Bayesian Networks I: BN Representation

For Self-learning Purposes

Jiashu Chen

February 6, 2024

Contents

Chapter 1

Joint Distribution

1.1 Covid Example: 12 Data Sample

#	Covid	Mask	Social Distancing
1	True	False	True
2	False	True	True
3	True	False	False
4	False	True	True
5	False	True	False
6	False	True	False
7	True	False	True
8	False	True	True
9	False	True	True
10	False	False	False
11	False	True	False
12	True	True	True

$$P(COVID = F \mid Mask = T, SocialDistancing = T) = ? \tag{1.1}$$

- Marginal Probability

$$P(M = T, D = T) = \sum_{C=T,F} P(C, M = T, D = T) = \frac{6}{12} \tag{1.2}$$

- Conditional Probability

$$\begin{aligned} P(C = F \mid M = T, D = T) &= \frac{P(C = F, M = T, D = T)}{P(M = T, D = T)} \\ &= \frac{5/12}{6/12} \end{aligned} \tag{1.3}$$

- Joint Probability Distribution

  The key is to find the joint probability distribution for C, M, D, i.e., $P(C, M, D)$, with 8 parameters p1, . . . , p8, where $\sum_{i=1}^{8} p_i = 1$

  So, the number of parameters to obtain $P(C, M, D)$ is $2^3 - 1 = 7$

Key Takeaways:
If we have the joint distribution, we can calculate any probability using conditional distribution and marginal distribution.

1.2 Probability

- Joint probability distribution

$$P(X_1, X_2, \dots, X_n)$$

- Marginal probability distribution

  Marginalize out some variables

$$P(X_1) = \sum_{x_2 \dots x_n} P(X_1, X_2, \dots, X_n)$$

- Conditional probability distribution

$$P(X_1 \mid X_2 \dots X_n) = \frac{P(X_1 X_2 \dots X_n)}{P(X_2 \dots X_n)}$$

- Number of parameters of a distribution

  Given discrete random variables $X_1 X_2 \dots X_n$ that take $\alpha_1, \alpha_2, \dots, \alpha_n$ values, respectively,

  – the number of parameters to represent $P(X_1 X_2 \dots X_n)$ is

$$\alpha_1 \alpha_2 \dots \alpha_n - 1$$

  – the number of parameters to represent $P(X_1 \mid X_2 \dots X_n)$ is

$$(\alpha_1 - 1)\alpha_2 \dots \alpha_n$$

1.3 Summary: What is the Goal?

The goal is to find the joint probability distribution from data. So that we could answer any probablistic inqueries.

$$P(C, M, D) = ?$$

#	Covid	Mask	Social Distancing	FlU	COugh	Fever	Ventilation	Season	ConGestion	Difficulty Breathing	DRug	Allergy
1	True	False	True	False	True	True	True	Spring	True	True	False	False
2	False	True	True	False	False	True	False	Summer	False	False	True	False
3	True	False	False	True	True	False	False	Fall	False	True	True	False
4	False	True	True	False	False	True	False	Winter	True	True	False	True
…	…	…	…	…	…	…	…	…	…	…	…	…
1000	True	True	True	False	True	False	True	Spring	False	False	True	True

However, in the real world, we often have much more variables. Suppose we have 12 variables in the covid example, all binary except for season with 4 values, then the number of required parameters to calculate joint probability is $2^{11} * 4 - 1 = 8191$

If we only have 1000 data points, then we have to set at least 7191 parameters to zero, which is problematic.

Chapter 2

Independence

So we understand the importance of joint probability distribution, the problem now is to deal with the large number of parameters while calculating joint probability distribution.

2.1 Importance of Independence

For Covid, Mask, Social Distancing, using conditional probability

$$P(C, M, D) = P(C \mid M, D)P(M, D)$$

Suppose M and D are independent, denoted by $M \perp D$,

$$P(M, D) = P(M)P(D)$$
$$\text{Thus, } P(C, M, D) = P(C \mid M, D)P(M)P(D)$$

$$Number\ of\ parameters = 2^2 + (2-1) + (2-1) = 6$$
$$P(C \mid M, D) : (2-1) * 2 * 2 = 2^2$$
$$P(M) : M\ takes\ two\ values = 2$$
$$P(D) : D\ takes\ two\ values = 2$$

2.2 Statistical Independence

Random variables X and Y are independent, i.e. $X \perp Y$, if

$$P(X, Y) = P(X)P(Y)$$
$$or P(X \mid Y) = P(X) or P(Y \mid X) = P(Y)$$

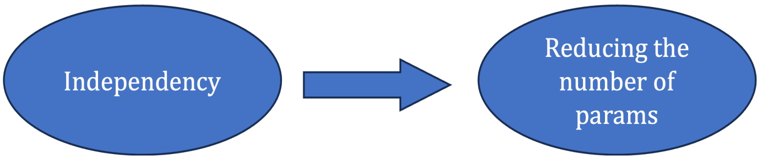
Covid Example

#	Covid	Mask	Social Distancing	FlU	COugh	Fever	Ventilation	Season	ConGestion	Difficulty Breathing	DRug	Allergy
1	True	False	True	False	True	True	True	Spring	True	True	False	False
2	False	True	True	False	False	True	False	Summer	False	False	True	False
3	True	False	False	True	True	False	False	Fall	False	True	True	False
4	False	True	True	False	False	True	False	Winter	True	True	False	True
…	…	…	…	…	…	…	…	…	…	…	…	…
1000	True	True	True	False	True	False	True	Spring	False	False	True	True

Suppose all 12 random variables are mutually independent:

$$C \perp M, C \perp D, \ldots, R \perp A$$
$$P(C, M, D, U, \ldots, A) = P(C)P(M)P(D) \cdots P(A)$$

$$Number\ of\ parameters = 11 + 3 = 14$$
The previous number was 8191.



Key Takeaways:
with M,D being independent, the number of parameters reduced from 7 to 6. Therefore, independence could reduce the number of parameters.

However, the independence assumption does not always hold. For instance, $Mask \not\perp Social\ distancing$

2.3 Conditional Independence

- Bother, Sister, Parent Example
  - Consider a brother(B) and a sister(S)
    $P(S = white \mid B = white) = 0.9$
    $P(S = white \mid B = black) = 0.1$
    Therefore, $S \not\perp B$
  - Consider brother(B), sister(S), and Parent(Pa)
    $P(S = white \mid Pa = white) = 0.95$
    $P(S = white \mid Pa = white, B = white) = 0.95$
    $P(S = white \mid Pa = white, B = black) = 0.95$
    Therefore, knowing the brother does not matter if we know the parent

    S is conditionally independent of B given Pa, i.e. $(S \perp B \mid Pa)$

- Conditional Independence Definition

  Consider the set of random variables X, Y, Z, We say that X is conditionally independent of Y given Z in a distribution P, denoted by $P \models (X \perp Y \mid Z)$ if

  $$P(X = x \mid Y = y, Z = z) = P(X = x \mid Z = z), \forall x, y, z$$

  - The variables in set Z are said to be observed
  - The set of all probability independencies in P is denoted by $I(P)$

    $$P \models (X \perp Y \mid Z)\ if\ and\ only\ if$$
    $$P(X, Y \mid Z) = P(X \mid Z)P(Y \mid Z)$$

- How can Conditional Independence Help
  - Using Covid, Fever, and PRC test example

$$P(F, P, C) = P(F \mid P, C)P(P, C)$$

- if $P(F \perp P \mid C) \in I(P)$, i.e. fever and prc test are conditionally independent given covid, then

$$P(F, P, C) = P(F \mid C)P(P, C)$$

- The number of parameters $= 2 + (2 * *2 - 1) = 5$

Key Takeaways:
Conditional independence can reduce the number of parameters.

