Generalizations & Applications of the BSM PDE

Black-Scholes Formula 是 PDE 的一个解

现在我们要将 PDE 的一些维度调整一下，然后找个捷径求出 European Call / Put 的定价公式

underlying asset 可以是股票，可以是带股息的股票，也可以是两个股票的比值，也可以是 foreign exchange rate

1. Write down the stochastic differential equation for underlying asset
2. Determine option's function (normally $V(S,t)$)
3. Ito's Lemma $-V + \Delta S + \beta B$

chose $\beta B = V - \Delta S$. Δ 去消 dS

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV = 0$$

$$\Downarrow$$

$$\frac{\partial V}{\partial t} + \frac{1}{2} a^2 X^2 \frac{\partial^2 V}{\partial X^2} + bX \frac{\partial V}{\partial X} - cV = 0$$

X is underlying asset，a 是 volatility of underlying asset

这两个比较重要 $\left\{\begin{array}{l} b \text{ is the risk-neutral expected return} \\ c \text{ is the discount rate / expected rate of the option} \end{array}\right.$

PDE: $\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$ (PS5 Q2)

↳ S 的 expected return 是 μ，两者不一样

下面我们看看 Black-Scholes 对 Call 的定价公式

$$V(X_0, 0) = e^{-cT}\left(S_0 e^{bT} N(d_1) - K N(d_2)\right)$$

where

$$d_1 = \frac{\ln\frac{X_0}{K} + (b + \frac{1}{2}a^2)T}{a\sqrt{T}}$$

$$d_2 = d_1 - a\sqrt{T}$$

注意！我们的 binomial tree 也依赖于同样的 parameters

$$u = e^{a\sqrt{\delta t}}$$

$$d = e^{-a\sqrt{\delta t}}$$

$$q = \frac{e^{b\delta t} - d}{u - d}$$

$$V_{j,i} = e^{-c\delta t}\left(q V_{j+1,i+1} + (1-q) V_{j+1,i}\right)$$

discount rate for the option

上面我们在干什么呢？ → Jump directly from PDE to binomial tree

PDE 在定 parameter 时是很有用的

下面我们 focus on foreign exchange option

我们有 $dB = rB\,dt$ (risk-free bond in home currency)

$dB_f = r_f B_f\,dt$ (risk-free bond in foreign currency)

$dS = \mu S\,dt + \sigma S\,dX$ (描述 exchange rate 的随机过程)

(value of GBP in \$)

e.g. $V(S,t) = \max(S-K, 0) \times £100$

没有 a, c, 我们需要推导 PDE

注意！我们不能直接交易 S (exchange rate)

我们要 hedge option：由 \$ value in £ 决定，但我们很难 hedge，所以要曲线救国 我们只能买 £，£用来干嘛呢？英国债券

于是我们引入 $S_f = SB_f$ (Value of B_f in \$)

$$dS_f = B_f dS + S dB_f \quad (\text{Ito's lemma})$$

$$dS_f = B_f dS + r_f B_f S dt$$

$$dS_f = B_f dS + S dB_f + dS dB_f$$

$$= (\hat{\mu} + r_f) S_f dt + \sigma S_f dX$$

$$\Pi = -V + \Delta S_f + \beta B$$

$$d\Pi = -dV + \Delta dS_f + \beta dB$$

$$d\Pi = -\left(\frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} \underset{\triangle}{dS} + \frac{1}{2}\sigma^2 \underset{\triangle}{S^2} \frac{\partial^2 V}{\partial S^2} dt\right) + \Delta B_f (dS + r_f S dt) + \beta dB$$

注意 V 还是 S 的函数

$$d\Pi = \left(-\frac{\partial V}{\partial t} - \hat{\mu} S \frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \Delta(\hat{\mu} + r_f) S_f + r\beta B\right) dt$$
$$+ \left(\underbrace{-\sigma S \frac{\partial V}{\partial S} + \Delta \sigma S_f}\right) dX$$

$$\Delta = -\frac{S}{S_f} \frac{\partial V}{\partial S} = -\frac{1}{B_f} \frac{\partial V}{\partial S}$$

$$d\Pi = -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r_f S \frac{\partial V}{\partial S}\right)dt + \beta r B dt$$

$$\beta B = V - \Delta S_f$$

$$0 = \left(-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r_f S \frac{\partial V}{\partial S} + rV - rS\frac{\partial V}{\partial S}\right)dt$$

$$\Downarrow$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - r_f) S \frac{\partial V}{\partial S} - rV = 0$$

Quiz：我们如何解释 $r-r_f$ 呢？

$r-r_f$ 是 risk-neutral expected return on the foreign currency

r_f 可以看作是 dividend yield → it's extra money you get

true expected \$ interest rate is r

Quiz 2 ：利用PDE来解 European option value

call $\quad e^{-rT}\left(S_0 e^{(r-r_f)T} N(d_1) - K N(d_2)\right)$

一定注意
写这两个

$$\begin{cases} d_1 = \dfrac{\ln \frac{S_0}{K} + (r - r_f + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\ d_2 = d_1 - \sigma\sqrt{T} \end{cases}$$

注意有时会变，一定要非常谨慎

Quiz 3：用上面的PDE决定 binomial tree 的 parameter

$u = e^{\sigma\sqrt{T}}$

$d = e^{-\sigma\sqrt{T}}$

$q = \dfrac{e^{r-r_f} - d}{u - d}$

$V_{j,i} = e^{-rT}\left(q V_{j+1,i+1} + (1-q) V_{j+1,i}\right)$

最后我们来讲一个 trick：我们可以降维：$V(S_1, S_2, T)$ (exchange option)

它有两种 underlying asset

$$V(S_1, S_2, T) = \max(S_2 - S_1, 0)$$

但我们可以简化

$$Y = \frac{S_2}{S_1}$$

$$S_1 W(Y, t) = V(S_1, S_2, t)$$

$$W(Y, T) = \max(Y - 1, 0) = \frac{1}{S_1} V(S_1, S_2, t)$$ 看起来像是 Call Option

降维有什么好处呢？

- more intuitive ✗ 显然不，都有 ratio 了
- easy to solve ✓
- boundary conditions are simpler ✗

以上是春假前的内容，接下来让我们好好看一下 two underlying assets 的情况：

We have two stocks which follow correlated Geometric Brownian Motions (X_1 and X_2)

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dX_1$$

$$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dX_2$$

$$E[dX_1 dX_2] = \rho dt$$