Dynamical Systems and Ergodic Theory

Jiaqi Cui
East China Normal University

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Abstract

This is my notes of the course "Dynamical Systems and Ergodic Theory" given by Manfred Einsiedler. https://video.ethz.ch/lectures/d-math/2024/spring/401-2374-24L.html

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0 Examples

Let X be a set and $T\colon X \to X$ be a map.

Definition 0.1. *fixed point, periodic point, period, orbit...*

Definition 0.2. *Assume X has a topology. The ω-limit of $x \in X$ is*

$$\omega^{\pm}(x) := \left\{ \lim_{k\to\infty} T^{n_k}x : n_k \nearrow \pm\infty \right\}.$$

We could also ask about the "distribution" of $x, Tx, T^2x, \cdots, T^nx$ inside X as $n \to \infty$.
More generally, a dynamical system can be defined as a group action.

Example 0.3. $X = \mathbb{R}$, $Tx = x + 1$. The ω-limits are empty set. Thus we will restrict to compact metric spaces.

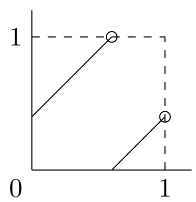
Example 0.4. $X = \mathbb{R} \cup \{\infty\}$ the one-point compactification of $\mathbb{R}$. $T(x) = \begin{cases} x+1,\ x \in \mathbb{R}; \\ \infty,\ x = \infty \end{cases}$. Then the ω-limits are all $\{\infty\}$.

Example 0.5. $X = \mathbb{R} \cup \{\pm\infty\}$. $T(x) = \begin{cases} x+1,\ x \in \mathbb{R}; \\ +\infty,\ x = +\infty; \\ -\infty,\ x = -\infty \end{cases}$. Then $\omega^+(x) = \begin{cases} +\infty,\ x \in \mathbb{R} \cup \{+\infty\}; \\ -\infty,\ x = -\infty \end{cases}$.

Example 0.6. North-South Dynamics

Example 0.7. $X = \mathbb{T} = \mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$ with the metric $d(x+\mathbb{Z}, y+\mathbb{Z}) = \min_{k\in\mathbb{Z}} |x - y + k|$. $R(x+\mathbb{Z}) := x+\alpha+\mathbb{Z}$ for a fixed $\alpha \in \mathbb{T}$. $R\colon \mathbb{T} \to \mathbb{T}$ is an isometry.

- If $\alpha = \frac{p}{q}$ is rational, then $R^q(x + \mathbb{Z}) = x + q\alpha + \mathbb{Z} = x + p + \mathbb{Z} = x + \mathbb{Z}$. Every point is periodic with period q.

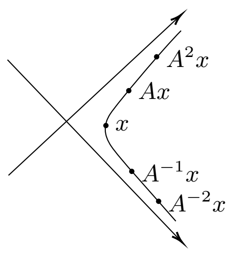


- If $\alpha \notin \mathbb{Q}/\mathbb{Z}$, then no point is periodic: say $R^n(x+\mathbb{Z}) = x+\mathbb{Z}$, then $n\alpha \in \mathbb{Z}$. Actually, all orbits are dense in this case.

Example 0.8. $X = \mathbb{T} = \mathbb{R}/\mathbb{Z}$. Fix $p \geqslant 2 \in \mathbb{N}$. $T(x) := px$. This map links to the base-p expansion of $x \in [0,1)$. Suppose $x = \sum_{k=1}^{\infty} \theta_k p^{-k}$ where $\theta_k\{0,\cdots,p-1\}$. Then $Tx = p\sum_{k=1}^{\infty}\theta_k p^{-k} + \mathbb{Z} = \sum_{k=1}^{\infty}\theta_{k+1}p^{-k} + \mathbb{Z}$.

Claim.
- There exist lots of periodic points– they are dense.
- There exist pre-periodic points that are not periodic, where x is pre-periodic if its orbit $|\mathcal{O}^+(x)| < \infty$.
- Many periods exist.
- The set of pre-periodic points is countable.
- There exist $x \in \mathbb{T}$ with $\omega^+(x) = \mathbb{T}$.
- There exist $x \in \mathbb{T}$ with $\omega^+(x)$ uncountable but not $\mathbb{T}$.
- There exist $x \in \mathbb{T}$ with $\omega^+(x)$ countable but not finite.

Example 0.9. $X = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. This is called a hyperbolic toral automorphism, because the orbit of any $x \neq 0 \in X$ is on a hyperbola.



