Nonlinear Regression Functions

Introduction to Econometrics,Fall 2018

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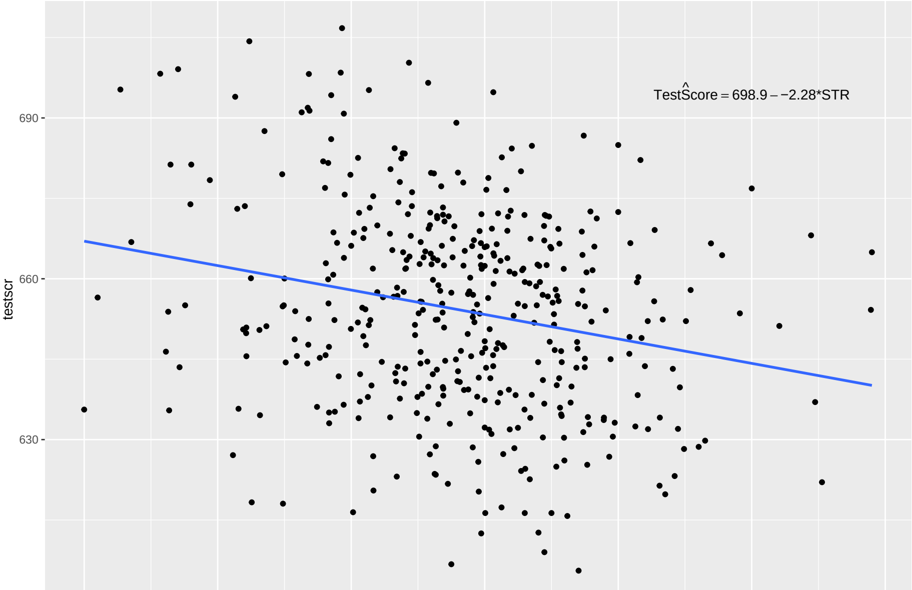
10/25/2018

Nonlinear Regression Functions:

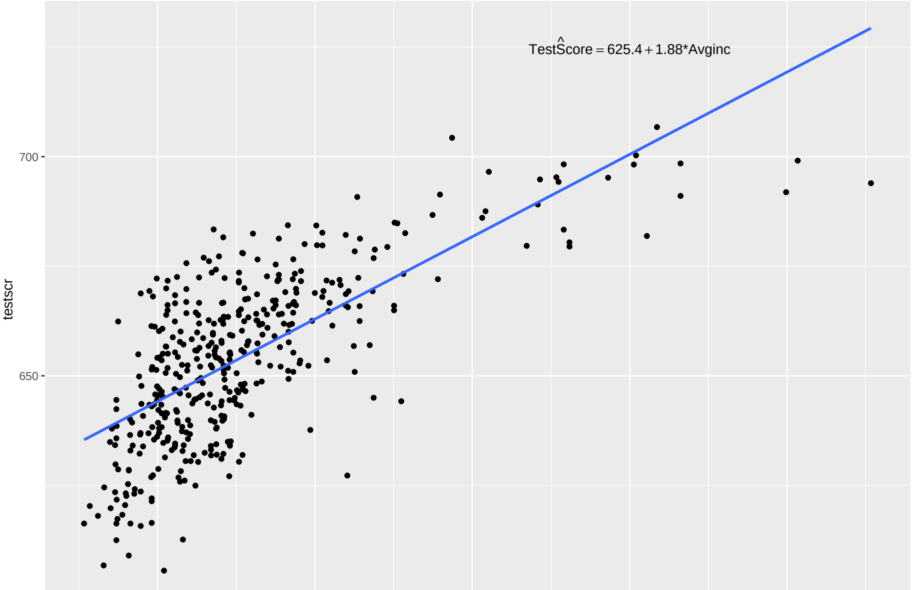
Introduction

- Everything so far has been linear in the X's
- But the linear approximation is not always a good one
- The multiple regression framework can be extended to handle regression functions that are nonlinear in one or more X.

The TestScore – STR relation looks linear (maybe)



But the TestScore – Income relation looks nonlinear



Nonlinear Regression Regression Functions – General Ideas (SW Section 8.1)

$$Y_i = \beta_0 + \beta_1 X_{1,i} + ... + \beta_k X_{k,i} + u_i$$

- The effect of a change in X_j by 1 is constant and equals βj :
- If a relation between Y and X is nonlinear:
    - The effect on Y of a change in X depends on the value of X – that is, the marginal effect of X is not constant.
    - A linear regression is misspecified – the functional form is wrong
    - The estimator of the effect on Y of X is biased(a special case of OVB)
    - The solution to this is to estimate a regression function that is nonlinear in X.

What are nonlinear regression functions: 2 Types

- There are 2 types of *nonlinear* regression models
  - Regression model that is a nonlinear function of the independent variables, $X_{1,i}, X_{2,i}, ..., X_{k,i}$ which is another version of multiple regression model and can be estimated by OLS.
  - Regression model that is a nonlinear function of the unknown coefficients, which can't be estimated by OLS, requires different estimation method.

OLS Assumptions Still Hold

General formula for a nonlinear population regression model:

$$Y_i = f(X_{1,i}, X_{2,i}, ..., X_{k,i}) + u_i$$

Assumptions:

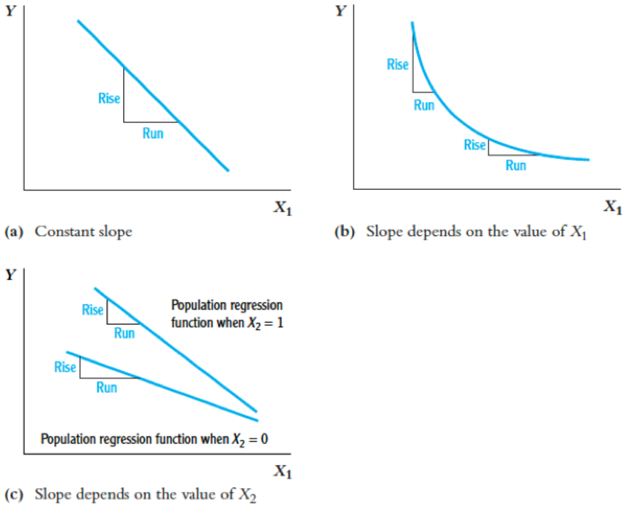
1. $E[u_i|X_{1,i}, X_{2,i}, ..., X_{k,i}] = 0$ implies that f is the conditional expectation of Y given the X's.
2. $(X_{1,i}, X_{2,i}, ..., X_{k,i})$ are i.i.d.
3. Large outliers are rare.
4. No perfect multicollinearity.

Two Cases:

- Two cases:
  - The effect of change in X1 on Y depends on X1
  - for example: the effect of a change in class size is bigger when initial class size is small
  - The effect of change in X1 on Y depends on another variable X2
  - For example: the effect of class size depends on the percentage of disadvantaged pupils in the class
- We start with case 1 using a regression model with only 1 independent variable

Different Slops

FIGURE 8.1 Population Regression Functions with Different Slopes



The Effect on Y of a Change in X in a Nonlinear Specifications

The Expected Change on Y of a Change in X_1 in the Nonlinear Regression Model (8.3)

KEY CONCEPT 8.1

The expected change in Y, ΔY, associated with the change in X_1, ΔX_1, holding $X_2, \ldots, X_k$ constant, is the difference between the value of the population regression function before and after changing X_1, holding $X_2, \ldots, X_k$ constant. That is, the expected change in Y is the difference:

$$\Delta Y = f(X_1 + \Delta X_1, X_2, \ldots, X_k) - f(X_1, X_2, \ldots, X_k). \qquad (8.4)$$

The estimator of this unknown population difference is the difference between the predicted values for these two cases. Let $\hat{f}(X_1, X_2, \ldots, X_k)$ be the predicted value of Y based on the estimator $\hat{f}$ of the population regression function. Then the predicted change in Y is

$$\Delta \hat{Y} = \hat{f}(X_1 + \Delta X_1, X_2, \ldots, X_k) - \hat{f}(X_1, X_2, \ldots, X_k). \qquad (8.5)$$

A General Approach to Modeling Nonlinearities Using Multiple Regression

- Identify a possible nonlinear relationship.
- Specify a nonlinear function and estimate its parameters by OLS.
- Determine whether the nonlinear model improves upon a linear model.
- Plot the estimated nonlinear regression function.
- Estimate the effect on Y of a change in X.

