Tight Guarantees for Multi-unit Prophet Inequalities and Online Stochastic Knapsack

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Abstract: Prophet inequalities are a useful tool for designing online allocation procedures and comparing their performance to the optimal offline allocation. In the basic setting of k-unit prophet inequalities, the well-known procedure of Alaei (2011) with its celebrated performance guarantee of $1-\frac{1}{\sqrt{k+3}}$ has found widespread adoption in mechanism design and general online allocation problems in online advertising, healthcare scheduling, and revenue management. Despite being commonly used to derive approximately-optimal algorithms for multi-resource allocation problems that suffer from the curse of dimensionality, the tightness of Alaei's procedure for a given k has remained unknown. In this paper we resolve this question, characterizing the optimal procedure and tight bound, and consequently improving the best-known guarantee for k-unit prophet inequalities for all $k>1$.

We also consider the more general online stochastic knapsack problem where each individual allocation can consume an arbitrary fraction of the initial capacity. Here we introduce a new "best-fit" procedure with a performance guarantee of $\frac{1}{3+e^{-2}}\approx 0.319$, which we also show is tight with respect to the standard LP relaxation. This improves the previously best-known guarantee of 0.2 for online knapsack. Our analysis differs from existing ones by eschewing the need to split items into "large" or "small" based on capacity consumption, using instead an invariant for the overall utilization on different sample paths. Finally, we refine our technique for the unit-density special case of knapsack, and improve the guarantee from 0.321 to 0.3557 in the multi-resource appointment scheduling application of Stein et al. (2020).

All in all, our results imply *tight* Online Contention Resolution Schemes for k-uniform matroids and the knapsack polytope, respectively, which has further implications.

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1. Introduction

Online resource allocation problems arise in many domains, such as online advertising, healthcare operations, and revenue management. In these problems the decision-maker faces a sequence of stochastically-generated queries, and must irrevocably assign each query to be served by a resource as it arrives online. The resources have limited capacities, and the objective is to maximize the cumulative reward collected from serving queries over a finite time horizon. Due to the curse of dimensionality caused by having multiple resources, this problem is deemed generally intractable, and it is of central interest in many academic communities to derive approximately-optimal algorithms.

A keystone ingredient for algorithm design throughout these communities has been to start with a *Linear Programming (LP) relaxation*, which is also known as the ex-ante, fluid, or deterministic relaxations. The LP relaxation decomposes the multi-resource allocation problem into single-resource subproblems, which allows for the capacity of each resource to be controlled separately. Deriving an approximation guarantee relative to the LP for each individual resource, which can be called a *prophet inequality*, then implies the same approximation guarantee in the original multi-resource problem (see e.g. Alaei et al. (2012); Alaei (2014); Gallego et al. (2015); Wang et al. (2018); Feng et al. (2020); we also formalize this reduction in Section 3.1). Once reduced to single-resource subproblems, the guarantees that are possible depend on whether a resource's capacity: 1) takes the form of k discrete units that are consumed one at a time; or 2) an arbitrary amount of its capacity may be needed to serve a single query.

In this paper we derive the *best-possible* guarantees for the single-resource subproblem in both cases 1) and 2) above, resolving long lines of investigation. Specifically, the k-*unit* setting described in 1) originated from Hajiaghayi et al. (2007), and a celebrated guarantee of $1-1/\sqrt{k+3}$ was derived by Alaei (2011) through a "Magician's problem". These guarantees have been recently improved in Wang et al. (2018); Chawla et al. (2020), and in this paper we fully resolve (see Table 1 and Figure 1) the question of k-unit prophet inequalities relative to the LP. Meanwhile, for 2) above which we will call the *knapsack* setting, we derive the tight guarantee of $\frac{1}{3+e^{-2}} \approx 0.319$ which improves earlier results of Dutting et al. (2020) and Alaei et al. (2013). We emphasize that our results can be *directly applied* to improve the guarantees in *all of the papers cited thus far*, among others (which we discuss in Section 2). The tightness of our results also characterizes the *limits* of any approach that seeks approximately-optimal algorithms through the LP relaxation.

Table 1 Our tight ratios for multi-unit prophet inequalities. The previous lower bounds are obtained as the maximum of the ratios in Alaei (2011); Chawla et al. (2020). The previously best-known upper bounds were 1/2 for $k=1$ and $1-e^{-k}k^k/k!$ for $k>1$, with the latter inherited from the "correlation gap" in the i.i.d. special case (see Yan, 2011).

value of k	1	2	3	4	5	6	7	8
Existing lower bounds	0.5000	0.5859	0.6309	0.6605	0.6821	0.6989	0.7125	0.7240
Our tight ratios	0.5000	0.6148	0.6741	0.7120	0.7389	0.7593	0.7754	0.7887
Existing upper bounds	0.5000	0.7293	0.7760	0.8046	0.8245	0.8394	0.8510	0.8604

In order to obtain the tight guarantees, it is *sufficient* to consider a reduced problem, called *Online Contention Resolution Scheme (OCRS)* that generalizes the Magician's problem in the k-unit setting and can capture the knapsack setting as well. Then, our k-unit result is derived through a new LP that re-interprets the OCRS problem, whose dual has a structured optimal solution, which allows us to solve the adversary's outer problem over instances to minimize the algorithm's guarantee. Meanwhile, our knapsack result is derived through a new "best-fit" algorithm for the OCRS problem, motivated by the bin packing problem, which allows us to establish the tight

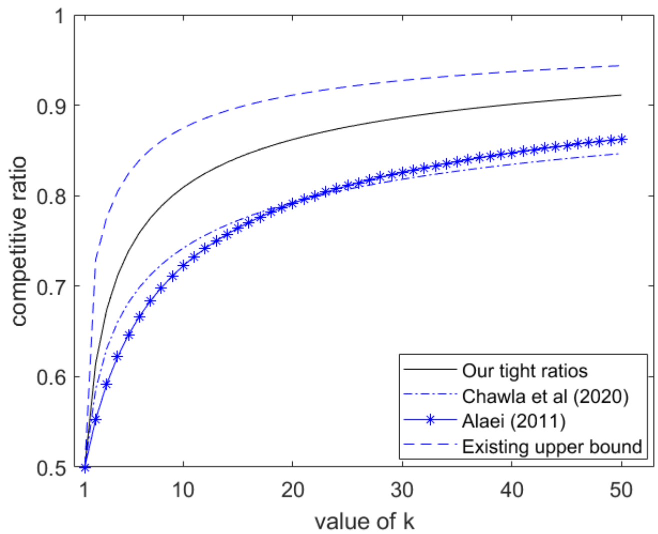

Figure 1 Our tight ratios in comparison to the previously best-known guarantees from Alaei (2011); Chawla et al. (2020) and the previously best-known upper bounds (see Yan, 2011).

guarantee of $\frac{1}{3+e^{-2}} \approx 0.319$ using an invariant. As a practical takeaway, our work suggests a stark contrast between how resources should be allocated in cases 1) and 2) above—in 1), the decision-maker wants to have a small amount of each resource remaining; whereas in 2), the decision-maker wants resources to either be fully utilized or have large amounts remaining. Our theoretical techniques are also general and can be adapted to other settings—in fact, we show that in the *unit-density* knapsack setting, our invariant method leads to an improved guarantee of 0.355, and hence our best-fit algorithm beats the state-of-the-art 0.321-approximate algorithm of Stein et al. (2020) in their healthcare scheduling problem. We now explain the OCRS problem and elaborate on our new techniques to solve the single-resource subproblems in cases 1) and 2).

1.1. New Techniques for k-unit Prophet Inequalities

We formalize what we mean by deriving an approximation guarantee relative to the LP for a single resource, which enables our approximation algorithms for the multi-resource problem. In fact, it suffices to solve a simpler problem where each query for the single resource has a binary state, which can be called the k-unit Online Contention Resolution Scheme (OCRS) problem. The formal reduction to this problem is presented in Section 3.2, and we now describe how we solve this problem optimally.

Definition 1 (k-unit OCRS Problem) *There is a sequence of queries $t = 1, \ldots, T$, each of which is active independently according to a known probability p_t. Whether a query is active is sequentially observed, and active queries can be immediately served or rejected, while inactive queries* *must be rejected. At most k queries can be served in total, and it is promised that $\sum_t p_t \le k$. The goal of an online algorithm is to serve every query t with probability at least γ conditional on it being active, for a constant $\gamma \in [0,1]$ as large as possible, potentially with the aid of randomization.*

It is easy to see[1] that despite $\boldsymbol{p}$ being fractionally feasible, a guarantee of $\gamma = 1$ in Definition 1 is generally impossible. The work of Alaei (2011) implies a solution to Definition 1 with $\gamma = 1 - \frac{1}{\sqrt{k+3}}$. Presented in the slightly different context of a "γ-Conservative Magician," Alaei's procedure has the further appealing property that it does not need to know vector $\boldsymbol{p}$ in advance, as long as each p_t is revealed when query t is observed, and it is promised that $\sum_t p_t \le k$. However, it has remained unknown whether Alaei's $\boldsymbol{p}$-agnostic procedure or its analyzed bound of $\gamma = 1 - \frac{1}{\sqrt{k+3}}$ is tight for an arbitrary positive integer k. In this paper we resolve this question, in the following steps.

1. Under the assumption that $\boldsymbol{p}$ is known, we formulate the optimal k-unit OCRS problem using a new LP. This LP tracks the probability distribution of the capacity utilization, which must lie in $\{0, \frac{1}{k}, \ldots, 1\}$, over time $t = 1, \ldots, T$. The decision variables correspond to subdividing and selecting sample paths at each time t, with total measure exactly γ, on which the algorithm will serve query t whenever it is active. This selection is constrained to sample paths with at least $\frac{1}{k}$ capacity remaining, which is enforced in the LP through tracking the capacity utilization. Finally, γ is also a decision variable, with the objective being to maximize γ.
2. For an arbitrary $\boldsymbol{p}$, we characterize an optimal solution to this LP based on the structure of its dual. The optimal selection prioritizes sample paths with the *least* capacity utilized, at every time t, *irrespective* of the values $p_{t+1}, p_{t+2}, \ldots$ in the future. Such a solution corresponds to the γ-Conservative Magician from Alaei (2011), except that γ, instead of being fixed to $1 - \frac{1}{\sqrt{k+3}}$, is set to an optimal value that depends on the vector $\boldsymbol{p}$.
3. We derive a closed-form expression for this optimal value of γ as a function of $\boldsymbol{p}$. We show that γ is minimized when $p_t = k/T$ for each t and $T \to \infty$, corresponding to a Poisson distribution of rate k. We characterize this infimum value of γ using an ODE and provide an efficient procedure for computing it numerically.

For any k, let γ_k^* denote the infimum value of γ described in Step 3 above. The conclusion is that setting $\gamma = \gamma_k^*$ is a feasible solution to Definition 1, with $\gamma_k^* > 1 - \frac{1}{\sqrt{k+3}}$ for all $k > 1$, achieved using the $\boldsymbol{p}$-agnostic procedure described above. Moreover, the guarantee of γ_k^* is the best-possible, since even a procedure that knows $\boldsymbol{p}$ in advance cannot do better than a γ-Conservative Magician with an optimized value of γ, which in the Poisson worst case can be as low as γ_k^*.

[1] For example, suppose that $k = 1$, $T = 2$, and $p_1 = p_2 = 1/2$. If we attempt to set $\gamma = 1$, then the first query would be served ex ante w.p. $1/2$, i.e., whenever it is active. This means that half the time no capacity would remain for query 2, i.e., half the time query 2 is active it does not get served. For this example, the optimal value of γ can be calculated to be $2/3$.

Comparison to Alaei et al. (2012). k-unit prophet inequalities have been analyzed using LP's before in Alaei et al. (2012), who formulate a primal LP encoding the adversary's problem of minimizing an online algorithm's optimal dynamic programming value. They then use an auxiliary "Magician's problem," analyzed through a "sand/barrier" process, to construct a feasible dual solution with $\gamma = 1 - \frac{1}{\sqrt{k+3}}$. By contrast, we directly formulate the k-unit OCRS problem using an LP under the assumption that the vector $\boldsymbol{p}$ is known. Our LP dual along with complementary slackness allows us to establish the structure of the optimal k-unit OCRS, showing that it indeed corresponds to a γ-Conservative Magician. However, in our case γ is set to a value dependent on $\boldsymbol{p}$, which we show is always at least γ_k^*, and strictly greater than $1 - \frac{1}{\sqrt{k+3}}$ for all $k > 1$.

Comparison to Wang et al. (2018). The values of γ_k^* we derive have previously appeared in Wang et al. (2018) through the stochastic analysis of a "reflecting" Poisson process. Our work differs by establishing *optimality* for these values γ_k^*, as the solutions to a sequence of optimization problems from our framework. Moreover, their paper assumes Poisson arrivals to begin with, while we allow arbitrary probability vectors $\boldsymbol{p}$ and show the limiting Poisson case to be the worst case.

The classical prophet inequality comparison. We should note that classically in the k-unit prophet inequality problem, the goal is to compute the worst-case performance of an online algorithm, who sequentially observes independent draws from known distributions and can keep k of them, and compare instead to a *prophet*, whose performance is the expected sum of the k highest realizations. The prophet's performance is upper-bounded by the LP relaxation, so our guarantees that are tight relative to the LP also imply the *best-known prophet inequalities to date* for all $k > 1$. We do give an example that demonstrates this guarantee to be "almost" tight even when comparing to the weaker prophet benchmark. Through our LP's and complementary slackness, we can convert the Poisson worst case for the k-unit OCRS problem into an explicit instance of k-unit prophet inequalities, on which the reward of any online algorithm relative to the LP relaxation is upper-bounded by γ_k^*. Moreover, by modifying such an instance, we also provide a new upper bound of 0.6269 relative to the prophet, when $k = 2$ (Proposition 1). Since $\gamma_2^* \approx 0.6148$, this shows that not much improvement beyond γ_k^* is possible relative to the prophet when $k = 2$.

1.2. New Techniques for the Knapsack Setting

Under the knapsack setting, we formalize the problem of comparing an online algorithm to the LP relaxation for a single resource, which enables our approximation algorithm for the general multi-resource problem. Following the reduction described in Section 3.2, it is enough to consider a simpler problem where the size of each query t is realized as d independently with a given probability $p_t(d)$. We call this problem the knapsack OCRS problem.

Definition 2 (Knapsack OCRS Problem) *There is a sequence of queries $t=1,\ldots,T$, and for each query t, the size is realized as $d\in[0,1]$ independently with a known probability $p_t(d)$. After the query's size is observed, the query must be immediately served or rejected. The total size of queries served cannot exceed 1, and it is promised that $\sum_t\sum_d p_t(d)\cdot d\le 1$. The goal of an online algorithm is to serve every query t with probability at least γ conditional on the size being realized to d, for each $d\in[0,1]$, and for a constant $\gamma\in[0,1]$ as large as possible.*

Similar to our approach for the multi-unit setting, we design solution for the knapsack OCRS by tracking the distribution of capacity utilization over time. For each size realization $d\in[0,1]$, we select for each query t a γ-measure of sample paths on which it should be served whenever the size is realized as d, under the constraint that these paths have a current utilization of at most $1-d$. However, different from the multi-unit setting, in the knapsack setting, we need to always maintain a γ-measure of sample paths on which utilization is 0, in case an item T with size realization $d_T=1$ and $p_T(d_T)=\varepsilon$ arrives at the end. Accordingly, in stark contrast to the γ-Conservative Magician, our knapsack procedure selects for each query and each size realization the sample paths with the *most* capacity utilized, on which that query still fits. We dub this procedure a "Best-fit Magician."[2] In the more general knapsack setting, capacity utilization can only be tracked in polynomial time after discretizing size realizations by $1/K$ for some large integer K; nonetheless, we will show (in Section 6.1) that this loses a negligible additive term of $O(1/K)$ in the guarantee.

To derive the maximum feasible guarantee γ for a Best-fit Magician, we note that the expected capacity utilization over the sample paths is $\gamma\cdot\sum_t\sum_d p_t(d)\cdot d$, which is always upper-bounded by γ, since $\sum_t\sum_d p_t(d)\cdot d\le 1$. Therefore, to lower-bound the measure of sample paths with 0 utilization, it suffices to upper-bound the measure of sample paths whose utilization is small but non-zero. To do so, we use the rule of the Best-fit Magician, namely, that an arriving query with a size realization d will only be served on a previously empty sample path if there is less than a γ-measure of sample paths with utilization in $(0,1-d]$. Based on this fact, we derive an *invariant* that holds after each query t, and upper-bounds the measure of sample paths with utilization in $(0,b]$ by a decreasing exponential function of the measure with utilization in $(b,1-b]$, for any small size $b\in(0,1/2]$. This allows us to show that a γ as large as $\frac{1}{3+e^{-2}}\approx 0.319$ allows for a γ-measure of sample paths to have 0 utilization at all times, and hence is feasible. The Best-fit Magician is also agnostic to knowing the probabilities $\{p_t(d)\}_{\forall t,\forall d}$ in advance, as long as it is promised that $\sum_t\sum_d p_t(d)\cdot d\le 1$. Nonetheless, we construct a counterexample showing it to be *optimal*, in that $\gamma=\frac{1}{3+e^{-2}}$ is an upper bound on the guarantee for the knapsack OCRS problem even if the probabilities $\{p_t(d)\}_{\forall t,\forall d}$ are known in advance.

[2] This is because it resembles the "best-fit" heuristic for bin packing (Garey et al., 1972).

To our knowledge, our analysis differs from all existing ones for knapsack in an online setting (Dutting et al., 2020; Stein et al., 2020; Feldman et al., 2021) by eschewing the need to split queries into "large" vs. "small" based on their size (usually, whether their size is greater than $1/2$). In fact, we show that any algorithm that *packs large and small queries separately* is limited to $\gamma \leq 0.25$ in our problem (Proposition 2), whereas our tight guarantee is $\gamma = \frac{1}{3+e^{-2}} \approx 0.319$.

Our result can be further improved in a case of *unit-density* online knapsack, where the random size and reward of a query are always identical. Indeed, since it is no longer possible for a small query to have a high reward, we no longer need to guarantee a uniform lower bound γ on the probability of serving any query with any size realization. Instead, we show that our invariant still holds for a decreasing sequence of service probabilities $\gamma_1 \geq \cdots \geq \gamma_T$, and devise a particular sequence that guarantees an expected reward that is at least 0.3557 times the optimal LP value. This then implies a 0.3557-approximation for the multi-resource appointment scheduling problem of Stein et al. (2020), improving upon their 0.321-approximation.

Comparison to Alaei et al. (2013). Another related setting is the online stochastic generalized assignment problem of Alaei et al. (2013), for which the authors establish a guarantee of $1 - \frac{1}{\sqrt{k}}$, when each query can realize a random size that is at most $1/k$. They eliminate the possibility of "large" queries by imposing k to be at least 2, showing that a constant-factor guarantee is impossible when $k = 1$. Although our problem can be generalized to random sizes, we need to assume that size is observed *before* the algorithm makes a decision, whereas in their problem size is randomly realized *after* the algorithm decides to serve a query. This distinction allows our problem to have a constant-factor guarantee that holds even when queries can have size 1. Moreover, our procedure starkly contrasts with theirs in that we prioritize selecting sample paths with the *most* capacity utilized on which a query fits, while they prioritize sample paths with the *least* capacity utilized.

1.3. Roadmap

Section 3 formalizes the model/notation for our general problem, illustrates the LP reduction approach, and establishes the equivalence between the single-resource problem and the OCRS problem. Section 4 presents all of our results for k-unit prophet inequalities, in which the sizes are always $1/k$. Section 5 considers the single-resource problem under the knapsack setting. Section 6 discusses the polynomial-time implementation of our algorithm under the knapsack setting, and presents our improvement in the unit-density special case.

2. Further Related Work

Online knapsack. The discrete version of the online stochastic knapsack problem as studied in this paper, which generalizes the k-unit prophet inequality problem, was originally posed in

Papastavrou et al. (1996) to model classical applications in operations research such as freight transportation and scheduling. We derive the best competitive ratios for this problem known to date, for both the general and unit-density settings, which extend to multiple knapsacks. We should point out though that in the unit-density setting with a *single* knapsack, a competitive ratio of $1/2$, better than our guarantee of 0.3557, is possible under any fixed sequence of adversarial arrivals (Han et al., 2015). However, such a guarantee fails[3] to extend to multiple knapsacks, whereas our guarantee of 0.3557, which holds relative to the LP, directly extends there, following the same reduction argument as in Stein et al. (2020).

Prophet inequalities. Prophet inequalities were originally posed in the statistics literature by Krengel and Sucheston (1978). Due to their implications for posted pricing and mechanism design, prophet inequalities have been a surging topic in algorithmic game theory since the seminal works of Hajiaghayi et al. (2007); Chawla et al. (2010); Yan (2011); Alaei (2011). Of particular interest in these works are bounds for k-item prophet inequalities, and in this paper we improve such bounds for all $k > 1$, and show that our bounds are tight relative to the LP relaxation, under an adversarial arrival order. More recently, prophet inequalities have also been studied under random order (Esfandiari et al., 2017; Correa et al., 2021; Arnosti and Ma, 2021), free order (Correa et al., 2021; Beyhaghi et al., 2021), or IID arrivals (Hill and Kertz, 1982; Correa et al., 2017), with k-unit prophet inequalities in particular being studied by Beyhaghi et al. (2021) under free order and Arnosti and Ma (2021) under random order. Prophet inequalities have also been studied under the batched setting (Alaei et al., 2022) with applications to descending-price auctions, and also been used as algorithmic subroutine for other revenue management problems (e.g. Cominetti et al. (2010); Alaei et al. (2021)). A survey of recent results in prophet inequalities can be found in Correa et al. (2019).

Online Contention Resolution Schemes (OCRS). A guarantee of γ for our problem in Definition 1 (resp. Definition 2) is identical to a γ-*selectable OCRS* for the k-uniform matroid (resp. knapsack polytope) as introduced in Feldman et al. (2021). However, we should clarify some assumptions about what is known beforehand and the choice of arrival order. Our OCRS hold against an *online adversary*, who can adaptively choose the next query to arrive, but does not know the realizations of queries yet to arrive. We show that the guarantee does not improve against the *weakest adversary*, who has to reveal the arrival order in advance. However, our OCRS do not satisfy the *greedy* property, and consequently do not hold against the *almighty adversary*, who knows the realizations of all queries before having to choose the order.

[3] An additional factor of $1/2$ would be lost, resulting in a guarantee of only $1/4$; see Ma et al. (2019). In fact, a guarantee of $1/2$ relative to the LP is impossible, due to the upper bound of 0.432 presented in our Proposition 3.

In Feldman et al. (2021), the authors derive a 1/4-selectable[4] greedy OCRS for general matroids, and a 0.085-selectable greedy OCRS for the knapsack polytope, both of which hold against an almighty adversary. We establish significantly improved selectabilities against the weaker online adversary, and, importantly, show that our guarantees are *tight* for our setting.

Magician's problem. Our algorithms do enjoy a property not featured in the OCRS setting though: they *need not know the universe* of elements in advance, holding even if the adversary can adaptively "create" the p_t (and d_t) of the next query t, under the promise that $\sum_t p_t d_t \le 1$. This property is inherited from the *Magician's problem*, introduced by Alaei (2011) as a powerful black box for approximately solving combinatorial auctions. Our work fully resolves[5] his k-unit Magician problem, showing his γ-Conservative Magician to be optimal, and, importantly, showing how to find the optimal value $\gamma = \gamma_k^*$, which is greater than the value of $\gamma = 1 - \frac{1}{\sqrt{k+3}}$, for all $k > 1$. This improves all of the guarantees for combinatorial auctions, summarized in Alaei (2014), that depend on this value of γ.

3. Online Generalized Assignment

In this section, we formalize the model and the notations for a generalized online multi-resource assignment problem, as well as illustrate the LP reduction approach that reduces the multi-resource problem into single-resource subproblems. We further show that the single-resource problem can be simplified equivalently into an OCRS problem. We note that in even more general settings where there is stochasticity after each decision in which resource gets consumed, the same reduction still holds (see e.g. the online assortment problem in Ma et al. (2021)). However, for simplicity, we restrict to the online generalized assignment problem in this paper which suffices to illustrate our contribution.

We consider an online generalized assignment problem over T discrete time periods. We assume that there are m resources and the initial capacity of each resource is without loss of generality scaled to 1. At the beginning of each period $t = 1, 2, \ldots, T$, a query arrives, denoted by query t, and is associated with a non-negative stochastic reward $\tilde{\mathbf{r}}_t = (\tilde{r}_{t1}, \ldots, \tilde{r}_{tm})$ and a positive stochastic size $\tilde{\mathbf{d}}_t = (\tilde{d}_{t1}, \ldots, \tilde{d}_{tm})$, where $(\tilde{\mathbf{r}}_t, \tilde{\mathbf{d}}_t)$ is assumed to follow a joint distribution $F_t(\cdot)$ that can be inhomogeneous in time and is independent across time. After the value of $(\tilde{\mathbf{r}}_t, \tilde{\mathbf{d}}_t)$ is revealed, the

[4] Lee and Singla (2018) have improved this to a 1/2-selectable OCRS for general matroids, against the weakest adversary. Our guarantees of γ_k^* are all greater than 1/2 and hold against the online adversary, but in the special case of k-uniform matroids.

[5] The main difference in the Magician's problem is that a query must be selected *before* it is known whether it is active, and, if so, is irrevocably served. The goal is to select each query with an ex-ante probability at least γ. Our problem can be reinterpreted as selecting a γ-measure of sample paths on which each query should be served whenever it is active, which is completely equivalent. Therefore, all of our results also hold for Alaei's Magician problem and its applications.

decision maker has to decide irrevocably whether to serve or reject this query. If served, the decision maker also needs to assign a resource $j \in \{1,\dots,m\}$ to serve query t. Then, query t will take up $\tilde{d}_{tj}$ capacity of resource j and a reward $\tilde{r}_{tj}$ will be collected. If rejected, then no reward is collected and no resource is consumed. The goal is to maximize the total collected reward subject to the capacity constraint of each resource.

We classify the elements described above into two categories: (i) the problem setup $\mathcal{H} = (F_1, F_2, \dots, F_T)$ denoting all distributions; and (ii) the realization of the rewards and sizes of the arriving queries $\boldsymbol{I} = ((\tilde{\mathbf{r}}_1, \tilde{\mathbf{d}}_1), (\tilde{\mathbf{r}}_2, \tilde{\mathbf{d}}_2), \dots, (\tilde{\mathbf{r}}_T, \tilde{\mathbf{d}}_T))$.

Any online policy π for the decision maker can be specified by a set of decision variables $\{x^\pi_{jt}\}_{j=1,\dots,m,t=1,\dots,T}$, where x^π_{jt} is a binary variable and denotes whether query t is served by resource j. Note that if π is a randomized policy, then x^π_{jt} is a random binary variable. A policy π is *feasible* if and only if π is non-anticipative; i.e., for each t, the value of x^π_{jt} can only depend on the problem setup $\mathcal{H}$ and the rewards and the sizes of the arriving queries up to query t, denoted by $\{(\tilde{\mathbf{r}}_1, \tilde{\mathbf{d}}_1), \dots, (\tilde{\mathbf{r}}_t, \tilde{\mathbf{d}}_t)\}$. Moreover, π needs to satisfy the following capacity constraint:

$$\sum_{t=1}^{T} \tilde{d}_{jt} \cdot x^\pi_{jt} \le 1, \quad \forall j = 1, \dots, m. \tag{1}$$

and the constraint $\sum_{j=1}^m x^\pi_{jt} \le 1$ for all $t = 1, \dots, T$. The total collected reward of policy π is denoted by $V^\pi(\boldsymbol{I}) = \sum_{t=1}^T \sum_{j=1}^m \tilde{r}_{jt} \cdot x^\pi_{jt}$, and the expected total collected reward of policy π is denoted by $\mathbb{E}_{\pi, \boldsymbol{I}\sim\mathcal{H}}[V^\pi(\boldsymbol{I})]$.

We compare to the prophet, who can make decisions based on the knowledge of the realizations of all the queries. Similarly, the optimal offline decision is specified by $\{x^{\text{off}}_{jt}(\boldsymbol{I})\}_{j=1,\dots,m,t=1,\dots,T}$, which is the optimal solution to the following offline problem

$$\begin{aligned} V^{\text{off}}(\boldsymbol{I}) = \ \max \quad & \sum_{t=1}^{T} \sum_{j=1}^{m} \tilde{r}_{jt} \cdot x_{jt} \\ \text{s.t.} \quad & \sum_{t=1}^{T} \tilde{d}_{jt} \cdot x_{jt} \le 1, \quad \forall j = 1, \dots, m \\ & \sum_{j=1}^{m} x_{jt} \le 1, \quad \forall t = 1, \dots, T \\ & x_{jt} \in \{0, 1\}, \quad \forall j, \forall t. \end{aligned} \tag{2}$$

The prophet's value is denoted by $\mathbb{E}_{\boldsymbol{I}\sim\mathcal{H}}[V^{\text{off}}(\boldsymbol{I})]$. For any feasible online policy π, its competitive ratio γ is defined as

$$\gamma = \inf_{\mathcal{H}} \frac{\mathbb{E}_{\pi, \boldsymbol{I}\sim\mathcal{H}}[V^\pi(\boldsymbol{I})]}{\mathbb{E}_{\boldsymbol{I}\sim\mathcal{H}}[V^{\text{off}}(\boldsymbol{I})]}. \tag{3}$$

In this paper, however, instead of directly comparing to $\mathbb{E}_{\boldsymbol{I}\sim\mathcal{H}}[V^{\text{off}}(\boldsymbol{I})]$, we compare to a linear programming (LP) upper bound of $\mathbb{E}_{\boldsymbol{I}\sim\mathcal{H}}[V^{\text{off}}(\boldsymbol{I})]$, which will be useful for decomposing into single-resource subproblems.. We consider the following LP as an upper bound:

$$
\begin{aligned}
\mathrm{UP}(\mathcal{H}) = \ \max \quad & \sum_{t=1}^{T} \mathbb{E}_{(\tilde{\mathbf{r}}_t, \tilde{\mathbf{d}}_t)\sim F_t}\Big[\sum_{j=1}^{m} \tilde{r}_{tj} \cdot x_{tj}(\tilde{\mathbf{r}}_t, \tilde{\mathbf{d}}_t)\Big] \\
\text{s.t.} \quad & \sum_{t=1}^{T} \mathbb{E}_{(\tilde{\mathbf{r}}_t, \tilde{\mathbf{d}}_t)\sim F_t}[\tilde{d}_{tj} \cdot x_{tj}(\tilde{\mathbf{r}}_t, \tilde{\mathbf{d}}_t)] \le 1, \quad \forall j = 1, \ldots, m \\
& \sum_{j=1}^{m} x_{tj}(\mathbf{r}_t, \mathbf{d}_t) \le 1, \quad \forall t, \forall (\mathbf{r}_t, \mathbf{d}_t) \\
& x_{tj}(\mathbf{r}_t, \mathbf{d}_t) \ge 0, \quad \forall t, \forall (\mathbf{r}_t, \mathbf{d}_t).
\end{aligned} \tag{4}
$$

Here $\mathcal{H} = (F_1, \ldots, F_T)$ denotes the problem setup, while $x_{tj}(\mathbf{r}_t, \mathbf{d}_t)$ denotes the probability of assigning resource j to serve query t conditional on its reward and size vectors realizing to $\mathbf{r}_t = (r_{t1}, \ldots, r_{tm})$ and $\mathbf{d}_t = (d_{t1}, \ldots, d_{tm})$, respectively. The following lemma shows that for a policy π to have a competitive ratio of at least γ, it suffices that $\mathbb{E}_{\pi,\boldsymbol{I}\sim\mathcal{H}}[V^{\pi}(\boldsymbol{I})]$ is at least γ times the optimal fractional allocation value $\mathrm{UP}(\mathcal{H})$ for every setup $\mathcal{H}$.

