Announcements

- HW1 grading is out
  - Regrading due by next Monday
  - Please submit requests to **Gradescope**

- Likely, Midterm 1 will be postponed to Oct 15'th
  - Time will be finalized during next lecture, but likely it's the 15'th

CS6161: Design and Analysis of Algorithms (Fall 2020)

Shortest Path Problems

Instructor: Haifeng Xu

Outline

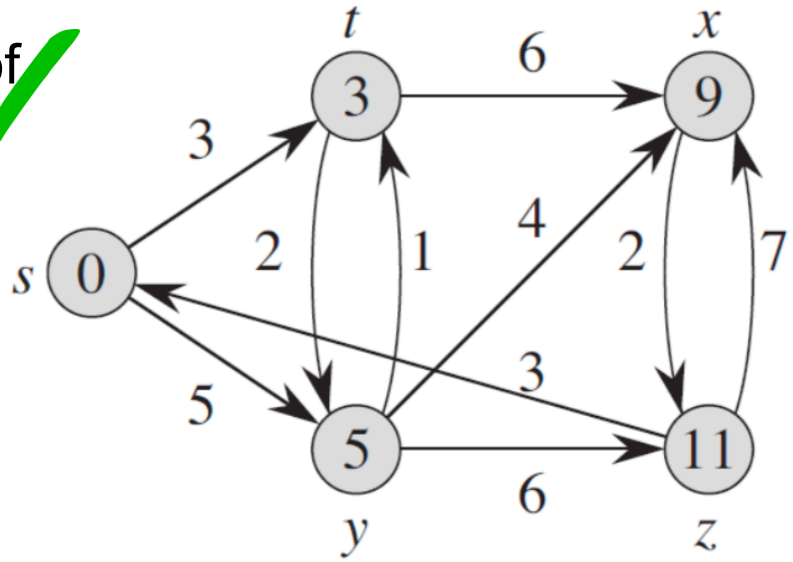
- Shortest Path Problems, and Properties
- Dijkstra's Algorithm

Shortest Paths Problems

- Typically for weighted graph, directed or undirected
  - Weights capture traversal cost or distance
- Try to find shortest way to go from one node to another

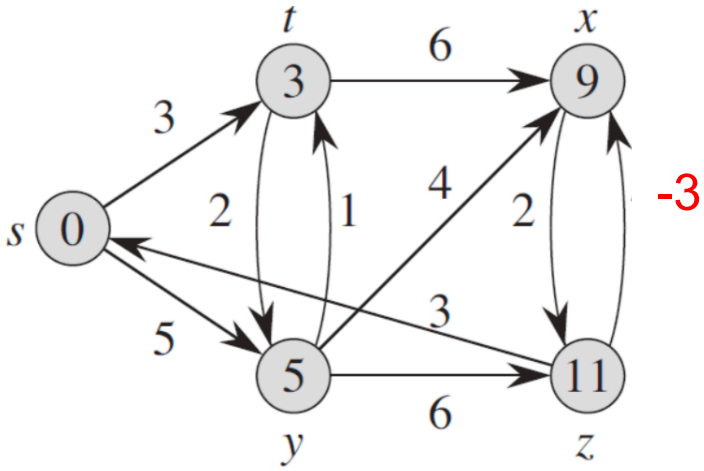
Q: How to represent a path?

- A sequence of edges; Or
- A sequence of nodes, and each pair of consecutive nodes have an edge ✓



Many Variants

- Single pair
- Single source, all destinations
- All pairs



Important Remarks

- One important condition is whether edges have negative costs
- Will focus on directed graphs from now on
  - Undirected graphs reduce to directed ones, by splitting each edge into two directed edges

Applications

Really a lot

- Traffic routing
- Network analysis
- Financial trading (e.g., can you find an arbitrage opportunity through a cycle trading?)
- Robotics, circuit design, …
- Metrics embedding
- A building block for many other algorithms (e.g., the DP for TSP)

Triangle Inequality of Shortest Distance

- Let $dist(s, v) =$ length of the shortest path from s to v

Lemma: for any s, v, t, we have $dist(s, v) + dist(v, t) \geq dist(s, t)$.

- Proof: the shortest path from s to v and the path from v to t form a path from s to t, and thus has cost at least $dist(s, t)$.

Corollary: for any $e = (v, t)$, we have $dist(s, v) + c(e) \geq dist(s, t)$.

