http://algs4.cs.princeton.edu

2.1 ELEMENTARY SORTS

- *rules of the game*
- *selection sort*
- *insertion sort*
- *shellsort*
- *shuffling*

Sorting problem

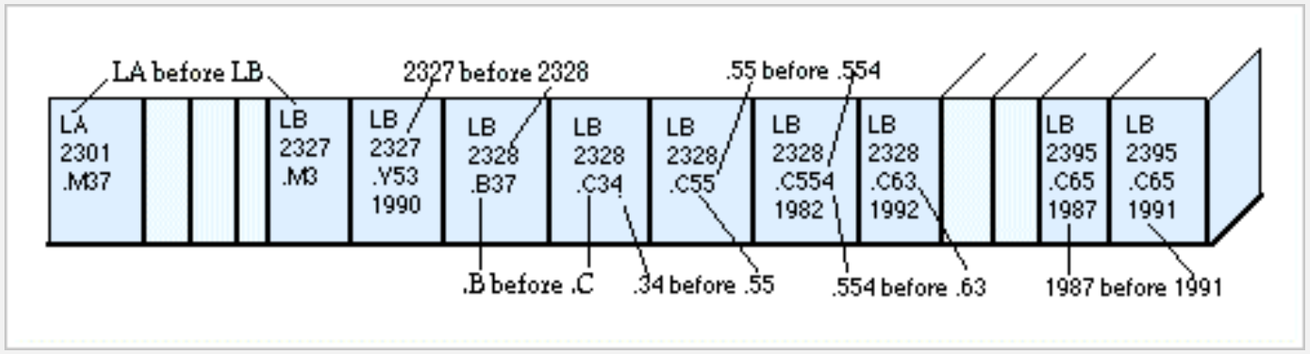
Ex. Student records in a university.

	Chen	3	A	991-878-4944	308 Blair
	Rohde	2	A	232-343-5555	343 Forbes
	Gazsi	4	B	766-093-9873	101 Brown
item →	Furia	1	A	766-093-9873	101 Brown
	Kanaga	3	B	898-122-9643	22 Brown
	Andrews	3	A	664-480-0023	097 Little
key →	Battle	4	C	874-088-1212	121 Whitman

Sort. Rearrange array of N items into ascending order.

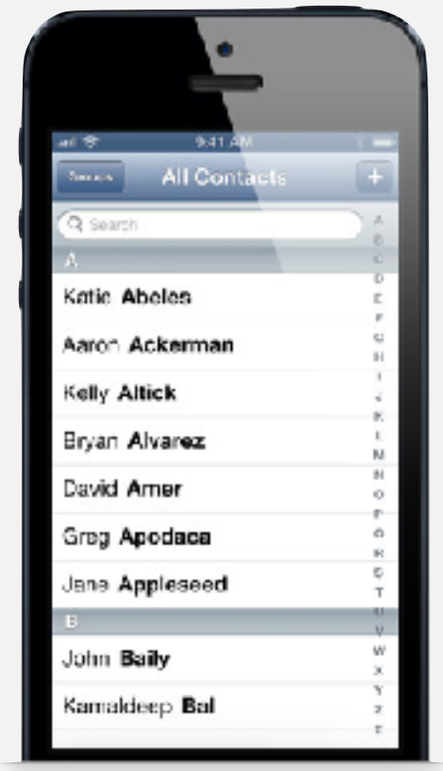
Andrews	3	A	664-480-0023	097 Little
Battle	4	C	874-088-1212	121 Whitman
Chen	3	A	991-878-4944	308 Blair
Furia	1	A	766-093-9873	101 Brown
Gazsi	4	B	766-093-9873	101 Brown
Kanaga	3	B	898-122-9643	22 Brown
Rohde	2	A	232-343-5555	343 Forbes

Sorting applications

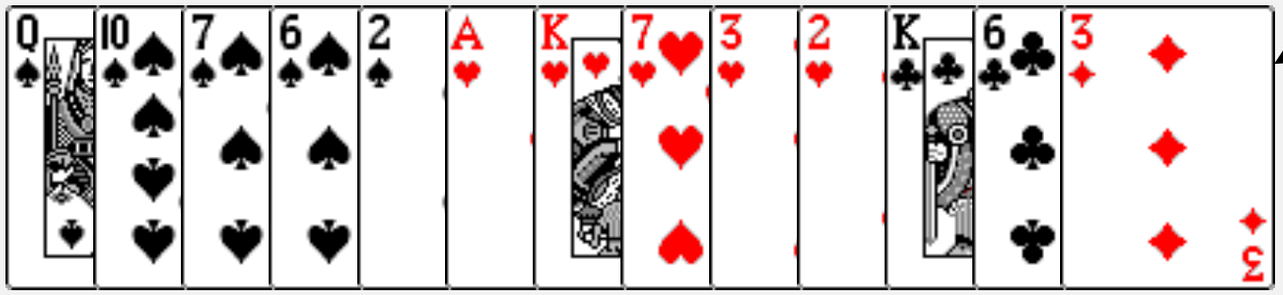


Library of Congress numbers

FedEx packages



contacts



playing cards

Please let me know if you know what this is :)

Hogwarts houses

Sample sort client 1

Goal. Sort any type of data.

Ex 1. Sort random real numbers in ascending order.

seems artificial (stay tuned for an application)

```
public class Experiment
{
   public static void main(String[] args)
   {
      int N = Integer.parseInt(args[0]);
      Double[] a = new Double[N];
      for (int i = 0; i < N; i++)
         a[i] = StdRandom.uniform();
      Insertion.sort(a);
      for (int i = 0; i < N; i++)
         StdOut.println(a[i]);
   }
}
```

```
% java Experiment 10
0.08614716385210452
0.09054270895414829
0.10708746304898642
0.21166190071646818
0.363292849257276
0.460954145685913
0.5340026311350087
0.7216129793703496
0.9003500354411443
0.9293994908845686
```

Sample sort client 2

Goal. Sort any type of data.

Ex 2. Sort strings in alphabetical order.

```
public class StringSorter
{
   public static void main(String[] args)
   {
      String[] a = StdIn.readAllStrings();
      Insertion.sort(a);
      for (int i = 0; i < a.length; i++)
         StdOut.println(a[i]);
   }
}
```

```
% more words3.txt
bed bug dad yet zoo ... all bad yes

% java StringSorter < words3.txt
all bad bed bug dad ... yes yet zoo
[suppressing newlines]
```

Sample sort client 3

Goal. Sort any type of data.

Ex 3. Sort the files in a given directory by filename.

```
import java.io.File;


public class FileSorter
{
   public static void main(String[] args)
   {
      File directory = new File(args[0]);
      File[] files = directory.listFiles();
      Insertion.sort(files);
      for (int i = 0; i < files.length; i++)
         StdOut.println(files[i].getName());
   }
}
```

```
% java FileSorter .
Insertion.class
Insertion.java
InsertionX.class
InsertionX.java
Selection.class
Selection.java
Shell.class
Shell.java
ShellX.class
ShellX.java
```

Total order

Goal. Sort any type of data (for which sorting is well defined).

A total order is a binary relation $\leq$ that satisfies:

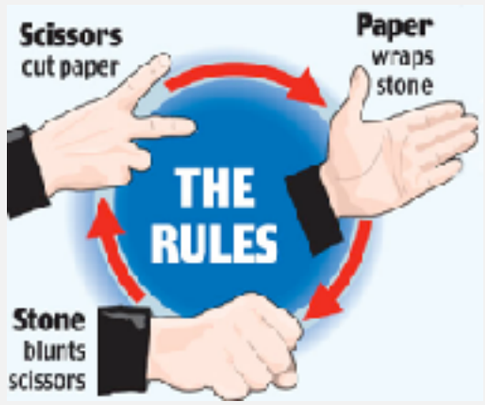
- Antisymmetry: if both $v \leq w$ and $w \leq v$, then $v = w$.
- Transitivity: if both $v \leq w$ and $w \leq x$, then $v \leq x$.
- Totality: either $v \leq w$ or $w \leq v$ or both.

Ex.

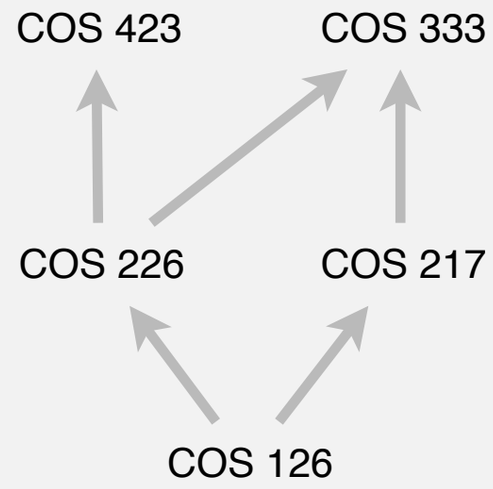
- Standard order for natural and real numbers.
- Chronological order for dates or times.
- Alphabetical order for strings.

No transitivity. Rock-paper-scissors.

No totality. PU course prerequisites.



violates transitivity



violates totality

Callbacks and/or higher-order functions.

Goal. Sort any type of data (for which sorting is well defined).

Q. How can sort() know how to compare data of type Double, String, and java.io.File without any information about the type of an item's key?

Callback: reference to executable code via interface (Java)

- Client passes array of objects to sort() function.
- The sort() function calls object's class' compareTo() method as needed.

Higher-order function: pass comparison function into sort method

Implementing callbacks.

- Java (interfaces), C: function pointers, C++: class-type functors, C#: delegates.

Implementing higher-order functions.

- Java8, Scala (may be *implicit*), Python, Perl, ML, Javascript.

Callbacks: roadmap

client

```
public class StringSorter
{
   public static void main(String[] args)
   {
      String[] a = StdIn.readAllStrings();
      Insertion.sort(a);
      for (int i = 0; i < a.length; i++)
         StdOut.println(a[i]);
   }
}
```

data-type implementation

```
public class String
implements Comparable<String>
{
   ...
   public int compareTo(String b)
   {
      ...
      return -1;
      ...
      return +1;
      ...
      return 0;
   }
}
```

Comparable interface (built in to Java)

```
public interface Comparable<Item>
{
   public int compareTo(Item that);
}
```

sort implementation

```
public static void sort(Comparable[] a)
{
   int N = a.length;
   for (int i = 0; i < N; i++)
      for (int j = i; j > 0; j--)
         if (a[j].compareTo(a[j-1]) < 0)
            swap(a, j, j-1);
         else break;
}
```

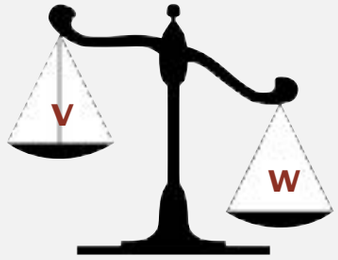
key point: no dependence on String data type

Comparable API

Implement compareTo() so that v.compareTo(w)

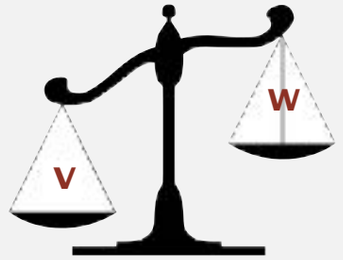
- Defines a total order.
- Returns a negative integer, zero, or positive integer

  if v is less than, equal to, or greater than w, respectively.
- Throws an exception if incompatible types (or either is null).



less than (return -1)

equal to (return 0)



greater than (return +1)

Built-in comparable types. Integer, Double, String, Date, File, ...

User-defined comparable types. Implement the Comparable interface.

Implementing the Comparable interface

MyDate data type. Simplified version of java.util.Date.

```
public class MyDate implements Comparable<MyDate> {

    public MyDate(int year, int month, int day) {
        this.year = year;
        this.month = month;
        this.day = day;
    }

   public int compareTo(MyDate that) {
        int cfy = Integer.compare(this.year, that.year);
        if (cfy!=0) return cfy;
        int cfm = Integer.compare(this.month, that.month);
        if (cfm!=0) return cfm;
        int cfd = Integer.compare(this.day, that.day);
        return cfd;
    }

    private final int year;
    private final int month;
    private final int day;
}
```

only compare dates to other dates

don't need @override annotation

An intransitive order

Q. Why doesn't the following compareTo() implement a total order?

```
public class Double implements Comparable<Double>
{
   private double x;


   …


   public int compareTo(Double that) {
      if      (this.x < that.x) return -1;
      else if (this.x > that.x) return +1;
      else                      return  0;
   }
}
```

A. Not transitive! Need to properly handle -0.0 vs. 0.0 and Double.NaN. That's because, although with (double) primitives, -0.0 = +0.0, that is not true for Double objects. In that case -0.0 < +0.0.

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2.1 Elementary Sorts

- *rules of the game*
- *selection sort*
- *insertion sort*
- *shellsort*
- *shuffling*

The four possibilities (review)

	Work then solve	Solve then work	Growth (~)
Equi-partition	Quick sort	Merge sort	N log N
Head-tail partition	Selection Sort	Insertion Sort*	$N^2/2$

* The number of cams for Insertion Sort is *min(N+X, $N^2/2$)* where *X* is # of inversions

Selection sort

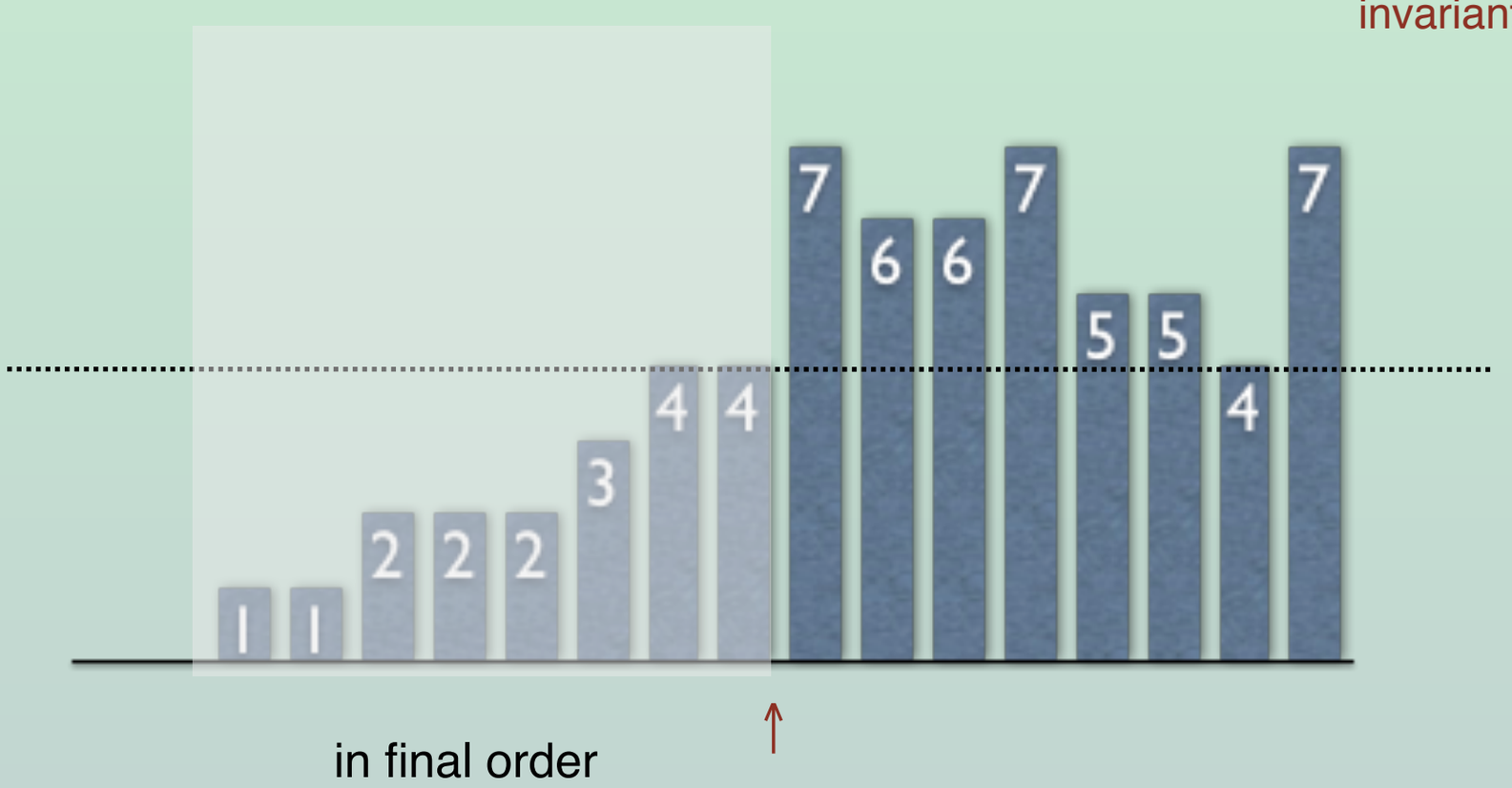
Algorithm. ↑ scans from left to right (iterate on *i* from 0…*N*-1)

- Iterate on *j* from *i*+1…*N*-1 to find smallest item.

Invariants (applied after each iteration completes)

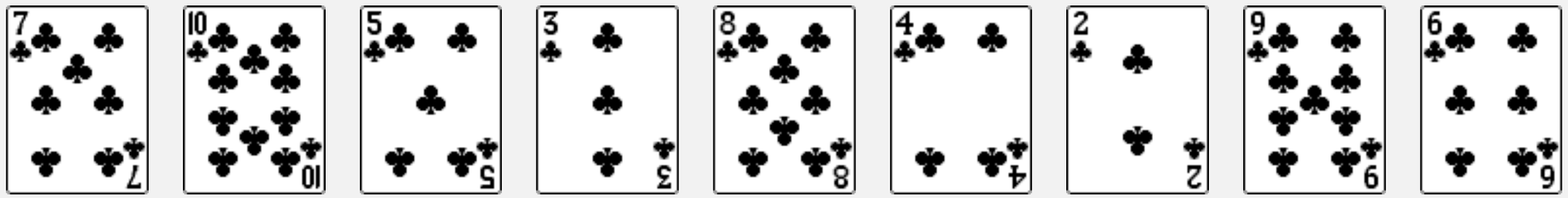
- Entries to the left of ↑ are fixed and in ascending order.
- No entry to right of ↑ is smaller than any entry to the left of ↑.

We can program these invariants as assertions in Java



Selection sort demo

- In iteration i, find index min of smallest remaining entry.
- Swap a[i] and a[min].



initial

Two useful sorting abstractions

Helper functions. Refer to data through compares and exchanges.

Less. Is item v less than w ?

```
private static boolean less(Comparable v, Comparable w)
{  return v.compareTo(w) < 0;  }
```

Exchange. Swap item in array a[] at index i with the one at index j.

```
private static void swap(Comparable[] a, int i, int j)
{
   Comparable swap = a[i];
   a[i] = a[j];
   a[j] = swap;
}
```

Selection sort inner loop

To maintain algorithm invariants:

- Move the pointer to the right.

```
i++;
```

- Identify index of minimum entry on right.

```
int min = i;
for (int j = i+1; j < N; j++)
   if (less(a[j], a[min]))
      min = j;
```

- Exchange into position.

```
swap(a, i, min);
```

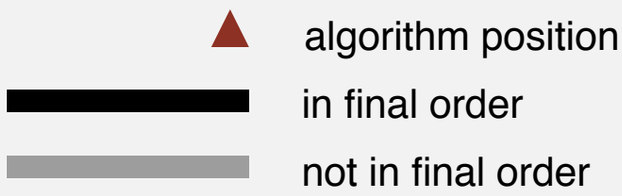
Selection sort: Java implementation

```
public class Selection
{
   public static void sort(Comparable[] a)
   {
      int N = a.length;
      for (int i = 0; i < N; i++)
      {
         int min = i;
         for (int j = i+1; j < N; j++)
            if (less(a[j], a[min]))
               min = j;
         swap(a, i, min);
      }
   }

   private static boolean less(Comparable v, Comparable w)
   {  /* as before */  }

   private static void swap(Comparable[] a, int i, int j)
   {  /* as before */  }
}
```