Two complementary approaches:

1. Polynomials in X
   - The population regression function is approximated by a quadratic, cubic, or higher-degree polynomial.
2. Logarithmic transformations
   - Y and/or X is transformed by taking its logarithm
   - this gives a "percentages" interpretation that makes sense in many applications

Polynomials in X

- Approximate the population regression function by a polynomial:

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X^2 ... + \beta_r X_i^r + u_i$$

- This is just the linear multiple regression model – except that the regressors are powers of X!
- Estimation, hypothesis testing, etc. proceeds as in the multiple regression model using OLS
- The coefficients are difficult to interpret, but the regression function itself is interpretable

Testing the null hypothesis that the population regression function is linear

$$H_0 : \beta_2 = 0, \beta_3 = 0, ..., \beta_r = 0 \; and \; H_1 : at \; least \; one \; \beta_j \neq 0$$

- it can be tested using the F-statistic

Which degree polynomial should I use?

- how many powers of X should be included in a polynomial regression? The answer balances a trade-off between flexibility and statistical precision.
- In many applications involving economic data, the nonlinear functions are smooth, that is, they do not have sharp jumps, or "spikes."
- If so, then it is appropriate to choose a small maximum degree for the polynomial, such as 2, 3, or 4.

Example: the TestScore-Income relation

- Quadratic specification:

$$TestScore_i = \beta_0 + \beta_1 Income_i + \beta_2 (Income_i)^2 + u_i$$

- Cubic specification:

$$TestScore_i = \beta_0 + \beta_1 Income_i + \beta_2 (Income_i)^2 + \beta_3 (Income_i)^3 + u_i$$

Estimation of the quadratic specification in R

Estimation of the quadratic specification in R

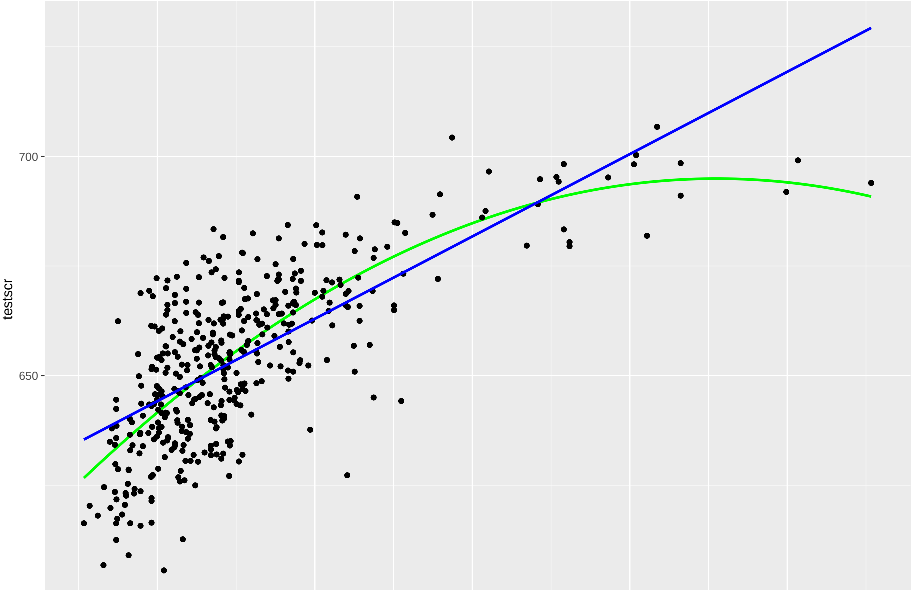
```
##
## Call:
##    felm(formula = testscr ~ avginc + I(avginc^2), data = ca
##
## Residuals:
##     Min       1Q   Median       3Q      Max
## -44.416  -9.048   0.440   8.348  31.639
##
## Coefficients:
##              Estimate Robust s.e t value Pr(>|t|)
## (Intercept) 607.30174    2.90175 209.288   <2e-16 ***
## avginc        3.85100    0.26809  14.364   <2e-16 ***
## I(avginc^2)  -0.04231    0.00478  -8.851   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 '
##
```

Interpreting the estimated regression function

- The OLS regression yields

$$\widehat{TestScore} = \underset{(2.9)}{607.3} + \underset{(0.27)}{3.85} Income - \underset{(0.0048)}{0.0423} (Income)^2$$

Linear and Quadratic Regression in figure



Quadratic vs Linear

- Is the quadratic model better than the linear model?
- We can test the null hypothesis that the regression function is linear against the alternative hypothesis that it is quadratic:

$$H_0 : \beta_2 = 0 \, and \, H_1 : \beta_2 \neq 0$$

- the t-statistic

$$t = \frac{(\hat{\beta}_2 - 0)}{SE(\hat{\beta}_2)} = \frac{-0.0423}{0.0048} = -8.81$$

- Since $8.81 > 2.58$ we reject the null hypothesis (the linear model) at a 1% significance level.

Interpreting the estimated regression function

- Predict Change in TestScore for a change in income
- from \$10,000 per capita to \$11,000 per capita:

$$\begin{aligned}\Delta TestScore &= 607.3 + 3.85 \times 11 - 0.0423 \times (11)^2 \\ &\quad - [607.3 + 3.85 \times 10 - 0.0423 \times (10)^2] \\ &= 2.96\end{aligned}$$

- from \$40,000 per capita to \$41,000 per capita:

$$\begin{aligned}\Delta TestScore &= 607.3 + 3.85 \times 41 - 0.0423 \times (41)^2 \\ &\quad - [607.3 + 3.85 \times 40 - 0.0423 \times (40)^2] \\ &= 0.42\end{aligned}$$

Logarithms

Logarithmic functions of Y and/or X

- Another way to specify a nonlinear regression model is to use the natural logarithm of Y and/or X.
- Ln(X) = the natural logarithm of X
- Logarithmic transforms permit modeling relations in "percentage" terms (like elasticities), rather than linearly.

Review of the Logarithmic functions

$$
\begin{aligned}
ln(1/x) &= -ln(x) \\
ln(ax) &= ln(a) + ln(x) \\
ln(x/a) &= ln(x) - ln(a) \\
ln(x^a) &= aln(x)
\end{aligned}
$$

Logarithms and percentages

- Because

$$ln(x + \Delta x) - ln(x) = ln\left(\frac{x + \Delta x}{x}\right)$$
$$\cong \frac{\Delta x}{x} \ (when \frac{\Delta x}{x} is \ small)$$

- for example

$$ln(1 + 0.01) = ln(101) - ln(100) = 0.00995 \cong 0.01$$

The three log regression specifications:

Case	Population regression function
I.linear-log	$Y_i = \beta_0 + \beta_1 ln(X_i) + u_i$
II.log-linear	$ln(Y_i) = \beta_0 + \beta_1 X_i + u_i$
III.log-log	$ln(Y_i) = \beta_0 + \beta_1 ln(X_i) + u_i$

- The interpretation of the slope coefficient differs in each case.
- The interpretation is found by applying the general "before and after" rule: "figure out the change in Y for a given change in X."

I. Linear-log population regression function

- the regression model is

$$Y_i = \beta_0 + \beta_1 ln(X_i) + u_i$$

- Change X:

$$\begin{aligned}\Delta Y &= [\beta_0 + \beta_1 ln(X + \Delta X)] - [\beta_0 + \beta_1 ln(X)] \\ &= \beta_1 [ln(X + \Delta X) - ln(X)] \\ &\cong \beta_1 \frac{\Delta X}{X}\end{aligned}$$

- Now $100\frac{\Delta X}{X} = percentage\ change\ in\ X$,so a 1% increase in X (multiplying X by 1.01) is associated with a $0.01\beta_1$ change in Y.

