INFO8006 Introduction to Artificial Intelligence

Exercises 4: Reasoning under uncertainty II

Learning outcomes

At the end of this exercise session you should be able to:

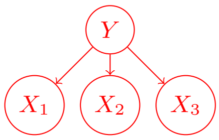
- Define and build a Bayesian network
- Compute probabilities in the context of a simple Bayesian network

Exercise 1: Bag of coins

We have a bag of three biased coins a, b, and c with probabilities of coming up heads of 20%, 60%, and 80%, respectively. One coin is drawn randomly from the bag (with equal probability of drawing each of the three coins), and then the coin is flipped three times to generate the outcomes $X1$, $X2$, and $X3$.

1. Draw the Bayesian network corresponding to this setup and define the necessary CPTs.
   We define the following random variables:

   - Y $\quad \text{Dom}_Y = \{a, b, c\}$: The coin drawn from the bag.
   - X_1 $\quad \text{Dom}_{X_1} = \{h, t\}$: The realisation of the first coin toss.
   - X_2 $\quad \text{Dom}_{X_2}\{h, t\}$: The realisation of the second coin toss.
   - X_3 $\quad \text{Dom}_{X_3}\{h, t\}$: The realisation of the third coin toss.



y	a	b	c
	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Table 1: $P(Y = y)$

$x \backslash y$	a	b	c
h	0.2	0.6	0.8
t	0.8	0.4	0.2

Table 2: $P(X_1 = x|Y = y)$

2. Calculate which coin was most likely to have been drawn from the bag if the observed flips come out heads twice and

tails once.

$$
\begin{aligned}
\hat{y} &= \arg\max_y P(y|h,h,t) \\
&= \arg\max_y \frac{P(y,h,h,t)}{P(h,h,t)} \\
&= \arg\max_y P(y,h,h,t) \\
&= \arg\max_y 3 \times P(y, X_1 = h, X_2 = h, X_3 = t) \\
&= \arg\max_y P(y, X_1 = h, X_2 = h, X_3 = t) \\
&= \arg\max_y P(X_1 = h|y)P(X_2 = h|y)P(X_3 = t|y)P(y) \\
&= \arg\max_y P(X_1 = h|y)P(X_2 = h|y)P(X_3 = t|y) \\
&= \left.\begin{matrix} y = a : 0.2^2 \times 0.8 = 0.032 \\ y = b : 0.6^2 \times 0.4 = 0.144 \\ y = c : 0.8^2 \times 0.2 = 0.128 \end{matrix}\right\} \hat{y} = b
\end{aligned}
$$

Exercise 2: Handedness (AIMA, Ex: 14.6)

Let H_x be a random variable denoting the handedness of an individual x, with possible values l or r. A common hypothesis is that left- or right-handedness is inherited by a simple mechanism; that is, perhaps there is a gene G_x, also with values l or r, and perhaps actual handedness turns out mostly the same (with some probability s) as the gene an individual possesses. Furthermore, perhaps the gene itself is equally likely to be inherited from either of an individual's parents, with a small nonzero probability m of a random mutation flipping the handedness.

1. Which of the three networks in Figure 1 claim that $P(G_{father}, G_{mother}, G_{child}) = P(G_{father})P(G_{mother})P(G_{child})$? *c.*

2. Which of the three networks make independence claims that are consistent with the hypothesis about the inheritance of handedness? *a* and *b* (*b* does not claim any independence).

