Announcements

- HW3 deadline extended to next Monday
  - HW4 will be out around next Monday
- Midterm grades should be out before the end of this week

CS6161: Design and Analysis of Algorithms (Fall 2020)

Linear Programming (I)

Instructor: Haifeng Xu

Mathematical Optimization (MO)

- The task of selecting the best configuration from a "feasible" set to optimize some objective

$$\begin{array}{ll} \text{minimize (or maximize)} & f(x) \\ \text{subject to} & x \in X \end{array}$$

- x: decision variable
- $f(x)$: objective function
- X: feasible set/region
- Optimal solution, optimal value

- Simple example: minimize x^2, s.t. $x \in [-1,1]$

Q: Why we can figure out optimal solution even there are infinitely many options?

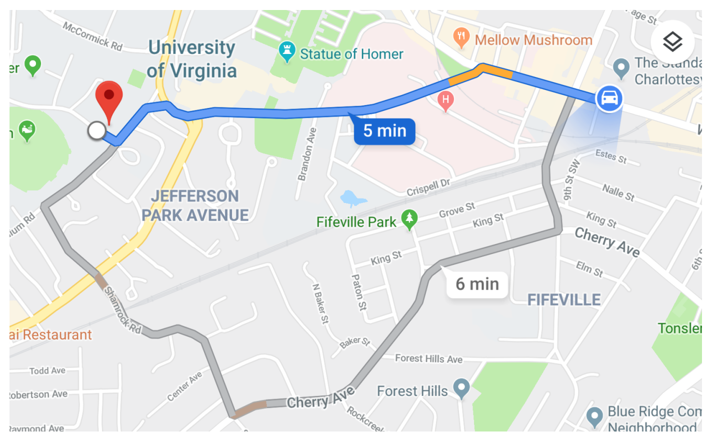
Because the problem has structures

MO Captures Many Algorithmic Problems

$$\begin{array}{ll} \text{minimize (or maximize)} & f(x) \\ \text{subject to} & x \in X \end{array}$$

➢ Example 1: shortest path

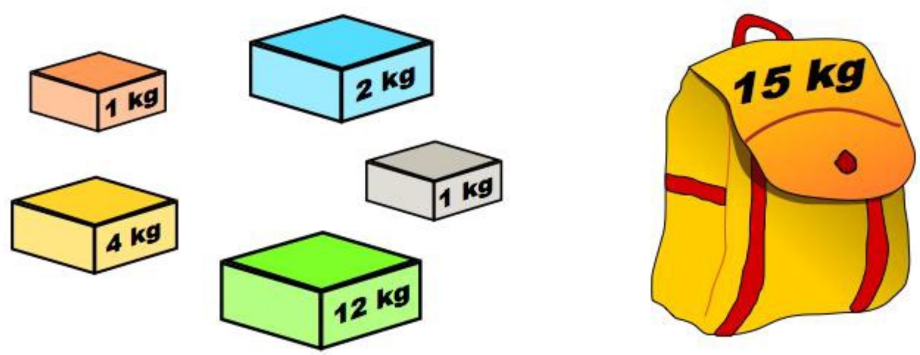
- X: the set of all paths from s
- $f(x)$ cost of path x
- Objective: minimize $f(x)$



MO Captures Many Algorithmic Problems

$$\begin{array}{ll} \text{minimize (or maximize)} & f(x) \\ \text{subject to} & x \in X \end{array}$$

- Example 2: knapsack problem
  - X: the set of all feasible ways to pack your knapsack
  - $f(x)$ the reward of packing strategy x
  - Objective: maximize $f(x)$



MO Captures Many Algorithmic Problems

minimize (or maximize) $f(x)$

subject to $x \in X$

- Example: max flow, min-cut, max-cut, maximum matching….

Did We Find a Universal Tool to Solve All Problems?

$$\begin{array}{ll} \text{minimize (or maximize)} & f(x) \\ \text{subject to} & x \in X \end{array}$$

- Difficult to solve without any assumptions on $f(x)$ and X
- A ubiquitous and well-understood case is *linear program*

Linear Program (LP) – General Form

minimize (or maximize)	$c^T \cdot x$	
subject to	$a_i \cdot x \le b_i$	$\forall i \in C_1$
	$a_i \cdot x \ge b_i$	$\forall i \in C_2$
	$a_i \cdot x = b_i$	$\forall i \in C_3$

- Decision variable: $x \in \mathbb{R}^n$
- Parameters:
  - $c \in \mathbb{R}^n$ define the linear objective
  - $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ defines the i'th linear constraint