Example 0.10. $X = (0,1) - \mathbb{Q}$, $Tx = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$. This relates to continued fraction expansion

$$x = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}$$

where $a_1, a_2, a_3, \cdots \in \mathbb{N}$. Note that

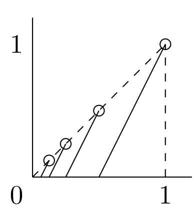
$$Tx = \cfrac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \cdots}}}.$$

Example 0.11 (Benford‘s law for powers of 2)**.** Given $j \in \{1,\cdots,9\}$, the limits

$$d_j := \lim_{N\to\infty} \frac{1}{N} \sharp \{2^n : 1 \leqslant n \leqslant N,\ 2^n \text{ starts in digital expension with } j\}$$

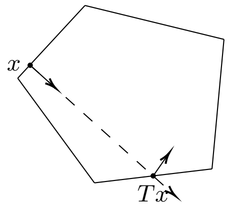
satisfy $d_1 > d_2 > \cdots > d_9 > 0$. In fact $d_1 = \log_{10} 2$.

Example 0.12. $X = [0,1]$, $T(x) = \begin{cases} 0, \ x = 0, 1; \\ nx - 1, \ x \in \left[\frac{1}{n}, \frac{1}{n-1}\right) \end{cases}$.



We claim that $\lim_{n\to\infty} T^n x = 0$, and if $x \in \mathbb{Q}$, there exists n with $T^n x = 0$ and if $x \notin \mathbb{Q}$, $T^n x > 0$ for all n. For $x = e$ this can be used to show that $e \notin \mathbb{Q}$.

Example 0.13 (Billiards)**.** X is the set of boundary points with a vector and T is the movement of a boundary point along its vector to the next boundary point with a reflected vector.



Given a triangle, are there always periodic points? When the triangle is acute, right or obtuse with rational angles, the answer is yes. For the other cases, people don't know.

Example 0.14 (Geodesic flow)**.** Given a nice manifold M and its unit tangent bundle. There exists a way of following any vector in the tangent. When M is a sphere, the orbits are great circles. When M is a torus, whether an orbit is closed depending on whether the initial vector is rational. When M is a orientable surface of genus greater than 2, the orbits will be more complicated and more sensitive to initial values.

1 Topological Dynamics

Assume X is a compact metric space and $T\colon X \to X$ is continuous or even a homeomorphism.

Definition 1.1. *A homeomorphism $T\colon X \to X$ is called (topological) transitive if there exists a point $x_0 \in X$ for which the orbit is dense, i.e. $\overline{\mathcal{O}(x_0)} = X$.*

Definition 1.2. *A comtinuous map $T\colon X \to X$ is called forward transitive if there exists $x_0 \in X$ with $\overline{\mathcal{O}^+(x_0)} = X$.*

Example 1.3. $T_p\colon \mathbb{T} \to \mathbb{T}$ for $p \geqslant 2$ an integer which maps x to px is forward transitive. We will construct x_0 using base-p-expansion. We first list all finite sequences in the symbols $0, 1, \cdots, p-1$, and consider the result as one sequence of digits $a_1, a_2, \cdots \in \{0, 1, \cdots, p-1\}$. Then we define $x_0 := \sum_{k=1}^{\infty} a_k p^{-k}$. For any $\left[\frac{j}{p^i}, \frac{j+1}{p^i}\right)$, we can find an l such that $T^l x_0$ is in this interval. Thus $\mathcal{O}^+(x_0)$ is dense in $\mathbb{T}$. For example, for $p = 2$, we write

$$0, 1, 00, 01, 10, 000, \cdots, 111, 0000, \cdots, 1111, \cdots$$

Then we write it as a number sequence

$$0, 1, 0, 0, 0, 1, 1, 0, 0, 0, 0, \cdots, 1, 1, 1, 0, 0, 0, 0, \cdots, 1, 1, 1, 1, \cdots$$

and then

$$x_0 = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{8} + 0 \cdot \frac{1}{16} + 0 \cdot \frac{1}{32} + 1 \cdot \frac{1}{32} + 1 \cdot \frac{1}{64} + \cdots$$

When we apply T on x_0 for n times, the first n numbers of the number sequence will become 0. Then for any $\frac{j+1}{2^i}$, we can find an n such that the base-2-expansion of $\frac{j+1}{2^i}$ will at the start of the number sequence. This means $T^n x_0 \in \left[\frac{j}{2^i}, \frac{j+1}{2^i}\right)$.

Example 1.4. $R_\alpha\colon \mathbb{T} \to \mathbb{T}$ maps x to $x + \alpha$.

- If $\alpha \in \mathbb{Q}/\mathbb{Z}$, R_α only has periodic orbits and so is not transitive.
- If $\alpha \notin \mathbb{Q}/\mathbb{Z}$, R_α is topological transitive. See later.

