Structural or Option-Theoretic Approach to Measuring Credit Risk

- 也叫 Merton 或 Merton-KMV approach —这是第二个框架

- 我们的 Insight 是什么呢？

  —一个公司的 liabilities 可以用 options on underlying firm asset value 解释

如果 asset value 跌到了 liability 以下，那公司就会 default

- This determines the probability of default
- The correlation between the asset values of two firms *i* and *j* determines the dependence (loosely, correlation) of the defaults by the two firms

一开始这个模型是用来评估 single firm 的，但我们发现这个模型可以拿来估计很多公司的 default prob. (correlation model)

下面我们考虑一个简化再简化的模型：

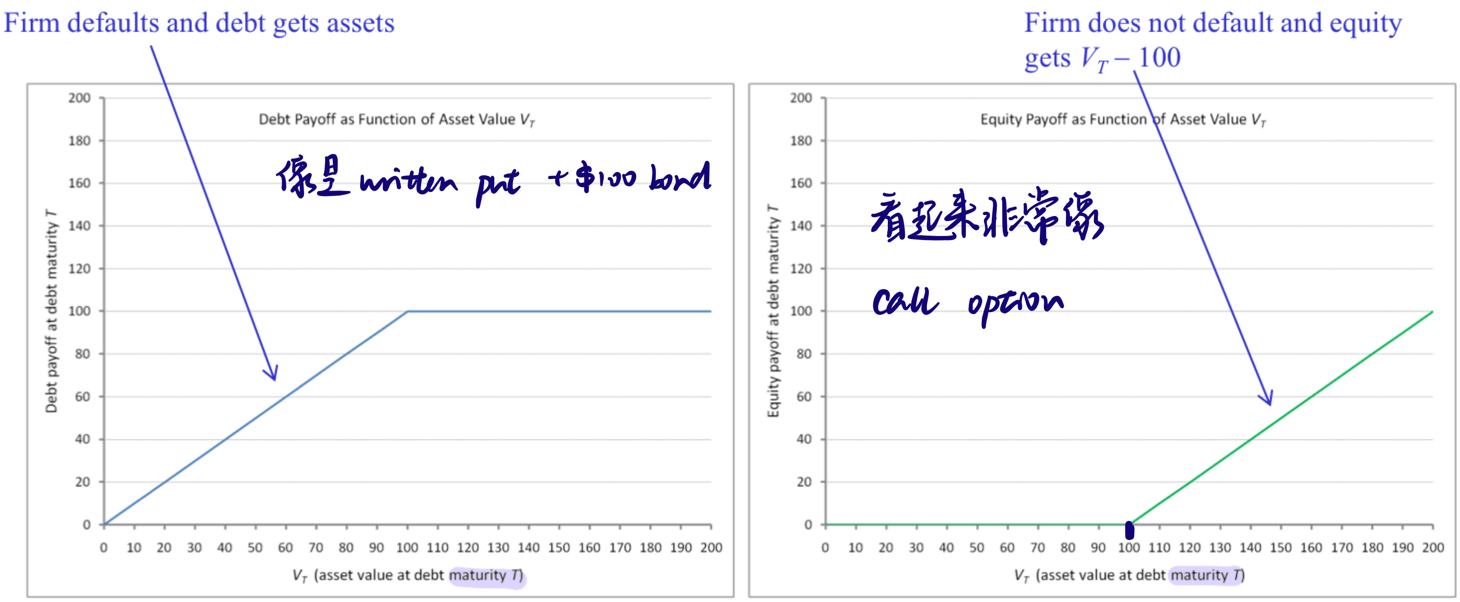
Assumptions:

- Firm has only a single class of debt, a zero-coupon maturing at time T
- Default occurs only at the debt maturity T
- Firm's behavior (investments and assets) will not be affected by how close it is to default
- No dividends or other intermediate payments to equity

At the price of these simplifying assumptions, required inputs are only:

- Time to maturity T
- Initial market value of assets V_0
- Present value (market value) of the liability
- Asset volatility σ_V

我们来看看图



那么 value of equity 可以用 BS formula 表示

Option's	$\Leftrightarrow$	Equity's
S_0		V_0 (current market value of assets)
K		F (face value of the debt)
σ		σ_V (asset volatility - s.d. of log returns)

$$* \ \ln\frac{S_0}{K} + rT = \ln\frac{S_0}{K}\cdot e^{rT} = \ln\frac{S_0}{K\cdot e^{-rT}}$$

$K\cdot e^{-rT}$		D_0 (discounted value of K)
$N(d_2)$		probability of not default
$1-N(d_2)$		P_0 (prob. of default)
V_0		E_0 (current market value of equity)

我们有 $V_0 = S_0 N(d_1) - e^{-rT} K N(d_2)$

则 $E_0 = V_0 N(d_1) - e^{-rT} F N(d_2) = V_0 N(d_1) - D_0 N(d_2)$

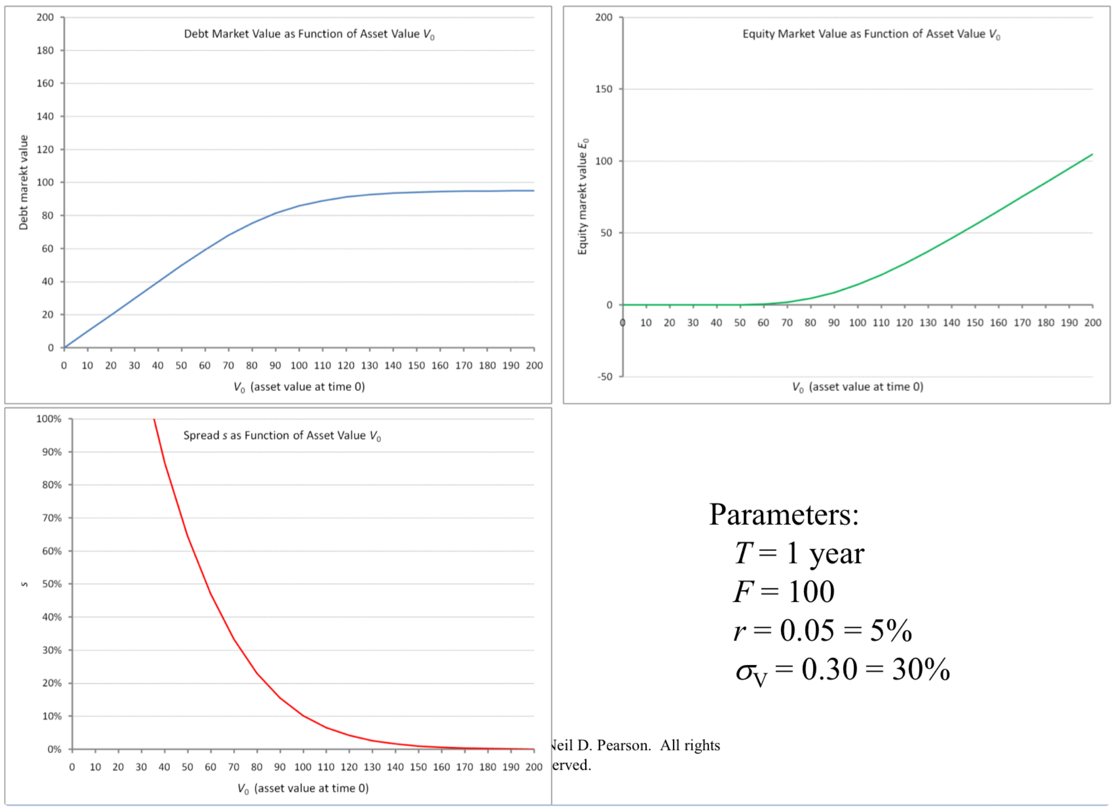
$$d_1 = \frac{\ln(\frac{V_0}{D_0}) + \frac{1}{2}\sigma_V^2 T}{\sigma_V\sqrt{T}} \qquad d_2 = d_1 - \sigma_V\sqrt{T}$$

market value of debt is $MV = V_0 - E_0$

credit spread $MV \cdot e^{(r+s)T} = F$

上面都是利用 equity 长得像 call option 搞出来的

上面的值长这样：



Asset

Liability & Equity

bond ← observe：可能不交易，我们不知道其MV

loan

loan

equity

用右边的算左边的

do not observe ⇐ ?

有什么问题呢：对于上市公司，equity 很容易求，但是 liability 很难搞 & 我们也不一定有，可能需要做一些假设

· T 和 D 都是很好求的.

注意: Equity Value E_0 和 volatility σ_E 一般都是可用的

$$E_0 = V_0 N(d_1) - D_0 N(d_2)$$

$$\sigma_E E_0 = N(d_1)\sigma_V V_0$$

?

第二个式子是哪来的呢?

$$\text{change in } \underset{\text{equity value}}{E} = \frac{\partial E}{\partial V} \times \text{change in } V$$

⇑ 就像这样

类似于 $\text{change in } \underset{\text{value of call option}}{C} = \underset{\Delta}{\frac{\partial C}{\partial S}} \times \text{change in } S$

$$\text{equity vol } \$ = \frac{\partial E}{\partial V} \times \text{asset vol } \$$$

$$\sigma_E \cdot E = \underset{N(d_1)}{\frac{\partial E}{\partial V}} \sigma_V \cdot V_0$$

(由 $E_0 = V_0 N(d_1) - D_0 N(d_2)$ 得到)

上面这个东西有两个用处:

- To estimate default probability of each obligor
- To estimate default dependence of different obligors

I 先看看怎么估一个公司的 default prob.

这个模型直接提供了 default prob. $P_D = 1 - N(d_2)$

这么做有什么问题么? P_D 是用 option value formula 定的. 用的是 risk-neutral prob. 但我们不想用 R-N prob.

$$d_2 = \frac{\ln\frac{V_0}{F} + (r - \frac{1}{2}\sigma_V^2)T}{\sigma_V\sqrt{T}}$$

$$d_2^{objective} = \frac{\ln\frac{V_0}{F} + (\mu - \frac{1}{2}\sigma_V^2)T}{\sigma_V\sqrt{T}}$$

$d_2^{objective}$ 实际上是"distance to default"

$$\underbrace{\ln V_0 + (r - \tfrac{1}{2}\sigma_V^2)T}_{E[\ln V_T]} - \underbrace{\ln F}_{\text{default boundary}}$$

* F 是 face value of debt

expected value of log value

$$\frac{\ln V_0 + (r - \frac{1}{2}\sigma_V^2)T - \ln F}{\sigma_V\sqrt{T}} \Rightarrow \text{difference to default measured in s.d.}$$

F is sum(short term debt + $\frac{1}{2}$ long term debt)