3. Which of the three networks is the best description of the hypothesis? *a.*

4. Write down the CPT for the Gchild node in network (a), in terms of s and m.

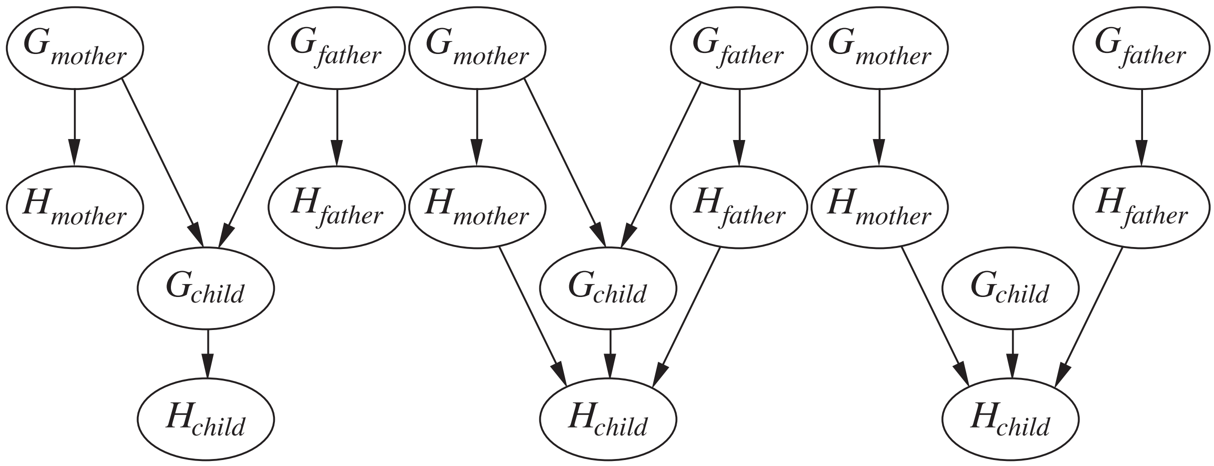
g_m \ g_f	l	r
l	$1-m$	0.5
r	0.5	m

Table 3: $P(\mathcal{G}_c = l|g_f, g_m)$

5. Suppose that $P(G_{father} = l) = P(G_{mother} = l) = q$. In network (a), derive an expression for $P(G_{child} = l)$ in terms of m and q only, by conditioning on its parent nodes.

$$
\begin{aligned}
P(G_{child} = l) &= \sum_{g_m}\sum_{g_f} P(G_{child} = l, g_m, g_f) && (1)\\
&= \sum_{g_m}\sum_{g_f} P(G_{child} = l|g_m, g_f)P(g_m)P(g_f) && (2)\\
&= q^2(1-m) + 0.5q(1-q) + 0.5q(1-q) + (1-q)^2 m && (3)\\
&= q + m - 2qm && (4)
\end{aligned}
$$

6. Under conditions of genetic equilibrium, we expect the distribution of genes to be the same across generations. Use this to calculate the value of q, and, given what you know about handedness in humans, explain why the hypothesis described at the beginning of this question must be wrong. hypothesis about the inheritance of handedness? $P(G_{child} = l) = P(G_{mother} = l) = P(G_{father} = l) \Leftrightarrow q = q + m - 2qm$ and so $q = 0.5$ which is not realisitic.



(a) (b) (c)

Figure 1: Possible Bayesian Networks of handedness inheritance

Exercise 3: D-separation

You are advised to take a look at d-separation before doing this exercise: `http://web.mit.edu/jmn/www/6.034/d-separation.pdf`.