2.4 Conditional Independence with Chain Rule

- We could factor joint distribution into CPDs using chain rule

$$\begin{aligned} P(X_1, \ldots, X_n) &= P(X_n \mid X_{n-1}, \ldots, X_1) \ldots P(X_3 \mid X_2, X_1)P(X_2 \mid X_1)P(X_1) \\ &= \textstyle\prod_{i=1}^n p(X_i \mid X_{i-1}, \ldots, X_1) \end{aligned}$$

  Suppose all random variables take binary values, then the number of parameters for

$$p(X_i \mid X_{i-1}, \ldots, X_1) = 2^{(i-1)}$$

- Consider joint distribution P(X1, X2, X3, X4). Using the chain rule, we could ge

$$P(X_1, X_2, X_3, X_4) = P(X_4 \mid X_3, X_2, X_1)P(X_3 \mid X_2, X_1)P(X_2 \mid X_1)P(X_1)$$

- If $(X_4 \perp X_1, X_2 \mid X_3) \in I(P)$, then

$$P(X_4 \mid X_3, X_2, X_1) = P(X_4 \mid X_3)$$

- The joint distribution becomes

$$P(X_1, X_2, X_3, X_4) = P(X_4 \mid X_3)P(X_3 \mid X_2, X_1)P(X_2 \mid X_1)P(X_1)$$

- The total number of parameters reduced from $2^4 - 1 = 15$ to $2 + 2^2 + 2 + 1 = 9$

2.5 Summary: What is the Goal?

- Now we know that to calculate joint probability distribution, we just need to find the conditional independencies and factorize the joint probability distribution. But, there are two more problems
  - How to find conditional independencies I(P) from data?
  - Given the conditional independencies, how to factorize the joint distribution?
- **To visualize the factorization of joint probability distribution is Bayesian Networks!**
- What is the goal?
  The goal is to find from data, the correct factorization of the joint probability distribution.
  - $P(C)P(M \mid C)P(D)$: 4 params
  - $P(C \mid M)P(M)P(D)$: 4 params
  - ...
  - $P(C \mid M)P(M \mid D)P(D)$: 5 params
  - $P(M \mid C, D)P(C)P(D)$: 6 params

  Different factorization leads to different number of parameters.

Chapter 3

Graphical Visualization

3.1 Parent Notation

- In previous example, If $(X_4 \perp X_1, X_2 \mid X_3) \in I(P)$, then

$$P(X_4 \mid X_3, X_2, X_1) = P(X_4 \mid X_3)$$

- For each X_i, we define the parents of X_i, denoted by Pax_i, as the set of variables that X_i is conditioned on in the factorization.

$$Pax_4 = X_3, Pax_3 = X_1, X_2, Pax_2 = X_1, Pax_1 = \emptyset$$

- Then the joint distribution could be written as

$$P(X_1, X_2, X_3, X_4) = \prod_{i=1}^{4} P(X_i \mid Pax_i)$$

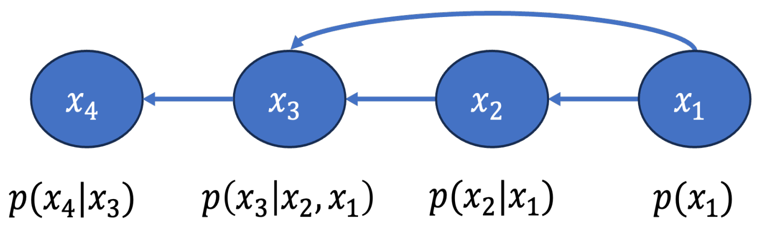
3.2 Graph

Now we could construct the directed graph G = (V, E) with vertices V = X_1, X_2, X_3, X_4 and where E is the set of directed edges from Pax_i to X_i for i = 1, 2, 3, 4

Directed Acyclic Graph (DAG)

Bayesian Networks: visualizing the way we factorize our joint probability distribution.

Lack of an edge indicates conditional independence.



3.3 Bayesian Networks

- Factorization Definition (Chain rule for Bayesian Networks)
  The distribution P factorizes according to the DAG G, if

$$P(X_1, \dots, X_n) = \prod_{i=1}^{n} P(X_i \mid Pax_i^G)$$

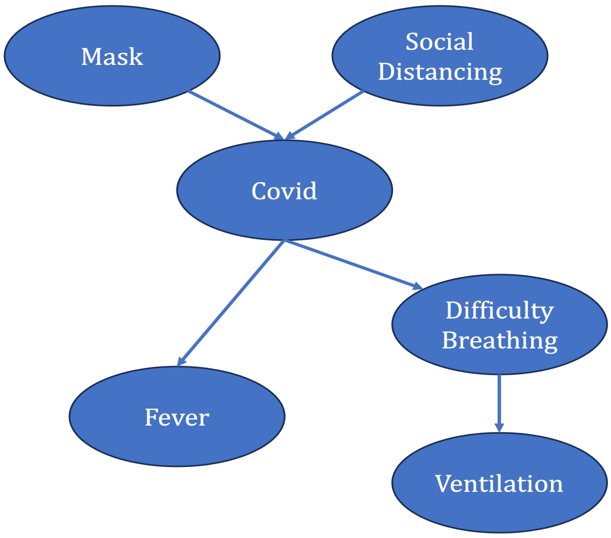
- Bayesian Networks Definition
  Given the random variables V = $X_1, X_2, \dots, X_n$, a Bayesian Network(BN) is a pair $B = (G; P_B)$, where
  - G is directed acyclic graph **(BN structure)** with node set V.
  - P_B is a probability function that factorizes according to G and is specified as a set of conditional probability distributions (CPDs) $P_B(X_i \mid Pax_i)$ for all $X_i \in V$ **(BN parameters)**.

3.4 Covid Example

- Assume following joint probability distribution for Mask, Social distancing, Covid, Fever, Difficulty Breathing, and Ventilation

$$P(M, D, C, B, F, V) = P(F \mid C)P(V \mid B)P(B \mid C)P(C \mid M, D)P(M)P(D)$$

- Corresponding Bayesian Networks



BN structure is a representation of how the joint probability distribution of a set of random variables can be factorized.

3.5 Summary: What is the Goal?

The goal is to find from data, the correct factorization of the joint probability distribution.