Lemma 1 *For all problem setups $\mathcal{H}$, it holds that $\mathbb{E}_{\boldsymbol{I}\sim\mathcal{H}}[V^{\mathit{off}}(\boldsymbol{I})] \le \mathit{UP}(\mathcal{H})$.*

The proof of Lemma 1 is relegated to the Section A1. In what follows, we will derive our results based on the optimal solution to $\mathrm{UP}(\mathcal{H})$.

3.1. Reduction to Single-resource Problems

In this section, we explain how to reduce the multi-resource problem into multiple single-resource problems. Denote by $\{x^*_{tj}(\mathbf{r}_t, \mathbf{d}_t)\}$ the optimal solution to $\mathrm{UP}(\mathcal{H})$. Then, at each period t, after query t has arrived and the value of $(\tilde{\mathbf{r}}_t, \tilde{\mathbf{d}}_t)$ is revealed, we randomly route query t to a resource j with probability $x^*_{tj}(\tilde{\mathbf{r}}_t, \tilde{\mathbf{d}}_t)$. Specifically, we choose a resource j with probability $x^*_{tj}(\tilde{\mathbf{r}}_t, \tilde{\mathbf{d}}_t)$ to be the only potential resource to serve query t and whether or not query t is served will depend on the individual serve/reject rule of resource j. Note that this random routing approach has also been applied in previous literature (Alaei et al., 2012, 2013; Wang et al., 2018; Stein et al., 2020) for reducing to the single-dimensional setting.

For each resource j and each t, we denote by $F_{tj}(\cdot)$ the distribution of the following random variable:

$$(\tilde{r}_{tj}, \tilde{d}_{tj}) \cdot \mathrm{Ber}(x^*_{tj}(\tilde{\boldsymbol{r}}_t, \tilde{\boldsymbol{d}}_t)).$$

Here, for each $x^*_{tj}(\tilde{\boldsymbol{r}}_t, \tilde{\boldsymbol{d}}_t)$, $\mathrm{Ber}(x^*_{tj}(\tilde{\boldsymbol{r}}_t, \tilde{\boldsymbol{d}}_t))$ is a Bernoulli random variable with mean $x^*_{tj}(\tilde{\boldsymbol{r}}_t, \tilde{\boldsymbol{d}}_t)$, where $\mathrm{Ber}(x^*_{tj}(\tilde{\boldsymbol{r}}_t, \tilde{\boldsymbol{d}}_t)) = 1$ (resp. $= 0$) denotes query t with reward and size $(\tilde{\boldsymbol{r}}_t, \tilde{\boldsymbol{d}}_t)$ is (resp. not) routed to resource j. Then, we denote $\mathcal{H}_j = (F_{1j}, \dots, F_{Tj})$ to be the problem setup faced by resource j. Note that

$$\sum_{t=1}^{T} \mathbb{E}_{(\tilde{r}_t, \tilde{d}_t) \sim F_{tj}}[\tilde{d}_t] = \sum_{t=1}^{T} \mathbb{E}_{(\tilde{\mathbf{r}}_t, \tilde{\mathbf{d}}_t) \sim F_t}[\tilde{d}_{tj} \cdot x^*_{tj}(\tilde{\mathbf{r}}_t, \tilde{\mathbf{d}}_t)] \leq 1.$$

Obviously, we have

$$\mathrm{UP}(\mathcal{H}_j) = \sum_{t=1}^{T} \mathbb{E}_{(\tilde{r}_t, \tilde{d}_t) \sim F_{tj}}[\tilde{r}_t] = \sum_{t=1}^{T} \mathbb{E}_{(\tilde{\mathbf{r}}_t, \tilde{\mathbf{d}}_t) \sim F_t}[\tilde{r}_{tj} \cdot x^*_{tj}(\tilde{\mathbf{r}}_t, \tilde{\mathbf{d}}_t)]$$

and we have $\sum_{j=1}^{m} \mathrm{UP}(\mathcal{H}_j) = \mathrm{UP}(\mathcal{H})$. Clearly, if an online policy π can collect an expected reward of at least $\gamma \cdot \mathrm{UP}(\mathcal{H}_j)$ on the problem setup $\mathcal{H}_j$, then, we can simply specify the policy π as the serve/reject rule for each resource after random routing, and the total reward collected from all resources is lower-bounded by $\gamma \cdot \sum_{j=1}^{m} \mathrm{UP}(\mathcal{H}_j) = \gamma \cdot \mathrm{UP}(\mathcal{H})$. The above discussion is summarized in the following lemma.

Lemma 2 *To derive an online multi-resource allocation policy with competitive ratio at least γ, it suffices to derive an online policy π such that $\mathbb{E}_{\pi, \boldsymbol{I} \sim \mathcal{H}_j}[V^{\pi}(\boldsymbol{I})] \geq \gamma \cdot UP(\mathcal{H}_j)$ is satisfied for all single resource instances $\mathcal{H}_j$ satisfying $\sum_{t=1}^{T} \mathbb{E}[\tilde{d}_t] \leq 1$.*

3.2. LP Duality and OCRS

We now analyze the problem faced by each resource. To be specific, we now assume that there is a single resource with an initial capacity 1, and each query t is associated with a random reward $\tilde{r}_t$ and a random size $\tilde{d}_t$, where $(\tilde{r}_t, \tilde{d}_t)$ follows a given joint distribution $F_t(\cdot)$ but is still realized independently of other queries. After $(\tilde{r}_t, \tilde{d}_t)$ is revealed, the online decision maker has to decide irrevocably whether or not to serve this query without violating the capacity constraint. Following Lemma 2, we restrict $\mathcal{H}$ to describe a single-resource problem instance and the constraint $\sum_{t=1}^{T} \mathbb{E}[\tilde{d}_t] \leq 1$ is satisfied.

We now show that in order to obtain the tight guarantee, it is sufficient to consider knapsack OCRS problem (Definition 1 for the k-unit setting and Definition 2 for the knapsack setting) and we obtain a new LP to work with. In a preliminary version (Jiang et al., 2022) of this work, we establish the sufficiency under a special case where the size of each query is deterministic. We generalize to the setting where the size can be random, through (infinite) LP duality.

We begin with considering the optimal online policy given by dynamic programming (DP). For notation brevity, we denote by $\omega_t = (r_t, d_t)$ a scenario for query t, and we denote by $r(\omega_t)$ and $d(\omega_t)$ the reward and size of query t under scenario ω_t. Denote by $V^*_t(c_t)$ the "value-to-go" function at period t with remaining capacity c_t. We have the following backward induction for the DP:

$$V^*_t(c_t) = V^*_{t+1}(c_t) + \mathbb{E}_{\omega_t \sim F_t}[\underbrace{\max\left\{0, 1_{\{c_t \geq d(\omega_t)\}} \cdot (r(\omega_t) + V^*_{t+1}(c_t - d(\omega_t)) - V^*_{t+1}(c_t))\right\}}_{\text{marginal increase}}], \qquad (5)$$

for all t and $0 \le c_t \le 1$, where $V^*_{T+1}(\cdot) = 0$. Having this notation, we are interested in characterizing the quantity

$$\inf_{\mathcal{H}} \frac{V_1^*(1)}{\mathrm{UP}(\mathcal{H})}. \tag{6}$$

Note that under the constraint $\sum_{t=1}^T \mathbb{E}_{\omega_t\sim F_t}[d(\omega_t)] \le 1$, we must have $\mathrm{UP}(\mathcal{H}) = \sum_{t=1}^T \mathbb{E}_{\omega_t\sim F_t}[r(\omega_t)]$ (because setting all the x-values to 1 forms a feasible solution to the LP describing $\mathrm{UP}(\mathcal{H})$). We scale the values of $\tilde{r}_t$ so that $\mathrm{UP}(\mathcal{H}) = \sum_{t=1}^T \mathbb{E}_{\omega_t\sim F_t}[r(\omega_t)] = 1$. We denote by $p_t(\omega_t)$ the probability that scenario ω_t happens for query t, and $p_t(d_t) = \sum_{\omega_t: d(\omega_t)=d_t} p_t(\omega_t)$ the probability that the size of query t is realized as d_t. With some further massaging, we can re-express (6) as the infimum value of an LP $\mathrm{Primal}(\boldsymbol{p},\boldsymbol{D})$, as well as its dual which we denote by $\mathrm{Dual}(\boldsymbol{p},\boldsymbol{D})$, over $\boldsymbol{p} = \{p_t(\omega_t)\}_{\forall t,\forall d_t}$ and $\boldsymbol{D} = \{d_t\}_{\forall t,\forall d_t}$ such that $\sum_{t=1}^T \sum_{d_t} p_t(d_t)\cdot d_t \le 1$. The following lemma formalizes this fact, where the proof is relegated to Section A2.

Lemma 3 *It holds that*

$$\inf_{\mathcal{H}} \frac{V_1^*(1)}{UP(\mathcal{H})} = \inf_{(\boldsymbol{p},\boldsymbol{D}):\sum_{t=1}^T \sum_{d_t} p_t(d_t)\cdot d_t \le 1} Primal(\boldsymbol{p},\boldsymbol{D}) = \inf_{(\boldsymbol{p},\boldsymbol{D}):\sum_{t=1}^T \sum_{d_t} p_t(d_t)\cdot d_t \le 1} Dual(\boldsymbol{p},\boldsymbol{D}).$$

where the formulations of Primal($\boldsymbol{p},\boldsymbol{D}$) and Dual($\boldsymbol{p},\boldsymbol{D}$) are given in and (7) and (8).

We work with the following LP

$$\mathrm{Primal}(\boldsymbol{p},\boldsymbol{D}) = \min \quad V_1(1) \tag{7}$$

$$\text{s.t.} \quad V_t(c_t) \ge V_{t+1}(c_t) + \mathbb{E}_{\omega_t\sim F_t}[1_{\{d(\omega_t)\le c_t\}}\cdot W_t(\omega_t,c_t)], \ \forall t, \forall c_t \tag{7a}$$

$$W_t(\omega_t,c_t) \ge r(\omega_t) + V_{t+1}(c_t - d(\omega_t)) - V_{t+1}(c_t), \ \forall t, \forall \omega_t, \forall c_t \ge d(\omega_t) \tag{7b}$$

$$\sum_{t=1}^T \sum_{\omega_t} r(\omega_t)\cdot p_t(\omega_t) = 1 \tag{7c}$$

$$V_{T+1}(c) = 0, \ \forall c$$

$$V_t(c) \in \mathbb{R}, W_t(\omega_t,c_t) \ge 0, r(\omega_t) \ge 0$$

In this LP $\mathrm{Primal}(\boldsymbol{p},\boldsymbol{D})$, decision variable $V_t(c_t)$ represents the dynamic programming value-to-go $V_t^*(c_t)$, decision variable $W_t(\omega_t,c_t)$ represents the marginal increase term given scenario ω_t for query t, and decision variable $r(\omega_t)$ must be scaled to respect $\mathrm{UP}(\mathcal{H}) = \sum_{t=1}^T \mathbb{E}_{\omega_t\sim F_t}[r(\omega_t)] = 1$. It is known that in an optimal solution, the decision variables $V_t(c_t)$ equal $V_t^*(c_t)$ of the DP formulation (Adelman, 2007). We obtain an LP dual with the interpretable set of constraints (8b) below.

$$\mathrm{Dual}(\boldsymbol{p},\boldsymbol{D}) = \max \ \theta \tag{8}$$

$$\text{s.t.} \quad \theta\cdot p_t(d_t) \le \sum_{c_t: c_t \ge d_t} \alpha_t(d_t,c_t), \quad \forall t, \forall d_t \tag{8a}$$

$$\alpha_t(d_t, c_t) \le p_t(d_t) \cdot \sum_{\tau \le t-1} \sum_{d_\tau} (\alpha_\tau(d_\tau, c_t + d_\tau) - \alpha_\tau(d_\tau, c_t)), \ \forall t, \forall d_t, \forall c_t < 1 \quad \text{(8b)}$$

$$\alpha_t(d_t, 1) \le p_t(d_t) \cdot (1 - \sum_{\tau \le t-1} \sum_{d_\tau} \alpha_\tau(d_\tau, 1)), \quad \forall t, \forall d_t \quad \text{(8c)}$$

$$\alpha_t(d_t, c_t) = 0, \quad \forall t, \forall d_t, \forall c_t > 1$$

$$\theta, \alpha_t(d_t, c_t) \ge 0, \quad \forall t, \forall d_t, \forall c_t$$

In fact, for general $\boldsymbol{p}$ and $\boldsymbol{D}$, Dual$(\boldsymbol{p}, \boldsymbol{D})$ characterizes the knapsack OCRS problem described in Definition 2, where each query t has a size d_t with probability $p_t(d_t)$. To see this, we interpret the variable θ as the guarantee γ in the knapsack OCRS problem, and we interpret the variable $\alpha_t(d_t, c_t)$ for each t, each d_t, and each $0 \le c_t \le 1$ as the *ex-ante* probability (the unconditional probability) that query t becomes active with size d_t and is served when the remaining capacity is c_t at the beginning of period t. Then, constraint (8a) corresponds to each query t being served with probability at least $\theta = \gamma$ conditional on it being active with size d_t. Moreover, for each t and each $0 \le c_t < 1$, the term $\sum_{\tau=1}^{t-1} \sum_{d_\tau} \alpha_\tau(d_\tau, c_t + d_\tau)$ denotes the ex-ante probability that the remaining capacity has "reached" the state c_t in the first $t-1$ periods and the term $\sum_{\tau=1}^{t-1} \sum_{d_\tau} \alpha_\tau(d_\tau, c_t)$ denotes the ex-ante probability that the remaining capacity has "left" the state c_t. Thus, the difference $\sum_{\tau=1}^{t-1} \sum_{d_\tau} (\alpha_\tau(d_\tau, c_t + d_\tau) - \alpha_\tau(d_\tau, c_t))$ denotes exactly the ex-ante probability that the remaining capacity at the beginning of period t is c_t. Similarly, the difference $1 - \sum_{\tau=1}^{t-1} \sum_{d_\tau} \alpha_\tau(d_\tau, 1)$ denotes the ex-ante probability that the remaining capacity at the beginning of period t is 1. Since each query t can only be served after the size is realized as some d_t, which happens independently with probability $p_t(d_t)$, we recover constraint (8b) and constraint (8c).

When each d_t in $\boldsymbol{D}$ only takes value 0 or $1/k$, Dual$(\boldsymbol{p}, \boldsymbol{D})$ characterizes the k-unit OCRS problem described in Definition 1, where each query t becomes active with probability $p_t(1/k)$. We provide further explanation on the characterization in Section 4.1.

Following the discussions above, the duality between Dual$(\boldsymbol{p}, \boldsymbol{D})$ and Primal$(\boldsymbol{p}, \boldsymbol{D})$ implies that OCRS is the dual problem of our online resource allocation problem, under both the k-unit setting (in which case our online resource allocation problem becomes the k-unit prophet inequality problem) and the knapsack setting. Consequently, the worst-case guarantee for the OCRS problem is equivalent to the worst-case guarantee of the optimal online policy relative to the LP relaxation.

4. Multi-unit Prophet Inequalities

We now consider the single-resource problem under the multi-unit setting. To be specific, we assume that $d_t = \frac{1}{k}$ for an integer k, for each t, i.e., the online decision maker can serve at most k queries to collect the corresponding rewards. Note that when $k = 1$, our problem reduces to the well-known prophet inequality (Krengel and Sucheston, 1978), and for general k, we get the so-called multi-unit prophet inequalities (Alaei, 2011). The main result of this section is that for each k, we derive the tight competitive ratio for the k-unit prophet inequality problem with respect to the LP upper bound, or equivalently the optimal solution γ_k^* to the k-unit OCRS problem. Note that our values γ_k^* strictly exceed $1-\frac{1}{\sqrt{k+3}}$ for all $k>1$, and hence we also improve the best-known prophet inequalities for all $k>1$.

The structure of our proof follows the three steps outlined in Section 1.1. In a preliminary version (Jiang et al., 2022) of this work, we illustrate our approach for a special case $k=2$.

4.1. LP Formulation of k-unit OCRS Problem

We first present a new LP formulation of the k-unit OCRS problem, assuming that the vector $\boldsymbol{p}$ is given in advance. This LP is derived from taking the dual of the LP formulation of the optimal dynamic programming (DP) policy. We name our LP as $\mathrm{Dual}(\boldsymbol{p},k)$, which is exactly $\mathrm{Dual}(\boldsymbol{p},\boldsymbol{D})$ in (8) under the k-unit setting. Then, the restriction $\sum_{t=1}^T\sum_{d_t}p_t(d_t)\cdot d_t\le 1$, which follows from Lemma 2, can be translated into $\sum_{t=1}^T p_t\le k$ by noting d_t takes a single value $1/k$ when query t becomes active. Therefore, Lemma 3 implies that it is enough to consider $\mathrm{Dual}(\boldsymbol{p},k)$ to obtain the tight guarantee for the k-unit prophet inequality problem.

$$\begin{aligned}
\mathrm{Dual}(\boldsymbol{p},k)= \max \quad & \theta && (9)\\
\text{s.t.} \quad & \theta\le\frac{\sum_{l=1}^k x_{l,t}}{p_t}\quad\forall t && (9a)\\
& x_{1,t}\le p_t\cdot\Big(1-\sum_{\tau<t}x_{1,\tau}\Big)\quad\forall t && (9b)\\
& x_{l,t}\le p_t\cdot\sum_{\tau<t}(x_{l-1,\tau}-x_{l,\tau})\quad\forall t,\forall l=2,\dots,k && (9c)\\
& x_{1,t}\ge 0, x_{2,t}\ge 0,\dots,x_{k,t}\ge 0.
\end{aligned}$$

Here, the variable θ can be interpreted as guarantee γ in the k-unit OCRS problem and $x_{l,t}$ can be interpreted as the ex-ante probability of serving query t as the l-th one. Then, constraint (9a) guarantees that each query t is served with an ex-ante probability $\theta\cdot p_t$. Moreover, it is easy to see that the term $\sum_{\tau<t}x_{l-1,\tau}$ can be interpreted as the probability that the number of served queries has “reached” $l-1$ during the first $t-1$ periods, while the term $\sum_{\tau<t}x_{l,\tau}$ can be interpreted as the probability that the number of served queries is larger than $l-1$. Then, the term $\sum_{\tau<t}(x_{l-1,\tau}-x_{l,\tau})$ denotes the probability that the number of served queries is $l-1$ at the beginning of period t. Similarly, the term $1-\sum_{\tau<t}x_{1,\tau}$ denotes the probability that no query is served at the beginning of period t. Further note that each query t can be served only after it becomes active, which happens independently with probability p_t, and hence we get constraint (9b) and (9c).

Although presented in a different context, the “γ-Conservative Magician” procedure of Alaei (2011) implies a feasible solution to $\mathrm{Dual}(\boldsymbol{p},k)$, for any $\boldsymbol{p}$ and any k satisfying $\sum_{t=1}^T p_t\le k$. We now describe this implied solution in Definition 3, which is based on a predetermined θ.

Definition 3 *A candidate solution to Dual($\boldsymbol{p}, k$) in (9) given θ (potentially infeasible)*

1. *For a fixed $\theta \in [0,1]$, we define $x_{1,t}(\theta) = \theta \cdot p_t$ from $t=1$ up to $t=t_2$, where t_2 is defined as the first time among $\{1,\dots,T\}$ such that $\theta > 1 - \sum_{t=1}^{t_2} \theta \cdot p_t$ and if such a t_2 does not exist, we denote $t_2 = T$. Then we define $x_{1,t}(\theta) = p_t \cdot (1 - \sum_{\tau=1}^{t-1} x_{1,\tau}(\theta))$ from $t = t_2 + 1$ up to $t = T$.*
2. *For $l = 2, 3, \dots, k-1$, we do the following:*
   (a) *Define $x_{l,t}(\theta) = 0$ from $t = 1$ up to $t = t_l$.*
   (b) *Define $x_{l,t}(\theta) = \theta \cdot p_t - \sum_{v=1}^{l-1} x_{v,t}(\theta)$ from $t = t_l + 1$ up to $t = t_{l+1}$, where t_{l+1} is defined as the first time among $\{1,\dots,T\}$ such that*
   $$\theta \cdot p_{t_{l+1}+1} - \sum_{v=1}^{l-1} x_{v,t_{l+1}+1}(\theta) > p_{t_{l+1}+1} \cdot \sum_{t=1}^{t_{l+1}} (x_{l-1,t}(\theta) - x_{l,t}(\theta))$$
   and if such a t_{l+1} does not exist, we denote $t_{l+1} = T$.
   (c) *Define $x_{l,t}(\theta) = p_t \cdot \sum_{\tau=1}^{t-1} (x_{l-1,\tau}(\theta) - x_{l,\tau}(\theta))$ from $t = t_{l+1} + 1$ up to $t = T$.*
3. *Define $x_{k,t}(\theta) = 0$ from $t = 1$ up to $t = t_k$ and define $x_{k,t}(\theta) = \theta \cdot p_t - \sum_{v=1}^{k-1} x_{v,t}(\theta)$ from $t = t_k + 1$ up to $t = T$.*

Note that in the above construction, the values of $\{t_l\}$ and $\{x_{l,t}(\theta)\}$ are uniquely determined by θ. Obviously, for an arbitrary θ, the solution $\{\theta, x_{l,t}(\theta)\}$ is not necessarily a feasible solution to Dual($\boldsymbol{p}, k$), much less an optimal one. Alaei et al. (2012) shows that if we set $\theta = 1 - \frac{1}{\sqrt{k+3}}$, then $\{\theta, x_{l,t}(\theta)\}$ is a feasible solution to Dual($\boldsymbol{p}, k$). Thus, they obtain a $1 - \frac{1}{\sqrt{k+3}}$-lower bound of the optimal competitive ratio. However, we now try to identify a θ^*, which is *dependent on* $\boldsymbol{p}$, such that $\{\theta^*, x_{l,t}(\theta^*)\}$ is an *optimal* solution to Dual($\boldsymbol{p}, k$).

Remark. In Definition 3 we describe the LP solution instead of the corresponding policy, because our proof of optimality is *implicit*, relying on duality and complementary slackness to eventually show that our derived bound of γ_k^* (which does not have a closed form) is tight. By contrast, later for knapsack (see Section 5) we described the corresponding policy, since we were able to *explicitly* establish the bound $\frac{1}{3+e^{-2}}$ and show through a counterexample that it is tight. Although here we are able to construct counterexamples as well, this will be done implicitly through LP duality.

4.2. Characterizing the Optimal LP Solution for a Given p

We begin by proving the condition on θ for $\{\theta, x_{l,t}(\theta)\}$ to be a feasible solution to Dual($\boldsymbol{p}, k$).

Lemma 4 *For any vector $\boldsymbol{p}$, there exists a unique $\theta^* \in [0,1]$ such that $\sum_{\tau=1}^{T-1} x_{k,\tau}(\theta^*) = 1 - \theta^*$. Moreover, for any $\theta \in [0, \theta^*]$, $\{\theta, x_{l,t}(\theta)\}$ is a feasible solution to Dual($\boldsymbol{p}, k$).*

The proof is relegated to Section B1. We now prove that $\{\theta^*, x_{l,t}(\theta^*)\}$ is an optimal solution to Dual($\boldsymbol{p}, k$). The dual of Dual($\boldsymbol{p}, k$) can be formulated as follows:

$$
\begin{aligned}
\text{Primal}(\boldsymbol{p},k) \;=\; \min\; & \sum_{t=1}^{T} p_t \cdot \beta_{1,t} && (10)\\
\text{s.t. } & \beta_{l,t} + \sum_{\tau>t} p_\tau \cdot (\beta_{l,\tau} - \beta_{l+1,\tau}) - \xi_t \geq 0, \quad \forall t=1,\dots,T,\ \forall l=1,2,\dots,k-1\\
& \beta_{k,t} + \sum_{\tau>t} p_\tau \cdot \beta_{k,\tau} - \xi_t \geq 0, \quad \forall t=1,\dots,T\\
& \sum_{t=1}^{T} p_t \cdot \xi_t = 1\\
& \beta_{l,t} \geq 0, \xi_t \geq 0, \quad \forall t=1,\dots,T, \forall l=1,\dots,k.
\end{aligned}
$$

Primal($\boldsymbol{p}, k$) in (10) is actually a re-formulation of Primal($\boldsymbol{p}, \boldsymbol{D}$) in (7), through variable substitution. We substitute $\beta_{l,t} = V_t(1-\frac{l-1}{k}) - V_{t+1}(1-\frac{l-1}{k})$ for each $l = 1,\dots,k$ and $\xi_t = \hat{r}_t$ for each t. Then the variable $W_t(\omega_t, c_t)$ can be canceled by summing constraint (7a) and constraint (7b) after taking expectation over all scenario ω_t, while constraint $W_t(\omega_t, c_t) \geq 0$ is equivalent to $\beta_{l,t}$ are non-negative for all l. Thus, for each l and t, the variable $\beta_{l,t}$ can be interpreted as the marginal increase of the total collected reward over time, when there are $l-1$ queries being served prior to time t, and the variable ξ_t can be interpreted as the reward of query t in the worst case.

To prove the optimality of $\{\theta^*, x_{l,t}(\theta^*)\}$, we will construct a feasible dual solution $\{\beta^*_{l,t}, \xi^*_t\}$ to Primal($\boldsymbol{p}, k$) such that complementary slackness conditions hold for the primal-dual pair $\{\theta^*, x_{l,t}(\theta^*)\}$ and $\{\beta^*_{l,t}, \xi^*_t\}$; then, the well-known primal-dual optimality criterion (Dantzig and Thapa, 2006) establishes that $\{\theta^*, x_{l,t}(\theta^*)\}$ and $\{\beta^*_{l,t}, \xi^*_t\}$ are the optimal primal-dual pair to Dual($\boldsymbol{p}, k$) and Primal($\boldsymbol{p}, k$), which completes our proof. The above arguments are formalized in the following theorem. The detailed construction of $\{\beta^*_{l,t}, \xi^*_t\}$ and the proof of Theorem 1 are relegated to Section B4.

Theorem 1 *The solution $\{\theta^*, x_{l,t}(\theta^*)\}$ is optimal for Dual($\boldsymbol{p}, k$), where θ^* is the unique solution to $\sum_{\tau=1}^{T-1} x_{k,\tau}(\theta^*) = 1 - \theta^*$.*

Theorem 1 shows that Definition 3 constructs an optimal solution to Dual($\boldsymbol{p}, k$), *as long as the θ is set as the optimal θ^**, as defined in Lemma 4. This optimal θ^* is uniquely defined based on $\boldsymbol{p}$. Lemma 4 further shows that any $\theta \leq \theta^*$ is feasible, and hence if we can find a θ that is no greater than the θ^* arising from any $\boldsymbol{p}$, then Definition 3 will correspond to a $\boldsymbol{p}$-agnostic procedure for the k-unit prophet inequality or OCRS problem with a guarantee of θ.

4.3. Characterizing the Worst-case Distribution

Our goal is now to find the $\boldsymbol{p}$ such that the optimal objective value θ^* of Dual$(\boldsymbol{p}, k)$ in (9) reaches its minimum. We would like to characterize the worst-case distribution and then compute the competitive ratio.

We first characterize the worst-case distribution for which the optimal objective value of Dual$(\boldsymbol{p}, k)$ reaches its minimum. Obviously, it is enough for us to consider only the $\boldsymbol{p}$ satisfying $\sum_{t=1}^{T} p_t = k$. We show in the following lemma that splitting one query into two queries can only make the optimal objective value of Dual$(\boldsymbol{p}, k)$ become smaller, and thus, in the worst-case distribution, each p_t should be infinitesimally small.

Lemma 5 *For any $\boldsymbol{p} = (p_1, \ldots, p_T)$ satisfying $\sum_{t=1}^{T} p_t = k$, and any $\sigma \in [0,1]$, $1 \le q \le T$, if we define a new sequence of arrival probabilities $\tilde{\boldsymbol{p}} = (\tilde{p}_1, \ldots, \tilde{p}_{T+1})$ such that*

$$\tilde{p}_t = p_t \quad \forall t < q, \qquad \tilde{p}_q = p_q \cdot \sigma, \qquad \tilde{p}_{q+1} = p_q \cdot (1-\sigma) \quad \text{and} \quad \tilde{p}_{t+1} = p_t \quad \forall q+1 \le t \le T,$$

then it holds that Dual$(\boldsymbol{p}, k) \ge$ Dual$(\tilde{\boldsymbol{p}}, k)$.