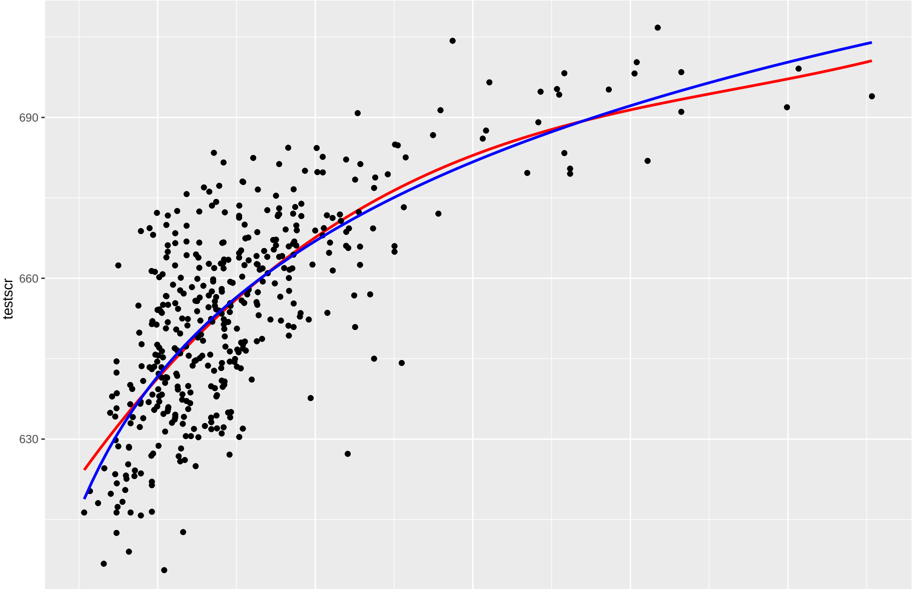
Example:the TestScore – log(Income) relation

- The OLS regression of ln(Income) on Testscore yields

$$\widehat{TestScore} = \underset{(3.8)}{557.8} + \underset{(1.4)}{36.42} \times ln(Income)$$

- so a 1% increase in Income is associated with an increase in TestScore of 0.36 points on the test.

Test scores: linear-log and cubic regression functions



Case II. Log-linear population regression function

- the regression model is

$$ln(Y_i) = \beta_0 + \beta_1 X_i + u_i$$

- Change X:

$$ln(\Delta Y + Y) - ln(Y) = [\beta_0 + \beta_1(X + \Delta X)] - [\beta_0 + \beta_1 X]$$
$$ln(1 + \frac{\Delta Y}{Y}) = \beta_1 \Delta X$$

- then

$$\frac{\Delta Y}{Y} \cong \beta_1 \Delta X$$

- Now $100\frac{\Delta Y}{Y} = percentage\ change\ in\ Y$,so a change in X by one unit is associated with a β_1% change in Y.

Case III. Log-linear population regression function

- the regression model is

$$ln(Y_i) = \beta_0 + \beta_1 ln(X_i) + u_i$$

- Change X:

$$ln(\Delta Y + Y) - ln(Y) = [\beta_0 + \beta_1 ln(X + \Delta X)] - [\beta_0 + \beta_1 ln(X)]$$
$$ln(1 + \frac{\Delta Y}{Y}) = ln(1 + \frac{\Delta X}{X})$$
$$\frac{\Delta Y}{Y} \cong \beta_1 \frac{\Delta X}{X}$$

- Now $100\frac{\Delta Y}{Y} = percentage\ change\ in\ Y$ and $100\frac{\Delta X}{X} = percentage\ change\ in\ X$
- so a 1% change in X by one unit is associated with a β_1% change in Y,thus $\beta_1$has the interpretation of an **elasticity**.

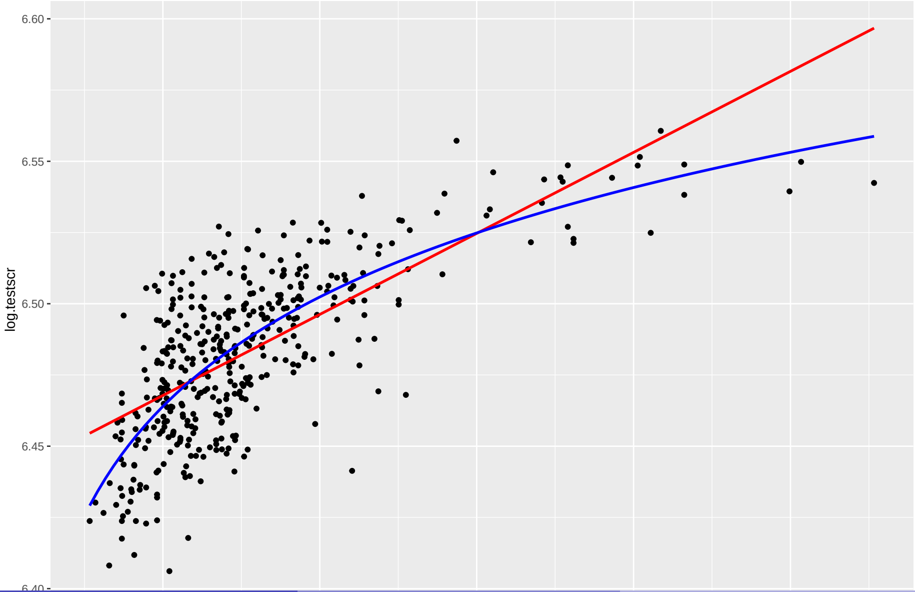
Test scores and income: log-log specifications

```
##
## t test of coefficients:
##
##              Estimate Std. Error  t value  Pr(>|t|)
## (Intercept) 6.3363494  0.0059105 1072.056 < 2.2e-16 ***
## loginc      0.0554190  0.0021395   25.903 < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 '
```

