

# Jiaqi's Thesis Progress Report (Updated Apr. 17)

Jiaqi Bi<sup>a</sup>

<sup>a</sup>*Western University,  
Schulich School of Medicine & Dentistry,  
Department of Epidemiology and Biostatistics*

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## 1. To Do List

1. Gibb's sampler

## 2. Notations

### List of Notations

|               |   |
|---------------|---|
| $i$           | Individual index                                    |
| $j$           | Family (Cluster) index                              |
| $p$           | Proband index                                       |
| $d_j$         | Number of events in family $j$                      |
| $t$           | Some time   |
| $a$           | Some Time for the proband                           |
| $T$           | Event Time  |
| $\delta_{ij}$ | Event indicator for individual $i$ in family $j$    |
| $w$           | The observed survival data $(t, \delta)$            |
| $n$           | Number of individuals                               |
| $J$           | Number of Families (Clusters)                       |
| $m$           | Index of the sampled completed dataset in the MCEM  |
| $M$           | Number of the sampled completed dataset in the MCEM |
| $z$           | Frailty term  |
| $q$           | $q$ -th element of Gauss Hermite Quadrature         |
| $\omega$      | $q$ -th weight of Gauss Hermite Quadrature          |
| $y_q$         | $q$ -th node of Gauss Hermite Quadrature            |
| $N_q$         | Total number of quadratures                         |
| $h(\cdot)$    | Hazard fuction                                      |
| $h_0(\cdot)$  | Baseline hazard function                            |
| $H(\cdot)$    | Cumulative hazard fuction                           |
| $S(\cdot)$    | Survival fuction                                    |
| $A_j(\cdot)$  | Ascertainment of family $j$ into the study          |

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Email address: jbi23@uwo.ca (Jiaqi Bi)

|                       |  |
|-----------------------|--|
| $L(\cdot)$            | Likelihood function  |
| $\ell(\cdot)$         | Log-likelihood function  |
| $\mathcal{L}(\cdot)$  | Laplace transform  |
| $\mathbf{x}$          | Covariates   |
| $\boldsymbol{\beta}$  | Model coefficients vector  |
| $\boldsymbol{\theta}$ | Parameter vector   |
| $\Lambda$             | The combination of $(\boldsymbol{\beta}, \lambda, \alpha)$         |
| $\lambda$             | Weibull shape parameter  |
| $\alpha$              | Weibull scale parameter  |
| $v$                   | General form of the parameter in an undefined frailty distribution |
| $k$                   | Gamma shape and rate parameters                                    |
| $\sigma^2$            | Log-Normal variance parameter                                      |
| $\psi$                | Missing data distribution parameters                               |

### 3. Weibull Parametric Approach

For the model efficiency of the analyses in a genetic research, a parametric survival analysis is usually chosen over semi-parametric survival analysis [1, 2]. From the beginning of the discussion, I have obtained the model, i.e., the hazard function is

$$h_{ij}(t_{ij}|\mathbf{x}_{ij}, z_j) = h_0(t_{ij}) \exp(\beta_1 x_{1,ij} + \beta_2 x_{2,ij}) z_j \quad (1)$$

There are total  $n_j$  individuals in family  $j$ , where  $i = 1, \dots, n_j$ , and total  $J$  families that  $j = 1, \dots, J$ .  $x_{1,ij}$  is the genotype, or say mutation gene status for individual  $i$  in family  $j$ .  $x_{2,ij}$  is the PRS for individual  $i$  in family  $j$ . The frailty term  $z_j$ , has a pdf of  $f(z)$ , which can be Gamma, log-normal, or other frailty distributions. The support of  $f(z)$  is always non-negative. The Weibull baseline hazard function is defined as

$$h_0(t_{ij}) = \alpha^\lambda \lambda t_{ij}^{\lambda-1} \quad (2)$$

where  $\lambda$  is the shape parameter and  $\alpha$  is the rate parameter. Let  $\xi_{ij} = \exp(\beta_1 x_{1,ij} + \beta_2 x_{2,ij})$ , the hazard function is

$$h_{ij}(t_{ij}|\mathbf{x}_{ij}, z_j) = \alpha^\lambda \lambda t_{ij}^{\lambda-1} \xi_{ij} z_j \quad (3)$$

The survival function  $S(t)$  can be obtained through cumulative hazard function  $H(t)$

$$H(t_{ij}|\mathbf{x}_{ij}, z_j) = \int_0^t h_{ij}(u|\mathbf{x}_{ij}, z_j) du \quad (4)$$

$$= \alpha^\lambda \xi_{ij} z_j \lambda \int_0^t u^{\lambda-1} du \quad (5)$$

$$= \alpha^\lambda \xi_{ij} z_j \lambda \cdot \frac{1}{\lambda} t_{ij}^\lambda = \alpha^\lambda \xi_{ij} z_j t_{ij}^\lambda \quad (6)$$