The proof is relegated to Section B5. Now, for each $\boldsymbol{p} = (p_1, \ldots, p_T)$ satisfying $\sum_{t=1}^{T} p_t = k$, we assume without loss of generality that p_t is a rational number for each t, i.e., $p_t = \frac{n_t}{N}$ where n_t is an integer for each t and N is an integer denoting the common denominator. We first split p_1 into $\frac{1}{N}$ and $\frac{n_1 - 1}{N}$ to form a new sequence of arrival probabilities. By Lemma 5, we know that such an operation can only decrease the optimal objective value of Dual$(\boldsymbol{p}, k)$. We then split $\frac{n_1-1}{N}$ into $\frac{1}{N}$ and $\frac{n_1-2}{N}$ and so on. In this way, we split p_1 into n_1 copies of $\frac{1}{N}$ to form a new sequence of arrival probabilities and Lemma 5 guarantees that the optimal objective value of Dual$(\boldsymbol{p}, k)$ can only become smaller. We repeat the above operation for each t. Finally, we form a new sequence of arrival probabilities, denoted by $\boldsymbol{p}^N = (\frac{1}{N}, \ldots, \frac{1}{N}) \in \mathbb{R}^{Nk}$, and we have Dual$(\boldsymbol{p}, k) \ge$ Dual$(\boldsymbol{p}^N, k)$. Intuitively, when $N \to \infty$, then the optimal objective value of Dual$(\boldsymbol{p}, k)$ reaches its minimum. Note that when $N \to \infty$, we always have $\sum_{t=1}^{Nk} p_t^N = k$, and then the Bernoulli arrival process approximates a Poisson process with rate 1 over the time interval $[0, k]$. The above argument implies that the worst-case arrival process is a Poisson process.

Under the Poisson process, for each fixed ratio $\theta \in [0,1]$, our solution in Definition 3 can be interpreted as a solution to an ordinary differential equation (ODE). We further note that for $\boldsymbol{p}^N$ and any $\theta \in [0,1]$, our solution in Definition 3 can be regarded as the solution obtained from applying Euler's method to solve this ODE by uniformly discretizing the interval $[0, k]$ into Nk discrete points. Then, for any fixed ratio $\theta \in [0,1]$, after showing the Lipschitz continuity of the function defining this ODE, we can apply the global truncation error theorem of Euler's method

(Theorem 212A in Butcher and Goodwin (2008)) to establish the solution under the Poisson process as the limit of the solution under $\boldsymbol{p}^N$ when $N \to \infty$. Based on this convergence, we can prove that the optimal value under the Poisson process is equivalent to $\lim_{N\to\infty} \mathrm{Dual}(\boldsymbol{p}^N, k)$, which is the optimal ratio we are looking for.

For general k, the values of γ_k^* have previously been shown in Wang et al. (2018) through the analysis of a "reflecting" Poisson process. However, we show that these values γ_k^* are *optimal*, deriving them instead from $\mathrm{Dual}(\boldsymbol{p}, \boldsymbol{D})$. Moreover, Wang et al. (2018) assume Poisson arrivals to begin with, whereas we allow for arbitrary probability vectors $\boldsymbol{p}$ and show that the limiting Poisson case is the worst case.

Specifically in the case of $k = 2$, we construct an example showing that relative to the weaker prophet benchmark $\mathbb{E}_{\boldsymbol{I}\sim\boldsymbol{F}}[V^{\mathrm{off}}(\boldsymbol{I})]$, it is not possible to do much better than γ_2^*. Our construction is based on adapting the tight example relative to the stronger benchmark $\mathrm{UP}(\mathcal{H})$. We note that this suggests that there is *some separation* between optimal ex-ante vs. non-ex-ante prophet inequalities when $k > 1$, which is not the case when $k = 1$ (because they are both 1/2). The formal proof of Proposition 1 below is relegated to Section B6.

Proposition 1 *For the* 2*-unit prophet inequality problem, it holds that* $\inf_{\mathcal{H}} \frac{\mathbb{E}_{\pi,\boldsymbol{I}\sim\boldsymbol{F}}[V^{\pi}(\boldsymbol{I})]}{\mathbb{E}_{\boldsymbol{I}\sim\boldsymbol{F}}[V^{\mathrm{off}}(\boldsymbol{I})]} \le 0.6269$ *for any online algorithm* π*.*

We now discuss how the construction in Definition 3 should be interpreted when the arrival process is a Poisson process. We find it is more convenient to work with the functions $\{\tilde{y}_{l,\theta}(\cdot)\}_{\forall l=1,\dots,k}$ over $[0,k]$, where $\tilde{y}_{l,\theta}(t)$ denotes the ex-ante probability that there is a query served as the l-th one during the period $[0,t]$. Note that the variable $x_{l,t}(\theta)$ denotes the ex-ante probability that there is a query accepted as the l-th query at time t, and hence we have $x_{l,t}(\theta) = d\tilde{y}_{l,\theta}(t)$. We denote $\tilde{y}_{0,\theta}(t) = 1$ for each $t \in [0,k]$. Then the functions $\{\tilde{y}_{l,\theta}(\cdot)\}_{\forall l=1,\dots,k}$ corresponding to the construction in Definition 3 under Poisson arrivals can be interpreted as follows.

Definition 4 *Ordinary Differential Equation (ODE) formula under Poisson arrival*

1. *For each fixed* $\theta \in [0,1]$*, we define* $\tilde{y}_{0,\theta}(t) = 1$ *for each* $t \in [0,k]$ *and* $t_1 = 0$*.*
2. *For each* $l = 1, 2, \dots, k-1$*, we do the following:*
   (a) $\tilde{y}_{l,\theta}(t) = 0$ *when* $t \le t_l$*.*
   (b) *When* $t_l \le t \le t_{l+1}$*, it holds that*

$$\frac{d\tilde{y}_{l,\theta}(t)}{dt} = \theta - \sum_{v=1}^{l-1} \frac{d\tilde{y}_{v,\theta}(t)}{dt} = \theta - 1 + \tilde{y}_{l-1,\theta}(t), \quad \forall t_l \le t \le t_{l+1}, \tag{11}$$

where t_{l+1} *is defined as the first time that* $\tilde{y}_{l,\theta}(t_{l+1}) = 1 - \theta$*. If such a* t_{l+1} *does not exist, we denote* $t_{l+1} = k$*.*

(c) *When $t_{l+1} \le t \le k$, it holds that*

$$\frac{d\tilde{y}_{l,\theta}(t)}{dt} = \tilde{y}_{l-1,\theta}(t) - \tilde{y}_{l,\theta}(t), \quad \forall t_{l+1} \le t \le k. \tag{12}$$

3. $\tilde{y}_{k,\theta}(t) = 0$ *if* $t \le t_k$ *and* $\frac{d\tilde{y}_{k,\theta}(t)}{dt} = \theta - 1 + \tilde{y}_{k-1,\theta}(t)$ *if* $t_k \le t \le k$.

Thus, by Theorem 1, the solution to the equation $\tilde{y}_{k,\theta}(k) = 1-\theta$ should be the minimum of the optimal objective value of Dual$(\boldsymbol{p}, k)$ in (9), which is the competitive ratio γ_k^* we are looking for. The above arguments are formalized in the following theorem and the proof is relegated to Section B7. Note that the following Theorem 2 is our ultimate result for the k-unit case, while Definition 4 characterizes the ODE formula mentioned in Section 1.1. In the remaining part of this section, we will describe the computational procedure for γ_k^*.

Theorem 2 *For each $\theta \in [0,1]$, denote by $\{\tilde{y}_{l,\theta}(\cdot)\}$ the functions defined in Definition 4. Then there exists a unique $\gamma_k^* \in [0,1]$ such that $\tilde{y}_{k,\gamma_k^*}(k) = 1-\gamma_k^*$ and it holds that*

$$\gamma_k^* = \inf_{\boldsymbol{p}} \ \textit{Dual}(\boldsymbol{p}, k) \quad \textit{s.t.} \ \sum_{t=1}^{T} p_t = k.$$

We now show that the ODE in Definition 4 admits an analytical solution that enables us to compute γ_k^* for each k. For each fixed θ, when $l=1$, it is immediate that

$$\tilde{y}_{1,\theta}(t) = \theta \cdot t, \ \text{ when } t \le t_2 = \frac{1-\theta}{\theta}, \ \text{ and } \ \tilde{y}_{1,\theta}(t) = 1 - \theta \cdot \exp(t_2 - t), \ \text{ for } t_2 \le t \le k.$$

Now suppose that there exists a fixed $2 \le l \le k$ such that for each $1 \le v \le l-1$, it holds that

$$\tilde{y}_{v,\theta}(t) = \zeta_v + \theta \cdot t + \sum_{q=0}^{v-2} \zeta_{v,q} \cdot t^q \cdot \exp(-t), \qquad \text{when } t_v \le t \le t_{v+1}$$
$$\tilde{y}_{v,\theta}(t) = 1 + \sum_{q=0}^{v-1} \psi_{v,q} \cdot t^q \cdot \exp(-t), \qquad \text{when } t_{v+1} \le t \le k$$

for some parameters $\{\zeta_v, \zeta_{v,q}, \psi_{v,q}\}$, which are specified by θ. Then by ODE (11) and (12), it must hold that

$$\tilde{y}_{l,\theta}(t) = \zeta_l + \theta \cdot t + \sum_{q=0}^{l-2} \zeta_{l,q} \cdot t^q \cdot \exp(-t), \qquad \text{when } t_l \le t \le t_{l+1}$$
$$\tilde{y}_{l,\theta}(t) = 1 + \sum_{q=0}^{l-1} \psi_{l,q} \cdot t^q \cdot \exp(-t), \qquad \text{when } t_{l+1} \le t \le k.$$

The parameters $\{\zeta_l, \zeta_{l,q}, \psi_{l,q}\}$ can be computed in the following steps:

1. Set $\zeta_{l,l-1} = 0$ and compute $\zeta_{l,q}$ iteratively from $q = l-2$ up to $q = 0$ by setting

$$\zeta_{l,q} = (q+1) \cdot \zeta_{l,q+1} - \psi_{l-1,q}.$$

2. Set the value of ζ_l such that $\tilde{y}_{l,\theta}(t_l) = 0$. If $l = k$, we set $t_{l+1} = k$; otherwise, we set t_{l+1} to be the solution to the following equation:

$$1 - \theta = \tilde{y}_{l,\theta}(t) = \zeta_l + \theta \cdot t + \sum_{q=0}^{l-2} \zeta_{l,q} \cdot t^q \cdot \exp(-t).$$

   Note that by definition $\tilde{y}_{l,\theta}(t)$ is monotone increasing with t, and hence we can do a bisection search on the interval $[t_l, k]$ to obtain the value of t_{l+1}.

3. Set $\psi_{l,q} = \frac{\psi_{l-1,q-1}}{q}$ for each $q = 1, \dots, l-1$. If $l < k$, the value of $\psi_{l,0}$ is determined such that

$$1 - \theta = 1 + \sum_{q=0}^{l-1} \psi_{l,q} \cdot t_{l+1}^q \cdot \exp(-t_{l+1}).$$

Thus, for each fixed θ, we can follow the above procedure to obtain the value of $\tilde{y}_{k,\theta}(k)$. Note that Lemma 10 established in Section B1 implies that the value of $\tilde{y}_{k,\theta}(k)$ is monotone increasing with θ, and hence we can do a bisection search on $\theta \in [0, 1]$ to obtain the value of γ_k^* as the unique solution of the equation $\tilde{y}_{k,\theta}(k) = 1 - \theta$. By Theorem 2, γ_k^* is the optimal value for the competitive ratio.

5. Results for the Knapsack Setting

In this section, we present our results for the knapsack problem. A combination of the random routing approach and the algorithm presented in this section would fully solve the multi-resource problem under the knapsack setting with the optimal competitive ratio guarantee. Following Lemma 2, we restrict $\mathcal{H}$ to describe a single-resource problem instance and the constraint $\sum_{t=1}^{T} \mathbb{E}[\tilde{d}_t] \le 1$ is satisfied. Note that now we have the LP upper bound $\mathrm{UP}(\mathcal{H}) = \sum_{t=1}^{T} \mathbb{E}[\tilde{r}_t]$. We are now ready to describe our policy.

The structure of our proof follows the steps outlined in Section 1.2. In a preliminary version (Jiang et al., 2022) of this work, we illustrate our approach for the special case where each $\tilde{d}_t$ can take at most two values, one of which is 0.

5.1. Algorithm and Interpretation

Our policy differs from existing ones for knapsack in an online setting (Dutting et al., 2020; Feldman et al., 2021; Stein et al., 2020) by eschewing the need to split queries into "large" vs. "small" based on whether its size is greater than 1/2. In fact, we can show that any algorithm which *considers large and small queries separately* in our problem is limited to $\gamma \le 1/4$, and hence could not match the $\frac{1}{3+e^{-2}}$ upper bound provided earlier. The result follows by considering a problem setup $\mathcal{H}$ where there are 4 queries and $(\tilde{r}_t, \tilde{d}_t)$ is realized as $(\hat{r}_t, \hat{d}_t)$ with probability p_t and is realized as $(0, 0)$ otherwise, for each query t, and letting

$$(\hat{r}_1, p_1, \hat{d}_1) = (r, 1, \epsilon), \quad (\hat{r}_2, p_2, \hat{d}_2) = (\hat{r}_3, p_3, \hat{d}_3) = (r, \frac{1-2\epsilon}{1+2\epsilon}, \frac{1}{2} + \epsilon), \quad (\hat{r}_4, p_4, \hat{d}_4) = (r/\epsilon, \epsilon, 1)$$

for $r > 0$ and some small $\epsilon > 0$. We formalize the above arguments as follows, where the formal proof is relegated to Section C1.

Proposition 2 *If the policy π serves only either "large" queries with a size larger than 1/2, or "small" queries with a size no larger than 1/2, then it holds that* $\inf_{\hat{\mathcal{H}}} \frac{\mathbb{E}_{\hat{\boldsymbol{I}}\sim\hat{\boldsymbol{F}}}[V^{\pi}(\hat{\boldsymbol{I}})]}{UP(\hat{\mathcal{H}})} \leq \frac{1}{4}$.

Our policy is formally stated in Algorithm 1. Our policy is based on a pre-specified parameter γ and a salient feature of our policy is that each query t, as long as its reward and size are realized as (d_t, r_t), is included in the knapsack with a probability γ. This guarantees that our policy enjoys a competitive ratio γ. Based on γ, for each t, we use $\tilde{X}_{t-1,\gamma}$ to denote the distribution of the capacity consumption under our policy at the beginning of period t, where $\tilde{X}_{0,\gamma}$ takes value 0 deterministically. Then, for each realization (r_t, d_t) of query t, we specify a threshold $\eta_{t,\gamma}(r_t, d_t)$ such that the probability of $\tilde{X}_{t-1,\gamma} \in (\eta_{t,\gamma}(r_t, d_t), 1 - d_t]$ is smaller than or equal to γ, and the probability that $\tilde{X}_{t-1,\gamma} \in [\eta_{t,\gamma}(r_t, d_t), 1 - d_t]$ is larger than or equal to γ. When query t is realized as $(\hat{r}_t, \hat{d}_t)$, we serve query t when the realized capacity consumption is among $(\eta_{t,\gamma}(\hat{r}_t, \hat{d}_t), 1 - d_t]$, or we serve query t with a certain probability (specified in step 4), when the realized capacity consumption equals $\eta_{t,\gamma}(\hat{r}_t, \hat{d}_t)$. It is clear to see that our policy guarantees that query t is served with a total probability γ. We finally update the distribution of capacity consumption in step 5.

In this section we establish competitive ratios while ignoring implementation runtime; in Section 6.1 we show how through discretization, a runtime polynomial in K is attainable while losing only an additive $O(1/K)$ in the competitive ratio, for any large integer K.

In fact, the policy π_γ constructs a feasible solution to Dual$(\boldsymbol{p}, \boldsymbol{D})$ in (8). Specifically, we denote variable θ in Dual$(\boldsymbol{p}, \boldsymbol{D})$ as γ in our policy π_γ. Then, for each t and each $0 \leq c_t \leq 1$, we denote the variable $\alpha_t(d_t, c_t)$ in Dual$(\boldsymbol{p}, \boldsymbol{D})$ as the ex-ante probability that query t is served when the *remaining capacity* is c_t at the beginning of period t, which also denotes the amount of probability mass of $\tilde{X}_{t-1,\gamma}$ that is moved from point $1 - c_t$ to the point $1 - c_t + d_t$ when defining $\tilde{X}_{t,\gamma}$ in step 5 in Algorithm 1. Obviously, when γ is feasible, the policy π_γ guarantees that each query t with size d_t is served with an ex-ante probability $\gamma \cdot p_t(d_t)$ where $p_t(d_t) = \sum_{r_t} p(r_t, d_t)$, which corresponds to constraint (8a). Moreover, from step 5, it is clear to see that

$$P(\tilde{X}_{t,\gamma} = 0) = P(\tilde{X}_{t-1,\gamma} = 0) - \sum_{d_t} \alpha_t(d_t, 1)$$

and

$$P(\tilde{X}_{t,\gamma} = 1 - c) = P(\tilde{X}_{t-1,\gamma} = 1 - c) + \sum_{d_t} (\alpha_t(d_t, c + d_t) - \alpha_t(d_t, c)), \quad \forall c < 1$$

which implies that

$$P(\tilde{X}_{t-1,\gamma} = 0) = 1 - \sum_{\tau=1}^{t-1} \sum_{d_\tau} \alpha_\tau(d_\tau, 1)$$

Algorithm 1 Best-fit Magician policy for single-knapsack (π_γ)

For a fixed γ, we initialize $\tilde{X}_{0,\gamma}$ as a random variable that takes the value 0 deterministically.

For $t = 1, 2, \ldots, T$, we do the following:

For each realization (r_t, d_t) of $(\tilde{r}_t, \tilde{d}_t)$, we denote by $p(r_t, d_t) = P((\tilde{r}_t, \tilde{d}_t) = (r_t, d_t))$. Then, we denote a threshold $\eta_{t,\gamma}(r_t, d_t)$ satisfying:

$$P(\eta_{t,\gamma}(r_t, d_t) < \tilde{X}_{t-1,\gamma} \leq 1 - d_t) \leq \gamma \leq P(\eta_{t,\gamma}(r_t, d_t) \leq \tilde{X}_{t-1,\gamma} \leq 1 - d_t). \tag{13}$$

Denote by $(\hat{r}_t, \hat{d}_t)$ the realization of query t and by X_{t-1} the consumed capacity at the end of period $t-1$. Then we serve query t if $\eta_{t,\gamma}(\hat{r}_t, \hat{d}_t) < X_{t-1} \leq 1 - \hat{d}_t$; we serve query t with probability $\frac{\gamma - P(\eta_{t,\gamma}(\hat{r}_t, \hat{d}_t) < \tilde{X}_{t-1,\gamma} \leq 1 - \hat{d}_t)}{P(\eta_{t,\gamma}(\hat{r}_t, \hat{d}_t) = \tilde{X}_{t-1,\gamma})}$ if $X_{t-1} = \eta_{t,\gamma}(\hat{r}_t, \hat{d}_t)$.

Based on $\tilde{X}_{t-1,\gamma}$, for each realization (r_t, d_t) of $(\tilde{r}_t, \tilde{d}_t)$ and each point $x \in (\eta_{t,\gamma}(r_t, d_t), 1 - d_t]$, we move the $p(r_t, d_t) \cdot P(\tilde{X}_{t-1,\gamma} = x)$ measure of probability mass from point x to the new point $x + d_t$ and when $x = \eta_{t,\gamma}(r_t, d_t)$, we move the $p(r_t, d_t) \cdot (\gamma - P(\eta_{t,\gamma}(r_t, d_t) < \tilde{X}_{t-1,\gamma} \leq 1 - d_t))$ measure of probability mass from point x to the new point $x + d_t$. We obtain a new distribution and we denote by $\tilde{X}_{t,\gamma}$ a random variable with this distribution.

and

$$P(\tilde{X}_{t-1,\gamma} = 1 - c) = \sum_{\tau=1}^{t-1} \sum_{d_\tau} (\alpha_\tau(d_\tau, c + d_\tau) - \alpha_\tau(d_\tau, c)), \quad \forall c < 1$$

Then, the constraint $\alpha_t(d_t, c) \leq p_t(d_t) \cdot P(\tilde{X}_{t-1,\gamma} = 1 - c)$ for each $0 \leq c \leq 1$ in the policy π_γ corresponds to constraint (8b) and constraint (8c).

5.2. Proof of Competitive Ratio and Tightness

In this section, we analyze the competitive ratio of our Best-fit Magician policy in Algorithm 1 and show that it is tight. When γ is fixed, our policy π_γ guarantees that the decision maker collects at least γ times the LP upper bound $\mathrm{UP}(\mathcal{H}) = \sum_{t=1}^{T} \mathbb{E}[\tilde{r}_t]$. Thus, the key point is to find the largest possible γ such that the policy π_γ is feasible for all problem setups $\mathcal{H}$, i.e., the random variables $\tilde{X}_{t,\gamma}$ are well defined for each t.

We now find such a γ. For any a and b, denote $\mu_{t,\gamma}(a, b] = P(a < \tilde{X}_{t,\gamma} \leq b)$ assuming $\tilde{X}_{t,\gamma}$ is well defined. A key observation of the Best-fit Magician is that an arriving query with a realized size d_t gets accepted in the previously empty sample path of $\tilde{X}_{t-1,\gamma}$ only if there is less than a γ-measure of sample paths with utilization in $(0, 1 - d_t]$. Then, we can establish an *invariant* that upper-bounds the measure of sample paths with utilization in $(0, b]$ by a decreasing exponential function of the measure with utilization in $(b, 1 - b]$. Our invariant holds for all $b \in (0, 1/2]$, at all times t.

Lemma 6 *For any $0 < b \leq \frac{1}{2}$ and any $0 < \gamma < 1$ such that $\tilde{X}_{t,\gamma}$ is well-defined, the following inequality*

$$\frac{1}{\gamma} \cdot \mu_{t,\gamma}(0,b] \leq \exp(-\frac{1}{\gamma} \cdot \mu_{t,\gamma}(b,1-b]) \tag{14}$$

holds for all $t = 0,1,\ldots,T$.

We omit proving Lemma 6 since we will prove a more general Lemma 7 in Section 6.2. For a fixed t, assume that the random variable $\tilde{X}_{t,\gamma}$ is well defined. Then, given the invariant (14) established in Lemma 6, we can lower-bound the measure of zero-utilization sample paths, i.e., $P(\tilde{X}_{t,\gamma} = 0)$, by the constraint

$$\begin{aligned}\sum_{\tau=1}^{t} \sum_{(r_\tau,d_\tau)} d_\tau \cdot p(r_\tau,d_\tau) &= \sum_{\tau=1}^{t} \sum_{(r_\tau,d_\tau)} d_\tau \cdot P((\tilde{r}_\tau,\tilde{d}_\tau) = (r_\tau,d_\tau)) \\ &= \sum_{\tau=1}^{t} \mathbb{E}_{(\tilde{r}_\tau,\tilde{d}_\tau)\sim F_\tau}[\tilde{d}_\tau] \leq 1,\end{aligned}$$

where the last inequality holds from our restriction stated at the beginning that $\sum_{t=1}^{T} \mathbb{E}[\tilde{d}_t] \leq 1$. Note that invariant (14) enables us a way to upper-bound the measure of "bad" sample paths with utilization within $(0,b]$ by the measure of sample paths with utilization within $(b,1-b]$ for some $0 < b < \frac{1}{2}$. Therefore, on the remaining sample paths the utilization must be either 0 or greater than $1-b$, which are good cases for us. From this, we show that a γ as large as $\frac{1}{3+e^{-2}} \approx 0.319$ allows for a γ-measure of sample paths to have zero utilization at time t, which implies that the random variable $\tilde{X}_{t+1,\gamma}$ is well defined. We iteratively apply the above arguments for each $t = 1$ up to $t = T$, and hence we prove the feasibility of our Best-fit Magician policy. The above arguments are formalized in the following theorem and the proof is relegated to Section C2.

Theorem 3 *When $\gamma = \frac{1}{3+e^{-2}}$, the threshold policy π_γ is feasible and has a competitive ratio at least $\frac{1}{3+e^{-2}}$.*

Finally, we show that the guarantee $\gamma = \frac{1}{3+e^{-2}}$ is tight. From Lemma 3, by bounding the optimal value of Dual$(\boldsymbol{p},\boldsymbol{D})$ when $\boldsymbol{p}$ and $\boldsymbol{D}$ (size of query t is deterministic) are specified as follows,

$$(p_1,d_1) = (1,\epsilon), \quad (p_t,d_t) = (\frac{1-2\epsilon}{(T-2)(\frac{1}{2}+\epsilon)}, \frac{1}{2}+\epsilon) \text{ for all } 2 \leq t \leq T-1 \text{ and } (p_T,d_T) = (\epsilon,1) \tag{15}$$

for some $\epsilon > 0$, we obtain the following upper bound of the guarantee of any online algorithm. We refer the formal proof to Section C3.

Theorem 4 *For any feasible online policy π, it holds that $\inf_{\hat{\mathcal{H}}} \frac{\mathbb{E}_{\hat{\boldsymbol{I}}\sim\hat{\boldsymbol{F}}}[V^\pi(\hat{\boldsymbol{I}})]}{UP(\hat{\mathcal{H}})} \leq \frac{1}{3+e^{-2}}$.*

6. Extensions for the Knapsack Setting

In this section, we discuss the polynomial-time implementation of our Best-fit Magician policy (Section 6.1), and present our improvement in the unit-density special case (Section 6.2).

6.1. A Polynomial-time Implementation Scheme

In this section, we discuss how our Best-fit Magician policy can be implemented in polynomial time. Note that the key step in our algorithm is to iteratively compute the distribution of $\tilde{X}_{t,\gamma}$. Our approach is to discretize the possible sizes to always be a multiple of $\frac{1}{KT}$, for some large integer K. Then the support set of $\tilde{X}_{t,\gamma}$ contains at most KT elements for each t, from which it can then be seen that our algorithm can be implemented in $O(KT^2)$ time.

For any problem setup $\mathcal{H}$ restricted to the single-resource setting, we perform this discretization by rounding each potential size d_t *up* to the nearest multiple of $\frac{1}{KT}$. We denote the rounded sizes using d'_t. We then define $F'_t(\cdot)$ as the distribution of (r_t, d'_t), which is a discretization of the distribution $F_t(\cdot)$, and we define $\mathcal{H}' = (F'_1, \dots, F'_T)$. The implementation of our algorithm on the problem setup $\mathcal{H}$ is equivalent to the implementation on $\mathcal{H}'$. To be more specific, we compute the distribution of $\tilde{X}_{t,\gamma}$ based on $F'_t(\cdot)$ for each t, and, when we face a query (r_t, d_t), we treat this query as a query with reward r_t and a size d'_t.

We now discuss the loss of the guarantee of our algorithm due to such discretization. Note that since $d_t \le d'_t$ for each t, if the algorithm attempts any solution that is feasible for $\mathcal{H}'$, then it is also feasible for $\mathcal{H}$. Moreover, since r_t is never changed, it is easy to see that the total reward collected by our algorithm in the problem setup $\mathcal{H}$ satisfies

$$\mathbb{E}_{\pi_\gamma, \boldsymbol{I}\sim\mathcal{H}}[V^{\pi_\gamma}(\boldsymbol{I})] = \mathbb{E}_{\pi_\gamma, \boldsymbol{I}\sim\mathcal{H}'}[V^{\pi_\gamma}(\boldsymbol{I})]$$

and, when $\gamma \le \frac{1}{3+e^{-2}}$, it holds that

$$\mathbb{E}_{\pi_\gamma, \boldsymbol{I}\sim\mathcal{H}'}[V^{\pi_\gamma}(\boldsymbol{I})] \ge \gamma \cdot \mathrm{UP}(\mathcal{H}').$$

It only remains to compare $\mathrm{UP}(\mathcal{H})$ and $\mathrm{UP}(\mathcal{H}')$. Note that the sizes in $\mathcal{H}'$ are bigger, but since at most $\frac{1}{KT}$ can be added to the size of each of the T queries, the total size added is at most $1/K$. We will use this to argue that $\mathrm{UP}(\mathcal{H}') \ge \frac{1}{1+1/K}\mathrm{UP}(\mathcal{H}) = (1-\frac{1}{K+1})\mathrm{UP}(\mathcal{H})$.

Denote by $\{x^*_t(r_t, d_t)\}$ the optimal solution to $\mathrm{UP}(\mathcal{H})$, and denote

$$\hat{x}_t(r_t, d'_t) = \frac{K}{K+1} \cdot \mathbb{E}_{(r_t, \tilde{d}_t)\sim F_t}\left[x^*_t(r_t, \tilde{d}_t) | \tilde{d}_t \text{ is rounded up to } d'_t\right]. \quad \forall (r_t, d'_t)$$

Note that if d_t is rounded up to d'_t, it must hold that $d'_t - d_t \le \frac{1}{KT}$. Then, we have

$$\begin{aligned}\sum_{t=1}^T \mathbb{E}_{(\tilde{r}_t, \tilde{d}'_t)\sim F'_t}[\tilde{d}'_t \cdot \hat{x}_t(\tilde{r}_t, \tilde{d}'_t)] &\le \frac{K}{K+1} \cdot \sum_{t=1}^T \mathbb{E}_{(\tilde{r}_t, \tilde{d}_t)\sim F_t}[\tilde{d}_t \cdot x^*_t(\tilde{r}_t, \tilde{d}_t)] + \sum_{t=1}^T \mathbb{E}_{(\tilde{r}_t, \tilde{d}'_t)\sim F'_t}[\frac{1}{KT} \cdot \hat{x}_t(\tilde{r}_t, \tilde{d}'_t)] \\ &\le \frac{K}{K+1} + \sum_{t=1}^T \frac{1}{KT} \cdot \frac{K}{K+1} = 1,\end{aligned}$$

where the last inequality holds by the feasibility of $\{x_t^*(r_t, d_t)\}$ and $\hat{x}_t(r_t, d_t') \in [0, \frac{K}{K+1}]$ for each (r_t, d_t'). We conclude that $\{\hat{x}_t(r_t, d_t')\}$ is a feasible solution to $\mathrm{UP}(\mathcal{H}')$. Thus, it holds that

$$\mathrm{UP}(\mathcal{H}') \geq \sum_{t=1}^{T} \mathbb{E}_{(\tilde{r}_t, \tilde{d}_t') \sim F_t'}[\tilde{r}_t \cdot x_t(\tilde{r}_t, \tilde{d}_t')] = \frac{K}{K+1} \cdot \sum_{t=1}^{T} \mathbb{E}_{(\tilde{r}_t, \tilde{d}_t) \sim F_t}[\tilde{r}_t \cdot x_t^*(\tilde{r}_t, \tilde{d}_t)] = \frac{K}{K+1} \cdot \mathrm{UP}(\mathcal{H}),$$

which implies that

$$\mathbb{E}_{\pi_\gamma, \boldsymbol{I} \sim \mathcal{H}}[V^{\pi_\gamma}(\boldsymbol{I})] \geq \frac{K}{K+1} \cdot \gamma \cdot \mathrm{UP}(\mathcal{H})$$

when $\gamma \leq \frac{1}{3+e^{-2}}$. In this way, we show how our algorithm can be implemented in $O(KT^2)$ time to achieve a guarantee of $\frac{K}{K+1} \cdot \frac{1}{3+e^{-2}}$.