Because $dist(v, t) \leq c(e)$

Triangle Inequality of Shortest Distance

- Let $dist(s, v) =$ length of the shortest path from s to v

Lemma: for any s, v, t, we have $dist(s, v) + dist(v, t) \geq dist(s, t)$.

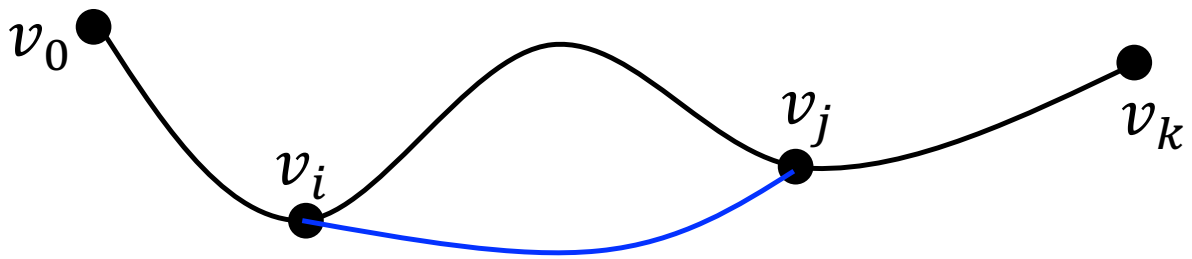
- Remark: triangle inequality is a natural property of distance functions, and is very useful in algorithm design

A Useful Lemma

Lem: If $P = (v_0, v_1, \cdots, v_k)$ is a shortest path from v_0 to v_k, then for any $0 \leq i < j \leq k$, $(v_i, \cdots, v_j)$ is a shortest path from v_i to v_j.

Proof: by contradiction

- If not, let $(v_i, u_1, \cdots, u_l, v_j)$ is a strictly shorter path from v_i to v_j
- Then substituting $(v_i, \cdots, v_j)$ in P by $(v_i, u_1, \cdots, u_l, v_j)$ will lead to a strictly shorter path from v_0 to v_k, contradicting the definition of P



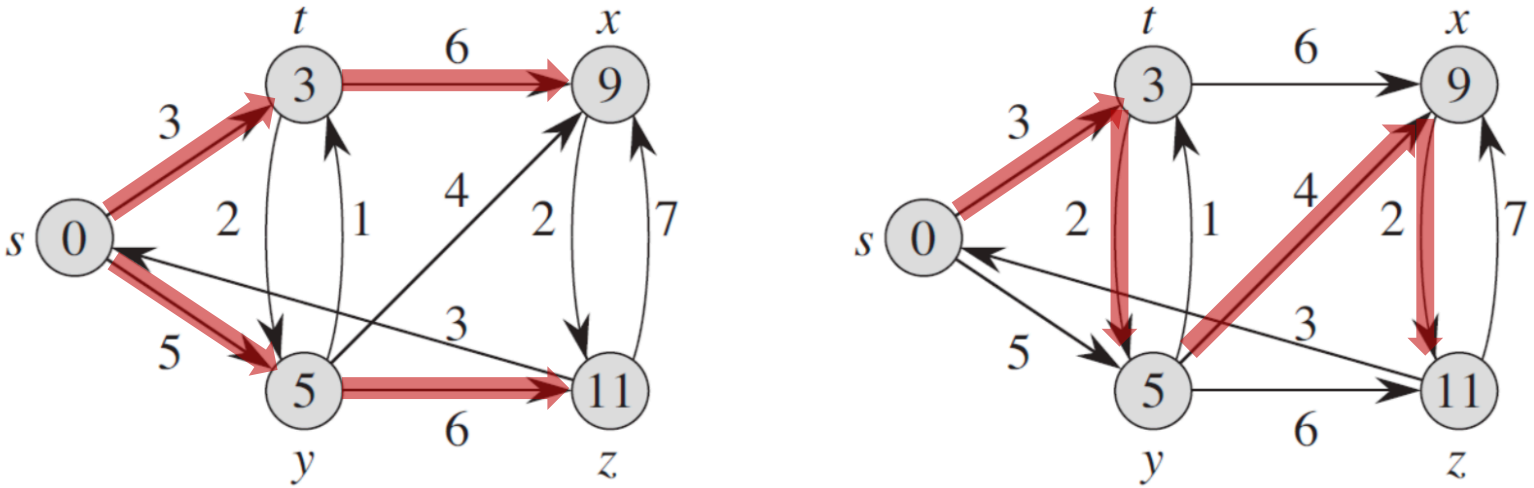
An Efficient Way to Store Shortest Paths

Shortest-Paths Trees

A shortest-paths tree rooted at s is a directed subgraph $G' = (V', E')$:

- V' = set of vertices reachable from s in G
- G' forms a rooted tree with root s
- For all $v \in V'$, the (unique) path from s to v in G' is a shortest path from s to v in G.

