CS6161: Design and Analysis of Algorithms (Fall 2020)

Linear Programming (II)

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Outline

- How to Write Dual Program of LP

- Duality, and Examples

Recall Linear Programs (LPs)

- A special type of mathematical optimization programs

$$\begin{array}{ll} \text{minimize (or maximize)} & f(x) \\ \text{subject to} & x \in X \end{array}$$

- x: decision variable
- $f(x)$: objective function
- X: feasible set/region
- Optimal solution, optimal value

- Simple example: minimize x^2, s.t. $x \in [-1,1]$

Q: Why we can figure out optimal solution even there are infinitely many options?

Because the problem has structures

Recall Linear Programs (LPs)

- A special type of mathematical optimization programs

maximize	$c^T \cdot x$	
subject to	$a_i \cdot x \leq b_i$	$\forall i = 1, \cdots, m$
	$x_j \geq 0$	$\forall j = 1, \cdots, n$

x_js are variables

Standard form

A natural application

- n products, m raw materials
- Every unit of product j uses a_{ij} units of raw material i
- There are b_i units of material i available
- Product j yields profit c_j per unit
- Factory wants to maximize profit subject to available raw materials

Recall Linear Programs (LPs)

- A special type of mathematical optimization programs

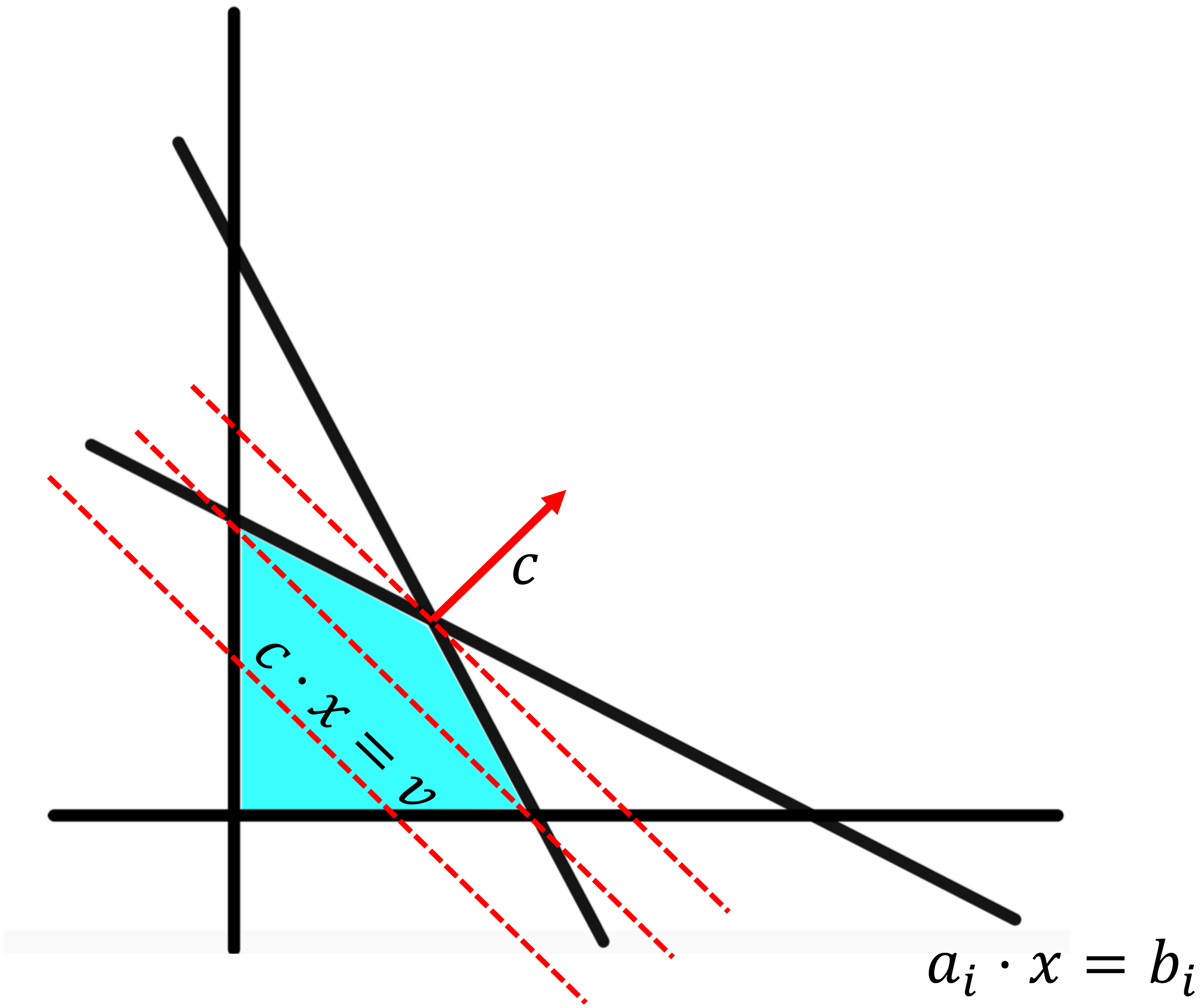
maximize	$c^T \cdot x$	
subject to	$a_i \cdot x \leq b_i$	$\forall i = 1, \cdots, m$
	$x_j \geq 0$	$\forall j = 1, \cdots, n$