Selection sort: animations

20 random items

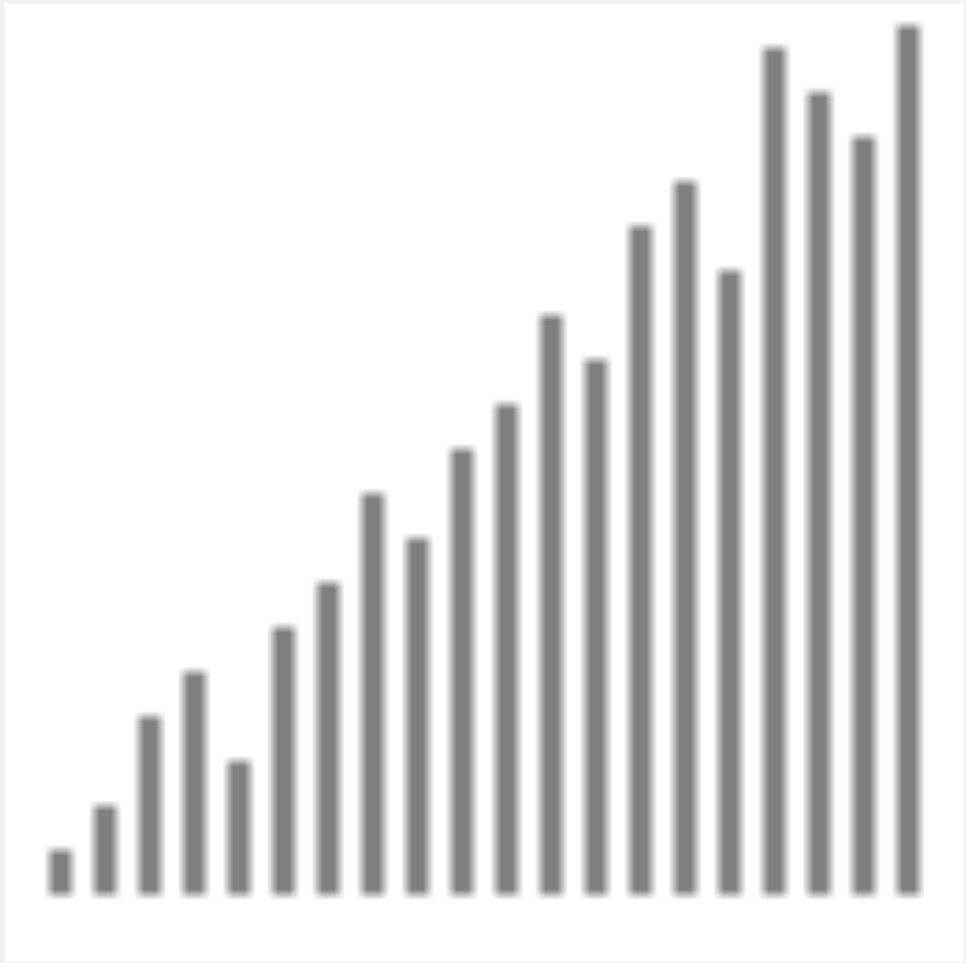
11.2 secs



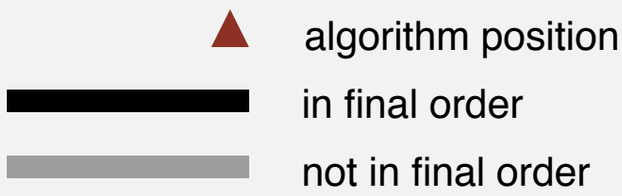
http://www.sorting-algorithms.com/selection-sort

Selection sort: animations

20 partially-sorted items



11.0 secs



http://www.sorting-algorithms.com/selection-sort

Selection sort: mathematical analysis

Proposition. Selection sort uses $(N-1) + (N-2) + \ ... \ + 1 + 0 \ \sim \ N^2 / 2$ compares and N exchanges.

		a[]										
i	min	0	1	2	3	4	5	6	7	8	9	10
		S	O	R	T	E	X	A	M	P	L	E
0	6	S	O	R	T	E	X	A	M	P	L	E
1	4	A	O	R	T	E	X	S	M	P	L	E
2	10	A	E	R	T	O	X	S	M	P	L	E
3	9	A	E	E	T	O	X	S	M	P	L	R
4	7	A	E	E	L	O	X	S	M	P	T	R
5	7	A	E	E	L	M	X	S	O	P	T	R
6	8	A	E	E	L	M	O	S	X	P	T	R
7	10	A	E	E	L	M	O	P	X	S	T	R
8	8	A	E	E	L	M	O	P	R	S	T	X
9	9	A	E	E	L	M	O	P	R	S	T	X
10	10	A	E	E	L	M	O	P	R	S	T	X
		A	E	E	L	M	O	P	R	S	T	X

entries in black are examined to find the minimum

entries in red are `a[min]`

entries in gray are in final position

Trace of selection sort (array contents just after each exchange)

Running time insensitive to input. Quadratic time, even if input is sorted.

Data movement is minimal. Linear number of exchanges.

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2.1 Elementary Sorts

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* The number of cams for Insertion Sort is *min(N+X, $N^2/2$)* where *X* is # of inversions

Insertion sort

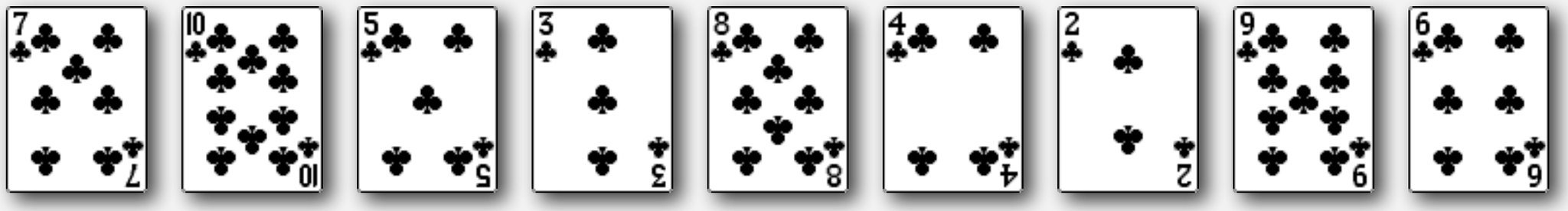
Algorithm. ↑ scans from left to right.

Invariants.

- Entries to the left of ↑ are in ascending order.
- Entries to the right of ↑ have not yet been seen.

Insertion sort demo

- In iteration i, swap a[i] with each larger entry to its left.

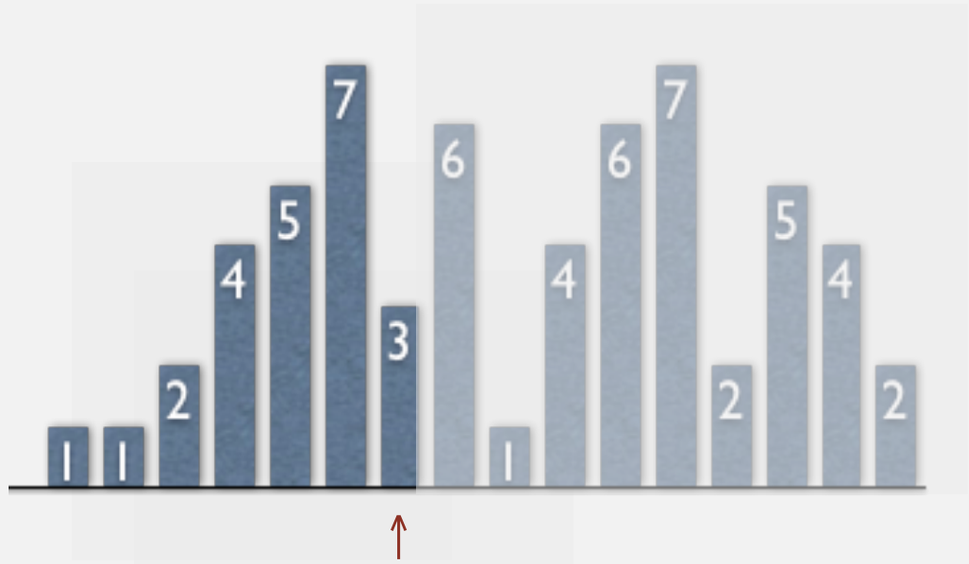


Insertion sort inner loop

To maintain algorithm invariants:

- Move the pointer to the right.

```
i++;
```



- Moving from right to left, exchange a[i] with each larger entry to its left.

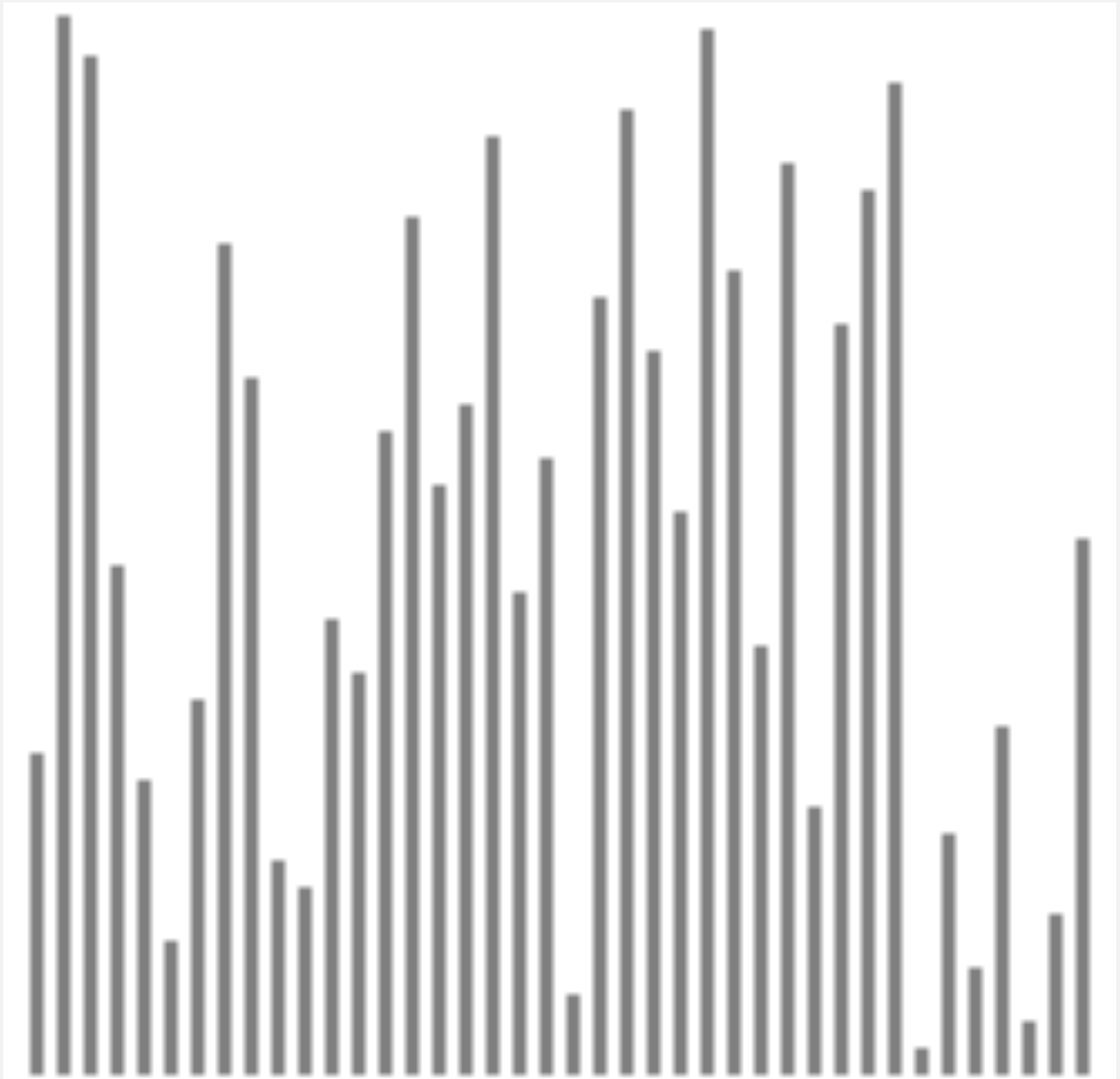
```
for (int j = i; j > 0; j--)
   if (less(a[j], a[j-1]))
         swap(a, j, j-1);
   else break;
```

Insertion sort: Java implementation

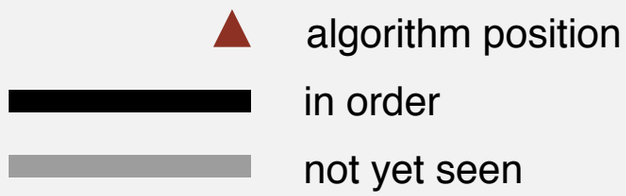
```
public class Insertion
{
   public static void sort(Comparable[] a)
   {
      int N = a.length;
      for (int i = 0; i < N; i++)
         for (int j = i; j > 0; j--)
            if (less(a[j], a[j-1]))
               swap(a, j, j-1);
            else break;
   }

   private static boolean less(Comparable v, Comparable w)
   {  /* as before */  }

   private static void swap(Comparable[] a, int i, int j)
   {  /* as before */  }
}
```

Insertion sort: animation

40 random items



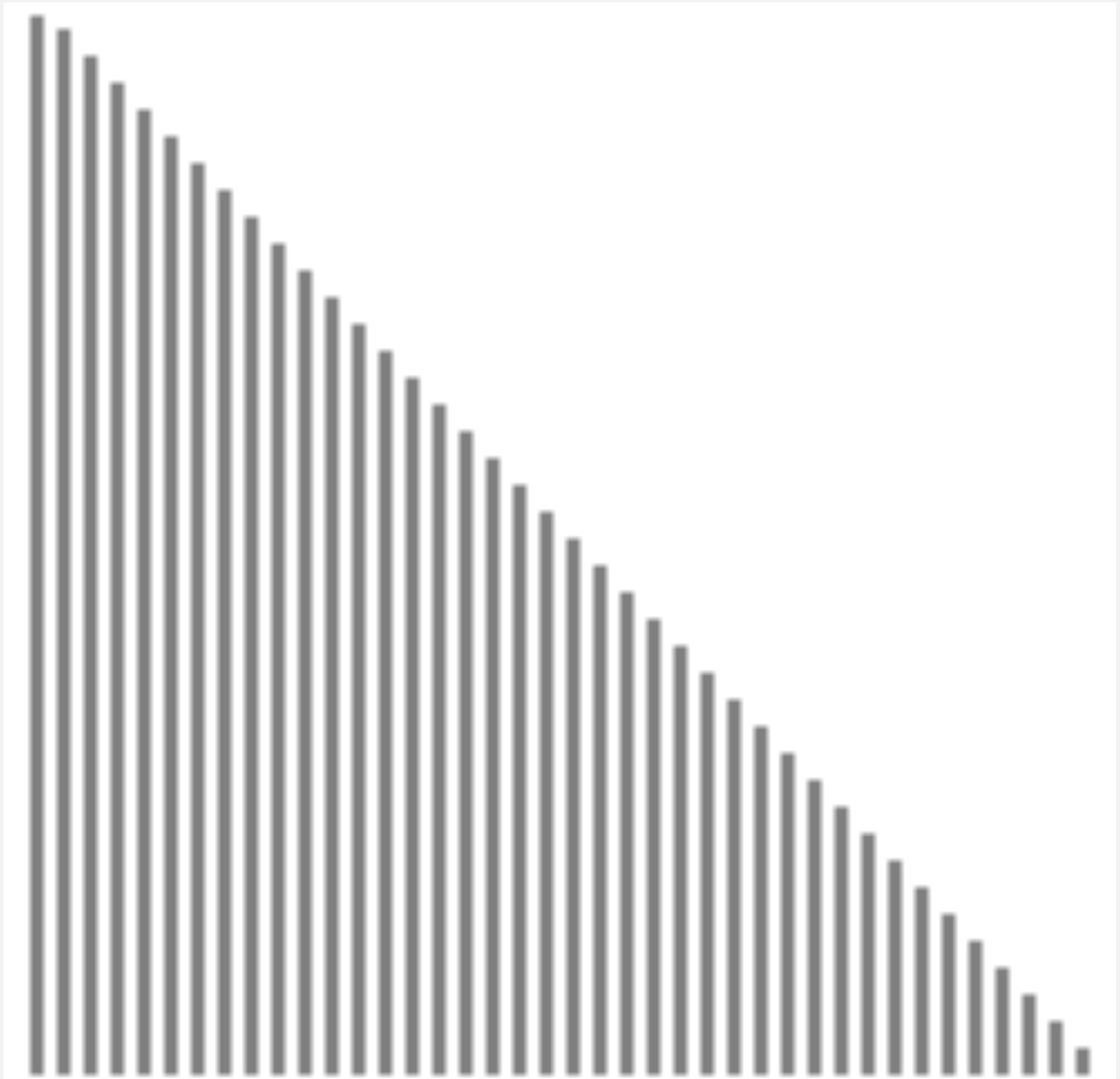
23.2 secs



http://www.sorting-algorithms.com/insertion-sort

Insertion sort: animation

40 reverse-sorted items



39.5 secs

algorithm position

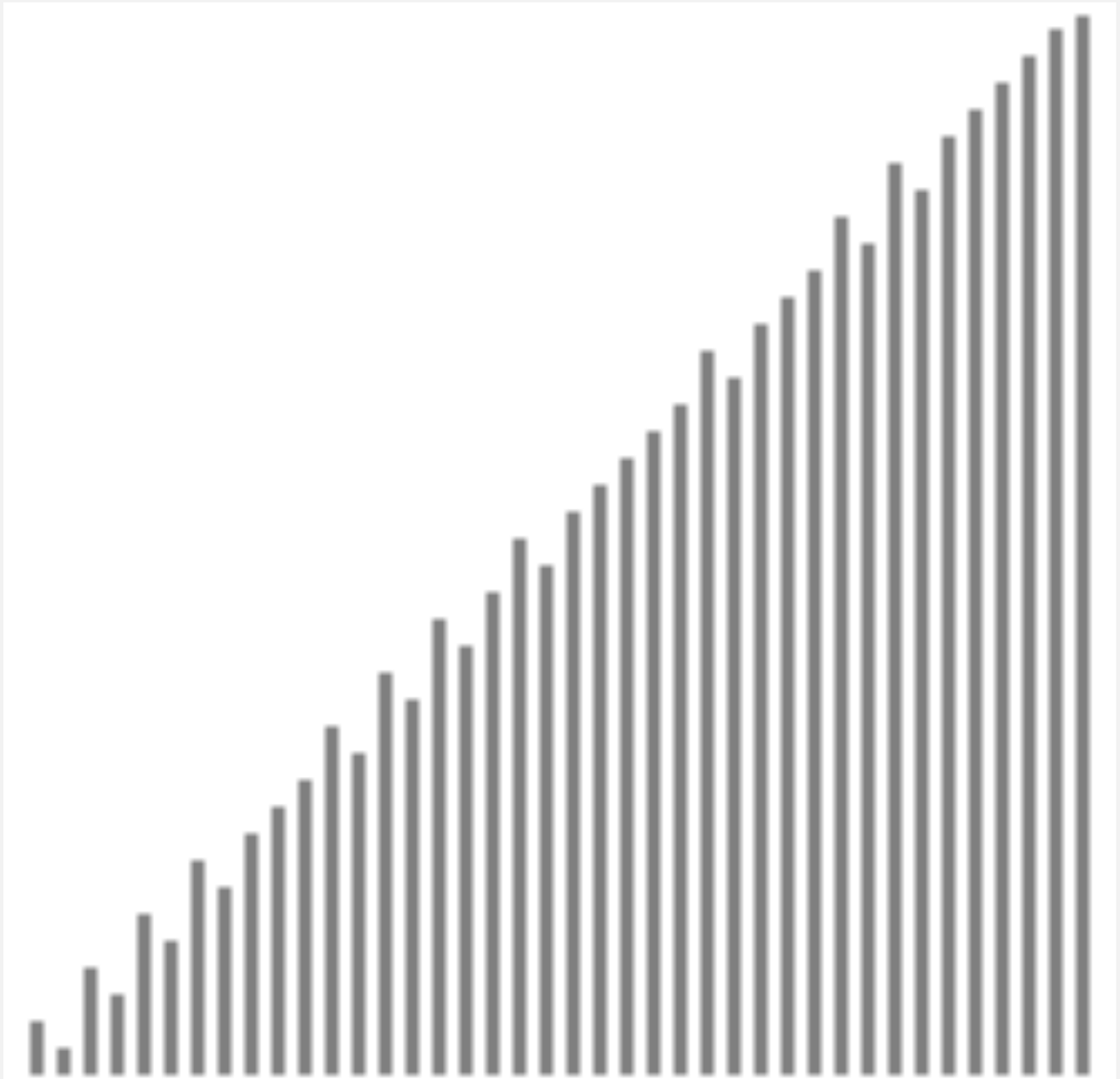
in order

not yet seen

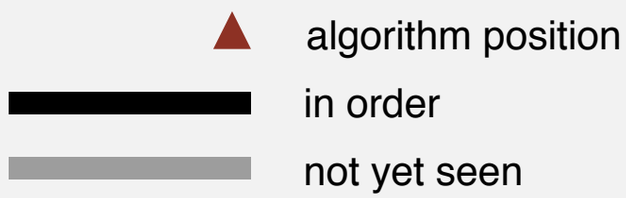
http://www.sorting-algorithms.com/insertion-sort

Insertion sort: animation

40 partially-sorted items



2.8 secs



http://www.sorting-algorithms.com/insertion-sort

Insertion sort: mathematical analysis

Proposition. To sort a randomly-ordered array with distinct keys, insertion sort uses $\sim \frac{1}{4} N^2$ compares and $\sim \frac{1}{4} N^2$ exchanges on average.