6.2. Improvement in the Unit-density Special Case

In this section, we consider the unit-density special case of our online knapsack problem where $\tilde{r}_t = \tilde{d}_t$ for each t. Then we can suppress the notation $(\tilde{r}_t, \tilde{d}_t)$ and simply use $\tilde{d}_t$ to denote the size and the reward of query t. We modify our previous Best-fit Magician policy to obtain an improved guarantee. Following Lemma 2, we further restrict $\mathcal{H}$ to describe a single-resource problem instance and the constraint $\sum_{t=1}^{T} \mathbb{E}[\tilde{r}_t] = \sum_{t=1}^{T} \mathbb{E}[\tilde{d}_t] \leq 1$ is satisfied. Note that now we have the LP upper bound $\mathrm{UP}(\mathcal{H}) = \sum_{t=1}^{T} \mathbb{E}[\tilde{r}_t] = \sum_{t=1}^{T} \mathbb{E}[\tilde{d}_t]$. To maximize the total collected reward, it is enough for us to maximize the expected capacity utilization.

We now motivate our policy. Note that for the general case where the reward can be arbitrarily different from the size, our Best-fit Magician policy guarantees that each query is served with a *common* ex-ante probability γ after the LP relaxation. In this way, each query is treated "equally" so that no "extreme" reward can be assigned to the query with the smallest ex-ante probability, which would worsen the guarantee of our algorithm. However, in the unit-density case where the reward of each query is restricted to equal its size, it is no longer essential for us to treat each query "equally." Instead, we will serve the *later-arriving queries* with a *smaller* probability to maximize capacity utilization. Our idea can be illustrated through the following example.

Example 1. We focus on the following example with 4 queries to illustrate how to maximize capacity utilization:

$$(p_1, d_1) = (\frac{2}{3}, \frac{1}{2}),\ (p_2, d_2) = (\frac{2}{3}, \frac{1}{2}),\ (p_3, d_3) = (1-\epsilon, \frac{1}{3}) \text{ and } (p_4, d_4) = (\frac{\epsilon}{3}, 1).$$

Note that if we apply the Best-fit Magician policy with a ratio γ, then the feasible condition of our policy is

$$P(\tilde{X}_{3,\gamma} = 0) = 1 - \frac{8\gamma}{9} - \frac{5\gamma(1-\epsilon)}{9} \geq \gamma, \tag{16}$$

which implies that $\gamma \leq \frac{9}{22}$ as $\epsilon \to 0$. Thus, we conclude that the Best-fit Magician policy can guarantee an expected capacity utilization of at most $\frac{9}{22}$ for this example. However, if we serve each query with a different probability, i.e., if we serve query t with probability γ_t whenever it arrives, then the feasibility condition (16) becomes $P(\tilde{X}_{3,\gamma_1,\gamma_2,\gamma_3}=0)\geq\gamma_4$. We can set $\gamma_4=0$, which enables us to set a larger value for γ_1,γ_2, and γ_3, which in turn leads to a larger capacity utilization.

To be more specific, if we denote by $\tilde{X}_t$ the distribution of capacity utilization at the end of period $t=1,\ldots,4$ under the serving probabilities $\gamma_1,\gamma_2,\gamma_3,\gamma_4$, then the distribution of $\tilde{X}_1$ is

$$P(\tilde{X}_1=\frac{1}{2})=p_1\cdot\gamma_1=\frac{2\gamma_1}{3}\quad\text{and}\quad P(\tilde{X}_1=0)=1-\frac{2\gamma_1}{3}.$$

We still proceed to serve query 2; however, we serve it with probability γ_2 whenever it arrives. Then, we obtain the following distribution of $\tilde{X}_2$:

$$P(\tilde{X}_2=1)=\gamma_1\cdot p_1\cdot p_2=\frac{4\gamma_1}{9},\quad P(\tilde{X}_2=\frac{1}{2})=\gamma_2\cdot p_2-\gamma_1\cdot p_1\cdot p_2+p_1\cdot\gamma_1-\gamma_1\cdot p_1\cdot p_2=\frac{2\gamma_2}{3}-\frac{2\gamma_1}{9}$$
$$P(\tilde{X}_2=0)=1-P(\tilde{X}_2=1)-P(\tilde{X}_2=\frac{1}{2})=1-\frac{2\gamma_1}{9}-\frac{2\gamma_2}{3}.$$

Finally, we serve query 3 with probability γ_3 whenever it arrives. Then, the distribution of $\tilde{X}_3$ can be obtained as follows:

$$P(\tilde{X}_3=1)=\frac{4\gamma_1}{9},\quad P(\tilde{X}_3=\frac{5}{6})=(\frac{2\gamma_2}{3}-\frac{2\gamma_1}{9})\cdot(1-\epsilon),\quad P(\tilde{X}_3=\frac{5}{6})=(\frac{2\gamma_2}{3}-\frac{2\gamma_1}{9})\cdot\epsilon,$$
$$P(\tilde{X}_3=\frac{1}{3})=(\gamma_3-\frac{2\gamma_2}{3}+\frac{2\gamma_1}{9})\cdot(1-\epsilon),\quad P(\tilde{X}_3=0)=1-\frac{4\gamma_1}{9}-\gamma_3+(\frac{2\gamma_1}{9}-\frac{2\gamma_2}{3}+\gamma_3)\cdot\epsilon.$$

Note that as long as $P(\tilde{X}_t=0)\geq\gamma_{t+1}$ for each $t=1,2,3$, the random variables $\{\tilde{X}_t\}_{t=1,2,3}$ are well defined and the above procedure is feasible. Obviously, the probabilities $P(\tilde{X}_t=0)$ are decreasing in t, which corresponds to the fact that the measure of the sample paths with no capacity consumed decreases over time. Thus, it is natural to set γ_t to decrease in terms of t. Specifically, we can set

$$\gamma_2=P(\tilde{X}_1=0)=1-\frac{2\gamma_1}{3}\quad\text{and}\quad\gamma_3=P(\tilde{X}_2=0)=1-\frac{2\gamma_1}{9}-\frac{2\gamma_2}{3}=\frac{1}{3}+\frac{2\gamma_1}{9}$$

as a function of γ_1. We can also set $\gamma_4=0$ since the expected capacity utilization of the last query is 0 as $\epsilon\to0$. Then, the only feasibility condition we need to satisfy is

$$P(\tilde{X}_3=0)\geq\gamma_4=0,$$

which implies that γ_1 can be set as large as 1 when $\epsilon\to0$. Then $\gamma_2=\frac{1}{3}$ and $\gamma_3=\frac{5}{9}$ as $\epsilon\to0$. Thus, we can guarantee a capacity utilization of $\frac{17}{27}$, which improves on the previous utilization of $\frac{9}{22}$ under the Best-fit Magician policy. $\square$

We now present our policy in Algorithm 2, which is based on a sequence of probabilities with which we serve each query based on its LP relaxation value, denoted by $\boldsymbol{\gamma}=(\gamma_1,\ldots,\gamma_T)$ and satisfying $1\geq\gamma_1\geq\gamma_2\geq\cdots\geq\gamma_T\geq0$. We will specify later how to determine the vector $\boldsymbol{\gamma}$ such that our policy is feasible and achieves the improved guarantee.

Algorithm 2 Modified Best-fit Magician policy for unit-density special case

1: For a fixed sequence $\boldsymbol{\gamma}$ satisfying $1 \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_T \geq 0$, we initialize $\tilde{X}_{0,\boldsymbol{\gamma}}$ as a random variable taking value 0 with probability 1.

2: **For** $t = 1, 2, \ldots, T$, we do the following:

3: For each realization d_t of $\tilde{d}_t$, we denote $p(d_t) = P(\tilde{d}_t = d_t)$. Then, we denote a threshold $\eta_{t,\boldsymbol{\gamma}}(d_t)$ satisfying

$$P(\eta_{t,\boldsymbol{\gamma}}(d_t) < \tilde{X}_{t-1,\boldsymbol{\gamma}} \leq 1 - d_t) \leq \gamma_t \leq P(\eta_{t,\boldsymbol{\gamma}}(d_t) \leq \tilde{X}_{t-1,\boldsymbol{\gamma}} \leq 1 - d_t). \tag{17}$$

4: Denote by $\hat{d}_t$ the realization of query t and by X_{t-1} the consumed capacity at the end of period $t-1$. Then we serve query t if $\eta_{t,\boldsymbol{\gamma}}(\hat{d}_t) < X_{t-1} \leq 1 - \hat{d}_t$; we serve query t with probability $\frac{\gamma_t - P(\eta_{t,\boldsymbol{\gamma}}(\hat{d}_t) < \tilde{X}_{t-1,\boldsymbol{\gamma}} \leq 1 - \hat{d}_t)}{P(\eta_{t,\boldsymbol{\gamma}}(\hat{d}_t) = \tilde{X}_{t-1,\boldsymbol{\gamma}})}$ if $X_{t-1} = \eta_{t,\boldsymbol{\gamma}}(\hat{d}_t)$.

5: Based on $\tilde{X}_{t-1,\boldsymbol{\gamma}}$, for each realization d_t of $\tilde{d}_t$ and each point $x \in (\eta_{t,\boldsymbol{\gamma}}(d_t), 1 - d_t]$, we move the $p(d_t) \cdot P(\tilde{X}_{t-1,\boldsymbol{\gamma}} = x)$ measure of probability mass from point x to the new point $x + d_t$ and when $x = \eta_{t,\boldsymbol{\gamma}}(d_t)$, we move the $p(d_t) \cdot (\gamma_t - P(\eta_{t,\boldsymbol{\gamma}}(d_t) < \tilde{X}_{t-1,\boldsymbol{\gamma}} \leq 1 - d_t))$ measure of probability mass from point x to the new point $x + d_t$. We obtain a new distribution and we denote by $\tilde{X}_{t,\boldsymbol{\gamma}}$ a random variable with this distribution.

Note that if the sequence $\boldsymbol{\gamma}$ is uniform, i.e., $\gamma_1 = \cdots = \gamma_T$, then the modified Best-fit Magician policy in Algorithm 2 is identical to the Best-fit Magician policy in Algorithm 1. We now discuss how to determine the sequence $\boldsymbol{\gamma}$ such that our policy in Algorithm 2 is feasible. Our approach relies crucially on the distribution of capacity utilization at the end of each period t, denoted by $\tilde{X}_{t,\boldsymbol{\gamma}}$, assuming that the sequence $\boldsymbol{\gamma}$ is feasible.

For any a and b, we denote $\mu_{t,\boldsymbol{\gamma}}(a, b] = P(a < \tilde{X}_{t,\boldsymbol{\gamma}} \leq b)$, where $\tilde{X}_{t,\boldsymbol{\gamma}}$ denotes the distribution of capacity utilization at the end of period t in Algorithm 2. Then, we can still establish the following invariant, which generalizes Lemma 6 from the uniform sequence to any sequence $\boldsymbol{\gamma}$ satisfying $1 \geq \gamma_1 \geq \cdots \geq \gamma_T \geq 0$.

Lemma 7 *For any $0 < b \leq \frac{1}{2}$ and any sequence $\boldsymbol{\gamma}$ satisfying $1 \geq \gamma_1 \geq \cdots \geq \gamma_T \geq 0$ such that $\tilde{X}_{t,\boldsymbol{\gamma}}$ is well defined, the following inequality*

$$\frac{1}{\gamma_1} \cdot \mu_{t,\boldsymbol{\gamma}}(0, b] \leq \exp(-\frac{1}{\gamma_1} \cdot \mu_{t,\boldsymbol{\gamma}}(b, 1-b]) \tag{18}$$

holds for any $t = 1, \ldots, T$.

Note that the "difficult" case for proving the invariant in Lemma 6 corresponds to when there is a probability mass moved from point 0 during the definition of $\tilde{X}_{t,\boldsymbol{\gamma}}$. Then, both $\mu_t(0, b]$ and $\mu_t(b, 1-b]$ can become larger, for some $b \in (0, 1/2)$. However, when $\boldsymbol{\gamma}$ is a non-increasing sequence, the amount of probability mass that is moved from 0 into either the interval $(0,b]$ or $(b,1-b]$ under γ_t is smaller than the one under γ_1, which makes the invariant easier to hold. The proof of Lemma 7 is relegated to Section D1. Using Lemma 7, we can modify the proof of Theorem 3 to obtain the following result, which will finally lead to our choice of the feasible sequence $\boldsymbol{\gamma}$ and the guarantee of our algorithm.

Theorem 5 *For any t, denote $\psi_t = \mathbb{E}_{\tilde{d}_t\sim F_t}[\tilde{d}_t]$. Then for any sequence $\boldsymbol{\gamma}$ satisfying $1\geq\gamma_1\geq\cdots\geq\gamma_T\geq 0$ such that $\tilde{X}_{t,\boldsymbol{\gamma}}$ is well defined, the following inequality*

$$P(\tilde{X}_{t,\boldsymbol{\gamma}}=0)\geq\min\{1-\gamma_1-\sum_{\tau=1}^{t}\gamma_\tau\cdot\psi_\tau,\ \ 1-2\cdot\sum_{\tau=1}^{t}\gamma_\tau\cdot\psi_\tau-\gamma_1\cdot\exp(-\frac{2}{\gamma_1}\cdot\sum_{\tau=1}^{t}\gamma_\tau\cdot\psi_\tau)\}\tag{19}$$

holds for each $t=1,\ldots,T$.

The proof is relegated to Section D2. Note that for each t, if the random variables $\tilde{X}_{\tau,\boldsymbol{\gamma}}$ are well defined for each $\tau\leq t$, and γ_{t+1} satisfies

$$0\leq\gamma_{t+1}\leq\min\{1-\gamma_1-\sum_{\tau=1}^{t}\gamma_\tau\cdot\psi_\tau,\ \ 1-2\cdot\sum_{\tau=1}^{t}\gamma_\tau\cdot\psi_\tau-\gamma_1\cdot\exp(-\frac{2}{\gamma_1}\cdot\sum_{\tau=1}^{t}\gamma_\tau\cdot\psi_\tau)\},\tag{20}$$

then (19) implies that $P(\tilde{X}_{t,\boldsymbol{\gamma}}=0)\geq\gamma_{t+1}$. Thus, we know that there always exists a threshold $\eta_{t+1,\boldsymbol{\gamma}}(d_t)$ such that (17) holds (since it can be set to 0), and the random variable $\tilde{X}_{t+1,\boldsymbol{\gamma}}$ is well defined. We apply the above argument iteratively for each $t=1$ up to $t=T$. In this way, we conclude that a sufficient condition for the non-increasing sequence $\boldsymbol{\gamma}$ to be feasible for our policy in Algorithm 2 is that (20) holds for each t.

Note that the expected utilization of our policy in Algorithm 2 is $\sum_{t=1}^{T}\gamma_t\cdot\psi_t$. The above analysis implies we can focus on solving the following optimization problem to determine the sequence $\boldsymbol{\gamma}$:

$$\begin{aligned}\mathrm{OP}(\boldsymbol{\psi}):=\ \max\ \ &\sum_{t=1}^{T}\gamma_t\cdot\psi_t\\ \text{s.t.}\ \ &\gamma_{t+1}\leq 1-\gamma_1-\sum_{\tau=1}^{t}\gamma_\tau\cdot\psi_\tau,\quad\forall t=1,\ldots,T-1\\ &\gamma_{t+1}\leq 1-2\cdot\sum_{\tau=1}^{t}\gamma_\tau\cdot\psi_\tau-\gamma_1\cdot\exp(-\frac{2}{\gamma_1}\cdot\sum_{\tau=1}^{t}\gamma_\tau\cdot\psi_\tau),\quad\forall t=1,\ldots,T-1\\ &1\geq\gamma_1\geq\cdots\geq\gamma_T\geq 0,\end{aligned}\tag{21}$$

where $\boldsymbol{\psi}=(\psi_1,\ldots,\psi_T)$ and it holds that $\sum_{t=1}^{T}\psi_t\leq 1$.

Our solution to $\mathrm{OP}(\boldsymbol{\psi})$ can be obtained from the following function over the interval $[0,1]$, the value of which is iteratively computed based on an initial value $\gamma_0\in(0,1)$:

$$h_{\gamma_0}(0)=\gamma_0\tag{22}$$

$$h_{\gamma_0}(t) = \min\left\{ \lim_{\tau\to t-} h_{\gamma_0}(\tau),\ 1-\gamma_0 - \int_{\tau=0}^{t} h_{\gamma_0}(\tau)d\tau,\right.$$
$$\left. 1-2\cdot\int_{\tau=0}^{t} h_{\gamma_0}(\tau)d\tau - \gamma_0\cdot\exp(-\frac{2}{\gamma_0}\cdot\int_{\tau=0}^{t} h_{\gamma_0}(\tau)d\tau)\right\}.$$

It is easy to see that the function $h_{\gamma_0}(\cdot)$ is non-increasing and non-negative over $[0,1]$ as long as $0<\gamma_0<1$. Thus, the function $h_{\gamma_0}(\cdot)$ specifies a feasible solution to OP($\boldsymbol{\psi}$) when each component of $\boldsymbol{\psi}$ is infinitesimally small and $T\to\infty$, where $h_{\gamma_0}(t)$ corresponds to γ_t for each $t\in[0,1]$.

We now show that for arbitrary $\boldsymbol{\psi}$ satisfying $\sum_{t=1}^{T}\psi_t\le 1$, we can still construct a feasible solution to OP($\boldsymbol{\psi}$) based on the function $h_{\gamma_0}(\cdot)$ for each fixed $0<\gamma_0<1$. Specifically, we define a set of indices $0=k_0\le k_1\le\cdots\le k_T\le 1$ such that $\psi_t = k_t - k_{t-1}$ for each $t=1,\dots,T$. Then, we define

$$\hat{\gamma}_t = \frac{\int_{\tau=k_{t-1}}^{k_t} h_{\gamma_0}(\tau)d\tau}{k_t - k_{t-1}} = \frac{\int_{\tau=k_{t-1}}^{k_t} h_{\gamma_0}(\tau)d\tau}{\psi_t} \tag{23}$$

for each $t=1,\dots,T$. We show in the following lemma that $\{\hat{\gamma}_t\}_{t=1}^{T}$ is a feasible solution to OP($\boldsymbol{\psi}$).

Lemma 8 *For each fixed $0<\gamma_0<1$, let $h_{\gamma_0}(\cdot)$ be the function defined in (22). Then, for any $\boldsymbol{\psi}$, the solution $\{\hat{\gamma}_t\}_{t=1}^{T}$ is a feasible solution to OP($\boldsymbol{\psi}$), where $\hat{\gamma}_t$ is as defined in (23) for each $t=1,\dots,T$.*

The formal proof is relegated to Section D3. Note that for each $\boldsymbol{\psi}$ satisfying $\sum_{t=1}^{T}\psi_t\le 1$, if $\{\hat{\gamma}_t\}_{t=1}^{T}$ is constructed according to (23), then it is easy to see that

$$\sum_{t=1}^{T}\hat{\gamma}_t\cdot\psi_t = \int_{t=0}^{k_T} h_{\gamma_0}(t)dt$$

and thus the guarantee of our policy in Algorithm 2 based on the sequence $\{\hat{\gamma}_t\}_{t=1}^{T}$ is $\frac{\int_{t=0}^{k_T} h_{\gamma_0}(t)dt}{k_T}$ for some $k_T\in(0,1]$, where k_T depends on the setup $\boldsymbol{\psi}$. Since the function $h_{\gamma_0}(\cdot)$ is non-increasing and non-negative over $[0,1]$ as long as $0<\gamma_0<1$, we know that the worst-case setup corresponds to $k_T=1$, i.e., $\sum_{t=1}^{T}\psi_t = 1$. Then, it is enough to focus on solving the following problem:

$$\max_{0<\gamma_0<1}\int_{t=0}^{1} h_{\gamma_0}(t)dt$$

to obtain the guarantee of our policy. Numerically, we can show that when $\gamma_0\approx 0.3977$, the above optimization problem reaches its maximum, which is 0.3557. We conclude that the guarantee of our policy is 0.3557. Note that the guarantee of our policy is developed with respect to the LP upper bound UP($\mathcal{H}$), and so it is straightforward to generalize our results to a multi-knapsack setting (Stein et al., 2020) where the size of each query can be knapsack-dependent, as explained in Section 3.1.

Also, we can show that no online algorithm can achieve a better guarantee than $\frac{1-e^{-2}}{2}\approx 0.432$ relative to UP($\mathcal{H}$), even in the single-knapsack setting. The counterexample can be constructed from a problem setup with T queries, where each query has a deterministic size $\frac{1}{2}+\frac{1}{T}$ and is active with probability $\frac{2}{T}$. The proof is relegated to Section D4.

Proposition 3 *In the single-knapsack unit-density case, no online algorithm can achieve a better guarantee than* $\frac{1-e^{-2}}{2} \approx 0.432$ *relative to the LP upper bound* $UP(\mathcal{H})$.

Our guarantee of 0.3557 relative to $\mathrm{UP}(\mathcal{H})$ in the unit-density case demonstrates the power of using our invariant-based analysis instead of a large/small analysis; we leave the possibility of tightening the guarantee relative to the upper bound of 0.432 as future work.

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Appendix. Proofs of Lemmas, Propositions and Theorems

A. Proofs in Section 3

A1. Proof of Lemma 1

Proof: Denote $\{x_{jt}^{\text{off}}(\boldsymbol{I})\}_{j=1,\dots,m,t=1,\dots,T}$ as the optimal solution to $V^{\text{off}}(\boldsymbol{I})$ in (2) and for each t, we denote $x_{jt}(\mathbf{r}_t,\mathbf{d}_t)=\mathbb{E}_{\boldsymbol{I}\sim\mathcal{H}}[x_{jt}^{\text{off}}(\boldsymbol{I})|\boldsymbol{I}_t=(\mathbf{r}_t,\mathbf{d}_t)]$, where $\boldsymbol{I}_t$ denotes the reward of query t in the sequence $\boldsymbol{I}$. Then, we must have

$$0\le x_{jt}(\mathbf{r}_t,\mathbf{d}_t)\le 1,\quad \forall t,\ \forall(\mathbf{r}_t,\mathbf{d}_t),\quad \text{and}\quad \sum_{j=1}^{m}x_{jt}(\mathbf{r}_t,\mathbf{d}_t)\le 1,\quad \forall t$$

and

$$1\ge \mathbb{E}_{\boldsymbol{I}\sim\mathcal{H}}[\sum_{t=1}^{T}d_{jt}\cdot x_{jt}^{\text{off}}(\boldsymbol{I})]=\sum_{t=1}^{T}\mathbb{E}_{(\mathbf{r}_t,\mathbf{d}_t)\sim F_t}[\mathbb{E}_{\boldsymbol{I}\sim\mathcal{H}}[d_{jt}\cdot x_{jt}^{\text{off}}(\boldsymbol{I})|\boldsymbol{I}_t=(\mathbf{r}_t,\mathbf{d}_t)]]=\sum_{t=1}^{T}\mathbb{E}_{(\mathbf{r}_t,\mathbf{d}_t)\sim F_t}[d_{jt}\cdot x_{jt}(\mathbf{r}_t,\mathbf{d}_t)]$$

Thus, we conclude that $\{x_{jt}(\mathbf{r}_t,\mathbf{d}_t)\}$ is a feasible solution to $\text{UP}(\mathcal{H})$ in (4) and it holds that

$$\mathbb{E}_{\boldsymbol{I}\sim\mathcal{H}}[V^{\text{off}}(\boldsymbol{I})]=\mathbb{E}_{\boldsymbol{I}\sim\mathcal{H}}[\sum_{t=1}^{T}\sum_{j=1}^{m}\tilde{r}_{jt}\cdot x_{jt}^{\text{off}}(\boldsymbol{I})]=\sum_{t=1}^{T}\mathbb{E}_{(\mathbf{r}_t,\mathbf{d}_t)\sim F_t}[\sum_{j=1}^{m}r_{jt}\cdot x_{jt}(\mathbf{r}_t,\mathbf{d}_t)]\le\text{UP}(\mathcal{H})$$

which completes our proof. $\square$

A2. Proof of Lemma 3

Proof: The only part that requires a proof is to establish the strong duality between Primal$(\boldsymbol{p}, \boldsymbol{D})$ and Dual$(\boldsymbol{p}, \boldsymbol{D})$. We first show that Primal$(\boldsymbol{p}, \boldsymbol{D})$ can be re-formulated as the following LP:

$$
\begin{aligned}
\mathrm{LP}(\boldsymbol{p}, \boldsymbol{D}) = \min \quad & V_1(1) && (24)\\
\text{s.t.} \quad & V_t(c_t) \geq V_{t+1}(c_t) + \sum_{d_t: d_t \leq c_t} p_t(d_t) \cdot W_t(d_t, c_t), \quad \forall t, \forall c_t && (24a)\\
& W_t(d_t, c_t) \geq r(d_t) + V_{t+1}(c_t - d_t) - V_{t+1}(c_t), \quad \forall t, \forall c_t \geq d_t && (24b)\\
& \sum_{t=1}^{T} \sum_{d_t} p_t(d_t) \cdot r(d_t) = 1 && (24c)\\
& V_{T+1}(c) = 0, \quad \forall c\\
& V_t(c_t), W_t(c_t) \geq 0, r(d_t) \geq 0 \quad \forall t, \forall c_t
\end{aligned}
$$

For each t and d_t, we denote by

$$
r(d_t) = \frac{1}{p_t(d_t)} \cdot \sum_{\omega_t: d(\omega_t) = d_t} r(\omega_t) \cdot p_t(\omega_t)
$$

We have that

$$
\sum_{t=1}^{T} \sum_{d_t} p_t(d_t) \cdot r(d_t) = \sum_{t=1}^{T} \sum_{d_t} \sum_{\omega_t: d(\omega_t) = d_t} r(\omega_t) \cdot p_t(\omega_t) = \sum_{t=1}^{T} \sum_{\omega_t} r(\omega_t) \cdot p_t(\omega_t) = 1
$$

Thus, we get constraint (24c) in LP$(\boldsymbol{p}, \boldsymbol{D})$ in (24). We also denote by

$$
W_t(d_t, c_t) = \frac{1}{p_t(d_t)} \cdot \sum_{\omega_t: d(\omega_t) = d_t} p_t(\omega_t) \cdot W_t(\omega_t, c_t) \tag{25}
$$

Then we have that

$$
\mathbb{E}_{\omega_t \sim F_t} \left[1_{\{d(\omega_t) \leq c_t\}} \cdot W_t(\omega_t, c_t) \right] = \sum_{d_t: d_t \leq c_t} p_t(d_t) \cdot W_t(d_t, c_t)
$$

which enables us to get constraint (24a) in LP$(\boldsymbol{p}, \boldsymbol{D})$. Finally, multiplying both sides of (7b) by $p(\omega_t)$ and summing over all ω_t satisfying $d(\omega_t) = d_t$, we get

$$
\begin{aligned}
p_t(d_t) \cdot W_t(d_t, c_t) &= \sum_{\omega_t: d(\omega_t) = d_t} p(\omega_t) \cdot W_t(\omega_t, c_t)\\
&\geq \sum_{\omega_t: d(\omega_t) = d_t} p(\omega_t) \cdot (r(\omega_t) + V_{t+1}(c_t - d(\omega_t)) - V_{t+1}(c_t))\\
&= p_t(d_t) \cdot (r(d_t) + V_{t+1}(c_t - d_t) - V_{t+1}(c_t))
\end{aligned}
$$

Thus, we get constraint (24b) in LP$(\boldsymbol{p}, \boldsymbol{D})$. Note that the constraint (7b) in Primal$(\boldsymbol{p}, \boldsymbol{D})$ can also be recovered from constraint (24b) in LP$(\boldsymbol{p}, \boldsymbol{D})$, by setting

$$
W_t(\omega_t, c_t) = W_t(d_t, c_t) \text{ and } r(\omega_t) = r(d_t)
$$

for all ω_t satisfying $d(\omega_t) = d_t$, and all d_t. Thus, we conclude that $\mathrm{LP}(\boldsymbol{p}, \boldsymbol{D})$ is a re-formulation of $\mathrm{Primal}(\boldsymbol{p}, \boldsymbol{D})$ and it holds that

$$\inf_{\hat{\mathcal{H}}} \frac{V_1^*(1)}{\mathbf{UP}(\hat{\mathcal{H}})} = \inf_{(\boldsymbol{p},\boldsymbol{D}):\sum_{t=1}^T \sum_{d_t} p_t(d_t)\cdot d_t \le 1} \mathrm{Primal}(\boldsymbol{p}, \boldsymbol{D}) = \inf_{(\boldsymbol{p},\boldsymbol{D}):\sum_{t=1}^T \sum_{d_t} p_t(d_t)\cdot d_t \le 1} \mathrm{LP}(\boldsymbol{p}, \boldsymbol{D})$$

It remains to show that $\mathrm{LP}(\boldsymbol{p}, \boldsymbol{D}) = \mathrm{Dual}(\boldsymbol{p}, \boldsymbol{D})$ for each $(\boldsymbol{p}, \boldsymbol{D})$ satisfying $\sum_{t=1}^T \sum_{d_t} p_t(d_t) \cdot d_t \le 1$.

We introduce the dual variable $\kappa_t(c_t)$ for constraint (24a), the dual variable $\alpha_t(d_t, c_t)$ for constraint (24b) and a dual variable θ for constraint (24c). Then we have the following LP as the dual of $\mathrm{LP}(\boldsymbol{p}, \boldsymbol{D})$.

$$\begin{aligned}
\max\quad & \theta \\
\text{s.t.}\quad & \theta \cdot p_t(d_t) \le \sum_{c_t \ge d_t} \alpha_t(d_t, c_t), \quad \forall t, \forall d_t \\
& \alpha_t(d_t, c_t) \le p_t(d_t) \cdot \kappa_t(c_t), \quad \forall t, \forall d_t, \forall c_t \ge d_t \\
& \kappa_t(c_t) = \kappa_{t-1}(c_t) + \sum_{d_{t-1}} (\alpha_{t-1}(d_{t-1}, c_t + d_{t-1}) - \alpha_{t-1}(d_{t-1}, c_t)), \quad \forall t, \forall c_t \le 1 \\
& \kappa_1(1) = 1 \\
& \alpha_t(d_t, c_t) = 0, \quad \forall t, \forall d_t, \forall c_t > 1 \\
& \theta, \alpha_t(c_t) \ge 0, \kappa_t(c_t) \ge 0, \quad \forall t, \forall c_t
\end{aligned} \tag{26}$$

Note that from the binding constraints, it must hold

$$\kappa_t(c_t) = \kappa_1(c_t) + \sum_{\tau \le t-1} \sum_{d_\tau} (\alpha_\tau(d_\tau, c_t + d_\tau) - \alpha_\tau(d_\tau, c_t))$$

where we set $\kappa_1(c) = 0$ for all $c < 1$. Thus, we further simplify the LP (26) by eliminating variable $\kappa_t(c_t)$, and derive LP $\mathrm{Dual}(\boldsymbol{p}, \boldsymbol{D})$ in (8).