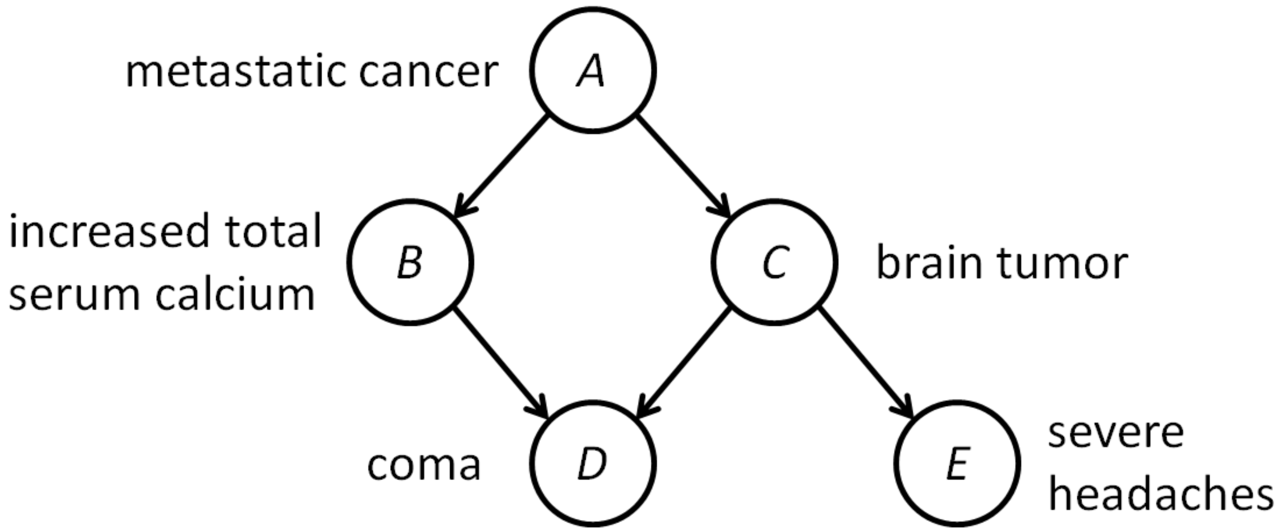


Figure 2: Bayesian Network of metastatic cancer

Consider the Bayesian network of Figure 2, which, if any, of the following are asserted by the network structure ?

1. $P(b,c) = P(b)P(c)$
2. $P(b,c|a) = P(b|a)P(c|a)$
3. $P(b,c|a,d) = P(b|a,d)P(c|a,d)$
4. $P(c|a,d,e) = P(c|a,b,d,e)$
5. $P(b,e|a) = P(b|a)P(e|a)$
6. $P(b,e) = \sum_{a \in A, c \in C, d \in D} P(a)P(b|a)P(c|a)P(e|c)P(d|b,c)$

1. $B \perp\!\!\!\perp C$? False.
2. $B \perp\!\!\!\perp C|A$? True.
3. $B \perp\!\!\!\perp C|A, D$? False.
4. $B \perp\!\!\!\perp C|A, D, E$? False (Markov blanket).

5. $B \perp\!\!\!\perp E | A$? True.

6. $P(b,e) = \sum_{a \in A, c \in C, d \in D} P(a,b,c,d,e) = \sum_{a \in A, c \in C, d \in D} P(a)P(b|a)P(c|a)P(e|c)P(d|b,c)$, so it is correct.

⋆ Use inference by variable elimination to compute $P(E|a,b)$.

$$P(E|a,b) = \frac{1}{\alpha} \sum_{c,d} P(a,b,c,d,E) \tag{5}$$

$$= \frac{1}{\alpha} \sum_{c,d} P(a)P(b|a)P(c|a)P(d|b,c)P(E|c) \tag{6}$$

$$= \frac{1}{\alpha} P(a)P(b|a) \sum_{c} P(c|a)P(E|c) \sum_{d} P(d|b,c) \tag{7}$$

We define the initial factors as: $F_1 = P(A=a)$, $F_2 = P(B=b|A=a)$, $F_3(C) = P(C|A=a)$, $F_4(C,D) = P(D|B=b,C)$, $F_5(E,C)P(E|C)$. And then we compute: $F_6(C) = \sum_d F_4(C,d)$, then $F_7'(C,E) = F_3(C) \times F_5(E,C) \times F_6(C)$ and $F_7(E) = \sum_c F_7'(c,E)$, then $F_8(E) = F_1 \times F_2 \times F_7(E)$, finally $P(E|a,b) = \frac{F_8(E)}{\sum_e F_8(e)}$

Exercise 4: Car Diagnosis (AIMA, Ex: 14.8)

