Announcements

- HW 4 deadline extended to tomorrow, 6 pm
  - Can use at most two late days

- HW5 (the last HW) will be out soon
  - Due roughly at the end of this month

- Reminder: Midterm 2 will be December 2nd 9 am to 3rd 9 am
  - Only covers second half of the course: max flow, min cut, NP-hardness, approximation Algo, game theory
  - Takes around 3 hours, but you will have the entire day
  - Format is the same as midterm 1, open notes as well

CS6161: Design and Analysis of Algorithms (Fall 2020)

Introduction to Game Theory I

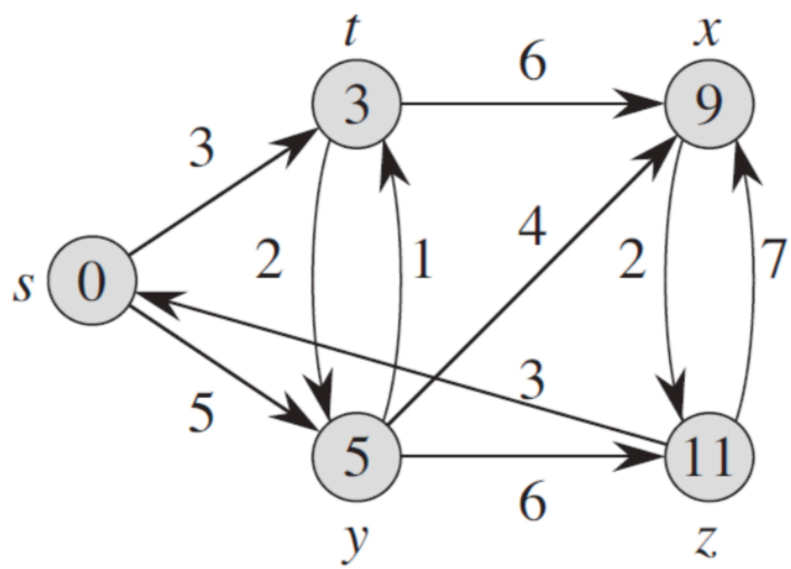
Instructor: Haifeng Xu

Outline

- Why Game Theory in Algorithm Design?
- Intro to Game Theory and Nash Equilibrium

In all algorithm design problems we see so far, the designer is the ONLY agent and controls everything

- Shortest path – only the designer takes a path, no other routers and no congestion



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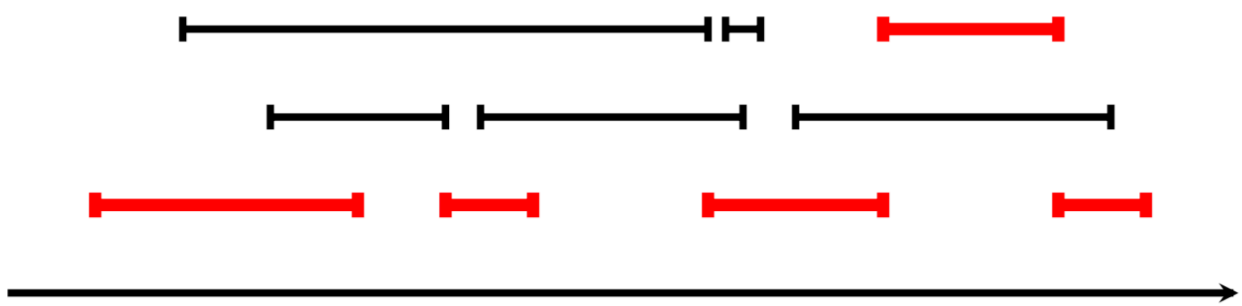

