Tightness without Counterexamples: A New Approach and New Results for Prophet Inequalities

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Prophet inequalities consist of many beautiful statements that establish tight performance ratios between online and offline allocation algorithms. Typically, tightness is established by constructing an algorithmic guarantee and a worst-case instance separately, whose bounds match as a result of some "ingenuity". In this paper, we instead formulate the construction of the worst-case instance as an optimization problem, which directly finds the tight ratio without needing to construct two bounds separately.

Our analysis of this complex optimization problem involves identifying structure in a new "Type Coverage" dual problem. It can be seen as akin to the celebrated Magician and OCRS (Online Contention Resolution Scheme) problems, except more general in that it can also provide tight ratios relative to the optimal offline allocation, whereas the earlier problems only establish tight ratios relative to the *ex-ante relaxation* of the offline problem.

Through this analysis, our paper provides a unified framework that derives new results and recovers many existing ones. First, we show that the "oblivious" method of setting a static threshold due to Chawla et al. (2020) is, surprisingly, best-possible among *all* static threshold algorithms, for any number k of starting units. We emphasize that this result is derived without needing to explicitly find any counterexample instances. We establish similar "no separation" results for static thresholds in the IID setting, which although previously known, required the construction of complicated counterexamples. Finally, our framework and in particular our Type Coverage problem yields a simplified derivation of the tight 0.745 ratio when $k = 1$ in the IID setting.

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1. Introduction

Prophet inequalities are fundamental problems in online decision making. In these problems, there is a decision maker with k slots and there are n agents arriving sequentially. Each agent has a valuation drawn from a known distribution, which will be convenient for us to denote in the following form. There are m possible *types* and each agent i belongs to one type j independently with a given probability $p_{i,j}$. Then, the valuation or value for an agent of type j is given by r_j, with $r_1 \geq \cdots \geq r_m \geq 0$. After the type of each agent is revealed, we know the reward of accepting this agent and the decision maker has to decide immediately whether to accept this agent by filling one slot with this agent to collect its value, or to reject it. The decision has to be made on-the-fly, i.e., the decision is irrevocable and has to be independent of the types of the not-arrived agents. The goal of the decision maker is to maximize the expected total collected values.

The typical benchmark for prophet inequalities is the *prophet*, who is aware of the types of all agents and fills the slots by the agents with highest values. By comparing with the prophet, the classical work Krengel and Sucheston (1978) obtains the tight guarantee 1/2 when there is one slot (i.e. $k=1$), and Correa et al. (2017) further improves the tight guarantee to 0.745 under the IID special case, where the value distribution is identical for each agent. Another widely used benchmark is the Ex-Ante relaxation of the prophet, which can be formulated via linear programming (LP). The benefits of the Ex-Ante benchmark are twofold: (i) it is generally easier to analyze (Alaei 2014), and possesses special structures such that the optimal policy and the tight guarantees can be derived even when $k>1$ (Jiang et al. 2022); (ii) it enables a decomposition technique where policies and guarantees for a single resource can easily extend to online allocations problems with multiple resources (i.e. multiple types of slots) (Alaei et al. 2012), even allowing for very general allocation procedures involving customer choice (Gallego et al. 2015).

Various online policies have been developed in the literature in order to achieve a good performance guarantee relative to the benchmarks. Some of the tight performance guarantees are derived from optimal dynamic policies, where the decision depends both on the realization of the current agent's value and the number of currently remaining slots; see, e.g., Correa et al. (2017) and Jiang et al. (2022). Simpler but more restrictive *static* threshold policies also enjoy strong performance guarantees (Hajiaghayi et al. 2007), where the decision is given by comparing the realized value of the agent against a threshold and the threshold is fixed for each agent, regardless of the remaining number of empty slots. A further restriction of *oblivious* static threshold policy was introduced in Chawla et al. (2020), where the threshold depends only on the probabilities $\{p_{i,j}\}_{\forall i\in[n],\forall j\in[m]}$ but not on the values $\{r_j\}_{\forall j\in[m]}$.

The main goal of this paper is to study the performance guarantees of general online policies, as well as (oblivious) static threshold policies, with respect to both the prophet benchmark and the Ex-Ante benchmark, in both the IID and non-IID settings. We denote by I the problem instance and $\mathcal{I}^{\mathrm{IID}}_{k,n}$ (resp, $\mathcal{I}_{k,n}$) the collection of all IID (resp. non-IID) problem instances with n agents and k slots. We also denote by $\mathsf{DP}(I)$ the value of optimal dynamic programming, $\mathsf{ST}(I)$ the value of the optimal non-oblivious threshold policy, $\mathsf{OST}(I)$ the value of the optimal oblivious threshold policy, $\mathsf{Proph}(I)$ the value of the prophet benchmark, and $\mathsf{ExAnte}(I)$ the value of the Ex-Ante benchmark, on the problem instance I (see Section 2 for formal definitions of these concepts).

1.1. New Contributions

The main difference between our approach and existing ones is that we are able to formulate the problem of finding the worst-case instance as solving an optimization problem, while the existing approaches rely on constructing an algorithmic guarantee and a worst-case instance separately, and showing that these two ratios match. As a result, we directly compute the tight guarantees by solving an optimization problem, without any explicit construction. In what follows, we describe our approach and compare it with existing ones.

We introduce a new LP framework, which explicitly computes the *tight* guarantee of an online policy versus both the prophet and the Ex-Ante benchmark. Note that in order to compute the guarantee of a policy, one needs to minimize the ratio of the expected value collected by the policy over the benchmark, over all instances, denoted by $\{p_{i,j}, r_j\}_{\forall i\in[n],\forall j\in[m]}$. The new idea of our framework is to break down this minimizing process into two stages. Given a fixed number of slots k and agents n, we first minimize over m and the type distributions $\{p_{i,j}\}_{\forall i\in[n],\forall j\in[m]}$, and we then minimize over the type valuations $\{r_j\}_{\forall j\in[m]}$. For example, if we are computing the tight guarantee for general online algorithms relative to the prophet over general non-IID instances, then our formulation would be

$$\inf_{m,\{p_{i,j}\}_{\forall i\in[n],\forall j\in[m]}} \inf_{\{r_j\}_{\forall j\in[m]}} \frac{\mathsf{DP}(I)}{\mathsf{Proph}(I)}. \tag{1}$$

The inner optimization problem over $\{r_j\}_{\forall j\in[m]}$ can be formulated as an LP. The dual of this LP gives us a new problem that is defined for each m and $\{p_{i,j}\}_{\forall i\in[n],\forall j\in[m]}$, whose optimal solution has a nice structure. This structure enables us to derive multiple new results, and recover several existing results in a clean way.

Type Coverage problem. The new dual LP can be interpreted as a "Type Coverage" problem, defined as follows. For each type j, let Q_j denote the expected number of agents of *type j or better* accepted by the benchmark. For example, if the benchmark is the prophet, then

$$Q_j = \mathbb{E}[\min\{\sum_{i=1}^{n} \mathrm{Ber}\Big(\sum_{j'=1}^{j} p_{i,j'}\Big), k\}], \tag{2}$$

where $\mathrm{Ber}(p)$ denotes an independent Bernoulli random variable of mean p. The RHS of (2) then counts the total agents among $i=1,\ldots,n$ who realize to a type j' whose value is at least as good as that of type j, and truncates this total by k to indicate the number of such agents accepted by the prophet. Finally, for an online policy to achieve a guarantee of θ in this dual Type Coverage problem, for every type $j\in[m]$, the number of agents of type j or better accepted by the online policy must be at least $\theta\cdot Q_j$.

Our framework shows that the guarantee under the worst-case instance in the original problem is equivalent to the guarantee under the worst-case instance for the Type Coverage problem, where now the parameters r_j have been eliminated. If the original prophet problem restricted to an *oblivious* static threshold policy, then the corresponding dual restriction is to have *any* static threshold policy. Meanwhile, if the original problem allowed non-oblivious static thresholds, then the dual allows *randomized* static thresholds (which can achieve a strictly greater θ for the Type Coverage problem, as we later show). These correspondences in the restrictions, which are not a priori obvious, are consequences of our framework and formalized in Section 2.2.

Another interpretation of our approach in (1) is that we are decoupling the adversary's optimization into an outer problem over *ordinal* distributions (of m ranked types) and an inner problem over *cardinal* valuations consistent with these ordinal rankings. General static thresholds can be set after the adversary decides these cardinal valuations, whereas oblivious static thresholds must be set before the adversary decides them.

Comparison vs. Magician/OCRS (Online Contention Resolution Scheme) problems. Magician (Alaei 2014) and OCRS (Feldman et al. 2021) are existing auxiliary problems used to derive Ex-Ante prophet inequalities. In these problems, each agent has an *active* probability and must be guaranteed to be accepted with a probability at least θ conditional on being active, by the online policy. The goal is to achieve the maximum θ. It has been shown in Jiang et al. (2022) that the optimal guarantees for the k-unit Magician/OCRS problems are actually the same, leading to tight Ex-Ante prophet inequalities.

Although ex-ante prophet inequalities are stronger and hence imply non-ex-ante prophet inequalities, no analogue to Magician/OCRS that can analyze *tight non-ex-ante prophet inequalities* has been known, until now. Our new Type Coverage problem essentially generalizes the Magician/OCRS problems to allow for agents to take *non-binary* states, which leads to a formulation of tight guarantees for non-ex-ante prophet inequalities. Our Type Coverage problem is also more general in that it can recover the Magician/OCRS problems in the appropriate setting, as we derive in Section 3.1. We now describe the other new results and simplifications it achieves.

(Oblivious) Static thresholds in Non-IID setting. First, for OST policies in the general non-IID setting, we recover the following positive result, which has been previously established in Chawla et al. (2020):

$$\inf_{I\in\mathcal{I}_{k,n}} \frac{\mathsf{OST}(I)}{\mathsf{ExAnte}(I)} \geq \mathrm{BernOpt}(k,n). \tag{3}$$

In (3), $\mathrm{BernOpt}(k,n)$ is an optimization problem over $n-1$ Bernoulli random variables that tweaks what Chawla et al. (2020) derived, with the formal specification of $\mathrm{BernOpt}(k,n)$ found in Corollary 1 in Section 3.2. We then show this specification to be *tight* for oblivious static thresholds:

$$\inf_{I\in\mathcal{I}_{k,n}} \frac{\mathsf{OST}(I)}{\mathsf{Proph}(I)} \leq \mathrm{BernOpt}(k,n), \tag{4}$$

even when competing against the weaker prophet benchmark. We derive this upper bound without explicitly constructing any counterexamples. Instead, we show that when the online policy is restricted to OST, the value of the dual Type Coverage problem cannot exceed a quantity (exact formulation given in (20)) which can be shown to be equivalent to $\mathrm{BernOpt}(k,n)$.

We then show that even non-oblivious static threshold policies cannot do better than the method of Chawla et al. (2020) in the worst case, although there can be a separation (see Lemma 5 in Section 3.3) between OST and ST for a given instance. To do so, we invoke the result of Chawla et al. (2020) that the worst-case instance is achieved when the Bernoulli random variables have equal probabilities. In this case, we show through a fairly technical argument (Theorem 6) that randomized static thresholds (corresponding to ST) cannot do better than a deterministic one (corresponding to OST) in the dual Type Coverage problem. In this way, we show that the ratios $\mathsf{ST}/\mathsf{Proph}$, $\mathsf{ST}/\mathsf{ExAnte}$, $\mathsf{OST}/\mathsf{Proph}$, and $\mathsf{OST}/\mathsf{ExAnte}$ are all equivalent in the worst case, for general non-IID distributions. This equivalence is new to the literature.

(Oblivious) Static thresholds in IID setting. When the distributions are IID, we show (in Section 4.1) that

$$\inf_{I\in\mathcal{I}^{\mathrm{IID}}_{k,n}}\frac{\mathsf{OST}(I)}{\mathsf{ExAnte}(I)}=\inf_{I\in\mathcal{I}^{\mathrm{IID}}_{k,n}}\frac{\mathsf{DP}(I)}{\mathsf{ExAnte}(I)}=\inf_{I\in\mathcal{I}^{\mathrm{IID}}_{k,n}}\frac{\mathsf{OST}(I)}{\mathsf{Proph}(I)}=\frac{\mathbb{E}[\min\{\mathrm{Bin}(n,k/n),k\}]}{k}. \tag{5}$$

That is, unlike the non-IID setting, even the optimal dynamic policy DP cannot do any better than static thresholds when compared to the Ex-Ante benchmark. We then establish a stronger equivalence between oblivious and non-oblivious static thresholds in the IID setting. Recall that in the non-IID setting, the equivalence was only true in the worst case, i.e., only after taking an infimum over problem instance I were the ratios $\mathsf{ST}/\mathsf{Proph}$, $\mathsf{ST}/\mathsf{ExAnte}$, $\mathsf{OST}/\mathsf{Proph}$, $\mathsf{OST}/\mathsf{ExAnte}$ equal. In the IID setting, we show (in Section 4.2) that ST is no better than OST with respect to both Proph and ExAnte, on *every* type distribution. As a result, OST's are *instance-optimal* for static threshold policies in the IID setting, and hence $\inf_{I\in\mathcal{I}^{\mathrm{IID}}_{k,n}}\frac{\mathsf{ST}(I)}{\mathsf{Proph}(I)}$ is also no better than (5).

These results are not new in that both the bounds $\inf_{I\in\mathcal{I}^{\mathrm{IID}}_{k,n}}\frac{\mathsf{OST}(I)}{\mathsf{ExAnte}(I)}\geq\frac{\mathbb{E}[\min\{\mathrm{Bin}(n,k/n),k\}]}{k}$ and $\inf_{I\in\mathcal{I}^{\mathrm{IID}}_{k,n}}\frac{\mathsf{DP}(I)}{\mathsf{ExAnte}(I)}\leq\frac{\mathbb{E}[\min\{\mathrm{Bin}(n,k/n),k\}]}{k}$ are folklore, and corresponding upper bounds on $\inf_{I\in\mathcal{I}^{\mathrm{IID}}_{k,n}}\frac{\mathsf{ST}(I)}{\mathsf{Proph}(I)}$ can also be constructed (see Arnosti and Ma (2021)). However, these upper bounds for static threshold policies relative to the weaker prophet benchmark require a complicated family of examples with messy calculations, whereas our framework elegantly establishes the tightness of the ratio $\frac{\mathbb{E}[\min\{\mathrm{Bin}(n,k/n),k\}]}{k}$ without needing to construct any counterexamples.

General dynamic policies in IID setting. When $k=1$, we obtain a simplified derivation of the classical 0.745 guarantee for $\mathsf{DP}/\mathsf{Proph}$ under the IID setting (Hill and Kertz 1982, Correa et al. 2017). Under the IID setting, the dual problem can be further simplified as a semi-infinite

LP by noting that $p_{i,j}$ equals some value p_j across all agents $i \in [n]$. The key step of our derivation is that the simplified semi-infinite LP admits a closed-form optimal solution, which helps us to establish the worst case as $n \to \infty$ and obtain the value of the optimal guarantee.

We note that a simplified proof of the 0.745 bound has also been recently introduced in Liu et al. (2021), using the clever notion of (ε, δ)-smallness. Our proof is of a different nature, where we directly solve the adversary's optimization problem over all instances.

1.2. Related Work and Comparisons of Results

Multi-unit prophet inequalities. Since first posed by Krengel and Sucheston (1978), prophet inequalities have been extensively studied (see a survey Correa et al. (2019a)) and in recent years, there have been a lot of working studying prophet inequalities with various feasibility constraints over the agents that can be accepted, including knapsack constraint (Feldman et al. 2021) and general matroid constraints (Kleinberg and Weinberg 2012). Our problem poses a uniform matroid constraint for the agents, and our problem is also referred to as the *multi-unit* prophet inequalities (Hajiaghayi et al. 2007). In a seminal work, Alaei (2014) obtains a $1 - \frac{1}{\sqrt{k+3}}$-guarantee when there are k slots in total. Jiang et al. (2022) improves the guarantee for every $k > 1$ and shows that their guarantee is tight with respect to the Ex-Ante benchmark. Our work further improves the existing results by formulating an LP to compute the ratio of a policy for every given value distribution densities. By minimizing the value distribution densities, we are able to obtain the tight guarantee, not only with respect to the Ex-Ante benchmark, but also with respect to the prophet benchmark. Our framework also enables us to study various types of policies, including the (oblivious) static threshold policies, at the same time, while the approaches developed in Alaei (2014), Jiang et al. (2022) are to analyze dynamic policies.

The connection to online pricing. Prophet inequalities have also been extensively used as algorithmic subroutine for online pricing and mechanism design, since the seminal works of Hajiaghayi et al. (2007), Chawla et al. (2010), Alaei (2014). Interested readers are referred to Lucier (2017) for a more detailed survey. Recently, Correa et al. (2019b) proves that designing posted price mechanisms is equivalent to designing online policies for prophet inequalities, in terms of performance guarantees. As a result, our approaches can be directly applied to designing online pricing policies with tight guarantees. Moreover, the static threshold policies we developed for prophet inequalities imply static threshold policies for online pricing, which are easy to describe and enjoys practical benefits - it is anonymous, non-adaptive, and order-oblivious (Chawla et al. 2020).

Static thresholds. Static threshold policies are important policies for both prophet inequalities and online pricing, and have been extensively studied. In particular, Hajiaghayi et al. (2007) shows that a static threshold policy enjoys a guarantee of $1-O(\sqrt{\frac{\log k}{k}})$, where k is the number of slots. In the IID setting, Yan (2011) shows that a static threshold policy achieves a guarantee of $1-\frac{k^k}{e^k k!}$, which is tight with respect to the Ex-Ante benchmark. This guarantee has been extended in Arnosti and Ma (2021) to the prophet secretary setting where the agents arrive in a random order, and has been shown to be tight among the static threshold policies. Static policy has also been used in online pricing problem with strategic customers to achieve a guarantee of $1-1/e$ (Chen et al. 2019). Beyond explicitly computing the tight guarantees for the optimal static threshold policies, our framework also reveals that the optimal static threshold policy can be value-oblivious under various settings, which implies that only the type distributions are required to be known when designing the optimal static threshold policy. The benefits of being oblivious have been illustrated in Arnosti and Ma (2021) when only a monotone transformation of values (rather than the values themselves) can be observed.

2. Problem Classes Considered and General Framework

We consider the prophet inequality problem where k out of $n>k$ agents can be accepted. That is, initially there are k *slots*, and n agents arrive in order $i=1,\dots,n$, each with a *valuation* $R_i \geq 0$ that is revealed only when agent i arrives. One must then immediately decide whether to use a slot to accept agent i, or to reject agent i forever, with this decision being irrevocable. Once all k slots have been used, no further agents can be accepted. The valuations R_i are drawn independently at random from known distributions. The objective is to make accept/reject decisions on-the-fly, in a way that maximizes the expected sum of valuations of accepted agents.

We assume that each R_i is drawn from a discrete distribution, input as follows. There is a universe of m possible valuations, sorted in the order $r_1 \geq \cdots \geq r_m \geq 0$. For each agent i, we let $p_{ij} \geq 0$ denote the probability that their valuation R_i realizes to r_j, for all $j=1,\dots,m$, with $\sum_{j=1}^m p_{ij}=1$. We refer to index j as the *type* of an agent, with smaller indices j said to be *better*. For simplicity, we also assume that agents arrive in exactly the order $i=1,\dots,n$, which is known in advance. Although many of the algorithms we discuss hold under certain adversarial manipulations of the arrival order, we do not attempt to make such distinctions comprehensively.

Definition 1 (Instance, IID vs. Non-IID). An *instance* I of the prophet inequality problem is defined by the number of slots k, agents n, types m, the valuations $r_1,\dots,r_m$, and the probability vectors $(p_{1j})_{j=1}^m,\dots,(p_{nj})_{j=1}^m$ under the fixed arrival order $i=1,\dots,n$. We let $\mathcal{I}_{k,n}$ denote the class of all instances with k slots and n agents, with $\mathcal{I}_k := \bigcup_{n=1}^\infty \mathcal{I}_{k,n}$. If p_{ij} is identical to some p_j across all agents i, for each type j, then we say that the instance is *IID* (independent and identically distributed), and we let $\mathcal{I}_{k,n}^{\text{IID}}$ denote the class of all IID instances with k slots and n agents, with $\mathcal{I}_k^{\text{IID}} := \bigcup_{n=1}^\infty \mathcal{I}_{k,n}^{\text{IID}}$. We sometimes refer to general instances as *non-IID*.

Under the assumptions mentioned earlier, given an instance, the optimal policy for making accept/reject decisions is easy to compute using dynamic programming (DP). However, as discussed in the Introduction, there is great interest and applicability in comparing the performance of DP to that of a *prophet* who knows the realizations of R_i in advance, over different instances. Furthermore, one can compare against the stronger *ex-ante* benchmark in which the number of agents of each type always equals its expectation. We now formally define these benchmarks.

Definition 2 (Prophet). The prophet's performance on an instance I, denoted $\mathsf{Proph}(I)$, is the expected sum of the k largest realized valuations. Formally, letting $[n]$ denote the set $\{1,\dots,n\}$,

$$\mathsf{Proph}(I) = \mathbb{E}\left[\max_{S\subseteq[n],|S|\le k}\sum_{i\in S}R_i\right].$$

Definition 3 (Ex-ante Relaxation). The ex-ante relaxation is defined by the following LP:

$$\begin{aligned}\mathsf{ExAnte}(I) = \max\ & \sum_{j=1}^m r_j a_j \\ \text{s.t.}\ & \sum_{j=1}^m a_j \le k \\ & 0 \le a_j \le \sum_{i=1}^n p_{ij} \qquad \forall j\in[m]\end{aligned}$$

in which variable a_j can be interpreted as the number of agents of type j accepted.

The following standard result is easy to show, holding because on every sample path, the prophet's selection of k largest valuations forms a feasible solution to the ex-ante LP.

Proposition 1 **(Folklore)**. $\mathsf{Proph}(I) \le \mathsf{ExAnte}(I)$ *for all instances* I.

Having explained the benchmarks $\mathsf{Proph}(I)$ and $\mathsf{ExAnte}(I)$, we are now ready to define the notion of a prophet inequality. Let $\mathsf{DP}(I)$ denote the expected sum collected by the optimal DP algorithm on an instance I. Let $\mathcal{I}$ denote a class of instances.

Definition 4. A *prophet inequality (resp. ex-ante prophet inequality)* is a statement of the form

$$\frac{\mathsf{DP}(I)}{\mathsf{Proph}(I)} \ge \alpha \qquad (\text{resp. } \frac{\mathsf{DP}(I)}{\mathsf{ExAnte}(I)} \ge \alpha) \qquad\qquad \forall I\in\mathcal{I} \tag{6}$$

for some constant $\alpha \le 1$. We refer to α as the *guarantee relative to the prophet (resp. ex-ante relaxation)*. We say that α is *tight* if it is the supremum value for which (6) holds.

In this paper we are interested in tight guarantees relative to both the prophet and ex-ante relaxation, for both the classes of IID and non-IID instances with a particular number of slots k. That is, we are interested in the values of $\alpha = \inf_{I\in\mathcal{I}}\frac{\mathsf{DP}(I)}{\mathsf{Proph}(I)}$ and $\alpha = \inf_{I\in\mathcal{I}}\frac{\mathsf{DP}(I)}{\mathsf{ExAnte}(I)}$ when $\mathcal{I}$ can be $\mathcal{I}_k$ or $\mathcal{I}_k^{\mathrm{IID}}$ for some k, which will affect the value of α. Many of our results also imply tight guarantees for more granular classes of instances, e.g. $\mathcal{I} = \mathcal{I}_{k,n}$ or $\mathcal{I}^{\text{IID}} = \mathcal{I}_{k,n}$, which restrict to having exactly n agents. Guarantees relative to the ex-ante relaxation are worse than those relative to the prophet (due to Proposition 1), and guarantees are also be worse for larger classes of instances (e.g. non-IID instead of IID). Finally, we sometimes consider the following subclass of policies that are not as powerful as the optimal DP, which would also make guarantees worse.

Definition 5 (Static Threshold Policies). A *static threshold* policy accepts the first k agents to arrive who have valuation at least r_J, or equivalently have a type j with $j \leq J$, for some fixed index $J \in [m]$. A static threshold policy is also allowed to set a tie-break probability $\rho \in (0,1]$, where each agent of type exactly J is accepted (while slots remain) according to an independent coin flip of probability ρ.

We emphasize that a static threshold policy is not allowed to change J on-the-fly based on the remaining number of slots or agents, making them less powerful than the optimal DP. Static policies of this type have been previously studied in Ehsani et al. (2018), Chawla et al. (2020), Arnosti and Ma (2021), who note that the tie-break probability is unnecessary (i.e. one can always set $\rho = 1$) under alternative models where the valuation distributions are continuous.

We further distinguish between *oblivious* static threshold algorithms that must set J, ρ without knowing the cardinal values of $r_1, \ldots, r_m$ (but knowing m and $\{p_{ij} : i \in [n], j \in [m]\}$), vs. an algorithm that can set J, ρ optimally with full knowledge of the instance I. This distinction was introduced by Chawla et al. (2020) and the benefits of being oblivious are elaborated on in Arnosti and Ma (2021). As an example of this distinction, a thresholding rule based on the *median* value of $\mathsf{Proph}(I)$ (as proposed in Samuel-Cahn (1984)) is oblivious, but a thresholding rule based on the *mean* value (as proposed in Kleinberg and Weinberg (2012)) is not.

2.1. Main Idea of General Framework

The idea of our general framework is to *explicitly formulate the adversary's optimization problem* of minimizing some prophet inequality ratio, e.g. $\mathsf{DP}(I)/\mathsf{Proph}(I)$, over all instances I belonging to some class. This allows us to *directly compute the tight guarantee* α in different settings, without having to separately construct a lower bound (usually based on analyzing a simple algorithm) and hoping that it is possible to construct a matching upper bound. Of course, the adversary's problem can be highly intractable, because it needs to optimize over the space of distributions, and encapsulate the best response from the (possibly restricted) algorithm on each instance I.

Our general framework overcomes this intractability by breaking down the adversary's problem into two stages. Treating the number of slots k and agents n as fixed, we assume that the adversary first optimizes over m and the type distributions $\{p_{ij} : i \in [n], j \in [m]\}$, and then optimizes over the specific valuations $\{r_j : j \in [m]\}$. The inner optimization problem over r_j's can then be formulated as an LP, which encapsulates the algorithm's best response using constraints that are linear when the p_{ij}'s are fixed. The dual of this LP gives rise to a new problem that is expressed solely in terms of the type distributions, whose optimal solution has a nice structure. This allows us to solve the outer optimization problem over type distributions in many settings.

Before proceeding we define some notation that will simplify the formulation of these problems.

Definition 6 (Δ_j, G_{ij}). Let $\Delta_j := r_j - r_{j+1}$ for all $j \in [m]$, with r_{m+1} understood to be 0. We will equivalently write the adversary's inner optimization problem using decision variables Δ_j. Recalling that $r_1 \geq \cdots \geq r_m \geq 0$, we have $\Delta_j \geq 0$ for all j, where Δ_j can be interpreted as the "valuation gain" when going from type $j+1$ to type j.

Also, let $G_{ij} := \sum_{j'=1}^{j} p_{ij'}$, the probability that agent i's valuation is at least r_j, for all $i \in [n]$ and $j = 0, \ldots, m$. Note that $0 = G_{i0} \leq \cdots \leq G_{im} = 1$ for all i.

Proposition 2. *For any instance I,*

$$\mathsf{Proph}(I) = \sum_{j=1}^{m} \Delta_j \cdot \mathbb{E}[\min\{\sum_{i=1}^{n} \mathrm{Ber}(G_{ij}), k\}] \qquad \textit{and} \qquad \mathsf{ExAnte}(I) = \sum_{j=1}^{m} \Delta_j \cdot \min\{\sum_{i=1}^{n} G_{ij}, k\}, \quad (7)$$

where $\mathrm{Ber}(G_{ij})$ *denotes an independent Bernoulli random variable with success probability* G_{ij}*.*

Proposition 2 is proven in Appendix A. Letting Q_j denote $\mathbb{E}[\min\{\sum_{i=1}^{n} \mathrm{Ber}(G_{ij}), k\}]$ (resp. $\min\{\sum_{i=1}^{n} G_{ij}, k\}$), Q_j can be interpreted as the expected number of agents of type j or better accepted by the prophet (resp. ex-ante relaxation), which explains the formulas in (7).

We are now ready to formulate the adversary's inner problem of minimizing $\mathsf{DP}(I)/\mathsf{Proph}(I)$ or $\mathsf{DP}(I)/\mathsf{ExAnte}(I)$ over decision variables Δ_j. By Proposition 2, as long as the type distributions given by G_{ij} are fixed, both $\mathsf{Proph}(I)$ and $\mathsf{ExAnte}(I)$ can be expressed as linear combinations of Δ_j, which the adversary normalizes to 1. Subject to this, the adversary then tries to minimize $\mathsf{DP}(I)$.

Definition 7 (Inner Problem for Minimizing DP). Consider the following linear program, with Q_j set to $\mathbb{E}[\min\{\sum_{i=1}^{n} \mathrm{Ber}(G_{ij}), k\}]$ (resp. $\min\{\sum_{i=1}^{n} G_{ij}, k\}$) for all $j \in [m]$.

$$\min V_1^k \tag{8a}$$

$$\text{s.t. } V_i^l = \sum_{j=1}^{m} p_{ij} U_{ij}^l + V_{i+1}^l \qquad \forall i \in [n], l \in [k] \tag{8b}$$

$$U_{ij}^l \geq \sum_{j'=j}^{m} \Delta_{j'} - (V_{i+1}^l - V_{i+1}^{l-1}) \qquad \forall i \in [n], j \in [m], l \in [k] \tag{8c}$$

$$\sum_{j=1}^{m} Q_j \Delta_j = 1 \tag{8d}$$

$$\Delta_j, U_{ij}^l \geq 0 \qquad \forall i \in [n], j \in [m], l \in [k] \tag{8e}$$

In constraint (8b), free variable V_i^l denotes the value-to-go of the DP when agent i arrives with exactly l slots remaining, with V_i^l understood to be 0 if $i = n+1$ or $l = 0$. Meanwhile, auxiliary variable U_{ij}^l denotes the utility gain when agent i realizes to type j with l slots remaining. The utility gain U_{ij}^l is lower-bounded by both 0 and the expression $\sum_{j'=j}^m \Delta_{j'} - V_{i+1}^l + V_{i+1}^{l-1}$ in (8c), which denotes the immediate gain from accepting agent i (who has valuation $r_j = \sum_{j'=j}^m \Delta_{j'}$) minus the loss $(V_{i+1}^l - V_{i+1}^{l-1})$ from proceeding to agent $i+1$ with $l-1$ instead of l slots remaining. Finally, constraint (8d) normalizes the value of $\mathsf{Proph}(I)$ (resp. $\mathsf{ExAnte}(I)$) to 1.

Therefore, in the linear program V_1^k will equal precisely the optimal performance $\mathsf{DP}(I)$ of dynamic programming (see e.g. Puterman 2014), and hence LP (8) correctly describes the adversary's inner problem of minimizing $\mathsf{DP}(I)/\mathsf{Proph}(I)$ (resp. $\mathsf{DP}(I)/\mathsf{ExAnte}(I)$) over all instances I with some given type distributions. We now take the dual of (8) to uncover a new problem.

Definition 8 (Dual of Inner Problem for Minimizing DP). Defining dual variables x_i^l, y_{ij}^l, θ for constraints (8b),(8c),(8d) respectively, the following LP is dual to (8).

$$\max \theta \tag{9a}$$

$$\text{s.t.} \quad \theta \cdot Q_j \le \sum_{i=1}^n \sum_{l=1}^k \sum_{j'=1}^j y_{ij'}^l \qquad \forall j \in [m] \tag{9b}$$

$$y_{ij}^l \le p_{ij} x_i^l \qquad \forall i \in [n], j \in [m], l \in [k] \tag{9c}$$

$$x_i^l = \begin{cases} 1, & i=1, l=k \\ 0, & i=1, l<k \\ x_{i-1}^l - \sum_{j=1}^m (y_{i-1,j}^l - y_{i-1,j}^{l+1}), & i>1 \end{cases} \qquad \forall i \in [n], l \in [k] \tag{9d}$$

$$y_{ij}^l \ge 0 \qquad \forall i \in [n], j \in [m], l \in [k] \tag{9e}$$

Before trying to interpret the LP (9), we notice the following structure. An optimal solution will always saturate constraints (9c) for better types j before setting $y_{ij'}^l > 0$ for types $j' > j$. This allows us to reformulate the LP using collapsed variables $y_i^l = \sum_j y_{ij}^l$, as formalized below.

Lemma 1 **(Dual Simplification)**. *LP (9) has the same optimal value as the following LP.*

$$\max \theta \tag{10a}$$

$$\text{s.t.} \quad \theta \cdot Q_j \le \sum_{i=1}^n \sum_{l=1}^k \min\{y_i^l, G_{ij} x_i^l\} \qquad \forall j \in [m] \tag{10b}$$

$$y_i^l \le x_i^l \qquad \forall i \in [n], l \in [k] \tag{10c}$$

$$x_i^l = \begin{cases} 1, & i=1, l=k \\ 0, & i=1, l<k \\ x_{i-1}^l - y_{i-1}^l + y_{i-1}^{l+1}, & i>1 \end{cases} \qquad \forall i \in [n], l \in [k] \tag{10d}$$

$$y_i^l \ge 0 \qquad \forall i \in [n], l \in [k] \tag{10e}$$

Lemma 1 is proven in Appendix A. We refer to (10) as an LP since the non-linear term $\min\{y_i^l, G_{ij} x_i^l\}$ can easily be represented using an auxiliary variable and linear constraints.

We note that LP (10) has the following interpretation. Free variable x_i^l denotes the probability of having exactly l slots remaining when agent i arrives, and y_i^l denotes the (unconditional) probability of accepting the agent in this state, which must lie in $[0, x_i^l]$ as enforced by (9c), (9e). Meanwhile, (9d) correctly updates the state probabilities x_i^l based on the acceptance probabilities y_j^l, which are understood to be 0 if $l = k+1$. Finally, $\min\{y_i^l, G_{ij}x_i^l\}$ represents the probability of accepting agent i with type j or better when there are l slots remaining, which is bottlenecked by $G_{ij}x_i^l$ (the probability of agent i having type j or better in state l) and y_i^l (the unconditional probability of accepting agent i in state l). Therefore, constraint (9b) says that the expected number of agents of type j or better accepted must be at least θ in comparison to Q_j, which is the number of agents of type j or better accepted by the prophet or ex-ante relaxation. The algorithm's guarantee is then given by the maximum θ that can be uniformly achieved across all types j.

We now formalize some more notation and summarize the developments of this section.

Definition 9 ($\mathbf{G}$, Simplified Duals for DP). Let $\mathbf{G}$ denote the collective information about the type distributions, which includes m and the values of G_{ij} that must satisfy $G_{i1} \leq \cdots \leq G_{im} = 1$ for all i. For any such valid $\mathbf{G}$, let $\text{innerLP}_{k,n}^{\mathsf{DP/Proph}}(\mathbf{G})$ (resp. $\text{innerLP}_{k,n}^{\mathsf{DP/ExAnte}}(\mathbf{G})$) denote the LP (10) where Q_j is set to $\mathbb{E}[\min\{\sum_{i=1}^n \text{Ber}(G_{ij}), k\}]$ (resp. $\min\{\sum_{i=1}^n G_{ij}, k\}$) for all $j \in [m]$.

Theorem 1 **(Reformulation of Tight Guarantees for DP)**. *For any fixed k and $n > k$,*

$$\inf_{I \in \mathcal{I}_{k,n}} \frac{\mathsf{DP}(I)}{\mathsf{Proph}(I)} = \inf_{\mathbf{G}} \ \text{innerLP}_{k,n}^{\mathsf{DP/Proph}}(\mathbf{G}) \qquad \textit{and} \qquad \inf_{I \in \mathcal{I}_{k,n}} \frac{\mathsf{DP}(I)}{\mathsf{ExAnte}(I)} = \inf_{\mathbf{G}} \ \text{innerLP}_{k,n}^{\mathsf{DP/ExAnte}}(\mathbf{G}).$$

Put in words, Theorem 1 says that the tight guarantee for the optimal DP relative to the prophet (resp. ex-ante relaxation) is given by $\inf_{\mathbf{G}} \ \text{innerLP}_{k,n}^{\mathsf{DP/Proph}}(\mathbf{G})$ (resp. $\inf_{\mathbf{G}} \ \text{innerLP}_{k,n}^{\mathsf{DP/ExAnte}}(\mathbf{G})$), which are based on our simplified dual formulation and have reduced the adversary's problem to be only over type distributions $\mathbf{G}$. Before delving into how to solve these problems over $\mathbf{G}$, we develop analogues of Theorem 1 in the settings where the algorithm in the primal problem is restricted to static threshold policies. We note that in the IID special case, there is a significant further simplification that expresses the adversary's entire problem as a single semi-infinite LP, which we derive in Section 4.

2.2. General Framework for (Oblivious) Static Threshold Algorithms

When the algorithm is restricted to static threshold policies, we can similarly formulate an inner primal LP and take its dual to uncover a new problem that depends on the type distributions but not on the specific valuations of agents. In fact, a nice *distinction* emerges in the dual depending on whether the algorithm must set the static threshold while oblivious to the specific valuations.

Definition 10 (Oblivious vs. Non-oblivious Static Thresholds). For any instance I, let $\mathsf{ST}(I)$ denote the expected performance of the best static threshold policy on I, whose parameters J,ρ can be set knowing instance I. By contrast, an *oblivious* static threshold (OST) algorithm must set the parameters J,ρ without knowing the specific agent valuations in the instance (but knowing everything else, including the type distributions).

The tight guarantee for OST algorithms relative to the prophet (resp. ex-ante relaxation) is defined by the following sequence of optimizations. First, for a fixed k and n, the adversary sets $\mathbf{G}$, which we recall is defined by m and $\{G_{ij} : i \in [n], j \in [m]\}$. Based on $\mathbf{G}$, the algorithm fixes the parameters J,ρ of the static threshold policy to be used. Finally, the adversary sets the valuations, defined by Δ_j, to minimize the policy's performance relative to $\mathsf{Proph}(I)$ (resp. $\mathsf{ExAnte}(I)$).

We first focus on tight guarantees for OST algorithms, which are simpler to capture using our framework. For a fixed $\mathbf{G}$ and parameters J,ρ chosen by the OST, we write the adversary's inner optimization problem over Δ_j.

Definition 11 (Inner Problem for Minimizing OST). Consider the following LP, where coefficient Q_j can be set to either $\mathbb{E}[\min\{\sum_{i=1}^n \mathrm{Ber}(G_{ij}), k\}]$ or $\min\{\sum_{i=1}^n G_{ij}, k\}$ like before.

$$\min\ V_1^k \tag{11a}$$

$$\text{s.t.}\quad V_i^l = \sum_{j<J} p_{ij} U_{ij}^l + p_{iJ}\rho U_{iJ}^l + V_{i+1}^l \qquad \forall i \in [n], l \in [k] \tag{11b}$$

$$U_{ij}^l = \sum_{j'=j}^{m} \Delta_{j'} - V_{i+1}^l + V_{i+1}^{l-1} \qquad \forall i \in [n], j \in [m], l \in [k] \tag{11c}$$

$$\sum_{j=1}^{m} Q_j \Delta_j = 1 \tag{11d}$$

$$\Delta_j \geq 0 \qquad \forall j \in [m] \tag{11e}$$

Compared to the inner LP (8) for the optimal DP algorithm, LP (11) differs in two ways. First, U_{ij}^l is now a free variable that could be negative, representing the change in utility when agent i takes type j and is *accepted* with l slots remaining, as set in (11c). Second, the policy is now *forced to accept* an agent with type $j < J$ w.p. 1 and an agent with type $j = J$ w.p. ρ, regardless of l and i, as reflected in constraints (11b). This allows the adversary to create a smaller objective value in LP (11), ultimately yielding a maximization problem for the dual in which the collapsed variable y_i^l (from the simplification in Lemma 1) must satisfy essentially the same static threshold rule as defined by J and ρ. This is formalized in the definition and theorem below.

Definition 12 (Simplified Duals for OST). For any type distributions $\mathbf{G}$ and static threshold policy J,ρ, let $\mathrm{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{Proph}}(\mathbf{G})$ (resp. $\mathrm{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{ExAnte}}(\mathbf{G})$) be identical to LP $\mathrm{innerLP}_{k,n}^{\mathsf{DP}/\mathsf{Proph}}(\mathbf{G})$ (resp. $\mathrm{innerLP}_{k,n}^{\mathsf{DP}/\mathsf{ExAnte}}(\mathbf{G})$), except with the additional constraints

$$y_i^l = \Big(\sum_{j<J} p_{ij} + p_{iJ}\rho\Big) x_i^l = ((1-\rho)G_{i,J-1} + \rho G_{iJ}) x_i^l \qquad \forall i \in [n], l \in [k]. \tag{12}$$

Theorem 2 (**Reformulating the Tight Guarantees for OST**). *For any fixed k and $n > k$, the tight guarantee for OST algorithms relative to the prophet (resp. ex-ante relaxation) is equal to* $\inf_{\mathbf{G}} \sup_{J,\rho} \mathrm{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{Proph}}(\mathbf{G})$ *(resp.* $\inf_{\mathbf{G}} \sup_{J,\rho} \mathrm{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{ExAnte}}(\mathbf{G})$*).*

The proof of Theorem 2 is similar to Section 2.1 and deferred to Appendix A.

We proceed to study tight guarantees for non-oblivious static thresholds. Earlier, the way in which the static threshold restriction directly translated into dual constraint (12) was crucially dependent on the fact that in the primal LP (11), J and ρ were set before the Δ_j's. We now show that if J and ρ are decided after the Δ_j's, then this translates into the dual algorithm being able to employ an *arbitrary convex combination* of static threshold rules, which can change the dual objective. First we formulate the adversary's inner problem for minimizing $\mathsf{ST}(I)$, which must set Δ_j's such that the performance of *any* static threshold policy defined by J, ρ is poor.

Definition 13 (Inner Problem for Minimizing ST). Consider the following LP, where coefficient Q_j can be set to either $\mathbb{E}[\min\{\sum_{i=1}^n \mathrm{Ber}(G_{ij}), k\}]$ or $\min\{\sum_{i=1}^n G_{ij}, k\}$ like before.

$$\min \alpha \tag{13a}$$
$$\text{s.t.} \quad V_i^l(J,\rho) = \sum_{j<J} p_{ij} U_{ij}^l(J,\rho) + p_{iJ}\rho U_{iJ}^l(J,\rho) + V_{i+1}^l(J,\rho) \qquad \forall i,l,J \in [m], \rho \in (0,1] \tag{13b}$$
$$U_{ij}^l(J,\rho) = \sum_{j'=j}^m \Delta_{j'} - V_{i+1}^l(J,\rho) + V_{i+1}^{l-1}(J,\rho) \qquad \forall i,j,l,J \in [m], \rho \in (0,1] \tag{13c}$$
$$\alpha \ge V_1^k(J,\rho) \qquad \forall J \in [m], \rho \in (0,1] \tag{13d}$$
$$\sum_{j=1}^m Q_j \Delta_j = 1 \tag{13e}$$
$$\Delta_j \ge 0 \qquad \forall j \in [m] \tag{13f}$$

We note that LP (13) is similar to LP (11), except a copy of the variables has been created for every possible static threshold J, ρ, all of which must perform no better than α. The simplified dual formulation corresponding to LP (13) will be easier to write using the following additional notation.

Definition 14 ($\mathbf{x}, \mathbf{y}, \mathcal{P}_n^k$). Let $\mathbf{x} := (x_i^l)_{i\in[n],l\in[k]}, \mathbf{y} := (y_i^l)_{i\in[n],l\in[k]}$, and let $\mathcal{P}_n^k$ denote the set of vectors $(\mathbf{x}, \mathbf{y})$ that satisfy the simplified dual problem's state-updating constraints (10c)–(10e).

Definition 15 (Simplified Duals for ST). For any $\mathbf{G}$, let $\mathrm{innerLP}_{k,n}^{\mathsf{ST}/\mathsf{Proph}}(\mathbf{G})$ (resp. $\mathrm{innerLP}_{k,n}^{\mathsf{ST}/\mathsf{ExAnte}}(\mathbf{G})$) denote the following LP, with coefficient Q_j set to $\mathbb{E}[\min\{\sum_{i=1}^n \mathrm{Ber}(G_{ij}), k\}]$ (resp. $\min\{\sum_{i=1}^n G_{ij}, k\}$) for all $j \in [m]$.

$$\max \theta \tag{14a}$$
$$\text{s.t.} \quad \theta \cdot Q_j \le \int_{J,\rho} \mu(J,\rho) \left(\sum_{i=1}^n \sum_{l=1}^k \min\{y_i^l(J,\rho), G_{ij} x_i^l(J,\rho)\} \right) \qquad \forall j \in [m] \tag{14b}$$

$$y_i^l(J,\rho) = ((1-\rho)G_{i,J-1} + \rho G_{iJ})x_i^l(J,\rho) \qquad \forall i,l,J \in [m], \rho \in (0,1] \tag{14c}$$

$$(\mathbf{x}(J,\rho), \mathbf{y}(J,\rho)) \in \mathcal{P}_n^k \qquad \forall J \in [m], \rho \in (0,1] \tag{14d}$$

$$\int_{J,\rho} \mu(J,\rho) = 1 \tag{14e}$$

$$\mu(J,\rho) \geq 0 \qquad \forall J \in [m], \rho \in (0,1] \tag{14f}$$

We note that there is no benefit for the primal algorithm using a convex combination of static thresholds (its expectation is maximized by choosing the best one), but since the dual problem has to uniformly cover each type j, there can be a benefit.

Theorem 3 **(Reformulating the Tight Guarantees for ST)**. *For any fixed k and $n > k$,*

$$\inf_{I \in \mathcal{I}_{k,n}} \frac{\mathsf{ST}(I)}{\mathsf{Proph}(I)} = \inf_{\mathbf{G}} \ \mathrm{innerLP}_{k,n}^{\mathsf{ST/Proph}}(\mathbf{G}) \qquad \textit{and} \qquad \inf_{I \in \mathcal{I}_{k,n}} \frac{\mathsf{ST}(I)}{\mathsf{ExAnte}(I)} = \inf_{\mathbf{G}} \ \mathrm{innerLP}_{k,n}^{\mathsf{ST/ExAnte}}(\mathbf{G}).$$

The proof of Theorem 3 is similar to Section 2.1 and also deferred to Appendix A.

3. General Framework applied to the Non-IID Setting

In this section we study tight guarantees over general non-IID instances, starting with those for the optimal DP. Recall that for any number of slots k and agents $n > k$, we have established in Theorem 1 that the tight guarantees relative to the prophet and ex-ante relaxation are given by $\inf_{\mathbf{G}} \mathrm{innerLP}_{k,n}^{\mathsf{DP/Proph}}(\mathbf{G})$ and $\inf_{\mathbf{G}} \mathrm{innerLP}_{k,n}^{\mathsf{DP/ExAnte}}(\mathbf{G})$ respectively. As a recap,by using the notation from Definition 14 that treats $\mathbf{x}, \mathbf{y}$ as vectors, $\mathrm{innerLP}_{k,n}^{\mathsf{DP/Proph}}(\mathbf{G})$ can be rewritten as

$$\mathrm{innerLP}_{k,n}^{\mathsf{DP/Proph}}(\mathbf{G}) = \max\ \theta \tag{15a}$$

$$\text{s.t.} \quad \theta \cdot \mathbb{E}[\min\{\sum_{i=1}^n \mathrm{Ber}(G_{ij}), k\}] \leq \sum_{i=1}^n \sum_{l=1}^k \min\{y_i^l, G_{ij}x_i^l\} \quad \forall j \in [m] \tag{15b}$$

$$(\mathbf{x}, \mathbf{y}) \in \mathcal{P}_n^k, \tag{15c}$$

and $\mathrm{innerLP}_{k,n}^{\mathsf{DP/ExAnte}}(\mathbf{G})$ can be rewritten as

$$\mathrm{innerLP}_{k,n}^{\mathsf{DP/ExAnte}}(\mathbf{G}) = \max\ \theta \tag{16a}$$

$$\text{s.t.} \quad \theta \cdot \min\{\sum_{i=1}^n G_{ij}, k\} \leq \sum_{i=1}^n \sum_{l=1}^k \min\{y_i^l, G_{ij}x_i^l\} \quad \forall j \in [m] \tag{16b}$$

$$(\mathbf{x}, \mathbf{y}) \in \mathcal{P}_n^k \tag{16c}$$

3.1. $\mathsf{DP/ExAnte}$ in Non-IID Setting

In general non-IID setting, we obtain the following result regarding $\mathsf{DP/ExAnte}$.

THEOREM 4. $\inf_{\mathbf{G}}$ innerLP$^{\mathsf{DP/ExAnte}}_{k,n}(\mathbf{G})$ *is equal to the optimal objective value of the following problem:*

$$\inf_{\substack{\sum_{i\in[n]} g_i\le k\\ 1\ge g_i\ge 0\ \ \forall i}} \quad \max\ \theta \tag{17a}$$

$$\text{s.t.} \quad \theta\cdot g_i \le \sum_{l=1}^{k}\min\{y_i^l, g_i x_i^l\} \quad \forall i\in[n] \tag{17b}$$

$$(\mathbf{x},\mathbf{y})\in\mathcal{P}_n^k \tag{17c}$$

The proof is deferred to Appendix A. The result derived in Theorem 4 has a nice interpretation. The variable g_i for each $i\in[n]$ can be interpreted as the marginal probability that agent i got accepted in the Ex-Ante relaxation. Denote by θ^* the optimal value of LP (17) after taking minimum over $\mathbf{g}=(g_1,\ldots,g_n)$. Denote by $\{\theta^*,\mathbf{x_g},\mathbf{y_g}\}$ a feasible solution to LP (17) for a fixed $\mathbf{g}$, where the value of $(\mathbf{x_g},\mathbf{y_g})$ depends on $\mathbf{g}$. Then constraint (17b) implies that each agent i got accepted by the policy specified by $(\mathbf{x_g},\mathbf{y_g})$ with a probability at least θ^* conditional on being accepted in the Ex-Ante relaxation, for any $\mathbf{g}$. Such an implication corresponds to the definition of θ^*-balancedness online contention resolution scheme (OCRS) in Feldman et al. (2021). Thus, Theorem 4 implies that an OCRS achieves the tight guarantee of the DP policy, with respect to the Ex-Ante relaxation. Note that this point has been previously proved in Jiang et al. (2022). Here, we prove the same result in an alternative way by exploiting the structures of our LP framework $\inf_{\mathbf{G}}$ innerLP$^{\mathsf{DP/ExAnte}}_{k,n}(\mathbf{G})$.

3.2. Optimal Oblivious Static Thresholds in Non-IID Setting

Recall that for any fixed k and $n>k$, we have established in Theorem 2 that the tight guarantees for OST algorithms relative to the prophet and ex-ante relaxation are given by $\inf_{\mathbf{G}}\sup_{J,\rho}$ innerLP$^{\mathsf{OST}(J,\rho)/\mathsf{Proph}}_{k,n}(\mathbf{G})$ and $\inf_{\mathbf{G}}\sup_{J,\rho}$ innerLP$^{\mathsf{OST}(J,\rho)/\mathsf{ExAnte}}_{k,n}(\mathbf{G})$ respectively, where the inner LP's correspond to (15) and (16) respectively but both have the added constraints

$$y_i^l = ((1-\rho)G_{i,J-1}+\rho G_{iJ})x_i^l \qquad \forall i\in[n], l\in[k]. \tag{18}$$

The inner LP's for OST's are substantially easier to analyze because under constraints (18), the RHS that is common to (15b) and (16b) can be rewritten as

$$\sum_{i=1}^{n}\sum_{l=1}^{k}\min\{y_i^l, G_{ij}x_i^l\} = \sum_{i=1}^{n}\min\{((1-\rho)G_{i,J-1}+\rho G_{iJ}), G_{ij}\}\sum_{l=1}^{k}x_i^l, \tag{19}$$

where term $\min\{((1-\rho)G_{i,J-1}+\rho G_{iJ}), G_{ij}\}$ for each agent i depends on the choices of J,ρ but not on the number of remaining slots l. Moreover, we have the following relationships.

Lemma 2. *Fix an OST J,ρ and define $\tau_i = (1-\rho)G_{i,J-1} + \rho G_{i,J}$ for all $i \in [n]$. Suppose vectors* $\mathbf{x},\mathbf{y}$ *satisfy* (18), *i.e.* $y_i^l = \tau_i x_i^l$ *for all i and l, as well as $(\mathbf{x},\mathbf{y}) \in \mathcal{P}_n^k$. Then for all $i \in [n]$, we have*

$$\sum_{l=1}^{k} x_i^l = \Pr[\sum_{i'<i} \mathrm{Ber}(\tau_{i'}) < k] \qquad \textit{and} \qquad \sum_{i'=1}^{i} \tau_{i'} \sum_{l=1}^{k} x_{i'}^l = \mathbb{E}[\min\{\sum_{i'=1}^{i} \mathrm{Ber}(\tau_{i'}), k\}].$$

Lemma 2 is formally proven in Appendix A, but natural under the interpretation that x_i^l denotes the probability of having exactly l slots remaining when agent i arrives. Note that each agent i "clears the bar" for acceptance independently with probability τ_i. An agent i is accepted if and only if they clear the bar and there is at least 1 slot remaining when they arrive, with the latter probability given by $\sum_{l=1}^{k} x_i^l$. The second part of Lemma 2 then follows because the number of agents accepted among $i' = 1,\ldots,i$ is equal to the number of them who clear the bar, truncated by k. Meanwhile, the first part of Lemma 2 follows because there is a slot remaining for agent i if and only if the number of previous agents $i' < i$ who cleared the bar is less than k.

Equipped with Lemma 2, we are now ready to prove our result that the tight guarantees for OST algorithms relative to the stronger ex-ante benchmark are no worse than relative to the prophet. We first show a lower bound of $\mathsf{OST}/\mathsf{ExAnte}$ in the following lemma.

Lemma 3. *For any type distributions* $\mathbf{G}$ *and static threshold policy J,ρ, we have*

$$\begin{aligned}&\mathrm{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{ExAnte}}(\mathbf{G})\\ &\geq \min\left\{\Pr\left[\sum_{i<n}\mathrm{Ber}((1-\rho)G_{i,J-1}+\rho G_{iJ})<k\right], \frac{\mathbb{E}[\min\{\sum_{i<n}\mathrm{Ber}((1-\rho)G_{i,J-1}+\rho G_{iJ}),k\}]}{k}\right\}.\end{aligned}$$

We then show an upper bound of $\mathsf{OST}/\mathsf{Proph}$ in the following lemma, which matches the lower bound established in Lemma 3.

Lemma 4. *For any fixed type distributions* $\mathbf{G}$*, the value of* $\inf_{\mathbf{G}'} \sup_{J,\rho} \mathrm{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{Proph}}(\mathbf{G}')$ *can be at most*

$$\sup_{J,\rho}\ \min\left\{\Pr\left[\sum_{i<n}\mathrm{Ber}((1-\rho)G_{i,J-1}+\rho G_{iJ})<k\right], \frac{\mathbb{E}[\min\{\sum_{i<n}\mathrm{Ber}((1-\rho)G_{i,J-1}+\rho G_{iJ}),k\}]}{k}\right\}. \tag{20}$$

Note that the Ex-Ante benchmark is a stronger benchmark than the prophet. Combining Lemma 3 and Lemma 4, we have the following result.

Theorem 5. *For any fixed k and n,*

$$\inf_{\mathbf{G}}\sup_{J,\rho}\ \mathrm{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{Proph}}(\mathbf{G}) = \inf_{\mathbf{G}}\sup_{J,\rho}\ \mathrm{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{ExAnte}}(\mathbf{G}) \tag{21}$$

$$= \inf_{\mathbf{G}}\sup_{J,\rho}\ \min\left\{\Pr\left[\sum_{i<n}\mathrm{Ber}((1-\rho)G_{i,J-1}+\rho G_{iJ})<k\right], \frac{\mathbb{E}[\min\{\sum_{i<n}\mathrm{Ber}((1-\rho)G_{i,J-1}+\rho G_{iJ}),k\}]}{k}\right\}.$$

The formal proofs of Lemma 3, Lemma 4 and Theorem 5 are referred to Appendix A.

It is direct to see that for the formula in (21), the first term in the min operator is decreasing over the threshold J,ρ, while the second term in the min operator is increasing over the threshold J,ρ. Thus, in order to achieve the supremum, the two terms within the min operator must be equivalent, which yields the following result.

COROLLARY 1. *For any number of slots k and agents $n>k$, the best-possible guarantees for OST algorithms relative to the prophet or ex-ante relaxation are identically equal to*

$$\min\ \alpha \tag{22a}$$

$$\text{s.t. } \alpha = \Pr\left[\sum_{i=1}^{n-1}\mathrm{Ber}(q_i) < k\right] = \frac{\mathbb{E}[\min\{\sum_{i=1}^{n-1}\mathrm{Ber}(q_i), k\}]}{k} \tag{22b}$$

$$q_i \in [0,1] \quad \forall i \in [n-1] \tag{22c}$$

Chawla et al. (2020) show that for a fixed k, the infimum value of problem (22) over $n>k$ occurs as $n\to\infty$, and is equal to $\Pr[\mathrm{Pois}(\lambda)<k] = \frac{\mathbb{E}[\min\{\mathrm{Pois}(\lambda),k\}]}{k}$, where λ is the unique real number that makes these quantities identical. They also show how to achieve this guarantee using an oblivious static threshold algorithm. Our framework shows that their guarantees are tight, regardless of whether one is comparing to the prophet or ex-ante relaxation, and moreover never required explicitly computing its value or constructing a family of counterexamples to establish a matching upper bound!

3.3. Oblivious vs. Non-oblivious Static Thresholds in Non-IID Setting

We further study the performances of oblivious static threshold policies versus general non-oblivious static threshold policies in the non-IID setting. We first show that there exists an instance $\mathbf{G}$ such that ST performs better than OST, with respect to both the Ex-Ante benchmark and the prophet.

LEMMA 5. *There exists an instance $\mathbf{G}$ such that*

$$\mathrm{innerLP}_{k,n}^{\mathsf{ST/Proph}}(\mathbf{G}) > \sup_{J,\rho}\mathrm{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)\mathsf{/Proph}}(\mathbf{G})$$

and

$$\mathrm{innerLP}_{k,n}^{\mathsf{ST/ExAnte}}(\mathbf{G}) > \sup_{J,\rho}\mathrm{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)\mathsf{/ExAnte}}(\mathbf{G})$$

The proof is referred to Appendix A. Note that in the proof of Lemma 5, we construct an instance where there are two agents, $n=2$, four types, $m=4$, and one slot, $k=1$. The distribution for the first agent is given by $\mathbf{G}_1=(0,\frac{1}{2},\frac{1}{2},1)$, and the distribution for the second agent is given by $\mathbf{G}_2=(\varepsilon,\varepsilon,1,1)$, for some $\varepsilon\to 0$. For this instance, we show that $\mathsf{ST/Proph}$ and $\mathsf{ST/ExAnte}$ achieve the value of 2/3, while the values for $\mathsf{OST/Proph}$ and $\mathsf{OST/ExAnte}$ are no greater than 1/2.

Therefore the method of Chawla et al. (2020) is not instance-optimal. However, we now show that it is optimal in the worst case, hence their bound is tight even for the more powerful class of ST policies. First we need the following analogue of Lemma 4 for non-oblivious static thresholds.

LEMMA 6. *For any fixed type distributions* $\mathbf{G}$*, the value of* $\inf_{\mathbf{G}'} \mathrm{innerLP}_{k,n}^{\mathsf{ST/Proph}}(\mathbf{G}')$ *can be at most the supremum of*

$$\min\left\{\int_{J,\rho} \Pr\left[\sum_{i<n} \mathrm{Ber}((1-\rho)G_{i,J-1}+\rho G_{iJ})<k\right]\mu(J,\rho), \int_{J,\rho} \frac{\mathbb{E}[\min\{\sum_{i<n}\mathrm{Ber}((1-\rho)G_{i,J-1}+\rho G_{iJ}),k\}]}{k}\mu(J,\rho)\right\}$$

over all measures μ *over* $J\in[m]$ *and* $\rho\in(0,1]$.

The proof is deferred to Appendix A.

In Theorem 5, we have shown that both the quantities $\inf_{\mathbf{G}}\sup_{J,\rho}\ \mathrm{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{Proph}}(\mathbf{G})$ and $\inf_{\mathbf{G}}\sup_{J,\rho}\ \mathrm{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{ExAnte}}(\mathbf{G})$ are identical to the quantity

$$\inf_{\mathbf{G}}\sup_{J,\rho}\ \min\left\{\Pr\left[\sum_{i<n}\mathrm{Ber}((1-\rho)G_{i,J-1}+\rho G_{iJ})<k\right], \frac{\mathbb{E}[\min\{\sum_{i<n}\mathrm{Ber}((1-\rho)G_{i,J-1}+\rho G_{iJ}),k\}]}{k}\right\}. \tag{23}$$

The results in Chawla et al. (2020) imply that this infimum is in fact achieved in the simple case where there is only $m=1$ type and $G_{i1}=1$ for all i, in which case (23) reduces to

$$\sup_{\rho\in(0,1]}\min\left\{\Pr\left[\mathrm{Bin}(n-1,\rho)<k\right], \frac{\mathbb{E}[\min\{\mathrm{Bin}(n-1,\rho),k\}]}{k}\right\}. \tag{24}$$

We now show that $\inf_{\mathbf{G}}\mathrm{innerLP}_{k,n}^{\mathsf{ST/ExAnte}}(\mathbf{G})$ cannot do any better, because even the larger quantity from Lemma 6, which on this one-type instance reduces to

$$\sup_{\mu:(0,1]\to\mathbb{R}_{\geq 0},\int_\rho \mu(\rho)=1}\min\left\{\int_\rho \Pr\left[\mathrm{Bin}(n-1,\rho)<k\right]\mu(\rho), \int_\rho\frac{\mathbb{E}[\min\{\mathrm{Bin}(n-1,\rho),k\}]}{k}\mu(\rho)\right\}, \tag{25}$$

is actually no larger than (24).

THEOREM 6. *For any fixed* n *and* k*, it holds that*

$$\begin{aligned}&\inf_{\mathbf{G}}\ \mathrm{innerLP}_{k,n}^{\mathsf{ST/Proph}}(\mathbf{G})=\inf_{\mathbf{G}}\ \mathrm{innerLP}_{k,n}^{\mathsf{ST/ExAnte}}(\mathbf{G})\\ &=\inf_{\mathbf{G}}\sup_{J,\rho}\ \mathrm{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{Proph}}(\mathbf{G})=\inf_{\mathbf{G}}\sup_{J,\rho}\ \mathrm{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{ExAnte}}(\mathbf{G}).\end{aligned}$$

4. Further Simplified General Framework, applied to the IID Setting

In this section we study tight guarantees for DP/OST/ST algorithms relative to the prophet/ex-ante relaxation, under the restriction that the type distributions $\mathbf{G}$ must be IID. That is, for each type $j\in[m]$, it is imposed that G_{ij} is identically equal to some value G_j across all agents $i\in[n]$.

When this is the case, the key dual constraints from Section 3 that compare to the prophet, e.g. (15b) in $\text{innerLP}^{\mathsf{DP/Proph}}_{k,n}(\mathbf{G})$, can be rewritten as

$$\theta \cdot \mathbb{E}[\min\{\sum_{i=1}^{n} \text{Ber}(G_j), k\}] \leq \sum_{i=1}^{n}\sum_{l=1}^{k} \min\{y_i^l, G_j x_i^l\} \qquad \forall j \in [m]. \tag{26}$$

where the values G_j no longer depend on i. Consequently, these constraints will always be hardest to satisfy when the IID type distribution becomes infinitely granular, i.e. $G_j = j/m$ for all $j \in [m]$ with $m \to \infty$, simply because there are more constraints. The overall problem of interest, $\inf_{\mathbf{G}} \text{innerLP}^{\mathsf{DP/Proph}}_{k,n}(\mathbf{G})$, can then have its outer optimization problem dropped, since the infimum always arises at the infinitely-granular $\mathbf{G}$. This leads to a drastic reduction where the tight guarantees are now described by a single semi-infinite LP, as formalized below. We note that the same reduction can be made for the dual constraints from before that compare to the ex-ante relaxation:

$$\theta \cdot \min\{\sum_{i=1}^{n} G_j, k\} \leq \sum_{i=1}^{n}\sum_{l=1}^{k} \min\{y_i^l, G_j x_i^l\} \qquad \forall j \in [m]. \tag{27}$$

Definition 16 (Semi-infinite LP's for DP in IID Setting). Let $\text{Bin}(n,q)$ denote a Binomial random variable with n independent trials of success probability q. Consider the semi-infinite families of constraints

$$\theta \cdot \mathbb{E}[\min\{\text{Bin}(n,q), k\}] \leq \sum_{i=1}^{n}\sum_{l=1}^{k} \min\{y_i^l, q x_i^l\} \qquad \forall q \in (0,1] \tag{28}$$

$$\theta \cdot \min\{nq, k\} \leq \sum_{i=1}^{n}\sum_{l=1}^{k} \min\{y_i^l, q x_i^l\} \qquad \forall q \in (0,1] \tag{29}$$

which correspond to the limiting cases of (26) and (27) as $G_j = j/m = q$ and $m \to \infty$. Then for any k and $n > k$, let $\text{iidLP}^{\mathsf{DP/Proph}}_{k,n}$ (resp. $\text{iidLP}^{\mathsf{DP/ExAnte}}_{k,n}$) denote the semi-infinite LP defined by: maximize θ, subject to constraints (28) (resp. (29)) and $(\mathbf{x},\mathbf{y}) \in \mathcal{P}^k_n$.

Static threshold policies J, ρ also simplify nicely in this semi-infinite dual problem in the IID setting. Namely, the previous dual constraints (18) for a static threshold policy, $y_i^l = ((1-\rho)G_{i,J-1} + \rho G_{iJ})x_i^l$, by setting $\tau = (1-\rho)G_{i,J-1} + \rho G_{iJ}$ which is identical across i, can be reduced to

$$y_i^l = \tau x_i^l \quad \forall i \in [n], l \in [k] \tag{30}$$

Note that $\tau \in (0,1]$ is a single number in the IID setting; we no longer need an index $J \in [m]$ combined with a tiebreak probability $\rho \in (0,1]$.

Definition 17 (Semi-infinite LP's for OST in IID Setting). For any k, $n > k$, and fixed static threshold policy defined by τ, let $\text{iidLP}^{\mathsf{OST}(\tau)/\mathsf{Proph}}_{k,n}$ (resp. $\text{iidLP}^{\mathsf{OST}(\tau)/\mathsf{ExAnte}}_{k,n}$) denote the semi-infinite LP defined by: maximize θ, subject to (28) (resp. (29)), constraints (30) for a static threshold policy in the IID setting, and $(\mathbf{x},\mathbf{y}) \in \mathcal{P}^k_n$.

Finally, for non-oblivious static thresholds, we define the analogue of (14) for the IID setting.

Definition 18 (ST in IID Setting). For any k and $n>k$, let $\mathrm{iidLP}^{\mathsf{ST/Proph}}_{k,n}$ (resp. $\mathrm{iidLP}^{\mathsf{ST/ExAnte}}_{k,n}$) denote the following optimization problem, when $Q(q)=\mathbb{E}[\min\{\mathrm{Bin}(n,q),k\}]$ (resp. $Q(q)=\min\{nq,k\}$) for all $q\in(0,1]$.

$$
\begin{aligned}
&\max \theta && && (31a)\\
&\text{s.t.}\ \ \theta\cdot Q(q)\le\int_{\tau\in(0,1]}\mu(\tau)\left(\sum_{i=1}^{n}\sum_{l=1}^{k}\min\{y_i^l(\tau),qx_i^l(\tau)\}\right) && \forall q\in(0,1] && (31b)\\
&\qquad y_i^l(\tau)=\tau x_i^l(\tau) && \forall i\in[n],l\in[k],\tau\in(0,1] && (31c)\\
&\qquad (\mathbf{x}(\tau),\mathbf{y}(\tau))\in\mathcal{P}_n^k && \forall\tau\in(0,1] && (31d)\\
&\qquad \int_{\tau\in(0,1]}\mu(\tau)=1 && && (31e)\\
&\qquad \mu(\tau)\ge 0 && \forall\tau\in(0,1] && (31f)
\end{aligned}
$$

We now formalize that all of these are the correct formulations, which compute tight guarantees for DP/OST/ST algorithms relative to the prophet/ex-ante relaxation in the IID special case.

Theorem 7 **(Reformulations in IID Setting)**. *For any k and $n>k$, when type distributions $\mathbf{G}$ are constrained to be IID, the adversary's optimization problems over $\mathbf{G}$ can be reformulated as*

$$
\begin{aligned}
&\inf_{\mathbf{G}:G_{1j}=\cdots=G_{nj}\forall j}\mathrm{innerLP}^{\mathsf{DP/Proph}}_{k,n}(\mathbf{G}) && =\mathrm{iidLP}^{\mathsf{DP/Proph}}_{k,n}\\
&\inf_{\mathbf{G}:G_{1j}=\cdots=G_{nj}\forall j}\mathrm{innerLP}^{\mathsf{DP/ExAnte}}_{k,n}(\mathbf{G}) && =\mathrm{iidLP}^{\mathsf{DP/ExAnte}}_{k,n}\\
&\inf_{\mathbf{G}:G_{1j}=\cdots=G_{nj}\forall j}\sup_{J,\rho}\ \mathrm{innerLP}^{\mathsf{OST}(J,\rho)\mathsf{/Proph}}_{k,n}(\mathbf{G}) && =\sup_{\tau}\ \mathrm{innerLP}^{\mathsf{OST}(\tau)\mathsf{/Proph}}_{k,n}\\
&\inf_{\mathbf{G}:G_{1j}=\cdots=G_{nj}\forall j}\sup_{J,\rho}\ \mathrm{innerLP}^{\mathsf{OST}(J,\rho)\mathsf{/ExAnte}}_{k,n}(\mathbf{G}) && =\sup_{\tau}\ \mathrm{innerLP}^{\mathsf{OST}(\tau)\mathsf{/ExAnte}}_{k,n}\\
&\inf_{\mathbf{G}:G_{1j}=\cdots=G_{nj}\forall j}\mathrm{innerLP}^{\mathsf{ST/Proph}}_{k,n}(\mathbf{G}) && =\mathrm{iidLP}^{\mathsf{ST/Proph}}_{k,n}\\
&\inf_{\mathbf{G}:G_{1j}=\cdots=G_{nj}\forall j}\mathrm{innerLP}^{\mathsf{ST/ExAnte}}_{k,n}(\mathbf{G}) && =\mathrm{iidLP}^{\mathsf{ST/ExAnte}}_{k,n}
\end{aligned}
$$

Theorem 7 is proven in Appendix A.

4.1. Equivalence of DP/ExAnte, OST/Proph, and OST/ExAnte in IID Setting

Similar to before, OST's are easier to analyze because under constraints (30), the RHS that is common to (28) and (29) can be rewritten as

$$
\sum_{i=1}^{n}\sum_{l=1}^{k}\min\{y_i^l,qx_i^l\}=\min\{\tau,q\}\sum_{i=1}^{n}\sum_{l=1}^{k}x_i^l, \tag{32}
$$

where now term $\min\{\tau, q\}$ depends only on the choice of τ and not on agent i nor the number of remaining slots l. Moreover, the analogues of the relationships in Lemma 2 are that assuming $y_i^l = \tau x_i^l$ for all i and l, for all $i \in [n]$, we have

$$\sum_{l=1}^{k} x_i^l = \Pr[\mathrm{Bin}(i-1,\tau) < k] \qquad \text{and} \qquad \tau \sum_{i'=1}^{i} \sum_{l=1}^{k} x_{i'}^l = \mathbb{E}[\min\{\mathrm{Bin}(i,\tau), k\}]. \tag{33}$$

This allows us to prove the following theorem in Appendix A. Although this result is well-known in the literature, as explained in Section 1.1, we emphasize that our framework simultaneously obtain both the lower bound and the upper bound of the ratios, without explicitly constructing a counterexample to show the upper bound and constructing a policy to show the lower bound.

THEOREM 8. *For any fixed k and n,*

$$\mathrm{iidLP}_{k,n}^{\mathsf{DP/ExAnte}} = \sup_{\tau} \ \mathrm{iidLP}_{k,n}^{\mathsf{OST}(\tau)\mathsf{/Proph}} = \sup_{\tau} \ \mathrm{iidLP}_{k,n}^{\mathsf{OST}(\tau)\mathsf{/ExAnte}} = \frac{\mathbb{E}[\min\{\mathrm{Bin}(n, k/n), k\}]}{k}.$$

4.2. Equivalence of Oblivious and Non-oblivious Static Thresholds in IID Setting

We now show that even the best static threshold performance $\mathsf{ST}(I)$ knowing the full instance I cannot beat the quantities in Theorem 8. In fact, we prove a stronger result which was not true in the non-IID setting (Section 3.2)—$\mathsf{ST}(I)$ is no better than oblivious static thresholds on *any* instance.

To prove this fact, we show that in the ST dual problem (31), one never benefits from using a convex combination of thresholds instead of a single threshold. It suffices to show that given any two thresholds $\underline{\tau}, \overline{\tau}$ with $\underline{\tau} < \overline{\tau}$, there exists a τ lying between them which contributes more to the RHS of (31b) than the average of $\underline{\tau}$ and $\overline{\tau}$, *simultaneously* for every $q \in (0,1]$. This is formalized in the theorem below.

THEOREM 9. *Given any $\underline{\tau}, \overline{\tau}$ with $\underline{\tau} < \overline{\tau}$, there exists a $\tau \in [\underline{\tau}, \overline{\tau}]$ such that*

$$\sum_{i=1}^{n} \sum_{l=1}^{k} \min\{y_i^l(\tau), q x_i^l(\tau)\} \geq \frac{1}{2} \left(\sum_{i=1}^{n} \sum_{l=1}^{k} \min\{y_i^l(\underline{\tau}), q x_i^l(\underline{\tau})\} + \sum_{i=1}^{n} \sum_{l=1}^{k} \min\{y_i^l(\overline{\tau}), q x_i^l(\overline{\tau})\} \right) \quad \forall q \in (0,1]. \tag{34}$$

Theorem 9 is proven in Appendix A. The main idea involves setting τ so that $\mathbb{E}[\min\{\mathrm{Bin}(n,\tau), k\}] = (\mathbb{E}[\min\{\mathrm{Bin}(n,\underline{\tau}), k\}] + \mathbb{E}[\min\{\mathrm{Bin}(n,\overline{\tau}), k\}])/2$, and showing through a careful sequence of stochastic comparisons that this implies

$$\sum_{i=1}^{n} \Pr[\mathrm{Bin}(i-1,\tau) < k] \geq \frac{1}{2} \left(\sum_{i=1}^{n} \Pr[\mathrm{Bin}(i-1,\underline{\tau}) < k] + \sum_{i=1}^{n} \Pr[\mathrm{Bin}(i-1,\overline{\tau}) < k] \right). \tag{35}$$

That is, if τ is set so that the number of agents accepted is the mean of that for $\underline{\tau}$ and $\overline{\tau}$, then its average probability of having a slot available across $i = 1, \ldots, n$ can only be higher than the mean of that for $\underline{\tau}$ and $\overline{\tau}$. We note that simply setting $\tau = (\underline{\tau} + \overline{\tau})/2$ does not work, which can be seen through a simple example where $k = 1$, $n = 3$, $\underline{\tau} = 1/3$, $\overline{\tau} = 1$. Then the LHS of (35) when $\tau = 2/3$ is $1 + 1/3 + 1/9 = 13/9$. Meanwhile, the RHS of (35) is $\frac{(1+2/3+4/9)+1}{2} = 14/9$, and hence setting $\tau = (\underline{\tau} + \overline{\tau})/2$ would not suffice.