- First, finding the conditional independence I(P).
- Use graphical visualization (BN), then determine which graph is the correct factorization.

Chapter 4

From Factorization to Independence: I-map

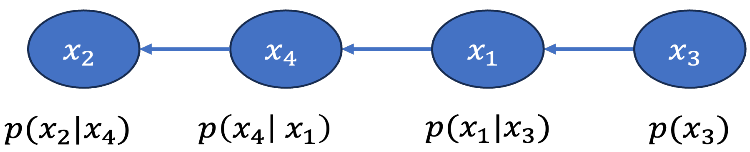
4.1 Local Independence from BN

- The algebraic way of finding independence from factorization is complicated. The easier way is to use Bayesian Networks.

  Assume we have the following factorization

$$P(X_1, X_2, X_3, X_4) = P(X_2 \mid X_4)P(X_4 \mid X1)P(X_1 \mid X_3)P(X_3)$$

  Equivalent BN



Then, $(X_2 \perp X_1, X_3 \mid X_4), (X_4 \perp X_3 \mid X_1) \in I(P)$

Lack of edges indicates conditional independence.

$(X_2 \perp NonDescendents_{X_2} \mid Pax_2), (X_4 \perp NonDescendents_{X_4} \mid Pax_4) \in I(P)$
$NonDescendents_{X_i}$ are all the nodes excluding the descendants of X_i.

These conditional independencies are known as the local independencies of the graph because they are conditioned on X_i's parents.

- Local Independence Definition

  Given the graph G, the set of **local (Markov) independencies**, denoted by $I_l(G)$, consists of

$$(X_i \perp NonDescendents_{X_i} \mid Pax_i^G)\ \forall i$$

- Independence-map (I_map) Definition

G is an I_map for P if $I_l^G \subseteq I(P)$

  So, G being an I_map for P means that P satisfies the local independencies of G.

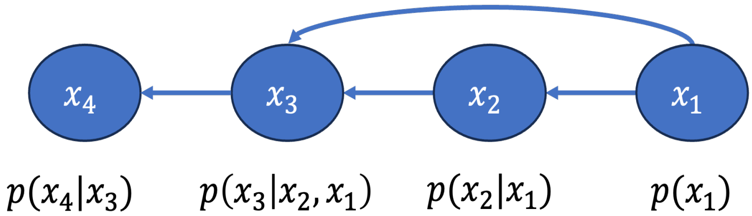
- I_map Theorem
  Let G be a DAG and P be an joint distribution over a set of random variables. P factorizes according to G, if and only if G is a I_map for P.

4.2 $I_l(G)$ *and* $I(P)$: Example

- Consider the joint distribution P over $X_1, X_2, \ldots, X_4$, where

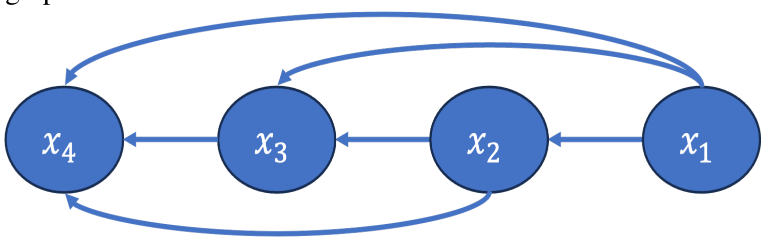
  I(P) = $(X_4 \perp X_2, X_1 \mid X_3)$ and its derivations

  - Factorization 1
    * Does P satisfy the following factorization?
      $$P(X_1, \ldots, X_4) = P(X_4 \mid X_3)P(X_3 \mid X_2, X_1)P(X_2 \mid X_1)P(X_1)$$
    * Equivalent graph G



imposing local independencies:
$I_l(G) = (X_4 \perp X_2, X_1 \perp X_3)$
Therefore, $I_l(G) \subseteq I(P)$, G is an i-map for P.

  - Factorization 2
    * Does P satisfy the following factorization?
      $$P(X_1, \ldots, X_4) = P(X_4 \mid X_3, X_2, X_1)P(X_3 \mid X_2, X_1)P(X_2 \mid X_1)P(X_1)$$
    * Equivalent graph G



imposing local independencies:
$I_l(G) = \emptyset$
Therefore, $I_l(G) \subseteq I(P)$, G is an i-map for P.

  - There could be more than one i_map for P.

4.3 Summary: What is the Goal?

The goal is to find from data, the correct factorization of the joint probability distribution.

- First, finding the conditional independence I(P)
- For each factorization, we could draw a bayesian nets, and we could write down local independencies I_l^G. If $I_l^G \subseteq I(P)$, then G is i_map for P
- There could be several i_map for P, the problem now is which one to choose?

Chapter 5

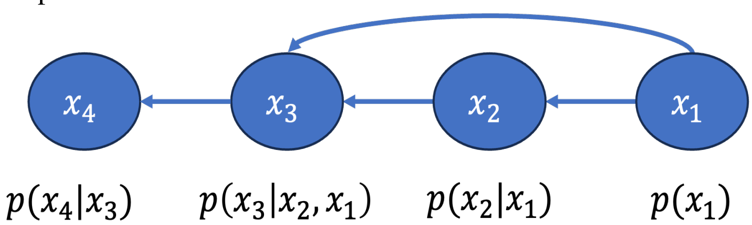
Minimal I-map

5.1 Multiple I-map

- Consider the joint distribution P over $X_1, X_2, \ldots, X_4$, where

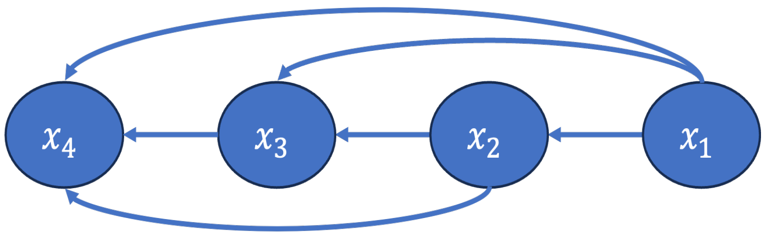
$$\text{I(P)} = (X_4 \perp X_2, X_1 \mid X_3) \text{ and its derivations}$$

  - Graph1



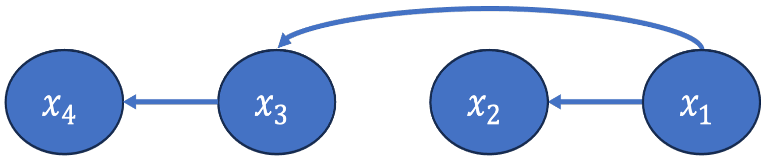
    $I_l(G_1) = (X_4 \perp X_2, X_1 \perp X_3)$
    Therefore, $I_l(G_1) \subseteq I(P)$, G_1 is an i-map for P.

  - Graph2



    $I_l(G_2) = \emptyset$
    Therefore, $I_l(G_2) \subseteq I(P)$, G_2 is an i-map for P.