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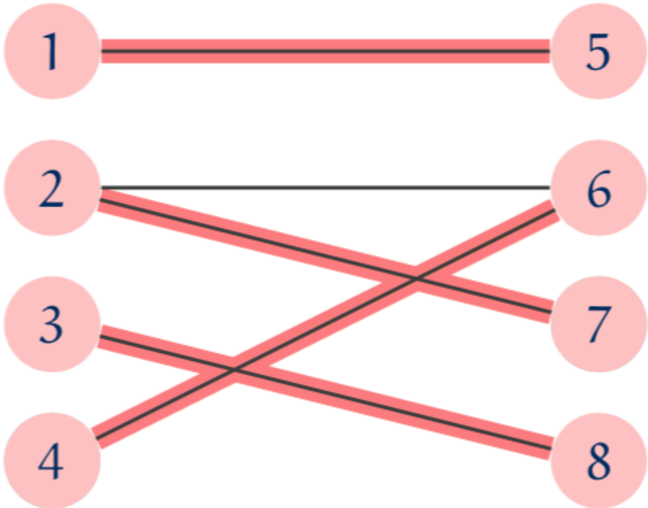

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- Max flow – can optimize smuggling of drug, but there will be interdiction measures from security agencies
- ….

Real Algorithmic Problems Involve Multiple Agents

- Shortest path – need to avoid congestion caused by other drivers
- Job scheduling – you want your job to be completed by the server, and may adjust its (start, end) time if it help
- Bipartite matching – you would want to be matched to a more preferred node, if that node also prefer you
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Previous algorithmic techniques cannot handle such problems with multiple strategic agents, a.k.a., games

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Game theory also studies optimization (i.e., algorithm design), but in settings with multiple agents, each maximizing their own objective

It turns out… things get much more complicated, even counter-intuitive, with multiple agents

- Even with two agents, each choosing among two actions

Example 1: Prisoner's Dilemma

- Two members A,B of a criminal gang are arrested
- They are questioned in two separate rooms
  - No communications between them

B A	B stays silent	B betrays
A stays silent	-1 $\boxed{-1}$	0 -3
A betrays	-3 $\boxed{0}$	-2 -2

Q: How should each prisoner act?

- Betray is the best action for each

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equilibrium

Q: How should each prisoner act?

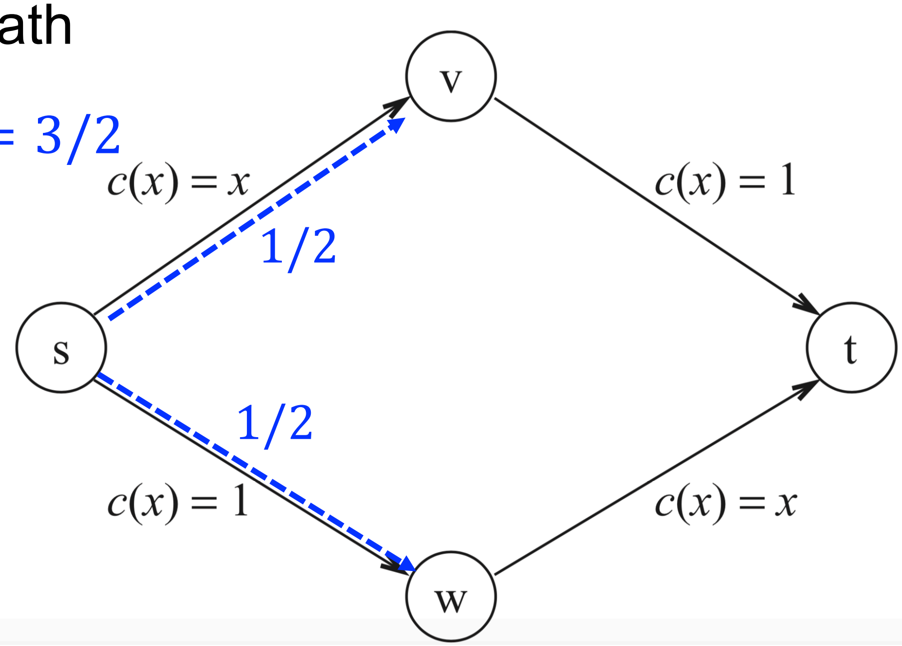
- Betray is the best action for each
- But, (-1,-1) is a better outcome for both
- Why? What goes wrong?
  - Selfish behaviors lead to inefficient social outcome

Example 2: Selfish Routing

- One unit flow from s to t which consists of (infinite) individuals, each controlling an infinitesimal small amount of flow
- Each individual wants to minimize his own travel cost

Q: What is the equilibrium status?

