ECON 815 - Firm Dynamics

Jason DeBacker

Fall 2017

Notes for the Firm's Problem in Partial Equilibrium

Capital Accumulation by a firm

- Objective: maximize the discounted present value of profits. Specifically, the firm solves:

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \prod_{u=0}^{t} \left(\frac{1}{1+r_u}\right) \left[\pi(z_t, k_t) - p_t^k I_t\right] \tag{1}$$

  - Note that in the expression above, there are no expectations, it is a deterministic problem.

- $I_t \equiv$ gross investment $= k_{t+1} - (1-\delta)k_t$
  - $\delta \equiv$ the rate physical depreciation of capital
  - $p_t^k \equiv$ the price of new capital
  - The law of motion for the capital stock can be found from the equation for investment:

$$k_{t+1} = (1-\delta)k_t + I_t \tag{2}$$

  - Note an important implication of this specification of the problem.
  - That is, capital does not become productive in the period the investment is made.
  - This is what Kydland and Prescott [1982] call a "time to build".
  - Here, it becomes productive in the next year (but one could modify this to be more than one period).

- Profit flow per period: $\pi(z, k)$
  - $k =$ capital stock
  - $z =$ shock to profits
    * These shocks could be to demand or production technology (it's reduced form)
    * E.g., could be positive due to a new product or a new way to make the same product cheaper

- Source of $\pi(z, k)$:
  - More generally, profits depend on more than capital and productivity, there are other inputs
  - $\pi(z, k) = \max_l R(z, k, l) - wl$
    * $k \equiv$ fixed factors of production (e.g., capital)
      · Already paid for by investment in previous period $\Rightarrow$ no cost in intra-temporal max problem
    * $l \equiv$ flexible factors of production (e.g., labor, electricity)
    * $w \equiv$ input prices
    * $R(\cdot) \equiv$ Revenue function: $p(q)q(p(q))$ (q is quantity, p is price, p may depend on q if have lots of market share and so the market is not competitive)
    * $q(z, k, l) \equiv$ production function

- Solving: $\max_l R(z,k,l) - wl \implies$ policy function $= l(z,k)$ (w here is a parameter, not a state variable)
- This policy function them implies we can write: $R(z,k,l(z,k)) - wl(z,k) = \pi(z,k)$
- Here we'll assume that $q(z,k,l) = zk^{\alpha_k} l^{\alpha_l}$
  * where, $\alpha_k + \alpha_l \leq 1$
  * If $\alpha_k + \alpha_l < 1$, then the firm has decreasing returns to scale and will generate economic profits
- Normalize the price of output such that $p(q) = 1$
  * This means that the model will be written in terms of units of output
  * It also means that the firm is a price taker
- Thus we have $R(z,k,l) = p(q) * q(z,k,l) = q(z,k,l) = zk^{\alpha_k} l^{\alpha_l}$
- The firm takes as given the capital stock it enters the period with and the productivity shock that is realized in that period (these are the state variables)
- Given these states, the firm is going to choose labor demand and investment. Only the investment decision is inter-temporal since only we are assuming only the capital stock is in the state vector.
- Thus, we can write the intra-temporal problem (i.e., the within period problem) of the firm as:

$$\begin{aligned} &\max_l R(z,k,l) - wl \\ &\max_l zk^{\alpha_k} l^{\alpha_l} - wl \end{aligned} \tag{3}$$

- The FOC for this problem is:

$$\alpha_l z k^{\alpha_k} l^{\alpha_l - 1} = w \tag{4}$$

- We can solve this for $l(z,k)$:

$$\begin{aligned} \alpha_l z k^{\alpha_k} l^{\alpha_l - 1} &= w \\ \implies l^{\alpha_l - 1} &= \frac{w}{\alpha_l z k^{\alpha_k}} \\ \implies l &= \left(\frac{w}{\alpha_l z k^{\alpha_k}}\right)^{\frac{1}{\alpha_l - 1}} \\ \implies l &= \left(\frac{\alpha_l}{w}\right)^{\frac{1}{1-\alpha_l}} z^{\frac{1}{1-\alpha_l}} k^{\frac{\alpha_k}{1-\alpha_l}} \end{aligned} \tag{5}$$

- Now we are able to solve for $\pi(z,k)$:

$$
\begin{aligned}
\pi(z,k) &= R(z,k,l(z,l)) - wl(z,k) \\
&= zk^{\alpha_k} l(z,k)^{\alpha_l} - wl(z,k) \\
&= l(z,k)\left[zk^{\alpha_k} l(z,k)^{\alpha_l - 1} - w\right] \\
&= l(z,k)\left[zk^{\alpha_k}\left(\left(\frac{\alpha_l}{w}\right)^{\frac{1}{1-\alpha_l}} z^{\frac{1}{1-\alpha_l}} k^{\frac{\alpha_k}{1-\alpha_l}}\right)^{\alpha_l - 1} - w\right] \\
&= l(z,k)\left[zk^{\alpha_k}\left(\frac{\alpha_l}{w}\right)^{-1} z^{-1} k^{-\alpha_k} - w\right] \\
&= l(z,k)\left[\left(\frac{\alpha_l}{w}\right)^{-1} - w\right] \\
&= \left(\frac{\alpha_l}{w}\right)^{\frac{1}{1-\alpha_l}} z^{\frac{1}{1-\alpha_l}} k^{\frac{\alpha_k}{1-\alpha_l}} \left[\left(\frac{\alpha_l}{w}\right)^{-1} - w\right] \\
&= z^{\frac{1}{1-\alpha_l}} k^{\frac{\alpha_k}{1-\alpha_l}} \left[\left(\frac{\alpha_l}{w}\right)^{\frac{1}{1-\alpha_l}}\left(\frac{\alpha_l}{w}\right)^{-1} - w\left(\frac{\alpha_l}{w}\right)^{\frac{1}{1-\alpha_l}}\right] \\
&= z^{\frac{1}{1-\alpha_l}} k^{\frac{\alpha_k}{1-\alpha_l}} \left[\left(\frac{\alpha_l}{w}\right)^{\frac{\alpha_l}{1-\alpha_l}} - w\left(\frac{\alpha_l}{w}\right)^{\frac{1}{1-\alpha_l}}\right] \\
&= z^{\frac{1}{1-\alpha_l}} k^{\frac{\alpha_k}{1-\alpha_l}} \left[\left(\frac{\alpha_l}{w}\right)^{\frac{\alpha_l}{1-\alpha_l}} - w\left(\frac{\alpha_l}{w}\right)^{\frac{1}{1-\alpha_l}}\frac{\alpha_l}{\alpha_l}\right] \\
&= z^{\frac{1}{1-\alpha_l}} k^{\frac{\alpha_k}{1-\alpha_l}} \left[\left(\frac{\alpha_l}{w}\right)^{\frac{\alpha_l}{1-\alpha_l}} - \frac{w}{\alpha_l}\left(\frac{\alpha_l}{w}\right)^{\frac{1}{1-\alpha_l}}\alpha_l\right] \\
&= z^{\frac{1}{1-\alpha_l}} k^{\frac{\alpha_k}{1-\alpha_l}} \left[\left(\frac{\alpha_l}{w}\right)^{\frac{\alpha_l}{1-\alpha_l}} - \left(\frac{\alpha_l}{w}\right)^{-1}\left(\frac{\alpha_l}{w}\right)^{\frac{1}{1-\alpha_l}}\alpha_l\right] \\
&= z^{\frac{1}{1-\alpha_l}} k^{\frac{\alpha_k}{1-\alpha_l}} \left[\left(\frac{\alpha_l}{w}\right)^{\frac{\alpha_l}{1-\alpha_l}} - \left(\frac{\alpha_l}{w}\right)^{\frac{\alpha_l}{1-\alpha_l}}\alpha_l\right] \\
&= z^{\frac{1}{1-\alpha_l}} k^{\frac{\alpha_k}{1-\alpha_l}} \left[\left(\frac{\alpha_l}{w}\right)^{\frac{\alpha_l}{1-\alpha_l}}(1-\alpha_l)\right] \\
&= (1-\alpha_l)\left(\frac{\alpha_l}{w}\right)^{\frac{\alpha_l}{1-\alpha_l}} z^{\frac{1}{1-\alpha_l}} k^{\frac{\alpha_k}{1-\alpha_l}}
\end{aligned} \tag{6}
$$

- This is then the profit function that will enter the firm's dynamic problem specified in Equation (1)
  * What we've done here is determined the optimal choice of labor for all states (all combinations of z and k).
  * Thus, by choosing capital for the next period, we're also determining labor.
  * Note that if labor were also a state variable (e.g. because there were adjustment costs in changing labor demand), then we couldn't do this and Equation (1) would be written to include the sequence of labor demand decision in the max operator.

<u>The Simplest Case</u>

- Assume that the firm's problem can be described by:

$$
\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \prod_{u=0}^{t}\left(\frac{1}{1+r_u}\right)\left[\pi(z_t,k_t) - p_t^k I_t\right] \tag{7}
$$

- The way in which this is written tell us a couple important things:

1. There is no uncertainty. We see this because there is no expectations operator.
2. There is not much in the way of dynamics in the firms' decisions. This is harder to see.

- Why is the firm problem not really dynamic? Time is denoted in the firm's objective and there is a time to build...
- The FOCs of the firm (one for each time period) are:

$$-p_t^k + \frac{1}{1+r_{t+1}}\left[\frac{\partial \pi(z_{t+1},k_{t+1})}{\partial k_{t+1}} + p_{t+1}^k(1-\delta)\right] = 0 \;\forall\; t = 0,\ldots,\infty \tag{8}$$

Written differently,

$$\frac{1}{1+r_{t+1}}\left[\frac{\partial \pi(z_{t+1},k_{t+1})}{\partial k_{t+1}} + p_{t+1}^k(1-\delta)\right] = p^{k_t} \tag{9}$$

where the left-hand side of the equation are the marginal benefits from more capital and the right hand side is the marginal cost of new capital. Yet another way to write this is to express the condition as a function of the user cost of capital.

$$\begin{aligned} \frac{\partial \pi(z_{t+1},k_{t+1})}{\partial k_{t+1}} &= p_t^k(1+r_{t+1}) - p_{t+1}^k(1-\delta) \\ \implies \underbrace{\frac{\partial \pi(z_{t+1},k_{t+1})}{\partial k_{t+1}}}_{\text{Marginal Revenue Product of Capital}} &= \underbrace{p_t^k - p_{t+1}^k + p_t^k r_{t+1} + p_{t+1}^k\delta}_{\text{User Cost of Capital}} \end{aligned} \tag{10}$$

- This FOC represents a single equation that can be solved for the single unknown, k_{t+1}.
- E.g., Given our representation of the profit function above, assuming economic aggregates are in the steady-state (i.e., $p_t^k = p_{t+1}^k$), and normalizing the price of the output/capital goods to one, we have:

$$\begin{aligned} \frac{\partial \pi(z_{t+1},k_{t+1})}{\partial k_{t+1}} &= \bar{r}+\delta \\ \implies \alpha_k\left(\frac{\alpha_l}{\bar{w}}\right)^{\frac{\alpha_l}{1-\alpha_l}} z_{t+1}^{\frac{1}{1-\alpha_l}} k_{t+1}^{\frac{\alpha_k+\alpha_l-1}{1-\alpha_l}} &= \bar{r}+\delta \\ \implies k_{t+1} = \left(\frac{\alpha_k}{\bar{r}+\delta}\right)^{\frac{1-\alpha_l}{1-\alpha_k-\alpha_l}} \left(\frac{\alpha_l}{w}\right)^{\frac{\alpha_l}{1-\alpha_k-\alpha_l}} z_{t+1}^{\frac{\alpha_l}{1-\alpha_k-\alpha_l}} & \\ \implies k_{t+1} = \left[\left(\frac{\alpha_k}{\bar{r}+\delta}\right)^{1-\alpha_l}\left(\frac{\alpha_l}{w}\right)^{\alpha_l} z_{t+1}^{\alpha_l}\right]^{1-\alpha_k-\alpha_l} & \end{aligned} \tag{11}$$

- A few observations about what this FOC implies:
  1. We have a closed form solution to k_{t+1}. No need for computational methods here!
  2. k_{t+1} is a function of the parameters of the model, the steady-state factor prices, and the firms' productivity shock in period $t+1$ (which is known).
     - Thus there are no real dynamics!
     - The firm's capital stock might change from period to period, but it is not a function of any past choices.
     - This is why the solution is so simple.
  3. We can look at this FOC and see that our economic intuition makes sense:
     - As the cost of capital ($\bar{r}+\delta$) increases, the optimal k_{t+1} goes down
     - As the productivity in $t+1$ increases, the optimal k_{t+1} goes up, etc.
  4. The problem is only determined with $\alpha_k + \alpha_l < 1$

- To see this, look at Equation (11) carefully
- Also, recall from micro the firms' problem with constant returns to scale (CRS)
- CRS $\implies$ the size of the firm is indeterminate

<u>Firm Dynamics</u>

- To make this problem truly dynamic, we need to introduce something into the firm's problem that makes the choice of capital next period a function of the capital stock of the firm in the current period.
- We do this through costs of adjusting the firm's capital stock
- We denote this function $c(I_t, k_t)$ or $c(k_{t+1}, k_t)$ (where the second specification acknowledges $I_t = k_{t+1} - (1-\delta)k_t$)
  - This general function is a way to represent various costs of adjusting the firm's capital stock.
  - It can represent things like irreversibility in investments, training workers to use new equipment, disruption costs to installing new equipment, etc.
  - We'll consider various forms for this function and their implications for investment.
- The problem of the firm thus becomes:

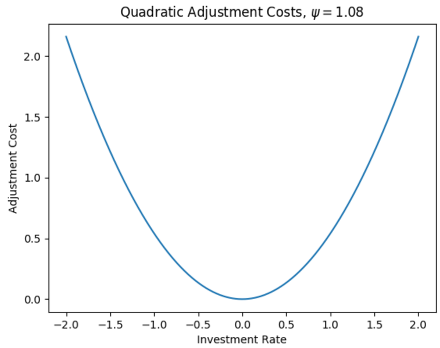
$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \prod_{u=0}^{t} \left(\frac{1}{1+r_u}\right) \left[\pi(z_t, k_t) - p_t^k I_t - c(k_{t+1}, k_t)\right] \tag{12}$$

- On the price of capital (things to consider):
  - p_t^k could be modeled as a stochastic process
  - Buying price > selling price
    - A bigger gap makes more cautious - not as responsive to increases in z_t
- More on costs of adjustment:
  - We generally think of these costs as being one of two types:

1. Convex costs of adjustment
   - The marginal cost of investment increases with the size of the investment.
   - Such costs mean that firms want to smooth, or spread out, capital adjustment over time.
   - "Rome not built in a day" - because expensive to adjust capital stock a lot.
   - A common form for, $c(k_{t+1}, k_t)$ is where these costs are quadratic (in the investment rate, $\frac{I}{k}$):

$$c(k_{t+1}, k_t) = \frac{\psi}{2}\left(\frac{k_{t+1} - (1-\delta)k_t}{k_t}\right)^2 k_t = \frac{\psi}{2}\left(\frac{I_t}{k_t}\right)^2 \underbrace{k_t}_{\text{scaling w/ k}} = \frac{\psi}{2}\frac{I_t^2}{k_t} \tag{13}$$

2. Non-convex costs of adjustment
   - The marginal cost of investment is not increasing in the size of the investment.
   - e.g., Fixed costs of adjustment
     - One-time cost to install as much capital as you want
     - $\Rightarrow$ bunching of activities (e.g. train all workers when shutdown plant to install new machines)
   - Such costs mean that firms want to bunch their investments together to spread fixed costs over a large investment.



    - "Rome not built everyday"
  - A gap between buying and selling price come in here too - this gap may be modeled as a cost of adjustment

- We can express the firm's problem, represented in the sequence problem in Equation (12) as a Bellman equation:

$$V(z,k) = max_{k'}\pi(z,k) - p^k(k' - (1-\delta)k) - c(k',k) + \frac{1}{1+r}V(z',k') \tag{14}$$

  - In this equation, primes denote one-period-ahead variables (so we can drop the t subscripts)
  - max over k' because choosing investment today which implies capital tomorrow
  - Other notes:
    * Consider the price of new capital, p^k to be constant over time
  - $V(z,k)$ is the "value function". It is the maximum value of the firm given the optimal choice of k'
  - The solution to the functional equation described in (14) is the value function, $V(z,k)$ and the policy function, $k' = h(z,k)$
  - These functions satisfy the necessary conditions of the firm's optimal choice of capital, which (assuming the costs of adjustment function is continuous) are given by:

$$\frac{\partial V(z,k)}{\partial k'} = -p^k - \frac{\partial c(k',k)}{\partial k'} + \frac{1}{1+r}\frac{\partial V(z',k')}{\partial k'} = 0\ \forall\ z,k \tag{15}$$

  - Equation (15) might represent the necessary conditions, but it doesn't help much in solving the firms problem since $\frac{\partial V(z',k')}{\partial k'}$ is an unknown and something we are solving for.
  - Fortunately, we can simplify things using Bellman's "principle of optimality"
  - But first, some notation:
    * Let $c_1(k',k)$ ($c_2(k',k)$) denote the derivative of the adjustment cost function with respect to the first (second) argument.
    * Let $V_1(z,k)$ ($V_2(z,k)$) denote the derivative of the value function with respect to the first (second) argument.
  - With our new notation, we can write the FOC as:

$$\begin{aligned}\frac{\partial V(z,k)}{\partial k'} &= -p^k - c_1(k',k) + \frac{1}{1+r}V_2(z',k') = 0\\ \implies p^k + c_1(k',k) &= \frac{1}{1+r}V_2(z',k')\end{aligned} \tag{16}$$

  - We can simplify the FOC by using the envelope condition (an application of Bellman's principle):

- $* \ V_2(z,k) = \pi_2(z,k) + (1-\delta)p^k - c_2(k',k) + \frac{\partial k'}{\partial k}\left[\underbrace{-p^k - c_1(k',k) + \frac{1}{1+r}V_2(z',k')}_{=0,\text{ by the FOC}}\right]$
- $* \implies V_2(z,k) = \pi_2(z,k) + (1-\delta)p^k - c_2(k',k)$
- $* \implies V_2(z',k') = \pi_2(z',k') + (1-\delta)p^k - c_2(k'',k')$
- $*$ Which means Equation (16) can be written as:

$$p^k + c_1(k',k) = \frac{1}{1+r}[\pi_2(z',k') + (1-\delta)p^k - c_2(k'',k')] \tag{17}$$

  - So we know the equation that characterizes k' for all k, z. Done?
  - Not quite, in general (and unlike the simply case about without dynamics) there is no closed form solution for the policy function, $k' = h(z,k)$.
  - What to do? Use the computer and solve this equation numerically! We'll use Value Function Iteration (VFI)
  - See example in the Jupyter Notebook(s) from class.

<u>Stochastic Profit Shocks</u>

- Adjustment costs added dynamics to the model.
- We can make the model more interesting by adding uncertainty.
- There are a number of way we we could introduce uncertainty:
  1. Stochastic price for new capital, p^k
  2. Stochastic price for output, p
  3. Shocks to the cost of adjusting capital, $c(k',k,\varepsilon)$, where ε is a random variable in the cost function
  4. Shocks to productivity/profitability, z
     - Note that this is isomorphic to stochastic output prices in the model as specified above
- We'll work with a model that has stochastic productivity shocks.
- Productivity shocks are assumed to follow and AR(1) process where:

$$\ln(z_{t+1}) = \rho\ln(z_t) + (1-\rho)\mu + \varepsilon_t, \tag{18}$$

  where $\varepsilon_t \sim N(0,\sigma_\varepsilon)$.
- The firm's problem can be summarized through the Bellman equation:

$$V(z,k) = \max_{k'} \pi(z,k) - p\underbrace{(k' - (1-\delta)k)}_{\text{gross invest}} - \underbrace{c(k',k)}_{\text{adj costs}} + \frac{1}{1+r}E_{z'|z}V(z',k'), \forall(z,k) \tag{19}$$

  - Note the expectation is over the future profitability shocks, which are stochastic.
- The FOC for the choice of k' is:

$$p^k + c_1(k',k) = \frac{1}{1+r}[E_{z'|z}\pi_2(z',k') + (1-\delta)p - c_2(k'',k')] \tag{20}$$

- This problem will also need to be solved by computational methods - e.g., VFI

- These computational methods will generally required a discretized state space. So how to represent the continuous AR(1) process?
  - We'll approximate this AR(1) process with a first-order Markov process
  - This Markov process will describe the transition probabilities from each point in the z space to each other point.
  - This is described in this Jupyter Notebook.

q-Theory

- "marginal q" = the shadow value of capital = the marginal value of the firm - i.e., $\frac{\partial V(z,k)}{\partial k}$
- Sometimes called "Tobin's q" after James Tobin.[1]
- Idea:
  - Recall the firm's FOC:
$$p^k + c_1(k', k) = \frac{1}{1+r} E_{z'|z} \underbrace{V_2(z', k')}_{\text{Marginal q}} \tag{21}$$
  - Marginal q represents the marginal benefit of investing in the firm
  - Thus q is a sufficient statistic for firm investment $\implies$ invest up until the point where the price of capital plus the cost of adjustment equals (expected) q
- Problem in practice: We can't observe marginal q!
- Solution (potentially): Hayashi [1982]
  - Idea:
    - Under certain conditions, Marginal q = Average q
    - And we can observe Average q = $\frac{V(z,k)}{k} = \frac{\text{Market value of the firm}}{\text{Book value of the firm}}$
  - Hayashi [1982] thus gives us the conditions under which we have some empirical analogue to marginal q
  - Now we can apply and test q-theory
    - Does q predict firm investment as the model says?
    - Are firms over or under valued? (i.e., is q > or < 1)
  - So why "potential" solution?
  - Hayashi [1982] relies on three important assumptions:
    1. Markets are competitive so that firms are price takers
    2. The profit function is linearly homogenous in z and k
    3. Adjustment costs are linearly homogenous in I and k
  - Proof of Hayashi [1982]:
  - Assume:
    - Output prices are exogenous and normalized to $p = 1$ (This satisfies assumption (1))
    - $\pi(z, k) = zk$ (This satisfies assumption (2))
    - Quadratic adjustment costs (This satisfies assumption (3))
    - Value function of the form $V(z, k) = \phi(z)k$ (this is a guess, but we'll prove it's true given the assumptions above)

[1] The letter "q" was first used to denote the shadow price of investment in Tobin [1969], although this approach modeling investment policy preceded the q notation in Brainard and Tobin [1968] and Kaldor [1966].

- $V(z,k) = \max_I \underbrace{zk}_{\text{profit fnc prop to k}} - p^k I - \underbrace{\frac{\psi}{2}\left(\frac{I}{k}\right)^2 k}_{\text{quad costs of adj}} + \frac{1}{1+r} E_{z'|z} V(z',k')$
- FOC:

$$\begin{aligned} \frac{\partial V(z,k)}{\partial I} &= -p^k - \psi\left(\frac{I}{k}\right) + \frac{1}{1+r} E_{z'|z} V_2(z',k') = 0 \\ \Rightarrow \left(\frac{I}{k}\right) &= \frac{1}{\psi}\left[\frac{1}{1+r} E_{z'|z} V_2(z',k') - p^k\right] \end{aligned} \tag{22}$$

