Dynamical Systems and Ergodic Theory

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Abstract

This is my notes of the course "Dynamical Systems and Ergodic Theory" given by Manfred Einsiedler. https://video.ethz.ch/lectures/d-math/2024/spring/401-2374-24L.html

Let X be a set and $T\colon X \to X$ be a map.

Definition 0.1. *fixed point, periodic point, period, orbit...*

Definition 0.2. *Assume X has a topology. The ω-limit of $x \in X$ is*

$$\omega^{\pm}(x) := \left\{ \lim_{k\to\infty} T^{n_k}x : n_k \nearrow \pm\infty \right\}.$$

We could also ask about the "distribution" of $x, Tx, T^2x, \cdots, T^nx$ inside X as $n \to \infty$.

More generally, a dynamical system can be defined as a group action.

Example 0.1. $X = \mathbb{R}$, $Tx = x + 1$. The ω-limits are empty set. Thus we will restrict to compact metric spaces.

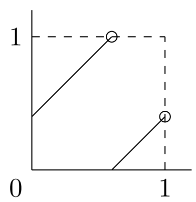
Example 0.2. $X = \mathbb{R} \cup \{\infty\}$ the one-point compactification of $\mathbb{R}$. $T(x) = \begin{cases} x+1, \ x \in \mathbb{R}; \\ \infty, \ x = \infty \end{cases}$. Then the ω-limits are all $\{\infty\}$.

Example 0.3. $X = \mathbb{R} \cup \{\pm\infty\}$. $T(x) = \begin{cases} x+1, \ x \in \mathbb{R}; \\ +\infty, \ x = +\infty; \\ -\infty, \ x = -\infty \end{cases}$. Then $\omega^+(x) = \begin{cases} +\infty, \ x \in \mathbb{R} \cup \{+\infty\}; \\ -\infty, \ x = -\infty \end{cases}$.

Example 0.4. North-South Dynamics

Example 0.5. $X = \mathbb{T} = \mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$ with the metric $d(x+\mathbb{Z}, y+\mathbb{Z}) = \min_{k\in\mathbb{Z}} |x - y + k|$. $R(x+\mathbb{Z}) := x + \alpha + \mathbb{Z}$ for a fixed $\alpha \in \mathbb{T}$. $R\colon \mathbb{T} \to \mathbb{T}$ is an isometry.

- If $\alpha = \frac{p}{q}$ is rational, then $R^q(x+\mathbb{Z}) = x + q\alpha + \mathbb{Z} = x + p + \mathbb{Z} = x + \mathbb{Z}$. Every point is periodic with period q.



- If $\alpha \notin \mathbb{Q}/\mathbb{Z}$, then no point is periodic: say $R^n(x+\mathbb{Z}) = x + \mathbb{Z}$, then $n\alpha \in \mathbb{Z}$. Actually, all orbits are dense in this case.

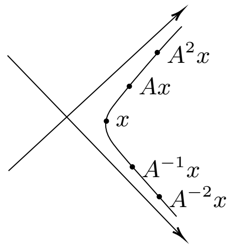
Example 0.6. $X = \mathbb{T} = \mathbb{R}/\mathbb{Z}$. Fix $p \geqslant 2 \in \mathbb{N}$. $T(x) := px$. This map links to the base-p expansion of $x \in [0,1)$. Suppose $x = \sum_{k=1}^{\infty} \theta_k p^{-k}$ where $\theta_k\{0, \cdots, p-1\}$. Then $Tx = p\sum_{k=1}^{\infty} \theta_k p^{-k} + \mathbb{Z} = \sum_{k=1}^{\infty} \theta_{k+1} p^{-k} + \mathbb{Z}$.

Claim.

- There exist lots of periodic points– they are dense.
- There exist pre-periodic points that are not periodic, where x is pre-periodic if its orbit $|O^+(x)| < \infty$.
- Many periods exist.
- The set of pre-periodic points is countable.

- There exist $x \in \mathbb{T}$ with $\omega^+(x) = \mathbb{T}$.
- There exist $x \in \mathbb{T}$ with $\omega^+(x)$ uncountable but not $\mathbb{T}$.
- There exist $x \in \mathbb{T}$ with $\omega^+(x)$ countable but not finite.

Example 0.7. $X = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. This is called a hyperbolic toral automorphism, because the orbit of any $x \neq 0 \in X$ is on a hyperbola.



Example 0.8. $X = (0,1) - \mathbb{Q}$, $Tx = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$. This relates to continued fraction expansion

$$x = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}$$

where $a_1, a_2, a_3, \cdots \in \mathbb{N}$. Note that

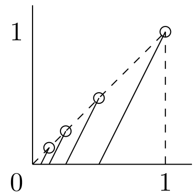
$$Tx = \cfrac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \cdots}}}.$$

Example 0.9 (Benford's law for powers of 2). Given $j \in \{1, \cdots, 9\}$, the limits

$$d_j := \lim_{N \to \infty} \frac{1}{N} \sharp \{2^n : 1 \leqslant n \leqslant N,\ 2^n \text{ starts in digital expension with } j\}$$

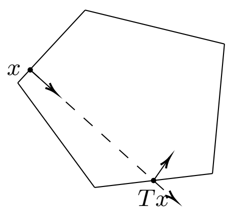
satisfy $d_1 > d_2 > \cdots > d_9 > 0$. In fact $d_1 = \log_{10} 2$.

Example 0.10. $X = [0,1]$, $T(x) = \begin{cases} 0, & x = 0, 1; \\ nx - 1, & x \in \left[\frac{1}{n}, \frac{1}{n-1}\right) \end{cases}$.



We claim that $\lim_{n\to\infty} T^n x = 0$, and if $x \in \mathbb{Q}$, there exists n with $T^n x = 0$ and if $x \notin \mathbb{Q}$, $T^n x > 0$ for all n. For $x = e$ this can be used to show that $e \notin \mathbb{Q}$.

Example 0.11 (Billiards). X is the set of boundary points with a vector and T is the movement of a boundary point along its vector to the next boundary point with a reflected vector.



Given a triangle, are there always periodic points? When the triangle is acute, right or obtuse with rational angles, the answer is yes. For the other cases, people don't know.

Example 0.12 (Geodesic flow). Given a nice manifold M and its unit tangent bundle. There exists a way of following any vector in the tangent. When M is a sphere, the orbits are great circles. When M is a torus, whether an orbit is closed depending on whether the initial vector is rational. When M is a orientable surface of genus greater than 2, the orbits will be more complicated and more sensitive to initial values.