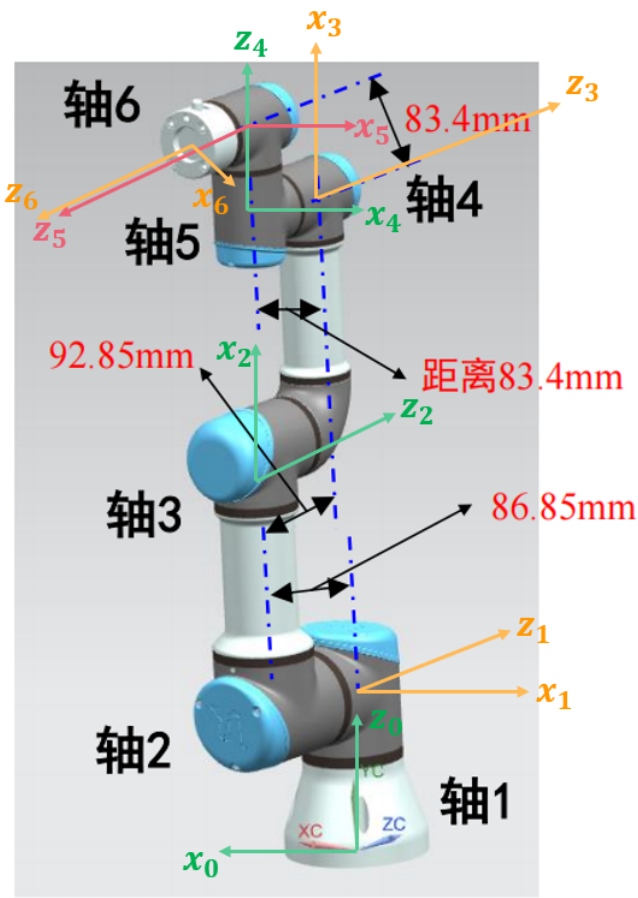
一、D-H参数表、推导$^0T_6$表达式

D-H参数表

针对优傲UR3机械臂建立各个关节坐标系，如下图所示：



给出对应的DH参数表：

	θ	d	a	α
1	θ_1	151.9	0	-90°
2	θ_2	-86.85	243.65	0°
3	θ_3	92.85	213	0°
4	θ_4	-83.4	0	90°
5	θ_5	83.4	0	90°
6	θ_6	300	0	0°

根据书上公式，求解A_n：

$$
\begin{aligned}
A_n &= \mathrm{Rot}(z,\theta_n)\times\mathrm{Trans}(0,0,d_n)\times\mathrm{Trans}(a_n,0,0)\times\mathrm{Rot}(x,\alpha_n)\\
&=\begin{bmatrix} C\theta_n & -S\theta_n & 0 & 0\\ S\theta_n & C\theta_n & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\times\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & d_n\\ 0 & 0 & 0 & 1\end{bmatrix}\times\begin{bmatrix} 1 & 0 & 0 & a_n\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\times\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & C\alpha_n & -S\alpha_n & 0\\ 0 & S\alpha_n & C\alpha_n & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\\
&=\begin{bmatrix} C\theta_n & -S\theta_n C\alpha_n & S\theta_n S\alpha_n & a_n C\theta_n\\ S\theta_n & C\theta_n C\alpha_n & -C\theta_n S\alpha_n & a_n S\theta_n\\ 0 & S\alpha_n & C\alpha_n & d_n\\ 0 & 0 & 0 & 1\end{bmatrix}
\end{aligned}
$$

代入具体参数得到:

$$
A_1=\begin{bmatrix} C_1 & 0 & -S_1 & 0\\ S_1 & 0 & C_1 & 0\\ 0 & -1 & 0 & d_1\\ 0 & 0 & 0 & 1\end{bmatrix}
$$

$$
A_2=\begin{bmatrix} C_2 & -S_2 & 0 & a_2C_2\\ S_2 & C_2 & 0 & a_2S_2\\ 0 & 0 & 1 & d_2\\ 0 & 0 & 0 & 1\end{bmatrix}
$$