Pf. Expect each entry to move halfway back.

		a[]										
i	j	0	1	2	3	4	5	6	7	8	9	10
		S	O	R	T	E	X	A	M	P	L	E
1	0	O	S	R	T	E	X	A	M	P	L	E
2	1	O	R	S	T	E	X	A	M	P	L	E
3	3	O	R	S	T	E	X	A	M	P	L	E
4	0	E	O	R	S	T	X	A	M	P	L	E
5	5	E	O	R	S	T	X	A	M	P	L	E
6	0	A	E	O	R	S	T	X	M	P	L	E
7	2	A	E	M	O	R	S	T	X	P	L	E
8	4	A	E	M	O	P	R	S	T	X	L	E
9	2	A	E	L	M	O	P	R	S	T	X	E
10	2	A	E	E	L	M	O	P	R	S	T	X
		A	E	E	L	M	O	P	R	S	T	X

entries in gray do not move

entry in red is `a[j]`

entries in black moved one position right for insertion

Trace of insertion sort (array contents just after each insertion)

Insertion sort: trace

a[ ]

i	j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
		A	S	O	M	E	W	H	A	T	L	O	N	C	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E
0	0	A	S	O	M	E	W	H	A	T	L	O	N	C	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E
1	1	A	S	O	M	E	W	H	A	T	L	O	N	C	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E
2	1	A	O	S	M	E	W	H	A	T	L	O	N	C	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E
3	1	A	M	O	S	E	W	H	A	T	L	O	N	C	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E
4	1	A	E	M	O	S	W	H	A	T	L	O	N	C	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E
5	5	A	E	M	O	S	W	H	A	T	L	O	N	C	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E
6	2	A	E	H	M	O	S	W	A	T	L	O	N	C	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E
7	1	A	A	E	H	M	O	S	W	T	I	O	N	C	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	I	E
8	7	A	A	E	H	M	O	S	T	W	I	O	N	C	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	I	E
9	4	A	A	E	H	L	M	O	S	T	W	O	N	C	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E
10	7	A	A	E	H	I	M	O	O	S	T	W	N	C	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	I	E
11	6	A	A	E	H	L	M	N	O	O	S	T	W	C	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E
12	3	A	A	E	G	H	I	M	N	O	O	S	T	W	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	I	E
13	3	A	A	E	E	G	H	L	M	N	O	O	S	T	W	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E
14	11	A	A	E	E	G	H	I	M	N	O	O	R	S	T	W	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	I	E
15	6	A	A	E	E	G	H	I	L	M	N	O	O	R	S	T	W	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E
16	10	A	A	E	E	G	H	I	L	M	N	N	O	O	R	S	T	W	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E
17	15	A	A	E	E	G	H	I	L	M	N	N	O	O	R	S	S	T	W	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E
18	4	A	A	E	E	E	G	H	I	L	M	N	N	O	O	R	S	S	T	W	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E
19	15	A	A	E	E	E	G	H	I	L	M	N	N	O	O	R	R	S	S	T	W	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E
20	19	A	A	E	E	E	G	H	I	L	M	N	N	O	O	R	R	S	S	T	T	W	I	O	N	S	O	R	T	E	X	A	M	P	L	E
21	8	A	A	E	E	E	G	H	I	I	L	M	N	N	O	O	R	R	S	S	T	T	W	O	N	S	O	R	T	E	X	A	M	P	L	E
22	15	A	A	E	E	E	G	H	I	I	L	M	N	N	O	O	O	R	R	S	S	T	T	W	N	S	O	R	T	E	X	A	M	P	L	E
23	13	A	A	E	E	E	G	H	I	I	L	M	N	N	N	O	O	O	R	R	S	S	T	T	W	S	O	R	T	E	X	A	M	P	L	E
24	21	A	A	E	E	E	G	H	I	I	L	M	N	N	N	O	O	O	R	R	S	S	S	T	T	W	O	R	T	E	X	A	M	P	L	E
25	17	A	A	E	E	E	G	H	I	I	I	M	N	N	N	O	O	O	O	R	R	S	S	S	T	T	W	R	T	E	X	A	M	P	I	E
26	20	A	A	E	E	E	G	H	I	I	L	M	N	N	N	O	O	O	O	R	R	R	S	S	S	T	T	W	T	E	X	A	M	P	L	E
27	26	A	A	E	E	E	G	H	I	I	I	M	N	N	N	O	O	O	O	R	R	R	S	S	S	T	T	T	W	E	X	A	M	P	I	E
28	5	A	A	E	E	E	E	G	H	I	I	L	M	N	N	N	O	O	O	O	R	R	R	S	S	S	T	T	T	W	X	A	M	P	L	E
29	29	A	A	E	E	E	E	G	H	I	I	I	M	N	N	N	O	O	O	O	R	R	R	S	S	S	T	T	T	W	X	A	M	P	I	E
30	2	A	A	A	E	E	E	E	G	H	I	I	L	M	N	N	N	O	O	O	O	R	R	R	S	S	S	T	T	T	W	X	M	P	L	E
31	13	A	A	A	E	E	E	E	G	H	I	I	I	M	M	N	N	N	O	O	O	O	R	R	R	S	S	S	T	T	T	W	X	P	I	E
32	21	A	A	A	E	E	E	E	G	H	I	I	L	M	M	N	N	N	O	O	O	O	P	R	R	R	S	S	S	T	T	T	W	X	L	E
33	12	A	A	A	E	E	E	E	G	H	I	I	L	L	M	M	N	N	N	O	O	O	O	P	R	R	R	S	S	S	T	T	T	W	X	E
34	7	A	A	A	E	E	E	E	E	G	H	I	I	L	L	M	M	N	N	N	O	O	O	O	P	R	R	R	S	S	S	T	T	T	W	X
		A	A	A	E	E	E	E	E	G	H	I	I	L	L	M	M	N	N	N	O	O	O	O	P	R	R	R	S	S	S	T	T	T	W	X

Insertion sort: analysis

Best case. If the array is in ascending order, insertion sort makes $N-1$ compares and 0 exchanges.

A E E L M O P R S T X

Worst case. If the array is in descending order (and no duplicates), insertion sort makes $\sim \frac{1}{2} N^2$ compares and $\sim \frac{1}{2} N^2$ exchanges.

X T S R P O M L F E A

Insertion sort: partially-sorted arrays

Def. An inversion is a pair of keys that are out of order.

A E E L M O T R X P S

T-R T-P T-S R-P X-P X-S

(6 inversions)

Def. An array is partially sorted if the number of inversions is $\leq c\,N$.

- Ex 1. A sorted array has 0 inversions.
- Ex 2. A subarray of size 10 appended to a sorted subarray of size N.

Proposition. For partially-sorted arrays, insertion sort runs in linear time.

Pf. Number of exchanges equals the number of inversions (X).

Note: Number of compares equals X + (N-1)

Insertion sort: practical improvements

Half exchanges. Shift items over (instead of exchanging).

- Eliminates unnecessary data movement.
- No longer uses only less() and exch() to access data.

A C H H I M N N P Q X Y K B I N A R Y

Binary insertion sort. Use binary search to find insertion point.

- Number of compares $\sim N \lg N$.
- But still a quadratic number of array accesses.

Insertion sort: practical improvements

Half exchanges. Shift items over (instead of exchanging).

- Eliminates unnecessary data movement.
- No longer uses only less() and swap() to access data.

A C H H I M N N P Q X Y K B I N A R Y

```
X x = xs[i];
int j = i;
while (j > 0 && less(x, xs[j - 1])) {
   xs[j] = xs[j - 1];
   j--;
}
xs[j] = x;
```

Binary insertion sort. Use binary search to find insertion point.

- Number of compares $\sim N \lg N$.
- But still a quadratic number of array accesses.

A C H H I M N N P Q X Y K B I N A R Y

binary search for first key > K

Comparison of simple sorts

Algorithm applied to random data	Comparisons	Swaps	Copies	Total Array Accesses (primitives)
Bubble*	$N^2/2$	$N^2/4$		$2\ N^2$
Selection	$N^2/2$	N		N^2
Insertion*	$N^2/4$	$N^2/4$		$3/2\ N^2$
Insertion with half-swaps	$N^2/4$		$N^2/4$	N^2
Insertion with binary search	N lg N		$N^2/4$	$1/2\ N^2$

* Bubble and Insertion sorts are O(N) when the data is already sorted

Link to sorting animations: https://www.toptal.com/developers/sorting-algorithms

http://algs4.cs.princeton.edu

2.1 Elementary Sorts

- *rules of the game*
- *selection sort*
- *insertion sort*
- *shellsort*
- *shuffling*

Shell sort

- Funny, I always assumed that the term “shell” had something to do with the way the algorithm works. No, it simply refers to the name of the originator, D. L. Shell, General Electric Company, Cincinnati, Ohio. (1959)
- Before we jump into shellsort, let’s think back to insertion sort. What, if anything, seems inefficient?
- Well, continuously swapping with the element to the left until our new element is in the right place.
- OK, but we kind of fixed that by using binary search on the sorted elements and then we tried a block move of all elements that need to move one to the right.
- And what caused insertion sort to be really efficient?
- Having the elements already in order, or close to being in order.
- What if we could try to get things closer to being in their proper order, distant swaps (fix many possible inversions), before starting on the insertion sort. This is the essence of shellsort.

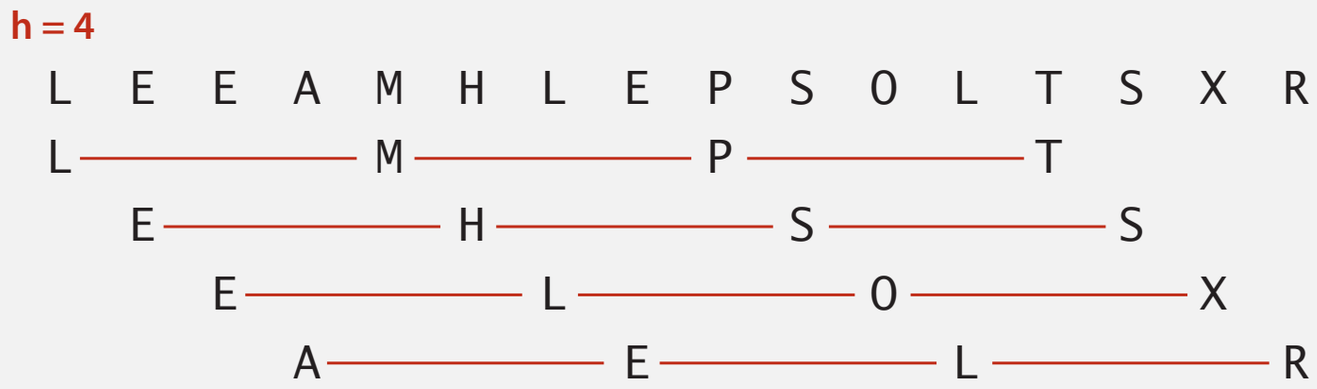
Let's take another look at insertion sort and generalize it

- I'm going to introduce a new type of insertion sort called an "h-sort."
- It works the same as insertion sort, except that it only looks at each h^{th} element of the array at a time.
- So, if *N* is, say, 31 and *h* is 3, we would sort the 1^{st}, 4^{th}, 7^{th}, … 28^{th}, 31^{st} elements as if the others didn't exist.
- Then, we'd sort the 2^{nd}, 5^{th}, 8^{th}, … 29^{th} elements as if the others didn't exist.
- Then, we'd sort the 3^{rd}, 6^{th}, 9^{th}, … 30^{th} elements as if the others didn't exist.
- Or, we can interleave these sorts.