x_js are variables

Standard form

- Any LP in general form can be converted to an equivalent LP in standard form

Geometric Interpretation



Dual Linear Program: General Form

Primal LP

max $\quad c^T \cdot x$

s.t.

$$\begin{aligned} a_i^T x &\le b_i, && \forall i \in C_1 \\ a_i^T x &= b_i, && \forall i \in C_2 \\ x_j &\ge 0, && \forall j \in D_1 \\ x_j &\in \mathbb{R}, && \forall j \in D_2 \end{aligned}$$

Dual LP

min $\quad b^T \cdot y$

s.t.

$$\begin{aligned} \bar{a}_j y &\ge c_j, && \forall j \in D_1 \\ \bar{a}_j\, y &= c_j, && \forall j \in D_2 \\ y_i &\ge 0, && \forall i \in C_1 \\ y_i &\in \mathbb{R}, && \forall i \in C_2 \end{aligned}$$

Note:

- There are good reasons to call this “Dual” and for why it has this form
- But for now, let’s just see, *mechanically*, how this dual is generated
  - In HW, you will be asked to write dual of an LP by exercising the rule

Dual Linear Program: General Form

Primal LP

$$
\begin{aligned}
\text{max} \quad & c^T \cdot x \\
\text{s.t.} \quad & \\
y_i: \quad & a_i^T x \leq b_i, && \forall i \in C_1 \\
y_i: \quad & a_i^T x = b_i, && \forall i \in C_2 \\
& x_j \geq 0, && \forall j \in D_1 \\
& x_j \in \mathbb{R}, && \forall j \in D_2
\end{aligned}
$$

Dual LP

$$
\begin{aligned}
\text{min} \quad & b^T \cdot y \\
\text{s.t.} \quad & \\
& \bar{a}_j y \geq c_j, && \forall j \in D_1 \\
& \bar{a}_j\, y = c_j, && \forall j \in D_2 \\
& y_i \geq 0, && \forall i \in C_1 \\
& y_i \in \mathbb{R}, && \forall i \in C_2
\end{aligned}
$$

- Each dual variable y_i corresponds to a primal constraint $a_i^T x \leq (\text{or } =) b_i$
  - Inequality constraint ⇒ nonnegative dual variable
  - Equality constraint ⇒ unconstrained dual variable

Dual Linear Program: General Form

Primal LP

$$\begin{aligned} \max \quad & c^T \cdot x \\ \text{s.t.} \quad & \\ y_i: \quad & a_i^T x \le b_i, && \forall i \in C_1 \\ y_i: \quad & a_i^T x = b_i, && \forall i \in C_2 \\ & x_j \ge 0, && \forall j \in D_1 \\ & x_j \in \mathbb{R}, && \forall j \in D_2 \end{aligned}$$

Dual LP

$$\begin{aligned} \min \quad & b^T \cdot y \\ \text{s.t.} \quad & \\ x_j: \quad & \bar{a}_j y \ge c_j, && \forall j \in D_1 \\ x_j: \quad & \bar{a}_j\, y = c_j, && \forall j \in D_2 \\ & y_i \ge 0, && \forall i \in C_1 \\ & y_i \in \mathbb{R}, && \forall i \in C_2 \end{aligned}$$

