The Probabilistic Solution

- 我们之前学了一些 Valuation 的方法

    - Binomial $\quad V_{j,i} = e^{-r\delta t}\left(q V_{j+1,i+1} + (1-q) V_{j+1,i}\right)$

    - European $\quad V_0 = e^{-rT}\left(S_0 e^{(r-d)T} N(d_1) - K N(d_2)\right)$

    - PDE $\quad \frac{\partial V}{\partial t} + \underbrace{(r-\delta)}_{\text{Risk-neutral expected return}} S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$

— 方法已经有了，但我们如何推导出 R-N prob. 呢？

— 我们又如何利用 R-N prob. 来定 options 的价呢？

* option values ⇒ expectations / solutions of PDEs

下面我们介绍两大知识点：Martingale + Girsanov's Theorem

"我们如何从 real world 进入 risk-neutral world 中呢？"

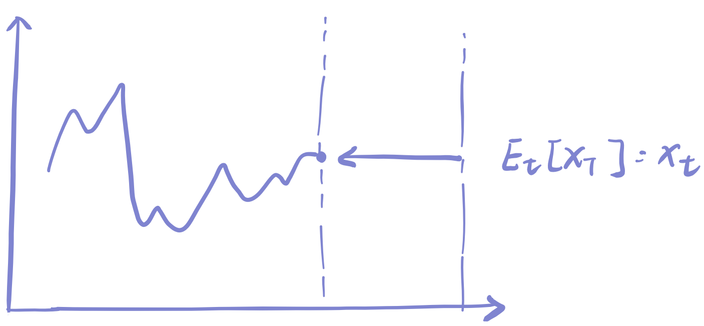
- Martingale

所以 martingale 是什么呢？

"No drift" 的 stochastic process

$E_t[X_T] = X_t \quad (T > t)$

↳ take expectation at time t



注意！ In finance very few things are martingales

any financial stochastic variables 基本都不是

（需要期望是 0，像债券，股票，期权基本都有正收益）

—要不然我们还买它们干嘛呢？

当 $r=0$ 时 option value $V_{j,i} = \underbrace{e^{-\overset{0}{r}\delta t}}_{1}\left(q V_{j+1,i+1} + (1-q) V_{j+1,i}\right)$

$V_{j,i} = 1 \times E_t[V_{t+\delta t}]$

Quiz：下列五个哪个是 martingale

- Stock price, S.
- Option price, V.
- Stock price, S, under risk-neutral measure Q when $R = 0$ ✓ → Risk-neutral world 一切资产的收益都是 r
- Bond price, B, under risk-neutral measure Q. 不
- The total received, X, in a gambling game where you toss a coin a finite number of times, receiving \$1 for each head and paying \$1 for each tail? → Martingale 名字就是 Martingale ✓

注意！在 Lecture 2 中，我们提到过

In the risk-neutral world the mean (μ) and variance (ν) of a normally distributed $r_{\Delta t}$ are:

$$\begin{aligned}\mu &= \left(r - \frac{1}{2}\sigma^2\right)\Delta t \quad \text{由上页得到} \\ \nu &= \sigma^2 \Delta t \quad \text{假设我们知道}\end{aligned}$$

那这是咋回事呢？注意 $r_{\delta t}$ 是 $\ln \frac{S_{\delta t}}{S_0}$ 其目的是想凑成 $E^{RN}[S_{\delta t}] = S_0 \cdot e^{r\delta t}$

这个本质是不变的

由上面的 quiz 可以看出，除非我们刻意设置 $R=0$，否则一般的金融产品都不是 martingale。—所以我们学它干嘛？

—即使 $R \neq 0$，martingale 也是存在的！

—在 Binomial model 里面就有 martingale → Martingale 是存在的

- Martingale 是什么，在哪里存在，为什么要用它

Binomial Model, with prob. q

$B_1 = B_0(1+R) \quad (= B_0 e^r)$

$E[S_1] = S_0(1+R) \quad (= S_0 e^r)$

$$\Downarrow$$

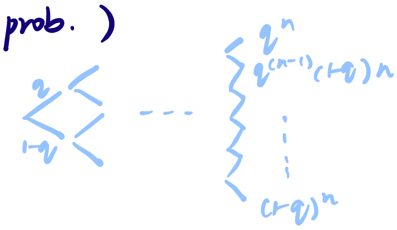
$E\left[\frac{S_1}{B_1}\right] = \frac{S_0(1+R)}{B_0(1+R)} = \frac{S_0}{B_0} \Rightarrow \frac{S_t}{B_t}$ is a martingale using prob. q

在 Risk-neutral world 中很多比率都是 martingale

*不是因为是 Martin 教的

那我们为啥要用它呢？因为它有很好的性质 → replicating property

If we have a probability measure Q (有一个这样的世界，不同 state 分配一个 prob.)



→每个都配给一个 prob.

在这个 Q 中，有两个 martingales : N, S

$$N_t = N_0 + \sum_{k=1}^{t} \Delta S_k \times \phi_k$$

我们总是能找到一个系数 ϕ_k，让一个 martingale 能用另一个 martingale 来表示

$$(\Delta N_k = \underbrace{\phi_k}_{\text{known parameter}} \cdot \Delta S_k)$$

In Binomial

$\phi_k \Leftrightarrow \Delta$ *忘了 Δ 是什么了吗？快翻 PPT 看一看吧！

Q *这里我也不是很确定，可能是指 Replicating portfolio 中 S 的系数

即用 $\Delta S + B$ 来换 option value. → 但这和 ϕ_k 有什么关系呢？ ϕ_k 应该是两个东西之间的关系

听到这里是存疑的，让我们继续探究

现在我们来找两个 martingale：

由刚才讲的得知： ① $\frac{S_t}{B_t} = Z_t$

weird one ② $\eta_t = E_t^Q[\underbrace{\text{pay off at } T}_{= X}]$

① 显然是 martingale，但②为什么是 martingale？

For η_t to be a martingale:

$$E_t^Q[\eta_T] = \eta_t$$

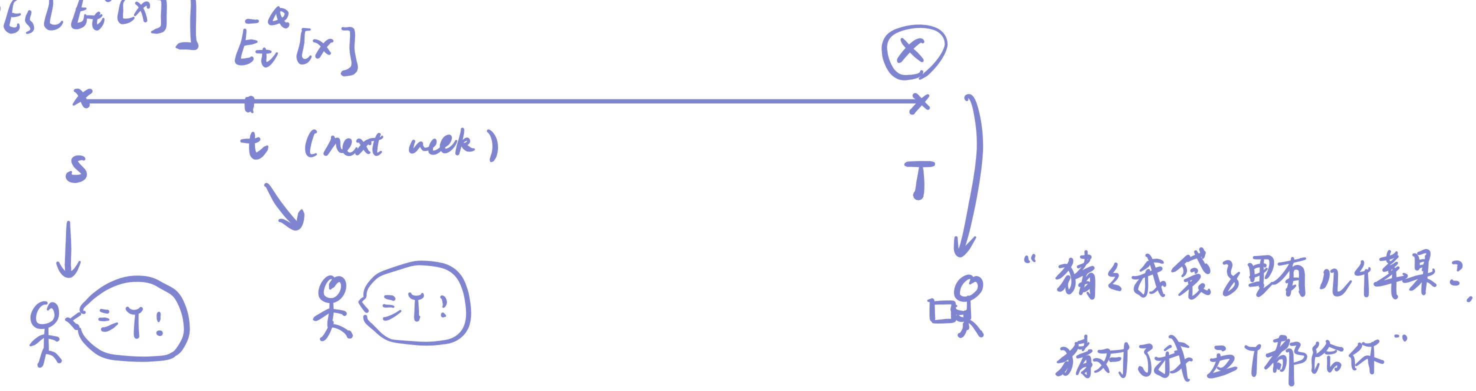
According to the defination: $s<t$

$$E_s^Q[E_t^Q[X]] = E_s^Q[X]$$

这个环套环是个啥呢？

$\to E_s[X]$

$\to E_s[E_t^Q[X]]$



— tower law

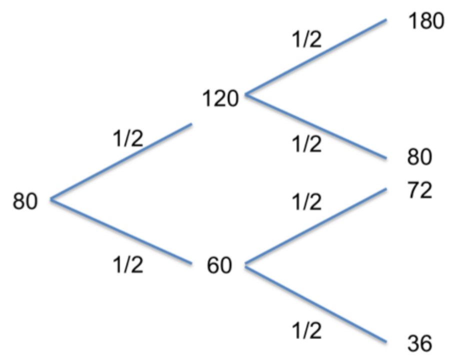
Quiz：这个 Quiz 很重要

What result would ensure that η_t was a martingale ($s > t$)?

- $\eta_t = \eta_s$.
- $\eta_t = E_t[\eta_s]$ ✓ ⇒ 我们只管它
- $X_t = E_t[X_s]$ → X 的定义只在 maturity

下面是一个解释的案例：（不是很重要，但如果不明白，那就很重要）

- Let's look at a stock price process:



and now consider

$$\eta_t = E_t[S_2]$$

for example.

- And so

$$\eta_0 = \frac{180 + 80 + 72 + 36}{4} = 92$$

- In the up state

$$\eta_1 = \frac{180 + 80}{2} = 130$$

and in the down state

$$\eta_1 = \frac{72 + 36}{2} = 54$$

- And so let's analyze η_t, 显然公式都是一样的

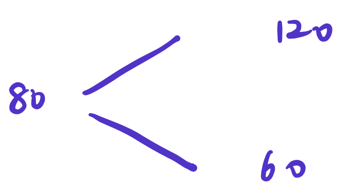
$$E_0[\eta_1] = \frac{130 + 54}{2} = 92$$

不重要了

which is of course, η_0 and so η_t is trivially a martingale. 就不是个 martingale 呀？

— 站在 $t=0$ 的时间点上，不管是到 t_1 还是 t_2，其期望都是92

— 看起来很简单，但是有一个核心问题：



—如果只看这一节，那 $E_0[\eta_1]$ 应该是 $\frac{1}{2}\times(120+60)=90$.

为什么不是呢？因为η_1是对 t_2 时 x 的期望！

这个例子有个什么问题呢？ $E_t[X_T] \neq X_t$

下面我们再回到 ①，② 两个 martingale 中，根据我们的定理：

$$\eta_t = \eta_0 + \sum_{k=1}^{t} \phi_k \Delta Z_k \qquad * \; Z_t = \frac{S_t}{B_t}$$

$$\left(\Delta\eta_t = \phi_t \Delta Z_t \right)$$

x 的任意性体现在就是 $\max(S-K, 0)$

由于 x 的任意性，我们取 $x = \frac{X}{B_T}$

$$\eta_t = E_t^Q\left[\frac{X}{B_T}\right]$$

然后我们来构建一个 portfolio：这是在干嘛呢？像二叉树一样，这是 replicate option payoff at T

at time t，我们买 ϕ_{t+1} units of stock and

Q 为什么是 $t+1$，t 又是如何

$\psi_{t+1} = \eta_t - \phi_{t+1}\frac{S_t}{B_t}$ of cash bond

以及 $t+1$ 是怎么观测的 不断调整我们的 Δ 使最终的 payoff 相当于我们 portfolio 的 payoff

$$\Pi_0 = \phi_1 S_0 + \psi_1 B_0 = \phi_1 S_0 + \left(\eta_0 - \phi_1 \frac{S_0}{B_0}\right) B_0 = \eta_0 B_0$$

基于个人理解：

$$= E_0^Q[x] \cdot \frac{1}{B_0 e^{rT}} \cdot B_0$$

$$= E_0^Q[x] \cdot e^{-rT}$$

在 $t=1$ 时这个 portfolio 值多少钱呢？（换一个 portfolio 的配置）

$$\Pi_1 = \phi_2 S_1 + \psi_2 B_1 = \phi_2 S_1 + \left(\eta_1 - \phi_2 \frac{S_1}{B_1}\right) B_1 = \eta_1 B_1$$

Self-financing！

（一开始的 portfolio 从 $S_0 \to S_1$，$B_0 \to B_1$ 会如何呢？

$$* \; \Delta\eta = \phi \Delta Z$$

$$\Pi_1 = \phi_1 S_1 + \psi_1 B_1 = \phi_1 S_1 + \left(\eta_0 - \phi_1 \frac{S_0}{B_0}\right) B_1 = \eta_0 B_1 + B_1 \phi_1 \underbrace{\left(\frac{S_1}{B_1} - \frac{S_0}{B_0}\right)}_{\Delta Z} = \eta_0 B_1 + B_1 \Delta\eta = B_1 \eta_1)$$

我们已经发现规律了，那么 $\Pi_2 = \eta_2 B_2$，$\Pi_3 = \eta_3 B_3$，$\cdots$ $\Pi_T = \eta_T B_T = \underline{\underline{X}}$

$$\left(E_T^Q\left[\frac{X}{B_T}\right] = \frac{X}{B_T} \right)$$

在 no-arbitrage 的情况下（上面的个人理解是正确的情况下）：

$$V_0 = \Pi_0 = B_0 \eta_0 = B_0 E_0^Q\left[\frac{\text{payoff}}{B_T}\right]$$

如果 $B_T = B_0 e^{rT}$（As usual），那么 $V_0 = e^{-rT} E_0^Q[\text{payoff}]$

这就是我们一直在用的公式呀！

Q 一个核心问题是：ϕ 是哪来的呢？"previsible" → 怎么看的？S_t，B_t，V_t 都可见

这类似于 Self-financing：我们 t_0 的 portfolio 到了 t_1，不需要加钱就可以改换成 t_1 的 portfolio

Quiz: What's true about the process $\frac{V_t}{B_t}$ here?

- It has a positive drift
- It is a martingale
- It mean reverts

$\frac{V_t}{B_t}$ 是啥呢？ $V_t = B_t E_t^Q \left[\frac{\overset{\nearrow V_T}{\text{payoff}}}{B_T}\right]$

$= B_t \eta_t \rightarrow \frac{V_t}{B_t}$ 是个 martingale

一个奇怪的问题：我们选择的 Q 使 $\frac{S}{B}$ 是 martingale，为啥不让 S 是 martingale

Consider a stock, bond and option in a one period world

Stock: 1 → 2 / 0.5　　Bond: 1 → 1.1 / 1.1　　Option: 4/11 → 1 / 0

$q = \frac{1.1 - 0.5}{2 - 0.5} = 0.4$

在这个设置下， $E_0^Q\left[\frac{S_1}{B_1}\right] = \frac{2 \times 0.4 + 0.5 \times 0.6}{1.1} = 1 = \frac{1}{1} = \frac{S_0}{B_0}$

$E_0^Q\left[\frac{V_1}{B_1}\right] = \frac{1 \times 0.4 + 0 \times 0.6}{1.1} = \frac{0.4}{1.1} = \frac{V_0}{B_0}$

但 q 不一定是 0.4 呀？我们不能让 S 是 martingale 呢？

Quiz: What choice of q and $1 - q$ would make the **stock price** a martingale?

- It's not possible
- $q = 2/3, 1 - q = 1/3$
- $q = 0.5, 1 - q = 0.5$
- $q = 1/3, 1 - q = 2/3$

$2q + 0.5(1-q) = 1$

$1.5q = 0.5 \Rightarrow q = \frac{1}{3}$

(展示了把 S 变成 martingale 是可行的)，因为 q 是什么是由我们定的

— 所以会出现什么结果呢？

q是 $\frac{1}{3}$ 了，S是 martingale 了，那么 Q下的两个 martingale 变成了 η_t 和 S_t

$$\eta_t = \eta_0 + \sum_{k=1}^{t} \phi_k \Delta S_k$$

好了，我们试图搞一个 replicating portfolio 出来：

$t=0$: ϕ_1 of stock

$\psi_1 = \eta_0 - \phi_1 S_0$ in the risk-free asset

Then $\Pi_1 = \phi_1 S_1 + \psi_1 B_1$

$= \phi_1 S_1 + (\eta_0 - \phi_1 S_0) B_1$

$= \phi_1 (S_1 - S_0 B_1) + \eta_0 B_1$

— 以前我们能搞出来一个 self-financing portfolio，现在就不行了

S/B is a martingale

$\Pi_0 = \eta_0 B_0$

$\downarrow$

$t=1$ $\Pi_1^{old} = \eta_1 B_1$

$\Pi_1^{new} = \eta_2 B_2$

$\downarrow$

$\vdots$

S is a martingale

$\Pi_0 = \eta_0 \ (= \eta_0 B_0)$

$\downarrow$

$\Pi_1 = \phi_1 (S_1 - S_0 B_1) + \eta_0 B_1$

$\neq \eta_1 B_1$

?

以上是 discrete time 的情形，下面我们讲一下 continuous time 的情形

和 discrete 的方法差不多：先找 Q，然后找 replicating portfolio，再定价

让它是 martingale

同样的 $E_s[M_t] = M_s$

$E[|M_t|] < \infty$ （在 finance 中我们可以不考虑这个）

在 Continuous time 中有一个现成的 martingale：

If $W(t)$ is a Brownian Motion, then it is a martingale

为啥呢？

① 从 x(T) 换了個甲

$$W(T) = W(t) + \int_t^T dW(s)$$

$$E_t[W(T)] = W(t) + 0 = W(t)$$

还有另一个和 discrete 同样的 martingale

②

$$Y_t = E_t^P[X]$$

因为 $E_s^P[E_t^P[X]] = E_s^P[X]$ (tower law)

轻松一下，一个没什么用的 Quiz：

Which of the following are martingales?

- Stock price, S
- Option price, V
- Bond price, B
- $\sigma S dW$ ✓
- Stock price, S under risk-neutral measure Q
- Stock price, S under risk-neutral measure Q when $R = 0$ ✓
- Bond price, B under risk-neutral measure Q.

在 Continuous 中也有两个 martingales 的关系：

$$N_t = N_0 + \int_0^t \phi_s \, dM_s$$

* ϕ is unique

这个关系为我们提供了便利 — 为啥呢？

在 continuous time 的世界里，stochastic process 多种多样：

$$ds = a\,dt + s\,dW$$

$$ds = s^2 e^{s^2} dW$$

$$ds = (a-bs)\,dt + s\,dW$$

上面这个关系告诉我们一个非常实用的定理：

✡ **In general we find that if X is a stochastic process with volatility σ_t, $dX_t = \mu_t dt + \sigma_t dW_t$ then**

dt 项没有系数

$$\boxed{\text{X is a martingale} \Leftrightarrow \text{X is driftless, } \mu_t = 0}$$

— 简单粗暴：有 dt 就不行

休息时间：应用这个定理的 quiz：

Which of the following are martingales?

- S where $dS = \mu S dt + \sigma S dW$.
- B where $dB = rBdt$.
- F where $dF = \sigma F dW$ ✓
- V where $dV = rdt + \sigma dW$
- V/B where $d(V/B) = \sigma(V/B)dW$ ✓
- r where $dr = k(\theta - r)dt + \sigma dW$

diffusion term 爱啥啥

注意！这里有一个隐藏的，但也是非常核心的结论：

If X_t is a martingale then it can be written as:

$$X_t = X_0 + \int_0^t \phi_s dW_s$$

where W is a Brownian motion and so $dX_t = \phi_t dW_t$.

这是在说什么呢？任何 martingale 的变化都可以用一个系数乘 dW 来表示

下面是非常困难的搞出 replicating portfolio 的过程：

$$\Delta \Pi_t = \phi_t \Delta S_t + \psi_t \Delta B_t$$

目的是 replicate payoff at T，那么有：

$$\Pi_T = \phi_T S_T + \psi_T B_T = X$$

这里有一个困难：在 discrete 中，我们只需 replicate portfolio，找出 q.（合是缺）
但 continuous time 中，q 是哪来的？（因为连续了）

下面到了 Girsanov's Theorem 的表演时间：

✡ If W is a Brownian motion under the probability measure P, then there exists an equivalent measure Q such that $\tilde{W}_t = W_t + \gamma t$ is a Brownian motion

注意！此时我们不管 P 是怎么到 Q 的，只需知道我们能这么做

—— "为什么你把你的房子烧着了？" "因为，我能"

上面的 P 可以看作一个 Real-World Probability，而 Q 是另一个世界，有另一套概率分布

under P: $dX = \mu_t dt + \sigma_t \underbrace{dW}$

↳ Brownian Motion

under Q: $dX = \mu_t dt + \sigma_t (d\tilde{W} - \gamma dt)$

$= \underbrace{(\mu_t - \gamma\sigma_t)}_{}dt + \sigma_t d\tilde{W}$

new drift，取个名叫 v_t

需要满足 $\gamma = \frac{\mu_t - v_t}{\sigma_t}$

我们想干嘛？在 discrete time 中我们让 q 满足 $\frac{S_t}{B_t}$ is a martingale

在 continuous time 中我们让 Q 满足 $\underset{\sim}{X}$ is a martingale

X 是啥一会儿说

怎么做呢？ $v_t = 0 \Rightarrow \gamma = \frac{\mu_t}{\sigma_t}$

下面我们还是找 replicating portfolio (ϕ_t, ψ_t)

continuous 我们也是让 $\frac{S_t}{B_t}$ 是 martingale

$dS = \mu S dt + \sigma S dW$

$dB = rB dt$

$d\left(\frac{S}{B}\right) = dS \cdot \frac{1}{B} + S \cdot d\left(\frac{1}{B}\right) + \underbrace{dS\, d\left(\frac{1}{B}\right)}_{0}$

困扰我们的是 $d\left(\frac{1}{B}\right)$:

用 Itô's Lemma $Y = \frac{1}{B}$

$dY = \frac{dY}{dB} dB + \frac{1}{2}\frac{d^2Y}{dB^2} dB^2 = -\frac{1}{B^2} dB + \frac{1}{2} \cdot \frac{2}{B^3} dB^2 = -\frac{r}{B} dt$

原式 $= (\mu S dt + \sigma S dW)\frac{1}{B} - r\cdot\frac{S}{B}dt$

$$= (\mu - r)\frac{S}{B}dt + \sigma\frac{S}{B}dW$$

$$\downarrow Z = \frac{S}{B}$$

$$dZ = (\mu - r)Z dt + \sigma Z dW$$

不忘初心，牢记使命：我们是想让 Z 是个 martingale，找能让它成立的 prob. measure Q

我们改换门庭，让 drift 消失：

$$dZ = (\mu - r)Z dt + \sigma Z(d\tilde{W} - \gamma dt)$$

$\Rightarrow \gamma = \frac{\mu - r}{\sigma}$，使 $dZ = \sigma Z d\tilde{W}$

这是 Sharpe ratio → market price of risk，把这个减掉，没risk了

Quiz: stock 现在 follow what process?

$$dS = \mu S dt + \sigma S(d\tilde{W} - \gamma dt)$$

$$= [\mu S - (\mu - r)S]dt + \sigma S d\tilde{W}$$

$$= \underline{rS}dt + \sigma S d\tilde{W}$$

为什么是 rS 呢？如果 $\frac{S}{B}$ 是 martingale，我们可以认为这是 risk-neutral world，此时 S 的 drift 是 rS

所以到底如何搞出来 Replicating Portfolio 呢？

有两个 martingale：$\begin{cases} \eta_t = E_t^Q\left[\frac{X}{B_T}\right] \\ Z_t = \frac{S_t}{B_t} \end{cases}$

market rep. theorem $d\eta_t = \phi_t dZ_t$

Portfolio 还是原来的配方： ϕ_t of stock

$\psi_t = \eta_t - \phi_t Z_t$ of bond

注意！和 Binomial 有什么区别呢？系数不是 $t+1$ 了、因为 continuous time 没有 $t+1$

下面就是紧张刺激的推导过程：

$\Pi_t = \phi_t S_t + \psi_t B_t$ $d\eta_t = \phi_t dZ_t = \phi_t \sigma Z d\tilde{W}$

$= \phi_t S_t + (\eta_t - \phi_t \frac{S_t}{B_t}) B_t$ $dB_t = rB_t dt$

$= \eta_t B_t$ (和 Binomial 结果是一样的)

接下来我们检查一下这个 portfolio 是否是 self-financing $\Leftrightarrow$ changes in Π only come from changes in S and B

$d\Pi_t = B_t d\eta_t + \eta_t dB_t + d\eta_t dB_t$ (先从上式直接 Itô)

$= B_t \phi_t dZ_t + \eta_t dB_t$ $\psi_t = \eta_t - \phi_t Z_t$

$= \phi_t B_t dZ_t + (\psi_t + \phi_t Z_t) dB_t$

$= \phi_t (B_t dZ_t + Z_t dB_t) + \psi_t dB_t$

$S_t = B_t \times Z_t \Rightarrow dS_t = B_t dZ_t + Z_t dB_t + \underbrace{dB_t dZ_t}_{0}$

$= \phi_t dS_t + \psi_t dB_t$

这样就有 self-financing 了！

At $t=0$ $\Pi_0 = \eta_0 B_0$ or at t $\Pi_t = \eta_t B_t$. This grows over time in a self-financing way until at maturity T : $\Pi_T = \eta_T B_T = X \rightarrow$ payoff

在 No-Arbitrage 的情况下，

$V_t = \pi_t = B_t \eta_t = B_t E_t^Q \left[\frac{payoff}{B_T}\right]$　　If $B_t = B_0 e^{rt}$, then

$V_t = e^{-r(T-t)} E_t^Q [payoff]$　和 discrete 情况是一样的

这个 Q 就可以看做是 RN

所以这到底有什么用？ 我们本来的难点就是算 $E_t^Q [payoff]$

现在这个问题并没有解决.

下面我们来算 $E_t^Q [payoff]$

有些方法：
- Analytically → 算出公式
- Monte Carlo → 下周要讲
- 用 PDE

我们先来试试 Valuing a Call Option：

Under Q:　$V_0 = e^{-rT} E_0^Q [payoff]$ ⇒ 所以 $\frac{V_t}{B_t}$ 也是 martingale

$dS = rS\,dt + \sigma S\,d\tilde{W}$

To value a call option:　payoff $= \max(S_T - K, 0)$

通过 SDE，我们知道 [Stochastic Calculus & Ito's Lemma. pg 8]

$S_T = S_0 \exp\left((r - \frac{1}{2}\sigma^2)T + \sigma W(T)\right)$

$W(T) = \sqrt{T}\, X$　　$X \sim N(0,1)$

我们把 S_T 算出来，代入 payoff，来求 V_0

$$V_0 = e^{-rT} E_0^Q \left[\max \left(S_0 e^{rT - \frac{1}{2}\sigma^2 T + \sigma\sqrt{T} \underset{\downarrow}{\underline{\underline{x}}}} - K, 0 \right) \right]$$

random normally distributed

下面我们算 Expectation. 我们唯一要求的期望就是 x 的期望，剩下的都是定值

$$V_0 = e^{-rT} \int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \times \max \left(S_0 e^{rT - \frac{1}{2}\sigma^2 T + \sigma\sqrt{T} x} - K, 0 \right) dx$$

prob.

很烦 → 把它丢了！

→ 为什么长这样呢？因为期望是 x 的所有概率 × payoff. $x \sim N(0,1)$

接下来我们要 choose a value of x below when payoff is 0

这个想法非常关键！

只需把正值积起来就可以了：

找到 x，使 $S_0 e^{rT - \frac{1}{2}\sigma^2 T + \sigma\sqrt{T} x} > K$

$$rT - \frac{1}{2}\sigma^2 T + \sigma\sqrt{T} x > \ln \frac{K}{S_0}$$

$$\sigma\sqrt{T}\, x > \ln \frac{K}{S_0} - \left(r - \frac{1}{2}\sigma^2\right) T$$

$$x > \underbrace{\frac{\ln \frac{K}{S_0} - \left(r - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}}_{L}$$

→ 和 d_1. d_2 长得很像

$$V_0 = e^{-rT} \int_{L}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \left(S_0 e^{rT - \frac{1}{2}\sigma^2 T + \sigma\sqrt{T} x} - K \right) dx$$

$$= A - B$$

$$= e^{-rT} \int_{L}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} S_0 e^{rT - \frac{1}{2}\sigma^2 T + \sigma\sqrt{T} x} dx - e^{-rT} \int_{L}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \cdot K \cdot dx$$

$$= \frac{S_0}{\sqrt{2\pi}} \int_L^\infty e^{-\frac{x^2}{2} - \frac{1}{2}\sigma^2 T + \sigma\sqrt{T} x} \, dx \qquad - \; K e^{-rT} \int_L^\infty \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \, dx$$

配完全平方数！

$$= \frac{S_0}{\sqrt{2\pi}} \int_L^\infty e^{-\frac{1}{2}(x-\sigma\sqrt{T})^2} \, dx$$

Let $y = x - \sigma\sqrt{T}$ $\quad dy = dx$

$x = L \quad y = L - \sigma\sqrt{T}$

$$= S_0 \underline{\int_{L-\sigma\sqrt{T}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \, dy}$$

Normal Density ☺ ①

Recall $N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dy$ $\qquad \int_L^\infty \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \, dx = N(-L)$

$$= N\left(\frac{\ln \frac{S_0}{K} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right)$$

$$= N(d_2)$$

$$B = K e^{-rT} N(d_2)$$

① : $\int_{L-\sigma\sqrt{T}}^\infty \rightarrow 1 - N(L - \sigma\sqrt{T}) = N(-(L - \sigma\sqrt{T}))$

$= N(-L + \sigma\sqrt{T})$

$$= N\left(\frac{\ln \frac{S_0}{K} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} + \sigma\sqrt{T} \right)$$

$$= N(d_1)$$

$A = S_0 N(d_1)$

$$= S_0 N(d_1) - K e^{-rT} N(d_2)$$

哦嗐！ ☺

这个方法比从 PDE 积分算出 B-S formula 简单多了

用 Martingale 还可以干嘛呢？还可以把 B-S 的 PDE 求出来！

We do some Itô's Lemma on V and B now:

under Q:

$$dS = rS\,dt + \sigma S\,d\tilde{W}$$

and $\frac{S_t}{B_t}$, $\frac{V_t}{B_t}$ are martingales

↳ 我们准备利用这一点

$$dB = rB\,dt$$

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2$$

$$= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}(rS\,dt + \sigma S\,d\tilde{W}) + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}dt$$

$$= \left(\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt + \sigma S\frac{\partial V}{\partial S}d\tilde{W}$$

$$d\left(\frac{V}{B}\right) = \frac{1}{B}dV + V d\left(\frac{1}{B}\right) + dV d\left(\frac{1}{B}\right)$$

$$= \frac{1}{B}\left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}(rS\,dt + \sigma S\,d\tilde{W}) + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt - \frac{rV}{B}dt$$

$$= \frac{1}{B}\left(\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - rV\right)dt + \sigma\frac{S}{B}\frac{\partial V}{\partial S}d\tilde{W}$$

- $\frac{V}{B}$ is a martingale ⇒ drift = 0

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - rV = 0$$

Horray! ☺

BSM PDE!

奥妙的地方是什么呢？ B_t is not special

You can choose any treated asset N_t as numeraire

在Q下 $\frac{A_t}{N_t}$ is a martingale → A_t 也是任意 asset

$N_t = B_t$ is the easiest case. 但什么都行. 因为反正它们的回报率都是r

- In the examples we have given the risk-free bond plays the role of the Numeraire asset. It is useful because it has no volatility but could we use any asset as the numeraire asset?
- In fact we can, we just have a revised pricing formula for any numeraire asset N_t so that now:

$$\frac{V_t}{N_t} = E_t^Q\left[\frac{V_T}{N_T}\right]$$

- This is very useful for foreign exchange problems and crucial for term structure work, as we will see later in the course.

接下来我们用 Exchange Option 和 Quanto 作为例子：

Exchange Option

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dW_1 \qquad * \; dW_1 dW_2 = \rho dt$$

$$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dW_2$$

$$dB = rB dt$$

$$V(S_1, S_2, T) = \max(S_2 - S_1, 0)$$

choose S_1 as "Numeraire". so there exists a measure $\mathbb{Q}$ where $\frac{V}{S_1}$, $\frac{S_2}{S_1}$ are martingales 能不能用 B作 numeraire 呢？能，但是太困难了

首先我们写出 $d\frac{S_2}{S_1} = \underbrace{(\mu_2 - \mu_1 + \sigma_1^2 - \rho\sigma_1\sigma_2)}_{\mu_3}\left(\frac{S_2}{S_1}\right)dt \underbrace{- \sigma_1\left(\frac{S_2}{S_1}\right)dW_1 + \sigma_2\left(\frac{S_2}{S_1}\right)dW_2}$

要考！

$\sigma_3 = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \rightarrow \sigma_3 \frac{S_2}{S_1} dW_3$

接下来我们 move to Q

$$d\frac{S_2}{S_1} = \mu_3\left(\frac{S_2}{S_1}\right)dt + \sigma_3\left(\frac{S_2}{S_1}\right)(d\tilde{W}_3 - \gamma dt)$$

$$\gamma = \frac{\mu_3}{\sigma_3}$$

$$\Rightarrow \quad d\frac{S_2}{S_1} = \sigma_3\left(\frac{S_2}{S_1}\right)d\tilde{W}_3$$

solve under Q

$$\frac{S_2(T)}{S_1(T)} = \frac{S_2(0)}{S_1(0)}\exp\left(-\frac{1}{2}\sigma_3^2 T + \sigma_3\sqrt{T}\,x\right) \qquad x \in N(0,1)$$

Also. $\frac{V}{S_1}$ is a martingale too.

$$\frac{V(0)}{S_1(0)} = E_0^Q\left[\frac{V(T)}{S_1(T)}\right]$$

$$V(0) = S_1(0)\, E_0^Q\left[\frac{\max(S_2(T) - S_1(T), 0)}{S_1(T)}\right]$$

$$= S_1(0)\, E_0^Q\left[\max\left(\underbrace{\frac{S_2(T)}{S_1(T)} - 1}_{\text{no drift}},\ 0\right)\right]$$

$$V(0) = S_1(0)\left[\frac{S_2(0)}{S_1(0)}N(d_1) - 1 \times N(d_2)\right]$$

其中 $$d_1 = \frac{\ln\frac{S_2(0)}{S_1(0)} + \frac{1}{2}\sigma_3^2 T}{\sigma_3\sqrt{T}}$$

$$d_2 = d_1 - \sigma_3\sqrt{T}$$

$$V(0) = S_2(0)N(d_1) - S_1(0)N(d_2)$$

当然我们可以用老办法（Lecture 11），和以前的做法都是一样的

Quanto Option （最难的部分）

$$dS = \mu S\,dt + \sigma S\,dW_1$$

$$de = \mu_e e\,dt + \sigma_e e\,dW_2 \quad \text{(exchange rate)}$$

$$dW_1 dW_2 = \rho\,dt$$

$$dB_\$ = r_\$ B_\$\,dt$$

$$dB_€ = r_€ B_€\,dt$$

Eventually $B_\$$ will be numeraire

· Problem ⇒ need traded assets denominated in \$

$\frac{A_t}{N_t}$ ⇒ A_t, N_t must be in the same currency

We are going to introduce Y: dollar value of Eurobond

Z: dollar value of Eurostoxx index

在这种设置下，我们有：

(Quiz) $\frac{Y}{B_\$}, \frac{Z}{B_\$}$ will be martingales 任何 USD denominated asset $\frac{V}{B}$ 都是

Note e is not a traded asset

↳ 买不了这个 exchange rate

$\frac{e}{B_\$}$ 就不是 martingale 了

$$dY = e\,dB_€ + B_€\,de = (\mu_e + r_€)Y\,dt + \sigma_e Y\,dW_2$$

$$dZ = S\,de + e\,dS + de\,dS = (\mu + \mu_e + \rho\sigma_e\sigma)Z\,dt + \sigma Z\,dW_1 + \sigma_e Z\,dW_2$$

$K = \frac{Y}{B_\$}$ is a martingale

$$dK = \frac{1}{B_\$} dY + d\left(\frac{1}{B_\$}\right) Y$$

$$= (r_€ + \mu_e - r_\$) K dt + \sigma_e K dW_2$$

$$\gamma_2 = \frac{r_€ + \mu_e - r_\$}{\sigma_e}$$

$\Rightarrow dK = \sigma_e K d\tilde{W}_2$, where $d\tilde{W} = dW_2 + \gamma_2 dt$

$L = \frac{Z}{B_\$}$ is also a martingale

$$dL = \frac{1}{B_\$} dZ + d\left(\frac{1}{B_\$}\right) Z$$

$$= (\mu + \mu_e + \rho\sigma_e\sigma - r_\$) L dt + \sigma L dW_1 + \sigma_e L dW_2$$

But under Q, $dW_2 \to d\tilde{W}_2 - \gamma_2 dt$

$$dL = (\mu + \mu_e + \rho\sigma_e\sigma - r_\$ - (r_€ + \mu_e - r_\$)) L dt + \sigma L dW_1 + \sigma_e L d\tilde{W}_2$$

$$= (\mu + \rho\sigma_e\sigma - r_€) L dt + \sigma L dW_1 + \sigma_e L d\tilde{W}_2$$

L is a martingale

$dW_1 \to d\tilde{W}_1 - \gamma_1 dt$

$\gamma_1 = \frac{\mu + \rho\sigma_e\sigma - r_€}{\sigma}$ $\Rightarrow dL = \sigma L d\tilde{W}_1 + \sigma_e L d\tilde{W}_2$

Finally, $\frac{V}{B_\$}$ is also a martingale

$$V_0 = e^{-rT} E_0^{Q}[V_T]$$

$$= e^{-rT} E_0^{Q}[\max(S_T - K, 0)]$$

We need to know what is S_T under Q

But recall

$$dS = \mu S dt + \sigma S dW_1$$

So under Q

$$dS = (\mu - \sigma\gamma_1) S dt + \sigma S d\tilde{W}_1$$

$$= (r_\epsilon - \rho\sigma_e\sigma) S dt + \sigma S d\tilde{W}_1$$

$$S_T = S_0 \exp\left((r_\epsilon - \rho\sigma_e\sigma - \tfrac{1}{2}\sigma^2) T + \sigma\sqrt{T}\, x \right) \quad x \in N(0,1)$$

或者

- And so,

$$= \frac{1}{B_\$} dV + V d\left(\frac{1}{B_\$}\right), \text{ where } dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S} dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} \underset{\sim}{dS^2}$$

$$d\left(\frac{V}{B_\$}\right) = \frac{1}{B_\$}\left(\frac{\partial V}{\partial t} + (r_€ - \rho\sigma_e\sigma)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - r_\$ V\right) dt + \sigma \frac{S}{B_\$}\frac{\partial V}{\partial S} d\tilde{W}_1$$

- But, under the risk neutral world V/B must be a martingale, in which case it must have zero drift and,

$$\frac{\partial V}{\partial t} + (r_€ - \rho\sigma_e\sigma)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - r_\$ V = 0$$

which, again, is the Black-Scholes-Merton PDE but now with a different first derivative term

Expectation & RN prob. 和 PDE 是一样的