不是算 default prob、而是直接用 $d_2^{objective}$

↳ 就这个东西重要

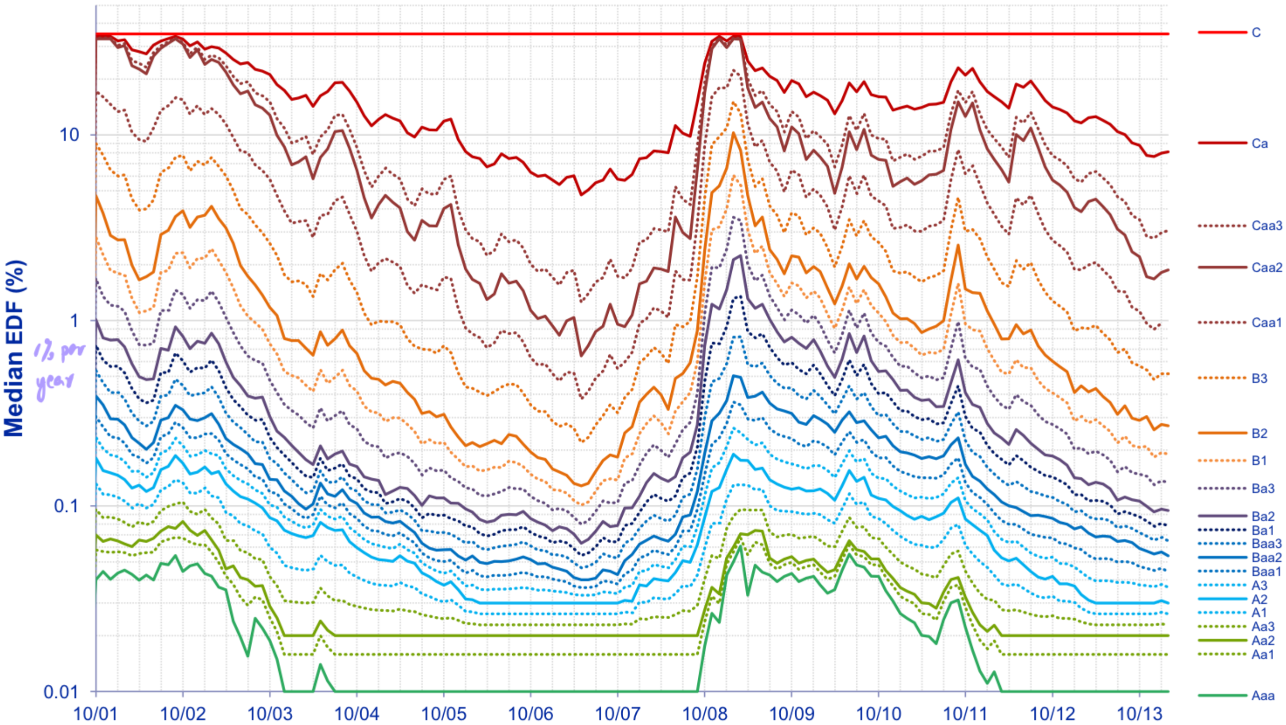
把 $d_2^{objective}$ 放在统计模型中, predict the likelihood of default

Structure model 有什么好处呢?

捕捉了 default prob、随着 business cycle 变化的情形. 如果用 historical default rate 来估计, 那么会失去这种趋势特征

下面用一个具体的图来表现一下

Median EDFs for MIS rating categories



MOODY'S ANALYTICS

我们看到 default prob. 变化非常大

在今天学的模型中，default prob. 由 DtD 算出

$$DtD = \frac{\ln V_0 + (\mu - \frac{1}{2}\sigma_v^2)T - \ln F}{\sigma_v \sqrt{T}}$$

μ, σ_v 是系数，不变，F 是债，几乎不变 (face value of debt)

$V_0 \to$ total asset $\begin{cases} E & ! \\ L & 不变 \end{cases}$

DtD 主要受 Equity value 的影响 所以 equity value 是能影响 default prob. 的！

这个图展示了 structure model 的优越性

Ⅱ 再看看如何估 joint default

由上面的模型我们知道每个公司的 default prob. 是由其 asset value 决定的
(V_i)

所以我们怎么估 default dependence 呢？用 asset value correlation！

但我们还需要 default prob. → 把我们的模型 calibrate 以匹配真正的default prob.

这个 default prob. 可以来自于
- structural model
- internal credit ratings } mapped to default prob.
- public credit ratings } mapped to default prob.
- other models

接下来讨论一下模型的校正：Calibrating the model

$$P_i = 1 - N(d_2^i) = N(-d_2^i)$$

$$\text{where} \quad d_2^i = \frac{\ln\frac{V_i}{D_i} + (\mu_A^i - \frac{\sigma_A^{i\,2}}{2})T}{\sigma_A^i\sqrt{T}}$$

* 这里的 D_i 就是 F
(default boundary)

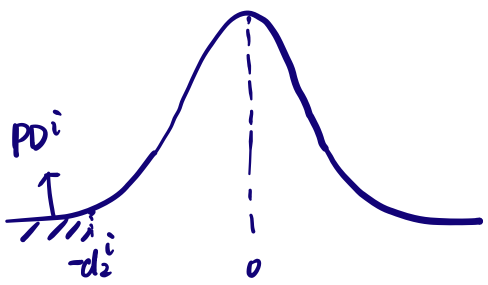
但 $\frac{V_i}{D_i}$，μ_A^i，σ_A^i 我们都不知道啊，有一个 default prob. 怎能知道三个系数呢？

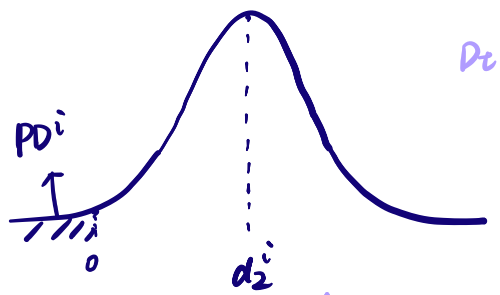
其实是很简单：　$P_i = N(-d_i)$ 是 default prob.

$N(-d_i)$ is the probability that a standard normal random $Z_i \sim N(0,1)$ is less than $-d_i$

也就是说是 probability that $Z_i \sim N(d_i, 1)$ is less than 0

画个图来说明一下：





default occurs when DtD = 0 ($d_2^i = 0$)

= distance to default

we can use $Z_i \sim N(0,1)$ or $Z_i \sim N(d_2^i, 1)$

我们假设我们知道 P_i （反正我们知道） $d_2^i = N^{-1}(P_i)$

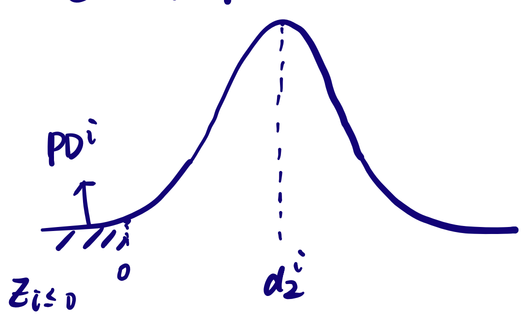
然后估计 correlation matrix

We also need the correlation matrix of the returns on the asset values V_i.

There are several reasonable estimates:

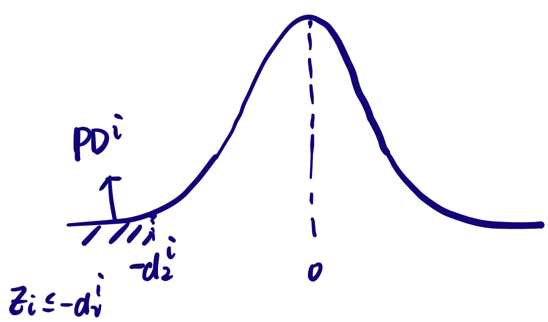
- Sum values of equity and debt to get the asset value, and then compute correlations from the asset value returns.
  - A limitation of this approach is that market values of debt are often unavailable
- Use stock return correlations
- Use correlations of changes in CDS spreads.
  - A limit is that the number of firms with available CDS spread data is limited

然后是汇总时间：



$Z_i \sim N(d_2^i, 1)$

或



$Z_i \sim N(0, 1)$

用 P_i (default prob.) 换成 d_2^i

然后用估出来的 correlation matrix ρ，估一串 $Z = (Z_1, Z_2, \cdots Z_k)'$，每个 $Z \sim N(0,1)$

它们的 correlation 是 ρ，如果 $Z_i \le -d_2^i$，那么 obligor default

有什么问题呢： P_c很低，需要很多 MC trials