16 and the survival function

$$S(t_{ij}|\mathbf{x}_{ij}, z_j) = \exp(-H(t_{ij}|\mathbf{x}_{ij}, z_j)) = \exp(-\alpha^\lambda \xi_{ij} z_j t_{ij}^\lambda) \quad (7)$$

Let  $\boldsymbol{\theta} = \{\boldsymbol{\beta}, \alpha, \lambda, v\}$ , where  $v$  is the parameter for the frailty distribution of the choice. In our example dataset,  $\boldsymbol{\beta} = (\beta_1, \beta_2)$ . Therefore, the likelihood assuming missing data and frailties are observed can be written as

$$L(\boldsymbol{\theta}) = \prod_{j=1}^J \prod_{i=1}^{n_j} (\alpha^\lambda \lambda t_{ij}^{\lambda-1} \xi_{ij} z_j)^{\delta_{ij}} \exp(-\alpha^\lambda \xi_{ij} z_j t_{ij}^\lambda) \quad (8)$$

$$= \prod_{j=1}^J \prod_{i=1}^{n_j} h(t_{ij}|\mathbf{x}_{ij}, z_j)^{\delta_{ij}} \exp(-H(t_{ij}|\mathbf{x}_{ij}, z_j)) \quad (9)$$

When there is no missing data but frailties are present, the frailty term can be integrated where the likelihood is taken to be the expectation with respect to the frailty  $z_j$ . The likelihood can be written as

$$L(\boldsymbol{\theta}) = \prod_{j=1}^J \prod_{i=1}^{n_j} \int_{z_j} (\alpha^\lambda \lambda t_{ij}^{\lambda-1} \xi_{ij} z_j)^{\delta_{ij}} \exp(-\alpha^\lambda \xi_{ij} z_j t_{ij}^\lambda) f(z_j) dz_j \quad (10)$$

$$= \prod_{j=1}^J \prod_{i=1}^{n_j} \int_{z_j} h(t_{ij}|\mathbf{x}_{ij}, z_j)^{\delta_{ij}} \exp(-H(t_{ij}|\mathbf{x}_{ij}, z_j)) f(z_j) dz_j \quad (11)$$

But when the missing data and the frailty both exist in the model, we will need to account for their joint distribution within the likelihood according to Herring et al. [3].

$$L(\boldsymbol{\theta}) = \prod_{j=1}^J \prod_{i=1}^{n_j} \int_{z_j, \mathbf{x}_{mis,ij}} (\alpha^\lambda \lambda t_{ij}^{\lambda-1} \xi_{ij} z_j)^{\delta_{ij}} \exp(-\alpha^\lambda \xi_{ij} z_j t_{ij}^\lambda) f(z_j, \mathbf{x}_{mis,ij}) dz_j d\mathbf{x}_{mis,ij} \quad (12)$$

$$= \prod_{j=1}^J \prod_{i=1}^{n_j} \int_{z_j, \mathbf{x}_{mis,ij}} h(t_{ij}|\mathbf{x}_{ij}, z_j)^{\delta_{ij}} \exp(-H(t_{ij}|\mathbf{x}_{ij}, z_j)) f(z_j, \mathbf{x}_{mis,ij}) dz_j d\mathbf{x}_{mis,ij} \quad (13)$$

17 The following section 5 and section 6 discuss how to handle the frailty within the likeli-  
 18 hood when there are no missing data, which are corresponding to the Equation 10 and 11.  
 19 The section 7 will discuss how to handle the frailty and the missing data jointly, which is  
 20 corresponding to the likelihood equation and 13.

## 21 4. Ascertainment Correction

22 Within a genetic study, those families are typically selected when there is an affected  
 23 person called a proband. This will yield a selection bias because this is no long a case-  
 24 control study, and can potentially defect the statistical power [4, 5]. It is crucial to address  
 25 the ascertainment bias. Consider  $A$  as the event of being ascertained,  $D$  as the data, we  
 26 then have  $P(D, A|\boldsymbol{\theta}) = P(A|D, \boldsymbol{\theta})P(D|\boldsymbol{\theta})$ . Also, we know  $A$  is included in  $D$ , from Baye's

27 rule

$$P(D|\boldsymbol{\theta}) = \frac{P(D, A|\boldsymbol{\theta})}{P(A|D, \boldsymbol{\theta})} \propto \frac{L(\boldsymbol{\theta}|D)}{P(A|D, \boldsymbol{\theta})} \quad (14)$$

28 For each family  $j$ , the ascertainment  $A_j$  is defined to be the probability of the proband  
 29  $p$  being ascertained by the age  $a_{p_j}$  at examination, i.e.,  $A_j = P(T_{p_j} < a_{p_j})$  where  $a_{p_j}$  is  
 30 proband's age at study entry. Applying the ascertainment correction for the log-likelihood  
 31 in family  $j$ :

$$\tilde{\ell}_j(\boldsymbol{\theta}) = \ell_j(\boldsymbol{\theta}) - \log A_j(\boldsymbol{\theta}) \quad (15)$$

32 where  $\tilde{\ell}$  is the log-likelihood with ascertainment correction, and  $\ell$  is the crude log-likelihood.  
 33 Define  $\mathbf{x}_{p_j}$  the covariates for proband in family  $j$ , so we can further write the formula for the  
 34 ascertainment correction within different frailty models.

