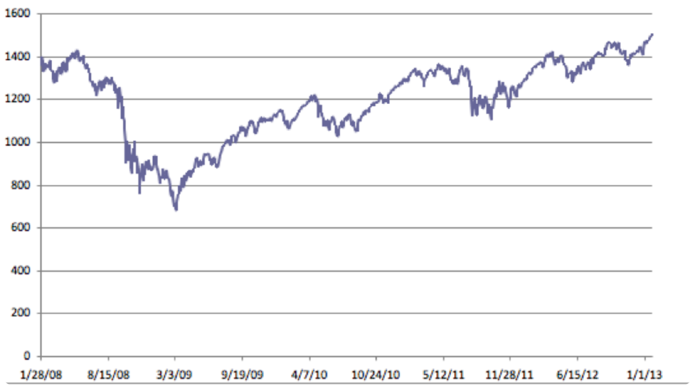
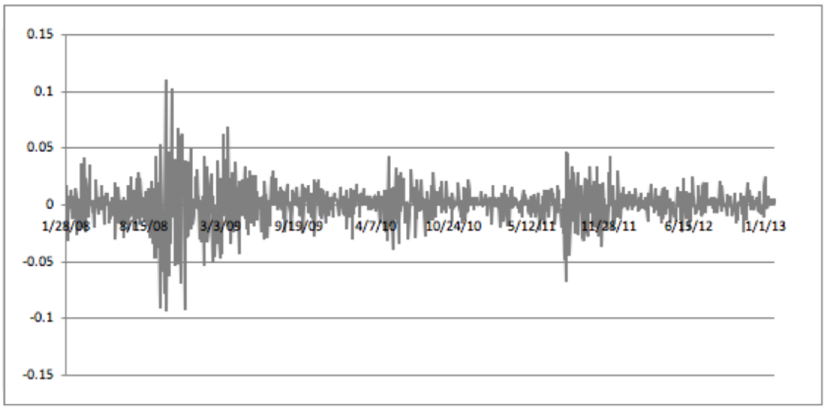
Brownian Motion &
A model for stock price movements

- 接下来我们要讨论 Continuous time 的条件下建立的模型 — 基于布朗运动
  因为我们希望 Stock 即使在很小的 Δt 内都有着无限的可能性. Binomial tree 显然做不到这点

- Consider a model for prices / returns. 但是 price 和 return 有着完全不同的性质:



price (correlated)



return (uncorrelated)

对于 returns: $\begin{cases} \mu \\ \sigma^2 \end{cases}$ roughly proportionally to time

- 接下来要注意的一点是: 我们的模型认为 stock mean returns = 0
  BUT! 显然是正的. It's risky, so it's return should $> r_f$

- Stochastic Process: a family of random variables, indexed by time

  It's continuous, 在一段时间里 r.v. 可以是某一区间内的任意值

那么它的数学模型是怎么得到的呢？

一 先来看看 binomial model：我们一般会选择 u, d，满足

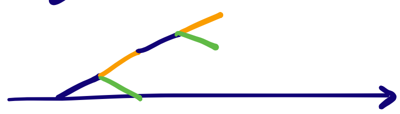
$$u = e^{\bar{\mu}\Delta t + \sigma\sqrt{\Delta t}}, \quad q \qquad \Rightarrow S_{t+\Delta t} = \begin{cases} uS_t, & q \\ dS_t, & 1-q \end{cases}$$

$$d = e^{\bar{\mu}\Delta t - \sigma\sqrt{\Delta t}}, \quad 1-q$$

在 CRR 中 μ 是 0，但理论上什么 μ 都可以，此时 log return

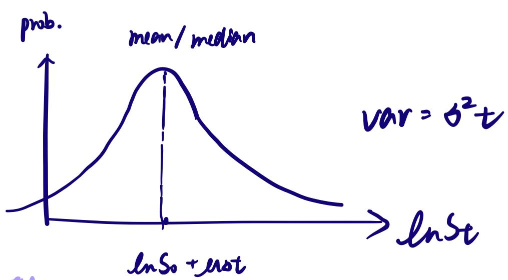
$$r_{\Delta t} = \ln\frac{S_{t+\Delta t}}{S_t} = \begin{cases} \bar{\mu}\Delta t + \sigma\sqrt{\Delta t}, & q \\ \bar{\mu}\Delta t - \sigma\sqrt{\Delta t}, & 1-q \end{cases}$$

$$\ln(S_{t+\Delta t}) = \ln S_t + \underbrace{\bar{\mu}\Delta t}_{\text{Drift}} \pm \underbrace{\sigma\sqrt{\Delta t}}_{\text{Diffusion}}$$



