Polar codes for symmetric binary-input memoryless channels

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Set-up

N independent copies of a symmetric binary-input discrete memoryless channel (W), eg. BEC

Motivation/Goal

Channel polarization is a method of constructing code sequences that **achieve symmetric capacity**:

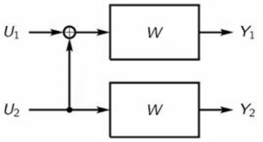
$$I(W) \triangleq \sum_{y\in\mathcal{Y}}\sum_{x\in\mathcal{X}} \frac{1}{2} W(y|x) \log \frac{W(y|x)}{\frac{1}{2}W(y|0) + \frac{1}{2}W(y|1)}$$

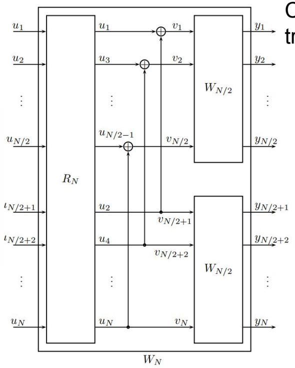
From the N identical B-DMC's each with capacity I(W), construct a set of N "polarized" binary-input channels, such that:

I(W) of the channels have capacity 1
1 - I(W) of the channels have capacity 0
(Overall capacity conserved)

Method: Encoding

2-Channel case
Given 2 BEC(p) channels W, construct:
W- = BEC(2p - p^2)
W+ = BEC(p^2)





Method: Encoding

Recursively apply 2-channel transformation, while reverse-shuffling inputs.

Model encoding as matrix transformation G_N:

$$G_N = B_N F^{\otimes n}$$

$$B_N = R_N(I_2 \otimes B_{N/2}).$$

R_N: reverse shuffle permutation
F = [[1, 0]
[1, 1]]

Method: Decoding

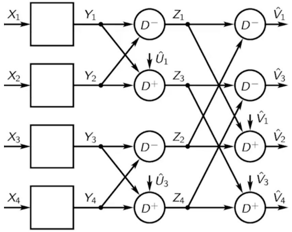
Decoding units:
D- for each W- channel
D+ for each W+ channel

$$D^-(y_1, y_2) = \begin{cases} y_1 \oplus y_2 & y_1 \neq ?, y_2 \neq ? \\ ? & \text{else} \end{cases}$$

$$D^+(z_1, z_2, v_1) = \begin{cases} z_2 & z_2 \neq ? \\ z_1 \oplus v_1 & z_1 \neq ? \\ ? & \text{else} \end{cases}$$

Iteratively produce $U_1, \ldots, U_N$, where $Y_1, \ldots, Y_N, U_1, \ldots U_{i-1}$ is used to decode U_i

Can also be model as a matrix transformation.

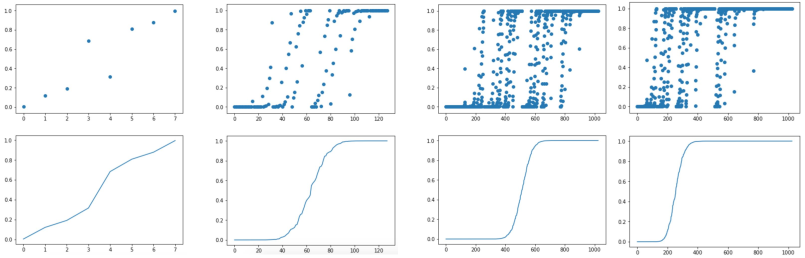


Our Implementation

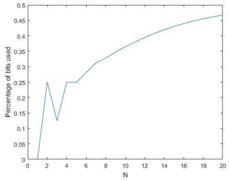
Implemented encoding, channel capacity identification for BEC, and decoding*
Ran convergence experiments with existing implementation

Variations

Rather than G_N any matrix such that no column permutations are upper triangular can polarize channels. The main difference is rate of error probability decay.



Capacity of polarized channels, starting with BEC P(erasure)=0.5 (columns 1, 2, 3) and P(erasure)=0.25 (4). As N, the number of channels grows, the portion of perfect channels approaches the channel capacity I(W).



Fraction of near-perfect channels approaches I(W) = 0.5 (for BEC(0.5)) as number of channels increases, where X axis N-value denotes 2^N channels
Note very slow convergence.

Applications

Computationally efficient capacity-achieving codes: encoding and decoding can be implemented in O(n log n). Unfortunately, need very large number of channels to actually reach capacity I(W) due to slow convergence.

References

Arikan, Erdal. "Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels." *IEEE Transactions on Information Theory* 55, no. 7 (2009): 3051-3073.

Korada, Satish Babu, Eren Sasoglu, and Rüdiger Urbanke. "Polar codes: Characterization of exponent, bounds, and constructions." *IEEE Transactions on Information Theory* 56, no. 12 (2010): 6253-6264.

https://www.youtube.com/watch?v=VhyoZSB9g0w
http://pfister.ee.duke.edu/courses/ecen655/polar.pdf