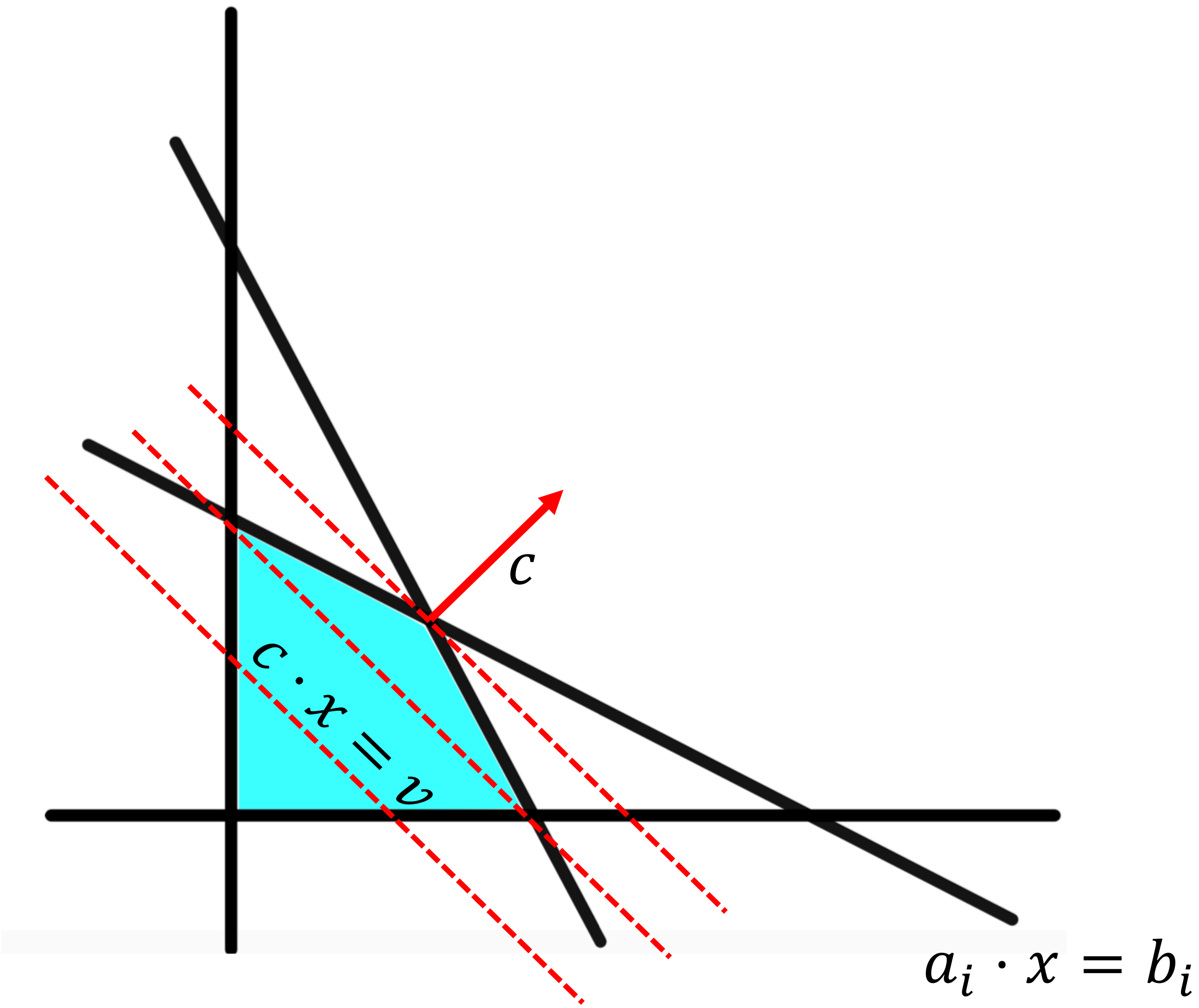
Linear Program (LP) – Standard Form

maximize	$c^T \cdot x$	
subject to	$a_i \cdot x \le b_i$	$\forall i = 1, \cdots, m$
	$x_j \ge 0$	$\forall j = 1, \cdots, n$

