Finite Difference Methods I

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我们已经知道如何推导PDE

B-S PDE $\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$

exchange $\frac{\partial W}{\partial t} + \frac{1}{2}\sigma_Y^2 Y^2 \frac{\partial^2 W}{\partial Y^2} = 0$

quanto $\frac{\partial V}{\partial t} + (r_f - d - \rho\sigma\sigma_e) \cdots$

⇒ European Call / Put formula

⇒ Get parameters for a binomial tree 这样我们就能给 American / Barrier 定价了

接下来我们要 Valuing
- American options
- Callable
- Autocall

using PDE directly

使用 Finite Difference Methods

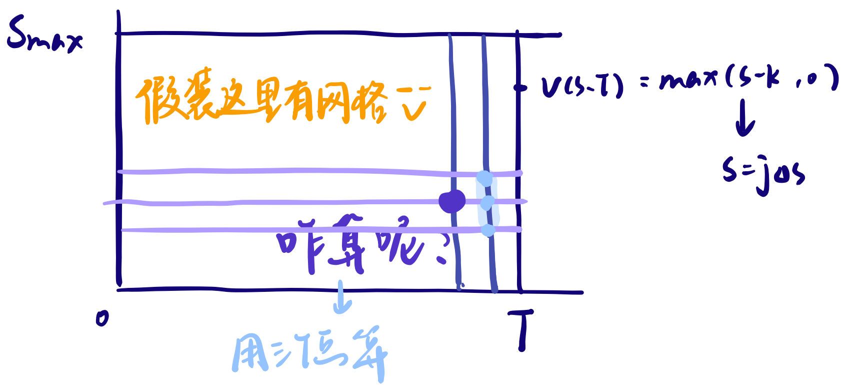
理论上和二叉树有很多相似之处

引:

Finite Difference Method	Binomial Tree
Need Payoff at T	Need Payoff at T
But more set-up cost ☹ ← Choose all step sizes ☺	Tree steps based on T, N, σ (e.g. $u = e^{\sigma\sqrt{\delta t}}$)
Use approximation to $\frac{\partial V}{\partial t}, \frac{\partial V}{\partial S}, \frac{\partial^2 V}{\partial S^2}$ 来解PDE	用二叉树推出价值. $V_0 = e^{-rT} E_0^{RN}[V_T]$

下面是详细介绍

首先的一切是估 parameters。怎么估的呢？用树上的数据。Taylor Expansion Formula。具体如何做呢？



这三个是用Taylor展开的

$$
\begin{aligned}
V(S+\Delta S, t) &= V(S,t) + \Delta S \frac{\partial V}{\partial S} + \frac{1}{2}(\Delta S)^2 \frac{\partial^2 V}{\partial S^2} + O(\Delta S^3) \quad (1) \\
V(S-\Delta S, t) &= V(S,t) - \Delta S \frac{\partial V}{\partial S} + \frac{1}{2}(\Delta S)^2 \frac{\partial^2 V}{\partial S^2} + O(\Delta S^3) \quad (2) \\
V(S, t+\Delta t) &= V(S,t) + \Delta t \frac{\partial V}{\partial t} + \frac{1}{2}(\Delta t)^2 \frac{\partial^2 V}{\partial t^2} + O(\Delta t^3) \quad (3)
\end{aligned}
$$

我们想算 $\frac{\partial V}{\partial t}$，$\frac{\partial V}{\partial S}$，$\frac{\partial^2 V}{\partial S^2}$

I
$$
\begin{aligned}
\frac{\partial V}{\partial S} &= \frac{V(S+\Delta S, t) - V(S,t)}{\Delta S} - \frac{1}{2}\Delta S \frac{\partial^2 V}{\partial S^2} + O((\Delta S)^2) \\
&= \frac{V(S+\Delta S, t) - V(S,t)}{\Delta S} + O(\Delta S)
\end{aligned}
$$

II
$$
\frac{\partial V}{\partial S} = \frac{V(S+\Delta S, t) - V(S-\Delta S, t)}{2\Delta S} + O((\Delta S)^2)
$$