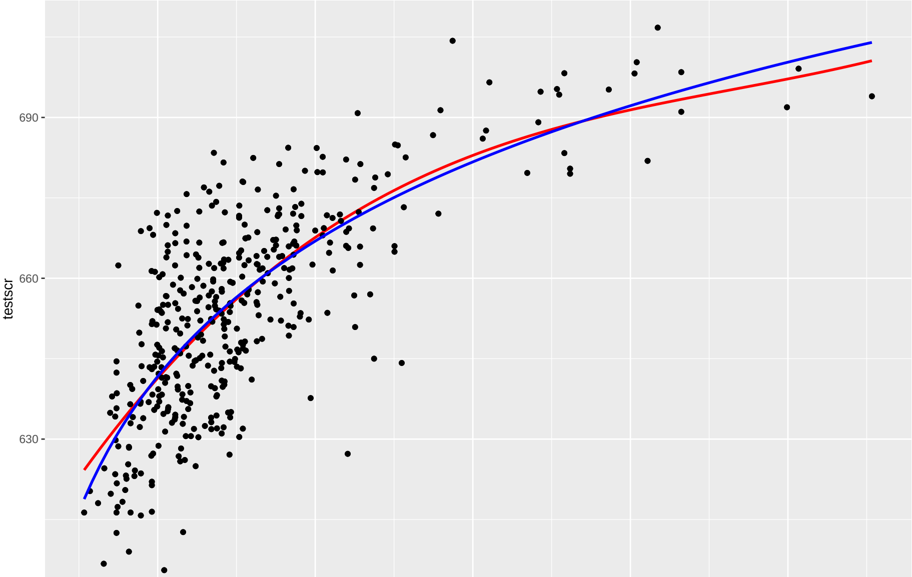
$$
\begin{aligned}
ln(\widehat{TestScore}) =& 6.336 + 0.055 \times ln(Income) \\
& \quad (0.006) \quad (0.002)
\end{aligned}
$$

- An 1% increase in Income is associated with an increase of .0554% in TestScore.

Test scores: The log-linear and log-log specifications:



linear-log and cubic regression functions



Choice of specification should be guided

- by Economic logic or theories(which interpretation makes the most sense in your application?),
- formal tests(seldom use in reality)
- and plotting predicted values

Summary

- We already have a very powerful tool for detecting misspecified functional form: the F test for joint exclusion restrictions.
- We can add quadratic terms of any significant variables to a model and to perform a joint test of significance. If the additional quadratics are significant, they can be added to the model.
- It can be difficult to pinpoint the precise reason for functional form misspecification.
- Fortunately, using logarithms of certain variables and adding quadratic functions are sufficient for detecting many important nonlinear relationships in economics.

Interactions Between Independent Variables

Interactions Between Two Binary Variables

- Assume we would like to study the earnings of worker in the labor market
- The population linear regression of Y_i is

$$Y_i = \beta_0 + \beta_1 D_{1i} + \beta_1 D_{2i} + u_i$$

- the Dependent Variable: **log earnings**(Y_i,where $Y_i = ln(Earnings)$)
- Independent Variables: two binary variables
  - $D_1 i = 1$ if the person graduate from college
  - $D_2 i = 1$ if the worker's gender is female
- So β_1 is the effect on log earnings of having a college degree, **holding gender constant**, and β_2 is the effect of being female, **holding schooling constant**.

Interactions Between Two Binary Variables

- The effect of having a college degree in this specification, holding constant gender, is the **same** for men and women. No reason that this must be so.
- the effect on Y_i of D_{1i}, holding D_{2i} constant, could depend on the value of D_{2i}
- there could be an interaction between having a college degree and gender so that the value in the job market of a degree is different for men and women.
- The new regression model of Y_i is

$$Y_i = \beta_0 + \beta_1 D_{1i} + \beta_2 D_{2i} + \beta_3 (D_{1i} \times D_{2i}) + u_i$$

- The new regressor, the product $D_{1i} \times D_{2i}$, is called an **interaction term** or an interacted regressor,

Interactions Between Two Binary Variables:

- The regression model of Y_i is

$$Y_i = \beta_0 + \beta_1 D_{1i} + \beta_2 D_{2i} + \beta_3 (D_{1i} \times D_{2i}) + u_i$$

- Then the conditional expectation of Yi for $D_{1i} = 0$, given a value of D_{2i}

$$E(Y_i | D_{1i} = 0, D_{2i} = d_2) = \beta_0 + \beta_1 \times 0 + \beta_2 d_2 + \beta_3 (0 \times d_2) = \beta_0 + \beta_2 d_2$$

- Then the conditional expectation of Yi for $D_{1i} = 1$, given a value of D_{2i}

$$E(Y_i | D_{1i} = 1, D_{2i} = d_2) = \beta_0 + \beta_1 \times 1 + \beta_2 d_2 + \beta_3 (1 \times d_2) = \beta_0 + \beta_1 + \beta_2 d_2$$

Interactions Between Two Binary Variables:

- The effect of this change is the difference of expected values,which is

$$E(Y_i|D_{1i}=1,D_{2i}=d_2)-E(Y_i|D_{1i}=0,D_{2i}=d_2)=\beta_1+\beta_3 d_2$$

- In the binary variable interaction specification, the effect of acquiring a college degree (a unit change in D_{1i}) depends on the person's gender.
- If the person is male,thus $D_{2i}=d_2=0$,then the effect is β_1
- If the person is female,thus $D_{2i}=d_2=1$,then the effect is $\beta_1+\beta_3$
- So the coefficient β_3 is the difference in the effect of acquiring a college degree for women versus men.

Application to the STR and the percentage of English learners

- Let $HiSTR_i$ be a binary variable for STR
  - $HiSTR_i = 1$ if the $STR > 20$
  - $HiSTR_i = 0$ otherwise
- Let $HiEL_i$ be a binary variable for English learner
  - $HiEL_i = 1$ if the $el_pct > 10 percent$
  - $HiEL_i = 0$ otherwise

Application to the STR and the percentage of English learners

$$\underset{}{ln(\widehat{TestScore})} = \underset{(1.4)}{664.1} - \underset{(1.9)}{1.9}HiSTR - \underset{(2.3)}{18.2}HiEL - \underset{(3.1)}{3.5}(HiSTR \times HiEL)$$

Interactions Between a Continuous and a Binary Variable

- one binary variable, whether the worker has a college degree
- the individual's years of work experience (X_i),
- the first population model is

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 D_i + u_i$$

Interactions Between a Continuous and a Binary Variable

- the second population model is

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 (D_i \times X_i) + u_i$$

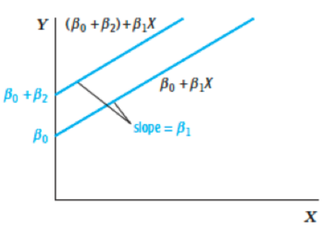
Interactions Between a Continuous and a Binary Variable

- the third population model is

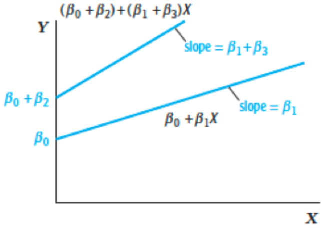
$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 D_i + \beta_3 (D_i \times X_i) + u_i$$

Interactions Between a Continuous and a Binary Variable

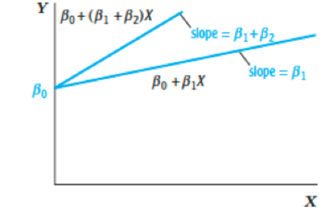
FIGURE 8.8 Regression Functions Using Binary and Continuous Variables