Claim. Every LP can be transformed to an *equivalent* standard form

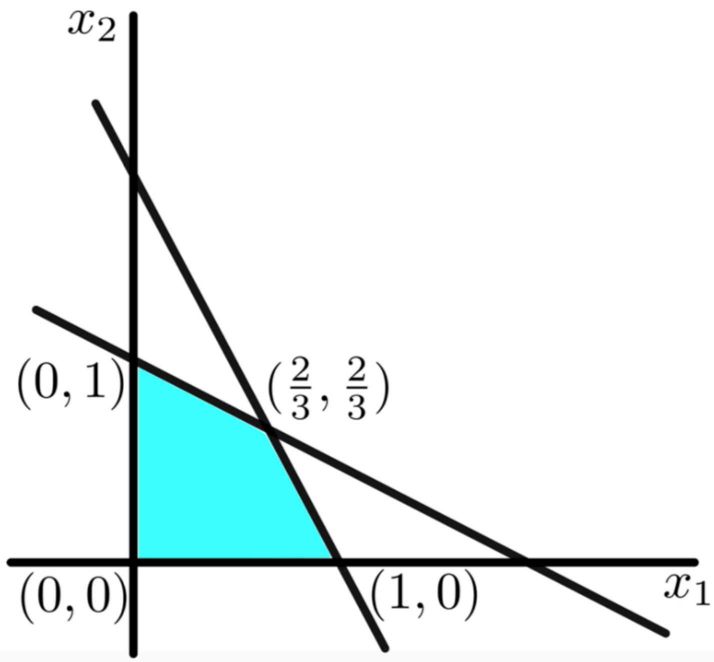
- minimize $c^T \cdot x \quad \Leftrightarrow \quad$ maximize $-c^T \cdot x$
- $a_i \cdot x \ge b_i \;\Leftrightarrow\; -a_i \cdot x \le -b_i$
- $a_i \cdot x = b_i \;\Leftrightarrow\; a_i \cdot x \le b_i$ and $-a_i \cdot x \le -b_i$
- Any unconstrained x_j can be replaced by $x_j^+ - x_j^-$ with $x_j^+, x_j^- \ge 0$

Geometric Interpretation



A 2-D Example

$$
\begin{array}{ll}
\text{maximize} & x_1 + x_2 \\
\text{subject to} & x_1 + 2x_2 \leq 2 \\
 & 2x_1 + x_2 \leq 2 \\
 & x_1, x_2 \geq 0
\end{array}
$$



Application: Optimal Production

- n products, m raw materials
- Every unit of product j uses a_{ij} units of raw material i
- There are b_i units of material i available
- Product j yields profit c_j per unit
- Factory wants to maximize profit subject to available raw materials

j: product index
i: material index

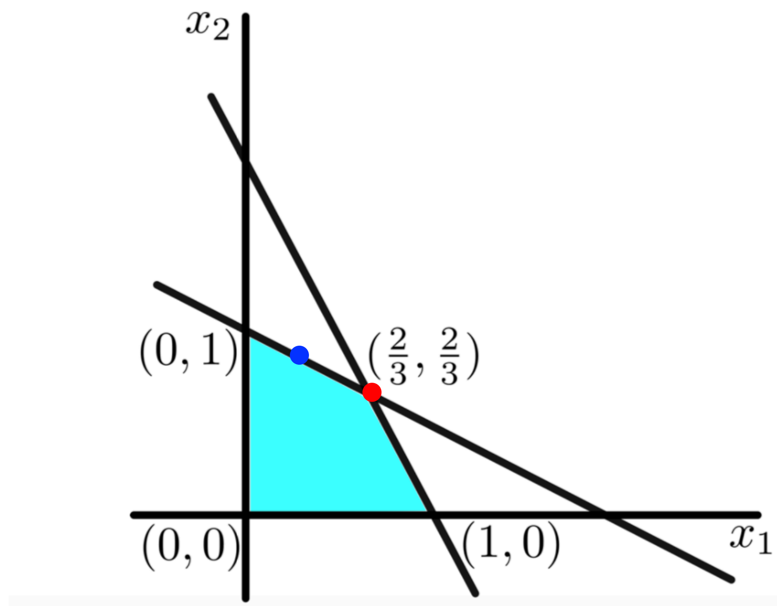
$$\begin{array}{lll} \text{maximize} & c^T \cdot x & \\ \text{subject to} & a_i \cdot x \leq b_i & \forall i = 1, \cdots, m \\ & x_j \geq 0 & \forall j = 1, \cdots, n \end{array}$$

where variable $x_j = \#$ units of product j

Terminology

- Hyperplane: The region defined by a linear equality $a_i \cdot x = b_i$
- Halfspace: The region defined by a linear inequality $a_i \cdot x \leq b_i$
- Polyhedron: The intersection of a set of linear inequalities
  - Feasible region of an LP is a polyhedron
- Polytope: *Bounded* polyhedron
- Vertex: A point x is a vertex of polyhedron P if $\nexists\, y \neq 0$ with $x + y \in P$ and $x - y \in P$