- Each dual variable y_i corresponds to a primal constraint $a_i^T x \le (\text{or } =) b_i$
  - Inequality constraint ⇒ nonnegative dual variable
  - Equality constraint ⇒ unconstrained dual variable
- Each dual constraint $\bar{a}_j y \ge (\text{or } =) c_j$ corresponds to a primal variable x_j
  - Unconstrained variable ⇒ equality dual constraint
  - Nonnegative variable ⇒ Inequality dual constraint

Dual Linear Program: General Form

Primal LP

$$\max \quad c^T \cdot x$$

s.t.

$$\begin{array}{lll} y_i: & a_i^T x \le b_i, & \forall i \in C_1 \\ y_i: & a_i^T x = b_i, & \forall i \in C_2 \\ & x_j \ge 0, & \forall j \in D_1 \\ & x_j \in \mathbb{R}, & \forall j \in D_2 \end{array}$$

Dual LP

$$\min \quad b^T \cdot y$$

s.t.

$$\begin{array}{lll} x_j: & \bar{a}_j y \ge c_j, & \forall j \in D_1 \\ x_j: & \bar{a}_j\, y = c_j, & \forall j \in D_2 \\ & y_i \ge 0, & \forall i \in C_1 \\ & y_i \in \mathbb{R}, & \forall i \in C_2 \end{array}$$

This is how $\bar{a}_j$ is generated:

Primal constraint: row a_i^{T} ←

x_1	x_2	x_3	x_4	
a_{11}	a_{12}	a_{13}	a_{14}	b_1
a_{21}	a_{22}	a_{23}	a_{24}	b_2
a_{31}	a_{32}	a_{33}	a_{34}	b_3
c_1	c_2	c_3	c_4	

Dual Linear Program: General Form

Primal LP

max $\quad c^T \cdot x$

s.t.

$$
\begin{array}{lll}
y_i: & a_i^T x \leq b_i, & \forall i \in C_1 \\
y_i: & a_i^T x = b_i, & \forall i \in C_2 \\
 & x_j \geq 0, & \forall j \in D_1 \\
 & x_j \in \mathbb{R}, & \forall j \in D_2
\end{array}
$$

Dual LP

min $\quad b^T \cdot y$

s.t.

$$
\begin{array}{lll}
x_j: & \bar{a}_j y \geq c_j, & \forall j \in D_1 \\
x_j: & \bar{a}_j\, y = c_j, & \forall j \in D_2 \\
 & y_i \geq 0, & \forall i \in C_1 \\
 & y_i \in \mathbb{R}, & \forall i \in C_2
\end{array}
$$

This is how $\bar{a}_j$ is generated:

Dual var y

	x_1	x_2	x_3	x_4	
y_1	a_{11}	a_{12}	a_{13}	a_{14}	b_1
y_2	a_{21}	a_{22}	a_{23}	a_{24}	b_2
y_3	a_{31}	a_{32}	a_{33}	a_{34}	b_3
	c_1	c_2	c_3	c_4	

Dual Linear Program: General Form

Primal LP

$$\max \quad c^T \cdot x$$

s.t.

$$\begin{array}{lll} y_i: & a_i^T x \leq b_i, & \forall i \in C_1 \\ y_i: & a_i^T x = b_i, & \forall i \in C_2 \\ & x_j \geq 0, & \forall j \in D_1 \\ & x_j \in \mathbb{R}, & \forall j \in D_2 \end{array}$$

Dual LP

$$\min \quad b^T \cdot y$$

s.t.

$$\begin{array}{lll} x_j: & \bar{a}_j y \geq c_j, & \forall j \in D_1 \\ x_j: & \bar{a}_j\, y = c_j, & \forall j \in D_2 \\ & y_i \geq 0, & \forall i \in C_1 \\ & y_i \in \mathbb{R}, & \forall i \in C_2 \end{array}$$

This is how $\bar{a}_j$ is generated:

Dual constraint: column $\bar{a}_j$

Dual var y

	x_1	x_2	x_3	x_4	
y_1	a_{11}	a_{12}	a_{13}	a_{14}	b_1
y_2	a_{21}	a_{22}	a_{23}	a_{24}	b_2
y_3	a_{31}	a_{32}	a_{33}	a_{34}	b_3
	c_1	c_2	c_3	c_4	

Dual Linear Program: Standard Form

Primal LP

$$\begin{aligned} \max \quad & c^T \cdot x \\ \text{s.t.} \quad & Ax \le b \\ & x \ge 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \min \quad & b^T \cdot y \\ \text{s.t.} \quad & A^T y \ge c \\ & y \ge 0 \end{aligned}$$

- $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$
- y_i is the dual variable corresponding to primal constraint $A_i x \le b_i$
- $A_j^T\ y \ge c_j$ is the dual constraint corresponding to primal variable x_j

Remark:
- This is easier to write, at least mechanically
- Result in an *equivalent* dual (may not look exactly the same)
- Thus, another way to write dual: (1) convert any LP to standard form; (2) use the above formula

Interpretation 1: Economic Interpretation

Recall the optimal production problem

- n products, m raw materials
- Every unit of product j uses a_{ij} units of raw material i
- There are b_i units of material i available
- Product j yields profit c_j per unit
- Factory wants to maximize profit subject to available raw materials

Interpretation 1: Economic Interpretation

Primal LP

$$\begin{aligned} \max \quad & c^T \cdot x \\ \text{s.t.} \quad & \textstyle\sum_{j=1}^{n} a_{ij}\, x_j \le b_i, \quad && \forall i \in [m] \\ & x_j \ge 0, && \forall j \in [n] \end{aligned}$$

Dual LP

$$\begin{aligned} \min \quad & b^T \cdot y \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^{m} a_{ij}\, y_i \ge c_j, \quad && \forall j \in [n] \\ & y_i \ge 0, && \forall i \in [m] \end{aligned}$$

j: product index
i: material index

Dual LP corresponds to the buyer's optimization problem, as follows:

- Buyer wants to directly buy the raw material
- Dual variable y_i is buyer's proposed price per unit of raw material i
- Dual price vector is feasible if factory is incentivized to sell materials
- Buyer wants to spend as little as possible to buy raw materials

Interpretation 1: Economic Interpretation

Primal LP

max $\quad c^T \cdot x$

s.t. $\quad \sum_{j=1}^{n} a_{ij}\, x_j \le b_i, \quad \forall i \in [m]$

$\qquad x_j \ge 0, \qquad \forall j \in [n]$

Dual LP

min $\quad b^T \cdot y$

s.t. $\quad \sum_{i=1}^{m} a_{ij}\, y_i \ge c_j, \quad \forall j \in [n]$

$\qquad y_i \ge 0, \qquad \forall i \in [m]$

units of products

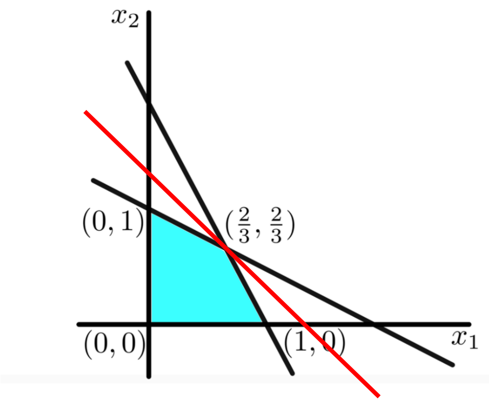
price of material

	x_1	x_2	x_3	x_4	
y_1	a_{11}	a_{12}	a_{13}	a_{14}	b_1
y_2	a_{21}	a_{22}	a_{23}	a_{24}	b_2
y_3	a_{31}	a_{32}	a_{33}	a_{34}	b_3
	c_1	c_2	c_3	c_4	

Interpretation II: Finding Best Upperbound

- Consider the simple LP from previous 2-D example

$$\begin{aligned} \text{maximize} \quad & x_1 + x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \le 2 \\ & 2x_1 + x_2 \le 2 \\ & x_1, x_2 \ge 0 \end{aligned}$$



- We found that the optimal solution was at $(\frac{2}{3}, \frac{2}{3})$ with an optimal value of $\frac{4}{3}$.
- What if, instead of finding the optimal solution, we sought to find an upperbound on its value by combining inequalities?
  - Each inequality implies an upper bound of 2
  - Multiplying each by 1 and summing gives $x_1 + x_2 \le 4/3$.