Proposition 1.5. *Let $T\colon X \to X$ be a homeomorphism. The followings are equivalent:*

1. *T is topological transitive;*
2. *if $U \subset X$ is open and $TU = U$, then either $U = \varnothing$ or $\overline{U} = X$;*
3. *if $U, V \subset X$ are non-empty and open, then there exists $n \in \mathbb{Z}$ so that $T^n U \cap V \neq \varnothing$;*
4. *the set $\left\{x_0 \in X : \overline{\mathcal{O}(x_0)} = X\right\}$ is a dense G_δ-set.*

Definition 1.6. *A set G is called a G_δ-set if it is a countable intersection of open sets.*

Theorem 1.7 (Baire Category Theorem)**.** *Let X be a complete metric space. Let $O_n \subset X$ be a sequence of dense open sets. Then $\bigcap_{n=1}^{\infty} O_n$ is a dense G_δ-set.*

Proof. We only prove that $\bigcup_{n=1}^{\infty} O_n$ is dense. For any open set U, we want to find a point in $U \cap \bigcup_{n=1}^{\infty} O_n$. First, $U \cap O_1$ is non-empty and open because O_1 is open and dense. Then we can find a open ball $B_{\varepsilon_1}(x_1) \subset U \cap O_1$. Repeat this process. We find a open ball $B_{\varepsilon_2}(x_2) \subset B_{\varepsilon_1}(x_1) \cap O_2$... WLOG we can suppose $\varepsilon_n \leqslant \frac{1}{n}$. Thus we can claim that $\{x_n\}$ is a Cauchy sequence and $x := \lim_{n\to\infty} x_n \in U \cap \bigcap_{n=1}^{\infty} O_n$: by construction, $x_m \in B_{\varepsilon_{m-1}}(x_{m-1}) \subset \cdots \subset B_{\varepsilon_n}(x_n)$ and then $d(x_m, x_n) < \varepsilon_n \leqslant \frac{1}{n}$. Note that X is complete, the limit $x = \lim_{n\to\infty} x_n \in X$ exists. For all n, taking the limit of m, we obtain $x = \lim_{m\to\infty} x_m \in \overline{B_{\varepsilon_n}(x_n)} \subset O_n$. Moreover, $x \in \overline{B_{\varepsilon_1}(x_1)} \subset U$. □

Corollary 1.8. *A countable intersection of dense G_δ-sets is a dense G_δ-set.*

Proof of Proposition 1.5. (1)⇒(2): Let $x_0 \in X$ with $\mathcal{O}(x_0)$ dense in X. Because T is a homeomorphism between compact metric space X and U is open, $\mathcal{O}(x_0) \cap U \neq \varnothing$. Then there exists $n \in \mathbb{Z}$ such that $T^n x_0 \in U$. Note that $TU = U$, we have $x_0 \in T^{-n}U = U = T^{-m}U$ for any $m \in \mathbb{Z}$. This shows $\mathcal{O}(x_0) \subset U$ and then U is dense in X.

(2)⇒(3): Define $\widetilde{U} = \bigcup_{n\in\mathbb{Z}} T^n U$. We note that $T\widetilde{U} = \widetilde{U}$ is non-empty and open. Then it is dense. Because T is a homeomorphism between compact metric space X and V is open again, $\widetilde{U} \cap V \neq \varnothing$ and then there exists $n \in \mathbb{Z}$ such that $T^n U \cap V \neq \varnothing$.

(3)⇒(4): For any $n \in \mathbb{N}$, $\bigcup_{x\in X} B_{\frac{1}{n}}(x)$ is a open cover. Because X is compact, there exists $k(n) \in \mathbb{N}$ such that $\bigcup_{i=1}^{k(n)} B_{\frac{1}{n}}(x_i)$ covers X. Denote $B_{\frac{1}{n}}(x_i)$, $i = 1, \cdots, k(n)$, $n \in \mathbb{N}$ by $O_1, O_2, \cdots$. To show density of a set, it suffices to show that the set intersects any of those open sets $O_1, O_2, \cdots$. For any $j \in \mathbb{N}$, we define $\widetilde{O_j} = \bigcup_{n\in\mathbb{Z}} T^n O_j$. By assumption, $\widetilde{O_j}$ intersects any other open set. This means $\widetilde{O_j}$ is dense. By Baire Category Theorem, $G := \bigcap_{j=1}^{\infty} \widetilde{O_j}$ is a dense G_δ-set and consists precisely of all points $x_o \in X$ with dense orbit. To see that, if x_0 has dense orbit, $\mathcal{O}(x_0)$ must intersect all open set $O_1, O_2, \cdots$. Then for any O_i, there is $n \in \mathbb{Z}$ such that $T^n x_0 \in O_i$. Thus $x_0 \in \widetilde{O_i}$ and then $x_0 \in G$. Conversely, for any $x_0 \in G$, $\mathcal{O}(x_0)$ intersects open balls with any small radius. Thus it is dense.