Red point: vertex
Blue point: not a vertex



Terminology

Convex set: A set S is convex if $\forall x, y \in S$ and $\forall p \in [0,1]$, we have

$$p \cdot x + (1 - p) \cdot y \ \in \ S$$

- Inherently related to convex functions

convex Non-convex

Terminology

Convex set: A set S is convex if $\forall x, y \in S$ and $\forall p \in [0,1]$, we have

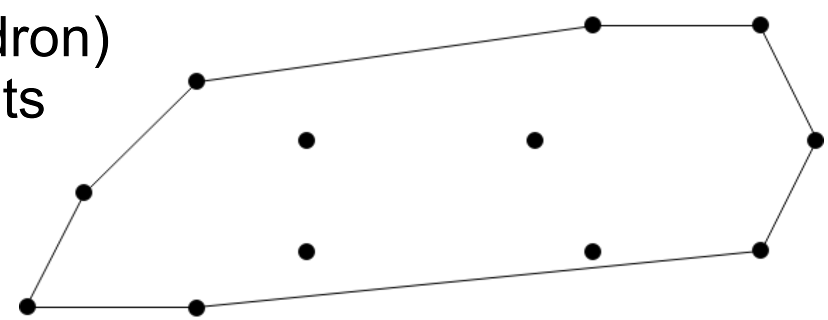
$$p \cdot x + (1-p) \cdot y \ \in \ S$$

Convex hull: the convex hull of points $\mathrm{x}_1, \cdots, x_m \in \mathbb{R}$ is

$$\mathrm{convhull}(x_1, \cdots, x_n) = \left\{ \mathrm{x} = \sum_{i=1}^{n} p_i x_i : \forall p \in \mathbb{R}_+^n \ \ s.t. \ \sum p_i = 1 \right\}$$

That is, $\mathrm{convhull}(x_1, \cdots, x_n)$ includes all points that can be written as expectation of $x_1, \cdots, x_n$ under some distribution p.

- Any polytope (i.e., a bounded polyhedron) is the convex hull of a finite set of points

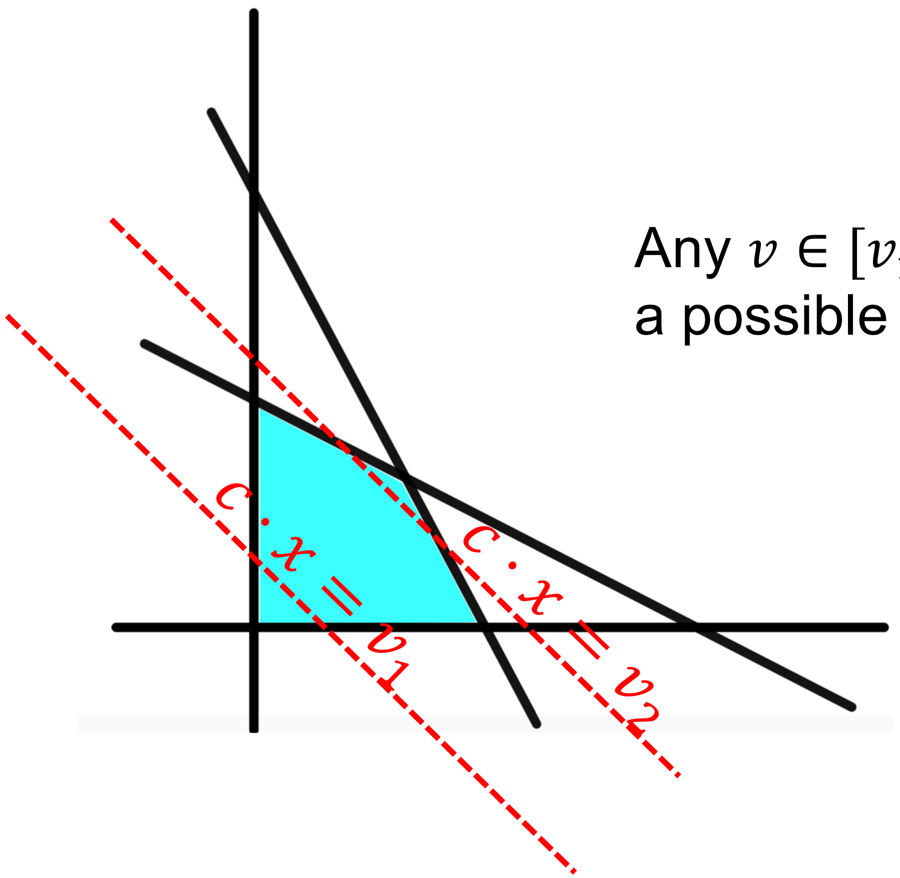


Geometric visualization of convex hull

Basic Facts about LPs and Polyhedrons

Fact: The feasible region of any LP (a polyhedron) is a convex set. All possible objective values form an interval (possibly unbounded).