Interpretation II: Finding Best Upperbound

Primal LP

max $\quad c^T \cdot x$

s.t. $\quad Ax \leq b$

$\quad x \geq 0$

Dual LP

min $\quad b^T \cdot y$

s.t. $\quad A^T y \geq c$

$\quad y \geq 0$

- Multiplying each row i by y_i and summing gives the inequality

$$y^T A x \leq y^T b$$

(now we see why $y_i \geq 0$ when $a_i x \leq b_i$ but $y_i \in \mathbb{R}$ when $a_i x = b_i$)

- When $c^T \leq y^T A$, we have

$$c^T x \leq y^T A x \leq y^T b$$

i.e., $y^T b$ is an upper bound on $c^T x$ for every feasible x

- The dual LP can be interpreted as finding the best upperbound on the primal that can be achieved this way.

Properties of Duals

- Duality is an inversion

Fact: Given any primal LP, the dual of its dual is itself.

Proof: homework exercise

Primal LP

$$
\begin{aligned}
\max \quad & c^T \cdot x \\
\text{s.t.} \quad & \\
& a_i^T x \le b_i, && \forall i \in C_1 \\
& a_i^T x = b_i, && \forall i \in C_2 \\
& x_j \ge 0, && \forall j \in D_1 \\
& x_j \in \mathbb{R}, && \forall j \in D_2
\end{aligned}
$$

Dual LP

$$
\begin{aligned}
\min \quad & b^T \cdot y \\
\text{s.t.} \quad & \\
& \bar{a}_j y \ge c_j, && \forall j \in D_1 \\
& \bar{a}_j\, y = c_j, && \forall j \in D_2 \\
& y_i \ge 0, && \forall i \in C_1 \\
& y_i \in \mathbb{R}, && \forall i \in C_2
\end{aligned}
$$

Outline

- How to Write Dual Program of LP
- Duality, and Examples

Weak Duality

Primal LP

$$\begin{aligned} \max \quad & c^t \cdot x \\ \text{s.t.} \quad & Ax \le b \\ & x \ge 0 \end{aligned}$$

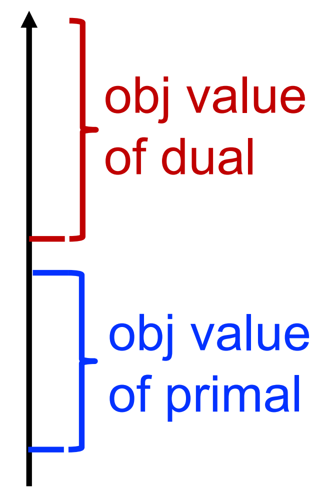
Dual LP

$$\begin{aligned} \min \quad & b^t \cdot y \\ \text{s.t.} \quad & A^t y \ge c \\ & y \ge 0 \end{aligned}$$

Theorem [Weak Duality]: For any primal feasible x and dual feasible y, we have $c^T \cdot x \le b^T \cdot y$

This should not be a surprise to you; Recall

Flow-Cut Weak Duality: Let f be any flow and (A, B) be any cut. Then $\mathrm{val}(f) \le \mathrm{cap}(A, B)$.



Weak Duality

Primal LP

max	$c^t \cdot x$
s.t.	$Ax \le b$
	$x \ge 0$

Dual LP

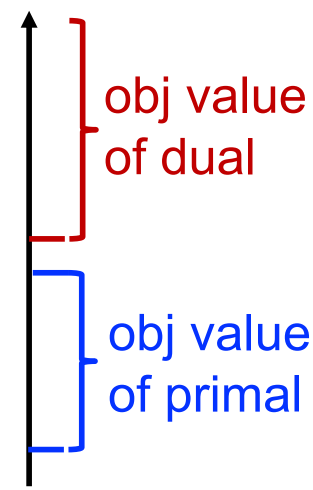
min	$b^t \cdot y$
s.t.	$A^t y \ge c$
	$y \ge 0$