- Half unit flow through each path
- Total social cost = $\frac{3}{2}\left(\frac{1}{2}+\frac{1}{2}\right) = 3/2$



Example 2: Selfish Routing

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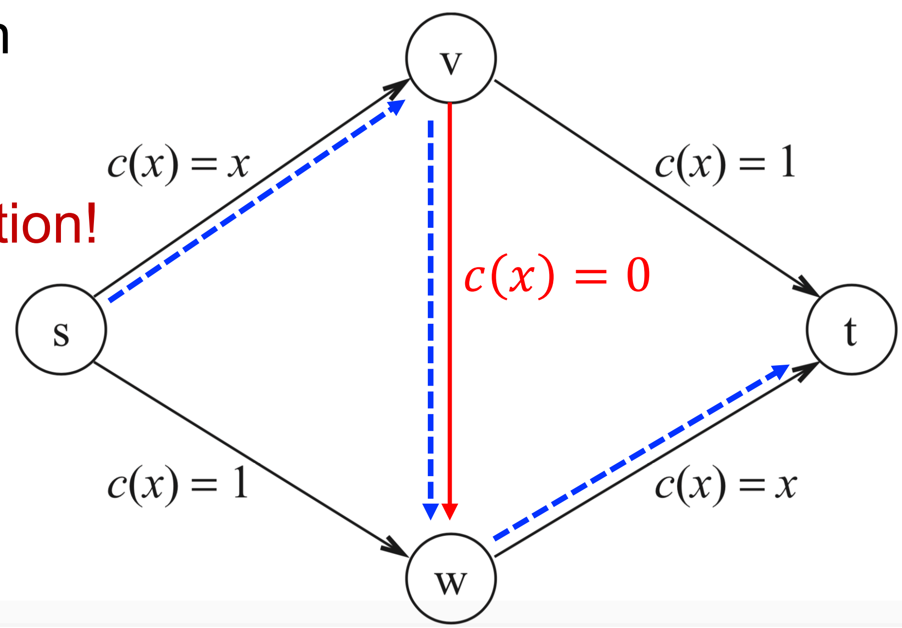
Q: If government builds a superior high way with 0 traveling cost, what is the new equilibrium status?

- Everyone takes the blue path
- Total social cost = 2
- More roads increase congestion!



Example 3: Traffic Light Game

- Two cars heading to orthogonal directions

B

A

	STOP	GO
STOP	(-3, -2)	(-3, 0)
GO	(0, -2)	(-100, -100)

Q: what are the equilibrium statuses?

