Extragalactic hw8

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Problem: stellar dynamics, analytic exercise 8

Ans:

(a)The distribution function f :

$$f = \begin{cases} k_1(-E)^p & E < 0 \\ 0 & E > 0 \end{cases} \tag{1}$$

f is non-zero only when $E < 0$, which fact will be used in the following calculation. Energy E takes the form:

$$E = \frac{1}{2}mv^2 + m\phi \tag{2}$$

Thus $E < 0$ corresponds to $v < v_{max} = \sqrt{2(-\phi)}$.

Density $\rho(\vec{x}) = mn(\vec{x})$, where n is number density and can be calculated by integrating $f(\vec{x}, \vec{v})$ over the velocity space:

$$\begin{aligned}
n &= \int f\, d^3\vec{v} \\
&= \int_0^\infty 4\pi v^2 f\, dv \\
&= \int_0^{v_{max}} 4\pi v^2 k_1(-\frac{1}{2}mv^2 - m\phi)^p\, dv + \int_{v_{max}}^\infty 4\pi v^2 * 0\, dv \\
&= 4\pi k_1(\frac{m}{2})^p \int_0^{v_{max}=\sqrt{2(-\phi)}} v^2(2(-\phi) - v^2)\, dv \\
&= 4\pi k_1(\frac{m}{2})^p(-2\phi)^{\frac{3+2p}{2}} \frac{\Gamma(3/2)\Gamma(p+1)}{2\Gamma(3/2+p+1)} \\
&\propto (-\phi)^{p+3/2}
\end{aligned}$$

$\therefore \rho = mn \propto (-\phi)^{p+3/2}$, or equivalently $\rho = k_2(-\phi)^{p+3/2}$.

(b) We want to show that potential below satisfies the relation $\rho = k_2(-\phi)^5$.

$$\phi = -\frac{GM}{R}\frac{1}{(1 + r^2/R^2)^{1/2}} \tag{3}$$

We know that the potential satisfies Poisson's equation:

$$\nabla^2\phi = 4\pi G\rho \tag{4}$$

For spherical equilibrium, potential only depend on radius r. Calculate LHS of Eqn.(4) with Eqn.(3) plugged in:

$$\begin{aligned}
\nabla^2\phi &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\phi}{\partial r}) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{GM}{2R}\frac{2r/R^2}{(1+r^2/R^2)^{3/2}}) \\
&= \frac{GM}{R^3}\frac{1}{r^2}\frac{\partial}{\partial r}(\frac{r^3}{(1+r^2/R^2)^{3/2}}) \\
&= \frac{GM}{R^3}(\frac{3(1+r^2/R^2)^{3/2}-r^2\frac{3}{2}(1+r^2/R^2)^{1/2}\frac{2}{R^2}}{(1+r^2/R^2)^3}) \\
&= \frac{3GM}{R^3(1+r^2/R^2)^{5/2}} \\
&\propto (-\phi)^5
\end{aligned}$$

$\therefore \rho \propto \nabla^2\phi \propto (-\phi)^5$

The proportionality can be easily fixed by assign k_1 properly.