$$
A_3=\begin{bmatrix} C_3 & -S_3 & 0 & a_3C_3\\ S_3 & C_3 & 0 & a_3S_3\\ 0 & 0 & 1 & d_3\\ 0 & 0 & 0 & 1\end{bmatrix}
$$

$$
A_4=\begin{bmatrix} C_4 & 0 & S_4 & 0\\ S_4 & 0 & -C_4 & 0\\ 0 & 1 & 0 & d_4\\ 0 & 0 & 0 & 1\end{bmatrix}
$$

$$
A_5=\begin{bmatrix} C_5 & 0 & S_5 & 0\\ S_5 & 0 & -C_5 & 0\\ 0 & 1 & 0 & d_5\\ 0 & 0 & 0 & 1\end{bmatrix}
$$

$$
A_6=\begin{bmatrix} C_6 & -S_6 & 0 & 0\\ S_6 & C_6 & 0 & 0\\ 0 & 0 & 1 & d_6\\ 0 & 0 & 0 & 1\end{bmatrix}
$$

根据书上公式，求解${}^0T_6$:

$$
{}^0T_6=A_1A_2A_3A_4A_5A_6=\begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14}\\ t_{21} & t_{22} & t_{23} & t_{24}\\ t_{31} & t_{32} & t_{33} & t_{34}\\ 0 & 0 & 0 & 1\end{bmatrix}
$$

其中，由于t_{ij}表达式较长，将结果展示如下:

$$
\begin{aligned}
t_{11} &= C_1C_{234}C_5C_6 - S_1S_5C_6 - C_1S_{234}S_6 \\
t_{12} &= -C_1C_{234}C_5S_6 - C_1S_{234}C_6 + S_1S_5S_6 \\
t_{13} &= S_1C_5 - C_1C_{234}S_5 \\
t_{14} &= a_3C_1C_{23} + a_2C_1C_2 - S_1d_2 - S_1d_3 - S_1d_4 + d_5C_1S_{234} + d_6(S_1C_5 - C_1S_5C_{234}) \\
t_{21} &= C_1S_5C_6 + S_1C_{234}C_5C_6 - S_1S_{234}S_6 \\
t_{22} &= -S_1S_{234}C_6 - C_1S_5S_6 + S_1C_{234}C_5S_6 \\
t_{23} &= -C_1C_5 - S_1S_5C_{234} \\
t_{24} &= C_1(d_2 + d_3 + d_4) + S_1(a_2C_2 + a_3C_{23} + S_{234}d_5) - d_6(C_1C_5 + S_1S_5C_{234}) \\
t_{31} &= -C_{234}S_6 - S_{234}C_5C_6 \\
t_{32} &= S_{234}C_5S_6 - C_{234}C_6 \\
t_{33} &= S_{234}S_5 \\
t_{34} &= -a_2S_2 - a_3S_{23} + d_1 + C_{234}d_5 + S_{234}S_5d_6
\end{aligned}
$$

具体推导过程如图所示:

[illegible]

$$
A_n = \begin{pmatrix} C_n & -S_nC_{\alpha n} & S_nS_{\alpha n} & a_nC_1 \\ S_n & C_nC_{\alpha n} & -C_nS_{\alpha n} & a_nS_n \\ 0 & S_{\alpha n} & C_{\alpha n} & d_n \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
A_1 = \begin{pmatrix} C_1 & 0 & -S_1 & 0 \\ S_1 & 0 & C_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}
A_2 = \begin{pmatrix} C_2 & -S_2 & 0 & a_2C_2 \\ S_2 & C_2 & 0 & a_2S_2 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}
A_3 = \begin{pmatrix} C_3 & -S_3 & 0 & a_3C_3 \\ S_3 & C_3 & 0 & a_3S_3 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
A_4 = \begin{pmatrix} C_4 & 0 & S_4 & 0 \\ S_4 & 0 & -C_4 & 0 \\ 0 & 1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}
A_5 = \begin{pmatrix} C_5 & 0 & S_5 & 0 \\ S_5 & 0 & -C_5 & 0 \\ 0 & 1 & 0 & d_5 \\ 0 & 0 & 0 & 1 \end{pmatrix}
A_6 = \begin{pmatrix} C_6 & -S_6 & 0 & 0 \\ S_6 & C_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