Quiz：对于 $\frac{\partial V}{\partial S}$ 来说哪个好呢？ II 好。因为 II 的误差比 I 的阶更小

$$
\frac{\partial^2 V}{\partial S^2} = \frac{V(S+\Delta S, t) - 2V(S,t) + V(S+\Delta S, t)}{(\Delta S)^2} + O((\Delta S)^2)
$$

$$
\begin{aligned}
\frac{\partial V}{\partial t} &= \frac{V(S, t+\Delta t) - V(S,t)}{\Delta t} + \frac{1}{2}\Delta t \frac{\partial^2 V}{\partial t^2} + O((\Delta t)^2) \\
&= \frac{V(S, t+\Delta t) - V(S,t)}{\Delta t} + \underbrace{O(\Delta t)}_{\text{很大}}
\end{aligned}
$$

但是 $\frac{\partial V}{\partial t}$ 很有趣. 我们并没有用 $V(s,t) - V(s,t-\Delta t)$

这个叫做 forward differencing

我们把 PDE 里所有的系数都估出来了，这样就能从外到里把网格中的值都求出来

我们以前算过 Boundary Condition

多说一嘴关于 American Put:
$$\begin{cases} V(s,T) = \max(K-s,0) \\ V(0,t) = K \\ V(s,t) = 0 \quad \text{as} \quad s \to \infty \end{cases}$$

接下来就可以画网格了: S 有 j_{max} 份 $\Delta s = \frac{S_u - S_L}{j_{max}}$

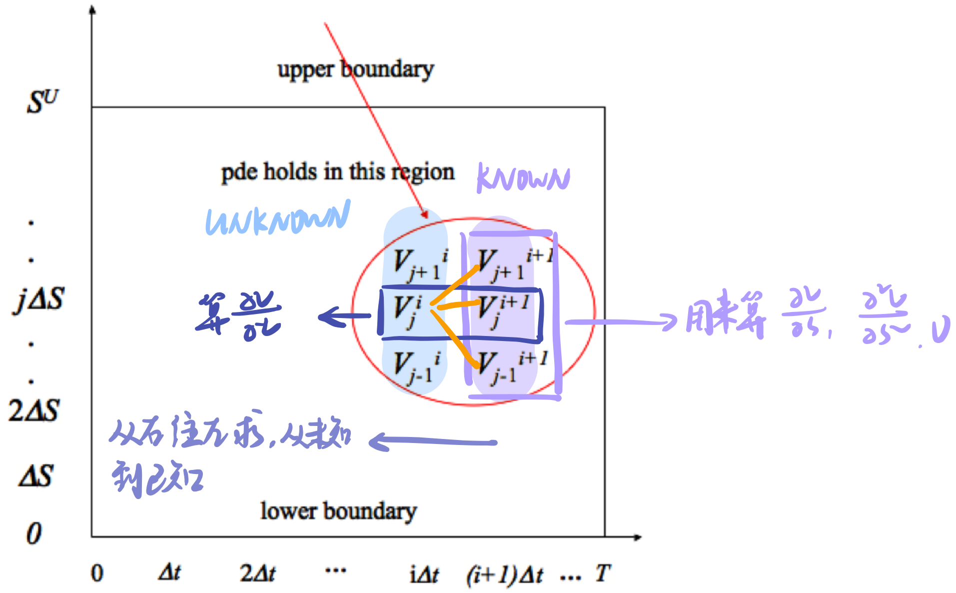
T 有 i_{max} 份 $\Delta t = \frac{T-0}{i_{max}}$

$t = i\Delta t \qquad s = j\Delta s$

$V(j\Delta s, i\Delta t) \Rightarrow V_j^i$ 正常 S_L 是0，但是 S_L 不一定是0

Quiz: What about down-and-out option? $S_L = B$!

Focus attention on i, j-th value V_j^i, and a little piece of the grid around that point



$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV = 0$$

$$\begin{aligned} 0 &= \frac{V_j^{i+1} - V_j^i}{\Delta t} + \frac{1}{2}\sigma^2 j^2 (\Delta S)^2 \frac{V_{j+1}^{i+1} - 2V_j^{i+1} + V_{j-1}^{i+1}}{(\Delta S)^2} \\ &\quad + (r-\delta) j \Delta S \frac{V_{j+1}^{i+1} - V_{j-1}^{i+1}}{2\Delta S} - rV_j^i \end{aligned}$$