Note: intervals are the only convex sets in $\mathbb{R}$



Any $v \in [v_1, v_2]$ must also be a possible objective value

Basic Facts about LPs and Polyhedrons

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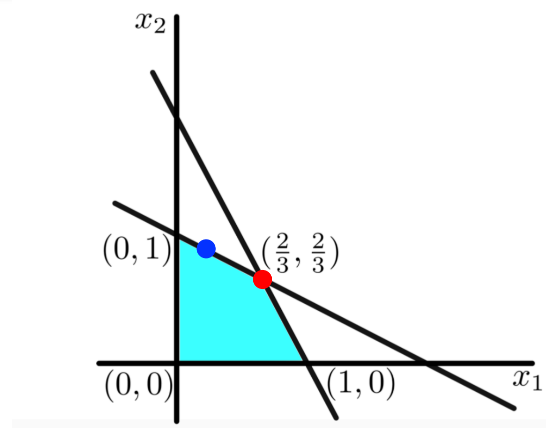
Note: intervals are the only convex sets in $\mathbb{R}$

Fact: The set of optimal solutions of any LP is a convex set.

- It is the intersection of feasible region and hyperplane $c^T \cdot x = OPT$

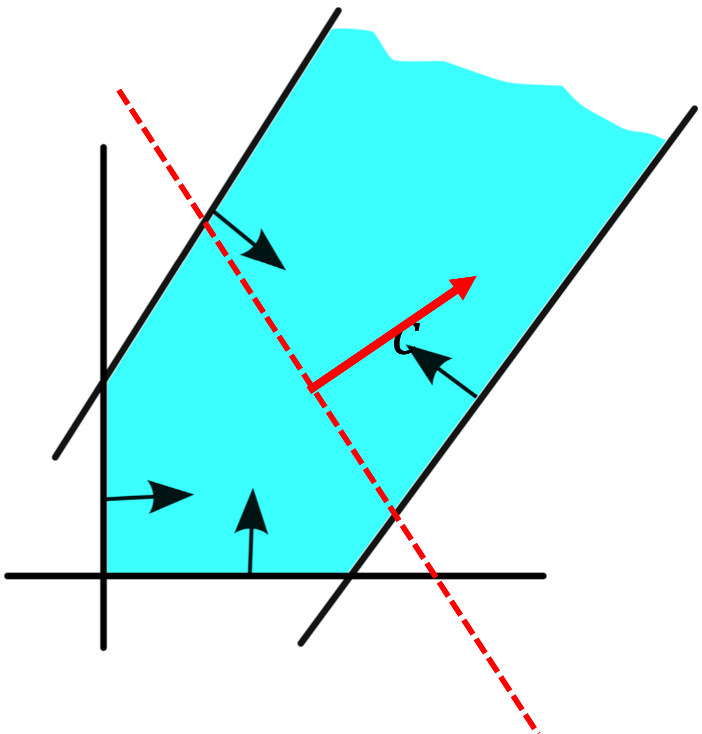
Fact: At a vertex, n linearly independent constraints are satisfied with equality (a.k.a., tight).