T_{11}:

$$
\begin{aligned}
& C_6\Big[C_5\big[C_4(C_1C_2C_3 - C_1S_2S_3) + S_4(-C_1C_2S_3 - C_1S_2C_3)\big] - S_1S_5\Big] + S_6\Big[C_4(-C_1C_2S_3 - C_1S_2C_3) - S_4(C_1C_2C_3 - C_1S_2S_3)\Big] \\
&= C_6\big[C_5[C_4C_1C_{23} + S_4(-C_1S_{23})] - S_1S_5\big] + S_6[-C_4C_1S_{23} - S_4C_1C_{23}] \\
&= C_6\big[C_5(C_1C_{234}) - S_1S_5\big] - S_6C_1S_{234} \\
&= C_1C_{234}C_5C_6 - \cancel{S_1C_1C_5S_5C_6}\ S_1S_5C_6 - S_6C_1S_{234}
\end{aligned}
$$

T_{12}:

$$
\begin{aligned}
& C_6\big[C_4(-C_1C_2S_3 - C_1C_3S_2) - S_4(C_1C_2C_3 - C_1S_2S_3)\big] - S_6\Big[C_5\big[C_4(C_1C_2C_3 - C_1S_2S_3) + S_4(-C_1C_2S_3 - C_1C_3S_2)\big] - S_1S_5\Big] \\
&= C_6[-C_4C_1S_{23} - S_4C_1C_{23}] - S_6\big[C_5[C_4C_1C_{23} + (-S_4C_1S_{23})] - S_1S_5\big] \\
&= -C_6C_1S_{234} - S_6(C_5C_1C_{234} - S_1S_5) \\
&= S_1S_5S_6 - C_1S_{234}C_6 - C_1C_{234}C_5S_6
\end{aligned}
$$

T_{13}:

$$
\begin{aligned}
& C_5S_1 - S_5\big[C_4(C_1C_2C_3 - C_1S_2S_3) + S_4(-C_1C_2S_3 - C_1S_2C_3)\big] \\
&= S_1C_5 - S_5[C_1C_{23}C_4 + -C_1S_{23}S_4] \\
&= S_1C_5 - S_5C_1C_{234}
\end{aligned}
$$

T_{14}:

$$
\begin{aligned}
& C_1C_2C_3a_3 + C_1C_2a_2 - C_1S_2S_3a_3 - S_1d_2 - S_1d_3 - S_1d_4 + d_5\big[-C_4(-C_1C_2S_3 - C_1C_3S_2) + S_4(C_1C_2C_3 - C_1S_2S_3)\big] + d_6\Big[C_5S_1 - S_5\big[C_4(C_1C_2C_3 - C_1S_2S_3) + S_4(-C_1C_2S_3 - C_1S_2C_3)\big]\Big] \\
&= a_3C_1C_{23} + a_2C_1C_2 - S_1d_2 - S_1d_3 - S_1d_4 + d_5[+C_4C_1S_{23} + S_4C_1C_{23}] + d_6\big[S_1C_5 - S_5[C_4C_1C_{23} - S_4S_{23}C_1]\big] \\
&= a_3C_1C_{23} + a_2C_1C_2 - S_1d_2 - S_1d_3 - S_1d_4 + d_5C_1S_{234} + d_6[S_1C_5 - S_5C_1C_{234}]
\end{aligned}
$$

P1

T_{21}:

$$C_6\left[C_1S_5 + C_5\left[C_4(C_2C_3S_1 - S_1S_2S_3) + S_4(C_2S_1S_3 - C_3S_1S_2)\right]\right] + S_6\left[C_4(-C_2S_1S_3 - C_3S_1S_2) - S_4(C_2C_3S_1 - S_1S_2S_3)\right]$$

$$= C_1S_5C_6 + C_6C_5\left[C_4S_1C_{23} - S_4S_1S_{23}\right] + S_6\left[-C_4S_1S_{23} - S_4S_1C_{23}\right]$$

$$= C_1S_5C_6 + S_1C_{234}C_5C_6 - S_1S_{234}S_6$$

T_{22}:

$$C_6\left[C_4(-C_2S_1S_3 - C_3S_1S_2) - S_4(C_2C_3S_1 - S_1S_2S_3)\right] - S_6\left[C_1S_5 + C_5\left[C_4(C_2C_3S_1 - S_1S_2S_3) + S_4(-C_2S_1S_3 - C_3S_1S_2)\right]\right]$$

$$= C_6\left[-C_4S_1S_{23} - S_1C_{23}S_4\right] - S_6\left[C_1S_5 + C_5(C_4S_1C_{23} - S_4S_1S_{23})\right]$$

$$= -S_1S_{23}C_4C_6 - S_1C_{23}S_4C_6 - C_1S_5S_6 + S_1C_{34}C_5S_6$$

$$= -S_1S_{234}C_6 - C_1S_5S_6 + S_1C_{234}C_5S_6$$

T_{23}:

$$-C_1C_5 - S_5\left[C_4(C_2C_3S_1 - S_1S_2S_3) + S_4(-C_2S_1S_3 - C_3S_1S_2)\right]$$

$$= -C_1C_5 - S_5\left[S_1C_{23}C_4 * - S_4S_1S_{23}\right]$$

$$= -C_1C_5 - \cancel{S_1C_{23}C_4S_5}\ S_5S_1C_{234}$$

T_{24}: $C_1d_2 + C_1d_3 + C_1d_4 + C_2C_3S_1 \cdot a_3 + C_2S_1a_2 - S_1S_2S_3a_3 + d_5\left[-C_4(-C_2S_1S_3 - C_3S_1S_2) + S_4(C_2C_3S_1 - S_1S_2S_3)\right]$

$$= C_1(d_2+d_3+d_4) + S_1a_3C_{23} + S_1a_2C_2 \quad + d_6\left[-C_1C_5 - S_5\left[C_4(C_2C_3S_1 - S_1S_2S_3) + S_4(-C_2S_1S_3 - C_3S_1S_2)\right]\right]$$

$$= \cdots\cdots \quad + S_1S_{234}d_5 \quad + d_5\left[+C_4S_1S_{23} + S_4S_1C_{23}\right] + d_6\left[-C_1C_5 - S_5\left[C_4S_1C_{23} - S_4S_1S_{23}\right]\right]$$

$$+ d_6\left[-C_1C_5 - S_5S_1C_{234}\right]$$

$$= C_1(d_2+d_3+d_4) + S_1(a_3C_{23} + a_2C_2) + S_1S_{234}d_5 - C_1C_5d_6 - S_1C_{234}S_5d_6$$

T_{31}

$$C_5C_6\left[C_4(-C_2S_3 - C_3S_2) + S_4(-C_2C_3 + S_2S_3)\right] + S_6\left[C_4(-C_2C_3 + S_2S_3) - S_4(-C_2S_3 - C_3S_2)\right]$$

$$= C_5C_6\left[C_4S_{23} - S_4C_{23}\right] + S_6\left[-C_4C_3 + S_4S_{23}\right]$$

$$= -S_{234}C_5C_6 * - S_6C_{34}$$

$$T_{32} = -C_5S_6\left[C_4(-C_2S_3 - C_3S_2) + S_4(-C_2C_3 + S_2S_3)\right] + C_6\left[C_4(-C_2C_3 + S_2S_3) - S_4(-C_2S_3 - C_3S_2)\right]$$

$$= -C_5S_6\left[-C_4S_{23} - S_4C_{23}\right] + \cancel{C_6[C_4 \; C_{23} +}\ C_6(-C_4C_{23} + S_4S_{23})$$

$$= S_{234}C_5S_6 - C_{234}C_6$$

$$T_{33}:\ -S_5\left[C_4(-C_2S_3 - C_3S_2) + S_4(-C_2C_3 + S_2S_3)\right]$$

$$= +S_5C_4S_{23} + S_4C_{23}S_5 = S_5S_{234}$$

T_{34}:

$$-C_2S_3a_3 - C_3S_2a_3 - S_2a_2 - S_5d_6\left[C_4(-C_2S_3 - C_3S_2) + S_4(-C_2C_3 + S_2S_3)\right] + d_1 + d_5\left[-C_4(-C_2C_3 + S_2S_3) + S_4(-C_2S_3 - C_3S_2)\right]$$

$$= -a_3S_{23} - a_2S_2 - S_5d_6\left[C_4(-S_{23}) - S_4C_{23}\right] + d_1 + d_5\left[C_4C_{23} - S_4S_{23}\right]$$

$$= -a_3S_{23} - a_2S_2 + S_{234}S_5d_6 + d_1 + d_5C_{234}$$

P_3

二、θ_1-θ_6显示表达式

此时，我们已知变换矩阵RHS和参数a_i，d_i。其中，RHS矩阵如下表示：

$$RHS = A_1A_2A_3A_4A_5A_6 = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

