STA355
UTSG

Midterm Review 2

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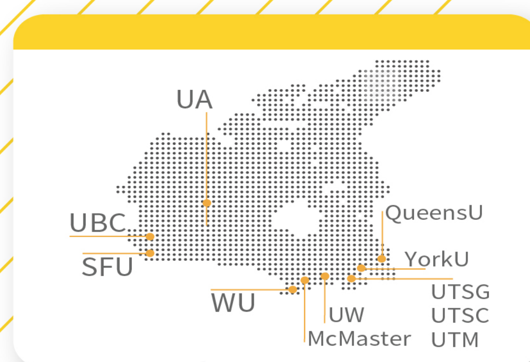

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今日内容:

1. 教授 practice problem
2. Past Tests

Practice Problem

Suppose that X is a positive continuous random variable with cdf $F(x)$ and pdf $f(x)$. The Lorenz curve is defined as $\mathcal{L}_F(t) = \frac{1}{E_F(X)} \int_0^t F^{-1}(s) ds$.

(a) Suppose you have a sample $X_1, \dots, X_n$ from F where F is unknown. Find an estimator of $\theta = \mathcal{L}_F(0.8)$.

Solution:

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(b) Let $E_F(X) = 4$ and $\mathcal{L}_F(t) = t^3$. Find the median of F.

Solution:

Practice Problem

Let X be a continuous random variable with cdf

$$F(x) = \begin{cases} 0 & x < \theta \\ \dfrac{x}{\theta} & 0 < x < \theta \\ 1 & x > \theta \end{cases}$$

Compute the expression of the Lorenz curve $\mathcal{L}_F(p) = \frac{1}{E_F(X)} \int_0^p F^{-1}(t) dt$.

Solution:

Practice Problem

Let $X_1, X_2, \cdots$ be a sequence of independent random variables with cdf

$$F_{X_n}(x) = \begin{cases} \left(1 - \dfrac{1}{1+nx}\right)^n & 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find the limiting distribution of X_n.

Solution:

Practice Problem

Suppose that $X_1, \cdots, X_n$ are independent random variables with pdf

$$f(x) = \begin{cases} \frac{1}{\theta} & \theta < x < 2\theta \\ 0 & \text{otherwise} \end{cases}$$

where $\theta > 0$.

(a) Find the method of moments estimator of θ.

Solution:

(b) Show that $\frac{X_{(n)}-\theta}{\theta}$ is a pivot for θ where $X_{(n)} = \max\{X_1, \cdots, X_n\}$.

Solution:

Practice Problem

Suppose that $X_1, X_2, \cdots$ is a sequence of independent random variables with mean μ and variance $\sigma^2 < \infty$; define $\bar{X}_n = n^{-1}(X_1 + \cdots + X_n)$. Describe the limiting behaviour (that is, either convergence in probability or convergence in distribution as well as the limit as $n \to \infty$) of the following random variables.

(a) $S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$.

Solution:

(b) $\sqrt{n}(\bar{X}_n - \mu)/S_n$.

Solution:

(c) $\sqrt{n}(\exp(\bar{X}_n) - \exp(\mu))/S_n$.

Solution:

Practice Problem

Suppose that $X_1, \cdots, X_n$ are independent positive random variables whose hazard function is

$$h(x) = \begin{cases} x^{-1} & \text{for } x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

and define $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ to be the order statistics.

(a) Show that the τ quantile of each X_i is $(1-\tau)^{-1}$.

Solution:

(b) Suppose that $n = 1000$ and define $D = X_{(501)} - X_{(499)}$. The distribution of D can be approximated by a distribution whose mean is μ. Which one of the following ((i)-(iv)) is the best approximation of μ ? Justify your answer to receive full marks.

(i) $2/1000$
(ii) $4/1000$
(iii) $8/1000$
(iv) $2/\sqrt{1000}$

Solution:

Practice Problem

Suppose that $X_1, \cdots, X_n$ are independent random variables with pdf

$$f(x;\theta) = \left(\frac{1}{2\pi x^3}\right)^{1/2} \exp\left(-\frac{(x-\theta)^2}{2\theta^2 x}\right) \text{ for } x \geq 0$$

where $\theta > 0$ is an unknown parameter.

Find the MLE of θ based on $X_1, \cdots, X_n$ and give an estimator of its standard error based on the observed Fisher information. (You may assume all the regularity conditions for asymptotic normality are satisfied and you do not need to show your estimator maximizes the likelihood function.)

Solution:

Practice Problem

Suppose that $X_1, \cdots, X_n$ are independent random variables on the interval $[0, \theta]$ with pdf

$$f(x;\theta) = \begin{cases} 3x^2/\theta^3 & \text{if } 0 \le x \le \theta \\ 0 & \text{otherwise} \end{cases}$$

where $\theta > 0$ is an unknown parameter.

(a) The cumulative distribution function of $X_{(n)} = \max(X_1, \cdots, X_n)$ is

$$G(x;\theta) = P_\theta\left(X_{(n)} \le x\right) = \left(\frac{x}{\theta}\right)^{3n} \quad \text{for } 0 \le x \le \theta$$

(You do not need to show this.)