(a) Different intercepts, same slope



(b) Different intercepts, different slopes



(c) Same intercept, different slopes

Interactions of binary variables and continuous variables can produce three different population regression functions:

Interactions Between Two Continuous Variables

- Now suppose that both independent variables (X_{1i} and X_{2i}) are continuous.
- X_{1i} is his or her years of work experience
- X_{2i} is the number of years he or she went to school.
- there might be an interaction between these two variables so that the effect on wages of an additional year of experience depends on the number of years of education.
- the population regression model

$$Y_i = \beta_0 + \beta_1 X_i + \beta_1 X_i + \beta_2 (X_{1i} \times X_{2i}) + u_i$$

Interactions Between Two Continuous Variables

- Thus the effect on Y of a change in X_1, holding X_2 constant, is

$$\frac{\Delta Y}{\Delta X_1} = \beta_1 + \beta_3 X_2$$

- A similar calculation shows that the effect on Y of a change ΔX_1 in X_2, holding X_1 constant, is

$$\frac{\Delta Y}{\Delta X_2} = \beta_2 + \beta_3 X_1$$

- That is, if X_1 changes by ΔX_1 and X_2 changes by ΔX_2, then the expected change in Y

$$\Delta Y = (\beta_1 + \beta_3 X_2)\Delta X_1 + (\beta_2 + \beta_3 X_1)\Delta X_2 + \beta_3 \Delta X_1 \Delta X_2$$

Application to the STR and the percentage of English learners

- The estimated interaction regression

$$\begin{aligned} ln(\widehat{TestScore}) =&686.3 - 1.12STR - 0.67PctEL + 0.0012(STR \times Pct \\ &(11.8) \quad (0.059) \quad (0.037) \qquad\qquad (0.019) \end{aligned}$$

- when the percentage of English learners is at the median($PctEL = 8.85$), the slope of the line relating test scores and the STR is

$$\frac{\Delta Y}{\Delta X_1} = \beta_1 + \beta_3 X_2 = -1.12 + 0.0012 \times 8.85 = -1.11$$

- when the percentage of English learners is at the 75th percentile($PctEL = 23.0$), the slope of the line relating test scores and the STR is

$$\frac{\Delta Y}{\Delta X_1} = \beta_1 + \beta_3 X_2 = -1.12 + 0.0012 \times 23.0 = -1.09$$

Application to the STR and the percentage of English learners

TABLE 8.3 Nonlinear Regression Models of Test Scores

Dependent variable: average test score in district; 420 observations.

Regressor	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Student–teacher ratio (*STR*)	−1.00** (0.27)	−0.73** (0.26)	−0.97 (0.59)	−0.53 (0.34)	64.33** (24.86)	83.70** (28.50)	65.29** (25.26)
STR^2					−3.42** (1.25)	−4.38** (1.44)	−3.47** (1.27)
STR^3					0.059** (0.021)	0.075** (0.024)	0.060** (0.021)
% English learners	−0.122** (0.033)	−0.176** (0.034)					−0.166** (0.034)
% English learners ≥ 10%? (Binary, *HiEL*)			5.64 (19.51)	5.50 (9.80)	−5.47** (1.03)	816.1* (327.7)	
$HiEL \times STR$			−1.28 (0.97)	−0.58 (0.50)		−123.3* (50.2)	
$HiEL \times STR^2$						6.12* (2.54)	
$HiEL \times STR^3$						−0.101* (0.043)	
% Eligible for subsidized lunch	−0.547** (0.024)	−0.398** (0.033)		−0.411** (0.029)	−0.420** (0.029)	−0.418** (0.029)	−0.402** (0.033)
Average district income (logarithm)		11.57** (1.81)		12.12** (1.80)	11.75** (1.78)	11.80** (1.78)	11.51** (1.81)
Intercept	700.2** (5.6)	658.6** (8.6)	682.2** (11.9)	653.6** (9.9)	252.0 (163.6)	122.3 (185.5)	244.8 (165.7)
***F*-Statistics and *p*-Values on Joint Hypotheses**							
(a) All *STR* variables and interactions = 0			5.64 (0.004)	5.92 (0.003)	6.31 (< 0.001)	4.96 (< 0.001)	5.91 (0.001)
(b) $STR^2, STR^3 = 0$					6.17 (< 0.001)	5.81 (0.003)	5.96 (0.003)