为求显式表达式，我尝试了2种方法，第一种即右乘A_6^{-1}，第二种为左乘A_1^{-1}，其思想都是通过"算两次""进行计算。

右乘A_6^{-1}

我们对等式两端求A_6的逆，即可以得到：

$$A_6^{-1} = \begin{bmatrix} C_6 & S_6 & 0 & 0 \\ -S_6 & C_6 & 0 & 0 \\ 0 & 0 & 1 & -d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$RHSA_6^{-1} = A_1 A_2 A_3 A_4 A_5$$

$$RHSA_6^{-1} = \begin{bmatrix} C_6 n_x - S_6 o_x & C_6 o_x + S_6 n_x & a_x & -a_x d_6 + p_x \\ C_6 n_y - S_6 o_y & C_6 o_y + S_6 n_y & a_y & -a_y d_6 + p_y \\ C_6 n_z - S_6 o_z & C_6 o_z + S_6 n_z & a_z & -a_z d_6 + p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

对应地，可以根据求出矩阵${}^0T_5 = A_1 A_2 A_3 A_4 A_5$

我们令${}^0T_5 = A$，即我们可以求得A全部元素：

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

尝试后，算出部分元素，如下图所示：

A_{31}: $C_5 \left[C_4(-C_2S_3 - C_3S_2) + S_4(-C_2C_3 + S_2S_3)\right]$

$= C_5(-S_{23}C_4 - S_4C_{23})$

$= -S_{234}C_5$

第二问

尝试 A_6^{-1} 失败 ---

A_{32}: $C_4(-C_2C_3 + S_2S_3) - S_4(-C_2S_3 - S_2C_3)$

$= -C_4C_{23} + S_4S_{23}$

$= -C_{234}$

A_{33}: $-S_5\left[C_4(-C_2S_3 - C_3S_2) + S_4(-C_2C_3 + S_2S_3)\right]$

$= -S_5(-C_4S_{23} - S_4C_{23})$

$= S_{234}S_5$

A_{34}: $\cancel{C_1d_2 + C_1d_4}$

$-C_2S_3a_3 - C_3S_2a_3 - S_2a_2 + d_1 + d_5\left(-C_4(-C_2C_3 + S_2S_3) + S_4(-C_2S_3 - C_3S_2)\right)$

$= -a_3S_{23} - S_2a_2 + d_1 + d_5(C_{23}C_4 - S_4S_{23})$

$= -a_3S_{23} - S_2a_2 + d_1 + C_{234}d_5$

A_{24} $C_1d_2 + C_1d_3 + C_1d_4 + C_2C_3S_1a_3 + C_2S_1a_2 - S_1S_2S_3a_3 + d_5\left(-C_4(-C_2S_1S_3 - C_3S_1S_2) + S_4(C_2C_3S_1 - S_1S_2S_3)\right)$

$= C_1d_2 + C_1d_3 + C_1d_4 + S_1C_{23}a_3 + S_1C_2a_2 + \underset{\substack{\parallel \\ d_5 S_1S_{234}}}{d_5(+C_4S_1S_{23} + S_4S_1C_{23})}$

A_{23} $-C_1C_5 - S_5\left[C_4(C_2C_3S_1 - S_1S_2S_3) + S_4(-C_2S_1S_3 - C_3S_1S_2)\right]$