## 35 5. Gamma Frailty

We can obtain the likelihood for Gamma frailty model following the instruction by Balan and Putter [6]. The Laplace transform of the frailty  $z \sim \text{Gamma}(k, k)$ , for the simplicity of the mathematical expression, the following Laplace transform will ignore the subscript, denote  $\mathcal{L}(f(z)) = \phi(s)$  where  $s = \sum_{i=1}^{n_j} H(t_{ij}|\mathbf{x}_{ij})$ :

$$\phi(s) = \int_0^\infty e^{-sz} f(z) dz \quad (16)$$

$$= \int_0^\infty e^{-sz} \frac{k^k}{\Gamma(k)} z^{k-1} e^{-kz} dz \quad (17)$$

36 Using the Gamma property:  $\int_0^\infty z^{n-1} e^{-az} dz = \frac{\Gamma(n)}{a^n}$ ,  $\phi(s)$  can be further written as

$$\phi(s) = \frac{k^k}{\Gamma(k)} \int_0^\infty e^{-(s+k)z} z^{k-1} dz = \frac{k^k}{\Gamma(k)} \cdot \frac{\Gamma(k)}{(s+k)^k} = \left(1 + \frac{s}{k}\right)^{-k} \quad (18)$$

37 The second derivative is  $\frac{d^2\phi(s)}{ds^2} = \int_0^\infty (-z)^2 e^{-sz} f(z) dz$ .

The third derivative is  $\frac{d^3\phi(s)}{ds^3} = \int_0^\infty (-z)^3 e^{-sz} f(z) dz$ , ... Therefore, its  $d$ -th derivative, denote  $\phi(s)^{(d)}$ :

$$\phi(s)^{(d)} = (-1)^d \int_0^\infty z^d e^{-sz} f(z) dz \quad (19)$$

$$= (-1)^d \frac{(k+d-1)!}{(k-1)!(s+k)^d} \left(1 + \frac{s}{k}\right)^{-k} \quad (20)$$

Let  $\boldsymbol{\theta} = (\beta_1, \beta_2, \alpha, \lambda, k)$  for Gamma frailty model, the log-likelihood is then written as

$$\ell(\boldsymbol{\theta}) = \sum_{j=1}^k \log \left[ \int_0^\infty \prod_{i=1}^{n_j} (h(t_{ij}|\mathbf{x}_{ij}, z_j))^{\delta_{ij}} \exp(-H(t_{ij}|\mathbf{x}_{ij}, z_j)) f(z_j) dz_j \right] \quad (21)$$

$$= \sum_{j=1}^J \log \left[ \int_0^\infty \prod_{i=1}^{n_j} (z_j h(t_{ij}|\mathbf{x}_{ij}))^{\delta_{ij}} \exp(-z_j H(t_{ij}|\mathbf{x}_{ij})) f(z_j) dz_j \right] \quad (22)$$

$$= \sum_{j=1}^J \log \left[ \prod_{i=1}^{n_j} (h(t_{ij}|\mathbf{x}_{ij}))^{\delta_{ij}} \int_0^\infty z_j^{d_j} \exp(-z_j \sum_{i=1}^{n_j} H(t_{ij}|\mathbf{x}_{ij})) f(z_j) dz_j \right] \quad (23)$$

$$= \sum_{j=1}^J \log \left[ \prod_{i=1}^{n_j} (h(t_{ij}|\mathbf{x}_{ij}))^{\delta_{ij}} \frac{(k + d_j - 1)!}{(k - 1)! (\sum_{i=1}^{n_j} H(t_{ij}|\mathbf{x}_{ij}) + k)^{d_j}} \left( 1 + \frac{\sum_{i=1}^{n_j} H(t_{ij}|\mathbf{x}_{ij})}{k} \right)^{-k} \right] \quad (24)$$

$$= \sum_{j=1}^J \log \left[ \prod_{i=1}^{n_j} ((h(t_{ij}|\mathbf{x}_{ij}))^{\delta_{ij}}) \frac{(k + d_j - 1)!}{k! k^{d_j - 1}} \left( 1 + \frac{\sum_{i=1}^{n_j} (H(t_{ij}|\mathbf{x}_{ij}))}{k} \right)^{-k - d_j} \right] \quad (25)$$

