Deduction Of Manipulator Kinematics

In this part, we will derive the general form of the transformation that relates the the frames attached to neighboring links. We then concatenate these individual transformations to solve for the position and orientation of link{n} relative to link 0.

Derivation of Link transformation

In this part, we wish to construct the transform that defines frame {i} relative o the frame {i -1}. In general, this transformation will be a function of four link parameters . For any given robot, this transformation will be a function of only one variable, the other three parameters being fixed by mechanical design. By defining a frame for each link, we have broken the kinematics problems into n subproblems. In order to solve each of these subproblems, namely ${}^{i-1}_{i}T$, we will further break each subproblem into four subsubproblems.

We begin by defining three intermediate frames for each links - {P}, {Q}, {R}.
And We define that:

Frame {R} differs from Frame {i-1} only by a rotation of α_{i-1}.

Frame{Q} differs from {R} by a translation a_{i-1}.

Frame{P} differs from {Q} by a rotation θ_i.

Frame {i} differs from {P} by a translation d_i.

We can write the transformation that transforms vectors defined in {i} to their description in {i-1},
We may write

$$ {}^{i-1}_{i}T = {}^{i-1}_{R}T \times {}^{R}_{Q}T \times {}^{Q}_{P}T \times {}^{P}_{i}T \times {}^{i}P $$

Or

$$ {}^{i-1}P = {}^{i-1}_{i}T \times {}^{i}P $$

Where

$$ {}^{i-1}_{i}T = {}^{i-1}_{R}T \times {}^{R}_{Q}T \times {}^{Q}_{P}T \times {}^{P}_{i}T $$

We could write it in another way:

$$ {}^{i-1}_{i}T = R_x(\alpha_{i-1}) \times D_x(a_{i-1}) \times R_z(\theta_i) \times D_z(d_i) $$

Where the notation R_x (α_{i-1}) is a rotation along axis x by an angle α_{i-1} , D_x (a_{i-1}) is a translation along an axis x by a distance a_{i-1} , R_x (θ_i) is a rotation along axis z by an angle θ_i , D_z (d_i) is a translation along an axis z by a distance d_i .

Or we also can write in this way:

$$ {}^{i-1}_{i}T = Screw_X(a_{i-1}, \alpha_{i-1}) \times Screw_Z(d_i, \theta_i) $$

Where the notation $Screw_Q$ (r,θ) stands for the combination of a translation along an aixs Q by a distance r and a rotation about the same axis by an angle θ. we can obtain the general form of transformation matrix:

$$ {}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

In next step we can use the link parameters to compute the individual transformations for each link.

Substituting the parameters into transformation matrix, we can obtain:

$$ {}^{0}_{1}T = \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 & 0 \\ S\theta_1 & C\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{2}T = \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 & a_1 \\ 0 & 0 & -1 & 0 \\ S\theta_2 & -C\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}T = \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 & a_2 \\ S\theta_3 & C\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{3}_{4}T = \begin{bmatrix} C\theta_4 & -S\theta_4 & 0 & 0 \\ 0 & 0 & -1 & -d_4 \\ S\theta_4 & C\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{4}_{5}T = \begin{bmatrix} C\theta_5 & -S\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ S\theta_5 & C\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{5}_{6}T = \begin{bmatrix} C\theta_6 & -S\theta_6 & 0 & 0 \\ 0 & 0 & -1 & -d_6 \\ S\theta_6 & C\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Concatenating link transformations

Once the link frames have been defined and the corresponding link parameters found, developing the kinematic equations is straightforward. From the values of the link parameters, the individual link-transformation matrices can be computed. Then, the link transformations can be multiplied together to find the single transformation that relates from frame {N} to frame{0}:

$$ {}^{0}_{\mathrm{N}}T = {}^{0}_{1}T\ {}^{1}_{2}T\ {}^{2}_{3}T \dots {}^{N-1}_{N}T $$

This transformation will be a function of all n joints variables. The Cartesian position and orientation of the last link can be computed by this transformation.

According this equation, I can compute the transformation for my own mechanical arm:

$$ {}^{0}_{6}T = {}^{0}_{1}T\ {}^{1}_{2}T\ {}^{2}_{3}T\ {}^{3}_{4}T\ {}^{4}_{5}T\ {}^{5}_{6}T $$

$$ {}^{0}_{6}T = \begin{bmatrix} A_{11} & A_{21} & A_{31} & A_{41} \\ A_{12} & A_{22} & A_{32} & A_{42} \\ A_{13} & A_{23} & A_{33} & A_{43} \\ A_{14} & A_{24} & A_{34} & A_{44} \end{bmatrix} $$

$$ \mathrm{A}_{11} = C_1C_{23}C_4C_5C_6 + S_1S_4C_5C_6 + C_1S_{23}S_5C_6 + \mathrm{C}_1C_{23}S_4S_6 - S_1C_4S_6 $$

$$ \mathrm{A}_{21} = -C_1C_{23}C_4C_5S_6 - S_1S_4C_5S_6 - C_1S_{23}S_5S_6 + \mathrm{C}_1C_{23}S_4C_6 - S_1C_4C_6 $$

$$ \mathrm{A}_{31} = C_1C_{23}C_4S_5 + S_1S_4S_5 - C_1S_{23}C_5 $$

$$ \mathrm{A}_{41} = C_1C_{23}C_4S_5d_6 + S_1S_4S_5d_6 - C_1S_{23}C_5d_6 + \mathrm{C}_1S_{23}d_4 + C_1C_2a_2 + a_1C_1 $$

$$ \mathrm{A}_{12} = S_1C_{23}C_4C_5C_6 - C_1S_4C_5C_6 + S_1S_{23}S_5C_6 + \mathrm{S}_1C_{23}S_4S_6 + C_1C_4S_6 $$

$$ \mathrm{A}_{22} = -S_1C_{23}C_4C_5S_6 + C_1S_4C_5S_6 - S_1S_{23}S_5S_6 + \mathrm{S}_1C_{23}S_4C_6 + C_1C_4C_6 $$

$$ \mathrm{A}_{32} = S_1C_{23}C_4S_5 - C_1S_4S_5 - S_1S_{23}C_5 $$

$$ \mathrm{A}_{42} = -C_1S_4S_5d_6 + S_1C_{23}C_4S_5d_6 - S_1S_{23}C_5d_6 + S_1S_{23}d_4 + S_1C_2a_2 + a_1S_1 $$

$$ \mathrm{A}_{13} = S_{23}C_4C_5C_6 - C_{23}S_5C_6 + S_{23}S_4S_6 $$

$$ \mathrm{A}_{23} = -S_{23}C_4C_5S_6 + C_{23}S_5S_6 + S_{23}S_4C_6 $$

$$A_{33} = S_{23} C_4 S_5 + C_{23} C_5 C_5$$

$$A_{43} = S_{23} C_4 S_5 d_6 + C_{23} C_5 d_6 - C_{23} d_4 + a_2 S_2$$

$$A_{43} = S_{23} C_4 S_5 d_6 + C_{23} C_5 d_6 - C_{23} d_4 + a_2 S_2$$

$$A_{43} = S_{23} C_4 S_5 d_6 + C_{23} C_5 d_6 - C_{23} d_4 + a_2 S_2$$

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$$A_{43} = S_{23} C_4 S_5 d_6 + C_{23} C_5 d_6 - C_{23} d_4 + a_2 S_2$$

$$A_{14} = 0$$

$$A_{24} = 0$$

$$A_{34} = 0$$

$$A_{44} = 1$$