$= -C_1C_5 - S_5(C_4S_1C_{23} - S_4S_1S_{23})$

$= -C_1C_5 - S_1C_{234}S_5$

$A_{22} = C_4(-C_2S_1S_3 - C_3S_1S_2) - S_4(C_2C_3S_1 - S_1S_2S_3)$

$= -C_4S_1S_{23} - S_4S_1C_{23}$

$= -S_1S_{234}$

$A_{12} = C_4(-C_1C_2S_3 - C_1S_2C_3) - S_4(C_1C_2C_3 - C_1S_2S_3)$

$= -C_1C_4S_{23} - S_4C_1C_{23}$

$= -C_1S_{234}$

A_{13}: $C_5S_1 - S_5(C_4C_1C_{23} - S_4C_1S_{23})$

$= S_1C_5 - C_1S_5C_{234}$

我发现如果右乘A_6，会为求解带来一个新的未知量，手算无法求解，遂改变方法。

左乘A_1^{-1}

我们对等式两端求A_1的逆，即可以得到：

$$A_1^{-1} = \begin{bmatrix} C_1 & S_1 & 0 & 0 \\ 0 & 0 & -1 & d_1 \\ -S_1 & C_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_1^{-1}RHS = A_2A_3A_4A_5A_6$$

$$A_1^{-1}RHS=\begin{bmatrix} C_1n_x+S_1n_y & C_1o_x+S_1o_y & C_1a_x+S_1a_y & C_1p_x+S_1p_y \\ -n_z & -o_z & -a_z & d_1-p_z \\ C_1n_y-S_1n_x & C_1o_y-S_1o_x & C_1a_y-S_1a_x & C_1p_y-S_1p_x \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

我们令$K=A_2A_3A_4A_5A_6$，我们用k_{ij}代替K矩阵的元素，并直接写出第三行的值。

$$K=\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
k_{11} &= C_{234}C_5C_6+S234S_6 \\
k_{12} &= -C_{234}C_5C_6-S_{234}C_6 \\
k_{13} &= -C_{234}S_5 \\
k_{14} &= a_3C_{23}+a_2C_2-C_{234}S_5d_6+S_{234}d_5 \\
k_{21} &= S_{234}S_6+S_{234}C_5C_6 \\
k_{22} &= C_{234}C_6-S_{234}C_5C_6 \\
k_{23} &= -S_{234}S_5 \\
k_{24} &= a_3S_{23}+a_2S_2-S_{234}S_5d_6-C_{234}d_5 \\
k_{31} &= S_5C_6 \\
k_{32} &= -S_5S_6 \\
k_{33} &= -C_5 \\
k_{34} &= -C_5d_6+d_2+d_3+d_4
\end{aligned}$$

计算公式如下图示：

尝试计算 A_1^{-1}，成功

$$C_6S_5 \quad -S_5S_6 \quad -C_5 \quad -C_5d_6+d_2+d_3+d_4$$

$$k_{11} \quad C_5C_6[C_4(C_2C_3-S_2S_3)+S_4(-C_2S_3-C_3S_2)]+S_6[C_4(-C_2S_3-C_3S_2)-S_4(C_2C_3-S_2S_3)]$$
$$=C_5C_6(C_4C_{23}-S_4S_{23})+S_6(-C_4S_{23}-S_4C_{23})$$
$$=C_5C_6C_{234}+S_6S_{234}$$

$$k_{12} \quad -C_5S_6[C_4(C_2C_3-S_2S_3)+S_4(-C_2S_3-C_3S_2)]+C_6[C_4(-C_2S_3-C_3S_2)-S_4(C_2C_3-S_2S_3)]$$
$$=-C_5S_6(C_4C_{23}-S_4S_{23})+C_6(-C_4S_{23}-S_4C_{23})$$
$$=-C_5S_6C_{234}-C_6S_{234}$$

$$k_{13} \quad -S_5[C_4(C_2C_3-S_2S_3)+S_4(-C_2S_3-C_3S_2)]$$
$$=-S_5(C_{23}C_4-S_4S_{23})$$
$$=-S_5C_{234}$$

$$k_{23}=-S_5(C_4(C_2S_3+C_3S_2)+S_4(C_2C_3-S_2S_3))$$
$$=-S_5(C_4S_{23}+S_4C_{23})$$
$$=-S_5S_{234}$$

$$k_{14}=C_2C_3a_3+C_2a_2-S_2S_3a_3-S_5d_6[C_4(C_2C_3-S_2S_3)+S_4(-C_2S_3-C_3S_2)]+d_5[-C_4(-C_2S_3-C_3S_2)+S_4(C_2C_3-S_2S_3)]$$
$$=a_3C_{23}+C_2a_2-S_5d_6C_{234}+d_5S_{234}$$

