Alternating Direction Implicit Method for 2D Heat Equation - EXPLAINED

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May 29, 2021

Consider 2D heat equation

$$\left(\frac{\partial}{\partial t} - \nabla^2\right) u(x,y,t) = f(x,y,t) \tag{1}$$

In 2D rectangular area with proper initial and boundary conditions.

1 Peaceman-Rachford Method

Discrete heat equation (1) at $(x_i, y_j, t_{k+1/2})$, given $\{u_{ij}^k\}_{i,j}$, fix j,solving $\{\bar{u}_{ij}\}_{i=0:m,j=1:m-1}$ as following,

$$\frac{\bar{u}_{i,j} - u_{i,j}^k}{\tau/2} - \left(\delta_x^2 \bar{u}_{i,j} + \delta_y^2 u_{i,j}^k\right) = f_{i,j}^{k+1/2} \tag{2}$$

or

$$-r\bar{u}_{i-1,j} + (1+2r)\bar{u}_{i,j} - r\bar{u}_{i+1,j} = ru_{i-1,j}^k + (1-2r)u_{i,j}^k + ru_{i+1,j}^k + \frac{\tau}{2} f_{i,j}^{k+1/2}$$

where $r = \frac{\tau}{2h^2}$. The upper scheme is an implicit scheme w.r.t. x direction.

Then, solve $\{u_{i,j}^{k+1}\}$ as following (fix i),

$$\frac{u_{i,j}^{k+1} - \bar{u}_{i,j}^k}{\tau/2} - \left(\delta_x^2 \bar{u}_{i,j} + \delta_y^2 u_{i,j}^{k+1}\right) = f_{i,j}^{k+1/2} \tag{3}$$

or

$$-ru_{i,j-1}^{k+1} + (1+2r)u_{i,j}^{k+1} - ru_{i,j+1}^{k+1} = r\bar{u}_{i-1,j} + (1-2r)\bar{u}_{i,j} + r\bar{u}_{i+1,j} + \frac{\tau}{2} f_{i,j}^{k+1/2}$$

is implicit scheme w.r.t. y direction.

(2) - (3), we get

$$\bar{u}_{i,j} = u_{i,j}^{k+1/2} - \frac{\tau^2}{4}\delta_y^2\delta_t u_{i,j}^{k+1/2} \tag{4}$$

which is meaningful at $i = 0, m$, so we get the matrix form of PR method,

$$\bar{u}_{0,j} = u_{0,j}^{k+1/2} - \frac{\tau^2}{4}\delta_y^2\delta_t u_{0,j}^{k+1/2}$$

$$\bar{u}_{m,j} = u_{m,j}^{k+1/2} - \frac{\tau^2}{4}\delta_y^2\delta_t u_{m,j}^{k+1/2}$$

$$\begin{bmatrix} 1+2r & -r & \cdots & \cdots & \cdots & \cdots \\ -r & 1+2r & -r & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cdots & \cdots & \cdots & -r & 1+2r & -r \\ \cdots & \cdots & \cdots & \cdots & -r & 1+2r \end{bmatrix} \begin{bmatrix} \bar{u}_{1,j} \\ \bar{u}_{2,j} \\ \vdots \\ \bar{u}_{m-2,j} \\ \bar{u}_{m-1,j} \end{bmatrix}$$

$$= \begin{bmatrix} 1-2r & r & \cdots & \cdots & \cdots & \cdots \\ r & 1-2r & r & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cdots & \cdots & \cdots & r & 1-2r & r \\ \cdots & \cdots & \cdots & \cdots & r & 1-2r \end{bmatrix} \begin{bmatrix} u_{1,j}^k \\ u_{2,j}^k \\ \vdots \\ u_{m-2,j}^k \\ u_{m-1,j}^k \end{bmatrix} + \begin{bmatrix} \frac{\tau}{2} f_{1,j}^{k+1/2} + r u_{0,j}^k \\ \frac{\tau}{2} f_{2,j}^{k+1/2} \\ \vdots \\ \frac{\tau}{2} f_{m-2,j}^{k+1/2} \\ \frac{\tau}{2} f_{m-1,j}^{k+1/2} + r u_{m,j}^k \end{bmatrix}$$

and, following the boundary conditions,

$$u_{i,0}^{k+1} = \alpha(x_i, y_0, t_{k+1})$$

$$u_{i,m}^{k+1} = \alpha(x_i, y_m, t_{k+1})$$

$$\begin{bmatrix} 1+2r & -r & \cdots & \cdots & \cdots & \cdots \\ -r & 1+2r & -r & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cdots & \cdots & \cdots & -r & 1+2r & -r \\ \cdots & \cdots & \cdots & \cdots & -r & 1+2r \end{bmatrix} \begin{bmatrix} u_{i,1}^{k+1} \\ u_{i,2}^{k+1} \\ \vdots \\ u_{i,m-2}^{k+1} \\ u_{i,m-1}^{k+1} \end{bmatrix}$$

