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3.2 BINARY SEARCH TREES

- *BSTs*
- *ordered operations*
- *deletion*

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3.2 Binary Search Trees

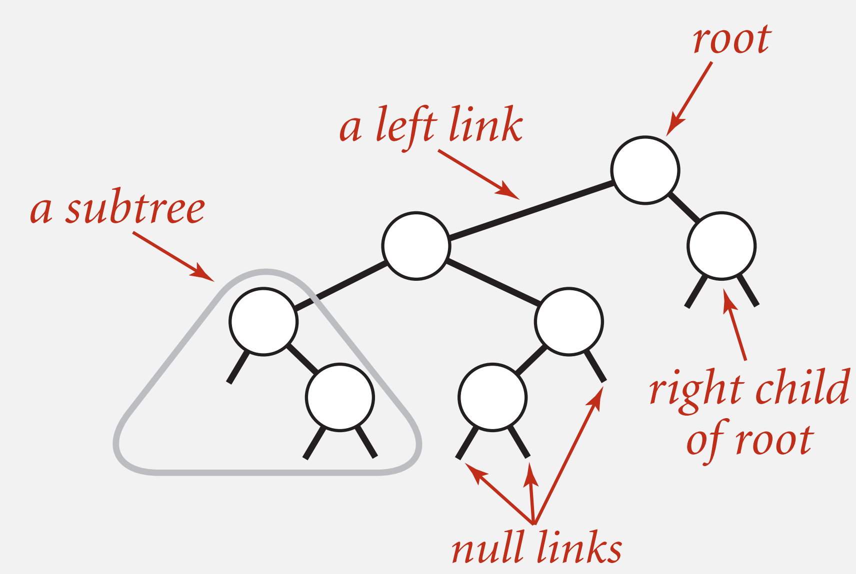
- *BSTs*
- *ordered operations*
- *deletion*

Binary search trees

Definition. A BST is a binary tree in symmetric order.

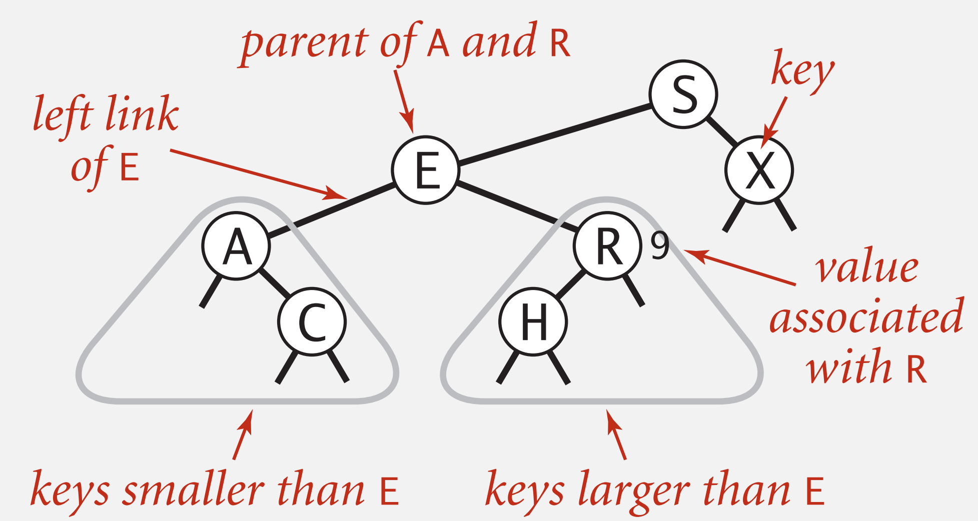
A binary tree is either:

- Empty.
- Two disjoint binary trees (left and right).



Symmetric order. Each node has a key, and every node's key is:

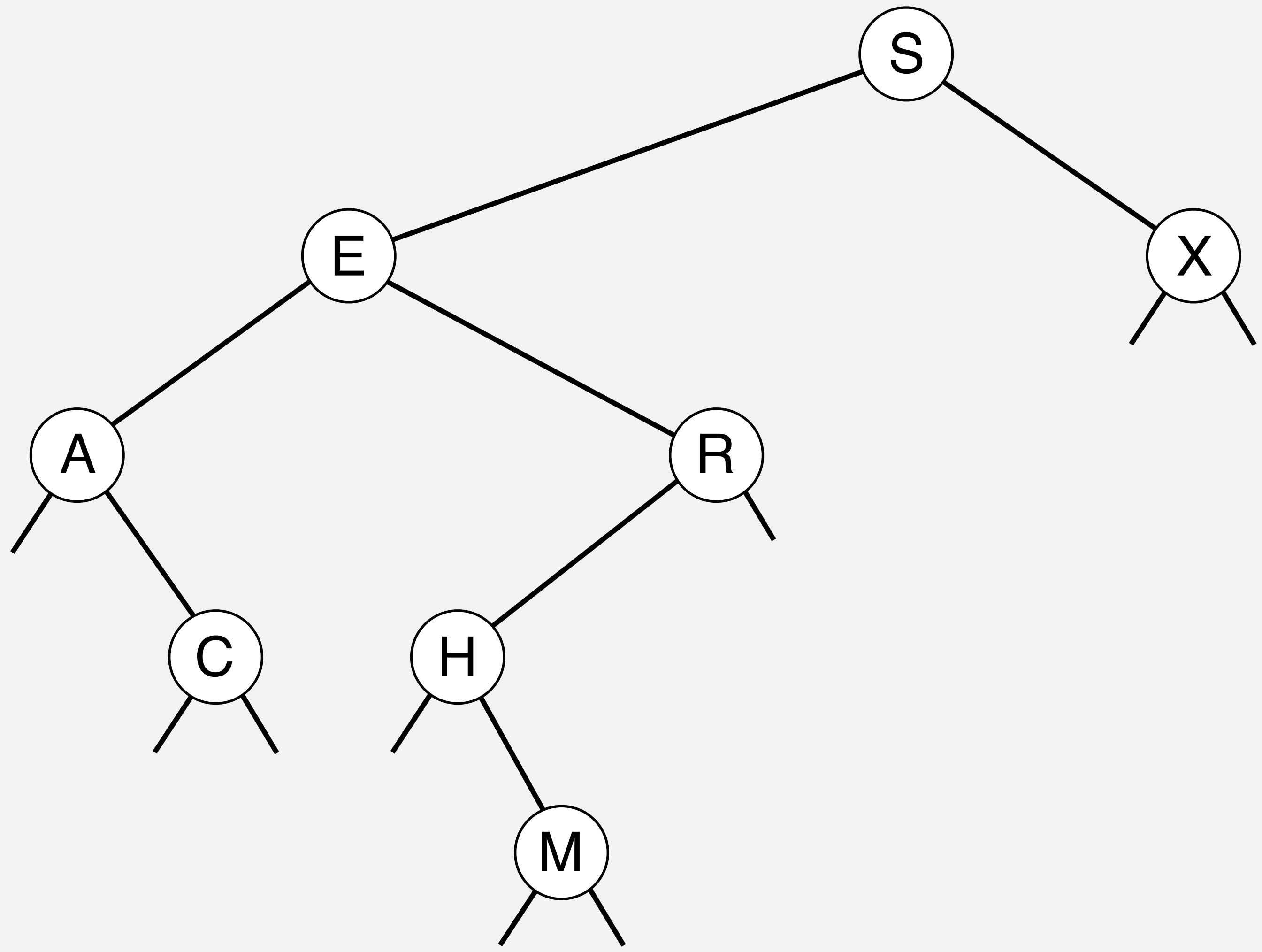
- Larger than all keys in its left subtree.
- Smaller than all keys in its right subtree.



Binary search tree demo

Search. If less, go left; if greater, go right; if equal, search hit.

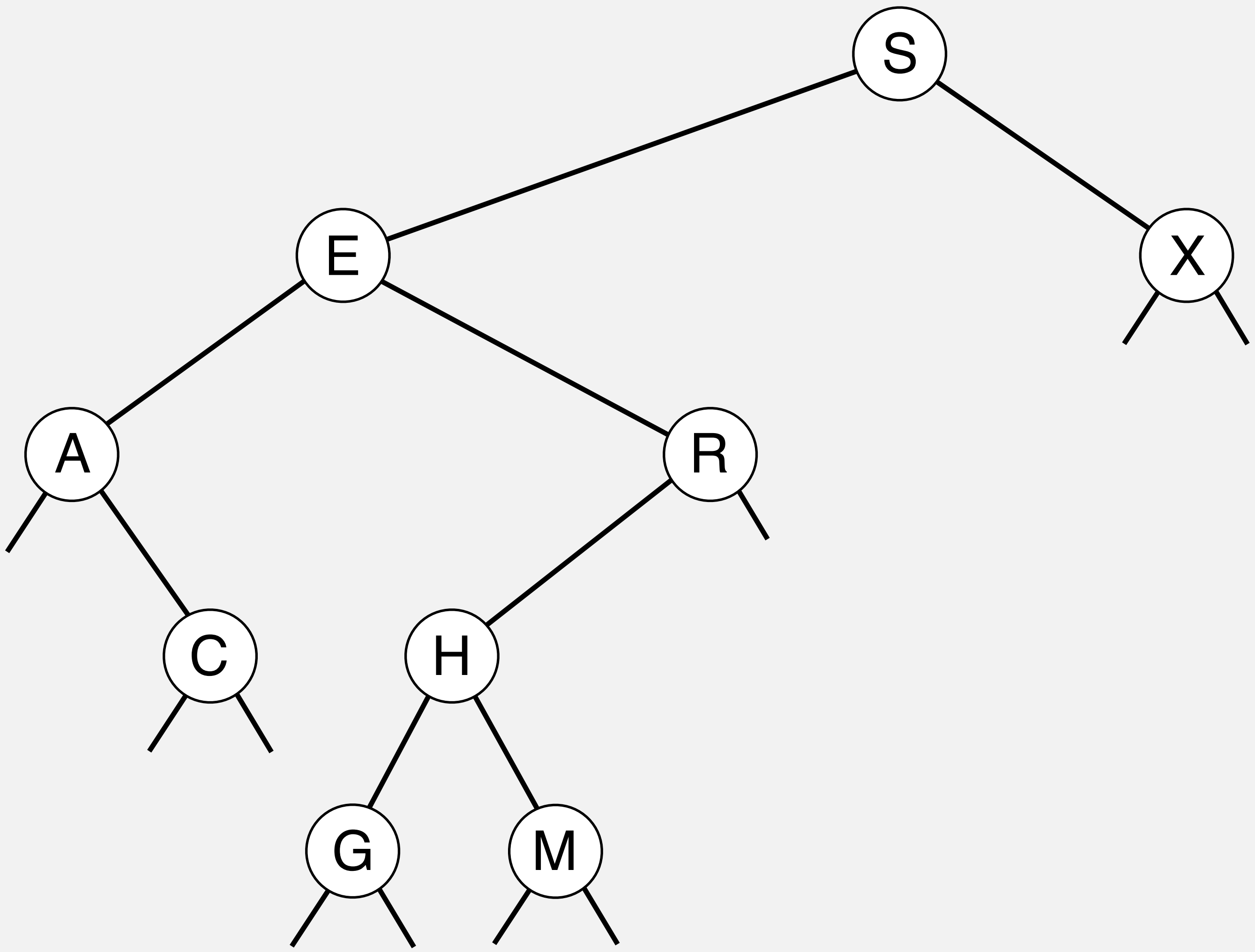
successful search for H



Binary search tree demo

Insert. If less, go left; if greater, go right; if null, insert.

insert G



BST representation in Java

Java definition. A BST is a reference to a root Node.

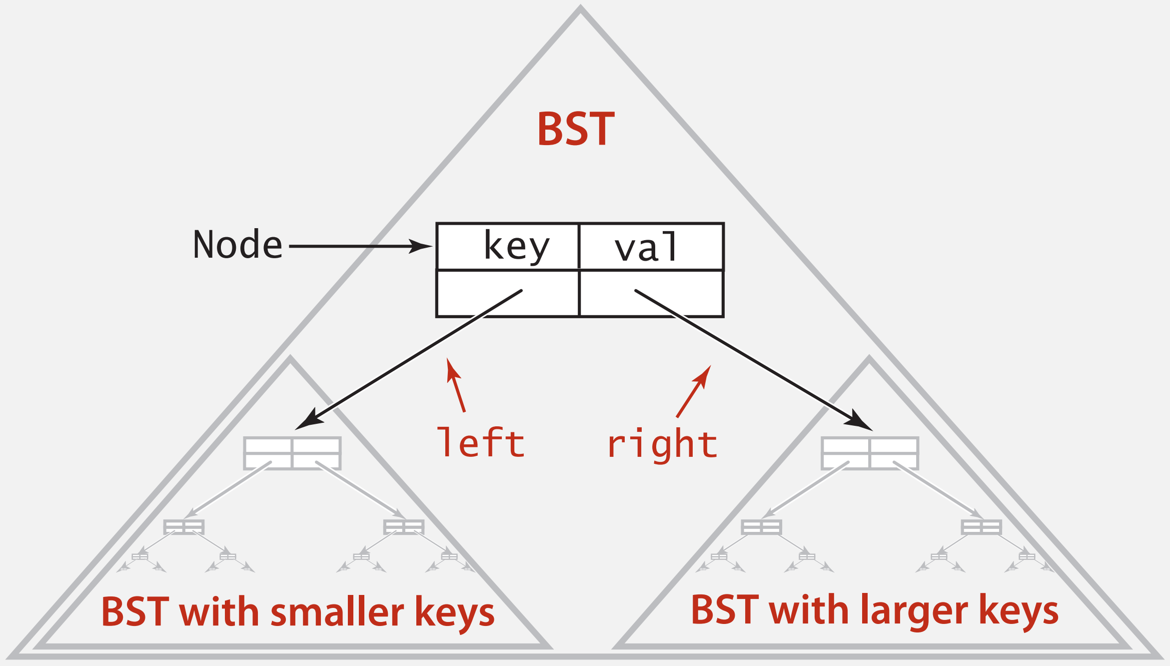
A Node is composed of four fields:

- A Key and a Value.
- A reference to the left and right subtree.

smaller keys larger keys

```
private class Node
{
  private Key key;
  private Value val;
  private Node left, right;
  public Node(Key key, Value val)
  {
    this.key = key;
    this.val = val;
  }
}
```

Key and Value are generic types; Key is Comparable



Binary search tree

BST implementation (skeleton)

```
public class BST<Key extends Comparable<Key>, Value>
{
  private Node root;

  private class Node
  { /* see previous slide */ }

  public void put(Key key, Value val)
  { /* see next slides */ }

  public Value get(Key key)
  { /* see next slides */ }

  public void delete(Key key)
  { /* see next slides */ }

  public Iterable<Key> iterator()
  { /* see next slides */ }
```

root of BST

BST search: Java implementation

Get. Return value corresponding to given key, or null if no such key.

```
public Value get(Key key)
{
   Node x = root;
   while (x != null)
   {
      int cmp = key.compareTo(x.key);
      if      (cmp  < 0) x = x.left;
      else if (cmp  > 0) x = x.right;
      else if (cmp == 0) return x.val;
   }
   return null;
}
```

Cost. Number of compares is equal to 1 + depth of node.

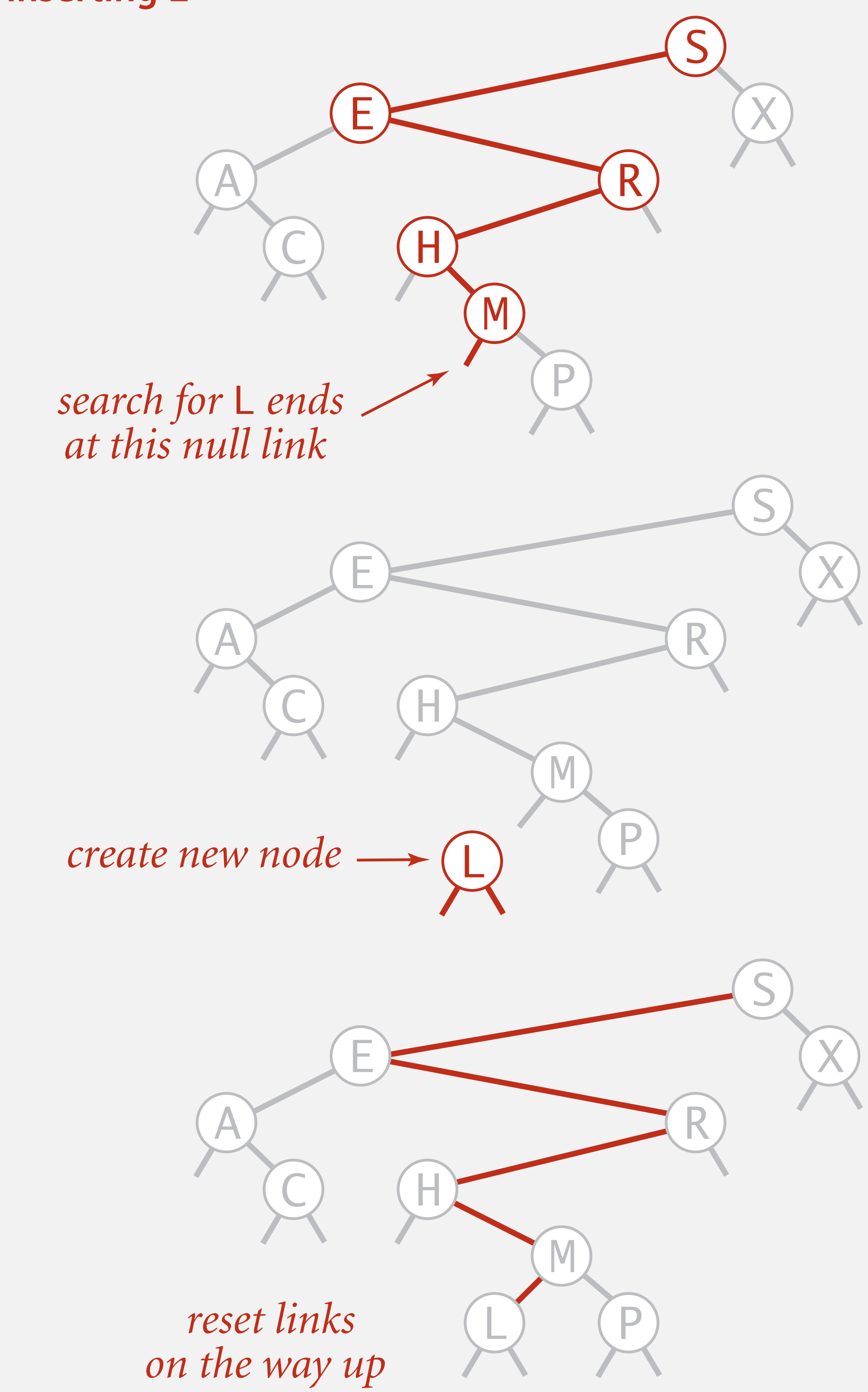
BST insert

Put. Associate value with key.

Search for key, then two cases:

- Key in tree $\Rightarrow$ reset value.
- Key not in tree $\Rightarrow$ add new node.

Note that in our simple implementation, there is no work to be done on the way up.



Insertion into a BST

BST insert: Java implementation

Put. Associate value with key.

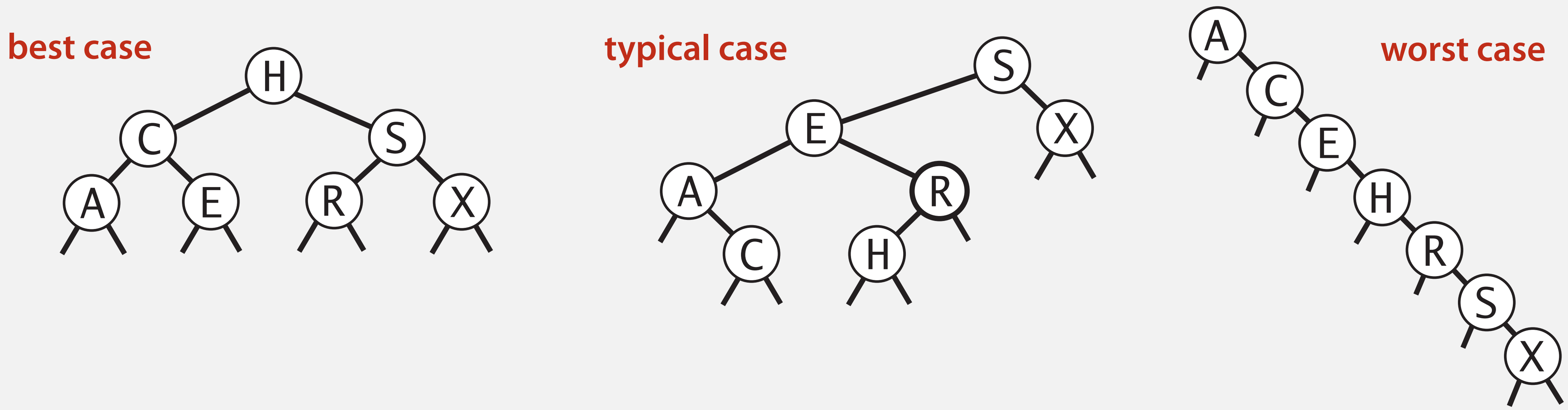
```
public void put(Key key, Value val)
{  root = put(root, key, val);  }

private Node put(Node x, Key key, Value val)
{
   if (x == null) return new Node(key, val);
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0)
      x.left  = put(x.left,  key, val);
   else if (cmp  > 0)
      x.right = put(x.right, key, val);
   else if (cmp == 0)
      x.val = val;
   return x;
}
```

concise, but tricky,
recursive code;
read carefully!

Cost. Number of compares is equal to 1 + depth of node.

Tree shape

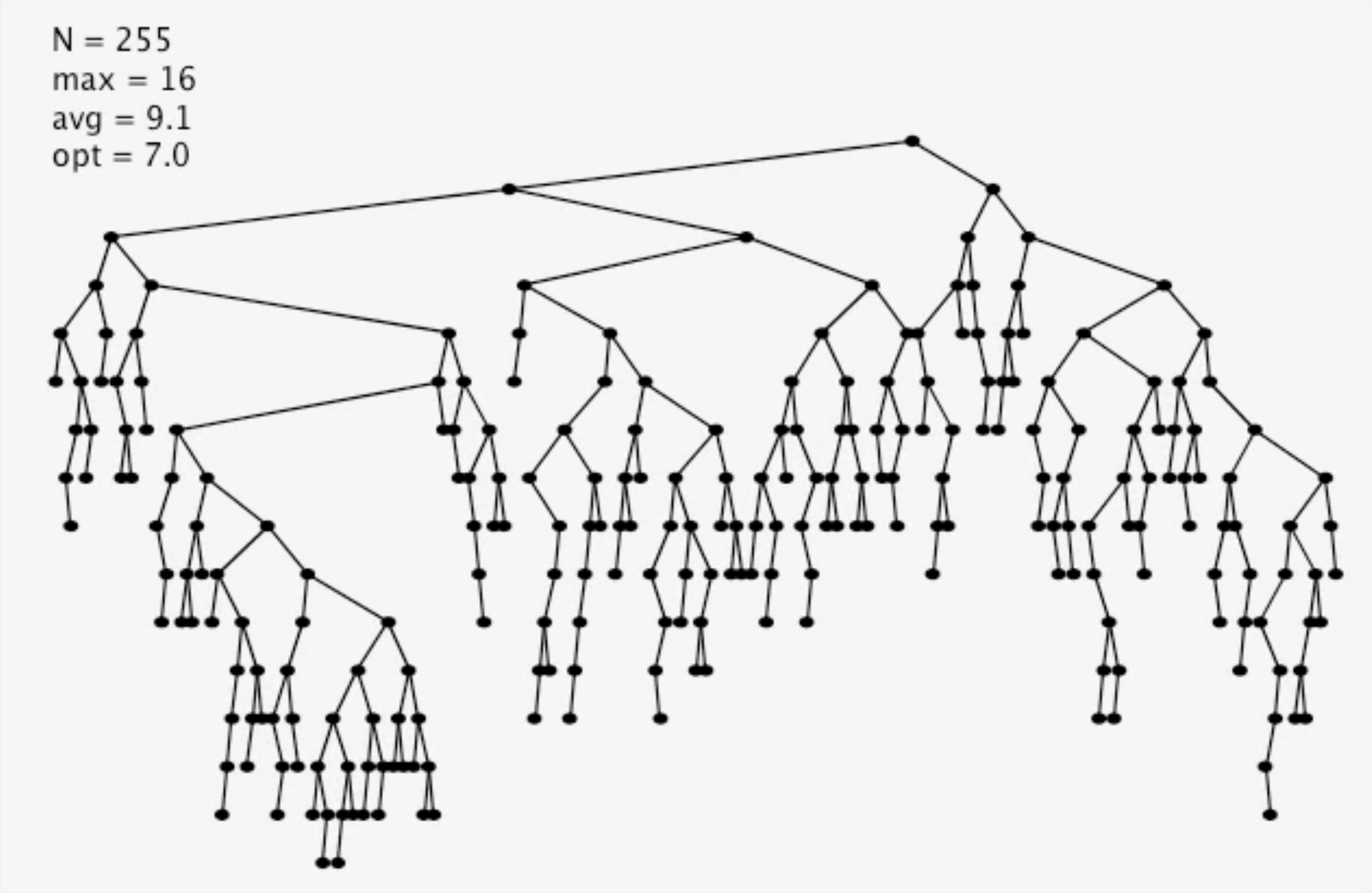
- Many BSTs correspond to same set of keys.
- Number of compares for search/insert is equal to 1 + depth of node.



Bottom line. Tree shape depends on order of insertion.

BST insertion: random order visualization

Ex. Insert keys in random order.



Sorting with a binary search tree

Q. What is this sorting algorithm?

0. Shuffle the array of keys.

1. Insert all keys into a BST.

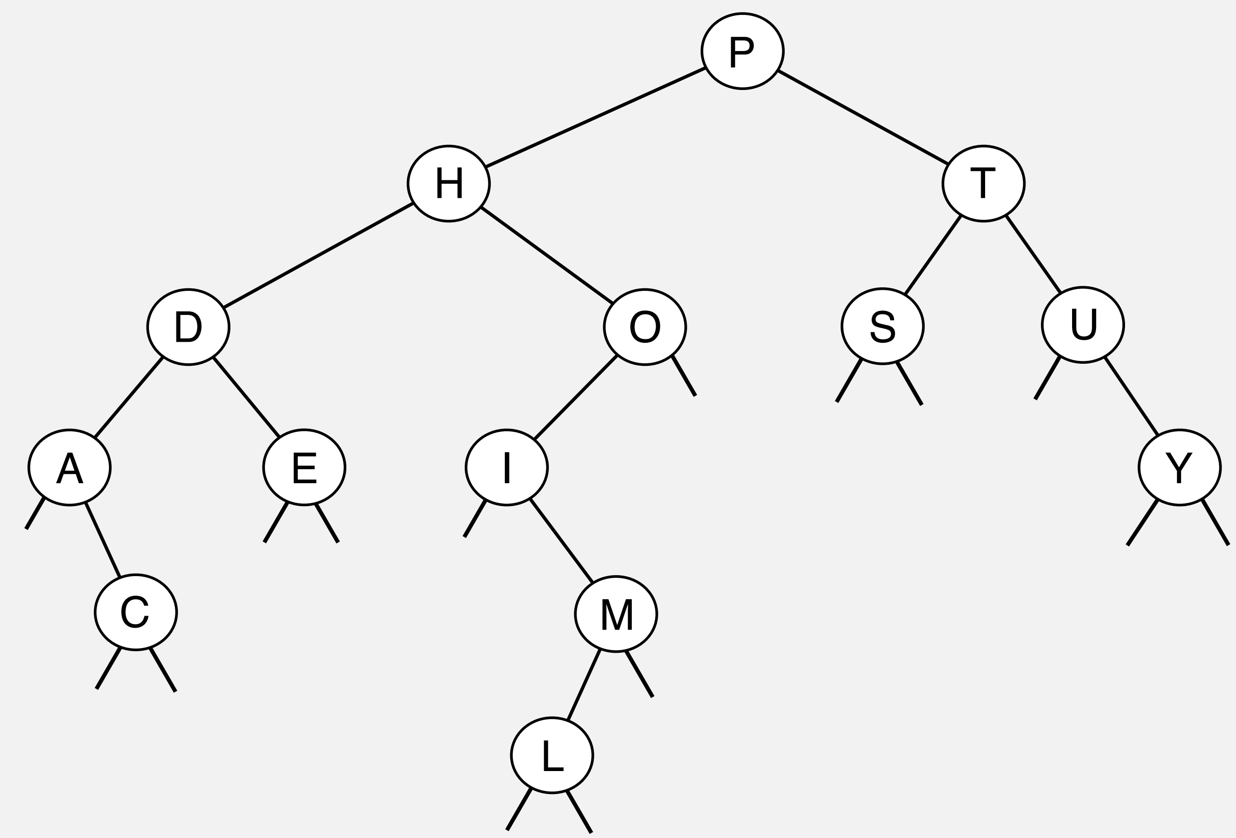
2. Do an inorder traversal of BST.

A. It's a sorting algorithm (assuming no duplicate keys). Just like in with the binary heap that led to heap sort.

Q. What are its properties?

Correspondence between BSTs and quicksort partitioning

0	1	2	3	4	5	6	7	8	9	10	11	12	13
P	S	E	U	D	O	M	Y	T	H	I	C	A	L
P	S	E	U	D	O	M	Y	T	H	I	C	A	L
H	L	E	A	D	O	M	C	I	P	T	Y	U	S
D	C	E	A	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	O	M	L	I	P	T	Y	U	S
A	C	D	E	H	I	M	L	O	P	T	Y	U	S
A	C	D	E	H	I	M	L	O	P	T	Y	U	S
A	C	D	E	H	I	L	M	O	P	T	Y	U	S
A	C	D	E	H	I	L	M	O	P	T	Y	U	S
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y
A	C	D	E	H	I	L	M	O	P	S	T	U	Y



Remark. Correspondence is 1–1 if array has no duplicate keys.

BSTs: mathematical analysis

Proposition. If N distinct keys are inserted into a BST in random order, the expected number of compares for a search/insert is ~ $2 \ln N$.

Pf. 1–1 correspondence with quicksort partitioning.

Proposition. [Reed, 2003] If N distinct keys are inserted in random order, expected height of tree is ~ $4.311 \ln N$.

How Tall is a Tree?

Bruce Reed
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ABSTRACT

Let H_n be the height of a random binary search tree on n nodes. We show that there exists constants $\alpha = 4.31107\ldots$ and $\beta = 1.95\ldots$ such that $\mathrm{E}(H_n) = \alpha \log n - \beta \log \log n + O(1)$, We also show that $\mathrm{Var}(H_n) = O(1)$.

But… Worst-case height is $N - 1$.

[exponentially small chance when keys are inserted in random order]

ST implementations: summary

implementation	guarantee		average case		operations on keys
	search	insert	search hit	insert	
sequential search (unordered list)	N	N	½ N	N	equals()
binary search (ordered array)	$\lg N$	N	$\lg N$	½ N	compareTo()
BST	N	N	$1.39 \lg N$	$1.39 \lg N$	compareTo()

Why not shuffle to ensure a (probabilistic) guarantee of 4.311 ln N?

Insurance: the "premium" is 2.921 lg N but the risk of not taking out insurance is N vs. 1.39 lg N

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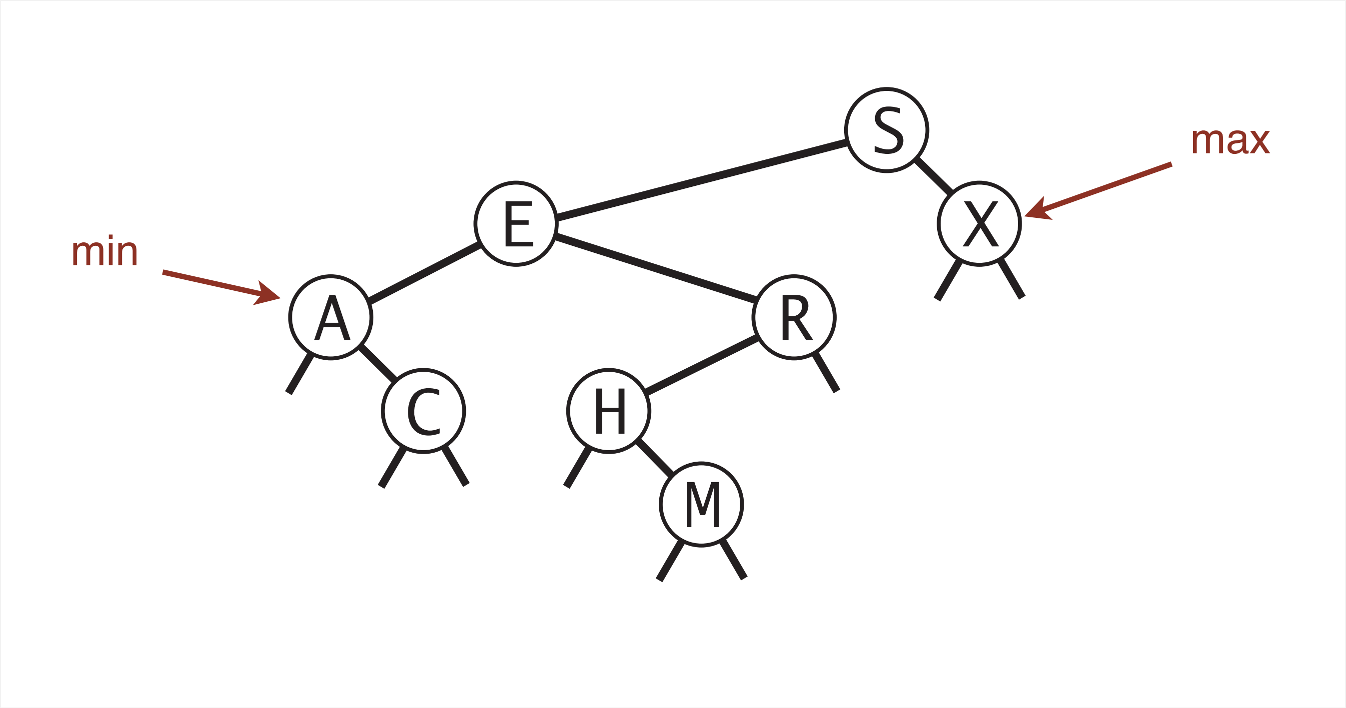
3.2 Binary Search Trees

- BSTs
- *ordered operations*
- deletion

Minimum and maximum

Minimum. Smallest key in table.

Maximum. Largest key in table.

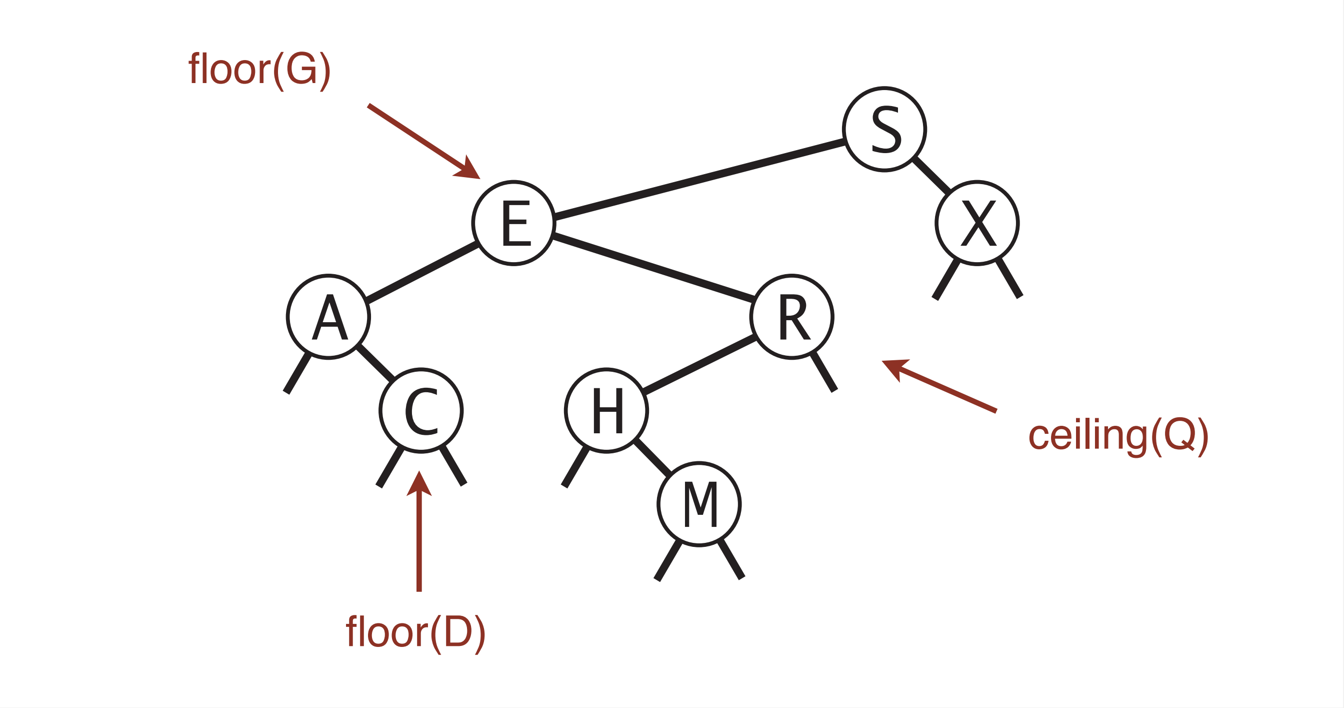


Q. How to find the min / max?

Floor and ceiling

Floor. Largest key ≤ a given key.

Ceiling. Smallest key ≥ a given key.



Q. How to find the floor / ceiling?

Computing the floor

Case 1. [k equals the key in the node]

The floor of k is k.

Case 2. [k is less than the key in the node]

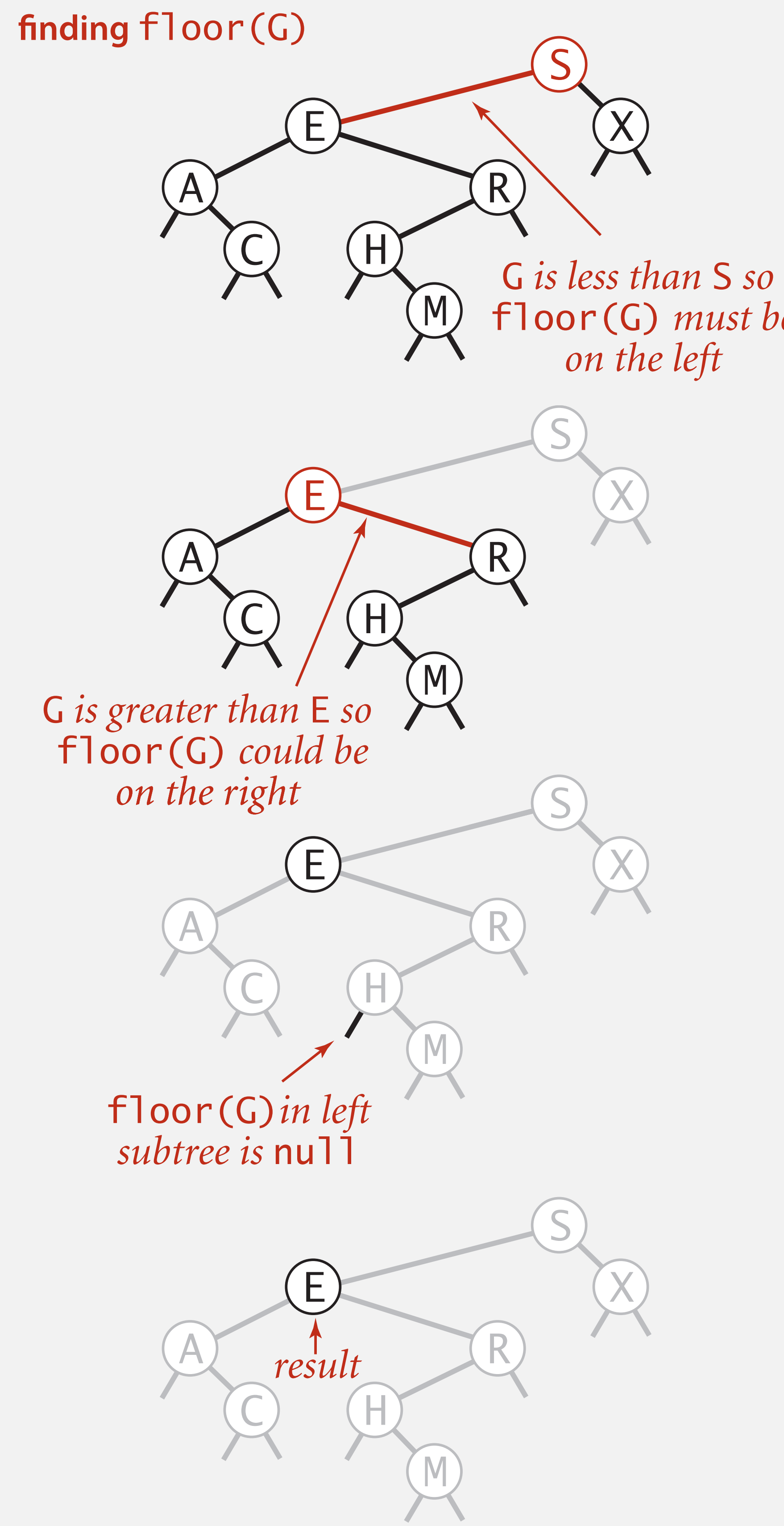
The floor of k is in the left subtree.

Case 3. [k is greater than the key in the node]

The floor of k is in the right subtree

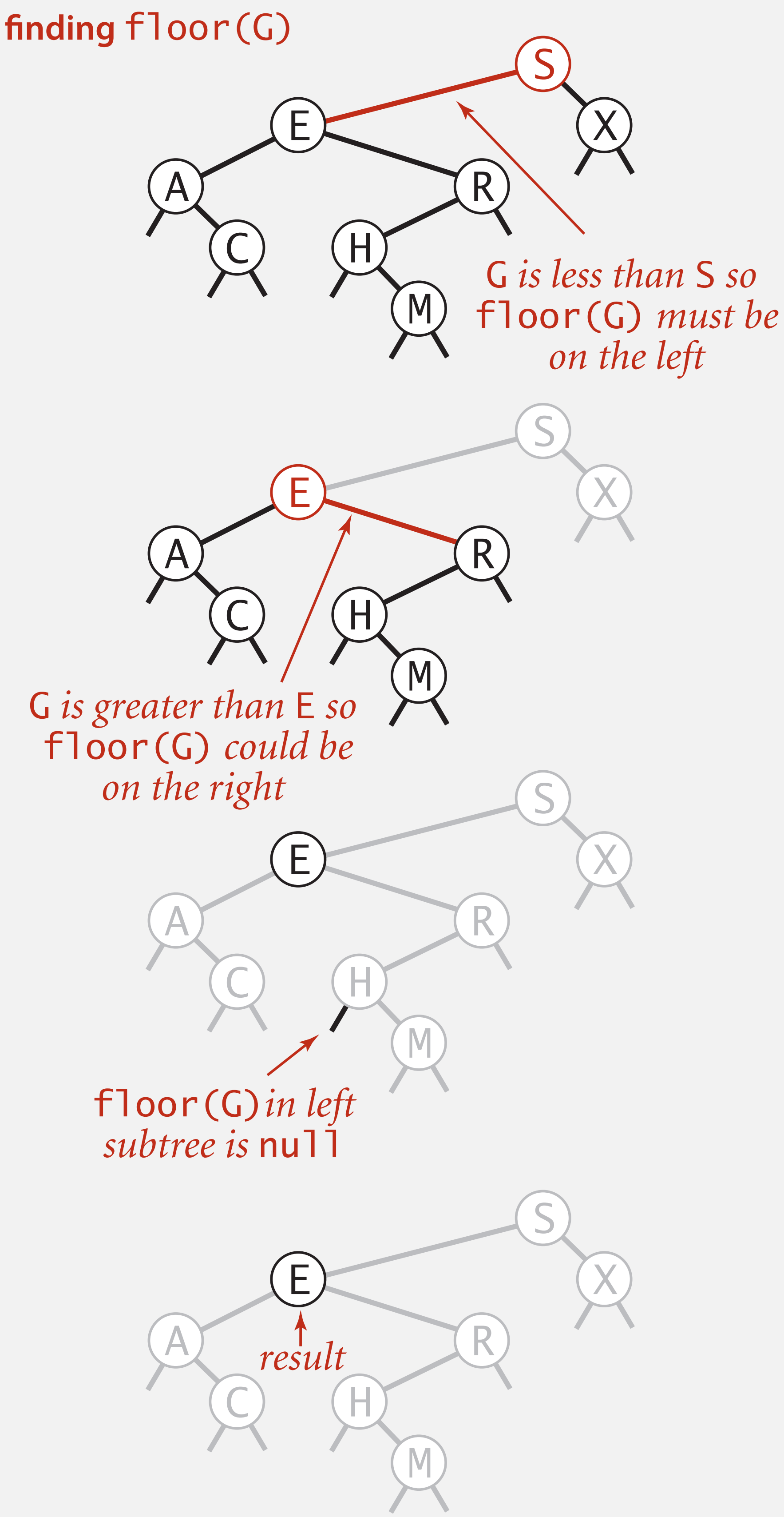
(if there is any key $\leq k$ in right subtree);

otherwise it is the key in the node.



Computing the floor

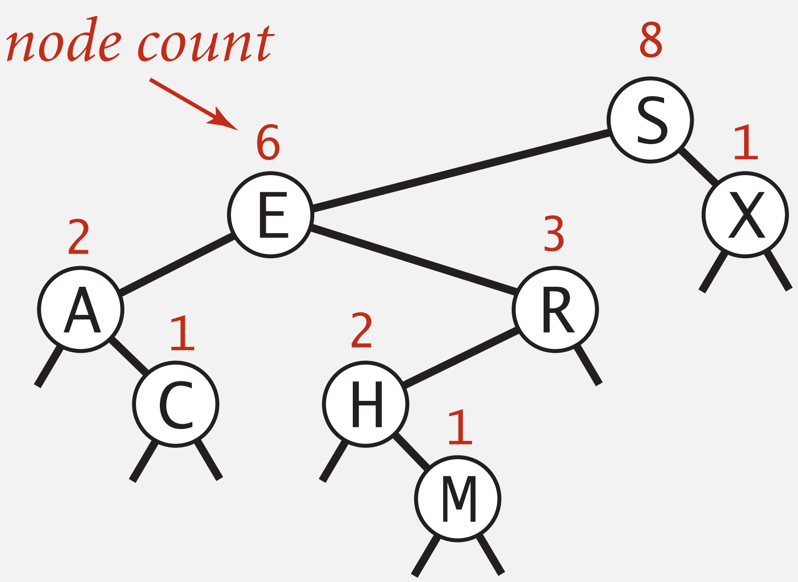
```
public Key floor(Key key)
{
   Node x = floor(root, key);
   if (x == null) return null;
   return x.key;
}
private Node floor(Node x, Key key)
{
   if (x == null) return null;
   int cmp = key.compareTo(x.key);

   if (cmp == 0) return x;

   if (cmp < 0)  return floor(x.left, key);

   Node t = floor(x.right, key);
   if (t != null) return t;
   else           return x;

}
```



Rank and select

Q. How to implement rank() and select() efficiently?

A. In each node, we store the number of nodes in the subtree rooted at that node; to implement size(), return the count at the root.



BST implementation: subtree counts

```
private class Node
{
   private Key key;
   private Value val;
   private Node left;
   private Node right;
   private int count;
}
```

number of nodes in subtree

```
public int size()
{ return size(root); }
```

```
private int size(Node x)
{
   if (x == null) return 0;
   return x.count;
}
```

ok to call
when x is null

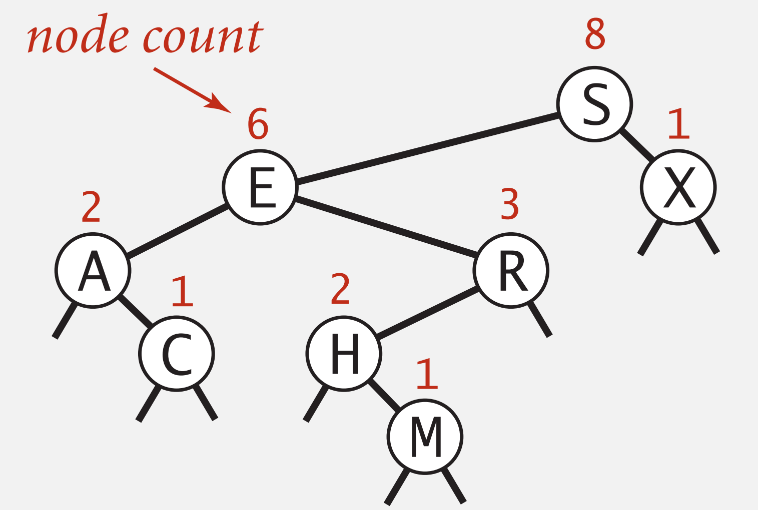
```
private Node put(Node x, Key key, Value val)
{
   if (x == null) return new Node(key, val, 1);
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0) x.left  = put(x.left,  key, val);
   else if (cmp  > 0) x.right = put(x.right, key, val);
   else if (cmp == 0) x.val = val;
   x.count = 1 + size(x.left) + size(x.right);
   return x;
}
```

initialize subtree
count to 1

Rank

Rank. How many keys < k ?

Easy recursive algorithm (3 cases!)




```
public int rank(Key key)
{  return rank(key, root);  }

private int rank(Key key, Node x)
{
   if (x == null) return 0;
   int cmp = key.compareTo(x.key);
   if      (cmp  < 0) return rank(key, x.left);
   else if (cmp  > 0) return 1 + size(x.left) + rank(key, x.right);
   else if (cmp == 0) return size(x.left);
}
```

How to do an in-order traversal?

Can you do it without using any extra memory?

No—you need extra memory to remember where you've been (like breadcrumbs)

What sort of data structure will you use to remember?

We need either a stack or a queue. Which one?

A *queue* if we traverse the tree smaller before larger and we want the elements in order of size, starting with the smallest.

In-order?

Why do we call it "in-order" (or "inorder")?

Well, there are two reasons (the second is the "real" reason):

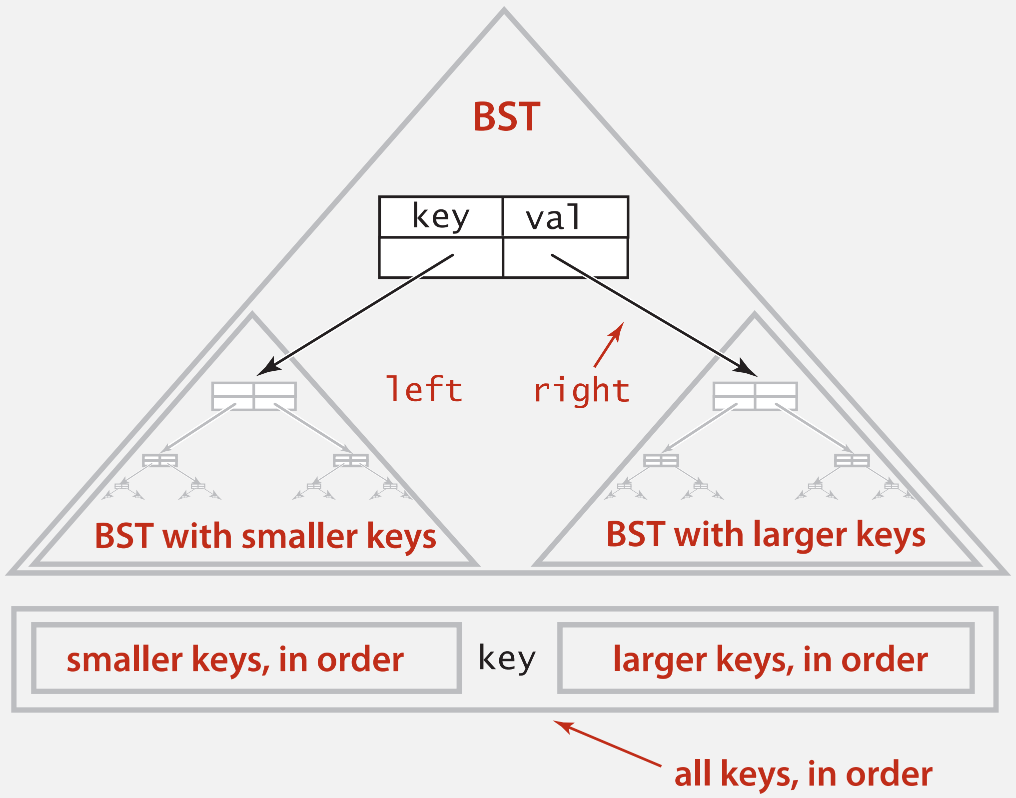
- It results in a list which is *in order*;
- It describes where/when in the recursive method we do the enqueuing:

```
Public void traverse(Node node, Queue queue) {
  queue.enqueue(node.key); // example of pre-order action
  traverse(node.left);
  queue.enqueue(node.key); // example of in-order action
  traverse(node.right);
  queue.enqueue(node.key); // example of post-order action
}
```

Inorder traversal

- Traverse left subtree.
- Enqueue key.
- Traverse right subtree.

```
public Iterable<Key> keys()
{
   Queue<Key> q = new Queue<Key>();
   inorder(root, q);
   return q;
}

private void inorder(Node x, Queue<Key> q)
{
   if (x == null) return;
   inorder(x.left, q);
   q.enqueue(x.key);
   inorder(x.right, q);
}
```



Property. Inorder traversal of a BST yields keys in ascending order.

BST: ordered symbol table operations summary

	sequential search	binary search	BST
search	N	$\lg N$	h
insert	N	N	h
min / max	N	1	h
floor / ceiling	N	$\lg N$	h
rank	N	$\lg N$	h
select	N	1	h
ordered iteration	$N \log N$	N	N

h = height of BST
(proportional to log N
if keys inserted in random order)

order of growth of running time of ordered symbol table operations

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3.2 Binary Search Trees

- *BSTs*
- *ordered operations*
- *deletion*

ST implementations: summary

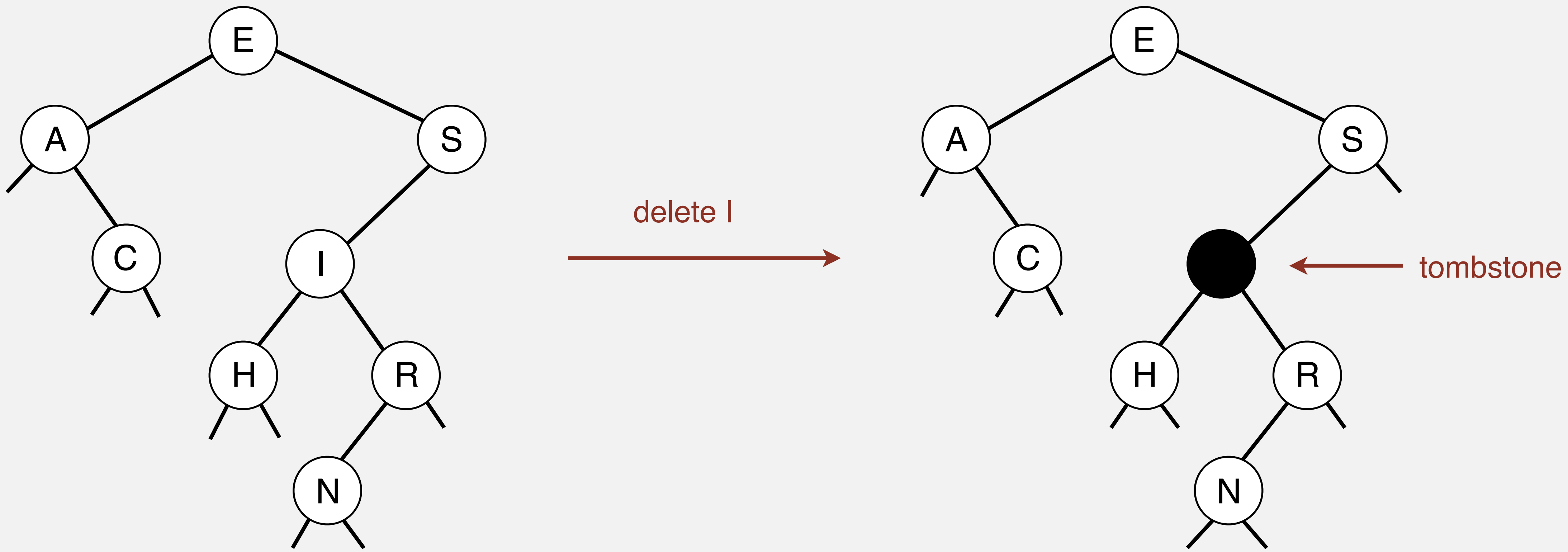
implementation	guarantee			average case			ordered ops?	operations on keys
	search	insert	delete	search hit	insert	delete		
sequential search (linked list)	N	N	N	$\frac{1}{2} N$	N	$\frac{1}{2} N$		equals()
binary search (ordered array)	$\lg N$	N	N	$\lg N$	$\frac{1}{2} N$	$\frac{1}{2} N$	✔	compareTo()
BST	N	N	N	$1.39 \lg N$	$1.39 \lg N$	? ? ?	✔	compareTo()

Next. Deletion in BSTs.

BST deletion: lazy approach

To remove a node with a given key:

- Set its value to null.
- Leave key in tree to guide search (but don't consider it equal in search).



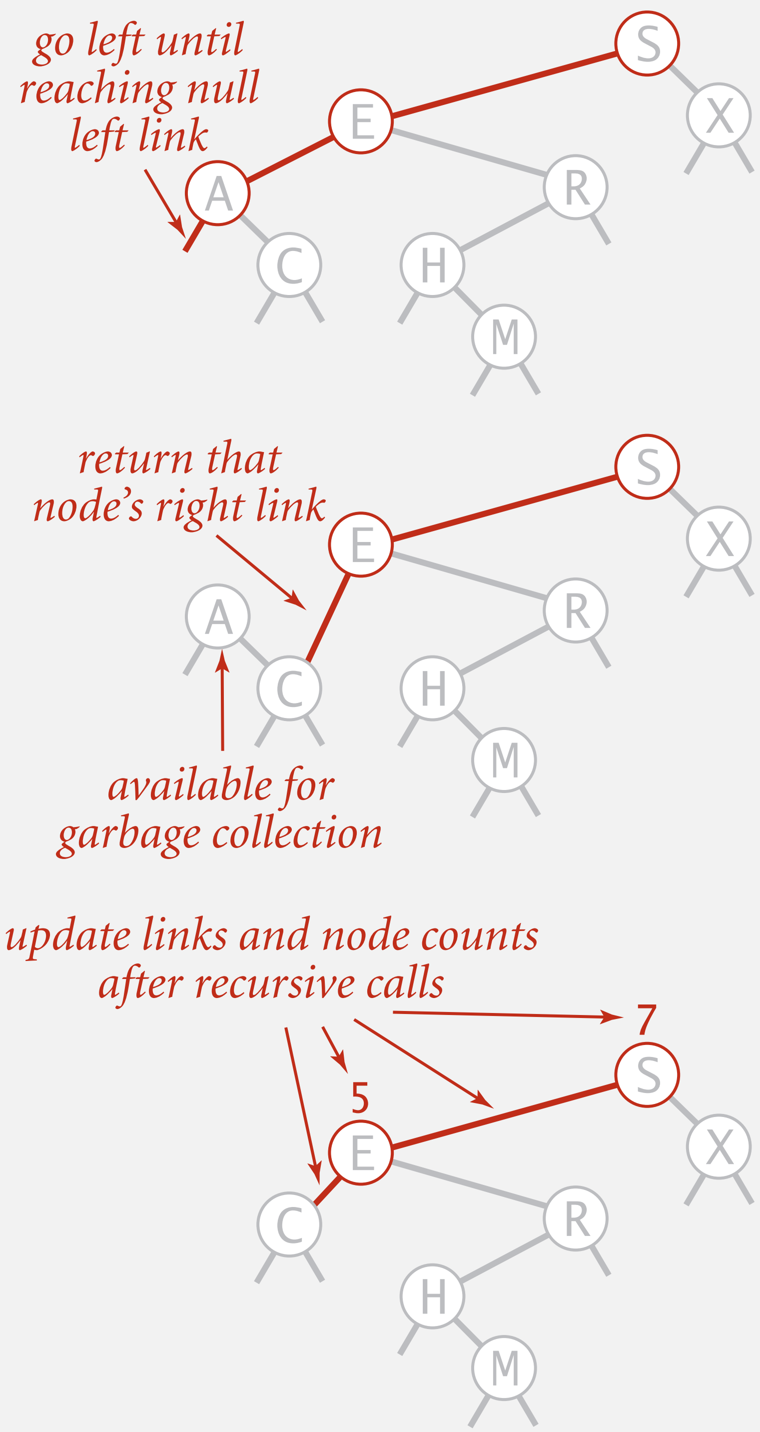
Cost. $\sim 2 \ln N'$ per insert, search, and delete (if keys in random order), where N' is the number of key-value pairs ever inserted in the BST.

Unsatisfactory solution. Tombstone (memory) overload.

Deleting the minimum

To delete the minimum key:

- Go left until finding a node with a null left link.
- Replace that node by its right link.
- Update subtree counts.



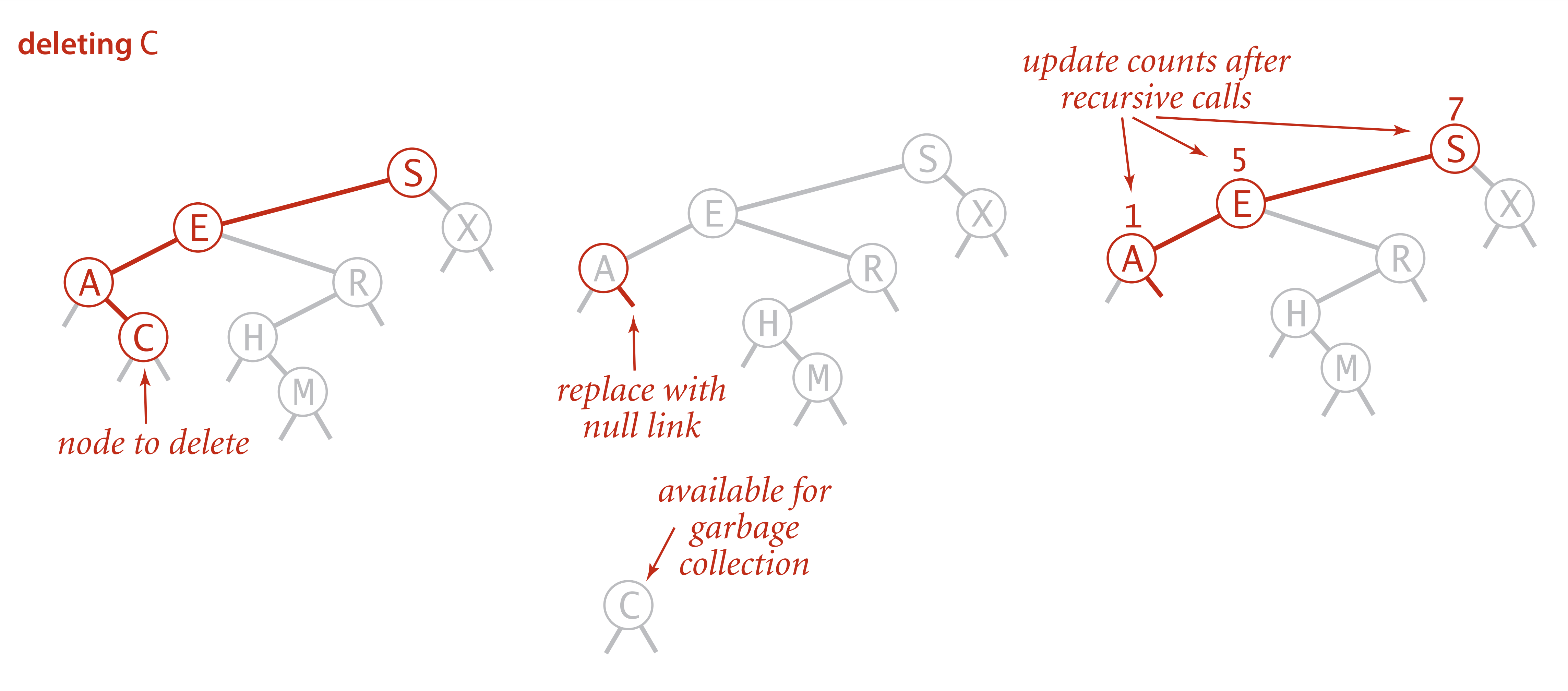

```
public void deleteMin()
{  root = deleteMin(root);  }

private Node deleteMin(Node x)
{
   if (x.left == null) return x.right;
   x.left = deleteMin(x.left);
   x.count = 1 + size(x.left) + size(x.right);
   return x;
}
```

Hibbard deletion

To delete a node with key k: search for node t containing key k.

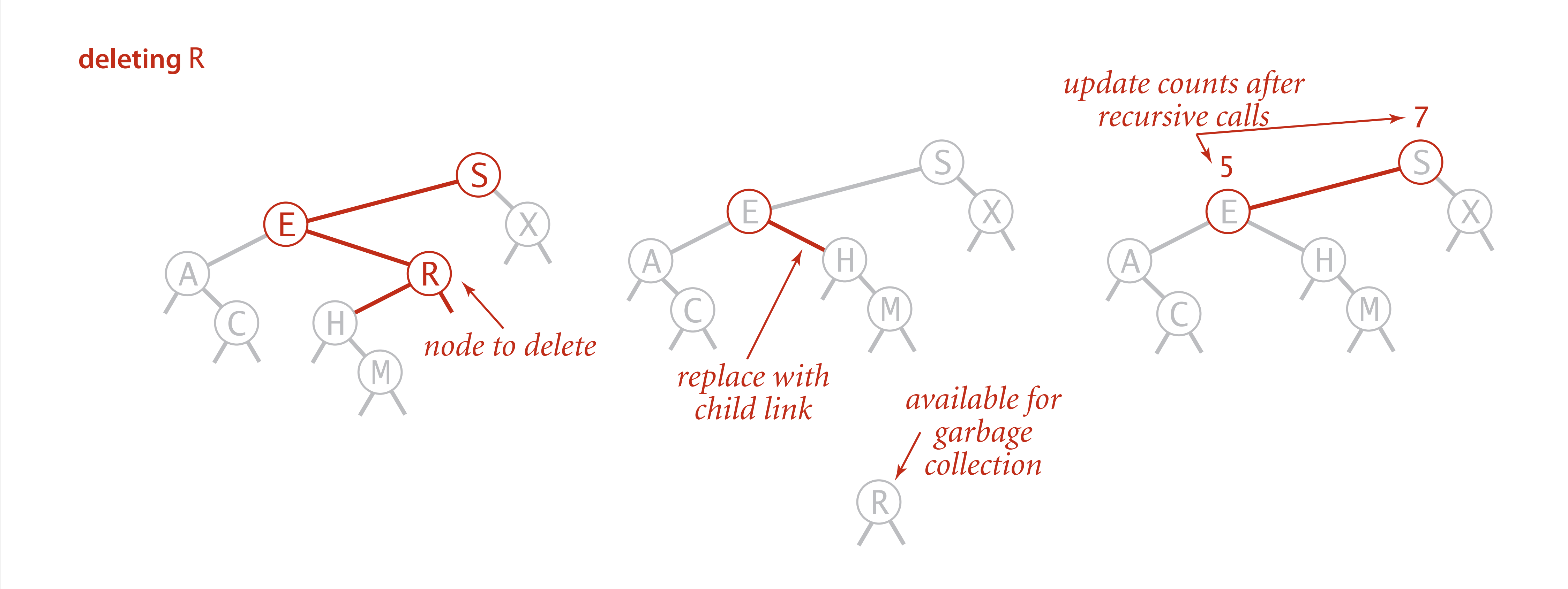
Case 0. [0 children] Delete t by setting parent link to null.



Hibbard deletion

To delete a node with key k: search for node t containing key k.

Case 1. [1 child] Delete t by replacing parent link.