- Use the assumed value function to find V_2: $V_2(z,k) = \frac{\partial V(z,k)}{\partial k} = \phi(z)$
- Which means we can write the FOC as: $\left(\frac{I}{k}\right) = \frac{1}{\psi}[\frac{1}{1+r} E_{z'|z}\phi(z') - p^k] \equiv h(z)$
  * b/c $E_{z'|z}$, investment rate is only a function of z
  * How much investment we chose depends upon how well z informs us about z'
- $\Rightarrow I = h(z)k$
- To get k':
  * $k' = (1-\delta)k + I$
  * $\implies k' = (1-\delta)k + h(z)k$
  * $\implies k' = ((1-\delta) + h(z))k$
- Now, subbing in the FOC sol'n into the functional equation:

$$\begin{aligned} \phi(z) * k &= zk - p^k \underbrace{h(z)k}_{I} - \underbrace{\frac{\psi}{2}(h(z))^2 k}_{\text{cost of adj}} + \frac{1}{1+r} E_{z'|z}\{\phi(z')((1-\delta) + h(z))k\} \\ [\phi(z)]k &= k[z - p^k h(z) - \frac{\psi}{2}(h(z))^2 + \frac{1}{1+r} E_{z'|z}\{\phi(z')(1-\delta+h(z))\}] \end{aligned} \tag{23}$$

  * This equation holds for all k b/c k multiplies both sides (i.e., k cancels out)
  * Thus we know this is a sol'n to the problem
  * Since $V(z,k) = \phi(z)k$, $q = \frac{\partial V(z,k)}{\partial k} = \phi(z) = \frac{V(z,k)}{k}$
  * $V(z,k)$ for firm should equal the market value of the firm (which we can observe from stock market data)
- Another problem with the marginal q = average q approach:
  * In practice, it is hard to measure average q because stock market volatility provides a lot of noise
  * This attenuates the relationship between measured average q and investment
  * See, for example, Erickson and Whited [2000]
- But, this result is useful in generating a parsimonious model for empirical work.
- From the FOC we have: $\left(\frac{I}{k}\right) = \frac{1}{\psi}\left[\frac{1}{1+r} E_{z'|z} V_2(z',k') - p^k\right]$
- Substitutiong in avg q for $E_{z'|z} V_2(z',k')$, we have: $\left(\frac{I}{k}\right) = \frac{1}{\psi}\left[\frac{1}{1+r}\text{Avg q} - p^k\right]$
- We could use this linear equation to estimate ψ or to test q-theory
  * E.g., Gilchrist and Himmelberg [1995] put cash flow into this equation
  * q-theory says cash flow shouldn't matter, so if it does, that means the model above is missing something (or there is measurement error in the investment rate or average q)

References

Finn E. Kydland and Edward C. Prescott. Time to build and aggregate fluctuations. *Econometrica*, 50(6): 1345–1370, 1982. ISSN 00129682, 14680262. URL http://www.jstor.org/stable/1913386.

James Tobin. A general equilibrium approach to monetary theory. *Journal of Money, Credit and Banking*, 1(1):15–29, 1969. ISSN 00222879, 15384616. URL http://www.jstor.org/stable/1991374.

William C. Brainard and James Tobin. Pitfalls in financial model building. *The American Economic Review*, 58(2):99–122, 1968. ISSN 00028282. URL http://www.jstor.org/stable/1831802.

Nicholas Kaldor. Marginal productivity and the macro-economic theories of distribution: Comment on samuelson and modigliani. *The Review of Economic Studies*, 33(4):309–319, 1966. ISSN 00346527, 1467937X. URL http://www.jstor.org/stable/2974428.

Fumio Hayashi. Tobin's marginal q and average q: A neoclassical interpretation. *Econometrica*, 50(1): 213–224, 1982. ISSN 00129682, 14680262. URL http://www.jstor.org/stable/1912538.

Timothy Erickson and Toni M. Whited. Measurement error and the relationship between investment and q. *Journal of Political Economy*, 108(5):1027–1057, 2000. ISSN 00223808, 1537534X. URL http://www.jstor.org/stable/10.1086/317670.

Simon Gilchrist and Charles P. Himmelberg. Evidence on the role of cash flow for investment. *Journal of Monetary Economics*, 36(3):541–572, December 1995. URL https://ideas.repec.org/a/eee/moneco/v36y1995i3p541-572.html.