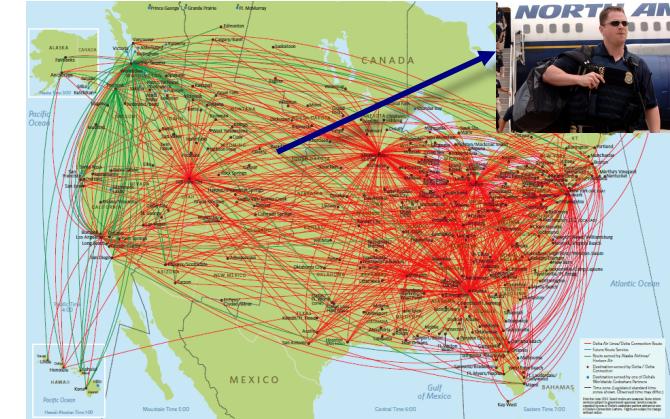
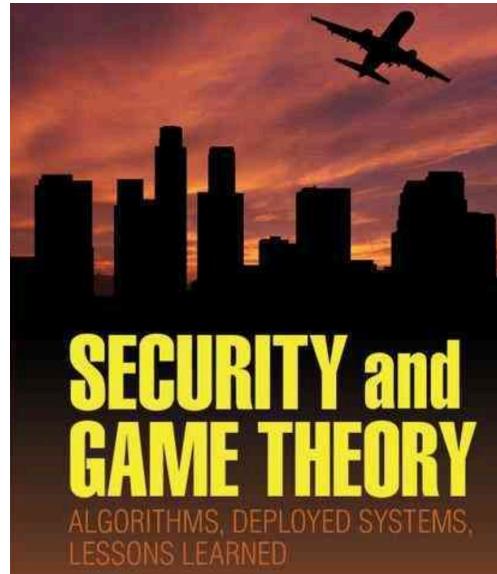
Since 1994, the Federal Communications Commission (FCC) has conducted auctions of licenses for electromagnetic spectrum. These auctions are open to any eligible company or individual that submits an application and upfront payment, and is found to be a qualified bidder by the Commission ([More About Auctions...](#))

Go to an Auction

Select an Au...

Strategic Games Are Ubiquitous

- Pricing
- Sponsored search
 - Drives 90%+ of Google's revenue
- FCC's Allocation of spectrum to radio frequency users
- National security, border patrolling, counter-terrorism



Optimize resource allocation against
attackers/adversaries

Strategic Games Are Ubiquitous

- Pricing
- Sponsored search
 - Drives 90%+ of Google's revenue
- FCC's Allocation of spectrum to radio frequency users
- National security, border patrolling, counter-terrorism
- Kidney exchange – decides who gets which kidney at when

The screenshot shows the UNOS website with a blue header bar. The header includes the UNOS logo, a navigation menu with links to Transplant, Solutions, Technology, Data, Policy, Community, Resources, News, and a search icon, and a breadcrumb trail indicating the user is on the 'Kidney paired donation' page under the 'Transplant' category.

Kidney paired donation

Kidney paired donation (KPD) is a transplant option for candidates who have a living donor who is medically able, but cannot donate a kidney to their intended candidate because they are incompatible (i.e., poorly matched).

[Download PDF](#)

[Learn about kidney paired donation](#)

**UNOS gratefully
acknowledges our
sponsors**

UNITED HEALTH FOUNDATION[®]



Strategic Games Are Ubiquitous

- Pricing
- Sponsored search
 - Drives 90%+ of Google's revenue
- FCC's Allocation of spectrum
- National security, border control
- Kidney exchange – decide who gets organs
- Entertainment games: poker, blackjack, Go, chess . . .
- Social choice problems such as voting, fair division, etc.



Strategic Games Are Ubiquitous

- Pricing
- Sponsored search
 - Drives 90%+ of Google's revenue
- FCC's Allocation of spectrum to radio frequency users
- National security, border patrolling, counter-terrorism
- Kidney exchange – decides who gets which kidney at when
- Entertainment games: poker, blackjack, Go, chess . . .
- Social choice problems such as voting, fair division, etc.

These are just a few example domains *where computer science has made significant impacts*; There are many others.

Outline

- Why Game Theory in Algorithm Design?
- Intro to Game Theory and Nash Equilibrium

Main Components of a Game

- **Players**: participants of the game, each may be an individual, organization, a machine or an algorithm, etc.
- **Strategies**: actions available to each player
- **Outcome**: the **profile** of player strategies
- **Payoffs**: a function mapping an **outcome** to a **utility** for each player

Normal Form Representation

For simplicity, consider two-player games:

- Player A with action set $\{a_1, a_2, \dots, a_n\}$
- Player B with action set $\{b_1, b_2, \dots, b_m\}$
- An outcome (a_i, b_j) results in utility (u_{ij}, v_{ij}) for A and B, respectively
- Matrix Representation (assume each player knows the matrix):

	b_1	b_2	...	b_m
a_1	(u_{11}, v_{11})	(u_{12}, v_{12})	...	(u_{1m}, v_{1m})
a_2	(u_{21}, v_{21})	(u_{22}, v_{22})	...	(u_{2m}, v_{2m})
...				
a_n	(u_{n1}, v_{n1})	(u_{n2}, v_{n2})	...	(u_{nm}, v_{nm})

But this representation easily generalizes to k players,
just with more involved notations

Normal Form Representation

For simplicity, consider two-player games:

- Player A with action set $\{a_1, a_2, \dots, a_n\}$
- Player B with action set $\{b_1, b_2, \dots, b_m\}$
- An outcome (a_i, b_j) results in utility (u_{ij}, v_{ij}) for A and B, respectively
- Matrix Representation (assume each player knows the matrix):

	b_1	b_2	...	b_m
a_1	(u_{11}, v_{11})	(u_{12}, v_{12})	...	(u_{1m}, v_{1m})
a_2	(u_{21}, v_{21})	(u_{22}, v_{22})	...	(u_{2m}, v_{2m})
...				
a_n	(u_{n1}, v_{n1})	(u_{n2}, v_{n2})	...	(u_{nm}, v_{nm})

But this representation easily generalizes to k players,
just with more involved notations

Normal Form Representation

For simplicity, consider two-player games:

- Player A with action set $\{a_1, a_2, \dots, a_n\}$
- Player B with action set $\{b_1, b_2, \dots, b_m\}$
- An outcome (a_i, b_j) results in utility (u_{ij}, v_{ij}) for A and B, respectively
- Matrix Representation (assume each player knows the matrix):

Remark:

- (1) This is the most basic form of representation
- (2) Can be more complicated (e.g., sequential actions, incomplete information), but will not cover in this course

Equilibrium

- An outcome (a_{i^*}, b_{j^*}) is an equilibrium if
 1. Given B plays b_{j^*} , a_{i^*} is optimal for A: $u_{i^*, j^*} \geq u_{i, j^*}$ for any $i \in [n]$
 2. Given A plays a_{i^*} , b_{j^*} is optimal for B: $v_{i^*, j^*} \geq v_{i^*, j}$ for any $j \in [m]$
- Also generalizes to k players – an action profile is an equilibrium if each action is the player's best action, given opponents' actions
- A special case of Nash equilibrium

Why should you care about “equilibrium”?

- It predicts what will happen in this game with rational players
 - E.g., in prisoner's dilemma, prisoners are predicted to both betray
- Just like **optimal solution describes what a rational algorithm designer will do** in a single-agent optimization problem

Examples

	B	B stays silent	B betrays
A			
A stays silent	-1	-3	0
A betrays	0	-3	-2

	L	M	R
U	4,3	5,1	6,2
M	2,1	8,4	3,6
D	3,0	9,6	2,5

Q: what is an equilibrium of this game? Is it unique?

Example: Rock-Paper-Scissor game

		Player 2		
		Rock	Paper	Scissor
Player 1	Rock	(0, 0)	(-1, 1)	(1, -1)
	Paper	(1, -1)	(0, 0)	(-1, 1)
	Scissor	(-1, 1)	(1, -1)	(0, 0)

Q: what is an equilibrium?

- No deterministic action pair can make both players happy

Example: Rock-Paper-Scissor game

		Player 2		
		Rock	Paper	Scissor
Player 1	Rock	(0, 0)	(-1, 1)	(1, -1)
	Paper	(1, -1)	(0, 0)	(-1, 1)
	Scissor	(-1, 1)	(1, -1)	(0, 0)

Q: what is an equilibrium?