Generally, not unique



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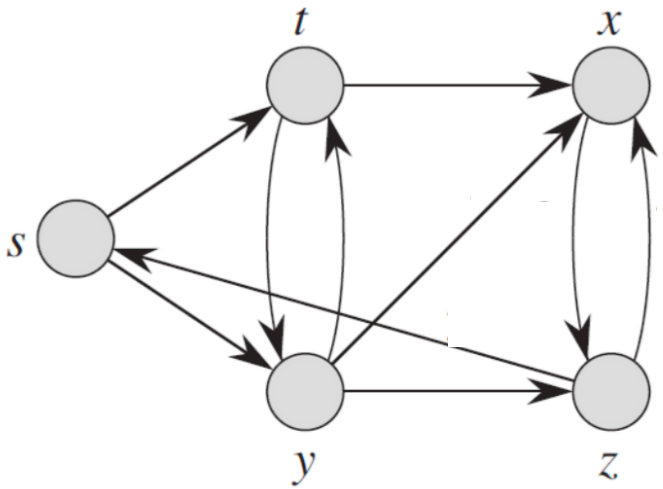
Remarks:

- This definition relies on previous lemma – in a shortest path P from s to v_k, any point v_i on P is visited through a shortest path as well
- This tree stores a shortest path from s to any reachable vertex

Actually, You Have Seen Shortest Path Algorithms Before!

- Recall BFS



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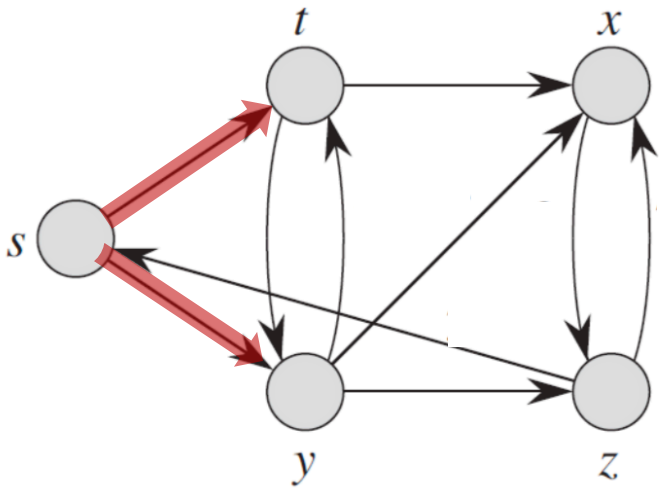

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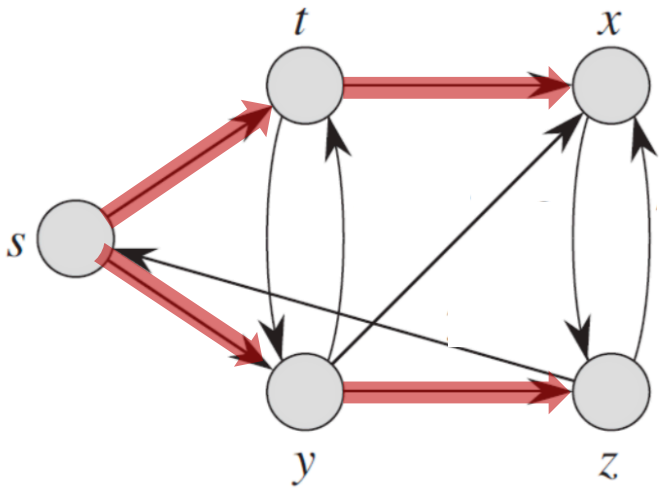
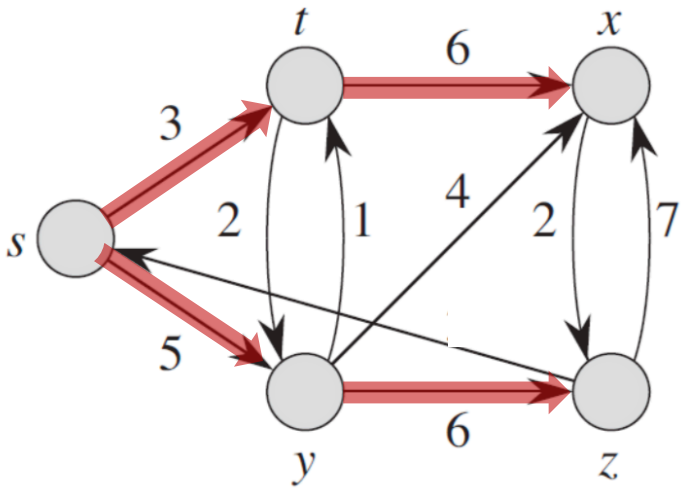