  - Graph3



    $(X_3 \perp X_2 \mid X_1) \in I_l(G_3)$ $(X_3 \perp X_2 \mid X_1) \notin I(P)$ Therefore, G_3 is not an imap for p.

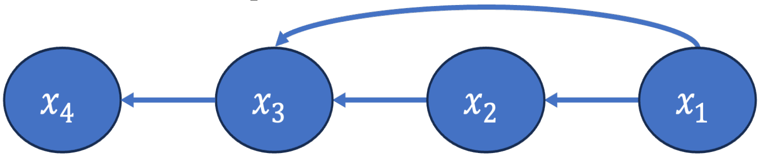
5.2 Minimal I-map

  - Definition

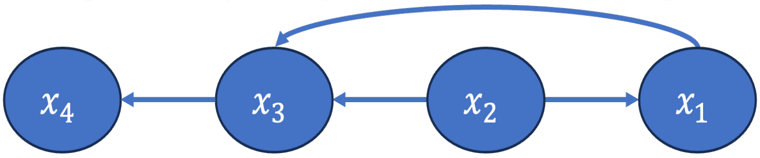
A graph G is a minimal I-map for P if it is an I-map for P, and if the removal of any edge from G makes it not an I-map.

– Minimal I-maps are not Unique

  ∗ G_1 is a minimal I-map



  ∗ By simply reversing one edge, G_2 is also a minimal I-map.



5.3 Summary: What is the Goal?

The goal is to find from data, the correct factorization of the joint probability distribution.

- First, finding the conditional independence I(P)
- For each factorization, we could draw a bayesian nets, and we could write down local independencies I_l^G. If $I_l^G \subseteq I(P)$, then G is i_map for P
- There could be several i_map for P, the problem now is which one to choose? The minimal I-maps.

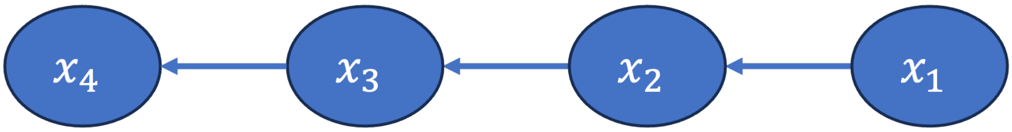
Chapter 6

Global Independence

6.1 Global Conditional Independence

- Consider the joint distribution P over $X_1, X_2, \ldots, X_4$, factorizing as

$$P(X_1, \ldots, X_4) = P(X_4 \mid X_3)P(X_3 \mid X_2)P(X_2 \mid X_1)P(X_1)$$



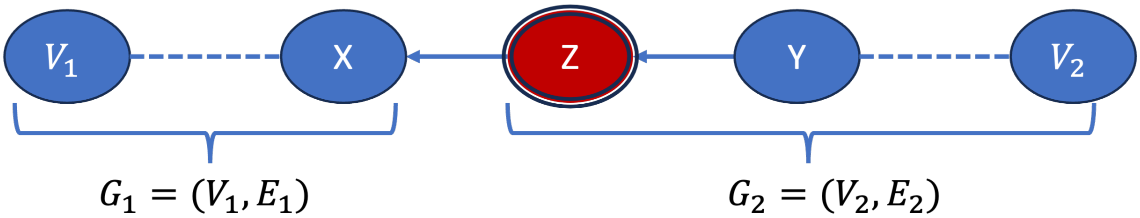
Equivalent graph G, imposing the local indepencies:

$$I_l(G) = (X_4 \perp X_1, X_2 \mid X_3), (X_3 \perp X_1 \mid X_2)$$

- What if we condition X_4 on X_2

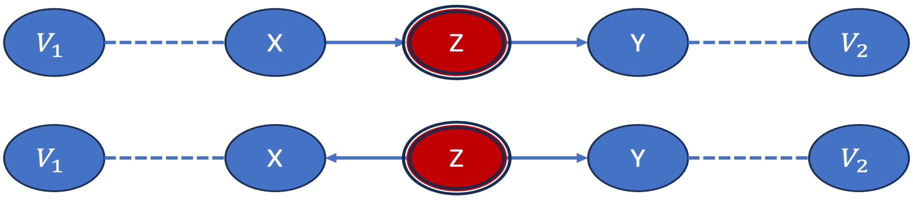
$$\begin{aligned}
P(X_4, X_1 \mid X_2) &= \frac{P(X_4, X_1, X_2)}{P(X_2)} \\
&= \frac{\sum_{X_3} P(X_4, X_3, X_2, X_1)}{P(X_2)} \\
&= \frac{\sum_{X_3} P(X_4 \mid X_3)P(X_3 \mid X_2)P(X_2 \mid X_1)P(X_1)}{P(X_2)} \\
&= \frac{\sum_{X_3} P(X_4 \mid X_3)P(X_3 \mid X_2)P(X_2)P(X_1 \mid X_2)}{P(X_2)} \\
&= \frac{\sum_{X_3} P(X_4, X_3, X_2)P(X_1 \mid X_2)}{P(X_2)} \\
&= \frac{P(X_4, X_2)P(X_1 \mid X_2)}{P(X_2)} \\
&= P(X_4 \mid X_2)P(X_1 \mid X_2)
\end{aligned} \tag{6.1}$$

- This type of conditional independence holds for a general graph including node V_1 and V_2 that are connected by some path, but become disconnected if node Z is removed.



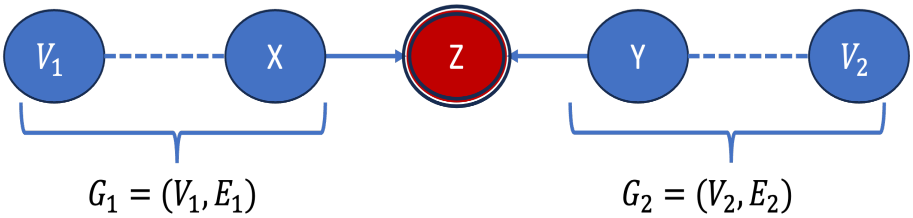
Hence, $V_1 \perp V_2 \mid Z$

It holds true for the following graphs:



The observed node Z blocks information flows between X and Y, and consequently V_1 and V_2

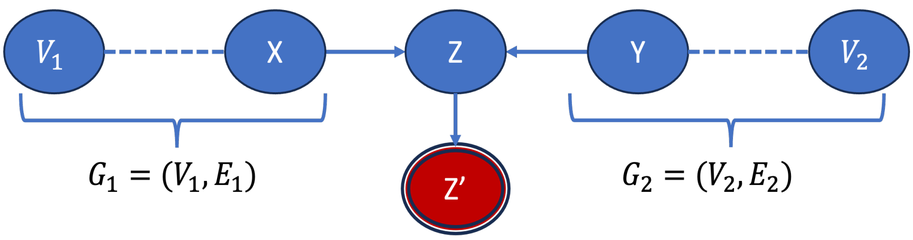
- What about X -> Z <- Y, known as the v-structure



$$
\begin{aligned}
P(V_1, V_2) &= \sum_{v \neq V_1, V_2} \prod_{v} P(v \mid Pa_v) \\
&= \sum_{v \neq V_1, V_2} \prod_{v \in V_1} P(v \mid Pa_v) \prod_{v \in V_2} P(v \mid Pa_v) P(Z \mid X, Y) \\
&= \sum_{v \neq V_1, V_2, Z} \prod_{v \in V_1} P(v \mid Pa_v) \prod_{v \in V_2} P(v \mid Pa_v) \sum_{Z} P(Z \mid X, Y) \\
&= \sum_{v \neq V_1, V_2, Z} P(V_1) P(V_2) \\
&= P(V_1) P(V_2)
\end{aligned}
\tag{6.2}
$$

Hence, $V_1 \perp V_2$

- The conclusion does not hold if Z had an observed descendant



6.2 Trails

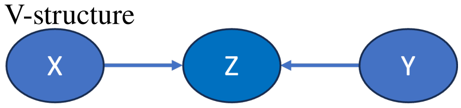
- In all of the cases, there are two nodes, v1 and v2.
  - information flows between them, they can be dependent.

  - information flow if blocked, they aren globally conditionally independent.

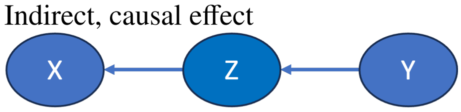
- A sequence of consecutively connected nodes x1 <=> x2 . . . <=> xn in a graph is called a trail.
- A trail is **active** if information flows and **inactive** if the flow is blocked.
- Three types of inactive trails
  - Indirect, causal effect

  - Common cause

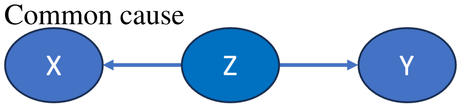
  - V-structure



  - Inactive trail <=> Global conditional independence
- Four types of active trails
  - Indirect, causal effect

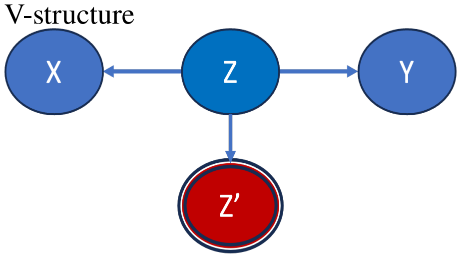


  - Common cause



  - V-structure

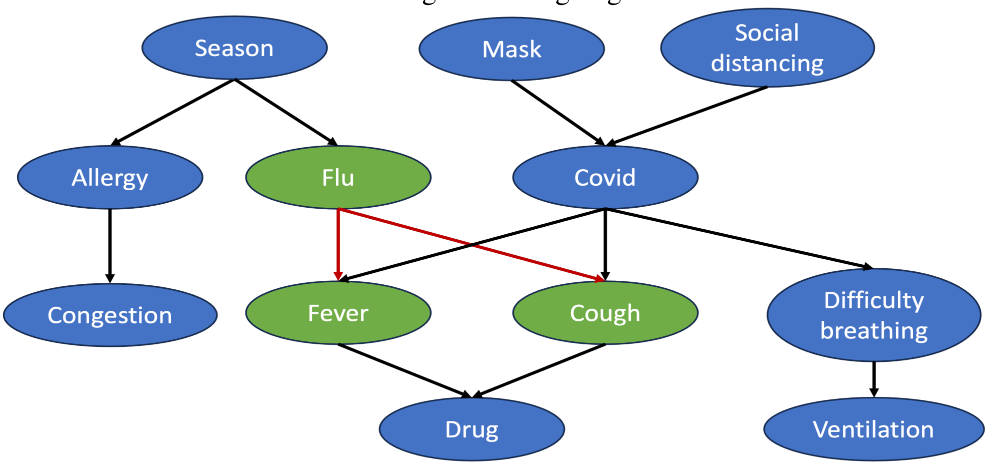
  - V-structure



- Common Cause Example: Consider the trail Fever <- Flu -> Cough in the covid example

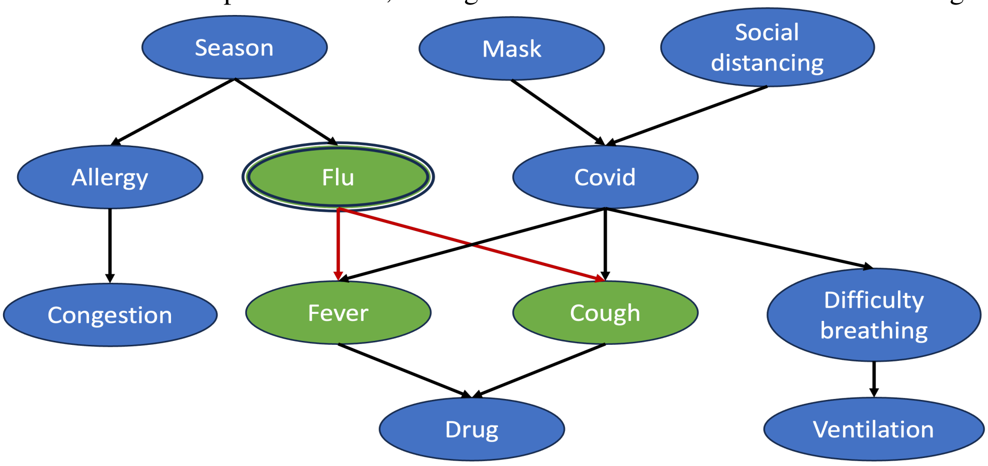
- If $Z = \emptyset$ => active trail

  Fever and cough are dependent: a person with fever gives us information about that person having flu or coughing.



- If Z = Flu => inactive trail

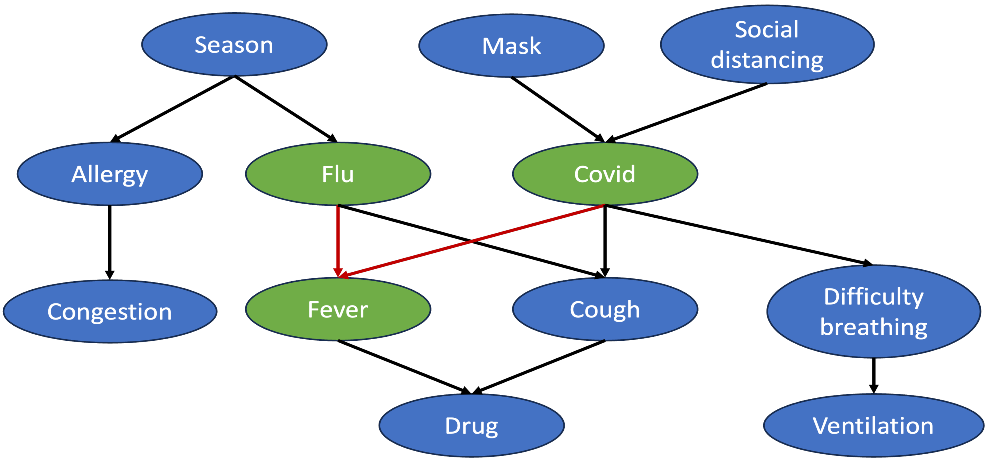
  Once we know the person has flu, fever gives no additional information about cough.