• 如果 $\Delta t \to 0$ 会是什么样的呢？

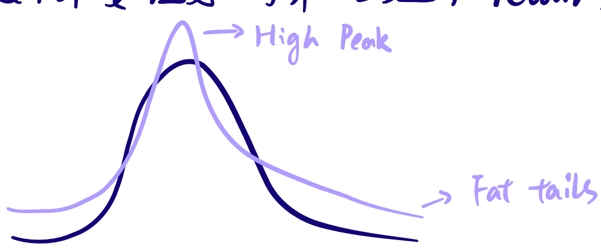
$\ln S_t \sim N$ →



S_t follows log normal distribution

→ 也说明 returns 是 normally distributed

• 接下来要注意的第二点是，returns 不是 normally distributed



kurtosis >> 3
↓
~N 的峰度

但我们还是假设 return 是正态分布的

下面我们研究一下 $\ln S_T$ 长什么样：（我们刻意选取 $q = 0.5$）

$$\ln S_T = \ln S_0 + \sum_{j=1}^{N} \bar{\mu}\delta t + \sum_{j=1}^{N} \sigma\sqrt{\delta t}\, z_j$$

ⓞ: $\ln S_T = \ln S_0 + \bar{\mu} T + \sigma \cdot \sqrt{\frac{T}{N}} \times \sum_{j=1}^{N} z_j \Rightarrow$

	MEAN	VAR.
N=1	0	1
N=2	0	2
N=3	0	3
N=4	0	4

$-1\ (\frac{1}{2})$ $\quad 1\ (\frac{1}{2})$

$-2\ (\frac{1}{4})$ $\quad 0\ (\frac{1}{2})$ $\quad 2\ (\frac{1}{4})$

⋮

根据CLT, $\frac{1}{\sqrt{N}}\sum_{j=1}^{N} z_j \sim N(0,1)$, 我们把它记作 ϕ

下面我们用 $\sqrt{\delta t}\,\phi$ 表示 ΔX

从这里开始,我们开始讨论 Brownian Motion

• 一个 Standard Brownian motion, X 具有以下特征:

这两个在下面的推导中非常重要!

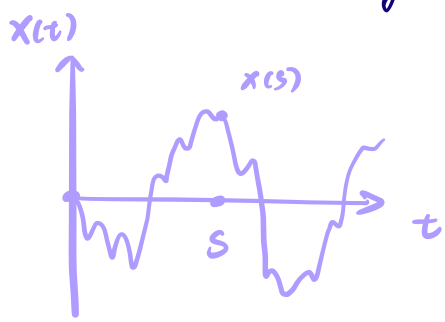
• I $X(0) = 0$ $\quad \sqrt{0} \times \phi = 0$ $\quad X(s) - X(t) = \sqrt{s-t}\,\phi$

• II For any times t and $s>t$, $X(s) - X(t) \sim N(0, s-t)$?

• $X(t_0)$, $X(t_1) - X(t_0) \cdots X(t_n) - X(t_{n-1})$ are independently distributed

• Sample paths are continuous

— 上面的严格来讲是 Winner Process



Brownian Motion 长这样: →

下面来介绍一个 Properties of Brownian Motion

- Increments are independent. not stock price maybe returns? EMH
- Sample paths are continuous.
- Increments normally distributed. ×
- Increments have mean of 0. ×但是 stock return should be positive } 前面提过了
- Increments have variances of s-t.
- Sample paths are non-differentiable with respect to time (See Aside at end) → × $\frac{dX}{dt}$
- Sample paths have infinite total variation over finite time periods.
- If the path hits a value then it hits it again in an arbitrarily short period afterwards.

接下来我们要写下来股价的模型了:

Arithmetic Brownian motion

$$Y(t) = Y(0) + \underbrace{\bar{\mu} t}_{\text{Drift}} + \underbrace{\sigma X(t)}_{\text{Diffusion}}$$

$$E[Y(s) - Y(t)] = \bar{\mu}(s-t) + \sigma E[X(s) - X(t)] = \bar{\mu}(s-t)$$

$$Var[Y(s) - Y(t)] = \sigma^2 var(X(s) - X(t)) = \sigma^2 (s-t)$$

但是可能会有 negative stock prices → 我们把它换成 e 的指数就可以了

Geometric Brownian Motion (GBM)

o : $\ln S_T = \ln S_0 + \bar{\mu} T + \sigma \cdot \sqrt{\frac{T}{N}} \times \sum_{j=1}^{N} z_j$

把 T 改为 t $\Downarrow$

$$\ln S_t = \ln S_0 + \bar{\mu} t + \sigma \sqrt{t} \times \phi$$

$$\Downarrow$$

$$S_t = S_0 \exp[\bar{\mu} t + \sigma X(t)]$$

这样有两个显而易见的好处
- $S_t > 0$
- Increments to $\ln S(t)$ are independent

还有一个值得注意的地方：$\ln S(s) - \ln S(t) \sim N$

$$\ln S(s) - \ln S(t) = \bar{\mu} s + \sigma X(s) - \bar{\mu} t - \sigma X(t)$$
$$= \bar{\mu}(s-t) + \sigma (X(s) - X(t))$$

$$\left.\begin{aligned} E[\ln S(s) - \ln S(t)] &= \bar{\mu}(s-t) \\ var[\ln S(s) - \ln S(t)] &= \sigma^2 (s-t) \end{aligned}\right\}$$ (根据特征Ⅱ, $X(s) - X(t) \sim N(0, s-t)$)

→ 所以我们到底想干什么呢？

我们想根据现在的 $S(t)$，知道下一刻的 $S(t+\delta t)$

下面来解决这个问题

我们知道了：$\ln S(s) - \ln S(t) = \bar{\mu} s + \sigma X(s) - \bar{\mu} t - \sigma X(t)$

$$= \bar{\mu}(s-t) + \sigma (X(s) - X(t))$$

接下来我们把 s 换成 $t+\Delta t$

$$\ln S(t+\Delta t) - \ln S(t) = \bar{\mu}\Delta t + \sigma (X(t+\Delta t) - X(t))$$

$$\ln\left(\frac{S(t+\Delta t)}{S(t)}\right) = \bar{\mu}\Delta t + \sigma\phi\sqrt{\Delta t}$$

$$\Downarrow$$

$$\frac{S(t+\Delta t)}{S(t)} = \exp[\bar{\mu}\Delta t + \sigma\phi\sqrt{\Delta t}] \qquad e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

$$= 1 + (\bar{\mu}\Delta t + \sigma\phi\sqrt{\Delta t}) + \frac{1}{2}\times(\bar{\mu}\Delta t + \sigma\phi\sqrt{\Delta t})^2 + \cdots$$

$$\Downarrow$$

$$\frac{S(t+\Delta t) - S(t)}{S(t)} = \sigma\phi\sqrt{\Delta t} + (\bar{\mu} + \frac{1}{2}\sigma^2\phi^2)\Delta t + O(\Delta t^{\frac{3}{2}})$$

左右两侧求期望 $E\left[\frac{S(t+\Delta t) - S(t)}{S(t)}\right] = \underset{E(\phi)=0}{0} + \bar{\mu}\Delta t + \underset{E(\phi^2) = E^2(\phi) + Var(\phi) = 1}{\frac{1}{2}\sigma^2\Delta t}$

$$= (\bar{\mu} + \frac{1}{2}\sigma^2)\Delta t$$

$$= \mu\Delta t$$

$\mu = \bar{\mu} + \frac{1}{2}\sigma^2 \rightarrow$ Expected Return

$Var\left[\frac{S(t+\Delta t) - S(t)}{S(t)}\right] = \sigma^2\Delta t$

我们可以改写 $S(t+\Delta t) - S(t)$ 了

$$S(t+\Delta t) - S(t) = \mu S(t)\Delta t + \sigma S(t)\phi\sqrt{\Delta t}$$

✡ SDE：$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dX(t)$

(Stochastic differential equation)

我们从 Brownian Motion 开始 → 试图描述 S_t 的表达式（$\ln S_t$）→ 改进表达式（放在指数上 → 我们想要更简洁的形式 → 泰勒展开 → 描述出 $S(t)$ 的表达式