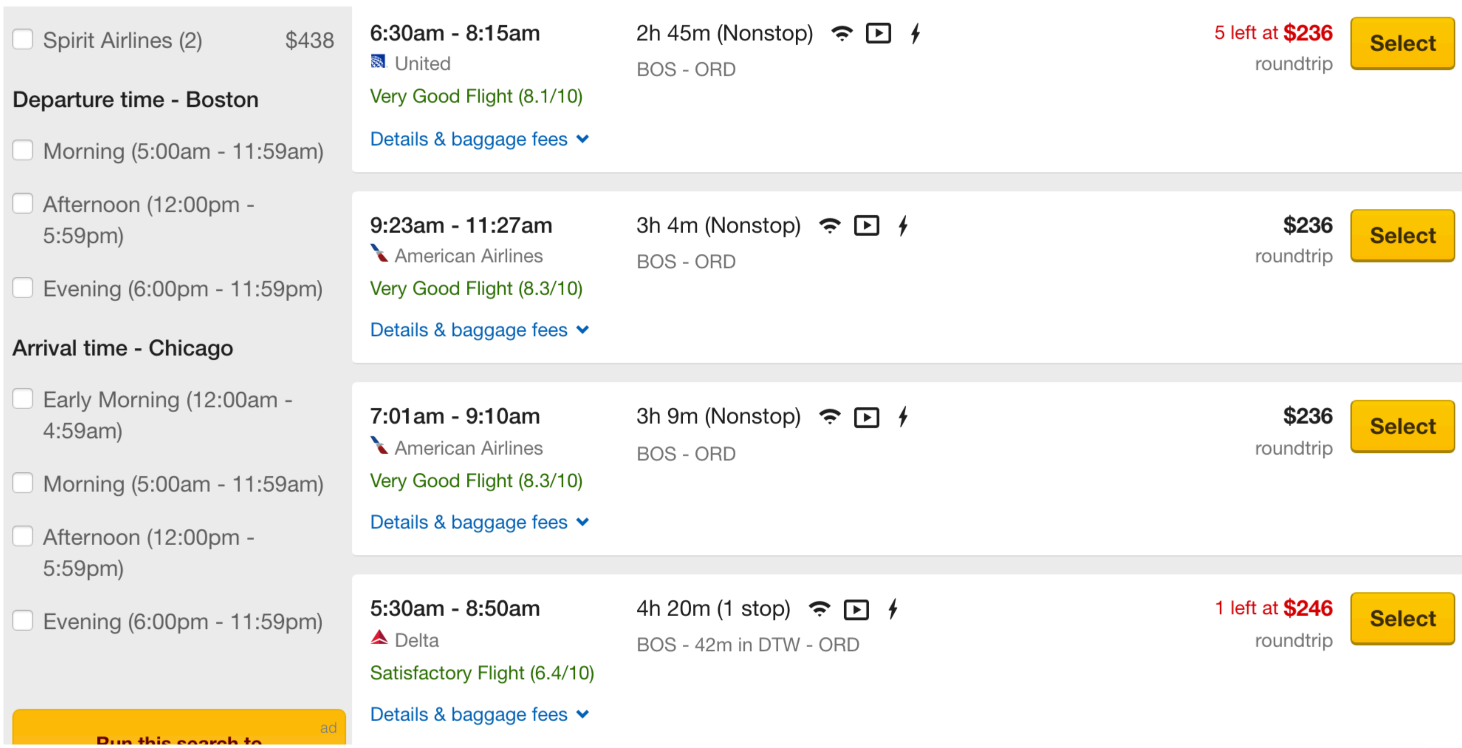
Answer: (STOP, GO) and (GO, STOP)

Key Characteristics of These Games

- Each agent wants to maximize her own payoff
- An agent's payoff depends on other agents' actions
- The interaction stabilizes at a state where no agent can increase his payoff via unilateral deviation
  - Remark: this does not mean they achieve globally best possible utility (e.g., prisoner's dilemma)