$$= \sum_{j=1}^J \log \left[ h(t_{ij}|\mathbf{x}_{ij})^{\delta_{ij}} \frac{(k + d_j - 1)!}{k! k^{d_j - 1}} \left( 1 + \frac{\sum_{i=1}^{n_j} (H(t_{ij}|\mathbf{x}_{ij}))}{k} \right)^{-k - d_j} \right] \quad (26)$$

$$= \sum_{j=1}^J \left[ \sum_{i=1}^{n_j} (\delta_{ij} \log h(t_{ij}|\mathbf{x}_{ij})) + \log \left( \frac{(k + d_j - 1)!}{k! k^{d_j - 1}} \left( 1 + \frac{\sum_{i=1}^{n_j} (H(t_{ij}|\mathbf{x}_{ij}))}{k} \right)^{-k - d_j} \right) \right] \quad (27)$$

Note we can still apply Laplace transform for the ascertainment correction, such that

$$A_j(\boldsymbol{\theta}) = 1 - S_{p_j}(a_{p_j}|\mathbf{x}_{p_j}) \quad (28)$$

$$= 1 - \int_0^\infty S_{p_j}(a_{p_j}|\mathbf{x}_{p_j}, z_j) f(z_j) dz_j \quad (29)$$

$$= 1 - \int_0^\infty \exp(-z_j \cdot H_{p_j}(a_{p_j}|\mathbf{x}_{p_j})) f(z_j) dz_j \quad (30)$$

$$= 1 - \left( 1 + \frac{H_{p_j}(a_{p_j}|\mathbf{x}_{p_j})}{k} \right)^{-k} \quad (31)$$

## 38 6. Log-Normal Frailty

The log-normal frailty is not the power-variance-function (PVF) family, so there is no closed form for Laplace transform or expressions for survivors. But we are able to estimate the Laplace transform using Gauss Hermite Quadrature. We typically standardize the log-normal frailty  $Z$  as

$$E(\log Z) = 0 \quad (32)$$

$$\text{Var}(\log Z) = \sigma^2 \quad (33)$$

39 That is,  $z \sim \text{log-Normal}(0, \sigma^2)$ . The probability density function  $f(z)$  is then

$$f(z) = \frac{1}{\sqrt{2\pi}\sigma} z^{-1} \exp\left(-\frac{\log(z)^2}{2\sigma^2}\right) \quad (34)$$

40 The Laplace transform is then

$$\phi(s) = \mathcal{L}(f_Z)(s) = \int_0^\infty \exp(-sz) \cdot f(z) dz \quad (35)$$

Using variable transformation, let  $y = \frac{\log(z)}{\sqrt{2}\sigma}$ , then  $z = \exp(\sqrt{2}\sigma y)$ , and  $dz = \sqrt{2}\sigma \exp(\sqrt{2}\sigma y) dy$ . Therefore, for  $d$ -th derivative:

$$\phi(s)^d = \int_{-\infty}^\infty z^d \exp(-sz) \cdot \frac{1}{\exp(\sqrt{2}\sigma y) \sigma \sqrt{2\pi}} \cdot \exp(-y^2) \cdot \sqrt{2}\sigma \exp(\sqrt{2}\sigma y) dy \quad (36)$$

$$= \int_{-\infty}^\infty \exp(\sqrt{2}\sigma y)^d \exp(-s \exp(\sqrt{2}\sigma y)) \cdot \frac{1}{\sqrt{\pi}} \exp(-y^2) dy \quad (37)$$

41 **Definition 1** (Gauss-Hermite Quadrature). *The integrand part can be solved using Gauss-*  
 42 *Hermite Quadrature. In numerical analysis, the method can be applied in the following form:*

$$\int_{-\infty}^\infty \exp(-x^2) f(x) dx \approx \sum_{i=1}^n \omega_i f(x_i) \quad (38)$$

43 where  $n$  is number of sample points used, and  $x_i$  is the roots of Hermite polynomial  $H_n(x)$   
 44 such that  $i = 1, \dots, n$ , and the weights  $\omega_i$  is

$$\omega_i = \frac{2^{n-1} n! \sqrt{n}}{n^2 [H_{n-1}(x_i)]^2} \quad (39)$$

45 Applying Definition 1, the integral of the Laplace transform is then

$$\phi(s)^d = \frac{1}{\sqrt{\pi}} \sum_{q=1}^{N_q} \omega_q \exp(-s \exp(\sqrt{2}\sigma y_q)) \exp(\sqrt{2}\sigma y_q)^d \quad (40)$$