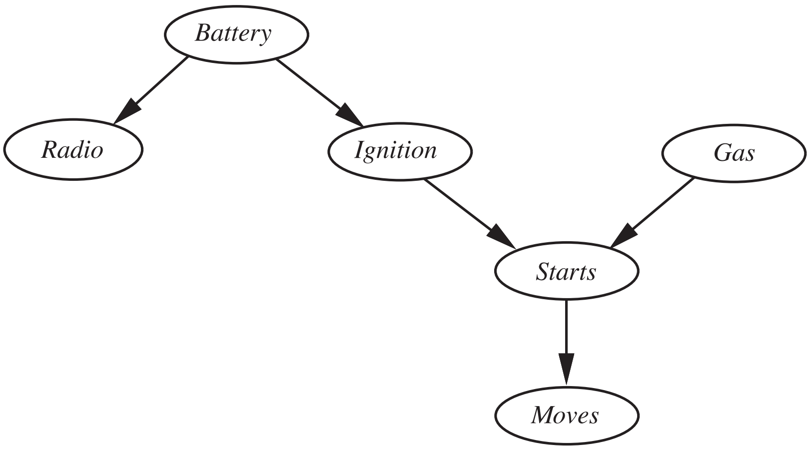


Figure 3: A Bayesian network describing some features of a car's electrical system and engine. Each variable is Boolean, and the true value indicates that the corresponding aspect of the vehicle is in working order.

Consider the network for car diagnosis shown in Figure 3.

1. Extend the network with the Boolean variables IcyWeather and StarterMotor. IcyWeather is not caused by any of the car-related variables, so needs no parents. It directly affects the battery and the starter motor. StarterMotor is an additional precondition for Starts. The new network is shown in Figure 4.

2. Give reasonable conditional probability tables for all the nodes. Reasonable probabilities may vary a lot depending on the kind of car and perhaps the personal experience of the assessor. The following values indicate the general order of magnitude and relative values that make sense:

   - A reasonable prior for IcyWeather might be 0.05 (perhaps depending on location and season).
   - $P(Battery|IcyWeather) = 0.95, P(Battery|\neg IcyWeather) = 0.997$.
   - $P(StarterMotor|IcyWeather) = 0.98,\ P(Battery|\neg IcyWeather) = 0.999$.
   - $P(Radio|Battery) = 0.9999, P(Radio|\neg Battery) = 0.05$.
   - $P(Ignition|Battery) = 0.998, P(Ignition|\neg Battery) = 0.01$.
   - $P(Gas) = 0.995$.
   - $P(Starts|Ignition, StarterMotor, Gas) = 0.9999, otherentries 0.0$.
   - $P(Moves|Starts) = 0.998$.

3. How many independent values are contained in the joint probability distribution for eight Boolean nodes, assuming that no conditional independence relations are known to hold among them? With 8 Boolean variables, the joint has $28 - 1 = 255$ independent entries.

4. How many independent probability values do your network tables contain? Given the topology shown in Figure S14.1, the total number of independent CPT entries is $1 + 2 + 2 + 2 + 2 + 1 + 8 + 2 = 20$.