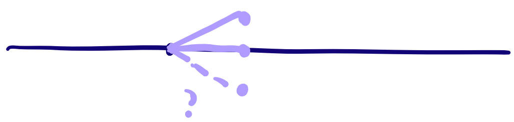
= 只有这个不知道

整理一下式子: $V_j^i = \frac{1}{1+r\Delta t}\left(AV_{j+1}^{i+1} + BV_j^{i+1} + CV_{j-1}^{i+1}\right)$

$$\left.\begin{aligned} A &= \left(\frac{1}{2}\sigma^2 j^2 + \frac{(r-\delta)j}{2}\right)\Delta t \\ B &= 1 - \sigma^2 j^2 \Delta t \\ C &= \left(\frac{1}{2}\sigma^2 j^2 - \frac{(r-\delta)j}{2}\right)\Delta t \end{aligned}\right\}$$

加起来是1，看起来像 prob.

注意这不包括 Boundary，用的是 Boundary 的值



但 Explicit 是有一些问题的. ABC 我们认为是 prob.

$\Rightarrow \quad B = 1 - \sigma^2 j^2 \Delta t \in (0,1)$

$\Delta t < \frac{1}{\sigma^2 j^2}$

如果没有限制条件的话，会偏差很大 large j 是单调变化的

→ We don't have complete freedom

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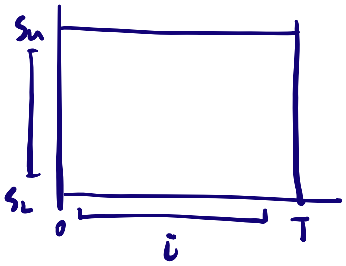
set up : T : maturity / valuation date

S_L : 0 (Lowest Value)

S_u : $2.5 \times S_0$ / $3 \times S_0$ → 需要 large enough



$i = 0, \cdots, i_{max}$ = # of days 或是它的整数倍

$j = 0, \cdots, j_{max}$ = # of % difference between S_u and S_L

$\Delta S = 1\%$ of S_0

比如 250 ($S_u = 2.5 S_0$, $S_L = 0$)

或是它的整数倍

但是会有 constraint

$$\Delta t < \frac{1}{\sigma^2 j^2} \Rightarrow \frac{T}{i_{max}} = \Delta t < \frac{1}{\sigma^2 j_{max}^2} \Rightarrow i_{max} > T \cdot \sigma^2 j_{max}^2$$

i 要比 j^2 还大，所以 i 的要求比较高 否则 $S^i_{j_{max}}$ 会比较离谱

然后算 Boundary Condition :

以 Call Option 为例

$$V_j^{i_{max}} = \max(j\Delta S - K, 0)$$

$$V_{j_{max}}^i = j\Delta S\, e^{-d(T - i\Delta t)} - K e^{-r(T - i\Delta t)}$$

$$V_0^i = 0$$

然后用上面的 ABC 搞一下就可以

这样就能解出来了！ 上面就是 General Method

接下来讲一下上面的问题：

$$0 = \frac{V_j^{i+1} - V_j^i}{\Delta t} + \frac{1}{2}\sigma^2 j^2 (\Delta S)^2 \frac{V_{j+1}^{i+1} - 2V_j^{i+1} + V_{j-1}^{i+1}}{(\Delta S)^2}$$
$$+ (r - \delta) j \Delta S \frac{V_{j+1}^{i+1} - V_{j-1}^{i+1}}{2\Delta S} - \underbrace{rV_j^{i+1}}$$

这样ABC就不是prob.了. 虽说上面那个也比较准

所以正确答案是：

$$V_j^i = \left(AV_{j+1}^{i+1} + BV_j^{i+1} + CV_{j-1}^{i+1}\right)$$

$$\left.\begin{aligned}
A &= \left(\frac{1}{2}\sigma^2 j^2 + \frac{(r-\delta)j}{2}\right)\Delta t \\
B &= 1 - r\Delta t - \sigma^2 j^2 \Delta t \\
C &= \left(\frac{1}{2}\sigma^2 j^2 - \frac{(r-\delta)j}{2}\right)\Delta t
\end{aligned}\right\} \text{加起来得 } 1$$

Quiz：利用 $i_{max} > T \cdot \sigma^2 j_{max}^2$ 解限制条件

Quiz：Stability. Options 有 non-Linearity error