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Combinatorial Geometry: Extended Abstract on Graph Theory

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ABSTRACT This extended abstract introduces the origin of Graph theory using two theorems raised by Euler in 18th century. We use modern mathematical method to prove these two theorems to get an insight of combinatorial geometry including two applications.

1 INTRODUCTION

The study field of Combinatorial Geometry coincides with the geometry and combinatorics. This field includes graph theory, number theory, and can be applied to computer sciences as it contains various algorithms. There are also lots of mathematical applications are unsolved, or honestly, will never be solved by hand.

2 FROM SEVEN BRIDGES TO WHAT FORMS A GRAPH?

Definition 2.1. *In mathematics, a graph is not just a picture. A Graph G is defined as vertices V connected by edges E, denoted by $G = \{V, E\}$. The term "degree" is defined as how many edges connected to the vertex. Without loss of generality, when $|V| > 1$, a* ***connected graph*** *is defined as no one vertex is left without connecting an edge.*

Then Euler proved that no one can walk through the seven bridges in old Prussia that connected Kneiphof and Lomse without passing through a bridge more than once as an island connected with odd number of bridges. This is now called Euler's theorem.

Theorem 2.2. *(Euler's Theorem) A connected graph is said to be Euclerian circuit if and only if every vertex has even degrees.*

Proof. Denote $G = \{V, E\}$ be the connected Graph with a circuit, then every vertex is connected and the path starts from the vertex $v_0 \in V$ and ends with $v_n \in V$ such that $v_0 = v_n$. Note a circuit is a closed graph that every edge E connects with a vertex V. Therefore, each vertex needs to be passed through to be a closed graph, i.e., a vertex has a "departure" edge and an "arriving" edge. Suppose each vertex has even degrees. Note graph G is a connected graph, so there is a connected path P from any $v \in V$ to any other $v \in V$. Since when P contains every edge in G will obviously form an Eulerian circuit, we are going to prove that when P does not have all edges. Removing edges in P gives a subgraph G' from G. Each component (a connected closed path) of G' has at least one vertex in common with P. Doing an induction on this step, we can find $v_n = v_1$, that is, each vertex has even degrees. □

Afterwards, Euler discovered another property for a connected graph.

Theorem 2.3. *Any connected plane graph G with V vertices, E edges and F faces (including the space face) will satisfy*

$$V - E + F = 2$$

Proof. We suppose G is the smallest counterexample that G does not satisfy the theorem. Suppose there are v vertices, e edges and f faces in G but $v - e + f \neq 2$. There exists an edge e that forms a circuit in G. Then removing e we will get v vertices still, $e-1$ edges, and the face will reduced to $f-1$, but there is no more circuit in the new G. Note this time G becomes a Tree graph, and Tree graph will always satisfy $V - E + F = 2$:

$$v - (e-1) + (f-1) = 2 \implies v - e + f = 2$$

This is a contradiction to our statement of G. □

3 Applications in Combinatorial Geometry

Example 3.1. Pick's Theorem is one of the most famous and interesting applications in combinatorial geometry, that can be proven using Euler's theorem. It indicates that in a grid plot, the area of a simple polygon can be calculated by counting the grids. Denote i be the interior grid counts, b be the side grid counts, then the area S:

$$S = i + \frac{b}{2} - 1$$

The example is shown in Figure 3.1. The polygon satisfies the above equation. We can consider that every plane polygon can be triangulated, so another way to prove this theorem can start from this approach.

Example 3.2. Four Color Theorem is the most difficult application that still cannot be solved by hand. Until 1976, this theorem is proven by Kenneth Appel and Wolfgang Haken using computers. Furthermore, scientists were interested in the theorem at the curved surface. Percy John Heawood proved a k-hole curved surface (S_k) has the upper boundary of minimum $Col(S_k)$ colors needed for two countries can be seperate with each other. Also, he proved that a ring surface will only need 7 colors. He proved the following equation:

$$Col(S_k) \leq \lfloor \frac{7 + \sqrt{1 + 48k}}{2} \rfloor$$

when $k = 1$, the surface is a ring:

$$Col(S_1) = \lfloor \frac{7 + \sqrt{1 + 48 \cdot 1}}{2} \rfloor = 7$$

Note that when $k = 0$ just satisfies the four color theorem in a plane surface, $Col(S_0) = 4$. Along with the four color theorem is properly proven, lots of artists start to celebrate by producing some attractive art works as shown in Figure 3.2.

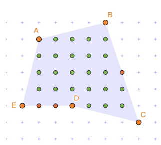

Figure 3.1: Pick's Theorem Example

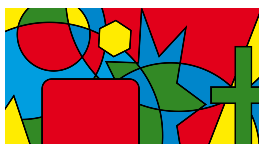

Figure 3.2: Art Work of Four Color Theorem

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