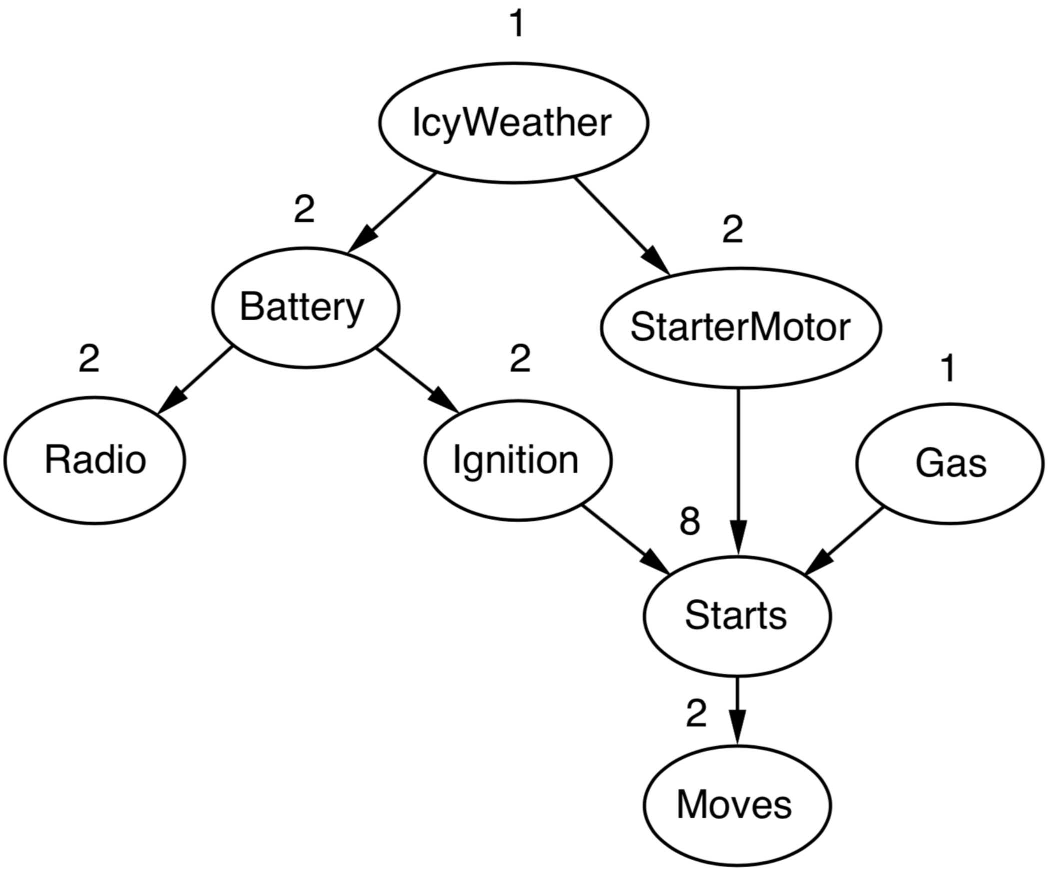


Figure 4: Extended Bayesian Network

Exercise 5 ⋆: Green Party President bis (Berkeley Spring 2014)

Consider the following Bayesian network.

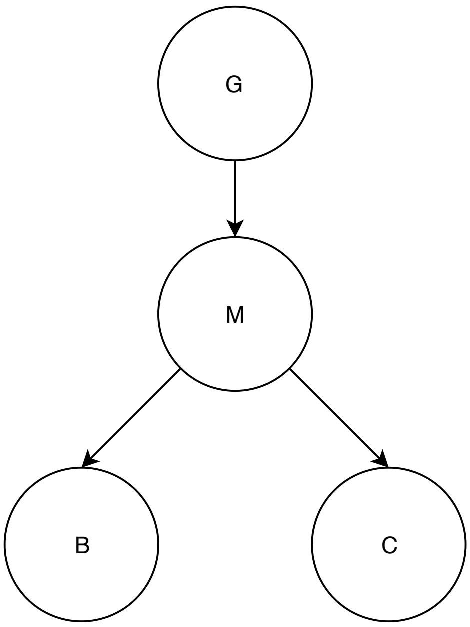


Figure 5: Green Party President Bayesian Network

1. The full joint distribution is given below. Fill in the missing values.

	g	$\neg g$
$P(G)$	0.1	0.9

	$P(m\|G)$	$P(\neg m\|G)$
g	0.667	0.333
$\neg g$	0.25	0.75

	$P(b\|M)$	$P(\neg b\|M)$
m	0.4	0.6
$\neg m$	0.2	0.8

	$P(c\|M)$	$P(\neg c\|M)$
m	0.25	0.75
$\neg m$	0.5	0.5

G	M	B	C	$P(G, M, B, C)$
0	0	0	0	27/100
0	0	0	1	?
0	0	1	0	27/400
0	0	1	1	27/400
0	1	0	0	81/800
0	1	0	1	27/800
0	1	1	0	27/400
0	1	1	1	9/400
1	0	0	0	1/75
1	0	0	1	?
1	0	1	0	1/300
1	0	1	1	1/300
1	1	0	0	3/100
1	1	0	1	1/100
1	1	1	0	?
1	1	1	1	1/150