$$k_{24}=C_2S_3a_3+C_3S_2a_3+S_2a_2-S_5d_6[C_4(C_2S_3+C_3S_2)+S_4(C_2C_3-S_2S_3)]+d_5[-C_4(C_2C_3-S_2S_3)+S_4(C_2S_3+C_3S_2)]$$
$$=a_3S_{23}+S_2a_2-S_5d_6[S_{23}C_4+S_4C_{23}]+d_5[-C_4C_{23}+S_4S_{23}]$$
$$=a_3S_{23}+a_2S_2-S_5d_6S_{234}-d_5C_{234}$$

$$k_{21} \quad C_5C_6[C_4(C_2S_3+C_3S_2)+S_4(C_2C_3-S_2S_3)]+S_6[C_4(C_2C_3-S_2S_3)-S_4(C_2S_3+C_3S_2)]$$
$$=C_5C_6(C_4S_{23}+S_4C_{23})+S_6(C_{23}C_4-S_4S_{23})$$
$$=C_5C_6S_{234}+S_6C_{234}$$

$$k_{22}:-C_5S_6[C_4(C_2S_3+C_3S_2)+S_4(C_2C_3-S_2S_3)]+C_6[C_4(C_2C_3-S_2S_3)-S_4(C_2S_3+C_3S_2)]$$
$$=-C_5S_6[C_4S_{23}+S_4C_{23}]+C_6[C_4C_{23}-S_4S_{23}]$$
$$=-C_5S_6S_{234}+C_{234}C_6$$

θ_1 的求解

于是我们联立下列等式：

$$\begin{cases} k_{33}=-C_5 & =C_1a_y-S_1a_x \\ k_{34}=-C_5d_6+d_2+d_3+d_4 & =C_1p_y-S_1p_x \end{cases}$$

消去C_5解得：

$$\begin{aligned}\theta_1 &= \arctan\frac{d_6a_y-p_y}{d_6a_x-p_x}+\arcsin\frac{d_2+d_3+d_4}{\sqrt{(d_6a_x-p_x)^2+(d_6a_y-p_y)^2}} \\ \text{或} &= \arctan\frac{d_6a_y-p_y}{d_6a_x-p_x}-\arcsin\frac{d_2+d_3+d_4}{\sqrt{(d_6a_x-p_x))^2+(d_6a_y-p_y)^2}}+\pi\end{aligned}$$

θ_5 的求解

已经求得θ_1的表达式，我们可以根据下式，求出θ_5的表达式：

$$-C_5 = C_1 a_y - S_1 a_x$$

求解：

$$\theta_5 = \pm \arccos(S_1 a_x - C_1 a_y)$$

θ_6的求解

进而，我们由：

$$S_5 C_6 = \sin\left(\pm \arccos(S_1 a_x - C_1 a_y)\right) C_6 = C_1 n_y - S_1 n_x$$

化简得到：

$$C_6 = \frac{C_1 n_y - S_1 n_x}{\sin\left(\pm \arccos(S_1 a_x - C_1 a_y)\right)}$$

求解：

$$\theta_6 = \pm \arccos(\frac{C_1 n_y - S_1 n_x}{\sin\left(\pm \arccos(S_1 a_x - C_1 a_y)\right)})$$

θ_{234}的求解

对于其余位置的元素，都是包含θ_{234}的计算，因此我们先求解θ_{234}，将其视为已知结果，求解剩余角度的表达式。我们通过下述等式：

$$k_{23} = -S_{234} S_5 = -a_z$$

得到：

$$S_{234} = \frac{a_z}{S_5}$$

解得：

$$\begin{aligned} \theta_{234} &= \arcsin(\frac{a_z}{S_5}) \\ \text{或} &= \pi - \arcsin(\frac{a_z}{S_5}) \end{aligned}$$