Shellsort overview

Idea. Move entries more than one position at a time by h-sorting the array.

an h-sorted array is h interleaved sorted subsequences

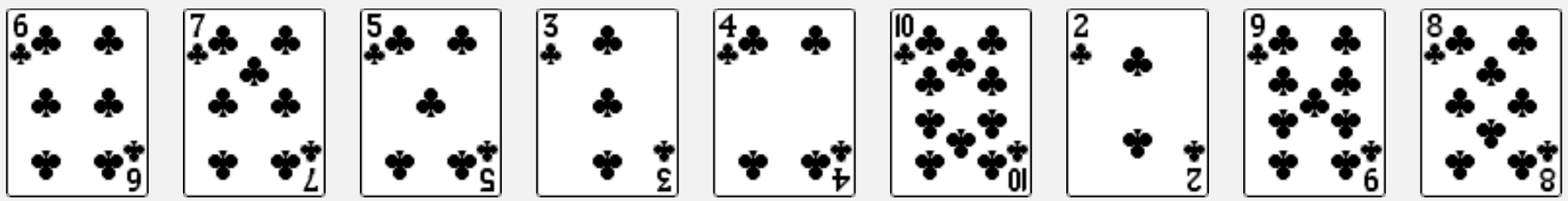


Shellsort. [Shell 1959] h-sort array for decreasing sequence of values of h.

input	S	H	E	L	L	S	O	R	T	E	X	A	M	P	L	E
13-sort	P	H	E	L	L	S	O	R	T	E	X	A	M	S	L	E
4-sort	L	E	E	A	M	H	L	E	P	S	O	L	T	S	X	R
1-sort	A	E	E	E	H	L	L	L	M	O	P	R	S	S	T	X

h-sorting demo

In iteration i, swap a[i] with each larger entry h positions to its left.



Shellsort example: increments 7, 3, 1

input

S O R T E X A M P L E

7-sort

S O R T E X A M P L E
M O R T E X A S P L E
M O R T E X A S P L E
M O L T E X A S P R E
M O L E E X A S P R T

3-sort

M O L E E X A S P R T
E O L M E X A S P R T
E E L M O X A S P R T
E E L M O X A S P R T
A E L E O X M S P R T
A E L E O X M S P R T
A E L E O P M S X R T
A E L E O P M S X R T
A E L E O P M S X R T

1-sort

A E L E O P M S X R T
A E L E O P M S X R T
A E L E O P M S X R T
A E E L O P M S X R T
A E E L O P M S X R T
A E E L M O P S X R T
A E E L M O P S X R T
A E E L M O P R S X T
A E E L M O P R S T X

result

A E E L M O P R S T X

Shellsort: Java implementation

```
public class Shell
{
   public static void sort(Comparable[] a)
   {
      int N = a.length;

      int h = 1;
      while (h < N/3) h = 3*h + 1; // 1, 4, 13, 40, 121, 364, ...

      while (h >= 1)
      {  // h-sort the array.
         for (int i = h; i < N; i++)
         {
            for (int j = i; j >= h && less(a[j], a[j-h]); j -= h)
               swap(a, j, j-h);
         }

         h = h/3;
      }
   }

   private static boolean less(Comparable v, Comparable w)
   { /* as before */ }
```

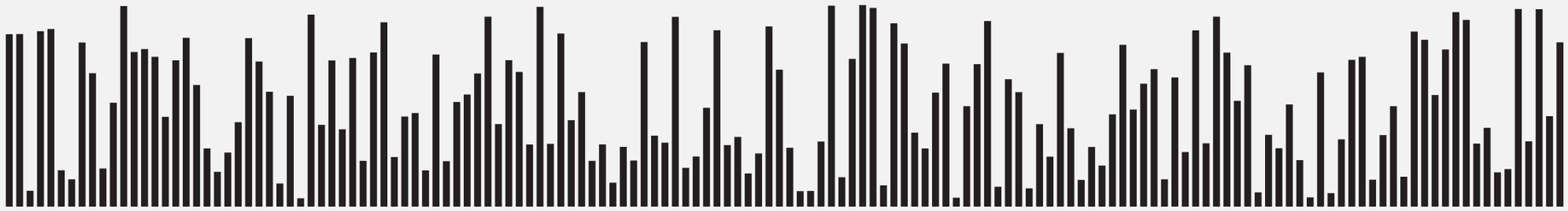
3x+1 increment sequence

insertion sort

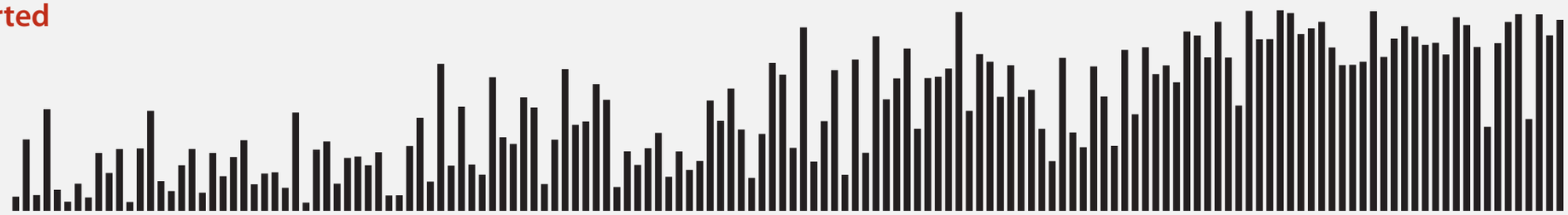
move to next increment

Shellsort: visual trace

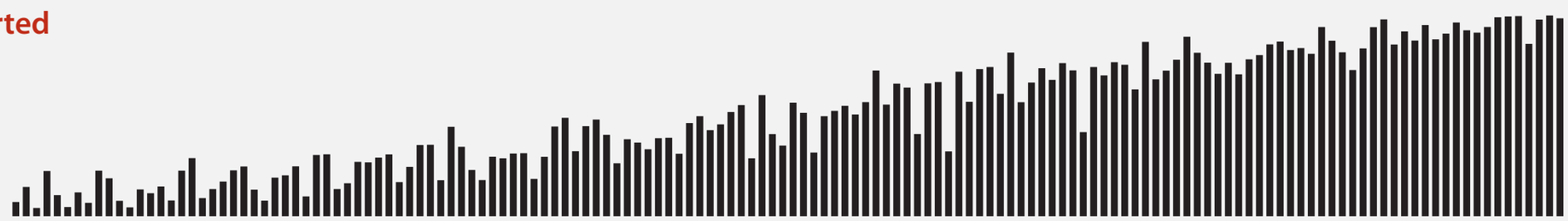
input



40-sorted



13-sorted

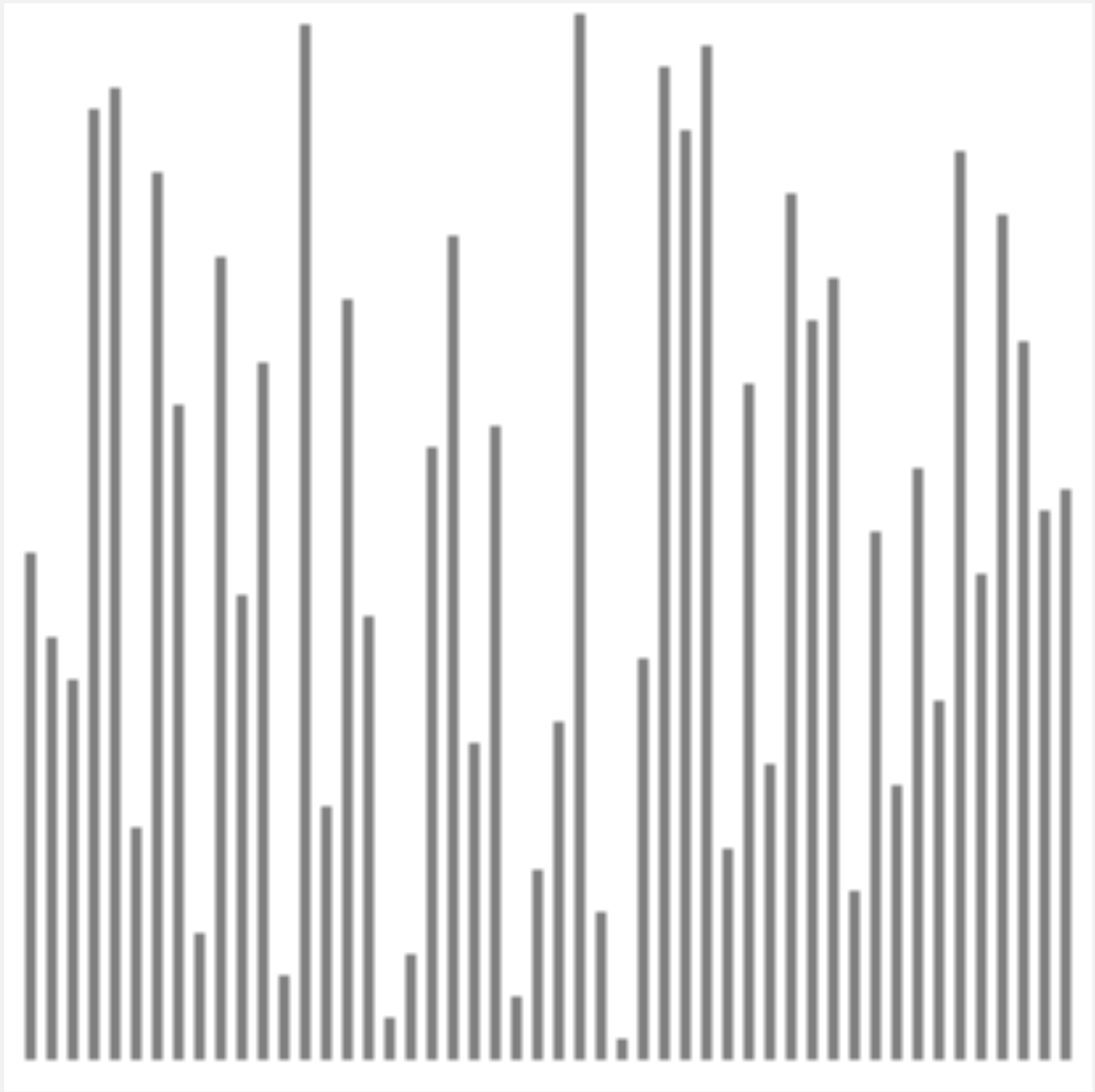


4-sorted

result

Shellsort: animation

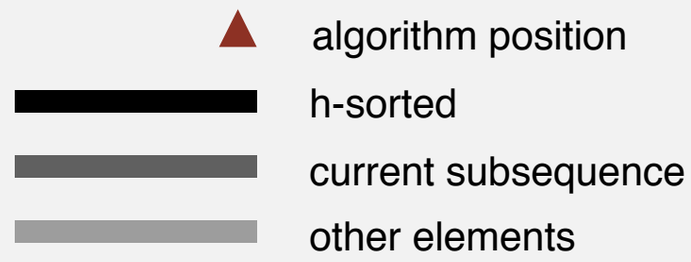
50 random items



13.9 secs

Strides (gaps):