- The path reaching any node v is the "shallowest"
- If edge weights are all 1, it is the shortest path and the BFS tree is the shortest-paths tree!

Actually, You Have Seen Shortest Path Algorithms Before!

- Recall BFS



What to do if edges have different weights?

Note: assume positive weights for now (negative weights are trickier)

Outline

- Shortest Path Problems, and Properties
- Dijkstra's Algorithm

Ideas

- Why does BFS work?
- BFS(s) explores nodes in increasing distance from s
- Let $dist(s, v)$ denote the shortest path length from s to v

```
ShortestPathAlgoIdea(s, G)
1    Initialize dist(s, v) = ∞ for each v ∈ V;
2    Initialize S = ∅;
3
4                                                    ;
5
6
```

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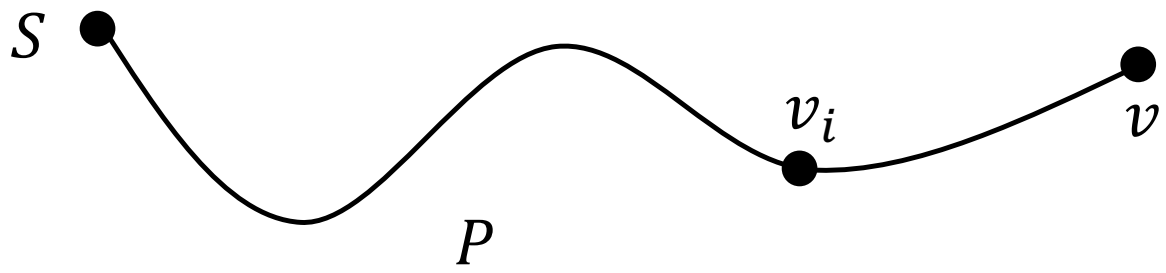
How to find the next closest vertex?

Finding the ith Closest Vertex

- At ith iteration, S already stores the $i-1$ closest vertices to s
- What do we know about the ith closest vertex?

Claim: Let P be any shortest path from s to v where v is the ith closest, then all intermediate vertices in P belongs to S.



$dist(s, v_i) < dist(s, v)$ due to positive edge weights

Finding the ith Closest Vertex

- At ith iteration, S already stores the $i-1$ closest vertices to s
- What do we know about the ith closest vertex?

Claim: Let P be any shortest path from s to v where v is the ith closest, then all intermediate vertices in P belongs to S.

Therefore, the next closest vertex must be adjacent to S

Well, but adjacent to which node in S?

Finding the ith Closest Vertex

- Let S contain the $(i-1)$ closest vertices to s
- For each $u \in V - S$, define $\pi(u) = \min_{a \in S}(dist(s,a) + c(a,u))$
  - $c(a,u) = \infty$ if (a,u) is not an edge

Lem: If u is the ith closest vertex to s, then:
(1) $\pi(u) = dist(s,u)$;
(2) $\pi(v) \geq \pi(u)$, for all $v \in V - S$
(3) If $\pi(v) = \pi(u)$ for some $v \in V - S$, then v is also the ith closest

Remark: this lemma tells us which node to be added at ith iteration

- Maintain array $\pi(u)$
- Add the $u \in V - S$ with minimum $\pi(u)$

Proof of the Lemma

- Let S contain the $(i-1)$ closest vertices to s
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Proof: for (1)

- By previous lemma, u is adjacent to some $a \in S$
- So $dist(s,u) = dist(s,a) + c(a,u)$ for the a achieving the minimum possible value
- So $dist(s,u) = \pi(u)$

Proof of the Lemma

- Let S contain the $(i-1)$ closest vertices to s
- For each $u \in V - S$, define $\pi(u) = \min_{a \in S}(dist(s,a) + c(a,u))$