Theorem [Weak Duality]: For any primal feasible x and dual feasible y, we have $c^T \cdot x \le b^T \cdot y$

Corollary:

- If primal is unbounded, dual is infeasible
- If dual is unbounded, primal is infeasible
- If primal and dual are both feasible, then

  OPT(primal) ≤ OPT(dual)



Weak Duality

Primal LP

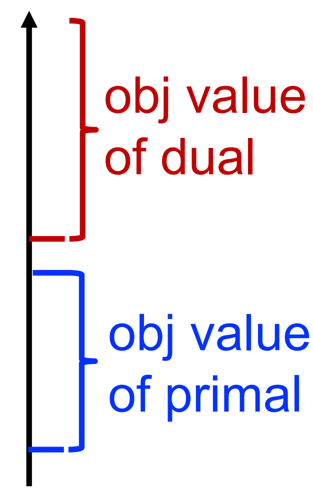
$$\begin{aligned} \max \quad & c^t \cdot x \\ \text{s.t.} \quad & Ax \le b \\ & x \ge 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \min \quad & b^t \cdot y \\ \text{s.t.} \quad & A^t y \ge c \\ & y \ge 0 \end{aligned}$$

Theorem [Weak Duality]: For any primal feasible x and dual feasible y, we have $c^T \cdot x \le b^T \cdot y$

Corollary: If x is primal feasible and y is dual feasible, and $c^T \cdot x = b^T \cdot y$, then both are optimal.



Interpretation of Weak Duality

Economic Interpretation:

If prices of raw materials are set such that there is incentive to sell raw materials directly, then factory's total revenue from sale of raw materials would exceed its profit from any production.

Upperbound Interpretation:

The method of rescaling and summing rows of the Primal indeed givens an upper bound of the Primal's objective value (well, self-evident…).

Proof of Weak Duality

Primal LP

max $\quad c^t \cdot x$

s.t. $\quad Ax \leq b$

$\quad x \geq 0$

Dual LP

min $\quad b^t \cdot y$

s.t. $\quad A^t y \geq c$

$\quad y \geq 0$

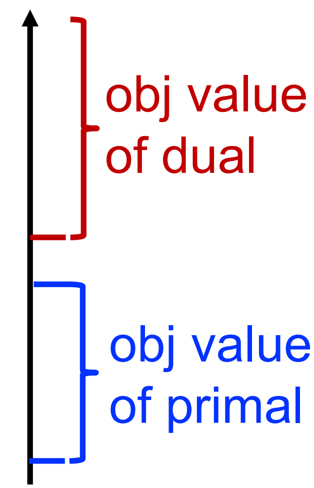
$$y^T \cdot b \geq y^T \cdot Ax = x^T \cdot A^T y \geq x^T \cdot c$$

Strong Duality

Theorem [Strong Duality]: If either the primal or dual is feasible and bounded, then so is the other and OPT(primal) = OPT(dual).

… I thought there was nothing worth publishing until the Minimax Theorem was proved.

John von Neumann



Interpretation of Strong Duality

Economic Interpretation:

There exist raw material prices such that the factory is indifferent between selling raw materials or products.

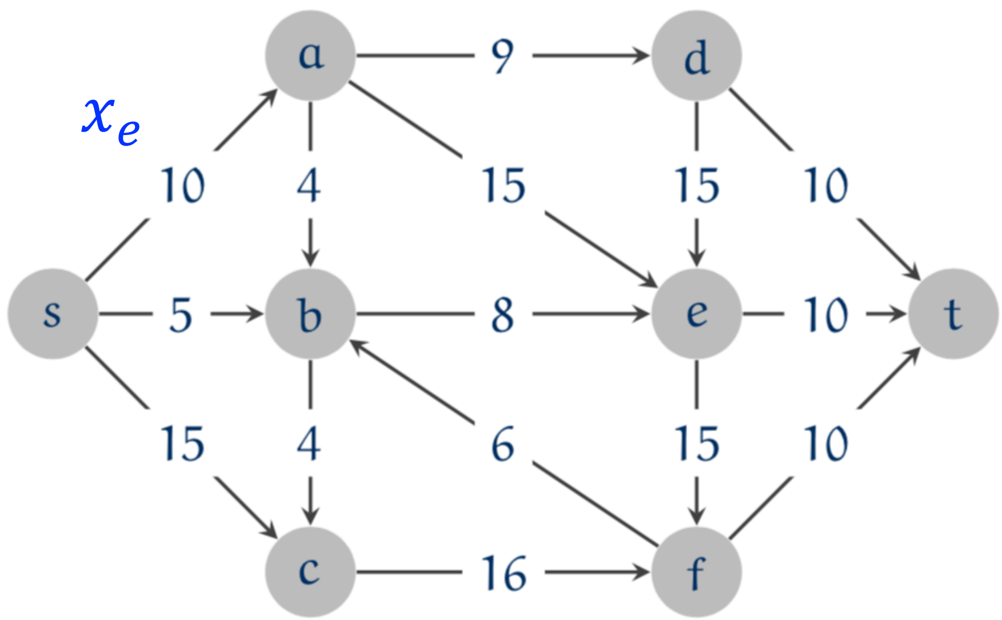
Upperbound Interpretation:

The method of scaling and summing constraints yields a tight upperbound for the primal objective value.

Duality Example: Flow vs Cut

- Recall the max flow problem

One flow variable x_e for each edge e:



Duality Example: Flow vs Cut

- Recall the max flow problem

One flow variable x_e for each edge e:

$O(v)$: set of edges outward from v
$I(v)$: set of edges inward to v

$$\max \quad \sum_{e \in O(s)} x_e - \sum_{e \in I(s)} x_e$$

$$\text{s.t.} \quad \sum_{e \in O(v)} x_e = \sum_{e \in I(v)} x_e \,, \ \forall v \neq s, t$$

$$x_e \leq c_e, \forall e \in E$$

$$x_e \geq 0, \forall e \in E$$

The LP formulation for max-flow problem

Duality Example: Flow vs Cut

- Recall the max flow problem

One flow variable x_e for each edge e:

max $\sum_{e \in O(s)} x_e - \sum_{e \in I(s)} x_e$ Objective: maximize total flow out from s

s.t. $\sum_{e \in O(v)} x_e = \sum_{e \in I(v)} x_e \,, \ \forall v \neq s, t$

$x_e \leq c_e, \forall e \in E$

$x_e \geq 0, \forall e \in E$

The LP formulation for max-flow problem

Duality Example: Flow vs Cut

- Recall the max flow problem

One flow variable x_e for each edge e:

$$\begin{aligned}
\text{max} \quad & \textstyle\sum_{e \in O(s)} x_e - \sum_{e \in I(s)} x_e \\
\text{s.t.} \quad & \textstyle\sum_{e \in O(v)} x_e = \sum_{e \in I(v)} x_e \,, \ \forall v \neq s, t \qquad \text{Flow conservation constraint} \\
& x_e \leq c_e, \forall e \in E \\
& x_e \geq 0, \forall e \in E
\end{aligned}$$

The LP formulation for max-flow problem

Duality Example: Flow vs Cut

- Recall the max flow problem

One flow variable x_e for each edge e:

$$\max \quad \sum_{e \in O(s)} x_e - \sum_{e \in I(s)} x_e$$

$$\text{s.t.} \quad \sum_{e \in O(v)} x_e = \sum_{e \in I(v)} x_e \,, \ \forall v \neq s, t$$

$$x_e \leq c_e, \forall e \in E$$

$$x_e \geq 0, \forall e \in E$$

Capacity constraint and non-negativity

The LP formulation for max-flow problem

Duality Example: Flow vs Cut

Primal LP

max $\sum_{e \in O(s)} x_e - \sum_{e \in I(s)} x_e$

s.t.

$\sum_{e \in O(v)} x_e = \sum_{e \in I(v)} x_e, \ \forall v \neq s, t$

$x_e \leq c_e, \forall e \in E$

$x_e \geq 0, \forall e \in E$

Dual LP

min $\sum_{e \in E} c_e z_e$

s.t.