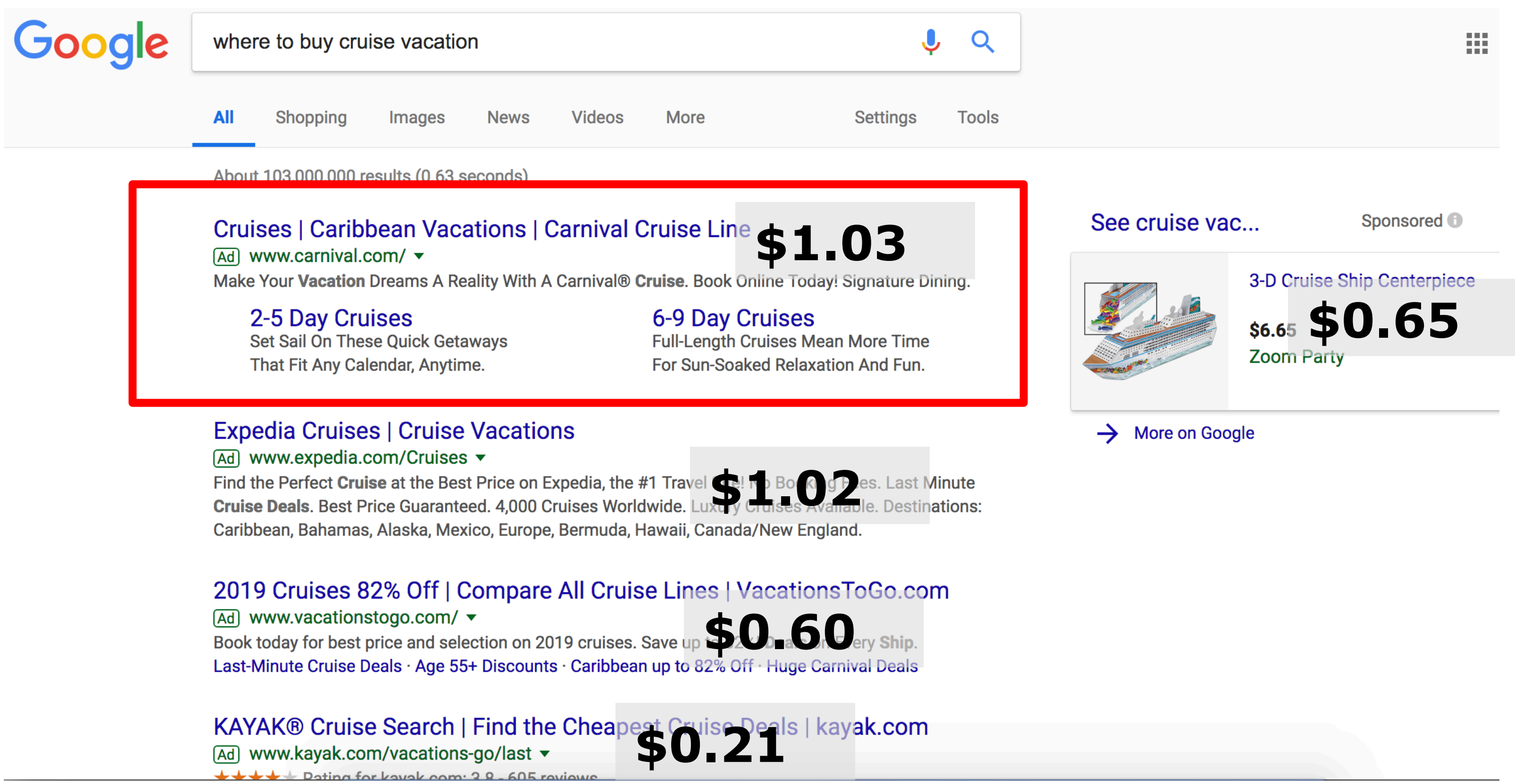
Strategic Games Are Ubiquitous

➢ Pricing



Strategic Games Are Ubiquitous

- Pricing
- Sponsored search
  - Drives 90%+ of Google's revenue