46 where  $q$  denotes the  $q$ -th element of Gauss Hermite Quadrature, i.e.,  $\omega_q$  denotes the  $q$ -th  
 47 weight,  $y_q$  denotes the  $q$ -th node, and  $N_q$  denotes the total number of quadratures. Thus,  
 48 substituting into the log-likelihood:

$$\ell_j(\boldsymbol{\theta}) = \sum_{i=1}^{n_j} \delta_{ij} \log(h(t_{ij}|\mathbf{x}_{ij})) + \log\left(\frac{1}{\sqrt{\pi}} \sum_{q=1}^{N_q} \left[ \omega_q \exp(\sqrt{2}\sigma y_q)^{d_j} \exp\left(-\sum_{i=1}^{n_j} H(t_{ij}|\mathbf{x}_{ij}) \exp(\sqrt{2}\sigma y_q)\right) \right]\right) \quad (41)$$

Similarly, the ascertainment correction in the log-normal frailty can be written as

$$A_j(\boldsymbol{\theta}) = 1 - \int_{-\infty}^{\infty} \exp(-zH(a_{p_j}|\mathbf{x}_{p_j}))f(z)dz \quad (42)$$

$$= 1 - \sum_{q=1}^{N_q} \omega_q \exp\left(-\left(\sum_{i=1}^{n_j} H(a_{p_j}|\mathbf{x}_{p_j})) \exp(\sqrt{2}\sigma y_{q_p})\right)\right) \quad (43)$$

## 7. Likelihood and Missing Data

### 7.1. Reviews on Missing Data

In this subsection, the notations are **distinct** to all other sections or subsections. The missing data problem was firstly brought by Rubin [7], and further targetted as a major statistical problem which many methodologists have developed different statistical tools to handle the missing data. Such as the practical book written by Rubin [8], and some comprehensive reviews on current missing data problems by Baraldi and Enders [9]. The missing data mechanism was introduced by Little and Rubin [10]. There are three missing data mechanisms, which are Missing Completely at Random (MCAR), Missing at Random (MAR), and Missing Not at Random (MNAR). There are some reviews on the missing data which rigorously present the statistical concept of three types of the missing mechanism [11].

**Definition 2.** (MCAR) Denote  $Y$  as the complete data matrix, and  $M$  as the missing data indicator matrix. Define  $y_{ij}$  and  $m_{ij}$  as  $i$ -th row (observation) and  $j$ -th column (variable) for the matrix  $Y$  and  $M$ . The conditional distribution of the missingness is said to be

$$f(m_i|y_i, \phi) = f(m_i|\phi) \quad (44)$$

That is, for the parameters of this distribution,  $m_i$  does not depend on any observed or missing data.

**Example 1.** (MCAR Example) There is a blind box with 500 indexed balls (No. 1 to 500) and their weights are unknown. We randomly draw 100 balls and measure their weights and record them in the Excel file. The Excel file contains two columns called Index and Weight, only those randomly selected balls will have Weights being filled. Those weights of unselected balls are called MCAR.

**Definition 3.** (MAR) Denote  $y_{i,obs}$  as the observed  $y$ , and  $y_{i,mis}$  as the missing  $y$ . Note that  $y_i = (y_{i,obs}, y_{i,mis})$ . The missing component is defined to be MAR if  $m$  only depends on  $y_{i,obs}$ . That is,

$$f(m_i|y_i, \phi) = f(m_i|y_{i,obs}, \phi) \quad (45)$$

**Example 2.** (MAR Example) In a psychological study, participants are asked to complete a survey so the scientist can profile their personalities. One question that asks participants to report their Mood status being good or bad. Male participants are typically too shy to answer this question, which yields some responses being missing. This is called the MAR, that the missingness on Mood status depends on the participant's gender, but not on the missing Mood itself.

**Definition 4.** (MNAR) In the MNAR, the missingness depends on the missing data itself, which is

$$f(m_i|y_i, \phi) = f(m_i|y_{i,mis}, y_{i,obs}, \phi) \quad (46)$$

In this case, the analysis needs to be conducted with caution. The missingness should be included in the likelihood construction.

**Example 3.** (MNAR Example) There is a study on participants' incomes. Person A makes \$200,000 per year, so they decide to report this amount without hesitancies. Person B makes \$10,000 per year, so they are not willing to provide this information, which this response is left as blank. This type of missing depends on the missing data itself, that Person B refuses to provide the response due to the response being comparatively low.