• V-structure Example: Consider the trail Flu -> Fever <- Cough in the covid example

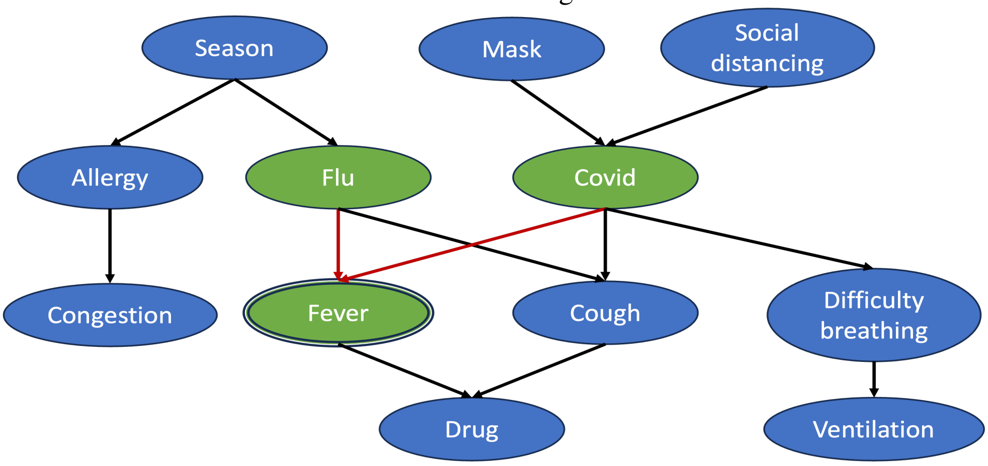
- If $Z = \emptyset$ => inactive trail

  having flu, does not make a person more or less likely to have covid. So, no information flows along the trail.

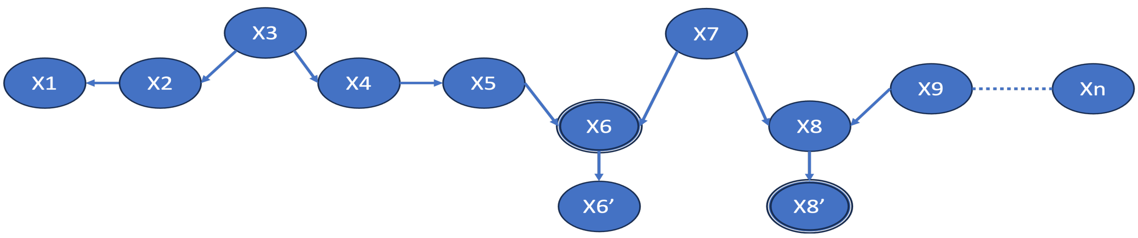


– If Z = Fever => active trail

Flu and covid become dependent. If a person is known to have fever, then she/he is likely infected with covid if not having flu and vice versa.



- Active trails



Put all the nodes on a horizontal line. Move common cause above the line and v-structure below the line. In an active trail, **exactly the nodes at the bottom lines are observed**. Only sinks could and should be observed.

- Definition

  – Active trail

  Let Z be the set of observed variables. Trail x1 <=> . . . <=> xn in G is active given Z if: 1. For any v-structure Xi-1 -> Xi <- Xi+1, either Xi or one of its descendants are in Z. 2. No other node along the trail is in Z.

  - In order to have global conditional independence, all trails must be inactive (no information flow).
  - Inactivity implies independence: Given observed Z, if there is no active trail between the nodes X and Y in graph G, then X and Y are independent given Z under any probability distribution that factorize according to G.

6.3 d-separation

- D(dependence)-separation definition

  In a graph G, two sets of nodes X and Y are d-separated given the set Z, denoted

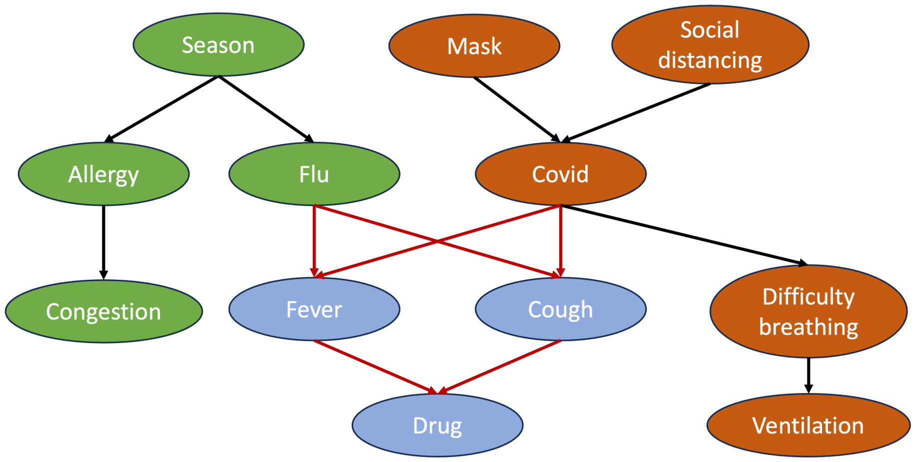
$$d - sepG(X; Y \mid Z)$$

  , if there is no active trail between any node $x \in X$ and any node $y \in Y$ given Z.

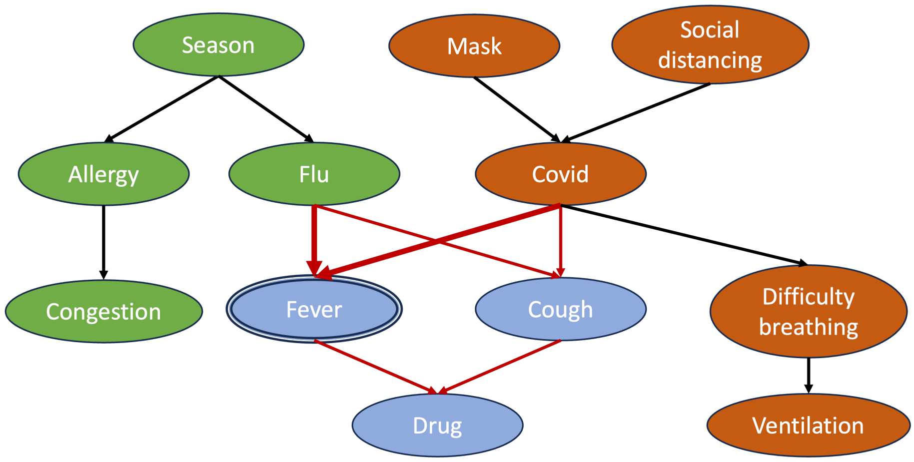
- Covid example

  Consider the sets X = Season, Allergy, Flu, Congestion and Y = Mask, Social distancing, COVID, Diffulty breathing, Ventilation.