G	M	B	C	$P(G, M, B, C)$
0	0	0	0	27/100
0	0	0	1	27/100
0	0	1	0	27/400
0	0	1	1	27/400
0	1	0	0	81/800
0	1	0	1	27/800
0	1	1	0	27/400
0	1	1	1	9/400
1	0	0	0	1/75
1	0	0	1	1/75
1	0	1	0	1/300
1	0	1	1	1/300
1	1	0	0	3/100
1	1	0	1	1/100
1	1	1	0	1/50
1	1	1	1	1/150

2. Compute the following quantities.

   (a) $P(b|m) = \frac{4}{10}$

   (b) $P(b|m, g) = \frac{4}{10}$

   (c) $P(b) = \frac{31}{120}$

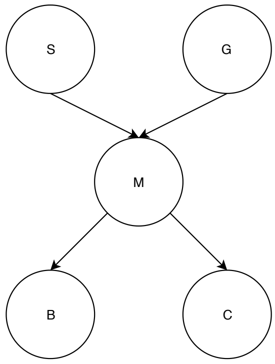
   (d) $P(c|b) = \frac{12}{31}$

3. Add a node S to the Bayesian network that reflects the possibility that a new scientific study could influence the probability of marijuana being legalised. Assume that the study does not directly influence B or C. Draw the new Bayesian network below. Which CPT(s) need to be modified?
$P(M|G)$ becomes $P(M|G, S)$, and will contain 8 entries instead of 4.

4. Consider your new Bayesian net. Which of the following are guaranteed to be true, and which are guaranteed to be false?

   (a) $B \perp\!\!\!\perp G$ Not indicated by the net.

   (b) $C \perp\!\!\!\perp G | M$ True.



(c) $G \perp\!\!\!\perp S$ True.

(d) $G \perp\!\!\!\perp S | M$ Not indicated by the net.

(e) $S \perp\!\!\!\perp G | B$ Not indicated by the net.

(f) $B \perp\!\!\!\perp C$ Not indicated by the net.

(g) $B \perp\!\!\!\perp C | G$ Not indicated by the net.

(h) $B \perp\!\!\!\perp C | M$ True.

Exercise 6 ⋆: Nuclear Power Plant (AIMA, Ex: 14.11)

In your local nuclear power station, there is an alarm that senses when a temperature gauge exceeds a given threshold. The gauge measures the temperature of the core. Consider the Boolean variables A (alarm sounds), FA (alarm is faulty), and FG (gauge is faulty) and the multivalued nodes G (gauge reading) and T (actual core temperature).

1. Draw a Bayesian network for this domain, given that the gauge is more likely to fail when the core temperature gets too high. See Figure 6

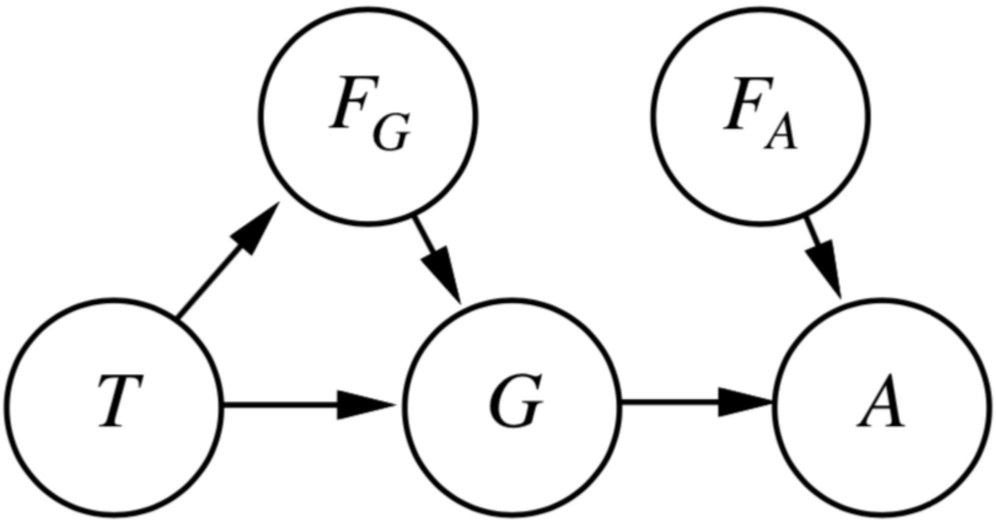


Figure 6: Power Plant Bayesians Network.