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Proof: for (2)

- $\pi(v) \geq dist(s,v)$ as $s \to a \to v$ is a path to v
- So $\pi(v) \geq dist(s,v) \geq dist(s,u)$ for any $v \in V - S$

Proof of the Lemma

- Let S contain the $(i-1)$ closest vertices to s
- For each $u \in V - S$, define $\pi(u) = \min_{a \in S}(dist(s,a) + c(a,u))$

Lem: If u is the ith closest vertex to s, then:
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Proof: for (3)

- If $\pi(v) = \pi(u)$, we must have $\pi(v) = dist(s,v)$ as well
  - Otherwise, $\pi(v) > dist(s,v)$, and u cannot be the ith closest
- Now, by definition, v is also the ith closest

Algorithm

ShortestPathAlgo(s, G)

1 Initialize $\pi(v) = \infty$ for each $v \in V$;

2 Initialize $S = \emptyset, \pi(s) = 0$;

3

4

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7

8

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5 $\text{dist}(\text{s}, \text{u}) = \pi(u)$;

6 $S = S \cup \{u\}$;

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- Correctness: by induction using previous lemmas

Algorithm: Running Time

ShortestPathAlgo(s, G)

1 Initialize $\pi(v) = \infty$ for each $v \in V$;

2 Initialize $S = \emptyset, \pi(s) = 0$;

3 while $S \neq V$ **do** → repeat n times

4 find vertex $u \in V - S$ with minimum $\pi(u)$;

5 $\text{dist}(s, u) = \pi(u)$;

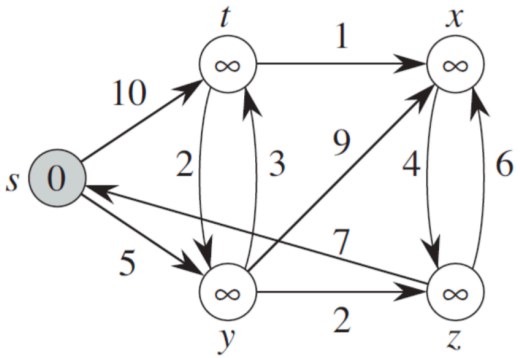
6 $S = S \cup \{u\}$;

7 foreach $v \in V - S$ do → $O(m)$ time

8 $\pi(v) = \min_{a \in S}(dist(s, a) + c(a, v))$

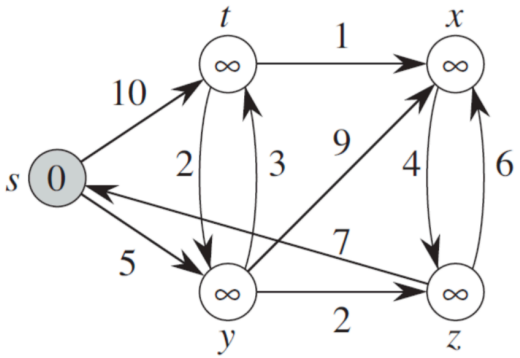
- Total: $O(nm)$ time.

Example Execution



(a)

Example Execution



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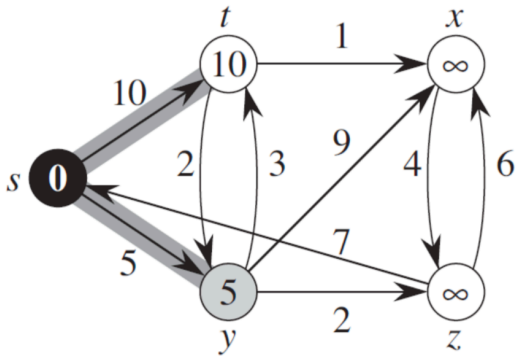

(b)

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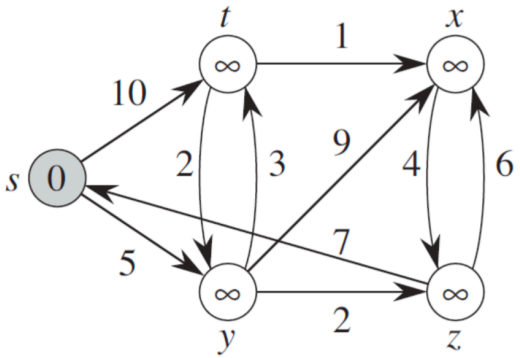

