UNIVERSITY OF TORONTO | Mathematics

Combinatorial Geometry: Extended Abstract on Graph Theory

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Outline

- Review of basic definition of graph theory
- Review of Euler's Theorem
- Review of extended Euler's theorem based on counting edges, faces, and vertices
- Application 1: Pick's Theorem
- Application 2: Four Color Theorem

Definition

Graph

A graph G is said to be a graph when there are vertices V connected by edges E, denote by $G = \{V, E\}$. The term **degree** is defined as how many edges connected to the vertex. WLOG, when $|V| > 1$, a **connected graph** is defined as no one vertex is left without connecting an edge.

Draws of a Graph

Euler's Theorem

Eulerian Circuit

A connected graph is said to be Eulerian iff every vertex has even degrees.

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Proof

- Say $G = \{V, E\}$ is a connected graph with a circuit.
- Every vertex must be passing through to make a circuit, i.e., start from $v_0 \in V$ and end with $v_0 \in V$.
- Conversely, say G has even degrees.
- Since G is connected, then there must be a path P from any $v_m \in V$ to any other $v_n \in V$.
- When P contains every edge in G will be a trivial proof, so we are considering when P does not have all edges in G.
- Removing edges in P gives a subgraph G' from G, and each component of G' has at least one vertex in common with P. Repeating this step (induction), we can find $v_n = v_1$. □

Another Extended Theorem by Euler

Another Euler's Theorem

Any connected plane graph G with V vertices, E edges and F faces (including the space face) will always satisfy

$$V - E + F = 2$$

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Proof by Contradiction

- Say G is the smallest counterexample that G does not satisfy the equation.
- Suppose there are V vertices, F faces, and E edges in G that $V - E + F \neq 2$.
- Note that a Tree graph will always satisfy the equation, so we assume that G is not a tree graph, and there must be a circuit in G
- When G has a circuit, then removing one edge makes $E - 1$ left, $F - 1$ left, and V left.
- Note new G becomes a tree graph: $V - (E - 1) + (F - 1) = 2 \implies V - E + F = 2$ $\square$

Applications

Pick's Theorem

In a grid plot, the area of a simple polygon can be calculated by counting the grids. Denote i be the interior grid counts, b be the side grid counts, and the area S is calculated as:

$$S = i + \frac{b}{2} - 1$$

Exmaples

- This theorem only holds if the polygon is regular, no curves or other strange lines!
- This theorem is provable using Euler's Theorem. (Not in high-dimension)

Application

Four Color Theorem

Unsolvable by hands, but has been proven using computer in 1976 by Kenneth Appel and Wolfgang Haken. In a map, all countries can be separated using at minimum 4 colors.

Some Ideas

A k-hole curved surface call it S_k has the upper boundary of minimum $Col(S_k)$ colors needed for countries can be separated with each other. Also, a ring surface will only need 7 colors

$$Col(S_k) \leq \lfloor \frac{7+\sqrt{1+48k}}{2} \rfloor \qquad Col(S_1) = \lfloor \frac{7+\sqrt{1+48}}{2} \rfloor = 7$$

What about $k = 0$

Percy John Heawood proved when $k = 1$ (ring), the minimum color needed is 7 colors. But not hard to observe that when $k = 0$, $Col(S_0) = 4$. This may be a breakthrough of solving four color theorem using hand.

Interesting Art Works of Four Color Theorem

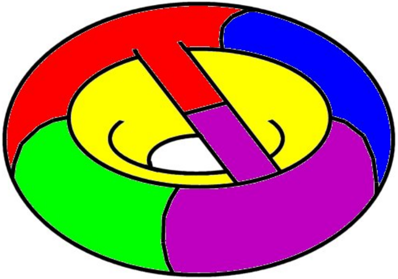

Figure: Four Color Theorem in a Ring

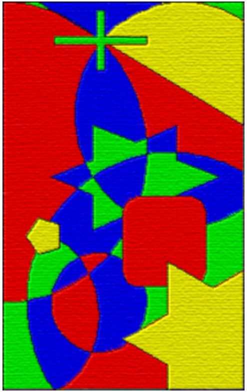

Figure: Art Work of Four Color Theorem

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Thank You!