2. Suppose there are just two possible actual and measured temperatures, normal and high; the probability that the gauge gives the correct temperature is x when it is working, but y when it is faulty. Give the conditional probability table associated with G. The CPT for G is shown below. Students should pay careful attention to the semantics of FG, which is true when the gauge is faulty, i.e., not working.

	T = Normal		T = High	
	F_G	$\neg F_G$	F_G	$\neg F_G$
G = Normal	y	x	1 - y	1 - x
G = High	1 - y	1 - x	y	x

3. Suppose the alarm works correctly unless it is faulty, in which case it never sounds. Give the conditional probability table associated with A. The CPT for A is as follows:

	G = Normal		G = High	
	F_A	$\neg F_A$	F_A	$\neg F_A$
A	0	0	0	1
$\neg A$	1	1	1	0

4. Suppose the alarm and gauge are working and the alarm sounds. Calculate an expression for the probability that the temperature of the core is too high, in terms of the various conditional probabilities in the network.

   Abbreviating $T = High$ and $G = High$ by T and G, the probability of interest here is $P(T|A, \neg F_G, \neg FA)$. Because the alarm's behaviour is deterministic, we can reason that if the alarm is working and sounds, G must be High.

   Because F_A and A are d-separated from T given G , we need only calculate $P(T|\neg F_G, G)$. There are several ways to go about doing this. The "opportunistic" way is to notice that the CPT entries give us $P(G|T, \neg F_G)$, which suggests using the generalized Bayes' Rule to switch G and T with $\neg F_G$ as background:

$$P(T|\neg F_G, G) = \alpha P(G|T, \neg F_G)P(T|\neg F_G)$$

   We then use Bayes' Rule again on the last term:

$$P(T|\neg F_G, G) = \alpha P(G|T, \neg F_G)P(\neg F_G|T)P(T)$$

   A similar relationship holds for $\neg T$:

$$P(\neg T|\neg F_G, G) = \alpha P(G|\neg T, \neg F_G)P(\neg F_G|\neg T)P(\neg T)$$

   Normalizing, we obtain

$$P(T|\neg F_G, G) = \frac{P(G|T, \neg F_G)P(\neg F_G|T)P(T)}{P(G|T, \neg F_G)P(\neg F_G|T)P(T) + P(G|\neg T, \neg F_G)P(\neg F_G|\neg T)P(\neg T)}$$

   The "systematic" way to do it is to revert the joint entries (noticing that the subgraph of T , G, and F_G is completely connected so no loss of efficiency is entailed). We have

$$P(T|\neg F_G, G) = \frac{P(T, \neg F_G, G)}{P(G, \neg F_G)} = \frac{P(T, \neg F_G, G)}{P(T, G, \neg F_G) + P(T, G, \neg F_G)}$$

   Now we use the chain rule formula to rewrite the joint entries as CPT entries:

$$P(T|\neg F_G, G) = \frac{P(T)P(\neg F_G|T)P(G|T, \neg F_G)}{P(T)P(\neg F_G|T)P(G|T, \neg F_G) + P(\neg T)P(\neg F_G|\neg T)P(G|\neg T, \neg F_G)}$$

   which of course is the same as the expression arrived at above. Letting $P(T) = p$, $P(F_G|T) = g$, and $P(F_G|\neg T) = h$, we get $P(T|\neg F_G, G) = \frac{p(1-g)(1-x)}{p(1-g)(1-x)+(1-p)(1-h)x}$

Supplementary materials

Book of Why