- Key insight: randomize over actions
- Common sense suggests $(1/3, 1/3, 1/3)$
- Deterministic action is called **pure strategy**
- A distribution over actions is called **mixed strategy**

Note: action and strategy will be used interchangeably

Mixed-Strategy Equilibrium

- Player A plays mixed strategy $\mathbf{p}(\sum p_i = 1)$ and B plays $\mathbf{q}(\sum q_i = 1)$
 - They sample actions independently
- Expected utility given joint strategy profile (\mathbf{p}, \mathbf{q}) :

$$u(\mathbf{p}, \mathbf{q}) = \sum_{ij} u_{i,j} p_i q_j \text{ for A,} \quad v(\mathbf{p}, \mathbf{q}) = \sum_{ij} v_{i,j} p_i q_j \text{ for B}$$

Def. A strategy profile (\mathbf{p}, \mathbf{q}) is a **mixed-strategy equilibrium** if

- (1) $u(\mathbf{p}, \mathbf{q}) \geq u(a_i, \mathbf{q}) = \sum_j u_{i,j} q_j$ for any $a_i \in A$
- (2) $v(\mathbf{p}, \mathbf{q}) \geq v(\mathbf{p}, b_j) = \sum_i v_{i,j} p_i$ for any $b_j \in B$

- In other words, neither player cannot do better by deviating to any pure action (**and thus any mixed action**)
- Key question: just like pure equilibrium, such a mixed equilibrium may not exist either!

Mixed-Strategy Equilibrium

Theorem (Nash, 1951): Every finite game admits at least one mixed-strategy equilibrium

- Finite = finite many actions and players
- A revolutionary result, become foundation of game theory
 - Since when pure equilibrium does not exist, we cannot predict how players will play
 - Nash's results suggest we can expect them to play mixed equilibrium, and it always exists!
 - For example, in rock-paper-scissor games, we should expect rational players to play uniformly at random
- Since then, people call (mixed) equilibrium the **Nash equilibrium (NE)**

An Illustrative Example of Nash Equilibrium (NE)

- Rock-paper-scissor: $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a best response to $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

		1/3	1/3	1/3
		Rock	Paper	Scissor
ExpU = 0	Rock	(0, 0)	(-1, 1)	(1, -1)
ExpU = 0	Paper	(1, -1)	(0, 0)	(-1, 1)
ExpU = 0	Scissor	(-1, 1)	(1, -1)	(0, 0)

An Important Property of Nash Equilibrium

Thm.: If (p, q) is a NE, then any a_i with $p_i > 0$ must be equally good – they are all optimal actions (also called best responses)

➤ Proof: follows from definition

$$u(\mathbf{p}, \mathbf{q}) \geq u(a_k, \mathbf{q})$$

		1/3	1/3	1/3
ExpU = 0	Rock	(0, 0)	(-1, 1)	(1, -1)
ExpU = 0	Paper	(1, -1)	(0, 0)	(-1, 1)
ExpU = 0	Scissor	(-1, 1)	(1, -1)	(0, 0)

An Important Property of Nash Equilibrium

Thm.: If (\mathbf{p}, \mathbf{q}) is a NE, then any a_i with $p_i > 0$ must be equally good – they are all optimal actions (also called best responses)

➤ Proof: follows from definition

$$\begin{aligned} \sum_i p_i [\sum_j u_{i,j} q_j] &= u(\mathbf{p}, \mathbf{q}) \geq u(a_k, \mathbf{q}) = \sum_j u_{k,j} q_j \text{ for any } a_k \\ \Rightarrow \sum_i p_i [\sum_j u_{i,j} q_j] &\geq \max_k \sum_j u_{k,j} q_j \end{aligned}$$

		1/3	1/3	1/3
ExpU = 0	Rock	(0, 0)	(-1, 1)	(1, -1)
ExpU = 0	Paper	(1, -1)	(0, 0)	(-1, 1)
ExpU = 0	Scissor	(-1, 1)	(1, -1)	(0, 0)

An Important Property of Nash Equilibrium

Thm.: If (\mathbf{p}, \mathbf{q}) is a NE, then any a_i with $p_i > 0$ must be equally good – they are all optimal actions (also called best responses)

➤ Proof: follows from definition

$$\begin{aligned} \sum_i p_i [\sum_j u_{i,j} q_j] &= u(\mathbf{p}, \mathbf{q}) \geq u(a_k, \mathbf{q}) = \sum_j u_{k,j} q_j \text{ for any } a_k \\ \Rightarrow \sum_i p_i [\sum_j u_{i,j} q_j] &\geq \max_k \sum_j u_{k,j} q_j \end{aligned}$$

So for any $p_i > 0$, we must have $\sum_j u_{i,j} q_j = \max_k \sum_j u_{k,j} q_j$

		1/3	1/3	1/3
ExpU = 0	Rock	(0, 0)	(-1, 1)	(1, -1)
ExpU = 0	Paper	(1, -1)	(0, 0)	(-1, 1)
ExpU = 0	Scissor	(-1, 1)	(1, -1)	(0, 0)

Thank You

Haifeng Xu

University of Virginia

hx4ad@virginia.edu