$y_v - y_u \leq z_e, \ \forall e = (u, v) \in E$

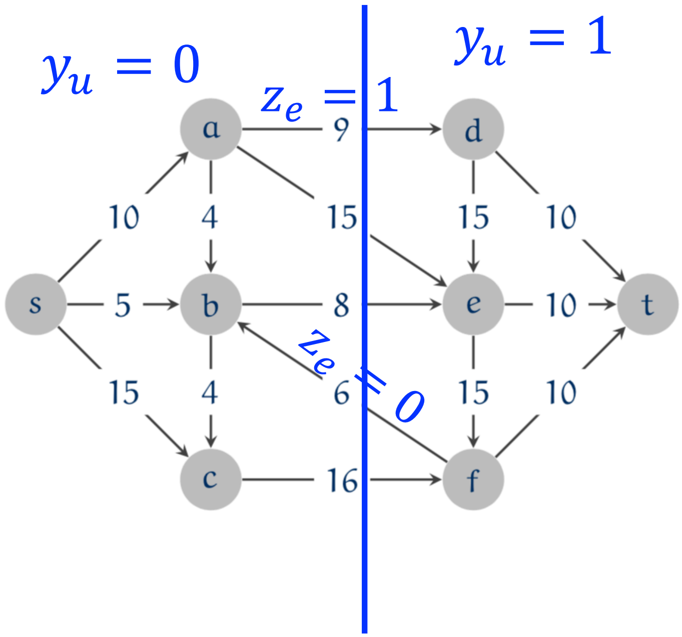
$y_s = 0$

$y_t = 1$

$z_e \geq 0, \ \forall e \in E$

- Dual variable describes fraction z_e of each edge to "fractionally" cut
- Dual constraints require that in total at least 1 edge is cut on every path from s to t
  - $\sum_{(u,v) \in P} z_{uv} \geq \sum_{(u,v) \in P} (y_v - y_u) = y_t - y_s = 1$

Duality Example: Flow vs Cut



Dual LP

$$\min \quad \sum_{e \in E} c_e z_e$$

s.t.

$$y_v - y_u \le z_e, \quad \forall e = (u, v) \in E$$

$$y_s = 0$$

$$y_t = 1$$

$$z_e \ge 0, \quad \forall e \in E$$

- Claim: every integral $s - t$ cut (S, T) is feasible for dual
- Proof:
  - Setting $y_u = 0$ for $u \in S$, $y_u = 1$ for $u \in T$,
  - $z_e = y_v - y_u = 1$ iff $e = (u, v)$ is cut; $z_e = 0$ otherwise

Duality Example: Flow vs Cut

Primal LP

max $\sum_{e \in O(s)} x_e - \sum_{e \in I(s)} x_e$

s.t.

$\sum_{e \in O(v)} x_e = \sum_{e \in I(v)} x_e, \quad \forall v \neq s, t$

$x_e \leq c_e, \forall e \in E$

$x_e \geq 0, \forall e \in E$

Dual LP

min $\sum_{e \in E} c_e z_e$

s.t.

$y_v - y_u \leq z_e, \ \forall e = (u, v) \in E$

$y_s = 0$

$y_t = 1$

$z_e \geq 0, \ \forall e \in E$

- Ford-Fulkerson implies: max flow = min integral cut $\geq$ min fractional cut
- Weak duality implies: max flow $\leq$ min fractional cut
- These two inequalities must achieve =, thus

  min integral cut = min fractional cut

  i.e., Dual always has an integral optimal solution
- Thus min (integral) cut can be computed by solving dual LP as well.

Duality Example: Flow vs Cut

Primal LP

max $\sum_{e\in O(s)} x_e - \sum_{e\in I(s)} x_e$

s.t.

$\sum_{e\in O(v)} x_e = \sum_{e\in I(v)} x_e\,, \quad \forall v \neq s, t$

$x_e \leq c_e, \forall e \in E$

$x_e \geq 0, \forall e \in E$

Dual LP

min $\sum_{e\in E} c_e\, z_e$

s.t.

$y_v - y_u \leq z_e, \ \ \forall e = (u, v) \in E$

$y_s = 0$

$y_t = 1$

$z_e \geq 0, \ \ \forall e \in E$

- To sum up: max-flow min-cut can both also be solved by LP
- Their equality is just a special case of LP duality
- There are many other such examples in combinatorial optimization (e.g., shortest paths, bipartite matching, etc.)

Thank You

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