In order to establish the strong duality between $\mathrm{LP}(\boldsymbol{p}, \boldsymbol{D})$ and $\mathrm{Dual}(\boldsymbol{p}, \boldsymbol{D})$, we only need to check the so-called Slater's condition such that all the inequality constraints in $\mathrm{LP}(\boldsymbol{p}, \boldsymbol{D})$ can be satisfied as strict inequality. Clearly, this can be done by setting $\theta = 0$ and iteratively setting $\alpha_t(d_t, c_t)$ to be strictly smaller than the RHS of the constraints (24b), (24c) for $t = 1, \ldots, T$. Thus, our proof is completed. □

B. Proofs in Section 4

B1. Proof of Lemma 4

We first present the following lemma, which show that instead of checking whether all the constraints of $\mathrm{Dual}(\boldsymbol{p}, \boldsymbol{D})$ are satisfied, it is enough to consider only one constraint.

Lemma 9 *For any $\theta \in [0, 1]$, $\{\theta, x_{l,t}(\theta)\}$ is a feasible solution to $\mathrm{Dual}(\boldsymbol{p}, k)$ if and only if $\sum_{\tau=1}^{T-1} x_{k,\tau}(\theta) \le 1 - \theta$.*

The proof is relegated to Section B2. We now prove the condition on θ such that $\sum_{\tau=1}^{T-1} x_{k,\tau}(\theta) \leq 1-\theta$. Due to Lemma 9, this condition implies the feasibility condition of $\{\theta, x_{l,t}(\theta)\}$. Specifically, we will first show that the term $\sum_{t=1}^{T-1} x_{k,t}(\theta)$ is continuously monotone increasing with θ in the next lemma, where the formal proof is in Section B3.

Lemma 10 *For any $1 \leq l \leq k$ and any $1 \leq t \leq T$, define $y_{l,t}(\theta) = \sum_{\tau=1}^{t} x_{l,t}(\theta)$. Then $y_{l,t}(\theta)$ is monotone increasing with θ and $y_{l,t}(\theta)$ is also Lipschitz continuous with θ.*

We are now ready to prove Lemma 4.

Proof of Lemma 4: Note that when $\theta = 0$, $y_{k,T-1}(0) = \sum_{\tau=1}^{T-1} x_{k,\tau}(0) = 0 < 1-\theta = 1$, and when $\theta = 1$, $y_{k,T-1}(1) = \sum_{\tau=1}^{T-1} x_{k,\tau}(1) > 0 = 1-\theta$. Further note that $1-\theta$ is continuously strictly decreasing with θ while Lemma 10 shows that $y_{k,T-1}(\theta) = \sum_{\tau=1}^{T-1} x_{k,\tau}(\theta)$ is continuously increasing with θ, there must exist a unique $\theta^* \in [0,1]$ such that $\sum_{\tau=1}^{T-1} x_{k,\tau}(\theta^*) = 1-\theta^*$ and for any $\theta \in [0,\theta^*]$, it holds that $\sum_{\tau=1}^{T-1} x_{k,\tau}(\theta) \leq 1-\theta$. Combining the above arguments with Lemma 9, we complete our proof. $\square$

B2. Proof of Lemma 9

We first prove that for any $\theta \in [0,1]$, $\{x_{l,t}(\theta)\}$ are non-negative.

Lemma 11 *For any $\theta \in [0,1]$, we have $x_{l,t}(\theta) \geq 0$ for any $l = 1,\dots,k$ and $t = 1,\dots,T$.*

Proof: We now use induction on l to show that for any l, we have that $x_{l,t}(\theta) \geq 0$ and $\sum_{v=1}^{l} x_{v,t}(\theta) \leq \theta \cdot p_t$ for any t. Since we focus on a fixed θ, we abbreviate θ in the expression $x_{l,t}(\theta)$ and substitute $x_{l,t}$ for $x_{l,t}(\theta)$ in the proof.

For $l = 1$, from definition, we have that for $1 \leq t \leq t_2$, it holds that $x_{1,t} \geq 0$ and $\sum_{v=1}^{1} x_{v,t} \leq \theta \cdot p_t$. We now use induction on t to show that for $t_2 + 1 \leq t \leq T$, we have that $0 \leq x_{1,t} \leq \theta \cdot p_t$. Note that from definition, we have that

$$x_{1,t_2+1} = p_{t_2+1} \cdot (1 - \sum_{\tau=1}^{t_2} \theta \cdot p_\tau) < p_{t_2+1} \cdot \theta$$

Also, note that $1 - \sum_{\tau=1}^{t_2-1} \theta \cdot p_\tau \geq \theta$ and $p_{t_2} \leq 1$, we have that

$$1 - \sum_{\tau=1}^{t_2} \theta \cdot p_\tau \geq 1 - \sum_{\tau=1}^{t_2-1} \theta \cdot p_\tau - \theta \geq 0$$

Thus, it holds that

$$0 \leq x_{1,t_2+1} = p_{t_2+1} \cdot (1 - \sum_{\tau=1}^{t_2} \theta \cdot p_\tau) < p_{t_2+1} \cdot \theta$$

Now, suppose for a t such that $t_2+1\le t\le T$, we have that $0\le x_{1,\tau}\le\theta\cdot p_\tau$ for any $t_2+1\le\tau\le t$. Then we have that

$$x_{1,t+1}\le p_{t+1}\cdot(1-\sum_{\tau=1}^{t}x_{1,\tau})\le p_{t+1}\cdot(1-\sum_{\tau=1}^{t_2}x_{1,\tau})=p_{t+1}\cdot(1-\sum_{\tau=1}^{t_2}\theta\cdot p_\tau)<p_{t+1}\cdot\theta$$

Also, note that $x_{1,t}\ge 0$ implies that $1-\sum_{\tau=1}^{t-1}x_{1,\tau}\ge 0$, we have that

$$x_{1,t+1}/p_{t+1}=1-\sum_{\tau=1}^{t-1}x_{1,\tau}-x_{1,t}=(1-p_t)\cdot(1-\sum_{\tau=1}^{t-1}x_{1,\tau})\ge 0$$

It holds that $0\le x_{1,t+1}\le p_{t+1}\cdot\theta$. Thus, from induction, for any t, we have proved that $0\le x_{1,t}\le p_t\cdot\theta$.

Suppose that for a l such that $1\le l\le k$, we have that $x_{l,t}\ge 0$ and $\sum_{v=1}^{l}x_{v,t}\le\theta\cdot p_t$ for any t. We now consider the case for $l+1$. From definition, $x_{l+1,t}=0$ when $1\le t\le t_{l+1}$ and when $t_{l+1}+1\le t\le t_{l+2}$, $x_{l+1,t}=\theta\cdot p_t-\sum_{v=1}^{l}x_{v,t}$. Thus, for $1\le t\le t_{l+2}$, we have proved that $x_{l+1,t}\ge 0$ and $\sum_{v=1}^{l+1}x_{v,t}\le\theta\cdot p_t$. We now use induction on t for $t>t_{l+2}$. When $t=t_{l+2}+1$, from definition, we have that

$$x_{l+1,t_{l+2}+1}=p_{t_{l+2}+1}\cdot\sum_{\tau=1}^{t_{l+2}}(x_{l,\tau}-x_{l+1,\tau})\le\theta\cdot p_{t_{l+2}+1}-\sum_{v=1}^{l}x_{v,t_{l+2}+1}\Rightarrow\sum_{v=1}^{l+1}x_{v,t_{l+2}+1}\le\theta\cdot p_{t_{l+2}+1}$$

Also, note that

$$0\le x_{l+1,t_{l+2}}=\theta\cdot p_{t_{l+2}}-\sum_{v=1}^{l}x_{v,t_{l+2}}\le p_{t_{l+2}}\cdot\sum_{\tau=1}^{t_{l+2}-1}(x_{l,\tau}-x_{l+1,\tau})$$

we get that

$$x_{l+1,t_{l+2}+1}/p_{t_{l+2}+1}=\sum_{\tau=1}^{t_{l+2}}(x_{l,\tau}-x_{l+1,\tau})\ge\sum_{\tau=1}^{t_{l+2}-1}(x_{l,\tau}-x_{l+1,\tau})-x_{l+1,t_{l+2}}\ge(1-p_{t_{l+2}})\cdot\sum_{\tau=1}^{t_{l+2}-1}(x_{l,\tau}-x_{l+1,\tau})$$

Thus, we proved that $0\le x_{l+1,t_{l+2}+1}$ and $\sum_{v=1}^{l+1}x_{v,t_{l+2}+1}\le\theta\cdot p_{t_{l+2}+1}$. Now suppose that for a t such that $t_{l+2}+1\le t\le T$, it holds that $0\le x_{l+1,t}$ and $\sum_{v=1}^{l+1}x_{v,t}\le\theta\cdot p_t$. Then we have that

$$\sum_{v=1}^{l+1}x_{v,t+1}/p_{t+1}=1-\sum_{\tau=1}^{t}x_{l+1,\tau}\le 1-\sum_{\tau=1}^{t-1}x_{l+1,\tau}=\sum_{v=1}^{l+1}x_{v,t}/p_t\le\theta$$

Also, note that $0\le x_{l+1,t}=p_t\cdot\sum_{\tau=1}^{t-1}(x_{l,\tau}-x_{l+1,\tau})$, we have that

$$x_{l+1,t+1}/p_{t+1}=\sum_{\tau=1}^{t}(x_{l,\tau}-x_{l+1,\tau})\ge\sum_{\tau=1}^{t-1}(x_{l,\tau}-x_{l+1,\tau})-x_{l+1,\tau}=(1-p_t)\cdot\sum_{\tau=1}^{t-1}(x_{l,\tau}-x_{l+1,\tau})\ge 0$$

Thus, we have proved that $0\le x_{l+1,t+1}$ and $\sum_{v=1}^{l+1}x_{v,t+1}\le\theta\cdot p_{t+1}$. From the induction on t, we can conclude that for any $1\le t\le T$, it holds that $0\le x_{l+1,t}$ and $\sum_{v=1}^{l+1}x_{v,t}\le\theta\cdot p_t$. Again, from the induction on l, we can conclude that for any $1\le l\le k$ and any $1\le t\le T$, it holds that $0\le x_{l,t}$ and $\sum_{v=1}^{l}x_{v,t}\le\theta\cdot p_t$, which completes our proof. □

Now we are ready to prove Lemma 9.

Proof of Lemma 9: When $\{x_{l,t}(\theta)\}$ is feasible to Dual$(\boldsymbol{p},k)$ in (9), we get from constraint (9b) and (9c) that

$$x_{1,T}(\theta)\leq p_T\cdot(1-\sum_{t=1}^{T-1}x_{1,t}(\theta)) \text{ and } x_{l,T}(\theta)\leq p_T\cdot\sum_{t=1}^{T-1}(x_{l-1,t}(\theta)-x_{l,t}(\theta)) \quad \forall l=2,\dots,k$$

Summing up the above inequalities, we get

$$\sum_{l=1}^{k}x_{l,T}(\theta)\leq p_T\cdot(1-\sum_{t=1}^{T-1}x_{k,t}(\theta))$$

Further note that by definition, we have $\sum_{l=1}^{k}x_{l,T}(\theta)=\theta\cdot p_T$. Thus, we show that $\{x_{l,t}(\theta)\}$ is feasible implies that $\sum_{t=1}^{T-1}x_{k,t}(\theta)\leq 1-\theta$.

Now we prove the reverse direction. Note that from the definition of $\{x_{l,t}(\theta)\}$, we have that $x_{l,t}(\theta)\leq p_t\cdot\sum_{\tau=1}^{t-1}(x_{l-1,\tau}(\theta)-x_{l,\tau}(\theta))$ holds for any $1\leq l\leq k-1$ and any $1\leq t\leq T$, where we set $\sum_{\tau=1}^{t-1}x_{0,\tau}(\theta)=1$ for any t for simplicity. Also, $\{x_{l,t}(\theta)\}$ are nonnegative as shown by Lemma 11. Thus, we have that

$$\{x_{l,t}(\theta)\}\text{ is feasible }\Leftrightarrow x_{k,t}(\theta)\leq p_t\cdot\sum_{\tau=1}^{t-1}(x_{k-1,\tau}(\theta)-x_{k,\tau}(\theta))\text{ holds for any }t_k+1\leq t\leq T$$

Moreover, note that from definition, for $t_k+1\leq t\leq T$, we have that $x_{l,t}(\theta)=p_t\cdot\sum_{\tau=1}^{t-1}(x_{l-1,\tau}(\theta)-x_{l,\tau}(\theta))$ when $1\leq l\leq k-1$. Thus, for $t_k+1\leq t\leq T$, we have that

$$x_{k,t}(\theta)\leq p_t\cdot\sum_{\tau=1}^{t-1}(x_{k-1,\tau}(\theta)-x_{k,\tau}(\theta))\Leftrightarrow\theta=\sum_{v=1}^{k}x_{v,t}/p_t\leq 1-\sum_{\tau=1}^{t-1}x_{k,\tau}(\theta)$$

From the nonnegativity of $\{x_{l,t}(\theta)\}$, we know that $\sum_{\tau=1}^{t-1}x_{k,\tau}(\theta)$ is monotone increasing with t. Thus, it holds that

$$\{x_{l,t}(\theta)\}\text{ is feasible }\Leftrightarrow\theta\leq 1-\sum_{t=1}^{T-1}x_{k,t}(\theta)$$

which completes our proof. $\square$

B3. Proof of Lemma 10

Proof: For any fixed $\theta\in[0,1]$ and any fixed $\Delta\geq 0$ such that $\theta+\Delta\in[0,1]$, we compare between $\{x_{l,t}(\theta)\}$ and $\{x_{l,t}(\theta+\Delta)\}$. Since we consider for a fixed θ and Δ, for notation brevity, we will omit θ and Δ by substituting $\{x_{l,t}\}$ for $\{x_{l,t}(\theta)\}$ and substituting $\{x'_{l,t}\}$ for $\{x_{l,t}(\theta+\Delta)\}$. Respectively, we denote $y_{l,t}=\sum_{\tau=1}^{t-1}x_{l,\tau}$ and $y'_{l,t}=\sum_{\tau=1}^{t-1}x'_{l,\tau}$. Also, we denote $\{t_l\}$ to be the time indexes associated with $\{x_{l,t}\}$ in the definition of $\{x_{l,t}\}$ and $\{t'_l\}$ to be the time indexes associated with $\{x'_{l,t}\}$. We will use induction to show that for each l, we have that $y_{l,t}\le y'_{l,t}$ and $\sum_{v=1}^{l} y'_{v,t}\le\sum_{v=1}^{l} y_{v,t}+\Delta\cdot\sum_{\tau=1}^{t} p_\tau$ hold for each t.

For the case $l=1$, obviously we have that $t'_2\le t_2$. When $1\le t\le t'_2$, from definition, it holds that $y_{1,t}\le y'_{1,t}\le y_{1,t}+\Delta\cdot\sum_{\tau=1}^{t} p_\tau$. We now use induction on t for $t'_2+1\le t\le t_2$. When $t=t'_2+1$, note that

$$x'_{1,t'_2+1}=p_{t'_2+1}\cdot(1-y'_{1,t'_2})\le(\theta+\Delta)\cdot p_{t'_2+1}\quad\text{and}\quad x_{1,t'_2+1}=\theta\cdot p_{t'_2+1}\le p_{t'_2+1}\cdot(1-y_{1,t'_2})$$

we have that

$$y_{1,t'_2+1}=y_{1,t'_2}+x_{1,t'_2+1}\le p_{t'_2+1}+(1-p_{t'_2+1})\cdot y_{1,t'_2}\le p_{t'_2+1}+(1-p_{t'_2+1})\cdot y'_{1,t'_2}=y'_{1,t'_2+1}$$

and

$$y'_{1,t'_2+1}=y'_{1,t'_2}+x'_{1,t'_2+1}\le y_{1,t'_2}+\Delta\cdot\sum_{t=1}^{t'_2} p_t+(\theta+\Delta)\cdot p_{t'_2+1}=y_{1,t'_2+1}+\Delta\cdot\sum_{t=1}^{t'_2+1} p_t$$

Now suppose for a fixed t satisfying $t'_2+1\le t\le t_2-1$, it holds $y_{1,t}\le y'_{1,t}\le y_{1,t}+\Delta\cdot\sum_{\tau=1}^{t} p_\tau$. From definition, note that

$$x'_{1,t+1}=p_{t+1}\cdot(1-y'_{1,t})\le(\theta+\Delta)\cdot p_{t+1}\quad\text{and}\quad x_{1,t+1}=\theta\cdot p_{t+1}\le p_{t+1}\cdot(1-y_{1,t})$$

we have

$$y_{1,t+1}=y_{1,t}+x_{1,t+1}\le p_{t+1}+(1-p_{t+1})\cdot y_{1,t}\le p_{t+1}+(1-p_{t+1})\cdot y'_{1,t}=y'_{1,t+1}$$

and

$$y'_{1,t+1}=y'_{1,t}+x'_{1,t+1}\le y_{1,t}+\Delta\cdot\sum_{\tau=1}^{t} p_\tau+(\theta+\Delta)\cdot p_{t+1}=y_{1,t+1}+\Delta\cdot\sum_{\tau=1}^{t+1} p_\tau$$

Thus, from induction on t, we conclude that $y_{1,t}\le y'_{1,t}\le y_{1,t}+\Delta\cdot\sum_{\tau=1}^{t} p_\tau$ holds for any $t'_2+1\le t\le t_2$. Finally, when $t\ge t_2+1$, note that

$$y_{1,t}=p_t+(1-p_t)\cdot y_{1,t-1}\quad\text{and}\quad y'_{1,t}=p_t+(1-p_t)\cdot y'_{1,t-1}$$

which implies that

$$y'_{1,t}-y_{1,t}=(1-p_t)\cdot(y'_{1,t-1}-y_{1,t-1})=\cdots=(y'_{1,t_2}-y_{1,t_2})\cdot\prod_{\tau=t_2+1}^{t}(1-p_\tau)$$

Thus, we prove that for any $1\le t\le T$, it holds that $y_{1,t}\le y'_{1,t}\le y_{1,t}+\Delta\cdot\sum_{\tau=1}^{t} p_\tau$.

Suppose that for a fixed $1\le l\le k$, $y_{l,t}\le y'_{l,t}$ and $\sum_{v=1}^{l} y'_{v,t}\le\sum_{v=1}^{l} y_{v,t}+\Delta\cdot\sum_{\tau=1}^{t} p_\tau$ hold for each t. We now consider the case for $l+1$. When $t\le\min\{t_{l+2},t'_{l+2}\}$, from definition, we have that

$$\sum_{v=1}^{l+1} y'_{v,t}=(\theta+\Delta)\cdot\sum_{\tau=1}^{t} p_\tau\quad\text{and}\quad\sum_{v=1}^{l+1} y_{v,t}=\theta\cdot\sum_{\tau=1}^{t} p_\tau$$

which implies that $\sum_{v=1}^{l+1} y'_{v,t} \leq \sum_{v=1}^{l+1} y_{v,t} + \Delta \cdot \sum_{\tau=1}^{t} p_\tau$. Also, we have

$$y'_{l+1,t} - y_{l+1,t} = \Delta \cdot \sum_{\tau=1}^{t} p_\tau - \left(\sum_{v=1}^{l} y'_{v,t} - \sum_{v=1}^{l} y_{v,t} \right) \geq 0$$

where the last inequality holds from induction condition. Thus, we prove that $y_{l+1,t} \leq y'_{l+1,t}$ and $\sum_{v=1}^{l+1} y'_{v,t} \leq \sum_{v=1}^{l+1} y_{v,t} + \Delta \cdot \sum_{\tau=1}^{t} p_\tau$ hold for each $1 \leq t \leq \min\{t_{l+2}, t'_{l+2}\}$. Moreover, note that t_{l+2} is defined as the first time that $\theta > 1 - y_{l+1,t_{l+2}}$ while t'_{l+2} is defined as the first time that $\theta + \Delta > 1 - y'_{l+1,t'_{l+2}}$. Since $y'_{l+1,t} \geq y_{l+1,t}$ when $t \leq \min\{t_{l+2}, t'_{l+2}\}$, we must have $t'_{l+2} \leq t_{l+2}$. Then we use induction on t for $t'_{l+2} + 1 \leq t \leq t_{l+2}$. When $t = t'_{l+2} + 1 \leq t_{l+2}$, from definition, we have

$$x'_{l+1,t'_{l+2}+1} = p_{t'_{l+2}+1} \cdot (y'_{l,t'_{l+2}} - y'_{l+1,t'_{l+2}}) \Rightarrow y'_{l+1,t'_{l+2}+1} = p_{t'_{l+2}+1} \cdot y'_{l,t'_{l+2}} + (1 - p_{t'_{l+2}+1}) \cdot y'_{l+1,t'_{l+2}}$$

and

$$x_{l+1,t'_{l+2}+1} \leq p_{t'_{l+2}+1} \cdot (y_{l,t'_{l+2}} - y_{l+1,t'_{l+2}}) \Rightarrow y_{l+1,t'_{l+2}+1} \leq p_{t'_{l+2}+1} \cdot y_{l,t'_{l+2}} + (1 - p_{t'_{l+2}+1}) \cdot y_{l+1,t'_{l+2}}$$

Note that $y'_{l,t'_{l+2}} \geq y_{l,t'_{l+2}}$ and $y'_{l+1,t'_{l+2}} \geq y_{l+1,t'_{l+2}}$, we get $y'_{l+1,t'_{l+2}+1} \geq y_{l+1,t'_{l+2}+1}$. Moreover, note that from the definition of t'_{l+2}, we have

$$\sum_{v=1}^{l+1} x'_{v,t'_{l+2}+1} \leq p_{t'_{l+2}+1} \cdot (\theta + \Delta) = \sum_{v=1}^{l+1} x_{v,t'_{l+2}+1} + \Delta \cdot p_{t'_{l+2}+1}$$

which implies that

$$\begin{aligned}
\sum_{v=1}^{l+1} y'_{v,t'_{l+2}+1} &= \sum_{v=1}^{l+1} y'_{v,t'_{l+2}} + \sum_{v=1}^{l+1} x'_{v,t'_{l+2}+1} \leq \sum_{v=1}^{l+1} y_{v,t'_{l+2}} + \Delta \cdot \sum_{j=1}^{t'_{l+2}} p_j + \sum_{v=1}^{l+1} x_{v,t'_{l+2}+1} + \Delta \cdot p_{t'_{l+2}+1} \\
&= \sum_{v=1}^{l+1} y_{v,t'_{l+2}+1} + \Delta \cdot \sum_{j=1}^{t'_{l+2}+1} p_j
\end{aligned}$$

Then suppose for a fixed t satisfying $t'_{l+2} + 1 \leq t \leq t_{l+2} - 1$, it holds that $y_{l+1,t} \leq y'_{l+1,t}$ and $\sum_{v=1}^{l+1} y'_{v,t} \leq \sum_{v=1}^{l+1} y_{v,t} + \Delta \cdot \sum_{\tau=1}^{t} p_\tau$. From definition, we have

$$x'_{l+1,t+1} = p_{t+1} \cdot (y'_{l,t} - y'_{l+1,t}) \Rightarrow y'_{l+1,t+1} = p_{t+1} \cdot y'_{l,t} + (1 - p_{t+1}) \cdot y'_{l+1,t}$$

and

$$x_{l+1,t+1} \leq p_{t+1} \cdot (y_{l,t} - y_{l+1,t}) \Rightarrow y_{l+1,t+1} \leq p_{t+1} \cdot y_{l,t} + (1 - p_{t+1}) \cdot y_{l+1,t}$$

Note that $y'_{l,t} \geq y_{l,t}$ and $y'_{l+1,t} \geq y_{l+1,t}$, we have $y'_{l+1,t+1} \geq y_{l+1,t+1}$. Also, from the definition of t'_{l+2}, we have

$$\sum_{v=1}^{l+1} x'_{v,t+1} \leq p_{t+1} \cdot (\theta + \Delta) = \sum_{v=1}^{l+1} x_{v,t+1} + \Delta \cdot p_{t+1}$$

which implies that

$$\sum_{v=1}^{l+1} y'_{v,t+1} = \sum_{v=1}^{l+1} y'_{v,t} + \sum_{v=1}^{l+1} x'_{v,t+1} \le \sum_{v=1}^{l+1} y_{v,t} + \Delta \cdot \sum_{\tau=1}^{t} p_\tau + \sum_{v=1}^{l+1} x_{v,t+1} + \Delta \cdot p_{t+1}$$
$$= \sum_{v=1}^{l+1} y_{v,t+1} + \Delta \cdot \sum_{\tau=1}^{t+1} p_\tau$$

Thus, from induction on t, we prove that $y_{l+1,t} \le y'_{l+1,t}$ and $\sum_{v=1}^{l+1} y'_{v,t} \le \sum_{v=1}^{l+1} y_{v,t} + \Delta \cdot \sum_{\tau=1}^{t} p_\tau$ hold for any $t'_{l+2}+1 \le t \le t_{l+2}$. Finally, when $t \ge t_{l+2}+1$, note that

$$y_{l+1,t} = p_t \cdot y_{l,t-1} + (1-p_t) \cdot y_{l+1,t-1} \quad \text{and} \quad y'_{l+1,t} = p_t \cdot y'_{l,t-1} + (1-p_t) \cdot y'_{l+1,t-1}$$

which implies that

$$y'_{l+1,t} - y_{l+1,t} = p_t \cdot (y'_{l,t-1} - y_{l,t-1}) + (1-p_t) \cdot (y'_{l+1,t-1} - y_{l+1,t-1})$$

It is direct to show inductively on t such that $y_{l+1,t} \le y'_{l+1,t}$ and $\sum_{v=1}^{l+1} y'_{v,t} \le \sum_{v=1}^{l+1} y_{v,t} + \Delta \cdot \sum_{\tau=1}^{t} p_\tau$ hold for any $t \ge t_{l+2}+1$.