$$= \begin{bmatrix} 1-2r & r & \cdots & \cdots & \cdots & \cdots \\ r & 1-2r & r & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cdots & \cdots & \cdots & r & 1-2r & r \\ \cdots & \cdots & \cdots & \cdots & r & 1-2r \end{bmatrix} \begin{bmatrix} \bar{u}_{i,1} \\ \bar{u}_{i,2} \\ \vdots \\ \bar{u}_{i,m-2} \\ \bar{u}_{i,m-1} \end{bmatrix} + \begin{bmatrix} \frac{\tau}{2} f_{1,j}^{k+1/2} + r \bar{u}_{i,0}^k \\ \frac{\tau}{2} f_{2,j}^{k+1/2} \\ \vdots \\ \frac{\tau}{2} f_{m-2,j}^{k+1/2} \\ \frac{\tau}{2} f_{m-1,j}^{k+1/2} + r \bar{u}_{i,m}^k \end{bmatrix}$$

2 D'Yakonov Method

$$\begin{cases} \left(1 - \frac{\tau}{2}\delta_x^2\right) \bar{u}_{ij} = \left(1 + \frac{\tau}{2}\delta_y^2\right) u_{i,j}^k + \frac{\tau}{2} f_{i,j}^{k+1/2}, & i,j = 1 : m-1 \\ \left(1 - \frac{\tau}{2}\delta_y^2\right) u_{i,j}^{k+1} = \left(1 + \frac{\tau}{2}\delta_x^2\right) \bar{u}_{i,j} + \frac{\tau}{2} f_{i,j}^{k+1/2}, & i,j = 1 : m-1 \end{cases} \tag{5}$$

From (5), we can cancel out intermittent variable $\bar{u}_{i,j}$, we get

$$\left(1 - \frac{\tau}{2}\delta_x^2\right)\left(1 - \frac{\tau}{2}\delta_y^2\right) u_{i,j}^{k+1} = \left(1 + \frac{\tau}{2}\delta_x^2\right)\left(1 + \frac{\tau}{2}\delta_y^2\right) u_{i,j}^k + \tau f_{i,j}^{k+1/2}, \quad i,j = 1 : m-1$$

Let $u_{i,j}^* = \left(1 - \frac{\tau}{2}\delta_y^2\right) u_{i,j}^{k+1}$, then

$$\begin{cases} \left(1 - \frac{\tau}{2}\delta_x^2\right) u_{i,j}^* = \left(1 + \frac{\tau}{2}\delta_x^2\right)\left(1 + \frac{\tau}{2}\delta_y^2\right) u_{i,j}^k + \tau f_{i,j}^{k+1/2}, & i,j = 1 : m-1 \\ \left(1 - \frac{\tau}{2}\delta_y^2\right) u_{i,j}^{k+1} = u_{i,j}^*, & i,j = 1 : m-1 \end{cases}$$

We can solve the equation as following precedures.

Firstly, fix j, we calculate the value of intermittent layer $u_{i,j}^*$, boundary condition

$$u_{0,j}^* = \left(1 - \frac{\tau}{2}\delta_y^2\right) u_{0,j}^{k+1}$$

$$u_{m,j}^* = \left(1 - \frac{\tau}{2}\delta_y^2\right) u_{m,j}^{k+1}$$

Then, using

$$\left(1 - \frac{\tau}{2}\delta_x^2\right) u_{i,j}^* = \left(1 + \frac{\tau}{2}\delta_x^2\right)\left(1 + \frac{\tau}{2}\delta_y^2\right) u_{i,j}^k + \tau f_{i,j}^{k+1/2}$$

or more precisely,

$$\begin{aligned} -ru_{i-1,j}^* + (1+2r)u_{i,j}^* - ru_{i+1,j}^* = {} & r^2 u_{i-1,j-1}^k + r(1-2r)u_{i,j-1}^k + r^2 u_{i+1,j-1}^k \\ & + r(1-2r)u_{i-1,j}^k + (1-2r)^2 u_{i,j}^k + r(1-2r)u_{i+1,j}^k \\ & + r^2 u_{i-1,j+1}^k + r(1-2r)u_{i,j+1}^k + r^2 u_{i+1,j+1}^k + \tau f_{i,j}^{k+1/2} \end{aligned}$$