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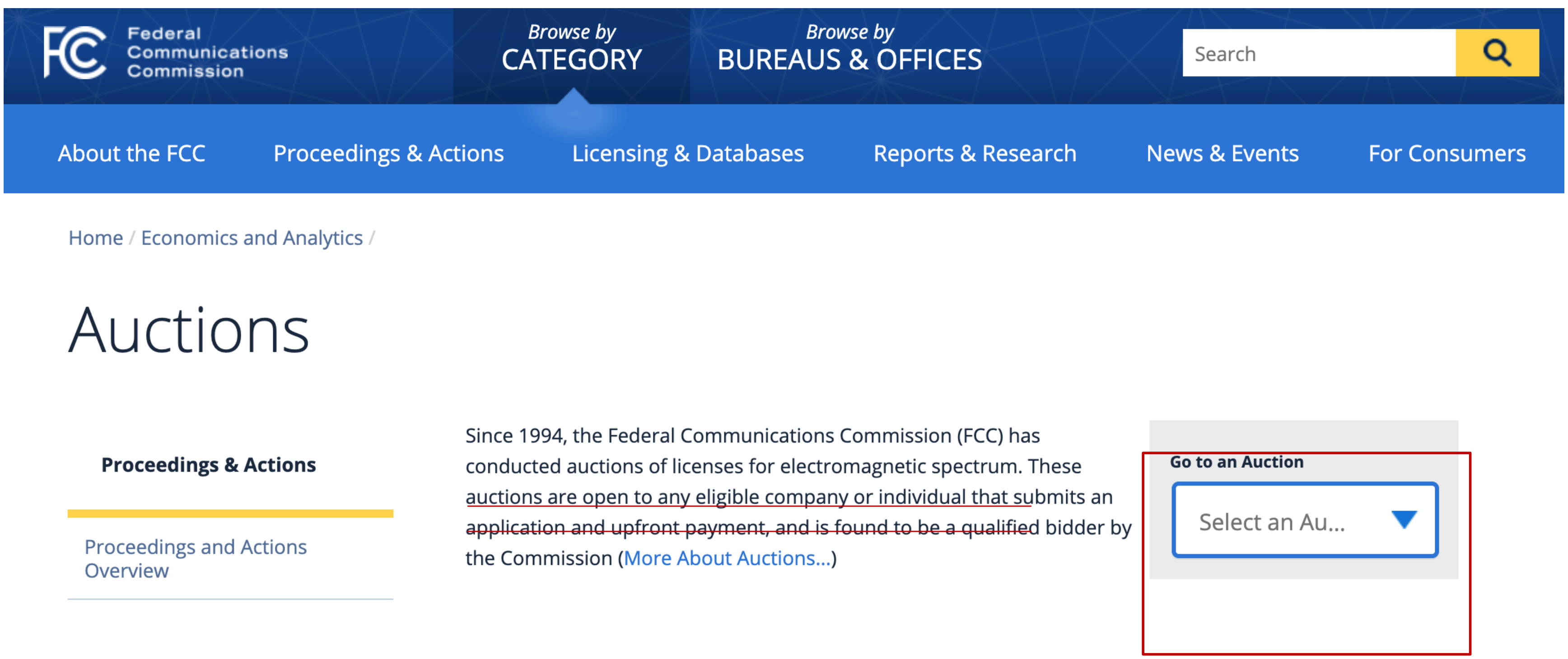
- FCC's Allocation of spectrum to radio frequency users