Thus, we have proved that for any $1 \le t \le T$, we have $y_{l+1,t} \le y'_{l+1,t}$ and $\sum_{v=1}^{l+1} y'_{v,t} \le \sum_{v=1}^{l} y_{v,t} + \Delta \cdot \sum_{\tau=1}^{t} p_\tau$. By the induction on l, we finally prove that for any $1 \le l \le k$, $y_{l,t} \le y'_{l,t}$ and $\sum_{v=1}^{l} y'_{v,t} \le \sum_{v=1}^{l} y_{v,t} + \Delta \cdot \sum_{\tau=1}^{t} p_\tau$ hold for any $1 \le t \le T$. In this way, we prove that $y_{l,t}(\theta)$ is monotone increasing with θ for any l, t. Moreover, note that since $\sum_{\tau=1}^{T} p_\tau \le k$, we have that $y_{l,t}(\theta+\Delta) \le y_{l,t}(\theta) + k \cdot \Delta$ hold for any θ, Δ and any l, t. Thus, $y_{l,t}(\theta)$ is a continuous function on θ, which completes our proof. $\square$

B4. Construction of $\{\beta^*_{l,t}, \xi^*_t\}$ and Proof of Theorem 1

Given Lemma 4, in order to prove Theorem 1, it is enough for us to construct a feasible solution $\{\beta^*_{l,t}, \xi^*_t\}$ to Primal$(\boldsymbol{p}, k)$ in (10) such that the primal-dual pair $\{\theta^*, x_{l,t}(\theta^*)\}$ and $\{\beta^*_{l,t}, \xi^*_t\}$ satisfies the complementary slackness conditions. Specifically, we will construct a feasible solution $\{\beta^*_{l,t}, \xi^*_t\}$ to Primal$(\boldsymbol{p}, k)$ satisfying the following conditions:

$$\beta^*_{1,t} \cdot \left(x_{1,t}(\theta^*) - p_t \cdot (1 - \sum_{\tau<t} x_{1,\tau}(\theta^*)) \right) = 0, \quad \forall t = 1, \dots, T \tag{27}$$
$$\beta^*_{l,t} \cdot \left(x_{l,t}(\theta^*) - p_t \cdot \sum_{\tau<t} (x_{l-1,\tau}(\theta^*) - x_{l,\tau}(\theta^*)) \right) = 0, \quad \forall t = 1, \dots, T, \forall l = 2, \dots, k$$
$$x_{l,t}(\theta^*) \cdot \left(\beta^*_{l,t} + \sum_{\tau>t} p_\tau \cdot (\beta^*_{l,\tau} - \beta^*_{l+1,\tau}) - \xi^*_t \right) = 0, \quad \forall t = 1, \dots, T, \forall l = 2, \dots, k$$
$$x_{k,t}(\theta^*) \cdot \left(\beta^*_{k,t} + \sum_{\tau>t} p_\tau \cdot \beta^*_{k,\tau} - \xi^*_t \right) = 0, \quad \forall t = 1, \dots, T$$

Note that from definitions, $\{x_{l,t}(\theta^*)\}$ satisfies the following conditions:

$$x_{l,t}(\theta^*) = 0 \leq p_t \cdot \sum_{\tau=1}^{t-1}(x_{l-1,\tau}(\theta^*) - x_{l,\tau}(\theta^*)), \quad \forall t \leq t_l$$
$$x_{l,t}(\theta^*) = \theta^* \cdot p_t - \sum_{v=1}^{l-1} x_{v,t}(\theta^*) \leq p_t \cdot \sum_{\tau=1}^{t-1}(x_{l-1,\tau}(\theta^*) - x_{l,\tau}(\theta^*)) \quad \text{for } t_l+1 \leq t \leq t_{l+1}$$

where $\{t_l\}$ are the time indexes associated with the definition of $\{x_{l,t}(\theta^*)\}$ and we define $t_1 = 0$, $t_{k+1} = T-1$. For simplicity, we also denote $\sum_{\tau=1}^{t-1} x_{0,\tau}(\theta^*) = 1$ for any t. Thus, in order for $\{\beta^*_{l,t}, \xi^*_t\}$ to satisfy the conditions in (27), it is enough for $\{\beta^*_{l,t}, \xi^*_t\}$ to be feasible to Primal$(\boldsymbol{p}, k)$ and satisfy the following conditions:

$$\beta^*_{l,t} = 0 \quad \text{for } t \leq t_{l+1} \tag{28}$$
$$\beta^*_{l,t} + \sum_{\tau=t+1}^{T} p_\tau \cdot (\beta^*_{l,\tau} - \beta^*_{l+1,\tau}) = \xi^*_t \quad \text{for } t \geq t_l + 1 \tag{29}$$

where we denote $\beta^*_{k+1,t} = 0$ for notation simplicity. We now show the construction of the solution $\{\beta^*_{l,t}, \xi^*_t\}$ to Primal$(\boldsymbol{p}, k)$. Define the following constants for each $l, q \in \{1, 2, \ldots, k\}$:

$$B_{l,q} = \sum_{t_l+1 \leq j_1 < j_2 < \cdots < j_q \leq t_{l+1}} \frac{p_{j_1} p_{j_2} \ldots p_{j_q}}{(1-p_{j_1})(1-p_{j_2})\ldots(1-p_{j_q})} \cdot \prod_{w=t_l+1}^{t_{l+1}} (1-p_w)$$

and we set $B_{l,0} = \prod_{w=t_l+1}^{t_{l+1}}(1-p_w)$. We also define the following terms for each $l, q \in \{1, 2, \ldots, k\}$ and each $t \in \{t_l+1, \ldots, t_{l+1}\}$, where $\{t_l\}$ are the time indexes defined in the construction of $\{\theta^*, x_{l,t}(\theta^*)\}$ and we define $t_1 = 0$, $t_{k+1} = T-1$:

$$A_{l,q}(t) = \sum_{t+1 \leq j_1 < j_2 < \cdots < j_q \leq t_{l+1}} \frac{p_{j_1} p_{j_2} \ldots p_{j_q}}{(1-p_{j_1})(1-p_{j_2})\ldots(1-p_{j_q})} \cdot \prod_{w=t+1}^{t_{l+1}} (1-p_w)$$

and we set $A_{l,0}(t) = \prod_{w=t+1}^{t_{l+1}}(1-p_w)$. Then our construction of the solution $\{\beta^*_{l,t}, \xi^*_t\}$ can be fully described as follows:

$$\begin{aligned}
&\xi^*_T = \beta^*_{l_1,T} = R, \quad \forall l_1 = 1, 2, \ldots, k \\
&\beta^*_{l_1,t} = 0, \quad \forall l_1 = 1, 2, \ldots, k, \forall t \leq t_{l_1+1} \\
&\xi^*_t = \phi_l \cdot p_T R, \quad \forall l = 1, 2, \ldots, k, \forall t_l + 1 \leq t \leq t_{l+1} \\
&\beta^*_{l_1,t} = p_T R \cdot \sum_{w=l_1}^{l_2-1} \delta_{w,l_2} \cdot A_{l_2, w-l_1}(t), \quad \forall l_1 = 1, 2, \ldots, k, \forall l_2 = l_1+1, \ldots, k, \forall t_{l_2} + 1 \leq t \leq t_{l_2+1}
\end{aligned} \tag{30}$$

where the parameters $\{\phi_l, \delta_{l_1,l_2}, R\}$ are defined as:

$$\delta_{l,k} = 1 \qquad \forall l = 1,2,\dots,k-1$$

$$\delta_{l,l} = 0 \qquad \forall l = 1,2,\dots,k$$

$$\delta_{l_1,l_2} = \sum_{w_0=l_1+1}^{l_2}\sum_{w_1=w_0}^{l_2+1}\sum_{w_2=w_1}^{l_2+2}\cdots\sum_{w_{k-1-l_2}=w_{k-2-l_2}}^{k-1} B_{l_2+1,w_1-w_0}\cdot B_{l_2+2,w_2-w_1}\dots B_{k-1,w_{k-1-l_2}-w_{k-2-l_2}}\cdot B_{k,k-w_{k-1-l_2}},$$
$$\forall l_2 = 1,2,\dots,k-1 \text{ and } l_1 = 1,2,\dots,l_2-1$$

$$\phi_k = 1$$

$$\phi_l = \sum_{q=l+1}^{k}\sum_{w=l+1}^{q}(\delta_{w-1,q}-\delta_{w,q})\cdot(1-\sum_{v=0}^{w-l-1}B_{q,v}) \quad \forall l = 1,2,\dots,k-1$$

and R is a positive constant such that $\sum_{t=1}^{T} p_t\cdot\xi_t^* = 1$. We then prove the feasibility of $\{\beta_{l,t}^*, \xi_t^*\}$ and the conditions (28), (29) are satisfied. Obviously, from definition, $\beta_{l,t}^*$ is nonnegative for each l and each t. We first prove that ξ_t^* is also nonnegative for each t.

Lemma 12 *For each $l_2 = 1,2,\dots,k$ and each $l_1 = 1,2,\dots,l_2-1$, we have that $\delta_{l_1,l_2} \geq \delta_{l_1+1,l_2}$.*

Proof: Note that when $l_2 = k$, we have that $\delta_{l,k} = 1$ for each $l = 1,2,\dots,k-1$, thus it holds that $\delta_{l,k} \geq \delta_{l+1,k}$. When $l_2 \leq k-1$, from definitions, we have that for each $l_1 = 1,2,\dots,l_2-1$

$$\delta_{l_1,l_2}-\delta_{l_1+1,l_2} = \sum_{w_1=l_1+1}^{l_2+1}\sum_{w_2=w_1}^{l_2+2}\cdots\sum_{w_{k-1-l_2}=w_{k-2-l_2}}^{k-1} B_{l_2+1,w_1-l_1-1}\cdot B_{l_2+2,w_2-w_1}\dots B_{k-1,w_{k-1-l_2}-w_{k-2-l_2}}\cdot B_{k,k-w_{k-1-l_2}}$$

which completes our proof. □

We then show that the term $1-\sum_{w=0}^{q} B_{l,w}$ is nonnegative for each l and each q. Note that the following lemma essentially implies that $\sum_{t=t_l+1}^{t_{l+1}} p_t\cdot A_{l,q}(t) = 1-\sum_{w=0}^{q} B_{l,w}$, by replacing i_1 with t_l and i_2 with t_{l+1} in (31), which establishes the nonnegativity of the term $1-\sum_{w=0}^{q} B_{l,w}$.

Lemma 13 *For each $q\in\{1,2,\dots,k\}$ and any $1\leq i_1+1\leq i_2\leq T$, it holds that*

$$\begin{aligned}&\sum_{t=i_1+1}^{i_2} p_t\cdot\sum_{t+1\leq j_1<j_2<\cdots<j_q\leq i_2}\frac{p_{j_1}p_{j_2}\dots p_{j_q}}{(1-p_{j_1})(1-p_{j_2})\dots(1-p_{j_q})}\cdot\prod_{v=t+1}^{i_2}(1-p_v)\\&=1-\sum_{w=0}^{q}\sum_{i_1+1\leq j_1<j_2<\cdots<j_w\leq i_2}\frac{p_{j_1}p_{j_2}\dots p_{j_w}}{(1-p_{j_1})(1-p_{j_2})\dots(1-p_{j_w})}\cdot\prod_{v=i_1+1}^{i_2}(1-p_v),\end{aligned}\tag{31}$$

Proof: We will do induction on q from $q=0$ to $q=k$ to prove (31). When $q=0$, we have that

$$\begin{aligned}\sum_{t=i_1+1}^{i_2} p_t\cdot\prod_{v=t+1}^{i_2}(1-p_v) &= \sum_{t=i_1+1}^{i_2}(1-(1-p_t))\cdot\prod_{v=t+1}^{i_2}(1-p_v) = \sum_{t=i_1+1}^{i_2}\left(\prod_{v=t+1}^{i_2}(1-p_v)-\prod_{v=t}^{i_2}(1-p_v)\right)\\&=1-\prod_{v=i_1+1}^{i_2}(1-p_v)\end{aligned}$$

Thus, we have (31) holds for $q=0$. Suppose (31) holds for $1,2,\ldots,q-1$, we consider the case for q. For any $1\le i_1+1\le i_2\le T$, we have that

$$\begin{aligned}
&\sum_{t=i_1+1}^{i_2} p_t\cdot \sum_{t+1\le j_1<j_2<\cdots<j_q\le i_2} \frac{p_{j_1}p_{j_2}\ldots p_{j_q}}{(1-p_{j_1})(1-p_{j_2})\ldots(1-p_{j_q})}\cdot\prod_{v=t+1}^{i_2}(1-p_v)\\
&=\sum_{t=i_1+1}^{i_2} p_t\cdot\sum_{j_1=t+1}^{i_2}\sum_{j_1<j_2<\cdots<j_q\le i_2} \frac{p_{j_1}p_{j_2}\ldots p_{j_q}}{(1-p_{j_1})(1-p_{j_2})\ldots(1-p_{j_q})}\cdot\prod_{v=t+1}^{i_2}(1-p_v)\\
&=\sum_{j_1=i_1+2}^{i_2}\sum_{t=i_1+1}^{j_1-1} p_t\cdot\frac{p_{j_1}}{1-p_{j_1}}\cdot\sum_{j_1<j_2<\cdots<j_q\le i_2}\frac{p_{j_2}\ldots p_{j_q}}{(1-p_{j_2})\ldots(1-p_{j_q})}\cdot\prod_{v=t+1}^{i_2}(1-p_v)\\
&=\sum_{j_1=i_1+2}^{i_2}\frac{p_{j_1}}{1-p_{j_1}}\cdot\sum_{j_1<j_2<\cdots<j_q\le i_2}\frac{p_{j_2}\ldots p_{j_q}}{(1-p_{j_2})\ldots(1-p_{j_q})}\cdot\prod_{v=j_1}^{i_2}(1-p_v)\cdot\sum_{t=i_1+1}^{j_1-1}p_t\cdot\prod_{v=t+1}^{j_1-1}(1-p_v)
\end{aligned}$$

where the second equality holds by exchanging the order of summation. Note that for induction purpose, we assume (31) holds for $q=0$, which implies that $\sum_{t=i_1+1}^{j_1-1}p_t\cdot\prod_{v=t+1}^{j_1-1}(1-p_v)=1-\prod_{v=i_1+1}^{j_1-1}(1-p_v)$. Then we have

$$\begin{aligned}
&\sum_{j_1=i_1+2}^{i_2}\frac{p_{j_1}}{1-p_{j_1}}\cdot\sum_{j_1<j_2<\cdots<j_q\le i_2}\frac{p_{j_2}\ldots p_{j_q}}{(1-p_{j_2})\ldots(1-p_{j_q})}\cdot\prod_{v=j_1}^{i_2}(1-p_v)\cdot\sum_{t=i_1+1}^{j_1-1}p_t\cdot\prod_{v=t+1}^{j_1-1}(1-p_v)\\
&=\sum_{j_1=i_1+2}^{i_2}p_{j_1}\cdot\sum_{j_1<j_2<\cdots<j_q\le i_2}\frac{p_{j_2}\ldots p_{j_q}}{(1-p_{j_2})\ldots(1-p_{j_q})}\cdot\prod_{v=j_1+1}^{i_2}(1-p_v)\cdot\left(1-\prod_{v=i_1+1}^{j_1-1}(1-p_v)\right)\\
&=\sum_{j_1=i_1+1}^{i_2}p_{j_1}\cdot\sum_{j_1<j_2<\cdots<j_q\le i_2}\frac{p_{j_2}\ldots p_{j_q}}{(1-p_{j_2})\ldots(1-p_{j_q})}\cdot\prod_{v=j_1+1}^{i_2}(1-p_v)\cdot\left(1-\prod_{v=i_1+1}^{j_1-1}(1-p_v)\right)
\end{aligned}$$

where the second equality holds by noting that when $j_1=i_1+1$, we have $1-\prod_{v=i_1+1}^{j_1-1}(1-p_v)=0$. Thus, it holds that

$$\begin{aligned}
&\sum_{t=i_1+1}^{i_2} p_t\cdot \sum_{t+1\le j_1<j_2<\cdots<j_q\le i_2} \frac{p_{j_1}p_{j_2}\ldots p_{j_q}}{(1-p_{j_1})(1-p_{j_2})\ldots(1-p_{j_q})}\cdot\prod_{v=t+1}^{i_2}(1-p_v)\\
&=\sum_{j_1=i_1+1}^{i_2}p_{j_1}\cdot\sum_{j_1<j_2<\cdots<j_q\le i_2}\frac{p_{j_2}\ldots p_{j_q}}{(1-p_{j_2})\ldots(1-p_{j_q})}\cdot\prod_{v=j_1+1}^{i_2}(1-p_v)\cdot\left(1-\prod_{v=i_1+1}^{j_1-1}(1-p_v)\right)
\end{aligned}$$

Note that for the induction purpose, we assume that (31) holds for $q-1$. Then, we have that

$$\begin{aligned}
&\sum_{j_1=i_1+1}^{i_2}p_{j_1}\cdot\sum_{j_1<j_2<\cdots<j_q\le i_2}\frac{p_{j_2}\ldots p_{j_q}}{(1-p_{j_2})\ldots(1-p_{j_q})}\cdot\prod_{v=j_1+1}^{i_2}(1-p_v)\\
&=\sum_{t=i_1+1}^{i_2}p_t\cdot\sum_{t+1\le j_1<j_2<\cdots<j_{q-1}\le i_2}\frac{p_{j_1}p_{j_2}\ldots p_{j_{q-1}}}{(1-p_{j_1})(1-p_{j_2})\ldots(1-p_{j_{q-1}})}\cdot\prod_{v=t+1}^{i_2}(1-p_v)\\
&=1-\sum_{w=0}^{q-1}\sum_{i_1+1\le j_1<j_2<\cdots<j_w\le i_2}\frac{p_{j_1}p_{j_2}\ldots p_{j_w}}{(1-p_{j_1})(1-p_{j_2})\ldots(1-p_{j_w})}\cdot\prod_{v=i_1+1}^{i_2}(1-p_v)
\end{aligned}$$

where the second equality holds from replacing the index j_{l+1} with j_l for $l=2,\dots,q$ and replace the index j_1 with t. Also, note that

$$\sum_{j_1=i_1+1}^{i_2} p_{j_1}\cdot \sum_{j_1<j_2<\dots<j_q\le i_2} \frac{p_{j_2}\dots p_{j_q}}{(1-p_{j_2})\dots(1-p_{j_q})}\cdot \prod_{v=j_1+1}^{i_2}(1-p_v)\cdot \prod_{v=i_1+1}^{j_1-1}(1-p_v)$$
$$= \sum_{i_1+1\le j_1<j_2<\dots<j_q\le i_2} \frac{p_{j_1}p_{j_2}\dots p_{j_q}}{(1-p_{j_1})(1-p_{j_2})\dots(1-p_{j_q})}\cdot \prod_{v=i_1+1}^{i_2}(1-p_v)$$

Thus, we have that

$$\sum_{t=i_1+1}^{i_2} p_t\cdot \sum_{t+1\le j_1<j_2<\dots<j_q\le i_2} \frac{p_{j_1}p_{j_2}\dots p_{j_q}}{(1-p_{j_1})(1-p_{j_2})\dots(1-p_{j_q})}\cdot \prod_{v=t+1}^{i_2}(1-p_v)$$
$$= \sum_{j_1=i_1+1}^{i_2} p_{j_1}\cdot \sum_{j_1<j_2<\dots<j_q\le i_2} \frac{p_{j_2}\dots p_{j_q}}{(1-p_{j_2})\dots(1-p_{j_q})}\cdot \prod_{v=j_1+1}^{i_2}(1-p_v)\cdot\left(1-\prod_{v=i_1+1}^{j_1-1}(1-p_v)\right)$$
$$= 1-\sum_{w=0}^{q}\sum_{i_1+1\le j_1<j_2<\dots<j_w\le i_2} \frac{p_{j_1}p_{j_2}\dots p_{j_w}}{(1-p_{j_1})(1-p_{j_2})\dots(1-p_{j_w})}\cdot \prod_{v=i_1+1}^{i_2}(1-p_v)$$

which completes our proof by induction on q. $\square$

Combining Lemma 12 and Lemma 13, we draw the following conclusion.

Lemma 14 *For each $l=1,2,\dots,k$ and each $t=1,2,\dots,T$, we have that $\beta^*_{l,t}\ge 0$ and $\xi^*_t\ge 0$.*

Proof: Note that from definition, $\beta^*_{l,t}\ge 0$ for each l and t. We then show the non-negativity of ξ^*_t for each t. Note that Lemma 12 shows that $\delta_{l_1,l_2}\ge\delta_{l_1+1,l_2}$ for each $l_2=1,2,\dots,k$ and each $l_1=1,2,\dots,l_2-1$. It only remains to show the non-negativity of the term $1-\sum_{w=0}^{q}B_{l,w}$, which can be directly established by Lemma 13. Specifically, by replacing i_1 with t_l and i_2 with t_{l+1} in (31), we have $1-\sum_{w=0}^{q}B_{l,w}=\sum_{t=t_l+1}^{t_{l+1}}p_t\cdot A_{l,q}(t)\ge 0$. $\square$

From the definition of $\{\beta^*_{l,t},\xi^*_t\}$, condition (28) holds obviously. We then prove that condition (29) is satisfied.

Lemma 15 *For each $l=1,2,\dots,k$ and each $t\ge t_l+1$, it holds that*

$$\beta^*_{l,t}+\sum_{j=t+1}^{T}p_j\cdot(\beta^*_{l,j}-\beta^*_{l+1,j})=\xi^*_t$$

where we denote $\beta^*_{k+1,t}=0$ for notation simplicity.

Proof: When $l=k$, from definition, we have $\beta^*_{l,j}=0$ for each $j\le t_{l+1}=T-1$ and $\beta^*_{l,T}=R$, thus the lemma holds directly. When $t=T$, it is also direct to show from definition that the lemma holds. We then focus on the case where $l\le k-1$ and $t\le T-1$.

For a fixed $l \leq k-1$ and a fixed $t_l+1 \leq t \leq T-1$, we denote an index $l_1 \geq l$ such that $t_{l_1}+1 \leq t \leq t_{l_1+1}$. We then consider the following cases separately based on the value of l_1.

(i). When $l_1 \leq k-1$, we have that

$$\beta^*_{l,t} = p_T R \cdot \sum_{w=l}^{l_1-1} \delta_{w,l_1} \cdot A_{l_1,w-l}(t) \tag{32}$$

also, for any $t+1 \leq j \leq t_{l_1+1}$, we have that

$$\beta^*_{l,j} - \beta^*_{l+1,j} = p_T R \cdot \sum_{w=l}^{l_1-1} (\delta_{w,l_1} - \delta_{w+1,l_1}) \cdot A_{l_1,w-l}(j)$$

which implies that

$$\sum_{j=t+1}^{t_{l_1+1}} p_j \cdot (\beta^*_{l,j} - \beta^*_{l+1,j}) = p_T R \cdot \sum_{w=l}^{l_1-1} (\delta_{w,l_1} - \delta_{w+1,l_1}) \cdot \sum_{j=t+1}^{t_{l_1+1}} p_j \cdot A_{l_1,w-l}(j)$$

Note that from (31), it holds that $\sum_{j=t+1}^{t_{l_1+1}} p_j \cdot A_{l_1,w-l}(j) = 1 - \sum_{q=0}^{w-l} A_{l_1,q}(t)$. Thus, we have that

$$\begin{aligned}\sum_{j=t+1}^{t_{l_1+1}} p_j \cdot (\beta^*_{l,j} - \beta^*_{l+1,j}) &= p_T R \cdot \sum_{w=l}^{l_1-1} (\delta_{w,l_1} - \delta_{w+1,l_1}) \cdot \left(1 - \sum_{q=0}^{w-l} A_{l_1,q}(t)\right) \\ &= p_T R \cdot \delta_{l,l_1} - p_T R \cdot \sum_{w=l}^{l_1-1} \delta_{w,l_1} \cdot A_{l_1,w-l}(t)\end{aligned} \tag{33}$$

where the last equality holds from $\delta_{l_1,l_1} = 0$. Similarly, for any $l_2 \geq l_1+1$ and any $t_{l_2}+1 \leq j \leq t_{l_2+1}$, we have that

$$\beta^*_{l,j} - \beta^*_{l+1,j} = p_T R \cdot \sum_{w=l}^{l_2-1} (\delta_{w,l_2} - \delta_{w+1,l_2}) \cdot A_{l_2,w-l}(j)$$

which implies that

$$\sum_{j=t_{l_2}+1}^{t_{l_2+1}} p_j \cdot (\beta^*_{l,j} - \beta^*_{l+1,j}) = p_T R \cdot \sum_{w=l}^{l_2-1} (\delta_{w,l_2} - \delta_{w+1,l_2}) \cdot \sum_{j=t_{l_2}+1}^{t_{l_2+1}} p_j \cdot A_{l_2,w-l}(j)$$

Note that from Lemma 13, we have that $\sum_{j=t_{l_2}+1}^{t_{l_2+1}} p_j \cdot A_{l_2,w-l}(j) = 1 - \sum_{q=0}^{w-l} B_{l_2,q}$. Thus, we have that

$$\begin{aligned}\sum_{j=t_{l_1+1}+1}^{t_{k+1}} p_j \cdot (\beta^*_{l,j} - \beta^*_{l+1,j}) &= \sum_{l_2=l_1+1}^{k} \sum_{j=t_{l_2}+1}^{t_{l_2+1}} p_j \cdot (\beta^*_{l,j} - \beta^*_{l+1,j}) \\ &= \sum_{l_2=l_1+1}^{k} p_T R \cdot \sum_{w=l}^{l_2-1} (\delta_{w,l_2} - \delta_{w+1,l_2}) \cdot (1 - \sum_{q=0}^{w-l} B_{l_2,q}) \\ &= p_T R \cdot \sum_{l_2=l_1+1}^{k} \sum_{w=l+1}^{l_2} (\delta_{w-1,l_2} - \delta_{w,l_2}) \cdot (1 - \sum_{q=0}^{w-l-1} B_{l_2,q})\end{aligned} \tag{34}$$

Combining (32), (33) and (34), we have that

$$\beta^*_{l,t} + \sum_{j=t+1}^{T} p_j \cdot (\beta^*_{l,j} - \beta^*_{l+1,j}) = p_T R \cdot \delta_{l,l_1} + p_T R \cdot \sum_{l_2=l_1+1}^{k} \sum_{w=l+1}^{l_2} (\delta_{w-1,l_2} - \delta_{w,l_2}) \cdot (1 - \sum_{q=0}^{w-l-1} B_{l_2,q}) \quad (35)$$

Note that

$$\xi^*_t = p_T R \cdot \sum_{l_2=l_1+1}^{k} \sum_{w=l_1+1}^{l_2} (\delta_{w-1,l_2} - \delta_{w,l_2}) \cdot (1 - \sum_{q=0}^{w-l_1-1} B_{l_2,q})$$

in order to show $\beta^*_{l,t} + \sum_{j=t+1}^{T} p_j \cdot (\beta^*_{l,j} - \beta^*_{l+1,j}) = \xi^*_t$, it is enough to prove that

$$\delta_{l,l_1} + \sum_{l_2=l_1+1}^{k} \sum_{w=l+1}^{l_2} (\delta_{w-1,l_2} - \delta_{w,l_2}) \cdot (1 - \sum_{q=0}^{w-l-1} B_{l_2,q}) = \sum_{l_2=l_1+1}^{k} \sum_{w=l_1+1}^{l_2} (\delta_{w-1,l_2} - \delta_{w,l_2}) \cdot (1 - \sum_{q=0}^{w-l_1-1} B_{l_2,q}) \quad (36)$$

Further note that

$$\begin{aligned}
&\sum_{l_2=l_1+1}^{k} \sum_{w=l+1}^{l_2} (\delta_{w-1,l_2} - \delta_{w,l_2}) \cdot (1 - \sum_{q=0}^{w-l-1} B_{l_2,q}) \\
&= \sum_{l_2=l_1+1}^{k} \sum_{w=l+1}^{l_2} (\delta_{w-1,l_2} - \delta_{w,l_2}) - \sum_{l_2=l_1+1}^{k} \sum_{w=l+1}^{l_2} \sum_{q=0}^{w-l-1} B_{l_2,q} \cdot (\delta_{w-1,l_2} - \delta_{w,l_2}) \\
&= \sum_{l_2=l_1+1}^{k} \delta_{l,l_2} - \sum_{l_2=l_1+1}^{k} \sum_{q=0}^{l_2-l-1} B_{l_2,q} \cdot \delta_{q+l,l_2}
\end{aligned}$$

and similarly, note that

$$\sum_{l_2=l_1+1}^{k} \sum_{w=l_1+1}^{l_2} (\delta_{w-1,l_2} - \delta_{w,l_2}) \cdot (1 - \sum_{q=0}^{w-l_1-1} B_{l_2,q}) = \sum_{l_2=l_1+1}^{k} \delta_{l_1,l_2} - \sum_{l_2=l_1+1}^{k} \sum_{q=0}^{l_2-l_1-1} B_{l_2,q} \cdot \delta_{q+l_1,l_2}$$

in order to prove (36), it is enough to show that

$$\sum_{l_2=l_1}^{k} \delta_{l,l_2} - \sum_{l_2=l_1+1}^{k} \sum_{q=0}^{l_2-l-1} B_{l_2,q} \cdot \delta_{q+l,l_2} = \sum_{l_2=l_1+1}^{k} \delta_{l_1,l_2} - \sum_{l_2=l_1+1}^{k} \sum_{q=0}^{l_2-l_1-1} B_{l_2,q} \cdot \delta_{q+l_1,l_2} \quad (37)$$

When $l_1 = l$, it is direct to check that (37) holds. The proof of (37) when $l_1 \geq l+1$ is relegated to Lemma 16. Thus, we prove that when $l_1 \leq k-1$, it holds that $\beta^*_{l,t} + \sum_{j=t+1}^{T} p_j \cdot (\beta^*_{l,j} - \beta^*_{l+1,j}) = \xi^*_t$.

(ii). When $l_1 = k$, we have that

$$\beta^*_{l,t} = p_T R \cdot \sum_{w=l}^{k-1} A_{k,w-l}(t)$$

and for each $t+1 \leq j \leq T-1$, it holds that

$$\beta^*_{l,j} - \beta^*_{l+1,j} = p_T R \cdot \left(\sum_{w=l}^{k-1} A_{k,w-l}(j) - \sum_{w=l+1}^{k-1} A_{k,w-l-1}(j) \right) = p_T R \cdot A_{k,k-1-l}(j)$$

Note that $\beta^*_{l,T} = \beta^*_{l+1,T} = R$, we have

$$\beta^*_{l,t} + \sum_{j=t+1}^{T-1} p_j \cdot (\beta^*_{l,j} - \beta^*_{l+1,j}) = p_T R \cdot \left(\sum_{w=l}^{k-1} A_{k,w-l}(t) + \sum_{j=t+1}^{T-1} p_j \cdot A_{k,k-1-l}(j) \right)$$

Note that from (31), it holds that $\sum_{j=t+1}^{T-1} p_j \cdot A_{k,k-1-l}(j) = 1 - \sum_{q=0}^{k-1-l} A_{k,q}(t)$. Thus, we have that

$$\beta^*_{l,t} + \sum_{j=t+1}^{T-1} p_j \cdot (\beta^*_{l,j} - \beta^*_{l+1,j}) = p_T R = \xi^*_t$$

which completes our proof. $\square$

Lemma 16 *For each $l = 1, 2, \ldots, k-1$ and each $l_1 = l, l+1, \ldots, k-1$, it holds that*

$$\sum_{l_2=l_1}^{k} \delta_{l,l_2} - \sum_{l_2=l_1+1}^{k} \sum_{q=0}^{l_2-l-1} B_{l_2,q} \cdot \delta_{q+l,l_2} = \sum_{l_2=l_1+1}^{k} \delta_{l_1,l_2} - \sum_{l_2=l_1+1}^{k} \sum_{q=0}^{l_2-l_1-1} B_{l_2,q} \cdot \delta_{q+l_1,l_2} \tag{38}$$

Proof: We now prove (38) by induction on l from $l = k-1$ to $l = 1$. When $l = k-1$, we must have $l_1 = k-1 = l$, then (38) holds obviously. Suppose that there exists a $1 \le l' \le k-2$ such that for any l satisfying $l'+1 \le l \le k-1$, (38) holds for each l_1 such that $l \le l_1 \le k-1$, then we consider the case when $l = l'$. For this case, we again use induction on l_1 from $l_1 = k-1$ to $l_1 = l+1 = l'+1$. When $l_1 = k-1$, we have that

$$\sum_{l_2=l_1}^{k} \delta_{l,l_2} - \sum_{l_2=l_1+1}^{k} \sum_{q=0}^{l_2-l-1} B_{l_2,q} \cdot \delta_{q+l,l_2} = \delta_{l,k-1} + \delta_{l,k} - \sum_{q=0}^{k-l-1} B_{k,q} \cdot \delta_{q+l,k}$$

and

$$\sum_{l_2=l_1+1}^{k} \delta_{l_1,l_2} - \sum_{l_2=l_1+1}^{k} \sum_{q=0}^{l_2-l_1-1} B_{l_2,q} \cdot \delta_{q+l_1,l_2} = \delta_{k-1,k} - B_{k,0} \cdot \delta_{k-1,k}$$

Further note that from definition, $\delta_{v,k} = 1$ for each $v \le k-1$ and $\delta_{l,k-1} = \sum_{w_0=l+1}^{k-1} B_{k,k-w_0} = \sum_{q=1}^{k-l-1} B_{k,q}$, it is obvious that (38) holds when $l_1 = k-1$. Now suppose that (38) holds for l_1+1 (we assume $l_1 \ge l+1$ since when $l_1 = l$, it is direct from definition that (38) holds), we consider the case for l_1. Note that

$$\text{LHS of (38)} = \delta_{l,l_1} - \sum_{q=0}^{l_1-l} B_{l_1+1,q} \cdot \delta_{q+l,l_1+1} + \sum_{l_2=l_1+1}^{k} \delta_{l,l_2} - \sum_{l_2=l_1+2}^{k} \sum_{q=0}^{l_2-l-1} B_{l_2,q} \cdot \delta_{q+l,l_2}$$

and

$$\text{RHS of (38)} = \delta_{l_1,l_1+1} - \sum_{l_2=l_1+1}^{k} B_{l_2,l_2-l_1-1} \cdot \delta_{l_2-1,l_2} + \sum_{l_2=l_1+2}^{k} \delta_{l_1,l_2} - \sum_{l_2=l_1+2}^{k} \sum_{q=0}^{l_2-l_1-2} B_{l_2,q} \cdot \delta_{q+l_1,l_2}$$

Since we suppose for induction that (38) holds for l_1+1, we have that

$$\text{(38) holds for } (l, l_1) \Leftrightarrow \delta_{l,l_1} - \sum_{q=0}^{l_1-l} B_{l_1+1,q} \cdot \delta_{q+l,l_1+1} = \delta_{l_1,l_1+1} - \sum_{l_2=l_1+1}^{k} B_{l_2,l_2-l_1-1} \cdot \delta_{l_2-1,l_2}$$

Further note that we have supposed for induction that (38) holds for $(l+1, l_1)$, which implies

$$\delta_{l+1,l_1} - \sum_{q=0}^{l_1-l-1} B_{l_1+1,q} \cdot \delta_{q+l+1,l_1+1} = \delta_{l_1,l_1+1} - \sum_{l_2=l_1+1}^{k} B_{l_2,l_2-l_1-1} \cdot \delta_{l_2-1,l_2}$$

Thus, it holds that

$$(38) \text{ holds for } (l, l_1) \Leftrightarrow \delta_{l,l_1} - \delta_{l+1,l_1} = \sum_{q=0}^{l_1-l} B_{l_1+1,q} \cdot (\delta_{q+l,l_1+1} - \delta_{q+l+1,l_1+1})$$

Finally, from definition, we have

$$\delta_{l,l_1} - \delta_{l+1,l_1} = \sum_{w_1=l+1}^{l_1+1} \sum_{w_2=w_1}^{l_1+2} \cdots \sum_{w_{k-1-l_1}=w_{k-2-l_1}}^{k-1} B_{l_1+1,w_1-l-1} \cdot B_{l_1+2,w_2-w_1} \ldots B_{k-1,w_{k-1-l_1}-w_{k-2-l_1}} \cdot B_{k,k-w_{k-1-l_1}}$$

and

$$\delta_{q+l,l_1+1} - \delta_{q+l+1,l_1+1} = \sum_{w_2=q+l+1}^{l_1+2} \cdots \sum_{w_{k-1-l_1}=w_{k-2-l_1}}^{k-1} B_{l_1+2,w_2-q-l-1} \ldots B_{k-1,w_{k-1-l_1}-w_{k-2-l_1}} \cdot B_{k,k-w_{k-1-l_1}}$$

which implies that

$$\delta_{l,l_1} - \delta_{l+1,l_1} = \sum_{q=0}^{l_1-l} B_{l_1+1,q} \cdot (\delta_{q+l,l_1+1} - \delta_{q+l+1,l_1+1}) \tag{39}$$

Thus, from induction, we prove that (38) holds for each $l_1 \geq l+1$. Note that (38) holds obviously for $l_1 = l$, (38) holds for each $l_1 \geq l$. From the induction on l, we know that (38) holds for each $1 \leq l \leq k-1$ and each $l \leq l_1 \leq k-1$, which completes our proof. □

Finally, we only need to prove feasibility of $\{\beta^*_{l,t}, \xi^*_t\}$ in the following lemma.