Formal proofs: exercise



Basic Facts about LPs and Polyhedrons

Fact: An LP either has an optimal solution, or is unbounded or infeasible



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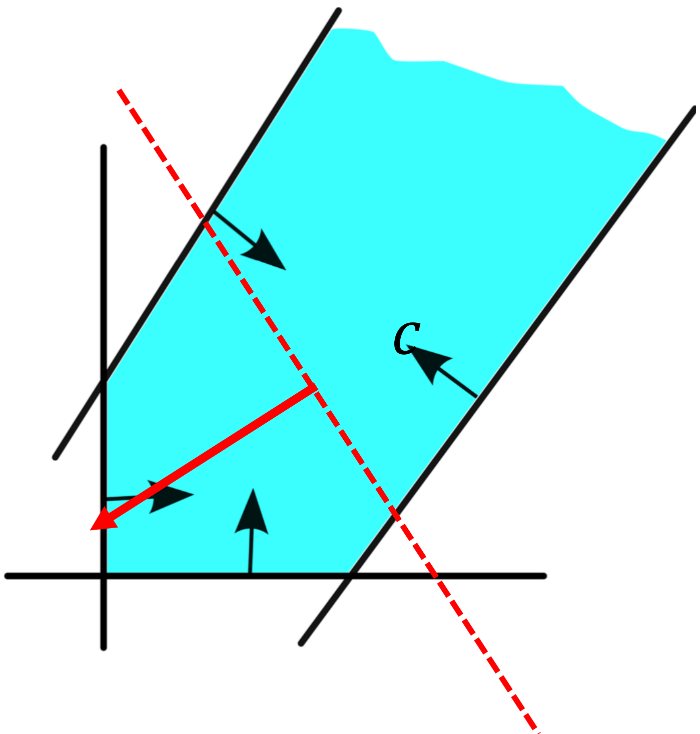

Fundamental Theorem of LP

Theorem: if an LP in standard form has an optimal solution, then it has a vertex optimal solution.

Proof

- Assume not, and take a non-vertex optimal solution $\bar{x}$ with the maximum number of tight constraints
- There is $y \neq 0$ s.t. $\bar{x} \pm y$ are feasible
- y is orthogonal to objective function and all tight constraints at $\bar{x}$
  - i.e. $c^T \cdot y = 0$, and $a_i^T \cdot y = 0$ whenever the i'th constraint is tight for $\bar{x}$

a) Arguments for $a_i^T \cdot y = 0$
  - $\bar{x} \pm y$ feasible $\Rightarrow a_i^T \cdot (\bar{x} \pm y) \leq b_i$
  - $\bar{x}$ is tight at constraint $i \Rightarrow a_i^T \cdot \bar{x} = b_i$
  - These together yield $a_i^T \cdot (\pm y) \leq 0 \Rightarrow a_i^T \cdot y = 0$

b) Similarly, $\bar{x}$ optimal implies $c^T(\bar{x} \pm y) \leq c^T \bar{x} \Rightarrow c^t y = 0$

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  - i.e. $c^T \cdot y = 0$, and $a_i^T \cdot y = 0$ whenever the i'th constraint is tight for $\bar{x}$
- Can choose y s.t. $y_j < 0$ for some j
- Let α be the largest constant such that $\bar{x} + \alpha y$ is feasible
  - Such an α exists (since $\bar{x}_j + \alpha y_j < 0$ if α very large)
- An additional constraint becomes tight at $\bar{x} + \alpha y$, contradiction

Fundamental Theorem of LP

Theorem: if an LP in standard form has an optimal solution, then it has a vertex optimal solution.

Corollary [counting non-zero variables]: If an LP in standard form has an optimal solution, then there is an optimal solution with at most m non-zero variables.

$$
\begin{array}{lll}
\text{maximize} & c^T \cdot x & \\
\text{subject to} & a_i \cdot x \leq b_i & \forall i = 1, \cdots, m \\
& x_j \geq 0 & \forall j = 1, \cdots, n
\end{array}
$$

- Meaningful when $m < n$
- E.g. for optimal production with $n = 10$ products and $m = 3$ raw materials, there is an optimal plan using at most 3 products.

Poly-Time Solvability of LP

Theorem: any linear program with n variables and m constraints can be solved in $\mathrm{poly}(m,n)$ time.

- Original proof gives an algorithm with very high polynomial degree
- Now, the fastest algorithm with guarantee takes $\sqrt{\min(n,m)} \cdot T$ where $T =$ time of solving linear equation systems of the same size
- In practice, Simplex Algorithm runs extremely fast though in (extremely rare) worst case it still takes exponential time
- Will not cover these algorithms as they have become mature technology
  - Instead, *we use them as building blocks to solve algorithmic problems*

Brief History of Linear Optimization

- The forefather of convex optimization problems, and the most ubiquitous.
- Developed by Kantorovich during World War II (1939) for planning the Soviet army's expenditures and returns. Kept secret.
- Discovered a few years later by George Dantzig, who in 1947 developed the simplex method for solving linear programs
- John von Neumann developed LP duality in 1947, and applied it to game theory
- Polynomial-time algorithms: Ellipsoid method (Khachiyan 1979), interior point methods (Karmarkar 1984).

Thank You

Haifeng Xu

University of Virginia

hx4ad@virginia.edu