Show that $X_{(n)}/\theta$ is a pivot for θ.

Solution:

(b) Use the pivot $X_{(n)}/\theta$ to find an exact $100p\%$ confidence interval of the form $\left[X_{(n)}, aX_{(n)}\right]$ where $a > 1$ will depend on p and n.

Solution:

Practice Problem

Consider the following data

$$1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0$$

which we assume are a random sample (outcomes of independent random variables $X_1, \cdots, X_{10}$) from a Bernoulli distribution whose pmf is

$$f(x;\theta) = \theta^x(1-\theta)^{1-x} \text{ for } x = 0,1$$

where θ is an unknown parameter with $0 < \theta < 1$.

(a) Given the data, compute the MLE of θ and give an estimate of its standard error.

Solution:

(b) Suppose that we know that θ can take only two possible values: $\theta = 1/2$ and $\theta = 3/4$ so that $\Theta = \{1/2, 3/4\}$. What is the MLE of θ in this case?

Solution:

Practice Problem

Suppose that $X_1, \cdots, X_n$ are independent continuous random variables with pdf

$$f(x;\theta) = \frac{1}{\theta} g(x/\theta) \text{ for } x \geq 0$$

where $\theta > 0$ is an unknown parameter but the function g (which is itself a pdf) is known.

(a) Show that $(X_1 + \cdots + X_n)/\theta$ is an exact pivot for θ. (Hint: Find the pdf of X_i/θ.)

Solution:

(b) Even though the random variable in part (a) is a pivot, its distribution is not necessarily easy to compute. Define

$$\mu_g = \int_0^\infty x g(x) dx \text{ and } \sigma_g^2 = \int_0^\infty (x - \mu_g)^2 g(x) dx.$$

Show that

$$\frac{1}{\sigma_g \sqrt{n}} \left(\frac{X_1 + \cdots + X_n}{\theta} - n\mu_g \right)$$

is approximately Normal with mean 0 and variance 1 when n is sufficiently large and use this to construct an approximate $100p\%$ confidence interval for θ, giving all appropriate details.

Solution:

Practice Problem

Suppose that $X_1, \cdots, X_n$ are independent continuous random variables with common pdf $f(x-\theta)$ (where $f(x) = f(-x)$). θ can be estimated using the Winsorized mean:

$$\hat{\theta} = \frac{1}{n}\left\{rX_{(r)} + \sum_{i=r+1}^{n-r} X_{(i)} + rX_{(n-r+1)}\right\}$$

where $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ are the order statistics

(a) Suppose we are given an R function winsor where winsor (x, r) computes the estimate of θ for data x for all given value of r. The R code below computes a jackknife variance estimate of $\hat{\theta}$ based on 100 observations using $r = 10$.

```
> r <- 10
> thetahat <- winsor(x,r)
> thetahat
[1] 0.07854802
> thetai <- NULL # leave-one-out estimates
> for (i in 1:100) {
+    thetai <- c(thetai,winsor(x[-i],r))
+    }
> thetadot <- mean(thetai)
> jackvar <- 99*sum((thetai-thetadot)^2)/100
> jackvar
[1] 0.01301084
```

Assuming that the data come from a symmetric distribution, give an approximate 95% confidence interval for θ. (The 0.975 quantile of a standard normal distribution is 1.96.)

Solution:

(b) Suppose that $X_1, \cdots, X_n$ are independent random variables from a continuos distribution F with quantiles $F^{-1}(t)$ for $0 < t < 1$. If we define $\hat{\theta}_n$ to be the Winsorized mean with $r/n = \tau$ then

$$\hat{\theta}_n \xrightarrow{p} \theta(F) = \alpha\{F^{-1}(\tau) + F^{-1}(1-\tau)\} + \beta \int_{\tau}^{1-\tau} F^{-1}(s) ds$$

Find expresssions for α and β in terms of τ.

Solution:

Practice Problem

Suppose that $X_1, \cdots, X_{100}$ are independent Exponential random variables whose pdf is

$$f(x;\lambda) = \lambda \exp(-\lambda x) \quad \text{for } x \geq 0$$

where $\lambda > 0$ is unknown. However, rather than observing all 100 observations, we observe only the smallest 50 observations, in other words, the order statistics $X_{(1)} \leq \cdots \leq X_{(50)}$.

(a) Define $\hat{\lambda} = \ln(2)/X_{(50)}$. From lecture, we know that we can approximate the distribution of $\hat{\lambda}$ by a $\mathcal{N}(\lambda, \sigma^2(\lambda))$ distribution. Find an expression for $\sigma^2(\lambda)$.

Solution:

(b) Find the MLE of λ based on $X_{(1)}, \cdots, X_{(50)}$. (Hint: You may use the fact (without proof) that $100X_{(1)}, 99\left(X_{(2)} - X_{(1)}\right), \cdots, 51\left(X_{(50)} - X_{(49)}\right)$ are independent Exponential random variables with parameter λ.)

Solution: