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视觉SLAM中的矩阵李群

基 础

王 京
UAV514

提纲

- 3D刚体运动表示
- 3D运动李群
- 增量和导数
- 在视觉SLAM中的应用

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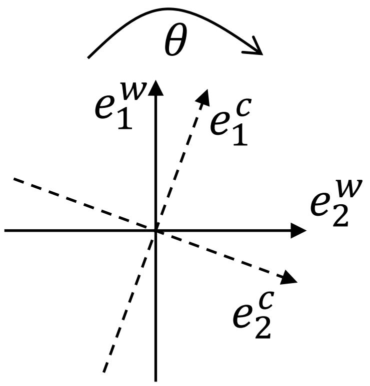
3D刚体运动

- 2D旋转
  - 刚体旋转等价于坐标系旋转，右手定则-旋转正方向
  - 坐标系$\mathcal{F}$的单位正交基向量$\boldsymbol{e}_1, \boldsymbol{e}_2 \in \mathbb{R}^2$
  - 旋转前坐标系表示$\mathcal{F}^w = [\boldsymbol{e}_1^w, \boldsymbol{e}_2^w]$
  - 旋转后坐标系表示$\mathcal{F}^c = [\boldsymbol{e}_1^c, \boldsymbol{e}_2^c]$
  - 正交基之间的线性组合关系

$$\boldsymbol{e}_1^c = \cos\theta \cdot \boldsymbol{e}_1^w + \sin\theta \cdot \boldsymbol{e}_2^w$$
$$\boldsymbol{e}_2^c = -\sin\theta \cdot \boldsymbol{e}_1^w + \cos\theta \cdot \boldsymbol{e}_2^w$$



$$[\boldsymbol{e}_1^c, \boldsymbol{e}_2^c] = [\boldsymbol{e}_1^w, \boldsymbol{e}_2^w] \cdot \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \stackrel{\text{def}}{=} [\boldsymbol{e}_1^w, \boldsymbol{e}_2^w] \cdot R_\theta$$

  - $\mathcal{F}^w$经过旋转R_{wc}得到了$\mathcal{F}^c$

$$\mathcal{F}^c = \mathcal{F}^w \cdot R_\theta = \mathcal{F}^w \cdot R_{wc}$$
$$R_{wc} = R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

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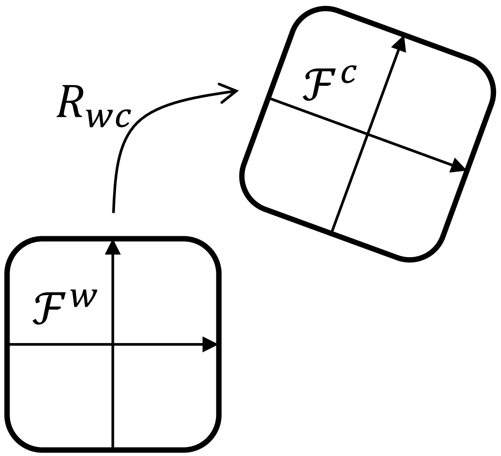
3D刚体运动

- 3D刚体固连坐标系$\mathcal{F}$的旋转
  - 刚体旋转等价于坐标系旋转
  - 坐标系$\mathcal{F}$的单位正交基$\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 \in \mathbb{R}^3$
  - 旋转前坐标系表示$\mathcal{F}^w = [\boldsymbol{e}_1^w, \boldsymbol{e}_2^w, \boldsymbol{e}_3^w]$
  - 旋转后坐标系表示$\mathcal{F}^c = [\boldsymbol{e}_1^c, \boldsymbol{e}_2^c, \boldsymbol{e}_3^c]$
  - 正交基之间的线性组合关系



$$\boldsymbol{e}_1^c = r_{11}\boldsymbol{e}_1^w + r_{21}\boldsymbol{e}_2^w + r_{31}\boldsymbol{e}_3^w$$
$$\boldsymbol{e}_2^c = r_{12}\boldsymbol{e}_1^w + r_{22}\boldsymbol{e}_2^w + r_{32}\boldsymbol{e}_3^w$$
$$\boldsymbol{e}_3^c = r_{13}\boldsymbol{e}_1^w + r_{23}\boldsymbol{e}_2^w + r_{33}\boldsymbol{e}_3^w$$

$$[\boldsymbol{e}_1^c, \boldsymbol{e}_2^c, \boldsymbol{e}_3^c] = [\boldsymbol{e}_1^w, \boldsymbol{e}_2^w, \boldsymbol{e}_3^w] \cdot \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \Leftrightarrow \mathcal{F}^c = \mathcal{F}^w \cdot R_{wc}$$

  - 含义：$\mathcal{F}^w$经过旋转R_{wc}得到了$\mathcal{F}^c$
  - （$R_{wc}R_{wc}^T = I$并且$\det(R_{wc}) = 1$）

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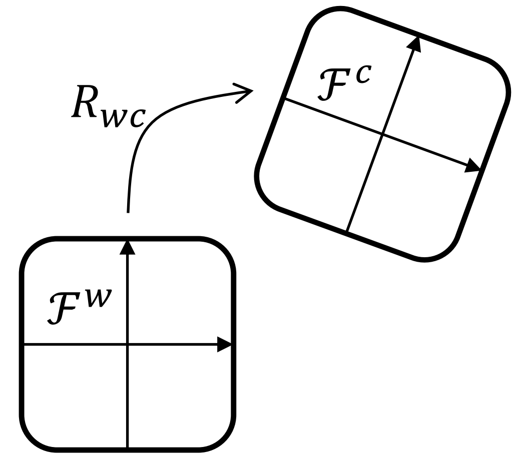
3D刚体运动

- 旋转作用下的固定点坐标变换
  - 3D空间中的固定点P
  - P在$\mathcal{F}^w$中坐标$p_w = [x_1, y_1, z_1]^T$
  - P在$\mathcal{F}^c$中坐标$p_c = [x_2, y_2, z_2]^T$
  - 不同坐标满足



$$P \coloneqq \mathcal{F}^w \cdot p_w = \mathcal{F}^c \cdot p_c$$

$$\mathcal{F}^w \cdot p_w = \mathcal{F}^c \cdot p_c = \mathcal{F}^w \cdot R_{wc} \cdot p_c \Rightarrow p_w = R_{wc} \cdot p_c$$

$$p_c = R_{wc}^{-1} \cdot p_w = R_{wc}^T \cdot p_w \stackrel{\text{def}}{=} R_{cw} \cdot p_w$$

$$\mathcal{F}^c = \mathcal{F}^w \cdot R_{wc} \quad R_{wc} = (\mathcal{F}^w)^{-1} \cdot \mathcal{F}^c$$

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3D刚体运动

- 刚体固连坐标系旋转 **vs** 固定点坐标变换
  - 刚体固连坐标系旋转

$$\mathcal{F}^c = \mathcal{F}^w \cdot R_{wc}$$

  - 固定点在不同坐标系下变换

$$p_c = R_{cw} \cdot p_w = R_{wc}^{-1} \cdot p_w \quad (= R_{wc}^T \cdot p_w)$$

  - 表示矩阵互为逆矩阵
  - 例：计算机体姿态欧拉角时，用R_{wc}(R_b^n?)
  - 北东地系下，偏航ψ-俯仰θ-滚转ϕ 顺序的机体欧拉角

$$\mathcal{F}^c = \mathcal{F}^w \cdot R_\psi R_\theta R_\phi = \mathcal{F}^w \cdot R_{wc}, \qquad p_c = R_{cw} p_w = R_{wc}^T p_w$$

$$R_{wc} = \begin{bmatrix} C_\psi & -S_\psi & \\ S_\psi & C_\psi & \\ & & 1 \end{bmatrix} \begin{bmatrix} C_\theta & & S_\theta \\ & 1 & \\ -S_\theta & & C_\theta \end{bmatrix} \begin{bmatrix} 1 & & \\ & C_\phi & -S_\phi \\ & S_\phi & C_\phi \end{bmatrix}$$

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3D刚体运动

- 考虑刚体平移
  - 除了旋转，坐标系原点$\mathcal{O}$有平移
  - 在$\mathcal{F}^w$下$\mathcal{O}_c$坐标即为$t_{wc} \in \mathbb{R}^3$

  - 固定点P的坐标变换？
  - 向量$V_{O_w \to P} = V_{O_w \to O_c} + V_{O_c \to P}$
  - 上式的 基向量x坐标 形式

$$\mathcal{F}^w \cdot p_w = \mathcal{F}^w \cdot t_{wc} + \mathcal{F}^c \cdot p_c$$

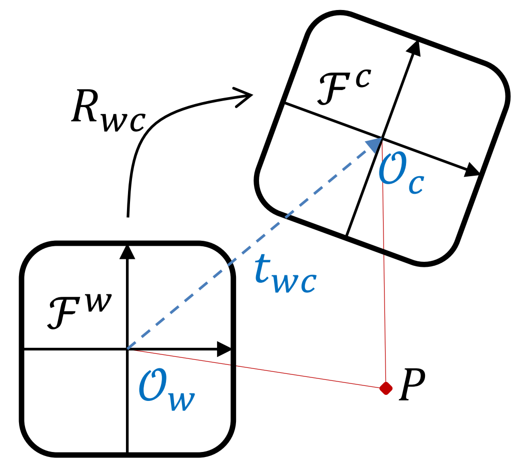
  - 由于$\mathcal{F}^c = \mathcal{F}^w \cdot R_{wc}$

$$p_w = t_{wc} + R_{wc} \cdot p_c$$

  - 齐次坐标：$\breve{p}_c = [p_c; 1]$

$$\breve{p}_w = \begin{bmatrix} R_{wc} & t_{wc} \\ 0 & 1 \end{bmatrix} \cdot \breve{p}_c \stackrel{\text{def}}{=} T_{wc} \cdot \breve{p}_c$$

  - $T_{wc}, T_{cw} \in SE(3), \quad T_{wc} = T_{cw}^{-1}$



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3D刚体运动

- 位姿更新的左乘、右乘
  - 假设$\mathcal{F}^w$是固定的参考坐标系，$\mathcal{F}^c$在运动相机上
  - 相机微小运动，固连坐标系运动到$\mathcal{F}^{c'}$，点坐标

$$\breve{p}_{c'} = T_{c'c}\breve{p}_c$$

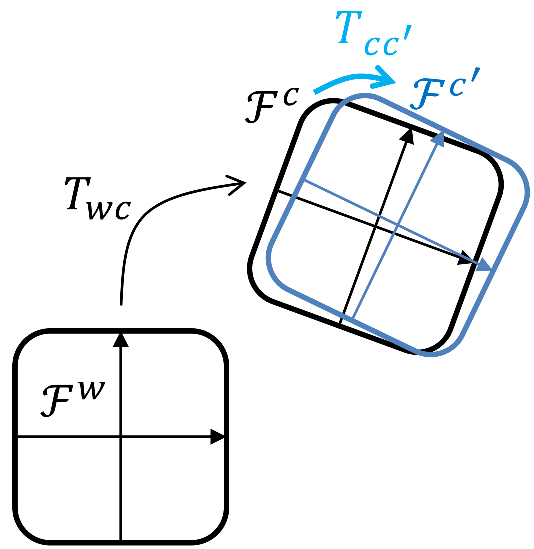
  - $\mathcal{F}^w$到$\mathcal{F}^{c'}$的固定点坐标变换矩阵

$$\breve{p}_{c'} = T_{c'c}\breve{p}_c = T_{c'c}T_{cw}\breve{p}_w = T_{c'w}\breve{p}_w$$

$$T_{c'w} = T_{c'c} \cdot T_{cw}$$

  - $\mathcal{F}^w$到$\mathcal{F}^{c'}$的刚体运动矩阵

$$T_{wc'} = T_{wc} \cdot T_{cc'}$$



  - 从$\mathcal{F}^w$到$\mathcal{F}^{c'}$
    - $T_{cw} \in SE(3)$的更新是左乘一个变化量（点坐标变换）
    - $T_{wc} \in SE(3)$的更新是右乘一个变化量（刚体坐标系运动）
    - 数学本质一样，物理含义不同，对左右乘的数学处理公式不同

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3D运动李群

- 李群基本数学定义
  - 群：集合G + 操作符 $\circ$: $G \circ G \rightarrow G$，满足：
    - 封闭性：$g_1 \circ g_2 \in G, \ \forall g_1, g_2 \in G$
    - 结合律：$(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3), \forall g_1, g_2, g_3 \in G$
    - 单位元：$\exists e \in G: \quad e \circ g = g \circ e = g, \forall g \in G$
    - 可逆性：$\exists g^{-1} \in G: \ g \circ g^{-1} = g^{-1} \circ g = e, \forall g \in G$

  - 李群：群 + 光滑
    - 群操作符的映射，是光滑映射
    - （整数群$\mathbb{Z}$不是李群）

  - 李群的李代数：向量空间 + 双线性操作符（李括号）
    - 数学空间，一个数域上的代数（algebra over a field）
    - 操作符4个性质：封闭性、双线性、alternating、雅克比等式（略）
    - 李群在单位元素处的切空间

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3D运动李群

- 常用李群举例
  - 一般线性群：$GL(n)$
    - 所有$n \times n$的可逆矩阵
    - 操作符为矩阵乘法
    - 单位元是单位矩阵$I_{n \times n}$
  - 正交群：$O(n) \subset GL(n)$
    - $O(n) = \{R \in GL(n) | R^T R = I\}$
  - 特殊正交群：$SO(n) \subset O(n) \subset GL(n)$
    - $SO(n) = \{R \in GL(n) | R^T R = I, \det(R) = +1\}$
  - 欧几里得群：$E(n) \subset GL(n+1)$
    - $E(n) = \left\{ \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} | R \in O(n), t \in \mathbb{R}^n \right\}$
  - 特殊欧几里得群： $SE(n) \subset GL(n+1)$
    - $SE(n) = \left\{ \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} | R \in SO(n), t \in \mathbb{R}^n \right\}$

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3D运动李群

- 反对称矩阵(skew-symmetric matrix)
  - 向量叉乘$a = [a_1, a_2, a_3]^T \in \mathbb{R}^3$和$b = [b_1, b_2, b_3]^T \in \mathbb{R}^3$

$$a \times b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \cdot b \stackrel{\text{def}}{=} [a]_\times b$$

  - $[a]_\times$是反对称矩阵，$[a]_\times^T = -[a]_\times$
  - 反对称矩阵与$\mathbb{R}^3$元素 一 一对应

  - 另一符号定义，$\widehat{w} \stackrel{\text{def}}{=} [w]_\times$
  - 向量$w = \phi \cdot u \in \mathbb{R}^3$, $\phi = |w|$为模长，性质：
    - $\widehat{w} w = w \times w = 0$
    - $\widehat{w}^2 = \widehat{w}\widehat{w} = \phi^2 \hat{u}\hat{u} = w w^T - \phi^2 I$
    - $\widehat{w}^3 = \widehat{w} \cdot \widehat{w}\widehat{w} = -\phi^2 \widehat{w}$

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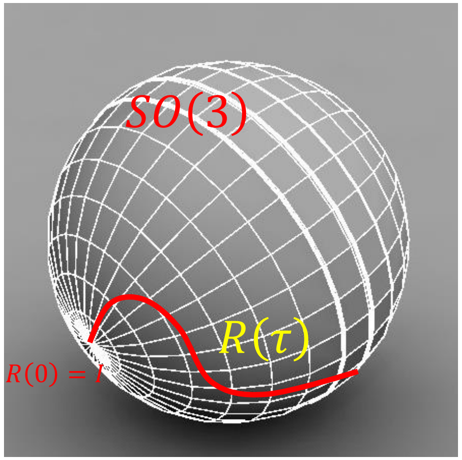
3D运动李群

- 3D特殊正交群$SO(3)$
  - $SO(3) = \{R \in GL(3) | R^T R = I, \det(R) = +1\}$
  - R代表3D旋转运动，是$SO(3)$的元素
  - 假设一个smooth path：
    - R随时间τ连续（光滑）变化，$R(\tau)$
    - $R(0) = I$，即0时刻没有旋转
  - 对$R(\tau)R(\tau)^T = I$关于τ求导

$$\left(R(\tau)\dot{R}(\tau)^T\right) = R(\tau)\dot{R}(\tau)^T + \dot{R}(\tau)R(\tau)^T = 0$$

$$\Rightarrow \dot{R}(\tau)R(\tau)^T = -R(\tau)\dot{R}(\tau)^T = -\left[\dot{R}(\tau)R(\tau)^T\right]^T$$



https://3dmagicmodels.com/shop/3d-models/sports-and-hobbies/outdoor-games/cricket-ball-3d-model/

  - $\dot{R}(\tau)R(\tau)^T$是一个反对称矩阵Ω，满足$\Omega^T = -\Omega$
  - $\exists \omega \in \mathbb{R}^3 \ s.t. \ \widehat{\omega} = \dot{R}(\tau)R(\tau)^T$

$$\dot{R}(\tau) = \widehat{\omega} \cdot R(\tau)$$

  - 注：$\left(R(\tau)^T R(\tau)\right)^{\cdot} = 0 \Rightarrow \dot{R}(\tau) = R(\tau) \cdot \widehat{\omega}$是另一种表示

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3D运动李群

- 3D特殊正交群$SO(3)$
  - $\dot{R}(\tau)=\widehat{\omega}\cdot R(\tau)\sim\dot{f}(x)=a\cdot f(x)$
  - 矩阵指数函数的导数
  - 矩阵（方阵）指数函数

$$\exp(M)=\sum_{n=0}^{+\infty}\frac{M^n}{n!}$$

  - 与标量指数区别：$\exp(A+B)=\exp(A)\cdot\exp(B)\Leftrightarrow AB=BA$
  - $\dot{R}(\tau)=\widehat{\omega}\cdot R(\tau)\Rightarrow R(\tau)=\exp(\widehat{\omega}\tau)\cdot R(0)$
  - 令$w=\omega\tau$，且$R(0)=I$

$$R=\exp(\widehat{w})$$

  - 角速度
    - $\dot{R}(\tau)=\widehat{\omega}\cdot R(\tau)$， $\omega=[\omega_1,\omega_2,\omega_3]^T\in\mathbb{R}^3$和刚体角速度有关

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3D运动李群

- 3D特殊正交群$SO(3)$
  - $R = \exp(\widehat{w})$

$$\exp(\widehat{w}) = \sum_{n=0}^{+\infty} \frac{\widehat{w}^n}{n!}$$

  - 展开

$$\exp(\widehat{w}) = \sum_{n=0}^{+\infty} \frac{\widehat{w}^n}{n!} = I + \widehat{w} + \frac{\widehat{w}^2}{2!} + \frac{\widehat{w}^3}{3!} + \frac{\widehat{w}^4}{4!} + \cdots$$

  - $w = |w| \cdot u \stackrel{\text{def}}{=} \phi u$，反对称矩阵性质$\widehat{w}^2 = \phi^2 \hat{u}\hat{u}, \widehat{w}^3 = -\phi^3 \hat{u}$
  - 简化

$$\exp(\widehat{w}) = I + \left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + \cdots\right)\hat{u} + \left(\frac{\phi^2}{2!} - \frac{\phi^4}{4!} + \frac{\phi^6}{6!} - \frac{\phi^8}{8!} + \cdots\right)\hat{u}\hat{u}$$
$$= I + \sin\phi \cdot \hat{u} + (1 - \cos\phi) \cdot \hat{u}\hat{u}$$

  - 即(Rodrigues公式) https://en.wikipedia.org/wiki/Rodrigues%27_rotation_formula

$$R = \exp(\widehat{w}) = I + \sin(|w|) \cdot \frac{\widehat{w}}{|w|} + (1 - \cos(|w|)) \cdot \frac{\widehat{w}\widehat{w}}{|w|^2}$$

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3D运动李群

- $SO(3)$的李代数$\mathfrak{so}(3)$

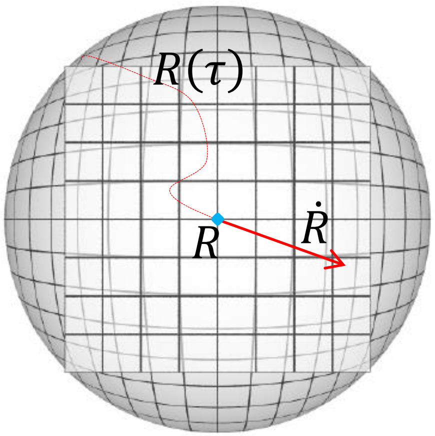
$$R = \exp(\hat{w}) = \exp(\hat{\omega}\tau), \qquad \dot{R} = \hat{\omega} \cdot R$$

  - $SO(3)$与$\hat{w}$之间是指数映射关系
  - $\dot{R}$代表R处的导数 ~ 切向量
  - 所有切向量组成：R处切空间

  - 在单位元处， $R = I, \dot{R} = \hat{\omega} \sim \hat{w}$
  - 各$\hat{w}$张成了单位元处的切空间

  - $\mathfrak{so}(3)$定义：
    - 向量空间： $\mathfrak{so}(3) = \{W = \hat{w} \in \mathbb{R}^{3\times3}, w \in \mathbb{R}^3\}$
    - 数域： 实数域$\mathbb{R}$
    - 李括号： $[W_1, W_2] = W_1W_2 - W_2W_1$
  - $\mathfrak{so}(3)$就是$SO(3)$在单位元处的切空间

最小表示$w \in \mathbb{R}^3 \rightarrow\rightarrow$ 李代数$\hat{w} \rightarrow\rightarrow$ 李群元素$R = \exp \hat{w}$



Strasdat, H. (2012). *Local accuracy and global consistency for efficient visual slam* (Doctoral dissertation, Department of Computing, Imperial College London).

3D运动李群

- $SE(3)$与李代数$\mathfrak{se}(3)$
  - $SE(3)$元素T

$$T = \left\{ \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \middle| R \in SO(3), t \in \mathbb{R}^3 \right\}$$

  - $\mathfrak{se}(3)$元素$\hat{\xi}$
    - 向量空间：$\mathfrak{se}(3) = \{\Xi = \hat{\xi} \in \mathbb{R}^{4\times4}, \xi = [w; v] \in \mathbb{R}^6, w \in \mathbb{R}^3, v \in \mathbb{R}^3\}$
    - 数域： 实数域$\mathbb{R}$
    - 李括号：$[\Xi_1, \Xi_2] = \Xi_1\Xi_2 - \Xi_2\Xi_1$

  - 其中的操作符^定义为

$$\hat{\xi} = \begin{bmatrix} \widehat{w} & v \\ 0 & 0 \end{bmatrix}, \qquad \xi = [w, v]^T$$

  - 下面说明这个操作符的来源及T与w, v的关系

？？？ 最小表示$\xi = [w; v] \to\to$ 李代数$\hat{\xi} \to\to$ 李群元素$T = \exp \hat{\xi}$

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3D运动李群

- $SE(3)$与李代数$\mathfrak{se}(3)$

$$T=\begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \Rightarrow T^{-1}=\begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix}$$

  – 类似$SO(3)$，假设随时间τ有$T(\tau)$并且 $T(0)=I$

  – 关于时间的导数与逆相乘

$$\dot{T}(\tau)T(\tau)^{-1}=\begin{bmatrix} \dot{R} & \dot{t} \\ 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} \dot{R}R^T & \dot{t}-\dot{R}R^T t \\ 0 & 0 \end{bmatrix}$$

    - 如果用$\mathrm{T}(\tau)^{-1}\dot{\mathrm{T}}(\tau)$？

  – $\exists\omega\in\mathbb{R}^3, s.t. \dot{R}R^T=[\omega]_\times$，因此

$$\dot{T}(\tau)=\begin{bmatrix} \widehat{\omega} & \dot{t}-\widehat{\omega}t \\ 0 & 0 \end{bmatrix}T(\tau) \Rightarrow T(\tau)=\exp\begin{bmatrix} \widehat{\omega}\tau & (\dot{t}-\widehat{\omega}t)\tau \\ 0 & 0 \end{bmatrix}\cdot T(0)$$

  – 令$w=\omega\tau, v=\dot{t}\tau-\widehat{w}t\in\mathbb{R}^3$，并且有$T(0)=I$，指数映射

$$T(\tau)=\exp\begin{bmatrix} \widehat{w} & v \\ 0 & 0 \end{bmatrix}\overset{\text{def}}{=}\exp\hat{\xi}$$

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3D运动李群

- $SE(3)$与李代数$\mathfrak{se}(3)$
  - 指数函数的泰勒级数

$$T=\exp(\hat{\xi})=\sum_{n=0}^{+\infty}\frac{\hat{\xi}^n}{n!}=\sum_{n=0}^{+\infty}\frac{1}{n!}\begin{bmatrix}\hat{w} & v\\ 0 & 0\end{bmatrix}^n$$

$$=\begin{bmatrix}\sum_{n=0}^{+\infty}\frac{1}{n!}\hat{w}^n & \sum_{n=0}^{+\infty}\frac{1}{(n+1)!}\hat{w}^n v\\ 0 & 1\end{bmatrix}$$

  - 化简形式

$$T=\exp(\hat{\xi})=\begin{bmatrix}\exp(\hat{w}) & J\cdot v\\ 0 & 1\end{bmatrix}=\begin{bmatrix}R & J\cdot v\\ 0 & 1\end{bmatrix}$$

  - 其中J见
    - TUM的MVG公开课[1]第二讲 公开课地址 https://vision.in.tum.de/teaching/ss2016/mvg2016
    - 或Barfoot的教材[4]

最小表示$\xi=[w;v]$ →→ 李代数$\hat{\xi}$ →→ 李群元素$T=\exp\hat{\xi}$

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3D运动李群

- $Sim(3)$与其李代数
  - 相似变换：平移+旋转+缩放（各向同性）
  - $Sim(3)$元素S

$$S = \begin{bmatrix} sR & t \\ 0 & 1 \end{bmatrix}, \qquad R \in SO(3), s \in \mathbb{R}, t \in \mathbb{R}^3$$

  - 类似$SO(3)$，假设随时间τ有$S(\tau)$并且$S(0) = I$

$$\dot{S} \cdot S^{-1} = \begin{bmatrix} \frac{\dot{s}}{s} RR^T + \dot{R}R^T & \dot{t} - \frac{\dot{s}}{s} RR^T t - \dot{R}R^T t \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\dot{s}}{s} I + \widehat{\omega} & \dot{t} - \frac{\dot{s}}{s} t - \widehat{\omega} t \\ 0 & 0 \end{bmatrix}$$

  - 其中$\exists \omega \in \mathbb{R}^3 \ s.t. \widehat{\omega} = \dot{R}R^T \in \mathfrak{so}(3)$

$$\dot{S} = \begin{bmatrix} \frac{\dot{s}}{s} I + \widehat{\omega} & \dot{t} - \frac{\dot{s}}{s} t - \widehat{\omega} t \\ 0 & 0 \end{bmatrix} \cdot S$$

  - 积分结果

$$S(\tau) = \exp\left(\begin{bmatrix} \frac{\dot{s}}{s} I + \widehat{\omega} & \dot{t} - \frac{\dot{s}}{s} t - \widehat{\omega} t \\ 0 & 0 \end{bmatrix} \cdot \tau \right) \cdot S(0)$$

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3D运动李群

- $Sim(3)$与其李代数

$$S(\tau) = \exp\left(\begin{bmatrix} \frac{\dot{s}}{s}\tau I + \widehat{\omega}\tau & \dot{t}\tau - \frac{\dot{s}}{s}\tau t - \widehat{\omega}\tau t \\ 0 & 0 \end{bmatrix}\right) \cdot S(0)$$

  - 令$\omega\tau = w, \frac{\dot{s}}{s}\tau = \sigma, v = \dot{t}\tau - \sigma t - \widehat{w}t$，并且$S(0) = I$
  - 定义$\zeta = [w; v; \sigma] \in \mathbb{R}^7$

– 李代数及hat操作符：

$$\hat{\cdot}_{Sim(3)} : \zeta = [w, v, \sigma] \in \mathbb{R}^7 \to \hat{\zeta} = \begin{bmatrix} \sigma I + \widehat{w} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

– 指数映射

$$S = \exp(\hat{\zeta}) = \sum_{n=0}^{+\infty} \frac{1}{n!} \begin{bmatrix} \sigma I + \widehat{w} & v \\ 0 & 0 \end{bmatrix}^n = \begin{bmatrix} e^{\sigma} \cdot R & J \cdot v \\ 0 & 1 \end{bmatrix}$$

  - 同样用泰勒级数计算。其中J参考Strasdat的博士论文5.2节

– $\zeta = [w; v; \sigma]$与s, R, t

$$s = e^{\sigma}, R = \exp(\widehat{w}), t = Jv$$

最小表示$\zeta = [w; v; \sigma] \to\to$ 李代数$\hat{\zeta} \to\to$ 李群元素$S = \exp\hat{\zeta}$

参考博士论文： Strasdat, H. (2012). *Local accuracy and global consistency for efficient visual slam* (Doctoral dissertation, Department of Computing, Imperial College London).

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提纲

- 3D刚体运动表示
- 3D运动李群
- 增量和导数
- 在视觉SLAM中的应用

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增量和导数

- BCH公式（**B**aker-**C**ampbell-**H**ausdorff，见wikepedia） https://en.wikipedia.org/wiki/Baker%E2%80%93Campbell%E2%80%93Hausdorff_formula
  - 对于标量的指数函数

$$\exp(a)\exp(b) = \exp(a+b)$$

  - 对于矩阵指数函数不同

$$\log(\exp X \exp Y) = \sum_{n>0} \frac{(-1)^{n-1}}{n} \sum_{\substack{r_i+s_i>0 \\ 1\le i\le n}} \frac{\left(\sum_{i=1}^{n}(r_i+s_i)\right)^{-1}}{r_1!\, s_1! \cdots r_n!\, s_n!} [X^{r_1}Y^{s_1}X^{r_2}Y^{s_2}\cdots X^{r_n}Y^{s_n}]$$

$$[X^{r_1}Y^{s_1}X^{r_2}Y^{s_2}\cdots X^{r_n}Y^{s_n}] = [\underbrace{X,[X,\ldots[X}_{r_1},\underbrace{[Y,[Y,\ldots[Y}_{s_1},\ldots\underbrace{[X,[X,\ldots[X}_{r_n},\underbrace{[Y,[Y,\ldots Y}_{s_n}]]\ldots]]$$

$$\begin{aligned}
Z(X,Y) &= \log(\exp X \exp Y) \\
&= X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] + [Y,[Y,X]]) \\
&\quad - \frac{1}{24}[Y,[X,[X,Y]]] \\
&\quad - \frac{1}{720}([Y,[Y,[Y,[Y,X]]]] + [X,[X,[X,[X,Y]]]]) \\
&\quad + \frac{1}{360}([X,[Y,[Y,[Y,X]]]] + [Y,[X,[X,[X,Y]]]]) \\
&\quad + \frac{1}{120}([Y,[X,[Y,[X,Y]]]] + [X,[Y,[X,[Y,X]]]]) + \cdots
\end{aligned}$$

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增量和导数

- $SO(3)$和$\mathfrak{so}(3)$增量
  - 1. $\mathfrak{so}(3)$加增量（谢晓佳@ZJU）
    - $R_0 = \exp(\widehat{w}_0), R = \exp(\widehat{w}_0 + \Delta\widehat{w})$
      - $R \approx \exp(J_l(\widehat{w}_0) \cdot \Delta\widehat{w}) \cdot \exp(w_0) = \exp(J_l(\widehat{w}_0) \cdot \Delta\widehat{w}) \cdot R_0$
      - $R \approx \exp(w_0) \cdot \exp(J_r(\widehat{w}_0) \cdot \Delta\widehat{w}) = R_0 \cdot \exp(J_r(\widehat{w}_0) \cdot \Delta\widehat{w})$
    - 左右导数 ($|w| = \phi$) 通过BCH公式求得
      - $J_l(\widehat{w}) = \frac{\sin\phi}{\phi} I + \left(1 - \frac{\sin\phi}{\phi}\right) aa^T + \frac{1-\cos\phi}{\phi} \hat{a}$
      - $J_r(\widehat{w}) = \frac{\sin\phi}{\phi} I + \left(1 - \frac{\sin\phi}{\phi}\right) aa^T - \frac{1-\cos\phi}{\phi} \hat{a}$
    - GTSAM 4.0(IMU-preintegration on manifold论文)中用到计算误差影响

  - 2. $SO(3)$乘增量
    - $R_0 = \exp(\widehat{w}_0), R = \Delta R \cdot R_0 = \exp(\Delta\widehat{w}) \cdot R_0$
      - 上面是左乘更新方式
      - 对于右乘更新，定义为$R = R_0 \cdot \exp(\Delta\widehat{w})$
    - VSLAM基本用此种更新方式
    - （主要介绍）

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增量和导数

- 点坐标关于增量的雅克比（大部分的**基础**）
  - 齐次坐标变换：$\breve{x} = T\breve{y} = [x; 1]$
  - 位姿T微小变化：$T_{\mathrm{trans}} = \exp(\hat{\xi}) \cdot T$
  - 变换后坐标：$\breve{x}_{\mathrm{trans}} = \exp(\hat{\xi}) \cdot T\breve{y} = \exp(\hat{\xi}) \cdot \breve{x}$
  - $\xi = [w; v] \to 0_{6\times1}$时，$\exp(\hat{\xi}) \approx I + \hat{\xi}$
  - 变换后坐标关于增量$\xi \to 0$的雅克比

$$\frac{\partial[\breve{x}_{\mathrm{trans}}]}{\partial\xi}\Big|_{\xi=0} = \frac{\partial\left[\exp(\hat{\xi}) \cdot \breve{x}\right]}{\partial\xi}\Big|_{\xi=0} \approx \frac{\partial\left[(I+\hat{\xi}) \cdot \breve{x}\right]}{\partial\xi}\Big|_{\xi=0}$$

$$= \frac{\partial[\hat{\xi} \cdot \breve{x}]}{\partial\xi}\Big|_{\xi=0} = \frac{\partial\left[\begin{bmatrix}\hat{w} & v \\ 0 & 0\end{bmatrix} \cdot \begin{bmatrix}x \\ 1\end{bmatrix}\right]}{\partial\xi}\Big|_{\xi=[w;v]=0} = \frac{\partial\begin{bmatrix}\hat{w}x + v \\ 0\end{bmatrix}}{\partial[w; v]}\Big|_{\xi=[w;v]=0}$$

$$= \frac{\partial\begin{bmatrix}-\hat{x}w + v \\ 0\end{bmatrix}}{\partial[w; v]} = \begin{bmatrix}-\hat{x} & I_{3\times3} \\ 0 & 0\end{bmatrix}$$

  - 对于非齐次坐标：$\frac{\partial x(\exp(\hat{\xi})T, y)}{\partial\xi}\Big|_{\xi=0} = \frac{\partial[x_{\mathrm{trans}}]}{\partial\xi}\Big|_{\xi=0} = [-\hat{x} \quad I_{3\times3}]$

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增量和导数

- 更麻烦的推导
  - $\xi = [\xi_1, \xi_2, \ldots, \xi_6]$的每一维分别考虑

$$\frac{\partial\left[\exp(\hat{\xi}) \cdot T\right]}{\partial \xi_k} \Big|_{\xi=0} = \frac{\partial\left[\exp(t\widehat{e_k}) \cdot T\right]}{\partial t} \Big|_{t=0}$$

    - 其中e_k是第k维单位向量，如$e_2 = [0,1,0,0,0,0]$
  - $\frac{\partial[\exp(t\widehat{e_k}) \cdot T]}{\partial t} \Big|_{t=0} = \left[\widehat{e_k} \cdot \exp(t\widehat{e_k}) \cdot T\right] \Big|_{t=0} = \widehat{e_k} \cdot T \stackrel{\text{def}}{=} G_k T$
    - 其中G_k是$\mathfrak{se}(3)$的generators

$$G_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad G_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad G_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad G_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  - 整体雅克比：4x4x6张量

$$\frac{\partial\left[\exp(\hat{\xi}) \cdot T\right]}{\partial \xi} \Big|_{\xi=0} = [G_1 T, G_2 T, G_3 T, G_4 T, G_5 T, G_6 T]$$

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增量和导数

- 更麻烦的推导-续
  - 整体雅克比：4x4x6张量

$$\frac{\partial[\exp(\hat{\xi})\cdot T]}{\partial \xi}\Big|_{\xi=0} = [G_1 T, G_2 T, G_3 T, G_4 T, G_5 T, G_6 T]$$

  - 点坐标变换关于增量的雅克比

$$\frac{\partial[\exp(\hat{\xi})\cdot T\breve{y}]}{\partial \xi}\Big|_{\xi=0} = [G_1 T\breve{y}, G_2 T\breve{y}, G_3 T\breve{y}, G_4 T\breve{y}, G_5 T\breve{y}, G_6 T\breve{y}]$$

$$= [G_1\breve{x}, G_2\breve{x}, G_3\breve{x}, G_4\breve{x}, G_5\breve{x}, G_6\breve{x}] = \begin{bmatrix} -\hat{x} & I_{3\times 3} \\ 0 & 0 \end{bmatrix}$$

  - Generators：$G_k \stackrel{\text{def}}{=} \widehat{e_k}$

$$G_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad G_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad G_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad G_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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提纲

- 3D刚体运动表示
- 3D运动李群
- 增量和导数
- 在视觉SLAM中的应用

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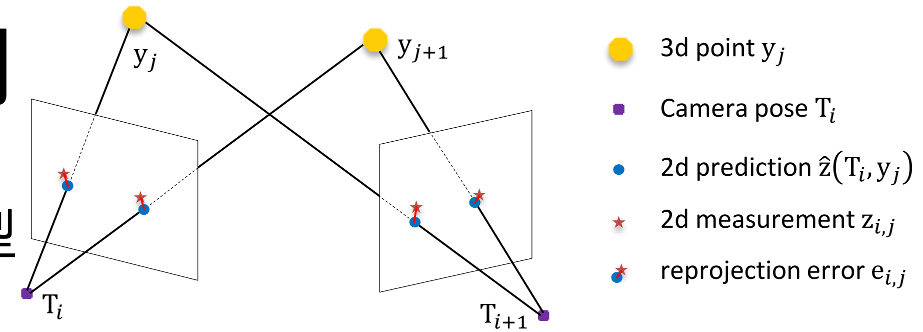
视觉SLAM中的应用



- 重投影误差的雅克比——相机模型
  - 相机内参矩阵$K = \begin{bmatrix} f_u & 0 & c_u \\ 0 & f_v & c_v \end{bmatrix}$
  - 相机坐标系中3D点坐标$x = [x_1, x_2, x_3]^T \in \mathbb{R}^3$
  - 相机投影模型

$$\hat{z}(x) = \mathrm{proj}(x) = K \cdot \frac{1}{x_3} x$$

  - 投影位置关于x的雅克比

$$\frac{\partial \hat{z}(x)}{\partial x} = \begin{bmatrix} \frac{f_u}{x_3} & 0 & -\frac{f_u x_1}{x_3^2} \\ 0 & \frac{f_v}{x_3} & -\frac{f_v x_2}{x_3^2} \end{bmatrix}$$

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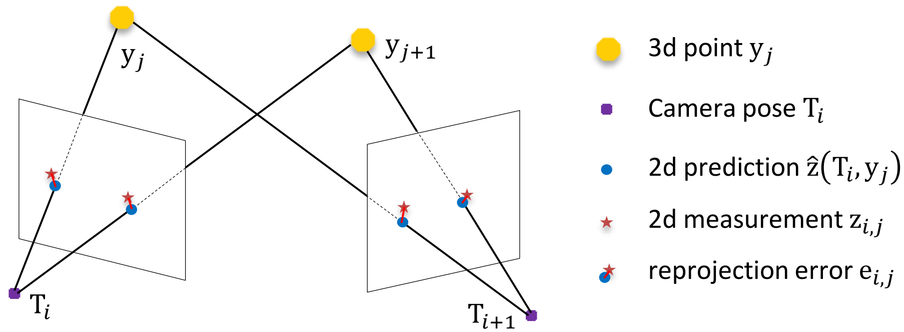
视觉SLAM中的应用



- 重投影误差的雅克比
  - 相机位姿T和3D点y
  - y在图像中观测位置z，估计位置$\hat{z}(T,y)$
  - 最小化重投影误差：

$$e(T,y) = z - \hat{z}(T,y) = z - \mathrm{proj}([I_{3\times 3} \quad 0_{3\times 1}] \cdot T\breve{y})$$

  - 误差关于位姿增量ξ的雅克比

$$\frac{\partial e\left(\exp\left(\hat{\xi}\right)T,y\right)}{\partial \xi}\Big|_{\xi=0} = -\frac{\partial \hat{z}(x)}{\partial x}\Big|_{\breve{x}=T\breve{y}} \cdot \frac{\partial x\left(\exp\left(\hat{\xi}\right)T,y\right)}{\partial \xi}\Big|_{\xi=0}$$

$$= -\begin{bmatrix} \frac{f_u}{x_3} & 0 & -\frac{f_u x_1}{x_3^2} \\ 0 & \frac{f_v}{x_3} & -\frac{f_v x_2}{x_3^2} \end{bmatrix} \cdot [-\hat{x} \quad I_{3\times 3}] \qquad x\text{：相机系中点坐标}$$

  - 误差关于3D点y的雅克比

$$\frac{\partial e(T,y)}{\partial y} = -\frac{\partial \hat{z}(x)}{\partial x} \cdot \frac{\partial x(T,y)}{\partial y} = -\frac{\partial \hat{z}(x)}{\partial x} \cdot R$$

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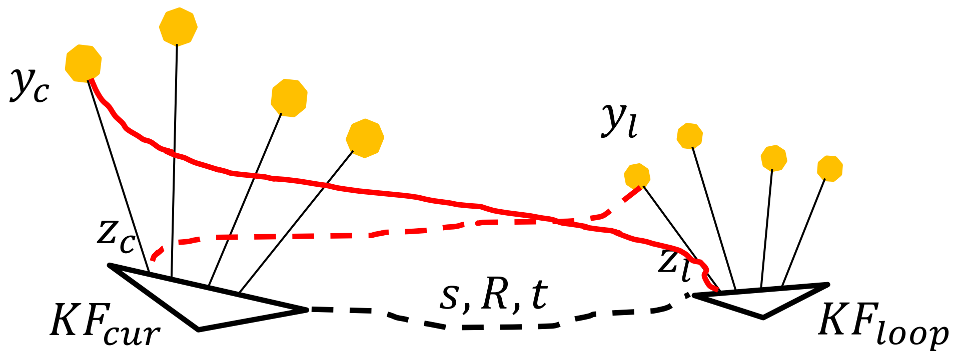
视觉SLAM中的应用

- 最小化重投影误差计算Sim3
  - 单目SLAM，没有绝对尺度
  - 误差累积可能导致尺度“漂移”
  - 现象：运行较长路径回到之前的地方，之前局部地图和当前局部地图的尺度不同



  - 上图
    - 两组局部地图点实际对应相同的3D点
    - 但在局部地图中尺度不一样。
  - SVD+RANSAC可以求解两组点的Sim3
  - ORBSLAM中，进一步OptimizeSim3()。但尺度会影响重投影误差？

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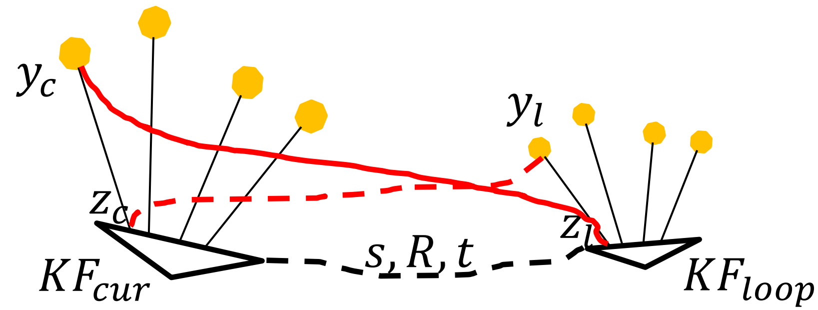
视觉SLAM中的应用



- 最小化重投影误差计算Sim3
  - 两者间变换S_{cl}

$$S_{cl} = \begin{bmatrix} s_{cl}R_{cl} & t_{cl} \\ 0 & 1 \end{bmatrix}, s_{cl} \in \mathbb{R}, R_{cl} \in SO(3), t_c \in \mathbb{R}^3$$

$$S_{lc} = S_{cl}^{-1} = \begin{bmatrix} \frac{1}{s_{cl}} R_{cl}^T & -\frac{1}{s_{cl}} R_{cl}^T t_{cl} \\ 0 & 1 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} s_{lc}R_{lc} & t_{lc} \\ 0 & 1 \end{bmatrix}$$

  - 点坐标变换$y_c = s_{cl}R_{cl}y_l + t_{cl}$

  - 优化计算策略
    - y_l通过S_{cl}变换到KF_{cur}并投影，重投影误差

$$e_c = z_c - \hat{z}_c = z_c - \mathrm{proj}(s_{cl}R_{cl}y_l + t_{cl})$$

    - y_c通过S_{lc}投影到KF_{loop}的重投影误差

$$e_l = z_l - \hat{z}_l = z_l - \mathrm{proj}(s_{lc}R_{lc}y_c + t_{lc})$$

    - 通过调整$S_{cl} \in Sim(3)$使所有这些残差之和最小
    - 每对地图点对应同一个被优化量S_{cl}，g2o中（正、逆投影）两条边

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视觉SLAM中的应用

- 最小化重投影误差计算Sim3

$$e_c = z_c - \hat{z}_c = z_c - \mathrm{proj}(s_{cl}R_{cl}y_l + t_{cl})$$
$$e_l = z_l - \hat{z}_l = z_l - \mathrm{proj}(s_{lc}R_{lc}y_c + t_{lc})$$

  - e_c关于y_l的雅克比

$$\frac{\partial e_c(S_{cl}, y_l)}{\partial y_l} = -\frac{\partial \hat{z}_c(x)}{\partial x} \cdot \frac{\partial x(S_{cl}, y_l)}{\partial y_l} \Big|_{x=s_{cl}R_{cl}y_l+t_{cl}} = -\frac{\partial \hat{z}_c(x)}{\partial x} \cdot s_{cl}R_{cl}$$

  - e_l关于y_c的雅克比

$$J_{y_c}^{e_l} \stackrel{\text{def}}{=} \frac{\partial e_l(S_{lc}, y_c)}{\partial y_c} = -\frac{\partial \hat{z}_l(x)}{\partial x} \cdot \frac{\partial x(S_{lc}, y_c)}{\partial y_c} \Big|_{x=s_{lc}R_{lc}y_c+t_{lc}} = -\frac{\partial \hat{z}_l(x)}{\partial x} \cdot s_{lc}R_{lc}$$

  - 位姿更新：$S_{cl} \leftarrow S_{cl}(\zeta) \stackrel{\text{def}}{=} \exp(\hat{\zeta})\, S_{cl}$
  - $\zeta = [w; v; \sigma] \in \mathbb{R}^7, w, v \in \mathbb{R}^3, \sigma \in \mathbb{R},\qquad \hat{\zeta} = \begin{bmatrix} \sigma I + \widehat{w} & v \\ 0 & 0 \end{bmatrix}$
  - e_c, e_l关于$\hat{\zeta}$的雅克比？

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视觉SLAM中的应用

- 最小化重投影误差计算Sim3
  - e_c关于S_{cl}增量$\hat{\zeta}$的雅克比
  - $\zeta \to 0$时， $\exp(\hat{\zeta}) \approx I + \hat{\zeta} = I + \begin{bmatrix} \sigma I + \widehat{w} & v \\ 0 & 0 \end{bmatrix}$

$$\frac{\partial e_c\left(S_{cl}(\hat{\zeta}), y_l\right)}{\partial \zeta}\Big|_{\zeta=0} = -\frac{\partial \hat{z}_c(x)}{\partial x}\Big|_{\breve{x}=S_{cl}\cdot\breve{y}_l} \cdot \frac{\partial x\left(S_{cl}(\hat{\zeta}), y_l\right)}{\partial \zeta}\Big|_{\zeta=0}$$

  - 令$S_{cl} \cdot \breve{y}_l = \breve{x}$，后一项

$$\frac{\partial x\left(S_{cl}(\hat{\zeta}), y_l\right)}{\partial \zeta}\Big|_{\zeta=0} = [I_{3\times3} \quad 0_{3\times1}] \cdot \frac{\partial \exp\left(\hat{\zeta}\right) S_{cl} \cdot \breve{y}_l}{\partial \zeta}\Big|_{\zeta=0}$$

$$= [I \quad 0] \cdot \frac{\partial \left(x + \begin{bmatrix} \sigma x + \widehat{w} x & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}\right)}{\partial [w; v; \sigma]}\Big|_{[w;v;\sigma]=0} = [-\hat{x} \quad I_{3\times3} \quad x]$$

  - 整体代入：尺度会影响重投影误差？

$$\frac{\partial e_c(S_{cl}(\zeta), y_c)}{\partial \zeta}\Big|_{\zeta=0} = \begin{bmatrix} -\frac{f_u}{x_3} & 0 & \frac{x_1 f_u}{x_3^2} & \frac{x_1 x_2}{x_3^2} f_u & -\left(1 + \frac{x_1^2}{x_3^2}\right) f_u & \frac{x_2}{x_3} f_u & 0 \\ 0 & -\frac{f_v}{x_3} & \frac{x_2 f_v}{x_3^2} & \left(1 + \frac{x_2^2}{x_3^2}\right) f_v & -\frac{x_1 x_2}{x_3^2} f_v & -\frac{x_1}{x_3} f_v & 0 \end{bmatrix}$$

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视觉SLAM中的应用

- 最小化重投影误差计算Sim3
  - e_c关于S_{cl}增量$\hat{\zeta}$的雅克比
  - 从Generators角度考虑
    - Sim3的Generators，前6个和SE3一样，第7个

$$G_7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  - 关于每个维度的偏导

$$\frac{\partial \exp(\hat{\zeta})}{\partial \zeta} = [G_1, G_2, G_3, G_4, G_5, G_6, G_7]$$

  - 代入计算可得相同结果

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视觉SLAM中的应用

- 最小化重投影误差计算Sim3
  - e_l关于S_{cl}增量$\hat{\zeta}$的雅克比（贺一家@中科院自动化所）
  - S_{lc}与S_{cl}变化相关联
  - S_{lc}位姿更新$S_{lc} \leftarrow \left(\exp(\hat{\zeta})\, S_{cl}\right)^{-1} = S_{cl}(\zeta)^{-1}$
  - 雅克比

$$\frac{\partial e_l(S_{cl}(\zeta)^{-1}, y_c)}{\partial \zeta}\Big|_{\zeta=0} = -\frac{\partial \hat{z}_l(x)}{\partial x}\Big|_{\breve{x}=S_{lc}\breve{y}_c} \cdot \frac{\partial x(S_{cl}(\zeta)^{-1}, y_c)}{\partial \zeta}\Big|_{\zeta=0}$$

$$= -\frac{\partial \hat{z}_l(x)}{\partial x}\Big|_{\breve{x}=S_{lc}\breve{y}_c} \cdot \begin{bmatrix} I_{3\times3} & 0_{3\times1} \end{bmatrix} \cdot \frac{\partial S_{cl}(\zeta)^{-1} \cdot \breve{y}_c}{\partial \zeta}\Big|_{\zeta=0}$$

  - 逆矩阵求导公式

$$\frac{\partial A^{-1}}{\partial a} = -A^{-1}\frac{\partial A}{\partial a}A^{-1}$$

  - 后一项化简

$$\frac{\partial S_{cl}(\zeta)^{-1} \cdot \breve{y}_c}{\partial \zeta}\Big|_{\zeta=0} = -\left[S_{lc} \cdot \frac{\partial S_{cl}(\zeta)}{\partial \zeta} \cdot S_{lc} \cdot \breve{y}_c\right]\Big|_{\zeta=0} = -S_{lc} \cdot \begin{bmatrix} -\widehat{y}_c & I_{3\times3} & y_c \\ 0 & 0 & 0 \end{bmatrix}$$

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视觉SLAM中的应用

- 最小化重投影误差计算Sim3
  - e_l关于S_{cl}增量ζ的雅克比（贺一家@中科院自动化所）
  - 整体代入

$$\frac{\partial e_l(S_{cl}(\zeta)^{-1}, y_c)}{\partial \zeta}\Big|_{\zeta=0} = -\frac{\partial \hat{z}_c(x)}{\partial x}\Big|_{x=s_{lc}R_{lc}y_c+t_{lc}} \cdot s_{lc}R_{lc} \cdot [\widehat{y_c} \quad -I_{3\times3} \quad -y_c]$$

$$= J_{y_c}^{e_l} \cdot [\widehat{y_c} \quad -I_{3\times3} \quad -y_c]$$

  - 红色部分是e_l关于y_c的雅克比$J_{y_c}^{e_l}$
  - 尺度会影响重投影误差？
    - 展开上式，尺度相关的最后一列不为0

$$-\begin{bmatrix} \frac{f_u}{x_3} & 0 & -\frac{f_u x_1}{x_3^2} \\ 0 & \frac{f_v}{x_3} & -\frac{f_v x_2}{x_3^2} \end{bmatrix} \cdot (t_{lc} - x) = -\begin{bmatrix} \frac{f_u}{x_3} & 0 & -\frac{f_u x_1}{x_3^2} \\ 0 & \frac{f_v}{x_3} & -\frac{f_v x_2}{x_3^2} \end{bmatrix} t_{lc} = \begin{bmatrix} \frac{f_u t_1}{x_3} - \frac{f_u x_1 t_3}{x_3^2} \\ \frac{f_v t_2}{x_3} - \frac{f_v x_2 t_3}{x_3^2} \end{bmatrix}$$

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参考资料

- [1] TUM的Multiple View Geometry公开课 https://www.youtube.com/playlist?list=PLTBdjV_4f-EJn6udZ34tht9EVIW7lbeo4
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- [5] Strasdat, H. (2012). *Local accuracy and global consistency for efficient visual slam* (Doctoral dissertation, Department of Computing, Imperial College London).

谢　谢