## 7.2. Without Considering the Kinship Structure

When assuming the data are missing at random, the missingness is only associated to the observed data. The frailty term and the missing data are therefore assumed independent. Denote  $w_{ij} = (t_{ij}, \delta_{ij})$  be the observed survival data. From the complete log-likelihood:

$$\ell_C(\boldsymbol{\theta}) = \sum_{j=1}^J \sum_{i=1}^{n_j} \delta_{ij} \log h(t_{ij}|\mathbf{x}_{ij}, z_j) - H(t_{ij}|\mathbf{x}_{ij}, z_j) \quad (47)$$

$$- \sum_{j=1}^J \log(1 - S_{p_j}(a_{p_j}|\mathbf{x}_{p_j}, z_j)) \quad (48)$$

$$= \sum_{j=1}^J \sum_{i=1}^{n_j} \delta_{ij} \log h(t_{ij}|\mathbf{x}_{ij}) z_j - H(t_{ij}|\mathbf{x}_{ij}) z_j \quad (49)$$

$$- \sum_{j=1}^J \log(1 - \exp(z_j H_{p_j}(a_{p_j}|\mathbf{x}_{p_j}))) \quad (50)$$

In the MCEM framework, define  $\boldsymbol{\theta}^{(r)}$  as  $r$ -th updates. The E-step can be written as

$$E(\ell_C(\boldsymbol{\theta})|\boldsymbol{\theta}^{(r)}) = \sum_{j=1}^J \sum_{i=1}^{n_j} \int_{\mathbf{x}_{mis}, z_j} \left( \delta_{ij} \log h(t_{ij}|\mathbf{x}_{ij}, z_j) - H(t_{ij}|\mathbf{x}_{ij}, z_j) \right) \quad (51)$$

$$\times f(\mathbf{x}_{ij,mis}, z_j|\mathbf{x}_{obs,ij}, \boldsymbol{\theta}^{(r)}) d\mathbf{x}_{ij,mis} dz_j \quad (52)$$

$$- \sum_{j=1}^J \int_{\mathbf{x}_{mis}, z_j} \log(1 - \exp(z_j H_{p_j}(a_{p_j}|\mathbf{x}_{p_j}))) \quad (53)$$

$$\times f(\mathbf{x}_{ij,mis}, z_j|\mathbf{x}_{obs,ij}, \boldsymbol{\theta}^{(r)}) d\mathbf{x}_{ij,mis} dz_j \quad (54)$$

such that we need to integrate out the joint density of the frailty term and the missing covariate from  $f(\mathbf{x}_{ij,mis}, z_j|\mathbf{x}_{obs,ij}, \boldsymbol{\theta}^{(r)})$ . There are selections of the frailty density, such as Gamma distribution, log-normal distribution, and etc which have been discused in the



previous chapter. In general, let's write  $f(z_j|v)$  for the frailty distribution may be chosen with some parameters  $v$ . Proposed by Herring et al. [3], the joint distribution of the frailty and the missing data can be adapted in our scenario that accounting for the ascertainment:

$$f(\mathbf{x}_{mis,ij}, z_j | \mathbf{x}_{obs,ij}, w_{ij}, T_{p_j} < a_{p_j}, \boldsymbol{\theta}^{(r)}) = f(\mathbf{x}_{mis,ij} | \mathbf{x}_{obs,ij}, w_{ij}, z_j, T_{p_j} < a_{p_j}, \boldsymbol{\theta}^{(r)}) \quad (55)$$

$$\times f(z_j | \mathbf{x}_{obs,ij}, w_{ij}, T_{p_j} < a_{p_j}, \boldsymbol{\theta}^{(r)}) \quad (56)$$

we define  $\Lambda = (\beta, \alpha, \lambda)$ , then further we can write

$$f(\mathbf{x}_{mis,ij}, z_j | \mathbf{x}_{obs,ij}, w_{ij}, T_{p_j} < a_{p_j}, \boldsymbol{\theta}^{(r)}) \quad (57)$$

$$= \frac{f(w_{ij} | \mathbf{x}_{mis,ij}, \mathbf{x}_{obs,ij}, z_j, T_{p_j} < a_{p_j}, \Lambda^{(r)}) f(\mathbf{x}_{mis,ij}, \mathbf{x}_{obs,ij} | \psi^{(r)}) f(z_j | v^{(r)})}{\int_{\mathbf{x}_{mis,ij}, z_j} f(w_{ij} | \mathbf{x}_{mis,ij}, \mathbf{x}_{obs,ij}, z_j, T_{p_j} < a_{p_j}, \Lambda^{(r)}) f(\mathbf{x}_{mis,ij}, \mathbf{x}_{obs,ij} | \psi^{(r)}) f(z_j | v^{(r)}) dz_j d\mathbf{x}_{mis,ij}} \quad (58)$$

$$\propto f(z_j | v^{(r)}) \prod_{i=1}^{n_j} f(w_{ij} | \mathbf{x}_{mis,ij}, \mathbf{x}_{obs,ij}, z_j, T_{p_j} < a_{p_j}, \Lambda^{(r)}) f(\mathbf{x}_{mis,ij} | \mathbf{x}_{obs,ij}, \psi^{(r)}) \quad (59)$$

Clearly, we know  $f(w_{ij} | \mathbf{x}_{mis,ij}, \mathbf{x}_{obs,ij}, z_j, T_{p_j} < a_{p_j}, \beta^{(r)})$  is the likelihood of one single observation  $i$  in family  $j$ , also we know the distribution of  $f(\mathbf{x}_{ij} | \psi)$ , as well as the frailty distribution  $f(z_j | v)$ . Therefore, in our case, we can apply Gibb's sampler where

1. We sample the missing data first, which we can obtain that

$$f(\mathbf{x}_{mis,ij} | \mathbf{x}_{obs,ij}, w_{ij}, z_j, T_{p_j} < a_{p_j}, \boldsymbol{\theta}^{(r)}) \propto f(w_{ij} | \mathbf{x}_{mis,ij}, \mathbf{x}_{obs,ij}, z_j, T_{p_j} < a_{p_j}, \Lambda^{(r)}) \quad (60)$$

$$\times f(\mathbf{x}_{mis,ij} | \mathbf{x}_{obs,ij}, \psi^{(r)}) \quad (61)$$

In this case, the missing data will be filled for each iteration  $r$ , that being said, all data will be "observed" in this case. Therefore, we will use  $\mathbf{x}_{ij}$  to simply denote completed covariates.

2. In order to approach the joint distribution, we now need to sample the frailty  $z_j$  from

$$f(z_j | \mathbf{x}_{ij}, w_{ij}, T_{p_j} < a_{p_j}, \boldsymbol{\theta}^{(r)}) \propto \prod_{i=1}^{n_j} f(w_{ij} | \mathbf{x}_{ij}, T_{p_j} < a_{p_j}, \Lambda^{(r)}) \times f(z_j | v^{(r)}) \quad (62)$$

3. The Gibb's sampler has been proven as an efficient sampling method to closely approach the desired joint distribution [12]. We can get a frailty distribution based on what we have sampled for the missing data, and we can obtain the missing data distribution based on what we have sampled from the frailty distribution.

These conditional densities can be explicitly written. We will obtain  $M$  completed dataset based on the Gibb's sampler. In general, without the specification of the frailty distribution,

the E-step in MCEM can be written as

$$Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(r)}) = \sum_{j=1}^J \frac{1}{M_j} \sum_{m=1}^{M_j} \sum_{i=1}^{n_j} \left( \delta_{ij} \log h(t_{ij}|\mathbf{x}_{ij}^{(m)}, z_j^{(m)}) - H(t_{ij}|\mathbf{x}_{ij}^{(m)}, z_j^{(m)}) \right) \quad (63)$$

$$+ \sum_{j=1}^J \frac{1}{M_j} \sum_{m=1}^{M_j} \log(1 - \exp(z_j H_{p_j}(a_{p_j}|\mathbf{x}_{p_j}))) \quad (64)$$

$$+ \sum_{j=1}^J \frac{1}{M_j} \sum_{m=1}^{M_j} \sum_{i=1}^{n_j} \log f(\mathbf{x}_{ij,mis}^{(m)}, z_j^{(m)}|\mathbf{x}_{obs,ij}, \boldsymbol{\theta}) \quad (65)$$

$$= \sum_{j=1}^J \frac{1}{M_j} \sum_{m=1}^{M_j} \sum_{i=1}^{n_j} \left( \delta_{ij} \log h(t_{ij}|\mathbf{x}_{ij}^{(m)}, z_j^{(m)}) - H(t_{ij}|\mathbf{x}_{ij}^{(m)}, z_j^{(m)}) \right) \quad (66)$$

$$+ \sum_{j=1}^J \frac{1}{M_j} \sum_{m=1}^{M_j} \log(1 - \exp(z_j H_{p_j}(a_{p_j}|\mathbf{x}_{p_j}))) \quad (67)$$

$$+ \sum_{j=1}^J \frac{1}{M_j} \sum_{m=1}^{M_j} \sum_{i=1}^{n_j} \log f(\mathbf{x}_{mis,ij}^{(m)}|\mathbf{x}_{obs,ij}, \psi) + \sum_{j=1}^J \frac{1}{M_j} \sum_{m=1}^{M_j} \sum_{i=1}^{n_j} \log f(z_j^{(m)}|v) \quad (68)$$

110 Note that in this case, the distribution of the missing data is univariate since we are imputing  
 111 each individual  $i$  in family  $j$ . However, in a genetic study, some missing covariates may need  
 112 some considerations of the multivariate structure such as the kinship matrix.

### 113 7.3. Considering the Kinship Matrix

114 When we want to include the kinship matrix into the consideration, the distribution of  
 115 the missing data becomes a multivariate distribution for family  $j$ . Denote  $f(\mathbf{x}_{mis,j}|\mathbf{x}_{obs,j}, \psi)$   
 116 as the multivariate distribution of the missing data in family  $j$ . It is important to obtain the  
 117 conditional distribution for each individual  $i$  conditioning on other individuals  $-i$  within fam-  
 118 ily  $j$ . Assume a  $n_j$  dimensional multivariate normal distribution of  $\mathbf{x}_{mis} = (\mathbf{x}_{mis,1}, \dots, \mathbf{x}_{mis,n_j})$   
 119 in family  $j$ , the index here will ignore the family index since if it's global, it will work too.  
 120 The multivariate distribution will be assumed a normal, because we are focusing on the  
 121 missing PRS which has a normal distributed behavior. The mean vector of this multivariate  
 122 normal distribution can be written as  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_{n_j})$  and the covariance matrix  $\boldsymbol{\Sigma}$ . Note  
 123 that  $\boldsymbol{\Sigma} = \tilde{\psi}_g^2 K + \tilde{\psi}_e^2$  and  $K$  is the kinship matrix with the diagonal of 1.  $\tilde{\psi}_g^2$  is the genetic  
 124 variance and  $\tilde{\psi}_e^2$  is the residual variance. If we want to find the conditional distribution of  
 125 each  $x_{mis,i}$  in family  $j$ , given others  $X_{-i}$  where  $X_{-i}$  is the vector of all other variables except  
 126  $x_{mis,i}$ . Partition the mean vector and the covariance matrix, suppose  $\mathbf{X} = (x_i, X_{-i})$ , then  
 127 partition  $\boldsymbol{\mu}$  and  $\boldsymbol{\Sigma}$  accordingly:

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_i \\ \boldsymbol{\mu}_{-i} \end{pmatrix} \quad (69)$$

128 and

$$\boldsymbol{\Sigma} = \begin{pmatrix} \Sigma_{ii} & \boldsymbol{\Sigma}_{i,-i} \\ \boldsymbol{\Sigma}_{-i,i} & \boldsymbol{\Sigma}_{-i,-i} \end{pmatrix} \quad (70)$$

where  $\Sigma_{ii}$  is actually the variance of  $x_i$ .  $\Sigma_{i,-i}$  and  $\Sigma_{-i,i}$  are transpose of each other, and are covariances between  $x_i$  and  $X_{-i}$ .  $\Sigma_{-i,-i}$  is the covariance matrix of  $X_{-i}$ . Also,  $\boldsymbol{\mu}$  can be estimated using a linear regression from a multivariate version with flexible covariance matrix introduced by Ziyatdinov et al. [13]. Then we can compute the conditional mean and variance from

$$E(x_i|X_{-i}) = \mu_i + \Sigma_{i,-i}\Sigma_{-i,-i}^{-1}(\mathbf{x}_{-i} - \boldsymbol{\mu}_{-i}) \quad (71)$$

and the variance

$$\text{Var}(x_i|\mathbf{x}_{-i}) = \Sigma_{ii} - \Sigma_{i,-i}\Sigma_{-i,-i}^{-1}\Sigma_{-i,i} \quad (72)$$

Once we can calculate these statistics, we are able to sample each  $x_i$  using a univariate normal distribution while still considering the kinship matrix. The MCEM will then perform using the same idea of the previous subsection.

## 8. Detailed Implementations of Gibb's Sampler

In section 7.2, the posterior distribution is not easy to sample from. So the Gibb's sampler needs an additional step called Matropolis-Hastings step. In the article by Herring et al. [3], the choice of the frailty distribution is multivariate normal, which satisfies the property of log-concavity for adaptive rejection algorithm. However, when the frailty distribution is designed to be Gamma or log-normal distribution, the log-concavity fails. One may use the Metropolis-Hastings (MH) step within Gibbs to determine the acceptance or rejection when sampling the posterior distributions [14]. There are some articles on the MH algorithm and how it works such as by Andrieu and Moulines [15], and some articles have discussed how Gibb's sampler are adapted using MCMC methods [16]. This section will discuss the MH-within-Gibbs algorithm when sampling the missing data and frailty. To sample the frailty using the Gibb's sampler with MH algorithm, the procedure is

1. We have a proposal sampling distribution  $q(z'_j|z_j)$ , which represents what we are sampling in the iteration  $r$
2. We first sample from this  $q(z'_j|z_j)$
3. For  $z_j$ ,  $q(z'_j|z_j) = f(z_j|v^{(r)})$
4. Then calculate the acceptance ratio:

$$\gamma = \min \left( 1, \frac{\prod_{i=1}^{n_j} f(w_{ij}|\mathbf{x}_{ij}, z'_j, T_{p_j} < a_{p_j}, \Lambda^{(r)}) \times q(z'_j|z_j)}{\prod_{i=1}^{n_j} f(w_{ij}|\mathbf{x}_{ij}, z_j, T_{p_j} < a_{p_j}, \Lambda^{(r)}) \times q(z_j|z'_j)} \right) \quad (73)$$

5. Now we are sampling  $\tilde{u} \sim \text{Unif}(0, 1)$ , and we accept  $z'$  if  $\tilde{u} \leq \gamma$ . Otherwise, reject and set  $z'_j = z_j$ .

The same idea for missing data.

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