(4)⇒(1): Dense set can not be empty. □

Definition 1.9. *A homeomorphism $T\colon X \to X$ is called topological mixing if for any two non-empty open sets $U, V \subset X$, there exists N such that $T^n U \cap V \neq \varnothing$ for all $n \in \mathbb{Z}$ with $|n| > N$.*

Definition 1.10. *A homeomorphism $T\colon X \to X$ is called minimal if every orbit is dense.*

Proposition 1.11. *Let $T\colon X \to X$ be a homeomorphism of a compact metric space X. The followings are equivalent:*

1. *T is minimal;*
2. *if $E = TE \subset X$ is closed, then either $E = \varnothing$ or $E = X$;*
3. *if $U \subset X$ is open and non-empty, then $\bigcup_{n\in\mathbb{Z}} T^n U = X$.*

Proof. (1)⇒(2): Suppose $x \in E$. We have $X = \overline{\mathcal{O}(x)} \subset \overline{TE} = E$.

(2)⇒(3): Denote $\widetilde{U} = \bigcup_{n\in\mathbb{Z}} T^n U$. $\widetilde{U}$ is open and $T\widetilde{U} = \widetilde{U}$. Hence $E := X - \widetilde{U}$ is closed and $TE = E$. Then we have $E = \varnothing$.

(3)⇒(1): Let $x_0 \in X$ and $U \neq \varnothing \subset X$ be open. Then there is $n \in \mathbb{Z}$ such that $x_0 \in T^n U$. This shows that $\mathcal{O}(x_0)$ intersects any non-empty open subset of X. Hence it is dense. □

Theorem 1.12. *Let X be a compact metric space and $T\colon X \to X$ be a homeomorphism. Then there exists a closed non-empty subset $Y \subset X$ so that $Y = TY$ and $T|_Y\colon Y \to Y$ is minimal.*

Proof. Define
$$\mathcal{E} := \{Y \subset X : Y \neq \varnothing, Y \text{ closed}, TY = Y\} \ni X.$$
We want to use Zorn's Lemma under $\subset$. We need to show that any chain in $\mathcal{E}$ has a lower bound in $\mathcal{E}$. Let $(Y_\alpha : \alpha \in \mathcal{I})$ be a chain. Define $Y = \bigcap_{\alpha\in\mathcal{I}} Y_\alpha$. Clearly, $TY = Y$ and Y is closed. Note that any intersection of finite Y_α is non-empty, by compactness of X, $Y \neq \varnothing$. These show that Y is the lower bound of chain $(Y_\alpha : \alpha \in \mathcal{I})$.

By Zorn's Lemma, there is a minimal element $Y \in \mathcal{E}$. Then $T|_Y\colon Y \to Y$ is minimal by Proposition 1.11. □

Corollary 1.13. *There exists $x_0 \in X$ so that $T^{n_k} x_0 \to x_0$ as $k \to \infty$ for some $n_k \nearrow \infty$.*

Proof. Let $Y \subset X$ be as in Theorem 1.12 and $x_0 \in Y$ be arbitrary. We have $\omega^+(x_0) \subset Y$ and $\omega^+(x_0)$ is T-invariant closed set. Note that $T|_Y : Y \to Y$ is minimal, by Proposition 1.11, $\omega^+(x_0) = \varnothing$ or Y. Note again that Y is a closed subset of the compact space X, it is also compact. Hence $\omega^+(x_0) \neq \varnothing$. This shows that $x_0 \in Y = \omega^+(x_0)$. □

Definition 1.14. *$x_0 \in X$ is called recurrent for $T\colon X \to X$ if $x_0 \in \omega_T^+(x_0)$.*

Example 1.15. Let $\alpha \notin \mathbb{Q}$ and $R\colon \mathbb{T} \to \mathbb{T}$ which maps x to $x + \alpha$. Then R is minimal: by 1.13, there is $x_0 \in \mathbb{T}$ which is recurrent. Let $\varepsilon > 0$. There is $n \in \mathbb{N}$ such that $\varepsilon > d(R^n x_0, x_0) = d(x_0 + n\alpha, x_0) = d(n\alpha, 0)$. Note that $n\alpha \neq 0$ in $\mathbb{T}$. For any $x \in \mathbb{T}$, $x, x \pm n\alpha, x \pm 2n\alpha, \cdots, x \pm \left\lfloor \frac{1}{d(n\alpha,0)} \right\rfloor n\alpha \in \mathcal{O}(x)$ is ε-dense in $\mathbb{T}$. This shows every orbit is dense.

Definition 1.16. *Let $T\colon X \to X$ be a homeomorphism. We say T is expansive if there exists (the expansive constant) $\delta > 0$ so that for any $x \neq y \in X$, there exists $n \in \mathbb{Z}$ so that $d(T^n x, T^n y) \geqslant \delta$.*

Example 1.17. $T_A\colon \mathbb{T}^2 \to \mathbb{T}^2$ given by matrix multiplication defined by $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ is expansive, topological transitive, mixing and not minimal.