  - If $Z = \emptyset$, all trails are inactive



  - If $Z = \{fever\}$, there exists an active trail.



- Definition: Global Markov Independence

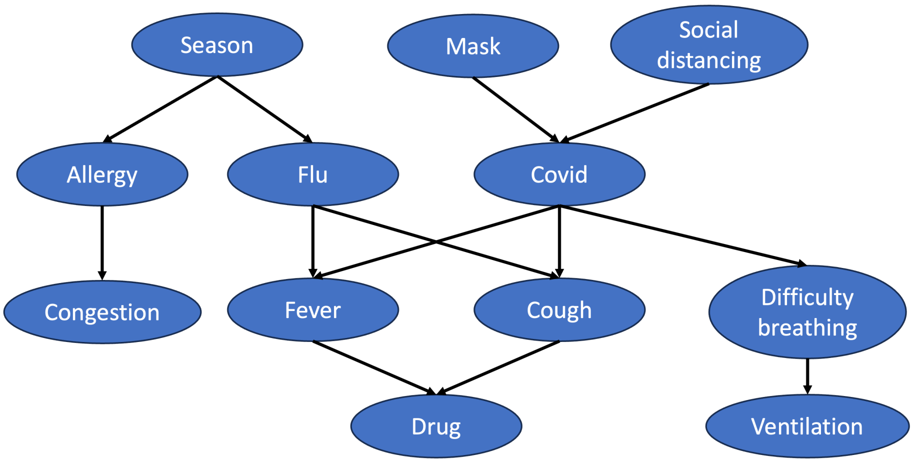
  The set of independencies resulting from d-separation is defined as the set of global markov independencies.

$$I(G) = (X \perp Y \mid Z) : d - sep_G(X; Y \perp Z)$$

- Theorem

If a distribution P factorizes according to G, then $I(G) \subseteq I(P)$.
$I_l(G) \subseteq I(P) => I(G) \subseteq I(P)$
i.e., if the local independencies are satisfied by P, so are the global.

- Example



  - Assume that the true joint probability distribution P of random variables satisfies the previously stated indepencies.
  - Is the graph an i-map for P? Does P factorize according to the graph? yes
  - Does the factorization impose any independence in addition to the previously stated one? yes, but P satisfied them all.

6.4 Summary: What is the Goal?

The goal is to find from data, the correct factorization of the joint probability distribution.

- First, finding the conditional independence I(P)
- For each factorization, we could draw a bayesian nets, and we could write down local independencies I_l^G. If $I_l^G \subseteq I(P)$, then G is I_map for P
- There could be several i_map for P, which one to choose? The minimal I-maps
- Does P satisfy all global independencies imposed by the minimal I-maps? yes

Chapter 7

P-map

7.1 $I(G) \stackrel{?}{=} I(P)$

- Recall from last chapter: If a distribution P factorizes according to G, then $I(G) \subseteq I(P)$. It means that all independencies implied by G are also included in P.
- The question now is can P have an independence not included in G? The answer is Yes.
- Can we conclude I(P) = I(G)? No, but almost yes.
- Theorem of Completeness

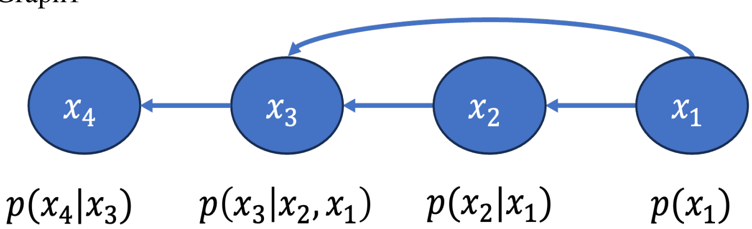
  For almost all distributions P that factorize over G, except for a set of measure zero in the space of CPD parameterizations, I(G) = I(P)

7.2 P-map

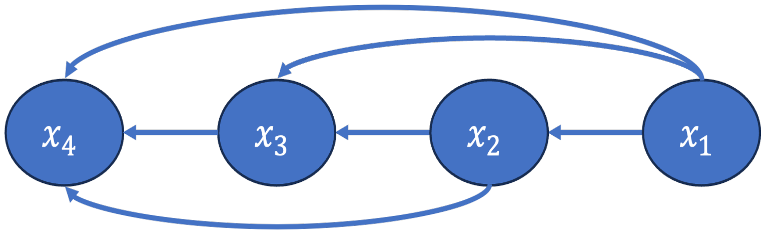
- Definition: Graph G is a perfect map (P-map) for P if I(P) = I(G).
- Consider the joint distribution P over $X_1, X_2, \ldots, X_4$, where

  I(P) = $(X_4 \perp X_2, X_1 \mid X_3)$ and its derivations

  - Graph1

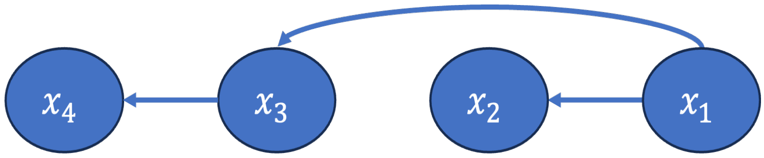
    

    $I_l(G_1) = (X_4 \perp X_2, X_1 \perp X_3)$
    Therefore, $I(G_1) = I(P)$, G_1 is an p-map for P.
  - Graph2



$I_l(G_2) = \emptyset$
Therefore, $I(G_2) \subseteq I(P)$, G_2 is not a p-map for P.

– Graph3



$(X_3 \perp X_2 \mid X_1) \in I_l(G_3)$ $(X_3 \perp X_2 \mid X_1) \notin I(P)$ Therefore, G_3 is not an i-map for p and not a p-map for p.

7.3 Summary: What is the Goal?

The goal is to find from data, the correct factorization of the joint probability distribution.

- First, finding the conditional independence I(P)
- For each factorization, we could draw a bayesian nets, and we could write down local independencies I_l^G. If $I_l^G \subseteq I(P)$, then G is I_map for P
- There could be several i_map for P, which one to choose? The minimal I-maps
- Does P satisfy all global independencies imposed by the minimal I-maps? yes
- Do the minimal I-maps satisfy all independencies imposed by P? Are the minimal I-maps a P-map? Depends on I(P)

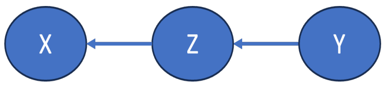
Chapter 8

Independence Equivalence

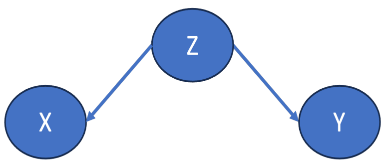
From previous chapters, we understand that there could be multiple minimal I-maps, is there a class of minimal I-maps?