Lemma 17 *For each $l = 1, 2, \ldots, k$ and each $t = 1, 2, \ldots, t_l$, it holds that*

$$\beta^*_{l,t} + \sum_{j=t+1}^{T} p_j \cdot (\beta^*_{l,j} - \beta^*_{l+1,j}) \geq \xi^*_t$$

where we denote $\beta^*_{k+1,t} = 0$ for notation simplicity.

Proof: Note that from Lemma 12, we have $\delta_{w,l_2} \geq \delta_{w+1,l_2}$, which implies that $\beta^*_{l,j} \geq \beta^*_{l+1,j}$ for each l and j. Thus, we have that for each $t = 1, 2, \ldots, t_l$, it holds that

$$\beta^*_{l,t} + \sum_{j=t+1}^{T} p_j \cdot (\beta^*_{l,j} - \beta^*_{l+1,j}) \geq \beta^*_{l,t_l+1} + \sum_{j=t_l+2}^{T} p_j \cdot (\beta^*_{l,j} - \beta^*_{l+1,j})$$

Further note that Lemma 15 implies that

$$\beta^*_{l,t_l+1} + \sum_{j=t_l+2}^{n} p_j \cdot (\beta^*_{l,j} - \beta^*_{l+1,j}) \geq \xi^*_{t_l+1}$$

Thus, it is enough to show that $\xi_i^* \le \xi_{t_l+1}^*$ for each $t = 1, 2, \dots, t_l$. From the definition of ξ_i^*, it is enough to show that $\phi_l \le \phi_{l+1}$. When $l = k-1$, we have $\phi_{l+1} = \phi_k = 1$ and $\phi_l = \phi_{k-1} = 1 - B_{k,0}$, which implies that $\phi_{k-1} \le \phi_k$. When $l \le k-2$, from definition, we have

$$\phi_l - \phi_{l+1} = \sum_{q=l+1}^{k} (\delta_{l,q} - \delta_{l+1,q}) \cdot (1 - B_{q,0}) - \sum_{q=l+2}^{k} \sum_{w=l+2}^{q} (\delta_{w-1,q} - \delta_{w,q}) \cdot B_{q,w-l-1}$$

Note that in the proof of Lemma 16, we proved (39), then when $k-1 \ge q \ge l+1$, we have

$$\delta_{l,q} - \delta_{l+1,q} = \sum_{w=0}^{q-l} B_{q+1,w} \cdot (\delta_{w+l,q+1} - \delta_{w+l+1,q+1}) = \sum_{w=l+1}^{q+1} B_{q+1,w-l-1} \cdot (\delta_{w-1,q+1} - \delta_{w,q+1})$$

Thus, it holds that

$$\begin{aligned}
\phi_l - \phi_{l+1} &= -\sum_{q=l+1}^{k-1} (\delta_{l,q} - \delta_{l+1,q}) \cdot B_{q,0} + \sum_{q=l+1}^{k-1} \sum_{w=l+1}^{q+1} B_{q+1,w-l-1} \cdot (\delta_{w-1,q+1} - \delta_{w,q+1}) \\
&\quad - \sum_{q=l+2}^{k} \sum_{w=l+2}^{q} (\delta_{w-1,q} - \delta_{w,q}) \cdot B_{q,w-l-1} \\
&= -\sum_{q=l+1}^{k-1} (\delta_{l,q} - \delta_{l+1,q}) \cdot B_{q,0} + \sum_{q=l+1}^{k-1} B_{q+1,0} \cdot (\delta_{l,q+1} - \delta_{l+1,q+1}) \\
&= -B_{l+1,0} \cdot \delta_{l,l+1} \le 0
\end{aligned}$$

which completes our proof. □

Together, Lemma 14, Lemma 15, and Lemma 17 establish the feasibility of $\{\beta_{l,t}^*, \xi_t^*\}$. Then, from the definition of $\{\beta_{l,t}^*, \xi_t^*\}$, obviously condition (28) is satisfied and from Lemma 15, condition (29) is satisfied. Thus, we finish the proof of Theorem 1.

B5. Proof of Lemma 5

Proof: Since we have $\mathrm{Dual}(\boldsymbol{p}, k) = \mathrm{Primal}(\boldsymbol{p}, k)$, it is enough to consider the primal LP $\mathrm{Primal}(\boldsymbol{p}, k)$ in (10) and prove that $\mathrm{Primal}(\boldsymbol{p}, k) \ge \mathrm{Primal}(\tilde{\boldsymbol{p}}, k)$. Suppose the optimal solution of $\mathrm{Primal}(\boldsymbol{p}, k)$ is denoted as $\{\beta_{l,t}^*, \xi_t^*\}$, as constructed in (30), we then construct a feasible solution $\{\tilde{\beta}_{l,t}, \tilde{\xi}_t\}$ to $\mathrm{Primal}(\tilde{\boldsymbol{p}}, k)$ as follows:

$$\begin{aligned}
&\tilde{\xi}_t = \xi_t^* \quad \forall 1 \le t < q, \qquad \tilde{\xi}_q = \tilde{\xi}_{q+1} = \xi_q^*, \qquad \tilde{\xi}_{t+1} = \xi_t^* \quad \forall q+1 \le t \le T \\
&\tilde{\beta}_{l,t} = \beta_{l,t}^* \quad \forall l = 1, \dots, k, \forall 1 \le t < q \\
&\tilde{\beta}_{l,q} = \tilde{\beta}_{l,q+1} = \beta_{l,q}^* \quad \forall l = 1, \dots, k \\
&\tilde{\beta}_{l,t+1} = \beta_{l,t}^* \quad \forall l = 1, \dots, k, \forall q+1 \le t \le T
\end{aligned}$$

Note that we have

$$\mathrm{Primal}(\boldsymbol{p}, k) = \sum_{t=1}^{T} p_t \cdot \beta_{1,t}^* = \sum_{t=1}^{T+1} \tilde{p}_t \cdot \tilde{\beta}_{1,t}$$

it is enough to prove that $\{\tilde{\beta}_{l,t}, \tilde{\xi}_t\}$ is feasible to Primal$(\tilde{\boldsymbol{p}}, k)$. Obviously, we have $\{\tilde{\beta}_{l,t}, \tilde{\xi}_t\}$ are non-negative and $\sum_{t=1}^{T+1} \tilde{p}_t \cdot \tilde{\xi}_t = \sum_{t=1}^{T} p_t \cdot \xi_t^* = 1$, then we only need to check whether the following constraint is satisfied:

$$\tilde{\beta}_{l,t} + \sum_{\tau > t} \tilde{p}_\tau \cdot (\tilde{\beta}_{l,\tau} - \tilde{\beta}_{l+1,\tau}) - \tilde{\xi}_t \geq 0, \quad \forall l = 1, \ldots, k, \forall t = 1, \ldots, T+1 \tag{40}$$

where we denote $\tilde{\beta}_{k+1,t} = 0$ for notation simplicity. Note that when $t \geq q+1$, we have that

$$\tilde{\beta}_{l,t} + \sum_{\tau > t} \tilde{p}_\tau \cdot (\tilde{\beta}_{l,\tau} - \tilde{\beta}_{l+1,\tau}) - \tilde{\xi}_t = \beta_{l,t}^* + \sum_{\tau > t} p_\tau \cdot (\beta_{l,\tau}^* - \beta_{l+1,\tau}^*) - \xi_t^* \geq 0, \quad \forall l = 1, \ldots, k$$

and when $1 \leq t \leq q-1$, we also have

$$\tilde{\beta}_{l,t} + \sum_{\tau > t} \tilde{p}_\tau \cdot (\tilde{\beta}_{l,\tau} - \tilde{\beta}_{l+1,\tau}) - \tilde{\xi}_t = \beta_{l,t}^* + \sum_{\tau > t} p_\tau \cdot (\beta_{l,\tau}^* - \beta_{l+1,\tau}^*) - \xi_t^* \geq 0, \quad \forall l = 1, \ldots, k$$

by noting $\tilde{p}_q + \tilde{p}_{q+1} = p_q$. Now we consider the case when $t = q$, then for each $l = 1, \ldots, k$, we have

$$\begin{aligned}
\tilde{\beta}_{l,q} + \sum_{j=q+1}^{T+1} \tilde{p}_j \cdot (\tilde{\beta}_{l-1,j} - \tilde{\beta}_{l,j}) - \tilde{\xi}_q &= \tilde{\beta}_{l,q} + \sum_{j=q+2}^{T+1} \tilde{p}_j \cdot (\tilde{\beta}_{l-1,j} - \tilde{\beta}_{l,j}) - \tilde{\xi}_q + \tilde{p}_{q+1} \cdot (\tilde{\beta}_{l-1,q+1} - \tilde{\beta}_{l,q+1}) \\
&= \beta_{l,q}^* + \sum_{j=q+1}^{T} p_j \cdot (\beta_{l-1,j}^* - \beta_{l,j}^*) - \xi_q^* + p_q \cdot (1-\sigma) \cdot (\beta_{l-1,q}^* - \beta_{l,q}^*) \\
&\geq p_q \cdot (1-\sigma) \cdot (\beta_{l-1,q}^* - \beta_{l,q}^*)
\end{aligned}$$

Thus, it is enough to show that $\beta_{l-1,q}^* \geq \beta_{l,q}^*$ to prove feasibility. Note that from Lemma 12, for each $l_2 = 1, 2, \ldots, k$ and each $l_1 = 1, 2, \ldots, l_2 - 1$, we have $\delta_{l_1,l_2} \geq \delta_{l_1+1,l_2}$, then, it is direct to show that $\beta_{l-1,q}^* \geq \beta_{l,q}^*$ from the construction (30), which completes our proof. □

B6. Proof of Proposition 1

Proof: We consider the following problem instance $\mathcal{H}$. At the beginning, there are two queries arriving deterministically with a reward 1. Then, over the time interval $[0,1]$, there are queries with reward $r_1 > 1$ arriving according to a Poisson process with rate λ. At last, there is one query with a reward $\frac{r_2}{\epsilon}$ arriving with a probability ϵ for some small $\epsilon > 0$.

Obviously, since $r_1 > 1$ and ϵ is set to be small, the prophet will first serve the last query as long as it arrives, and then serve the queries with a reward r_1 as much as possible, and at least serve the first two queries. Then, we have that

$$\mathbb{E}_{\boldsymbol{I} \sim \boldsymbol{F}}[V^{\text{off}}(\boldsymbol{I})] = \hat{V} := r_2 + 2 \cdot \exp(-\lambda) + (r_1 + 1) \cdot \lambda \cdot \exp(-\lambda) + 2r_1 \cdot (1 - (\lambda+1) \cdot \exp(-\lambda) + O(\epsilon)$$

Moreover, for any online algorithm π, we consider the following situations separately based on the number of the first two queries that π will serve.

(i). If π will always serve the first two queries, then it is obvious that $\mathbb{E}_{\pi, \boldsymbol{I} \sim \boldsymbol{F}}[V^\pi(\boldsymbol{I})] = 2$.

(ii). If π serves only one of the first two queries, then the optimal way for π to serve the second query will depend on the value of r_1 and r_2. To be more specific, if $r_1 \geq r_2$, then the optimal way is to serve the query with reward r_1 as long as it arrives, and if $r_1 < r_2$, then the optimal way is to reject all the arriving queries with reward r_1 and only serve the last query. Thus, it holds that

$$\mathbb{E}_{\pi,\boldsymbol{I}\sim\boldsymbol{F}}[V^{\pi}(\boldsymbol{I})] \leq V(1) := 1 + \exp(-\lambda)\cdot r_2 + (1-\exp(-\lambda))\cdot \max\{r_1, r_2\} + O(\epsilon)$$

(iii). If π rejects all the first two queries, then conditioning on there are more than one queries with reward r_1 arriving during the interval $[0,1]$, the optimal way for π is to serve both queries with reward r_1 if $r_1 \geq r_2$ and only serve one query with reward r_1 if $r_1 < r_2$. Then, it holds that

$$\mathbb{E}_{\pi,\boldsymbol{I}\sim\boldsymbol{F}}[V^{\pi}(\boldsymbol{I})] \leq V(2) := \exp(-\lambda)\cdot r_2 + \lambda\cdot\exp(-\lambda)\cdot(r_1+r_2) + (1-(\lambda+1)\cdot\exp(-\lambda)\cdot(r_1+\max\{r_1,r_2\})$$

Thus, we conclude that for any online algorithm π, it holds that

$$\frac{\mathbb{E}_{\pi,\boldsymbol{I}\sim\boldsymbol{F}}[V^{\pi}(\boldsymbol{I})]}{\mathbb{E}_{\boldsymbol{I}\sim\boldsymbol{F}}[V^{\mathrm{off}}(\boldsymbol{I})]} \leq g(r_1, r_2, \lambda) := \frac{\max\{V(1), V(2), 2\}}{\hat{V}}$$

where we can neglect the $O(\epsilon)$ term by letting $\epsilon \to 0$. In this way, we can focus on the following optimization problem

$$\inf_{r_1>1, r_2>1, \lambda} g(r_1, r_2, \lambda)$$

to obtain the upper bound of the guarantee of any online algorithm relative to the prophet's value. We can numerically solve the above problem and show that when $r_1 = r_2 = 1.4119$, $\lambda = 1.2319$, the value of $g(r_1, r_2, \lambda)$ reaches its minimum and equals 0.6269, which completes our proof. $\square$

B7. Proof of Theorem 2

Proof: For each $\boldsymbol{p} = (p_1, \ldots, p_T)$ satisfying $\sum_{t=1}^{T} p_t = k$, since each irrational number can be arbitrarily approximated by a rational number, we assume without loss of generality that p_t is a rational number for each t, i.e., $p_t = \frac{n_t}{N}$ where n_t is an integer for each t and N is an integer to denote the common denominator. We first split p_1 into $\frac{1}{N}$ and $\frac{n_1-1}{N}$ to form a new sequence $\tilde{\boldsymbol{p}} = (\frac{1}{N}, \frac{n_1-1}{N}, \frac{n_2}{N}, \ldots, \frac{n_T}{N})$. From Lemma 5, we know $\mathrm{Dual}(\boldsymbol{p}, k) \geq \mathrm{Dual}(\tilde{\boldsymbol{p}}, k)$. We then split $\frac{n_1-1}{N}$ into $\frac{1}{N}$ and $\frac{n_1-2}{N}$ and so on. In this way, we split p_1 into n_1 copies of $\frac{1}{N}$ to form a new sequence $\tilde{\boldsymbol{p}} = (\frac{1}{N}, \ldots, \frac{1}{N}, \frac{n_2}{N}, \ldots, \frac{n_T}{N})$ and Lemma 5 guarantees that $\mathrm{LP}(\boldsymbol{p}, k) \geq \mathrm{LP}(\tilde{\boldsymbol{p}}, k)$. We repeat the above operation for each t. Finally, we form a new sequence of arrival probabilities, denoted as $\boldsymbol{p}^N = (\frac{1}{N}, \ldots, \frac{1}{N}) \in \mathbb{R}^{Nk}$, and we have $\mathrm{Dual}(\boldsymbol{p}, k) \geq \mathrm{Dual}(\boldsymbol{p}^N, k)$.

From the above argument, we know that for each $\boldsymbol{p} = (p_1, \ldots, p_T)$ satisfying $\sum_{t=1}^{T} p_t = k$, there exists an integer N such that $\mathrm{Dual}(\boldsymbol{p}, k) \geq \mathrm{Dual}(\boldsymbol{p}^N, k)$, which implies that

$$\inf_{\boldsymbol{p}:\sum_t p_t = k} \mathrm{Dual}(\boldsymbol{p}, k) = \liminf_{N\to\infty} \mathrm{Dual}(\boldsymbol{p}^N, k)$$

Thus, it is enough to consider $\liminf_{N\to\infty} \mathrm{Dual}(\boldsymbol{p}^N, k)$.

We denote $\tilde{\boldsymbol{y}}_\theta(t) = (\tilde{y}_{1,\theta}(t), \dots, \tilde{y}_{k,\theta}(t))$. We define a function $\boldsymbol{f}_\theta(\cdot) = (f_{1,\theta}(\cdot), \dots, f_{k,\theta}(\cdot))$, where we denote $\tilde{y}_{0,\theta}(t) = 1$ and for each $l = 1, \dots, k-1$

$$f_{l,\theta}(\tilde{y}_{1,\theta}, \dots, \tilde{y}_{k,\theta}, t) = \begin{cases} 0, & \text{if } \tilde{y}_{l-1,\theta}(t) \le 1-\theta \\ \tilde{y}_{l-1,\theta}(t) - (1-\theta), & \text{if } \tilde{y}_{l,\theta}(t) \le 1-\theta \le \tilde{y}_{l-1,\theta}(t) \\ \tilde{y}_{l-1,\theta}(t) - \tilde{y}_{l,\theta}(t), & \text{if } \tilde{y}_{l,\theta}(t) \ge 1-\theta \end{cases}$$

and

$$f_{k,\theta}(\tilde{y}_{1,\theta}, \dots, \tilde{y}_{k,\theta}, t) = \begin{cases} 0, & \text{if } \tilde{y}_{k-1,\theta}(t) \le 1-\theta \\ \tilde{y}_{k-1,\theta}(t) - (1-\theta), & \text{if } \tilde{y}_{k-1,\theta}(t) \ge 1-\theta \end{cases}$$

Moreover, variable $(\tilde{y}_1, \dots, \tilde{y}_k, t)$ belongs to the feasible set of the function $f_{l,\theta}(\cdot)$ if and only if $y_{v-1} \ge y_v$ for $v = 1, \dots, k-1$. Then, for each $\theta \in [0,1]$, the function $\tilde{\boldsymbol{y}}_\theta(t)$ in Definition 4 should be the solution to the following ordinary differential equation (ODE):

$$\frac{d\tilde{\boldsymbol{y}}_\theta(t)}{dt} = \boldsymbol{f}_\theta(\tilde{\boldsymbol{y}}_\theta, t) \quad \text{for } t \in [0,k] \text{ with starting point } \tilde{\boldsymbol{y}}_\theta(0) = (0, \dots, 0) \tag{41}$$

For each integer N and $\boldsymbol{p}^N = (\frac{1}{N}, \dots, \frac{1}{N})$ where $\|\boldsymbol{p}^N\|_1 = k$, for any fixed $\theta \in [0,1]$, we denote $\{x_{l,t}(\theta, N)\}$ as the variables constructed in Definition 3 under the arrival probabilities $\boldsymbol{p}^N$, where $l = 1, \dots, k$ and $t = 1, \dots, Nk$. We further denote $y_{l,\theta,N}(\frac{t}{N}) = \sum_{\tau=1}^{t} x_{l,\tau}(\theta, N)$ and denote $\boldsymbol{y}_{\theta,N}(\cdot) = (y_{1,\theta,N}(\cdot), \dots, y_{k,\theta,N}(\cdot))$. It is direct to check that for each $t = 1, \dots, Nk$, it holds that

$$(\boldsymbol{y}_{\theta,N}(\frac{t}{N}) - \boldsymbol{y}_{\theta,N}(\frac{t-1}{N}))/(\frac{1}{N}) = \boldsymbol{f}_\theta(\boldsymbol{y}_{\theta,N}, \frac{t-1}{N})$$

Thus, $\{\boldsymbol{y}_{\theta,N}(t)\}_{\forall t \in [0,k]}$ can be viewed as the result obtained from applying Euler's method (Butcher and Goodwin, 2008) to solve ODE (41), where there are Nk discrete points uniformly distributed within $[0,k]$. Note that for each $\theta \in [0,1]$, the function $\boldsymbol{f}_\theta(\cdot)$ is Lipschitz continuous with a Lipschitz constant 2 under infinity norm. Moreover, it is direct to note that for each $\theta \in [0,1]$ and each $t \in [0,k]$, it holds that $\|\boldsymbol{f}_\theta(\tilde{\boldsymbol{y}}, t)\|_\infty \le 1$. Then, for each $\theta \in [0,1]$, each $t_1, t_2 \in [0,k]$ and each $l = 1, \dots, k$, we have

$$\left|\frac{d\tilde{y}_{l,\theta}(t_1)}{dt} - \frac{d\tilde{y}_{l,\theta}(t_2)}{dt}\right| \le 2 \cdot \|\tilde{\boldsymbol{y}}_\theta(t_1) - \tilde{\boldsymbol{y}}_\theta(t_2)\|_\infty \le 2 \cdot |t_1 - t_2|$$

Thus, we know that

$$\left|\tilde{y}_{l,\theta}(t_1) - \tilde{y}_{l,\theta}(t_2) - \frac{d\tilde{y}_{l,\theta}(t_2)}{dt} \cdot (t_1 - t_2)\right| \le 2 \cdot (t_1 - t_2)^2$$

We can apply the global truncation error of Euler's method (Theorem 212A (Butcher and Goodwin, 2008)) to show that $\boldsymbol{y}_{\theta,N}(k)$ converges to $\tilde{\boldsymbol{y}}_\theta(k)$ when $N \to \infty$. Specifically, we have

$$\|\boldsymbol{y}_{\theta,N}(k) - \tilde{\boldsymbol{y}}_\theta(k)\|_\infty \le (\exp(2k) - 1) \cdot \frac{1}{N}, \quad \forall \theta \in [0,1] \tag{42}$$

Now we define $Y(\theta) = \tilde{y}_{k,\theta}(k)$ as a function of $\theta \in [0,1]$ and for each N, we define $Y_N(\theta) = y_{k,\theta,N}(k)$ as a function of $\theta \in [0,1]$. (42) implies that the function sequence $\{Y_N\}_{\forall N}$ converges uniformly to the function Y when $N \to \infty$. Note that for each N, the function $Y_N(\theta)$ is continuously monotone increasing with θ due to Lemma 10, then from uniform limit theorem, $Y(\theta)$ must be a continuously monotone increasing function over θ. Thus, the equation $Y(\theta) = 1-\theta$ has a unique solution, denoted as γ_k^*. For each N, we denote θ_N^* as the unique solution to the equation $Y_N(\theta) = 1-\theta$, where we have that $\theta_N^* = \mathrm{Dual}(\boldsymbol{p}^N, k)$. Since $\{Y_N\}_{\forall N}$ converges uniformly to the function Y, it must hold that $\gamma_k^* = \lim_{N\to\infty} \theta_N^*$, which completes our proof. □

C. Proofs in Section 5

C1. Proof of Proposition 2

Proof: The proof is the same as the proof of Proposition 3.1 in Jiang et al. (2022).

Consider a problem setup $\hat{\mathcal{H}}$ with 4 queries and

$$(\hat{r}_1, p_1, d_1) = (r, 1, \epsilon), \quad (\hat{r}_2, p_2, d_2) = (\hat{r}_3, p_3, d_3) = (r, \frac{1-2\epsilon}{1+2\epsilon}, \frac{1}{2}+\epsilon), \quad (\hat{r}_4, p_4, d_4) = (r/\epsilon, \epsilon, 1)$$

for $r > 0$ and some $\epsilon > 0$. Obviously, if the policy π only serves queries with a size greater than $1/2$, then the expected total reward is $V_L^\pi = r + O(\epsilon)$. If the policy π only serves queries with a size no greater than $1/2$, then the expected total reward is $V_S^\pi = r$. Thus, the expected total reward of the policy π is

$$V^\pi = \max\{V_L^\pi, V_S^\pi\} = r + O(\epsilon)$$

Moreover, it is direct to see that $\sum_{t=1}^4 p_t \cdot d_t = 1$, then, we have $\mathrm{UP}(\hat{\mathcal{H}}) = 4r$. Thus, the guarantee of π is upper bounded by $1/4 + O(\epsilon)$, which converges to $1/4$ as $\epsilon \to 0$. □

C2. Proof of Theorem 3

Proof: It is enough to prove that the threshold policy π_γ is feasible when $\gamma = \frac{1}{3+e^{-2}}$. In the remaining proof, we set $\gamma = \frac{1}{3+e^{-2}}$. For a fixed t, and any a and b, denote $\mu_{t,\gamma}(a,b] = P(a < \tilde{X}_{t,\gamma} \le b)$ assuming $\tilde{X}_{t,\gamma}$ is well-defined, it is enough to prove that $\mu_{t,\gamma}(0,1] \le 1-\gamma$ thus the random variable $\tilde{X}_{t+1,\gamma}$ is well-defined.

We define $U_t(s) = \mu_{t,\gamma}(0,s]$ for any $s \in (0,1]$. Note that by definition, we have $\mathbb{E}[\tilde{X}_{t,\gamma}] = \gamma \cdot \sum_{\tau=1}^t p_\tau \cdot d_\tau \le \gamma$. From integration by parts, we have that

$$\gamma \ge \mathbb{E}[\tilde{X}_{t,\gamma}] = \int_{s=0}^1 s dU_t(s) = U_t(1) - \int_{s=0}^1 U_t(s) ds \tag{43}$$

We then bound the term $\int_{s=0}^1 U_t(s)ds$. Now suppose $U_t(1) \ge \gamma$, otherwise $U_t(1) < \gamma$ immediately implies that $U_t(1) \le 1-\gamma$, which proves our result. Then there must exists a constant $u^* \in (0,1)$ such that $\gamma \cdot u^* - \gamma \cdot \ln(u^*) = U_t(1)$. We further define

$$s^* = \begin{cases} \min\{s \in (0,1/2] : U_t(s) \ge \gamma \cdot u^*\}, & \text{if } U_t(\frac{1}{2}) \ge \gamma \cdot u^* \\ \frac{1}{2}, & \text{if } U_t(\frac{1}{2}) < \gamma \cdot u^* \end{cases}$$

Denote $U_t(s^*-) = \lim_{s\to s^*} U_t(s)$, it holds that

$$\begin{aligned}\int_{s=0}^{1} U_t(s)ds &= \int_{s=0}^{s^*-} U_t(s)ds + \int_{s=s^*}^{1/2} U_t(s)ds + \int_{s=1/2}^{1-s^*} U_t(s)ds + \int_{s=1-s^*}^{1} U_t(s)ds \\ &\le s^*\cdot(U_t(s^*-)+U_t(1)) + \int_{s=s^*}^{1/2} U_t(s)ds + \int_{s=1/2}^{1-s^*} U_t(s)ds \\ &\le s^*\cdot(2\gamma u^* - \gamma\cdot\ln(u^*)) + \int_{s=s^*}^{1/2}(2U_t(s) - \gamma\cdot\ln(\frac{U_t(s)}{\gamma}))ds\end{aligned}$$

where the last inequality holds by noting that $U_t(s^*-)\le\gamma u^*$ and for any $s\in[s^*,1/2]$, from Lemma 6, we have $\frac{U_t(s)}{\gamma}\le\exp(-\frac{U_t(1-s)-U_t(s)}{\gamma})$, which implies that $\frac{U_t(1-s)}{\gamma}\le\frac{U_t(s)}{\gamma}-\ln(\frac{U_t(s)}{\gamma})$. Note that for any $s\in[s^*,1/2]$, we have that $\gamma\cdot u^*\le U_t(s^*)\le U_t(s)\le U_t(1/2)\le\gamma$, where $U_t(1/2)\le\gamma$ holds directly from Lemma 6. Further note that the function $2x-\gamma\cdot\ln(x/\gamma)$ is a convex function, thus is quasi convex. Then, for any $s\in[s^*,1/2]$, it holds that

$$2U_t(s) - \gamma\cdot\ln(\frac{U_t(s)}{\gamma}) \le \max\{2\gamma u^* - \gamma\cdot\ln(u^*), 2\gamma\}$$

Thus, we have that

$$\int_{s=0}^{1} U_t(s)ds \le s^*\cdot(2\gamma u^* - \gamma\cdot\ln(u^*)) + (1/2 - s^*)\cdot\max\{2\gamma u^* - \gamma\cdot\ln(u^*), 2\gamma\}$$

If $2\gamma u^* - \gamma\cdot\ln(u^*)\le 2\gamma$, we have $\int_{s=0}^{1} U_t(s)ds \le 2s^*\gamma + \gamma - 2s^*\gamma = \gamma$. From (43), we have that $U_t(1)\le 2\gamma < 1-\gamma$.