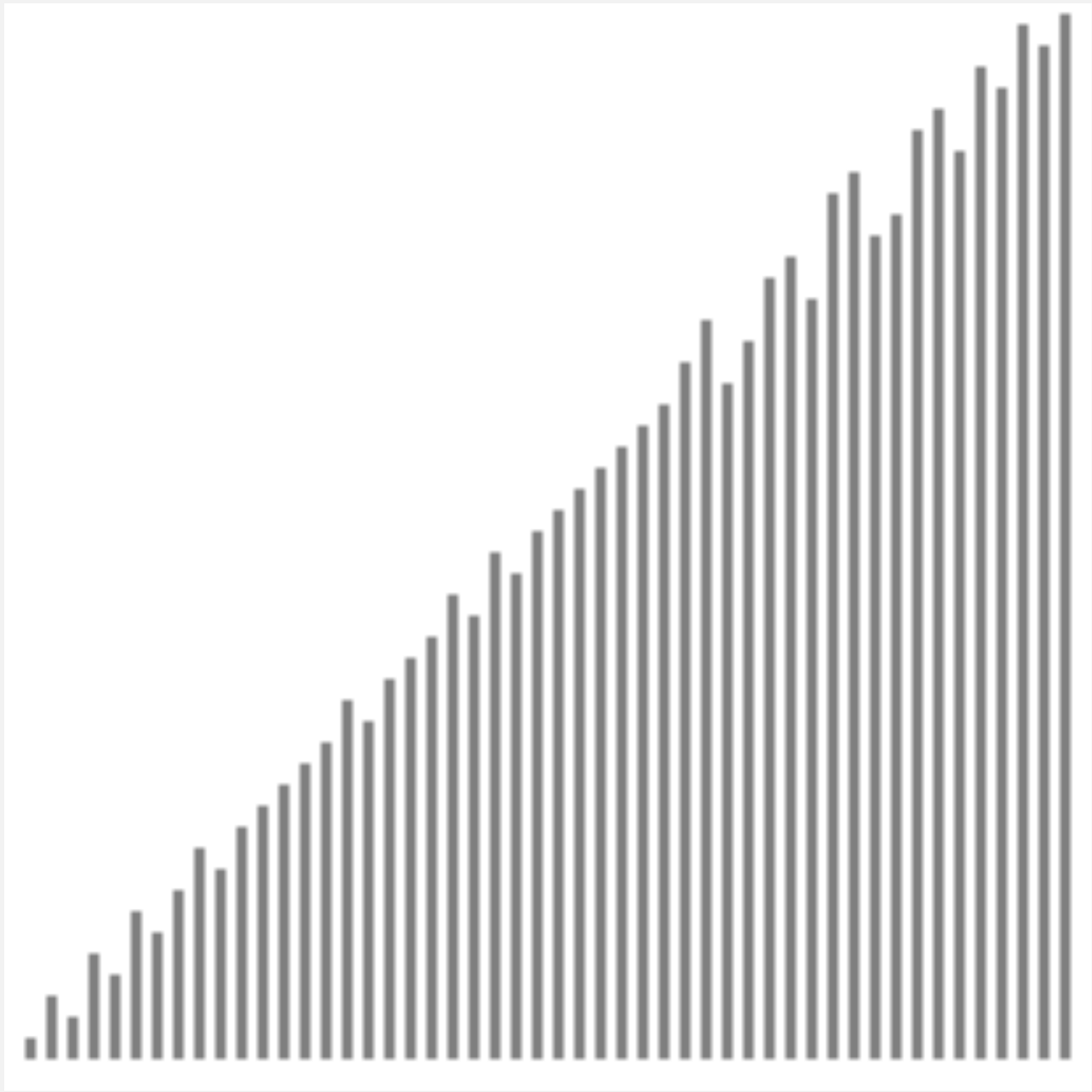
40, 13, 4, 1



http://www.sorting-algorithms.com/shell-sort

Shellsort: animation

50 partially-sorted items



8.3 secs

Strides (gaps):

40, 13, 4, 1

algorithm position

h-sorted

current subsequence

other elements

http://www.sorting-algorithms.com/shell-sort

Shellsort: which increment sequence to use?

Powers of two. 1, 2, 4, 8, 16, 32, ...

No.

Powers of two minus one. 1, 3, 7, 15, 31, 63, …

Maybe.

Shell's original idea.

→ 3x + 1. 1, 4, 13, 40, 121, 364, …

OK. Easy to compute.

Relatively prime

Sedgewick. 1, 5, 19, 41, 109, 209, 505, 929, 2161, 3905, …

Good. Tough to beat in empirical studies.

merging of $(9 \times 4^i) - (9 \times 2^i) + 1$

and $4^i - (3 \times 2^i) + 1$

Shellsort: intuition

Proposition. An h-sorted array remains h-sorted after g-sorting it.

7-sort

```
S O R T E X A M P L E
M O R T E X A S P L E
M O R T E X A S P L E
M O L T E X A S P R E
M O L E E X A S P R T
```

3-sort

```
M O L E E X A S P R T
E O L M E X A S P R T
E E L M O X A S P R T
A E L E O X M S P R T
A E L E O X M S P R T
A E L E O P M S X R T
A E L E O P M S X R T
A E L E O P M S X R T
A E L E O P M S X R T
```

still 7-sorted

Challenge. Prove this fact—it's more subtle than you'd think!

An intuitive answer to whether or not h-sorts interfere

- If you *h*-sort an array and then *g*-sort it, how can we know that these operations are independent and the the *g*-sort isn't undoing the work of the *h*-sort?
- One way to look at this is to consider how these sorts compose:
- If you *h*-sort then *g*-sort, the result can be said to be *(ah + bg)*-sorted where:
- *a*, *b* >= 0.
- So, your array will be *h*-sorted *and g*-sorted (neither of these has been undone);
- Your array is also *(h+g)*-sorted, *(h+2g)*-sorted, and so on.
- An intuitive way to think of this is that once two elements have been placed in relative order, they will never be moved out of that order, *even if they are re-visited.*

What about re-visiting previous compares?

- If you *h*-sort an array and then *g*-sort it, you will be re-visiting previous compares if there is an integer factor *k* such that *h = kg* or *g = kh*.
- Therefore, *h* and *g* should, if possible, be *relatively prime,* i.e. no such factor *k*.

Shellsort: analysis

Proposition. The order of growth of the worst-case number of compares used by shellsort with the 3x+1 increments is $N^{3/2}$.

Property. The expected number of compares to shellsort a randomly-ordered array using 3x+1 increments is….

N	compares	2.5 N ln N	0.25 N ln^2 N	N$^{1.3}$
5,000	93K	106K	91K	64K
10,000	209K	230K	213K	158K
20,000	467K	495K	490K	390K
40,000	1022K	1059K	1122K	960K
80,000	2266K	2258K	2549K	2366K

Remark. Accurate model has not yet been discovered (!)

Why are we interested in shellsort?

Example of simple idea leading to substantial performance gains.

Useful in practice.

R, bzip2, /linux/kernel/groups.c

- Fast unless array size is huge (used for small subarrays).
- Tiny, fixed footprint for code (used in some embedded systems).
- Hardware sort prototype.

uClibc

Simple algorithm, nontrivial performance, interesting questions.

- Asymptotic growth rate?
- Best sequence of increments? ← open problem: find a better increment sequence
- Average-case performance?

Lesson. Some good algorithms are still waiting discovery.

Elementary sorts summary

Today. Elementary sorting algorithms.

algorithm	best	average	worst
selection sort	N^2	N^2	N^2
insertion sort	N	N^2	N^2
Shellsort: $(3^k-1)/2$	$N \log_p N$	?	$N^{3/2}$
goal	N	$N \log N$	$N \log N$

order of growth of running time to sort an array of N items

best case for Shellsort is where p is the number of passes and is approximately 3

Next week. $N \log N$ sorting algorithms (in worst case).

More on Shellsort

https://en.wikipedia.org/wiki/Shellsort

http://algs4.cs.princeton.edu

2.1 Elementary Sorts

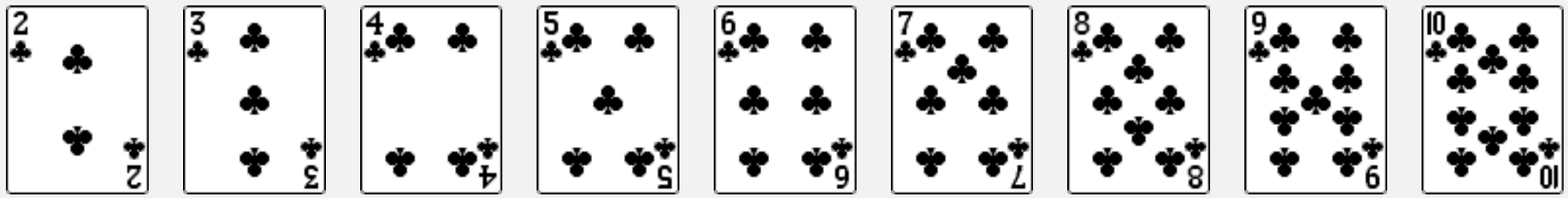
- *rules of the game*
- *selection sort*
- *insertion sort*
- *shellsort*
- *shuffling*

How to shuffle an array

Goal. Rearrange array so that result is a uniformly random permutation.

all permutations

equally likely



How to shuffle?

Shuffling is the inverse of sorting.

Shuffling requires an injection of entropy

- Where are you going to get that entropy from?
- Pseudo-random number generator?
- The user?
- A real hardware random number generator?

Several ways to shuffle

Shuffle method	Time	Space	Random?
Select one of all possible permutations	O(1)	O(N!)	Y
Sort on random double	O(N lgN)	O(2N)	Y
Modified Fisher-Yates (Knuth Algorithm “P”)	O(N)	O(N)	Y
Microsoft Browser Choice	O(N lgN)	O(N)	N

How not to shuffle…

- Use a comparator function that looks like this:

```
function RandomSort(Object a, Object b) {
  return 0.5 - Math.random();
}
```

- What's wrong with this?
  - Hint: we talked about the "rules" of sorting earlier in this module.