θ_3的求解

我们联立下述两个等式

$$\begin{cases} k_{14} = a_3 C_{23} + a_2 C_2 - S_5 d_6 C_{234} + d_5 S_{234} = & c_1 p_x + s_1 p_y \\ k_{24} = a_3 S_{23} + a_2 S_2 - S_5 d_6 S_{234} - d_5 C_{234} = & d_1 - p_z \end{cases}$$

化简得到：

$$\begin{cases} a_3 C_{23} + a_2 C_2 = c_1 p_x + s_1 p_y \\ a_3 S_{23} + a_2 S_2 = d_1 - p_z \end{cases}$$

进而我们得到：

$$a_2^2 + a_3^2 + 2 a_2 a_3 C_3 = (c_1 p_x + s_1 p_y)^2 + (d_1 - p_z)^2$$

$$C_3 = \frac{(c_1 p_x + s_1 p_y)^2 + (d_1 - p_z)^2 - a_2^2 - a_3^2}{2 a_2 a_3}$$

解得：

$$\theta_3 = \pm \arccos(\frac{(c_1 p_x + s_1 p_y)^2 + (d_1 - p_z)^2 - a_2^2 - a_3^2}{2a_2 a_3})$$

θ_2的求解

根据上述化简后的公式：

$$a_3 C_{23} + a_2 C_2 = c_1 p_x + s_1 p_y$$

可以得到：

$$a_3 C_2 C_3 - a_3 S_2 S_3 + a_2 C_2 = c_1 p_x + s_1 p_y$$

进而有：

$$(C_3 a_3 + a_2) C_2 - S_3 a_3 S_2 = c_1 p_x + s_1 p_y$$

解得：

$$\begin{aligned} \theta_2 &= arctan\frac{S_3 a_3}{C_3 a_3 + a_2} + \arcsin\frac{c_1 p_x + s_1 p_y}{\sqrt{(d_6 a_x - p_x)^2 + (d_6 a_y - p_y)^2}} \\ \text{或} &= arctan\frac{S_3 a_3}{C_3 a_3 + a_2} - \arcsin\frac{c_1 p_x + s_1 p_y}{\sqrt{(d_6 a_x - p_x)^2 + (d_6 a_y - p_y)^2}} + \pi \end{aligned}$$

θ_4的求解

我们通过已知量得解，可以直接得到θ_4的表达式

$$\theta_4 = \theta_{234} - \theta_2 - \theta_3$$

于是，$\theta_1 - \theta_6$的显示表达式全部求出，我整理如下：

$$\begin{aligned} \theta_1 &= \arctan\frac{d_6 a_y - p_y}{d_6 a_x - p_x} + \arcsin\frac{d_2 + d_3 + d_4}{\sqrt{(d_6 a_x - p_x)^2 + (d_6 a_y - p_y)^2}} \\ \text{或} &= \arctan\frac{d_6 a_y - p_y}{d_6 a_x - p_x} - \arcsin\frac{d_2 + d_3 + d_4}{\sqrt{(d_6 a_x - p_x))^2 + (d_6 a_y - p_y)^2}} + \pi \end{aligned}$$

$$\begin{aligned} \theta_2 &= arctan\frac{S_3 a_3}{C_3 a_3 + a_2} + \arcsin\frac{c_1 p_x + s_1 p_y}{\sqrt{(d_6 a_x - p_x)^2 + (d_6 a_y - p_y)^2}} \\ \text{或} &= arctan\frac{S_3 a_3}{C_3 a_3 + a_2} - \arcsin\frac{c_1 p_x + s_1 p_y}{\sqrt{(d_6 a_x - p_x)^2 + (d_6 a_y - p_y)^2}} + \pi \end{aligned}$$

$$\theta_3 = \pm \arccos(\frac{(c_1 p_x + s_1 p_y)^2 + (d_1 - p_z)^2 - a_2^2 - a_3^2}{2a_2 a_3})$$

$$\theta_4 = \theta_{234} - \theta_2 - \theta_3$$

$$\theta_5 = \pm \arccos(S_1 a_x - C_1 a_y)$$

$$\theta_6 = \pm \arccos(\frac{C_1 n_y - S_1 n_x}{\sin(\pm \arccos(S_1 a_x - C_1 a_y))})$$