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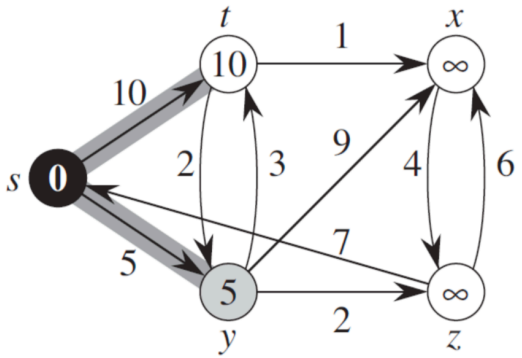

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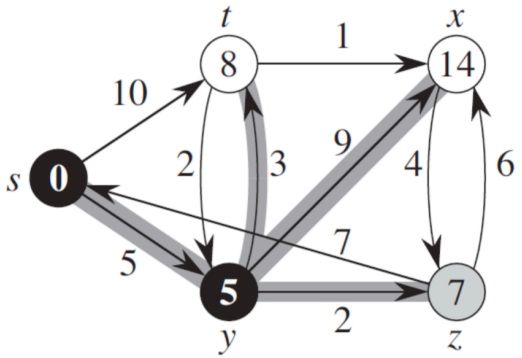
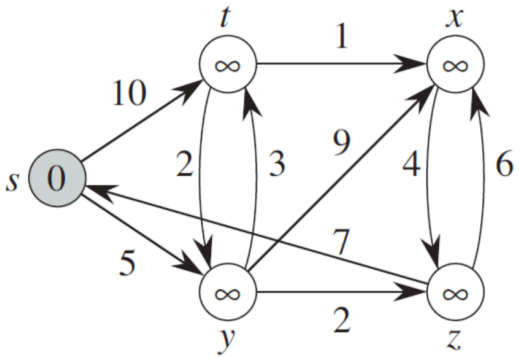


(c)

Example Execution



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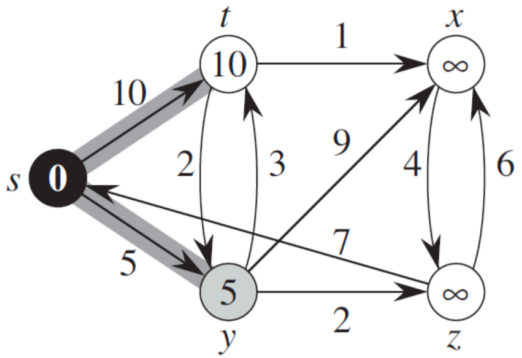

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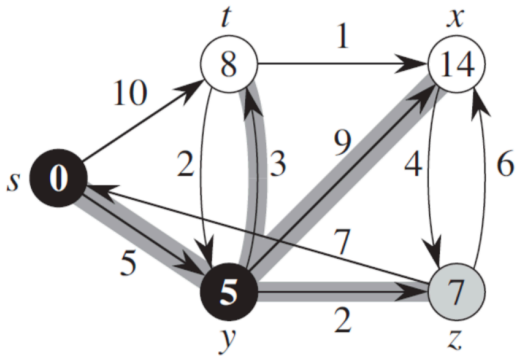

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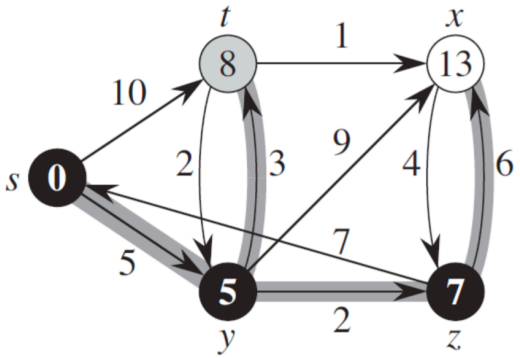

(d)

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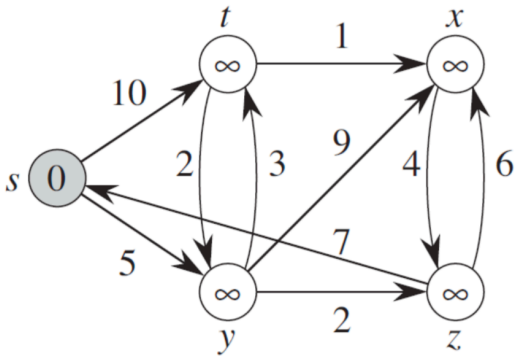