If $2\gamma u^* - \gamma\cdot\ln(u^*) > 2\gamma$, we have $\int_{s=0}^{1} U_t(s)ds \le \gamma u^* - \frac{\gamma}{2}\cdot\ln(u^*)$. From (43) and the definition of u^*, we have that

$$U_t(1) = \gamma u^* - \gamma\cdot\ln(u^*) \le \gamma + \gamma u^* - \frac{\gamma}{2}\cdot\ln(u^*)$$

which implies that $u^*\ge\exp(-2)$. Note that the function $x-\ln(x)$ is non-increasing on $(0,1)$, we have $U_t(1)\le\gamma\cdot\exp(-2)+2\gamma = 1-\gamma$, which completes our proof. □

C3. Proof of Theorem 4

Proof: It is enough to consider a special case of our problem where the size of each query is deterministic. Then the proof is the same as the proof of Theorem 3.1 in Jiang et al. (2022).

We denote by d_t the size of query t. Then, we have $\boldsymbol{p} = (p_1,\ldots,p_t)$ and $\boldsymbol{D} = (d_1,\ldots,d_T)$. Due to Lemma 3, it is enough to construct a $\boldsymbol{p}$ and $\boldsymbol{D}$ satisfying $\sum_{t=1}^{T} p_t\cdot d_t\le 1$, such that $\text{Dual}(\boldsymbol{p},\boldsymbol{D})\le \frac{1}{3+e^{-2}}$. For each $\epsilon>0$, we consider the following $\boldsymbol{p}$ and $\boldsymbol{D}$:

$$(p_1,d_1) = (1,\epsilon),\quad (p_t,d_t) = (\frac{1-2\epsilon}{(T-2)(\frac{1}{2}+\epsilon)}, \frac{1}{2}+\epsilon)\ \text{ for all } 2\le t\le T-1 \ \text{ and }\ (p_T,d_T) = (\epsilon,1)$$

It is direct to check that $\sum_{t=1}^{T} p_t \cdot d_t = 1$. Now denote by $\{\theta^*, \alpha_t^*(c_t)\}$ the optimal solution to Dual($\boldsymbol{p}, \boldsymbol{D}$) (we omit d_t in the expression $\alpha(d_t, c_t)$ since now d_t takes a single value). We further denote $\mu_t = \sum_{c_t: d_t \le c_t < 1} \alpha_t^*(c_t)$ for each $2 \le t \le T-1$. Then, constraint (8a) implies that

$$\alpha_t^*(1) \ge \theta^* \cdot p_t - \mu_t, \quad \forall 2 \le t \le T-1$$

Thus, we have that

$$\theta^* \cdot p_T \le \alpha_T^*(1) \le p_T \cdot (1 - \alpha_1^*(1) - \sum_{t=2}^{T-1} \alpha_t^*(1)) \Rightarrow \theta^* \cdot (1 + \sum_{t=2}^{T-1} p_t) \le 1 - \alpha_1^*(1) + \sum_{t=2}^{T-1} \mu_t$$

We then consider the term $\sum_{t=2}^{T-1} \mu_t$ to upper bound θ^*. Note that since $d_t > 1/2$ for all $2 \le t \le T-1$, constraint (8b) implies that

$$\alpha_t^*(c_t) \le p_t \cdot (\alpha_1^*(c_t + d_1) - \sum_{\tau=2}^{t-1} \alpha_\tau^*(c_t)), \quad \forall d_t \le c_t < 1$$

Further note that $\alpha_1^*(c) = 0$ when $c < 1$, we must have that $\alpha_t^*(c_t) = 0$ when $c_t \neq 1 - d_1$. Then, we have $\mu_t = \alpha_t^*(1 - d_1)$ and

$$\mu_t \le p_t \cdot (\alpha_1^*(1) - \sum_{\tau=2}^{t-1} \mu_\tau), \quad \forall 2 \le t \le T-1$$

We then inductively show that for any $2 \le t \le T-1$, we have

$$\sum_{\tau=2}^{t} \mu_t \le (1 - \prod_{\tau=2}^{t} (1 - p_\tau)) \cdot \alpha_1^*(1) \tag{44}$$

When $t = 2$, (44) holds obviously. Now suppose that (44) holds for $t-1$, we have

$$\sum_{\tau=2}^{t} \mu_t = \sum_{\tau=2}^{t-1} \mu_\tau + \mu_t \le \alpha_1^*(1) \cdot p_t + (1 - p_t) \cdot \sum_{\tau=2}^{t-1} \mu_\tau \le \alpha_1^*(1) \cdot p_t + (1 - p_t) \cdot (1 - \prod_{\tau=2}^{t-1} (1 - p_\tau)) \cdot \alpha_1^*(1)$$
$$= \alpha_1^*(1) \cdot (1 - \prod_{\tau=2}^{t} (1 - p_\tau))$$

Thus, (44) holds for all $2 \le t \le T-1$ and we have

$$\sum_{t=2}^{T-1} \mu_t \le \alpha_1^*(1) \cdot (1 - \prod_{t=2}^{T-1} (1 - p_t)) = \alpha_1^*(1) \cdot (1 - (1 - \frac{1 - 2\epsilon}{(T-2)(\frac{1}{2} + \epsilon)})^{T-2}) \le \alpha_1^*(1) \cdot (1 - \exp(-2)) + O(\epsilon)$$

which implies that

$$3\theta^* + O(\epsilon) = \theta^* \cdot (1 + \sum_{t=2}^{T-1} p_t) \le 1 - \alpha_1^*(1) + \sum_{t=2}^{T-1} \mu_t \le 1 - \alpha_1^*(1) \cdot \exp(-2) + O(\epsilon)$$

Moreover, note that constraint (8a) requires $\alpha_1^*(1) \ge \theta^*$, we have that $\theta^* \le \frac{1}{3 + e^{-2}} + O(\epsilon)$. The proof is finished by taking $\epsilon \to 0$.

D. Proofs in section 6

D1. Proof of Lemma 7

Proof: We prove (18) by induction on t. When $t=0$, since $\mu_{0,\gamma}(0,b]=0$ for any $0<b\le 1/2$, (18) holds trivially. Now suppose that (18) holds for $t-1$, we consider the case for t. Denote $\mathcal{F}_t$ as the support of $\tilde{d}_t$ and for each $d_t\in\mathcal{F}_t$, we denote $\eta_{t,\gamma}(d_t)$ as the threshold defined in (17). Then we define the following division of $\mathcal{F}_t$:

$$\mathcal{F}_{t,1}:=\{d_t\in\mathcal{F}_t:\eta_{t,\gamma}(d_t)=0 \text{ and } d_t\le b\}$$

$$\mathcal{F}_{t,2}=\{d_t\in\mathcal{F}_t:\eta_{t,\gamma}(d_t)=0 \text{ and } b<d_t\le 1-b\}$$

$$\mathcal{F}_{t,3}=\{d_t\in\mathcal{F}_t:\eta_{t,\gamma}(d_t)>0 \text{ and } d_t\le 1-b\}$$

$$\mathcal{F}_{t,4}=\{d_t\in\mathcal{F}_t:\eta_{t,\gamma}(d_t)=0 \text{ and } 1-b<d_t\}$$

$$\mathcal{F}_{t,5}=\{d_t\in\mathcal{F}_t:\eta_{t,\gamma}(d_t)>0 \text{ and } 1-b<d_t\}$$

Note that for each $d_t\in\mathcal{F}_t$, $\eta_{t,\gamma}(d_t)=0$ implies that a measure $p(d_t)\cdot(\gamma_t-\mu_{t-1,\gamma}(0,1-d_t])$ of empty sample paths will be moved to d_t due to the inclusion of realization d_t when defining $\tilde{X}_{t,\gamma}$. More specifically, the movement of sample paths due to the inclusion of each realization $d_t\in\mathcal{F}_t$ can be described as follows:

(i). For each $d_t\in\mathcal{F}_{t,1}$, obviously, $p(d_t)\cdot(\gamma_t-\mu_{t-1,\gamma}(0,1-d_t])$ measure of sample paths, which is upper bounded by $p(d_t)\cdot(\gamma_t-\mu_{t-1,\gamma}(0,1-b])$, will be moved from 0 to the range $(0,b]$, while $p(d_t)\cdot\mu_{t-1,\gamma}(0,b]$ measure of sample paths will be moved out of the range $(0,b]$. Moreover, at most $p(d_t)\cdot\mu_{t-1,\gamma}(0,b]$ measure of sample paths will be moved into the range $(b,1-b]$.

(ii). For each $d_t\in\mathcal{F}_{t,2}$, $p(d_t)\cdot\mu_{t-1,\gamma}(0,b]$ measure of sample paths will be moved out of the range $(0,b]$. Moreover, $p(d_t)\cdot(\gamma_t-\mu_{t-1,\gamma}(0,1-d_t])$ measure of sample paths, which is upper bounded by $p(d_t)\cdot(\gamma_t-\mu_{t-1,\gamma}(0,b])$, will be moved from 0 into the range $(b,1-b]$, while at most $p(d_t)\cdot\mu_{t-1,\gamma}(0,b]$ measure of sample paths will be moved from $(0,b]$ into $(b,1-b]$. Thus, the measure of new sample path that is moved into the range $(b,1-b]$ is upper bounded by $\gamma_t\cdot p(d_t)$.

(iii). For each $d_t\in\mathcal{F}_{t,3}$, then at most $p(d_t)\cdot\mu_{t-1,\gamma}(0,b]$ measure of sample paths is moved out of the range $(0,b]$, and at most $p(d_t)\cdot\mu_{t-1,\gamma}(0,b]$ measure of sample paths is moved into the range $(b,1-b]$.

(iv). For each $d_t\in\mathcal{F}_{t,4}$ or $d_t\in\mathcal{F}_{t,5}$, since $d_t>1-b$, obviously, no new sample path will be added to the range $(b,1-b]$ due to the inclusion of such realization d_t when defining $\tilde{X}_{t,\gamma}$, while the measure of the sample paths within the range $(0,b]$ can only become smaller.

To conclude, denoting

$$\hat{p}_1=\sum_{d_t\in\mathcal{F}_{t,1}}p(d_t) \quad\text{and}\quad \hat{p}_2=\sum_{d_t\in\mathcal{F}_{t,2}}p(d_t) \quad\text{and}\quad \hat{p}_3=\sum_{d_t\in\mathcal{F}_{t,3}}p(d_t)$$

we have that

$$\mu_{t,\boldsymbol{\gamma}}(0,b] \leq \mu_{t-1,\boldsymbol{\gamma}}(0,b] + (\gamma_t - \mu_{t-1,\boldsymbol{\gamma}}(0,1-b] - \mu_{t-1,\boldsymbol{\gamma}}(0,b]) \cdot \hat{p}_1 - \mu_{t-1,\boldsymbol{\gamma}}(0,b] \cdot \hat{p}_2 - \mu_{t-1,\boldsymbol{\gamma}}(0,b] \cdot \hat{p}_3 \tag{45}$$

and

$$\mu_{t,\boldsymbol{\gamma}}(b,1-b] \leq \mu_{t-1,\boldsymbol{\gamma}}(b,1-b] + \mu_{t-1,\boldsymbol{\gamma}}(0,b] \cdot \hat{p}_1 + \gamma_t \cdot \hat{p}_2 + \mu_{t-1,\boldsymbol{\gamma}}(0,b] \cdot \hat{p}_3 \tag{46}$$

Moreover, it holds that $\hat{p}_1 + \hat{p}_2 + \hat{p}_3 \leq 1$. We now consider the following two cases separately.

Case 1: If $\hat{p}_1 > 0$, then we must have $\gamma_t \geq \mu_{t-1,\boldsymbol{\gamma}}(0,1-b]$. Notice that $\hat{p}_1 \leq 1 - \hat{p}_2$, from (45), we have

$$\begin{aligned}
\mu_{t,\boldsymbol{\gamma}}(0,b] &\leq \mu_{t-1,\boldsymbol{\gamma}}(0,b] + (\gamma_t - \mu_{t-1,\boldsymbol{\gamma}}(0,1-b]) \cdot \hat{p}_1 - \mu_{t-1,\boldsymbol{\gamma}}(0,b]) \cdot \hat{p}_1 - \mu_{t-1,\boldsymbol{\gamma}}(0,b] \cdot \hat{p}_2 - \mu_{t-1,\boldsymbol{\gamma}}(0,b] \cdot \hat{p}_3 \\
&\leq \mu_{t-1,\boldsymbol{\gamma}}(0,b] + (\gamma_t - \mu_{t-1,\boldsymbol{\gamma}}(0,1-b]) \cdot (1-\hat{p}_2) - \mu_{t-1,\boldsymbol{\gamma}}(0,b]) \cdot \hat{p}_1 - \mu_{t-1,\boldsymbol{\gamma}}(0,b] \cdot \hat{p}_2 - \mu_{t-1,\boldsymbol{\gamma}}(0,b] \cdot \hat{p}_3 \\
&= (\gamma_t - \mu_{t-1,\boldsymbol{\gamma}}(b,1-b]) \cdot (1-\hat{p}_2) - \mu_{t-1,\boldsymbol{\gamma}}(0,b]) \cdot \hat{p}_1 - \mu_{t-1,\boldsymbol{\gamma}}(0,b] \cdot \hat{p}_3 \\
&\leq (\gamma_1 - \mu_{t-1,\boldsymbol{\gamma}}(b,1-b]) \cdot (1-\hat{p}_2) - \mu_{t-1,\boldsymbol{\gamma}}(0,b]) \cdot \hat{p}_1 - \mu_{t-1,\boldsymbol{\gamma}}(0,b] \cdot \hat{p}_3
\end{aligned} \tag{47}$$

where the last inequality holds from $\gamma_1 \geq \gamma_t$. Moreover, from (46), we have that

$$\begin{aligned}
\exp(-\frac{1}{\gamma_1} \cdot \mu_{t,\boldsymbol{\gamma}}(b,1-b]) &\geq \exp(-\frac{1}{\gamma_1} \cdot \mu_{t-1,\boldsymbol{\gamma}}(b,1-b] - \frac{\gamma_t \hat{p}_2}{\gamma_1}) \cdot \exp(-\frac{1}{\gamma_1} \cdot \mu_{t-1,\boldsymbol{\gamma}}(0,b] \cdot \hat{p}_1 - \frac{1}{\gamma_1} \cdot \mu_{t-1,\boldsymbol{\gamma}}(0,b] \cdot \hat{p}_3) \\
&\geq \exp(-\frac{1}{\gamma_1} \cdot \mu_{t-1,\boldsymbol{\gamma}}(b,1-b] - \hat{p}_2) \cdot \exp(-\frac{1}{\gamma_1} \cdot \mu_{t-1,\boldsymbol{\gamma}}(0,b] \cdot \hat{p}_1 - \frac{1}{\gamma_1} \cdot \mu_{t-1,\boldsymbol{\gamma}}(0,b] \cdot \hat{p}_3) \\
&\geq \exp(-\frac{1}{\gamma_1} \cdot \mu_{t-1,\boldsymbol{\gamma}}(b,1-b] - \hat{p}_2) \cdot (1 - \frac{1}{\gamma_1} \cdot \mu_{t-1,\boldsymbol{\gamma}}(0,b] \cdot \hat{p}_1 - \frac{1}{\gamma_1} \cdot \mu_{t-1,\boldsymbol{\gamma}}(0,b] \cdot \hat{p}_3) \\
&= \exp(-\frac{1}{\gamma_1} \cdot \mu_{t-1,\boldsymbol{\gamma}}(b,1-b] - \hat{p}_2) \\
&\quad - \exp(-\frac{1}{\gamma_1} \cdot \mu_{t-1,\boldsymbol{\gamma}}(b,1-b] - \hat{p}_2) \cdot (\frac{1}{\gamma_1} \cdot \mu_{t-1,\boldsymbol{\gamma}}(0,b] \cdot \hat{p}_1 + \frac{1}{\gamma_1} \cdot \mu_{t-1,\boldsymbol{\gamma}}(0,b] \cdot \hat{p}_3) \\
&\geq \exp(-\frac{1}{\gamma_1} \cdot \mu_{t-1,\boldsymbol{\gamma}}(b,1-b] - \hat{p}_2) - \frac{1}{\gamma_1} \cdot \mu_{t-1,\boldsymbol{\gamma}}(0,b] \cdot \hat{p}_1 - \frac{1}{\gamma_1} \cdot \mu_{t-1,\boldsymbol{\gamma}}(0,b] \cdot \hat{p}_3
\end{aligned} \tag{48}$$

where the second inequality holds from $\gamma_1 \geq \gamma_t$, the third inequality holds from $\exp(-x) \geq 1 - x$ for any $x \geq 0$ and the last inequality holds from $\exp(-x) \leq 1$ for any $x \geq 0$. Further note that

$$\exp(-\frac{1}{\gamma_1} \cdot \mu_{t-1,\boldsymbol{\gamma}}(b,1-b] - \hat{p}_2) = \exp(-\frac{1}{\gamma_1} \cdot \mu_{t-1,\boldsymbol{\gamma}}(b,1-b]) \cdot \exp(-\hat{p}_2) \geq (1 - \frac{1}{\gamma_1} \cdot \mu_{t-1,\boldsymbol{\gamma}}(b,1-b]) \cdot (1-\hat{p}_2)$$

From (47) and (48), we have

$$\begin{aligned}
\exp(-\frac{1}{\gamma_1} \cdot \mu_{t,\boldsymbol{\gamma}}(b,1-b]) &\geq (1 - \frac{1}{\gamma_1} \cdot \mu_{t-1,\boldsymbol{\gamma}}(b,1-b]) \cdot (1-\hat{p}_2) - \frac{1}{\gamma_1} \cdot \mu_{t-1,\boldsymbol{\gamma}}(0,b] \cdot \hat{p}_1 - \frac{1}{\gamma_1} \cdot \mu_{t-1,\boldsymbol{\gamma}}(0,b] \cdot \hat{p}_3 \\
&\geq \frac{1}{\gamma_1} \cdot \mu_{t,\boldsymbol{\gamma}}(0,b]
\end{aligned}$$

Case 2: If $\hat{p}_1 = 0$, then we have

$$\mu_{t,\gamma}(0,b] \leq \mu_{t-1,\gamma}(0,b] - \mu_{t-1,\gamma}(0,b] \cdot \hat{p}_2 - \mu_{t-1,\gamma}(0,b] \cdot \hat{p}_3 \tag{49}$$

and

$$\mu_{t,\gamma}(b,1-b] \leq \mu_{t-1,\gamma}(b,1-b] + \gamma_t \cdot \hat{p}_2 + \mu_{t-1,\gamma}(0,b] \cdot \hat{p}_3 \leq \mu_{t-1,\gamma}(b,1-b] + \gamma_1 \cdot \hat{p}_2 + \mu_{t-1,\gamma}(0,b] \cdot \hat{p}_3$$

Thus, it holds that

$$\begin{aligned}
\exp(-\frac{1}{\gamma_1} \cdot \mu_{t,\gamma}(b,1-b]) &\geq \exp(-\frac{1}{\gamma_1} \cdot \mu_{t-1,\gamma}(b,1-b] - \hat{p}_2) \cdot \exp(-\frac{1}{\gamma_1} \cdot \mu_{t-1,\gamma}(0,b] \cdot \hat{p}_3) \\
&\geq \exp(-\frac{1}{\gamma_1} \cdot \mu_{t-1,\gamma}(b,1-b] - \hat{p}_2) \cdot (1 - \frac{1}{\gamma_1} \cdot \mu_{t-1,\gamma}(0,b] \cdot \hat{p}_3) \\
&\geq \exp(-\frac{1}{\gamma_1} \cdot \mu_{t-1,\gamma}(b,1-b] - \hat{p}_2) - \frac{1}{\gamma_1} \cdot \mu_{t-1,\gamma}(0,b] \cdot \hat{p}_3 \\
&\geq \exp(-\frac{1}{\gamma_1} \cdot \mu_{t-1,\gamma}(b,1-b]) \cdot (1 - \hat{p}_2) - \frac{1}{\gamma_1} \cdot \mu_{t-1,\gamma}(0,b] \cdot \hat{p}_3 \\
&\geq \frac{1}{\gamma_1} \cdot \mu_{t-1,\gamma}(0,b] \cdot (1 - \hat{p}_2) - \frac{1}{\gamma_1} \cdot \mu_{t-1,\gamma}(0,b] \cdot \hat{p}_3
\end{aligned} \tag{50}$$

where the third inequality holds from $\exp(-a) \leq 1$ for any $a \geq 0$ and the last inequality holds from induction hypothesis. Our proof is completed immediately by combining (49) and (50). $\square$

D2. Proof of Theorem 5

Proof: For each fixed t, we define $U_t(s) = \mu_{t,\gamma}(0,s] = P(0 < \tilde{X}_{t,\gamma} \leq s)$ for any $s \in (0,1]$. Note that by Algorithm 2, we have $\mathbb{E}[\tilde{X}_{t,\gamma}] = \sum_{\tau=1}^{t} \gamma_\tau \cdot \psi_\tau$. From integration by parts, we have that

$$\sum_{\tau=1}^{t} \gamma_\tau \cdot \psi_\tau = \mathbb{E}[\tilde{X}_{t,\gamma}] = \int_{s=0}^{1} U_t(s) ds = U_t(1) - \int_{s=0}^{1} U_t(s) ds \tag{51}$$

We then bound the term $\int_{s=0}^{1} U_t(s) ds$. If $U_t(1) < \gamma_1$, then we immediately have

$$P(\tilde{X}_{t,\gamma} = 0) > 1 - \gamma_1 \geq 1 - \gamma_1 - \sum_{\tau=1}^{t} \gamma_\tau \cdot \psi_\tau$$

which proves (19). Thus, in the remaining part of the proof, it is enough for us to only focus on the case $U_t(1) \geq \gamma_1$.

If $U_t(1) \geq \gamma_1$, then there must exists a constant $u^* \in (0,1)$ such that

$$\gamma_1 \cdot u^* - \gamma_1 \cdot \ln(u^*) = U_t(1).$$

We further define

$$s^* = \begin{cases} \min\{s \in (0,1/2] : U_t(s) \geq \gamma_1 \cdot u^*\}, & \text{if } U_t(\frac{1}{2}) \geq \gamma_1 \cdot u^* \\ \frac{1}{2}, & \text{if } U_t(\frac{1}{2}) < \gamma_1 \cdot u^* \end{cases}$$

Following the proof of Theorem 3, we can show that

$$\int_{s=0}^{1} U_t(s)ds \le s^* \cdot (2\gamma_1 \cdot u^* - \gamma_1 \cdot \ln(u^*)) + (1/2 - s^*) \cdot \max\{2\gamma_1 \cdot u^* - \gamma_1 \cdot \ln(u^*), 2\gamma_1\}$$

We further simplify the above expression separately by comparing the value of $2\gamma_1 \cdot u^* - \gamma_1 \cdot \ln(u^*)$ and $2\gamma_1$.

Case 1: If $2\gamma_1 \cdot u^* - \gamma_1 \cdot \ln(u^*) \le 2\gamma_1$, we have $\int_{s=0}^{1} U_t(s)ds \le 2s^*\gamma_1 + \gamma_1 - 2s^*\gamma_1 = \gamma_1$. From (51), we have that

$$U_t(1) \le \gamma_1 + \sum_{\tau=1}^{t} \gamma_\tau \cdot \psi_\tau$$

Case 2: If $2\gamma_1 \cdot u^* - \gamma_1 \cdot \ln(u^*) > 2\gamma_1$, we have $\int_{s=0}^{1} U_t(s)ds \le \gamma_1 \cdot u^* - \frac{\gamma_1}{2} \cdot \ln(u^*)$. From (51) and the definition of u^*, we have that

$$U_t(1) = \gamma_1 \cdot u^* - \gamma_1 \cdot \ln(u^*) \le \sum_{\tau=1}^{t} \gamma_\tau \cdot \psi_\tau + \gamma_1 \cdot u^* - \frac{\gamma_1}{2} \cdot \ln(u^*)$$

The above inequality implies that

$$u^* \ge \exp(-\frac{2}{\gamma_1} \cdot \sum_{\tau=1}^{t} \gamma_\tau \cdot \psi_\tau)$$

Note that the function $x - \ln(x)$ is non-increasing on $(0,1)$, hence we have

$$U_t(1) \le 2 \cdot \sum_{\tau=1}^{t} \gamma_\tau \cdot \psi_\tau + \gamma_1 \cdot \exp(-\frac{2}{\gamma_1} \cdot \sum_{\tau=1}^{t} \gamma_\tau \cdot \psi_\tau)$$

Combing the above two cases, we conclude that

$$U_t(1) \le \max\{\gamma_1 + \sum_{\tau=1}^{t} \gamma_\tau \cdot \psi_\tau, \quad 2 \cdot \sum_{\tau=1}^{t} \gamma_\tau \cdot \psi_\tau + \gamma_1 \cdot \exp(-\frac{2}{\gamma_1} \cdot \sum_{\tau=1}^{t} \gamma_\tau \cdot \psi_\tau)\}$$

Note that $P(\tilde{X}_{t,\gamma} = 0) = 1 - U_t(1)$, we conclude that

$$P(\tilde{X}_{t,\gamma} = 0) \ge \min\{1 - \gamma_1 - \sum_{\tau=1}^{t} \gamma_\tau \cdot \psi_\tau, \quad 1 - 2 \cdot \sum_{\tau=1}^{t} \gamma_\tau \cdot \psi_\tau - \gamma_1 \cdot \exp(-\frac{2}{\gamma_1} \cdot \sum_{\tau=1}^{t} \gamma_\tau \cdot \psi_\tau)\}$$

which completes our proof. □

D3. Proof of Lemma 8

Proof: Since the function $h_{\gamma_0}(\cdot)$ is non-increasing and non-negative over $[0,1]$, it is direct to see that

$$1 \ge \hat{\gamma}_1 \ge \cdots \ge \hat{\gamma}_T \ge 0$$

Note that for each $t = 1, \ldots, T$, we have

$$\int_{\tau=0}^{k_t} h_{\gamma_0}(\tau) d\tau = \sum_{\tau=1}^{t} \hat{\gamma}_\tau \cdot \psi_\tau$$

and $\gamma_0 \geq \hat{\gamma}_1$. Then, for each $t = 1, \ldots, T-1$ and each $\tau \in [k_t, k_{t+1}]$, it holds that

$$h_{\gamma_0}(\tau) \leq 1 - \gamma_0 - \int_{\tau'=0}^{\tau} h_{\gamma_0}(\tau')d\tau' \leq 1 - \gamma_0 - \int_{\tau'=0}^{k_t} h_{\gamma_0}(\tau')d\tau' \leq 1 - \hat{\gamma}_1 - \sum_{\tau'=1}^{t} \hat{\gamma}_{\tau'} \cdot \psi_{\tau'}$$

which implies that

$$\hat{\gamma}_{t+1} \leq 1 - \hat{\gamma}_1 - \sum_{\tau'=1}^{t} \hat{\gamma}_{\tau'} \cdot \psi_{\tau'}$$

since $\hat{\gamma}_{t+1}$ is defined as the average of function $h_{\gamma_0}(\cdot)$ over $[k_t, k_{t+1}]$ in (23).

Similarly, note that the function $2x + \gamma_0 \cdot \exp(-\frac{2}{\gamma_0} \cdot x)$ is monotone increasing when $x \geq 0$. Then, for each $t = 1, \ldots, T-1$ and each $\tau \in [k_t, k_{t+1}]$, we have

$$\begin{aligned}
h_{\gamma_0}(\tau) &\leq 1 - 2 \cdot \int_{\tau'=0}^{\tau} h_{\gamma_0}(\tau')d\tau' - \gamma_0 \cdot \exp(-\frac{2}{\gamma_0} \cdot \int_{\tau'=0}^{\tau} h_{\gamma_0}(\tau')d\tau') \\
&\leq 1 - 2 \cdot \int_{\tau'=0}^{k_t} h_{\gamma_0}(\tau')d\tau' - \gamma_0 \cdot \exp(-\frac{2}{\gamma_0} \cdot \int_{\tau'=0}^{k_t} h_{\gamma_0}(\tau')d\tau') \\
&= 1 - 2 \cdot \sum_{\tau'=1}^{t} \hat{\gamma}_{\tau'} \cdot \psi_{\tau'} - \gamma_0 \cdot \exp(-\frac{2}{\gamma_0} \cdot \sum_{\tau'=1}^{t} \hat{\gamma}_{\tau'} \cdot \psi_{\tau'})
\end{aligned}$$

which implies that

$$\hat{\gamma}_{t+1} \leq 1 - 2 \cdot \sum_{\tau'=1}^{t} \hat{\gamma}_{\tau'} \cdot \psi_{\tau'} - \gamma_0 \cdot \exp(-\frac{2}{\gamma_0} \cdot \sum_{\tau'=1}^{t} \hat{\gamma}_{\tau'} \cdot \psi_{\tau'})$$

since $\hat{\gamma}_{t+1}$ is defined as the average of function $h_{\gamma_0}(\cdot)$ over $[k_t, k_{t+1}]$ in (23). Thus, we conclude that $\{\hat{\gamma}_t\}_{t=1}^{T}$ is a feasible solution to OP($\boldsymbol{\psi}$). $\square$

D4. Proof of Proposition 3

It is enough for us to consider a problem setup $\mathcal{H}$ with T queries, where each query has a deterministic size $\frac{1}{2} + \frac{1}{T}$ and is active with probability $\frac{2}{T}$. It is clear that $\mathbf{UP}(\mathcal{H}) = 1$. However, any online algorithm π can serve at most one query, given at least one query has arrived. Then, the expected capacity utilization of any online algorithm π is upper bound by

$$(\frac{1}{2} + \frac{1}{T}) \cdot (1 - (1 - \frac{2}{T})^T) = \frac{1 - e^{-2}}{2} + O(\frac{1}{T})$$

This implies an upper bound $\frac{1-e^{-2}}{2}$ as $T \to \infty$.