8.1 I-equivalence

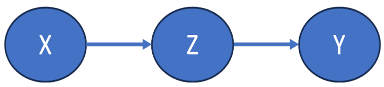
- Using the Bayes rule, 3 types of strutures have equal distributions. For all three graphs, $I(G) = X \perp Y \mid Z$
  - $P(X,Y,Z) = P(X)P(Z \perp X)P(Y \perp Z)$



  - $P(X,Y,Z) = P(Z)P(X \perp Z)P(Y \perp Z)$

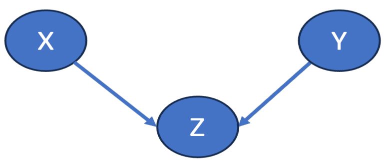


  - $P(X,Y,Z) = P(Y)P(Z \perp Y)P(X \perp Z)$



- V-Structure

  Suppose we have I(G) = $X \perp Y$, $P(X,Y,Z) = P(X)P(Y)P(Z \perp X,Y)$, it cannot be converted to another structure using the bayes rule.



- Definition: Independence-equivalence

  Two graphs G1 and G2 defined over the same variables are I-equivalent if I(G1) = I(G2). If two graphs have the same skeleton and set of v-structures, then they are i-equivalent.

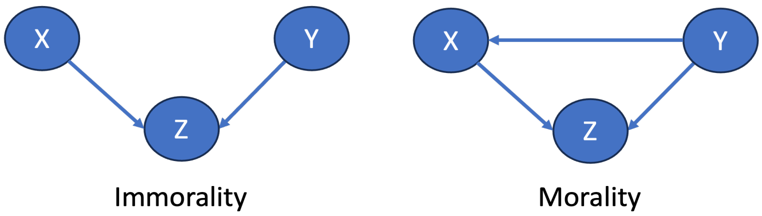
- General trail

  In general, the arrows in a trail can be reversed as long as a new v-structure is not produced.

- Immorality

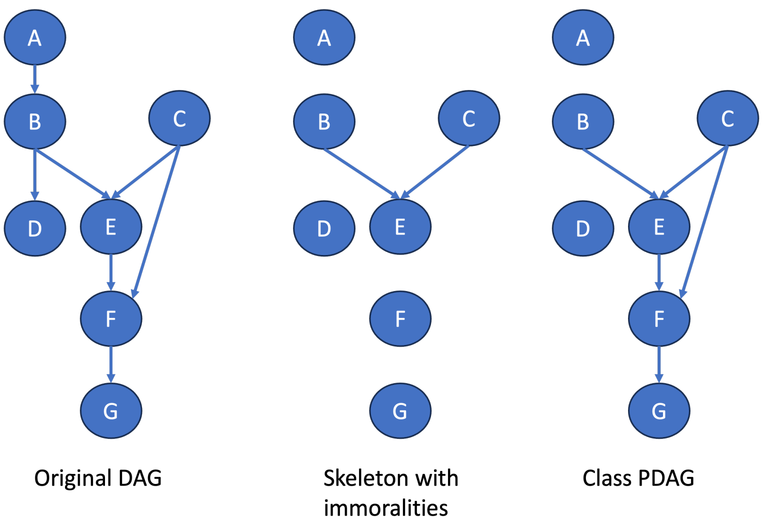
  A v-structure X -> Z <- Y is an immorality if X and Y are not linked. If there is a link, it is called a covering edge.

  The graphs G1 and G2 are i-equivalent if and only if they have the same skeleton and the same set of immoralities.



8.2 I-equivalence class: PDAG

- Class PDAG of a DAG g is a PDAG that
  - has the same skeleton as G
  - includes a directed edge only if all of the i-equivalent graphs to G also have that directed edge.
- How to obtain the class PDAG of a DAG G
  - obtain the skeleton of G.
  - find the immoralities of G and draw them in the skeleton.
  - find the orientations of the other edges by obeying the folloing rules: 1. do not create extra immorality. 2. do not create a directed cycle.
- Example



8.3 Summary: What is the Goal?

The goal is to find from data, the correct factorization of the joint probability distribution.

- First, finding the conditional independence I(P)

- For each factorization, we could draw a bayesian nets, and we could write down local independencies I_l^G. If $I_l^G \subseteq I(P)$, then G is I_map for P
- There could be several i_map for P, which one to choose? The minimal I-maps
- Does P satisfy all global independencies imposed by the minimal I-maps? yes
- Do the minimal I-maps satisfy all independencies imposed by P? Are the minimal I-maps a P-map? Depends on I(P)
- Is there a class of minimal I-maps? Yes

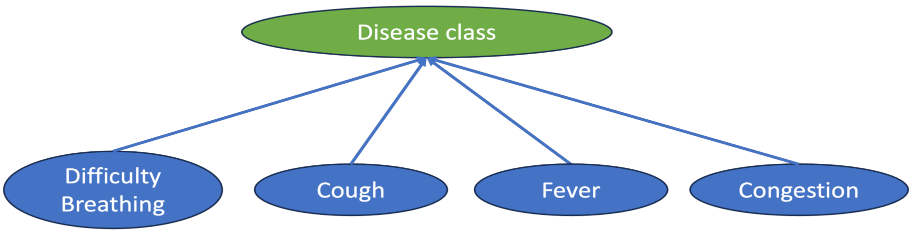
Chapter 9

Naive Bayes Model

In the Covid example, suppose we want to detect the disease type based on the symptoms. What is the simpliest BN that can do this?

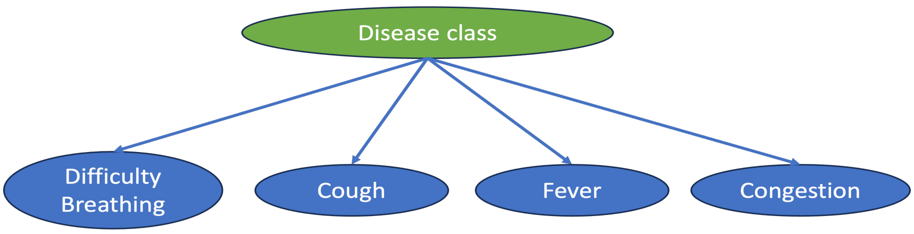
Assume Disease class Y = Allergy, Flu, COVID

- Option 1

  This BN will lead to 32 parameters, P(Disease|B,C,F,G).

- Option 2

  This BN will lead to 15 parameters.

  – Local dependencies: $I(G) = \{(G \perp F, O, B \mid Y), (F \perp O, G, B \mid Y),$ $(O \perp F, G, B \mid Y), (B \perp F, O, G \mid Y)\}$
  – So the symptoms are mutually independent given the disease class.
  – By specifying the CPDs, we have the probability of each symptoms given the disease.
  – Total number of params is 15: 2 for Y and 3 for each sympton.
  – P(Y, X1, . . . , Xn) are factorized as $P(Y) \prod_{i=1}^{n} P(X_i \mid Y)$
  – With joint probability distribution, we could calculate the probability of each disease given the symptoms.