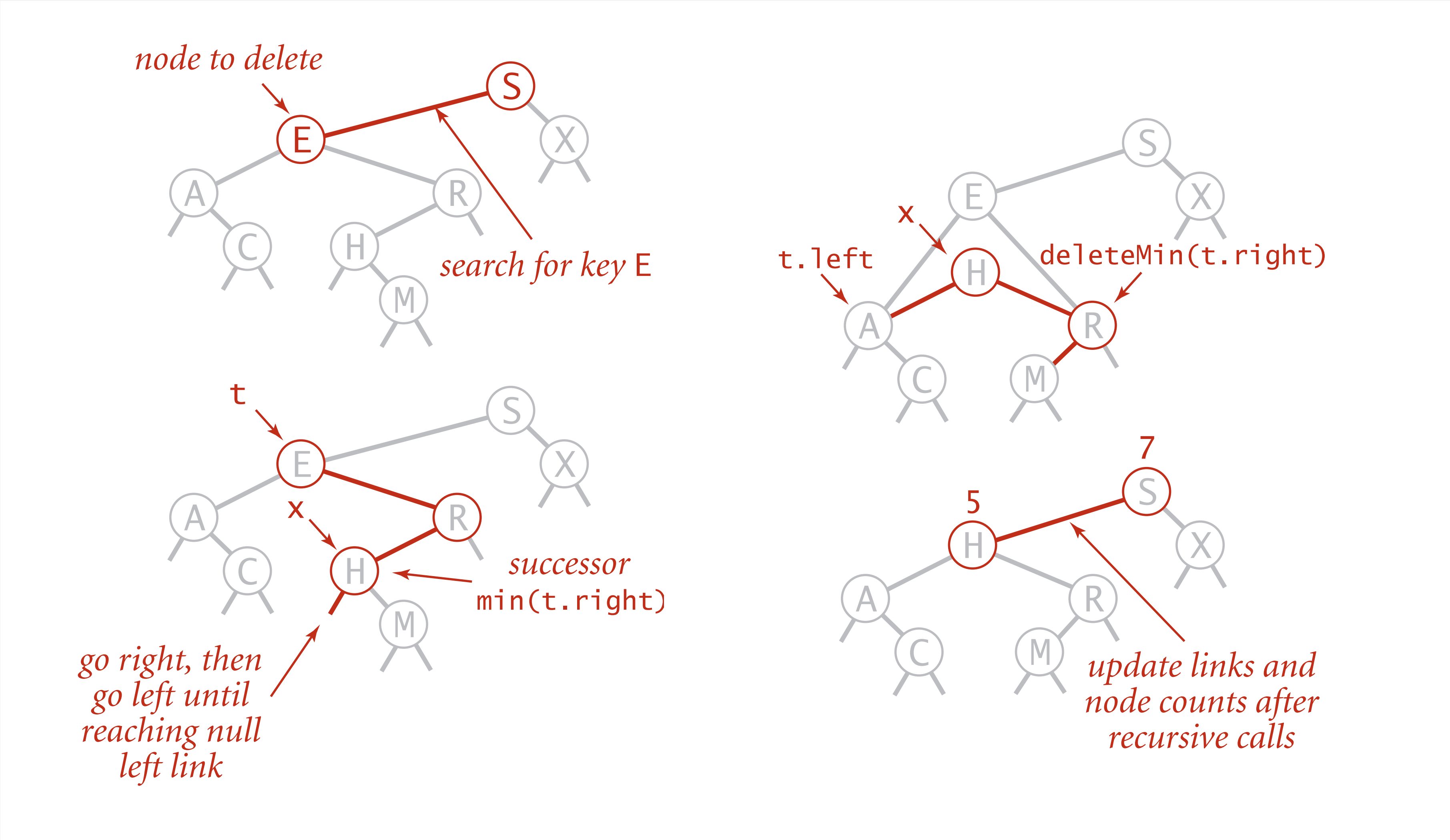


Hibbard deletion

To delete a node with key k: search for node t containing key k.

Case 2. [2 children]

- Find successor x of t. ← x has no left child
- Delete the minimum in t's right subtree. ← but don't garbage collect x
- Put x in t's spot. ← still a BST



Hibbard deletion: Java implementation

```
public void delete(Key key)
{  root = delete(root, key);  }

private Node delete(Node x, Key key) {
   if (x == null) return null;
   int cmp = key.compareTo(x.key);
   if      (cmp < 0) x.left  = delete(x.left,  key);
   else if (cmp > 0) x.right = delete(x.right, key);
   else {
      if (x.right == null) return x.left;
      if (x.left  == null) return x.right;

      Node t = x;
      x = min(t.right);
      x.right = deleteMin(t.right);
      x.left = t.left;
   }
   x.count = size(x.left) + size(x.right) + 1;
   return x;
```

search for key

no right child

no left child

replace with successor

update subtree counts

Notice anything?

Remember what I said about algorithms making arbitrary decisions leading to arbitrary paths through the code?

Take another look at the code:

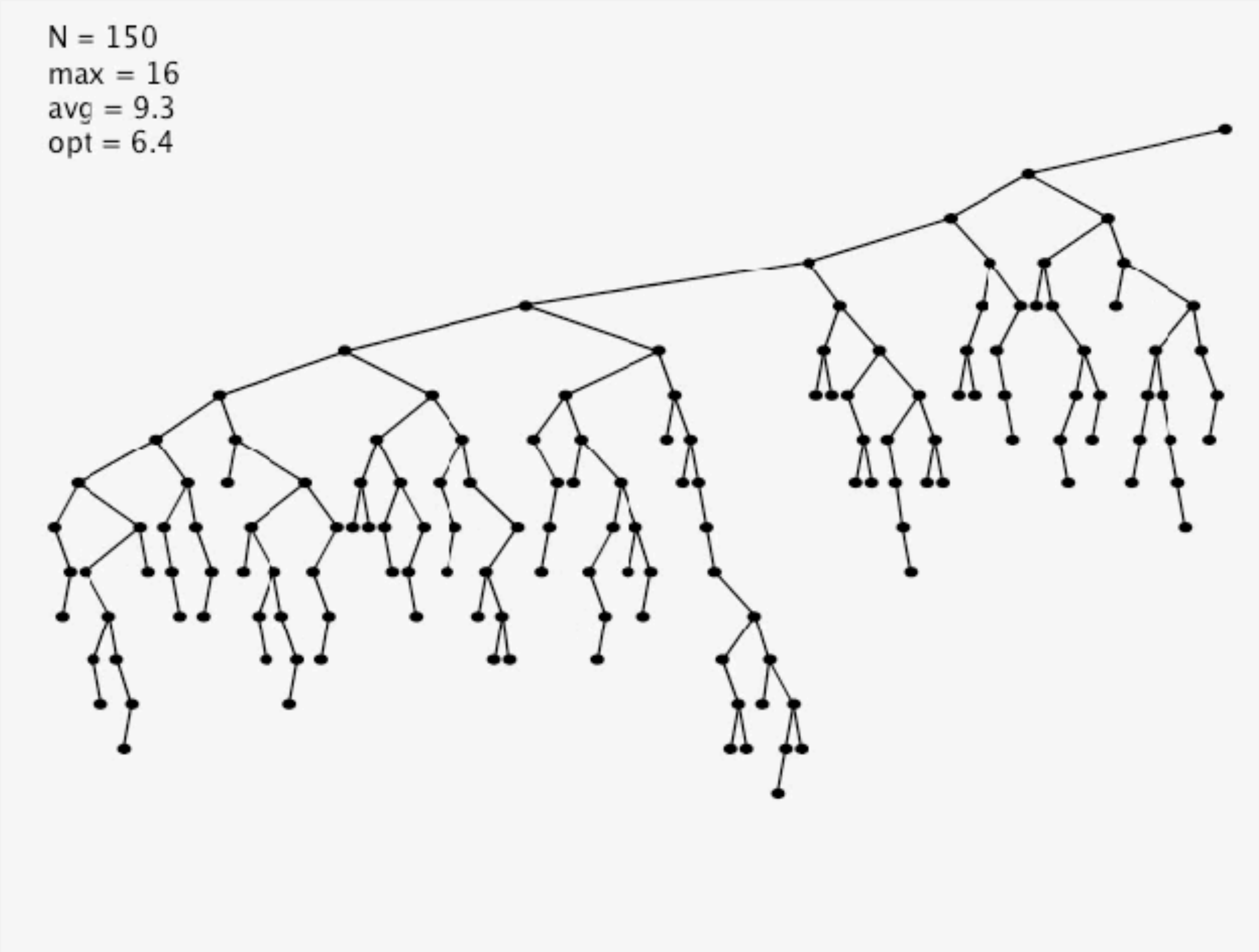
```
private Node delete(Node x, Key key) {
   if (x == null) return null;
    int cmp = key.compareTo(x.key);
    if      (cmp < 0) x.left  = delete(x.left,  key);
    else if (cmp > 0) x.right = delete(x.right, key);
    else {
       if (x.right == null) return x.left;
       if (x.left  == null) return x.right;
       Node t = x;
       x = min(t.right);
       x.right = deleteMin(t.right);
       x.left = t.left;
    }
    x.count = size(x.left) + size(x.right) + 1;
    return x;
}
```

It's "rightist"

Here too!

Hibbard deletion: analysis

Unsatisfactory solution. Not symmetric.



Surprising consequence. Trees not random (!) $\Rightarrow$ $\sqrt{N}$ per op (like random walk).

Longstanding open problem. Simple and efficient delete for BSTs.

ST implementations: summary

implementation	guarantee			average case			ordered ops?	operations on keys
	search	insert	delete	search hit	insert	delete		
sequential search (linked list)	N	N	N	$½ N$	N	$½ N$		equals()
binary search (ordered array)	$\lg N$	N	N	$\lg N$	$½ N$	$½ N$	✔	compareTo()
BST	N	N	N	$1.39 \lg N$	$1.39 \lg N$	$\sqrt{N}$	✔	compareTo()

other operations also become √N
if deletions allowed

Next lecture. Guarantee logarithmic performance for all operations.

What can we do about it?

Try to think of some strategies to avoid this problem.

Consider:

- What is it about the binary search tree that tends to cause this bad behavior?
- What modifications to the BST could perhaps be made to prevent this bad behavior?
- Remember how we improved from the linear list to the binary tree essentially by adding degrees of freedom?
- Hint: with linear list each node (element) has one successor;
- With binary tree, each node has two successors;
- …