We get $\{u_{i,j}^*\}$. Secondly, fix i, using boundary condition

$$u_{i,0}^{k+1} = \alpha(x_i, y_0, t_{k+1})$$

$$u_{i,m}^{k+1} = \alpha(x_i, y_m, t_{k+1})$$

Solving the tridiagnoal equation

$$\left(1 - \frac{\tau}{2}\delta_y^2\right) u_{i,j}^{k+1} = u_{i,j}^*$$

or

$$-ru_{i,j-1}^{k+1} + (1+2r)u_{i,j}^{k+1} - ru_{i,j+1}^{k+1} = u_{i,j}^{*}$$

Matrix form of D'Yakonov Method

Step 1,

$$\begin{bmatrix} 1+2r & -r & \cdots & \cdots & \cdots & \cdots \\ -r & 1+2r & -r & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cdots & \cdots & \cdots & -r & 1+2r & -r \\ \cdots & \cdots & \cdots & \cdots & -r & 1+2r \end{bmatrix} \begin{bmatrix} u_{1,j}^{*} \\ u_{2,j}^{*} \\ \vdots \\ u_{m-2,j}^{*} \\ u_{m-1,j}^{*} \end{bmatrix}$$

$$= \begin{bmatrix} r(1-2r) & r^2 & \cdots & \cdots & \cdots & \cdots \\ r^2 & r(1-2r) & r^2 & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cdots & \cdots & \cdots & r^2 & r(1-2r) & r^2 \\ \cdots & \cdots & \cdots & \cdots & r^2 & r(1-2r) \end{bmatrix} \begin{bmatrix} u_{1,j-1}^{k} \\ u_{2,j-1}^{k} \\ \vdots \\ u_{m-2,j-1} \\ u_{m-1,j-1} \end{bmatrix}$$

$$+ \begin{bmatrix} (1-2r)^2 & r(1-2r) & \cdots & \cdots & \cdots & \cdots \\ r(1-2r) & (1-2r)^2 & r(1-2r) & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cdots & \cdots & \cdots & r(1-2r) & (1-2r)^2 & r(1-2r) \\ \cdots & \cdots & \cdots & \cdots & r(1-2r) & (1-2r)^2 \end{bmatrix} \begin{bmatrix} u_{1,j}^{k} \\ u_{2,j}^{k} \\ \vdots \\ u_{m-2,j} \\ u_{m-1,j} \end{bmatrix}$$

$$+ \begin{bmatrix} r(1-2r) & r^2 & \cdots & \cdots & \cdots & \cdots \\ r^2 & r(1-2r) & r^2 & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cdots & \cdots & \cdots & r^2 & r(1-2r) & r^2 \\ \cdots & \cdots & \cdots & \cdots & r^2 & r(1-2r) \end{bmatrix} \begin{bmatrix} u_{1,j+1}^{k} \\ u_{2,j+1}^{k} \\ \vdots \\ u_{m-2,j+1} \\ u_{m-1,j+1} \end{bmatrix}$$

$$+ \begin{bmatrix} \tau f_{1,j}^{k+1/2} + ru_{0,j-1}^{k} + (1-2r)u_{0,j}^{k} + ru_{0,j+1}^{k} \\ \tau f_{2,j}^{k+1/2} \\ \vdots \\ \tau f_{m-2,j}^{k+1/2} \\ \tau f_{m-1,j}^{k+1/2} + ru_{m,j-1}^{k} + (1-2r)u_{m,j}^{k} + ru_{m,j+1}^{k} \end{bmatrix}$$

Step 2,

$$\begin{bmatrix} 1+2r & -r & \cdots & \cdots & \cdots & \cdots \\ -r & 1+2r & -r & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cdots & \cdots & \cdots & -r & 1+2r & -r \\ \cdots & \cdots & \cdots & \cdots & -r & 1+2r \end{bmatrix} \begin{bmatrix} u_{i,1}^{k+1} \\ u_{i,2}^{k+1} \\ \vdots \\ u_{i,m-2}^{k+1} \\ u_{i,m-1}^{k+1} \end{bmatrix} = \begin{bmatrix} u_{i,1}^{*} \\ u_{i,2}^{*} \\ \vdots \\ u_{i,m-2}^{*} \\ u_{i,m-1}^{*} \end{bmatrix}$$

Conclusion

The meaning of **Alternating Direction** and **Implicit** are obvious while observing the matrix form and notice the index. Solve the above equations layer by layer (span of time). The matrix form of both schemes are original work. If you need to use or dispense it, please contact me via my personal email.