How to shuffle an array

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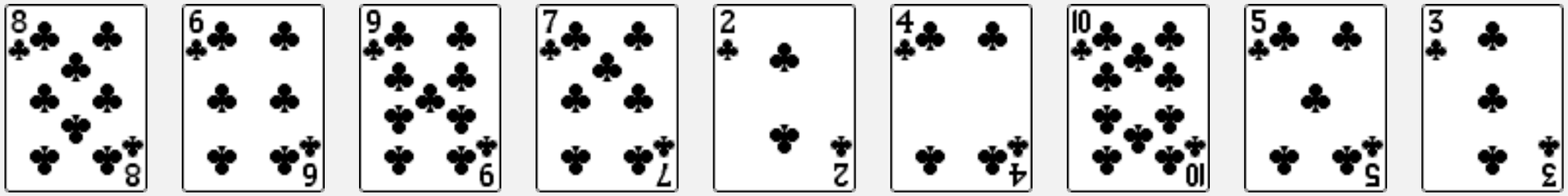
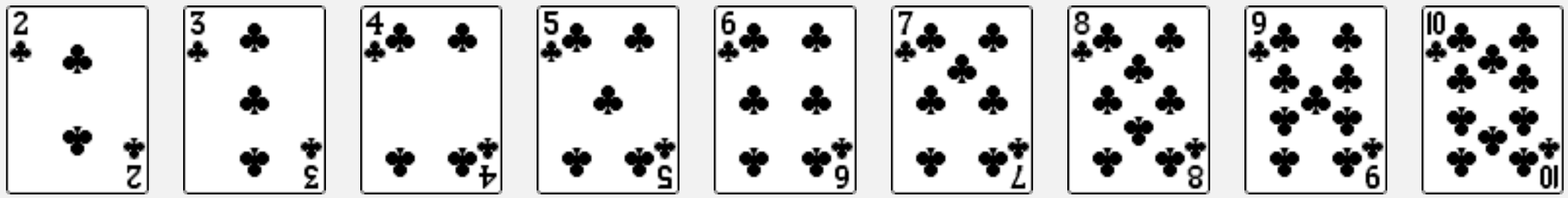
equally likely



Shuffle sort

- Generate a random real number for each array entry.
- Sort the array.

useful for shuffling columns in a spreadsheet



0.8003 0.9706 0.9157 0.9649 0.1576 0.4854 0.1419 0.4218 0.9572

Shuffle sort

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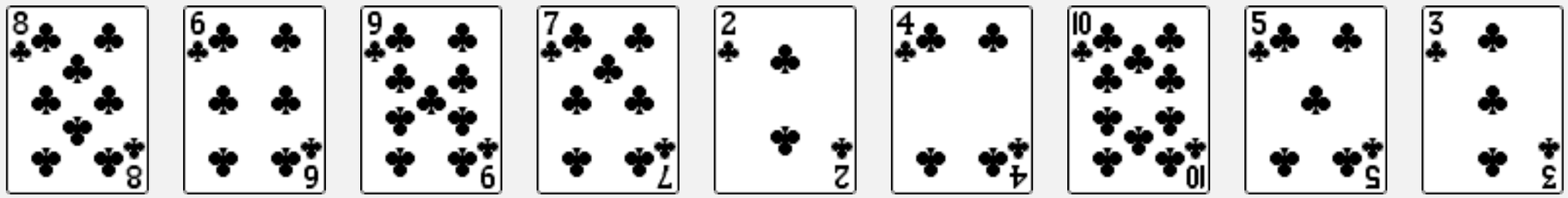

0.1419 0.1576 0.4218 0.4854 0.8003 0.9157 0.9572 0.9649 0.9706

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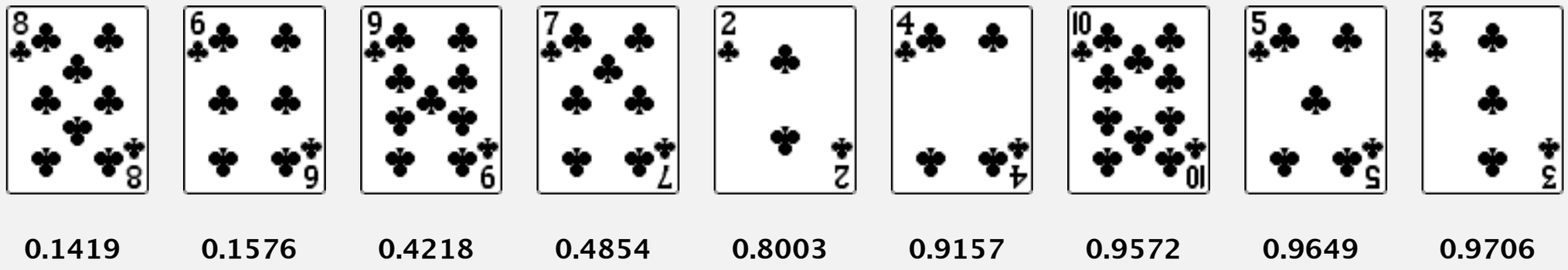


Proposition. Shuffle sort produces a uniformly random permutation.

assuming real numbers uniformly at random (and no ties)

War story (Microsoft)

Microsoft antitrust probe by EU. Microsoft agreed to provide a randomized ballot screen for users to select browser in Windows 7.

http://www.browserchoice.eu

appeared last

50% of the time

War story (Microsoft)

Microsoft antitrust probe by EU. Microsoft agreed to provide a randomized ballot screen for users to select browser in Windows 7.

Solution? Implement shuffle sort by making comparator always return a random answer.

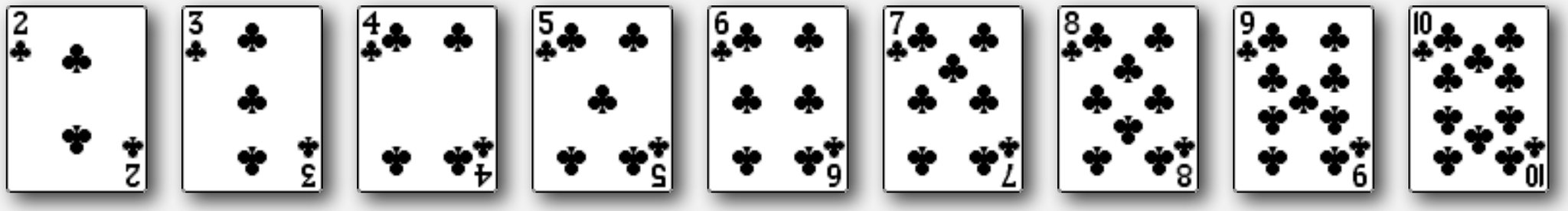
```
public int compareTo(Browser that)
{
   double r = Math.random();
   if (r < 0.5) return -1;
   if (r > 0.5) return +1;
   return 0;
}
```

browser comparator

(should implement a total order)

Knuth shuffle demo

- In iteration i, pick integer r between 0 and i uniformly at random.
- Swap a[i] and a[r].



Knuth shuffle

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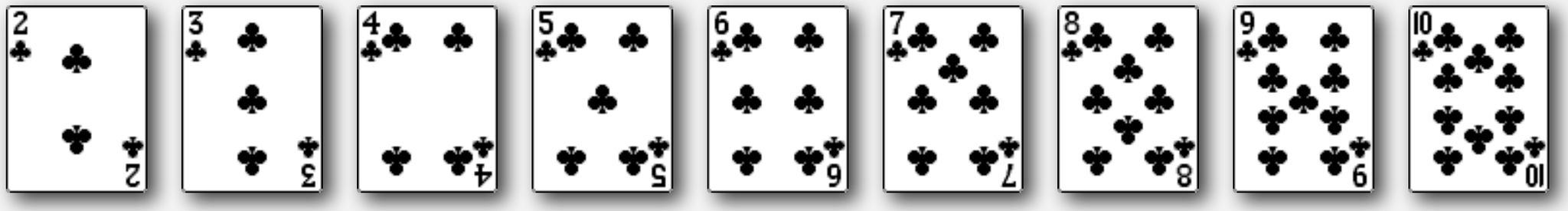


Proposition. [Fisher-Yates 1938] Knuth shuffling algorithm produces a uniformly random permutation of the input array in linear time.

assuming integers uniformly at random

Knuth shuffle

- In iteration i, pick integer r between 0 and i uniformly at random.
- Swap a[i] and a[r].

common bug: between 0 and N – 1

correct variant: between i and N – 1

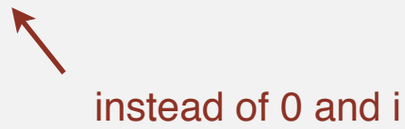
```
public class StdRandom
{
   ...
   public static void shuffle(Object[] a)
   {
      int N = a.length;
      for (int i = 0; i < N; i++)
      {
         int r = StdRandom.uniform(i + 1);
         swap(a, i, r);
      }
   }
}
```

between 0 and i

Broken Knuth shuffle

Q. What happens if integer is chosen between 0 and N-1 ?

A. Not uniformly random!



permutation	Knuth shuffle	broken shuffle
A B C	1/6	4/27
A C B	1/6	5/27
B A C	1/6	5/27
B C A	1/6	5/27
C A B	1/6	4/27
C B A	1/6	4/27

probability of each result when shuffling { A, B, C }

War story (online poker)

Texas hold'em poker. Software must shuffle electronic cards.

How We Learned to Cheat at Online Poker: A Study in Software Security

http://www.datamation.com/entdev/article.php/616221

War story (online poker)

Shuffling algorithm in FAQ at www.planetpoker.com

```
for i := 1 to 52 do begin
   r := random(51) + 1;
   swap := card[r];
   card[r] := card[i];
   card[i] := swap;
end;
```

← between 1 and 51

Bug 1. Random number r never 52 $\Rightarrow$ 52nd card can't end up in 52nd place.

Bug 2. Shuffle not uniform (should be between 1 and i).

Bug 3. random() uses 32-bit seed $\Rightarrow$ 2^{32} possible shuffles.

Bug 4. Seed = milliseconds since midnight $\Rightarrow$ 86.4 million shuffles.

> *" The generation of random numbers is too important to be left to chance. "*
>
> *— Robert R. Coveyou*

War story (online poker)

Best practices for shuffling (if your business depends on it).

- Use a hardware random-number generator that has passed both the FIPS 140-2 and the NIST statistical test suites.
- Continuously monitor statistic properties: hardware random-number generators are fragile and fail silently.
- Use an unbiased shuffling algorithm.

RANDOM.ORG

Bottom line. Shuffling a deck of cards is hard!