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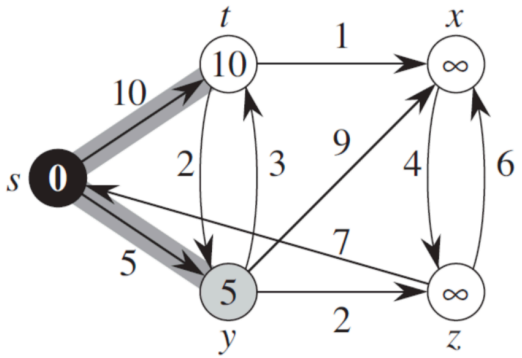

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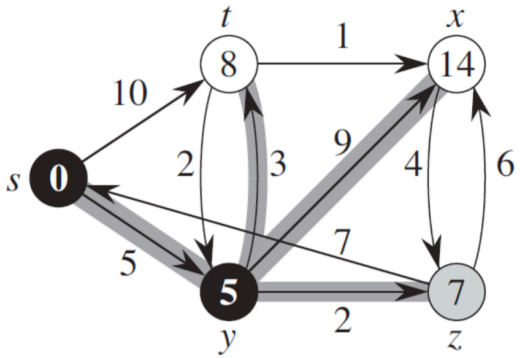

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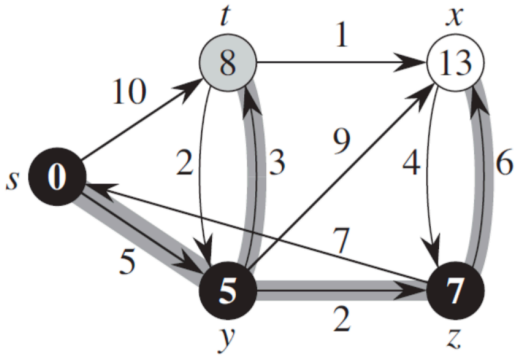

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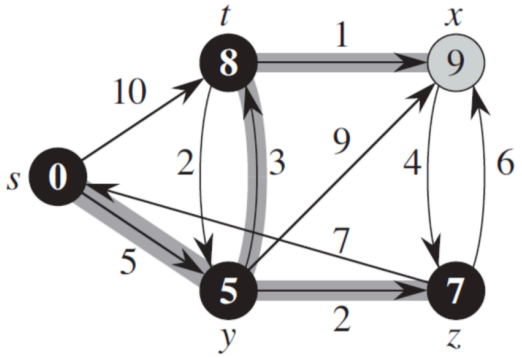
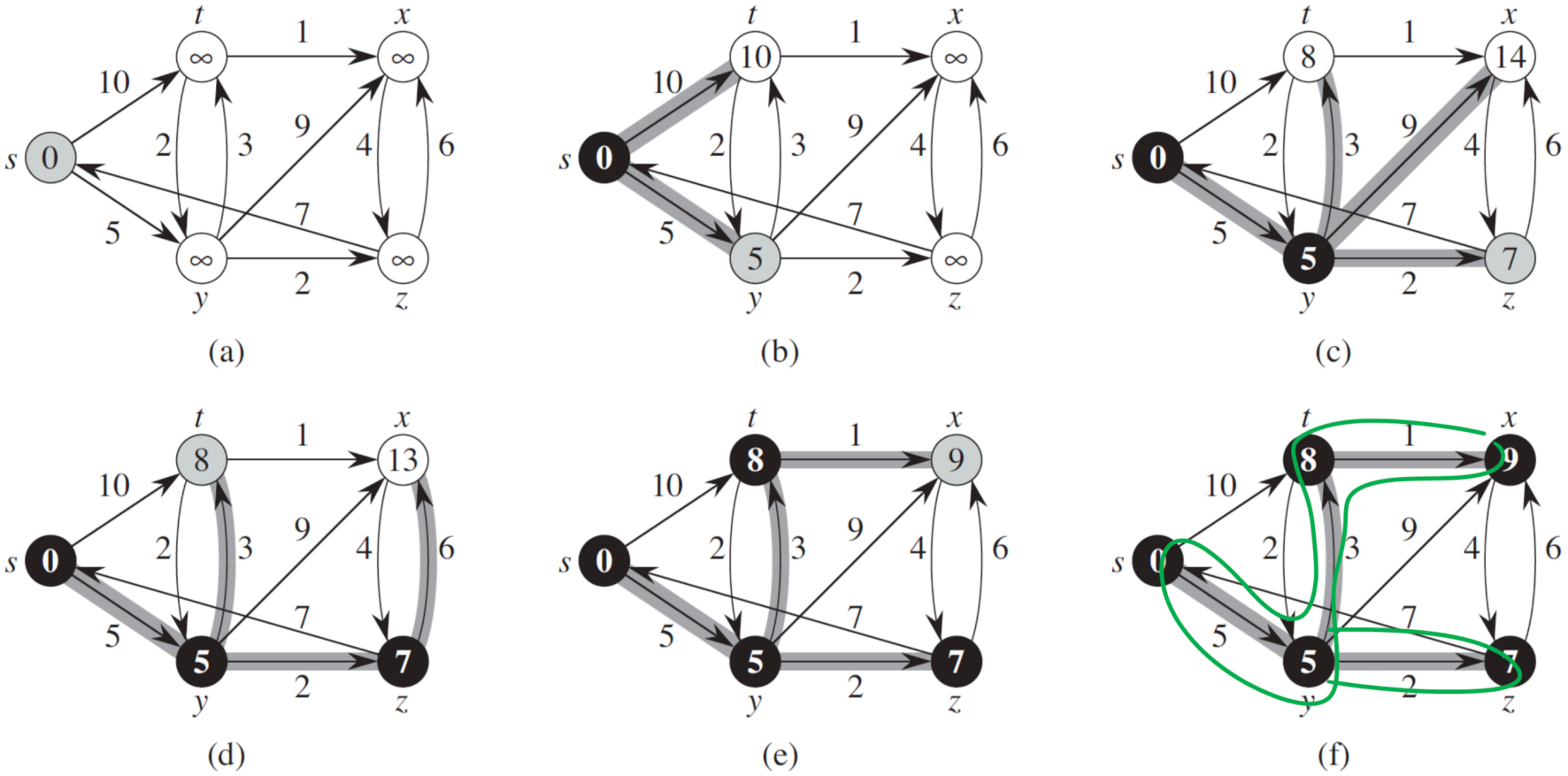