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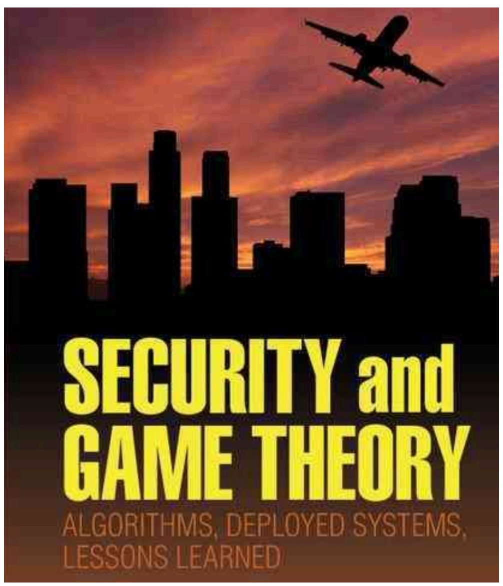

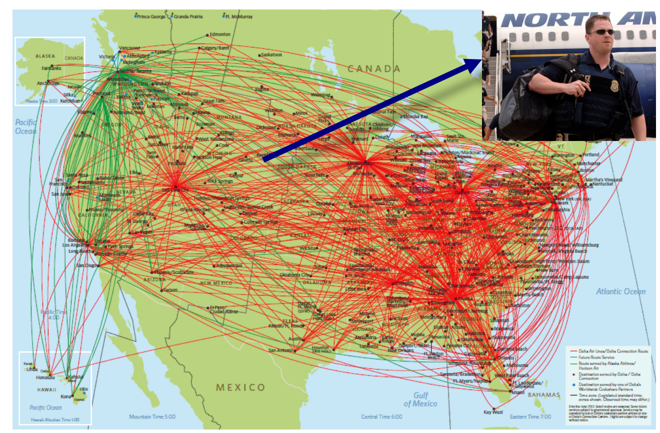

Optimize resource allocation against attackers/adversaries

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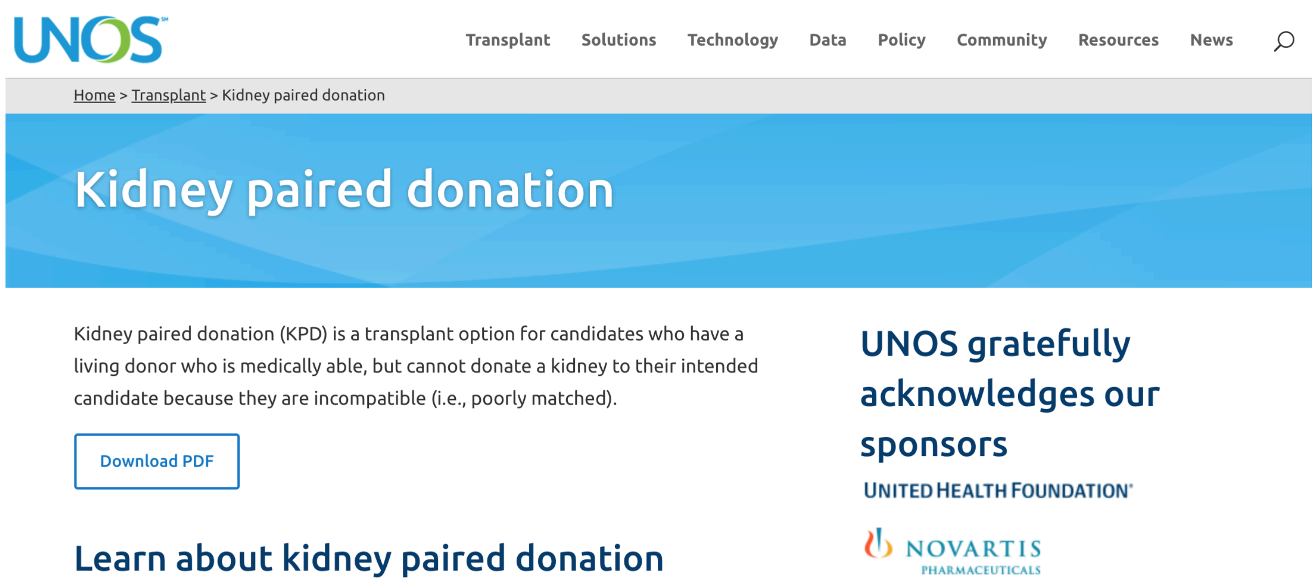
- Kidney exchange – decides who gets which kidney at when



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These are just a few example domains *where computer science has made significant impacts*; There are many others.

Outline

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Main Components of a Game

- Players: participants of the game, each may be an individual, organization, a machine or an algorithm, etc.
- Strategies: actions available to each player
- Outcome: the profile of player strategies
- Payoffs: a function mapping an outcome to a utility for each player

Normal Form Representation

For simplicity, consider two-player games:

- Player A with action set $\{a_1, a_2, \dots, a_n\}$
- Player B with action set $\{b_1, b_2, \dots, b_m\}$
- An outcome $(a_i\ , b_j)$ results in utility $(u_{ij}\ , v_{ij})$ for A and B, respectively
- Matrix Representation (assume each player knows the matrix):

	b_1	b_2	$\dots$	b_m
a_1	$(u_{11}\ , v_{11})$	$(u_{12}\ , v_{12})$	$\dots$	$(u_{1m}\ , v_{1m})$
a_2	$(u_{21}\ , v_{21})$	$(u_{22}\ , v_{22})$	$\dots$	$(u_{2m}\ , v_{2m})$
...				
a_n	(u_{n1}, v_{n1})	$(u_{n2}\ , v_{n2})$	$\dots$	$(u_{nm}\ , v_{nm})$

But this representation easily generalizes to k players, just with more involved notations

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- Matrix Representation (assume each player knows the matrix):

<u>Remark:</u>

(1) This is the most basic form of representation

(2) Can be more complicated (e.g., sequential actions, incomplete information), but will not cover in this course

Equilibrium

- An outcome (a_{i^*}, b_{j^*}) is an equilibrium if
  1. Given B plays b_{j^*}, a_{i^*} is optimal for A: $u_{i^*,j^*} \geq u_{i,j^*}$ for any $i \in [n]$
  2. Given A plays a_{i^*}, b_{j^*} is optimal for B: $v_{i^*,j^*} \geq v_{i^*,j}$ for any $j \in [m]$
- Also generalizes to k players – an action profile is an equilibrium if each action is the player's best action, given opponents' actions
- A special case of Nash equilibrium

Why should you care about "equilibrium"?

- It predicts what will happen in this game with rational players
  - E.g., in prisoner's dilemma, prisoners are predicted to both betray
- Just like optimal solution describes what a rational algorithm designer will do in a single-agent optimization problem

Examples

B A	B stays silent	B betrays
A stays silent	-1 -1	0 -3
A betrays	-3 0	-2 -2

	L	M	R
U	4,3	5,1	6,2
M	2,1	8,4	3,6
D	3,0	9,6	2,5

Q: what is an equilibrium of this game? Is it unique?

Example: Rock-Paper-Scissor game

Player 2

Player 1

	Rock	Paper	Scissor
Rock	(0, 0)	(-1, 1)	(1, -1)
Paper	(1, -1)	(0, 0)	(-1, 1)
Scissor	(-1, 1)	(1, -1)	(0, 0)

Q: what is an equilibrium?

- No deterministic action pair can make both players happy

Example: Rock-Paper-Scissor game

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Q: what is an equilibrium?

Note: action and strategy will be used interchangeably

- Key insight: randomize over actions
- Common sense suggests (1/3,1/3,1/3)
- Deterministic action is called pure strategy
- A distribution over actions is called mixed strategy

Mixed-Strategy Equilibrium

- Player A plays mixed strategy $\boldsymbol{p}(\sum p_i = 1)$ and B plays $\boldsymbol{q}(\sum q_i = 1)$
  - They sample actions independently
- Expected utility given joint strategy profile $(\boldsymbol{p}, \boldsymbol{q})$:

  $u(\boldsymbol{p}, \boldsymbol{q}) = \sum_{ij} u_{i,j}\, p_i q_j$ for A, $\quad v(\boldsymbol{p}, \boldsymbol{q}) = \sum_{ij} v_{i,j}\, p_i q_j$ for B

Def. A strategy profile $(\boldsymbol{p}, \boldsymbol{q})$ is a mixed-strategy equilibrium if

(1) $u(\boldsymbol{p}, \boldsymbol{q}) \geq u(a_i, \boldsymbol{q}) = \sum_j u_{i,j}\, q_j$ for any $a_i \in A$

(2) $v(\boldsymbol{p}, \boldsymbol{q}) \geq v(\boldsymbol{p}, b_j) = \sum_i v_{i,j}\, p_i$ for any $b_j \in B$

- In other words, neither player cannot do better by deviating to any pure action (and thus any mixed action)
- Key question: just like pure equilibrium, such a mixed equilibrium may not exist neither!

Mixed-Strategy Equilibrium

Theorem (Nash, 1951): Every finite game admits at least one mixed-strategy equilibrium

- Finite = finite many actions and players
- A revolutionary result, become foundation of game theory
  - Since when pure equilibrium does not exist, we cannot predict how players will play
  - Nash's results suggest we can expect them to play mixed equilibrium, and it always exists!
  - For example, in rock-paper-scissor games, we should expect rational players to play uniformly at random
- Since then, people call (mixed) equilibrium the Nash equilibrium (NE)

An Illustrative Example of Nash Equilibrium (NE)

- Rock-paper-scissor: $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a best response to $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

		$1/3$	$1/3$	$1/3$
		Rock	Paper	Scissor
ExpU = 0	Rock	(0, 0)	(-1, 1)	(1, -1)
ExpU = 0	Paper	(1, -1)	(0, 0)	(-1, 1)
ExpU = 0	Scissor	(-1, 1)	(1, -1)	(0, 0)

An Important Property of Nash Equilibrium

Thm.: If $(\boldsymbol{p}, \boldsymbol{q})$ is a NE, then any a_i with $p_i > 0$ must be equally good – they are all optimal actions (also called best responses)

- Proof: follows from definition

$$u(\boldsymbol{p}, \boldsymbol{q}) \geq u(a_k, \boldsymbol{q})$$

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$$\Rightarrow \sum_i p_i \left[\sum_j u_{i,j}\, q_j\right] = u(\boldsymbol{p}, \boldsymbol{q}) \geq \max_k \sum_j u_{k,j}\, q_j$$

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$$\Rightarrow \sum_i p_i \left[\sum_j u_{i,j} \, q_j\right] = u(\boldsymbol{p}, \boldsymbol{q}) \geq \max_k \sum_j u_{k,j} \, q_j$$

So for any $p_i > 0$, we must have $\sum_j u_{i,j} \, q_j = \max_k \sum_j u_{k,j} \, q_j$

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Thank You

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