(e)

Example Execution



More Efficient Implementation

Critical Optimization: For unexplored vertex $v \in V - S$, explicitly maintain $\pi(v)$ instead of computing directly from formula:

$$\pi(v) = \min_{u \in S}(dist(s,u) + c(u,v)).$$

Main Idea:

- For each $v \notin S, \pi(v)$ can only decrease, because S only increase
- Suppose u is added to S and there is an edge (u,v) leaving u. Then, it suffices to update

$$\pi(v) = \min\{\pi(v), dist(s,u) + c(u,v)\}$$

Dijkstra's Algorithm

- Eliminate $\pi(v)$ and let $dist(s,v)$ maintain it
- Update $dist$ values after adding u by scanning only edges leaving u

```
ShortestPathAlgo(s, G)
1    Initialize dist(s, v) = ∞ for each v ∈ V;
2    Initialize S = ∅, dist(s, s) = 0;
3    while S ≠ V do
4        find vertex u ∈ V − S with minimum d(s, u);
5        S = S ∪ {u};
6        foreach v ∈ Adj(u) do
7            dist(s, v) = min{dist(s, v), dist(s, u) + c(u, v)}
```

All nodes adjacent to u

- Step (6) (7) are fast now, but step (4) becomes a bottleneck
- How to update values and retrieve min values efficiently?

Dijkstra's Algorithm

- Eliminate $\pi(v)$ and let $dist(s,v)$ maintain it
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5 $S = S \cup \{u\}$;
6 **foreach** $v \in Adj(u)$ **do**
7 $dist(s,v) = \min\{dist(s,v), dist(s,u) + c(u,v)\}$

All nodes adjacent to u

- Use priority queue!
  - Step(4) in $O(1)$ time
  - Each update of $dist$ in $O(\log n)$ time

Total: $O(m \log n)$ time

More on Dijkstra and Priority Queues

- Dijkstra's algorithm gives shortest paths from s to all vertices in V.
- There is an even faster implementation in $O(n \log n + m)$ time using Fibonacci Heaps.
  - If $m = \Omega(n \log n)$, running time is linear in input size.
- Data structures are complicated to analyze/implement. Recent work has obtained data structures that are easier to analyze and implement, and perform well in practice. Rank-Pairing Heaps (European Symposium on Algorithms, September 2009)

Thank You

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