Theorem 9 shows that when designing the convex combination of thresholds given by $\mu(\tau)$ in problem (31), if there is mass on two distinct thresholds $\underline{\tau}, \overline{\tau}$, then it is always better to move them to be a single mass at some intermediate threshold. This means that ultimately it is better to place all the mass at one point, leading to the following corollary.

COROLLARY 2. *It holds that*

$$\mathrm{iidLP}_{k,n}^{\mathsf{ST/Proph}} = \sup_{\tau} \mathrm{iidLP}_{k,n}^{\mathsf{OST}(\tau)/\mathsf{Proph}}$$

and

$$\mathrm{iidLP}_{k,n}^{\mathsf{ST/ExAnte}} = \sup_{\tau} \mathrm{iidLP}_{k,n}^{\mathsf{OST}(\tau)/\mathsf{ExAnte}}.$$

We note that the same argument works on a particular instance, defined by the type distribution given by $\{G_j : j \in [m]\}$ that is common across agents (in which case constraints (31b) only need to be checked at the points $q = G_j$ for some $j \in [m]$). Therefore, OST's are *instance-optimal* for static threshold policies in the IID setting.

4.3. $\mathsf{DP/Proph}$ in IID Setting: A Simplified Derivation of the Tight $\alpha = 0.745$ for $k = 1$

We now assume $k = 1$ and show how our framework can be used to recover the tight guarantee 0.745 for the optimal online policy relative to the prophet, by further exploiting the optimality structure of $\mathrm{iidLP}_{1,n}^{\mathsf{DP/Proph}}$. In the following part, we first derive a relaxation of $\mathrm{iidLP}_{1,n}^{\mathsf{DP/Proph}}$, denoted as $\mathsf{LP}_n^{\mathsf{relax}}$, and we obtain an optimal solution of $\mathsf{LP}_n^{\mathsf{relax}}$ with a closed form. We then establish that the relaxation does not improve the objective value, i.e. $\mathrm{iidLP}_{1,n}^{\mathsf{DP/Proph}} = \mathsf{LP}_n^{\mathsf{relax}}$, showing that an optimal solution of $\mathsf{LP}_n^{\mathsf{relax}}$ can be transformed into an optimal solution of $\mathrm{iidLP}_{1,n}^{\mathsf{DP/Proph}}$. Finally, based on the closed form solution of $\mathsf{LP}_n^{\mathsf{relax}}$, we show that the worst case occurs as $n \to \infty$ and we obtain 0.745 as the tight guarantee. We now elaborate on the key techniques in this three-step derivation, with all proofs of statements relegated to Appendix A.

First step. We rewrite $\mathrm{iidLP}_{1,n}^{\mathsf{DP/Proph}}$ with superscript l omitted:

$$\begin{aligned}
\mathrm{iidLP}_{1,n}^{\mathsf{DP/Proph}} = \max\ & \theta && \text{(36a)}\\
\text{s.t.}\ & (1-(1-\kappa)^n)\theta \le \sum_{i=1}^{n} \min\{y_i, \kappa(1 - \sum_{i'=1}^{i-1} y_{i'})\} \quad \forall \kappa \in (0,1] && \text{(36b)}\\
& \sum_{i=1}^{n} y_i \le 1 \\
& y_i \ge 0 \quad \forall i \in [n]
\end{aligned}$$

This is a semi-infinite program. The constraint (36b) is equivalent to

$$(1-(1-\kappa)^n)\theta \le \sum_{i\in S} y_i + \kappa \cdot \sum_{i\in[n]\setminus S}(1-\sum_{i'=1}^{i-1} y_{i'}) \qquad \forall \kappa\in[0,1], S\subseteq[n], \tag{37}$$

which can be relaxed to

$$(1-(1-\kappa)^n)\theta \le \sum_{i=1}^{I} y_i + \kappa \sum_{i=I+1}^{n}(1-\sum_{i'=1}^{i-1} y_{i'}) \qquad \forall \kappa\in(0,1], I=0,1\ldots,n. \tag{38}$$

We later show that this relaxation does not improve the objective value of $\mathrm{iidLP}_{1,n}^{\mathsf{DP/Proph}}$.

Letting $Y_i = \sum_{i'=1}^{i} y_{i'}$ for all $i\in[n]$, we now consider the optimization problem

$$\max \ \theta \tag{39a}$$

$$\text{s.t.} \ \max_{\kappa\in(0,1]}\left\{(1-(1-\kappa)^n)\theta - \kappa\sum_{i=I}^{n-1}(1-Y_i)\right\} \le Y_I \quad \forall I=0,\ldots,n \tag{39b}$$

$$Y_0 \le 0 \le Y_1 \le \cdots \le Y_n \le 1. \tag{39c}$$

Notice that the LHS of (39b) involves solving a concave maximization problem with κ being the decision variable, which has a closed-form solution. Thus the semi-infinite constraint (39b) can be replaced by an equivalent nonlinear but finite-dimensional constraint. Thus, we have obtained a finite-dimensional nonlinear program as a relaxation of $\mathrm{iidLP}_{1,n}^{\mathsf{DP/Proph}}$. This discussion is formalized in Lemma 7 below. Its proof, which requires a variable substitution of the form $z_I = \frac{1}{n\theta}\sum_{i=I}^{n-1}(1-Y_i)$, is elementary.

Lemma 7 **(Relaxation after Eliminating κ and Substituting Variables)**. *It holds that* $\mathrm{iidLP}_{1,n}^{\mathsf{DP/Proph}} \le \mathsf{LP}_n^{\mathsf{relax}}$ *where*

$$\begin{aligned}\mathsf{LP}_n^{\mathsf{relax}} := \max \ & \theta \\ \text{s.t.} \ & (n-1)z_I^{n/(n-1)} \le n z_{I+1} + \frac{1}{\theta} - 1 \qquad \forall I=0,\ldots,n-1 \\ & z_0 = 1, z_n = 0 \\ & z_i\in[0,1] \qquad \forall i\in[n-1].\end{aligned}$$

The optimization problem in Lemma 7 then has the following structured optimal solution where the z_I's are decreasing from $z_0=1$ to $z_n=0$.

Lemma 8 **(Closed-Form Solution for $\mathsf{LP}_n^{\mathsf{relax}}$)**. *Denote $\{\theta, z_I\}_{I=0}^n$ such that $z_0=1$, $z_n=0$ and*

$$z_{I+1} = \frac{n-1}{n} z_I^{n/(n-1)} - \frac{1}{n\theta} + \frac{1}{n}, \quad \forall I=0,\ldots,n-1$$

Then $\{\theta, z_I\}_{I=0}^n$ is an optimal solution of $\mathsf{LP}_n^{\mathsf{relax}}$.

Second step. We must show that the solution constructed in Lemma 8 can be converted into a feasible solution of $\mathrm{iidLP}_{1,n}^{\mathsf{DP/Proph}}$ with identical objective value. The challenge lies in verifying that the reverse substitution $y_i = n\theta(z_{i+1} - 2z_i + z_{i-1})$ (with the z_i's defined according to Lemma 8) satisfies all of the constraints in $\mathrm{iidLP}_{1,n}^{\mathsf{DP/Proph}}$, which can be distilled down to showing inequality (37). A priori, (37) is only satisfied when S takes the form $\{1,\dots,I\}$ (since those are the constraints we kept in the first relaxation (38)); in fact (37) is even non-obvious when $\kappa = 0$ and S consists of a singleton i (in which case (37) is equivalent to analytically checking that $y_i \geq 0$). To streamline the proof of (37), we define a set function $f(S)$ that substitutes the pessimal value of κ into (37) for each set S, and show this set function to be supermodular. This allows us to ultimately show that it is maximized when S takes the form of an interval $\{1,\dots,I\}$, for which we already knew by construction that (37) is satisfied as equality. This is all formalized in the proof of the lemma below.

Lemma 9. *It holds that* $\mathrm{iidLP}_{1,n}^{\mathsf{DP/Proph}} = \mathsf{LP}_n^{\mathsf{relax}}$ *for each* $n \geq 1$.

Third step. Having established $\mathrm{iidLP}_{1,n}^{\mathsf{DP/Proph}} = \mathsf{LP}_n^{\mathsf{relax}}$, the proof is completed by showing that the objective value of $\mathsf{LP}_n^{\mathsf{relax}}$ is minimized as $n \to \infty$. Although monotonicity in n is difficult to prove in general, we bypass this difficulty by comparing the values of LP_n with LP_{2n}, which our closed-form solution allows us to do.

Lemma 10. *For any* $n \geq 1$*, it holds that* $\mathsf{LP}_n^{\mathsf{relax}} \geq \mathsf{LP}_{2n}^{\mathsf{relax}}$.

Thus, in order to obtain the tight worst case guarantee, it remains to analyze the behavior of the optimal solution $\{\theta, z_I\}_{I=0}^{n}$ of $\mathsf{LP}_n^{\mathsf{relax}}$ when $n \to \infty$. By Lemma 8, we have the recursive equation

$$z_{I+1} = \frac{n-1}{n} z_I^{n/(n-1)} - \frac{1}{n\theta} + \frac{1}{n}, \quad \forall I = 0,\dots,n-1$$

with $z_0 = 1$ and $z_n = 0$. By treating x as I/n and $H(x)$ as z_I,the recursive equation above motivates the following differential equation,

$$\frac{1}{\theta^*} - 1 = H(x)(\ln H(x) - 1) - H'(x), \qquad \forall x \in [0,1] \tag{40}$$

with the boundary conditions $H(0) = 1$ and $H(1) = 0$, with θ^* being the precise constant that allows these relationships to hold. Then, we have the following result, which is the guarantee for $\mathsf{DP/Proph}$ in the IID setting with $k = 1$.

Theorem 10. *It holds that* $\inf_n \mathrm{iidLP}_{1,n}^{\mathsf{DP/Proph}} = \lim_{n\to\infty} \mathrm{iidLP}_{1,n}^{\mathsf{DP/Proph}} = \theta^*$*, where* θ^* *is defined in (40).*

Note that from the definition of the function $H(x)$ and θ^*, it holds that

$$\int_1^0 \frac{1}{1 - \frac{1}{\theta^*} - H(1 - \ln H)} dH = 1.$$

This recovers the same integral relationship as in Hill and Kertz (1982), Correa et al. (2017), Liu et al. (2021) which is used to establish the numerical guarantee of $\alpha = \theta^* \approx 0.745$.

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Appendix A: Missing Proofs

Proof of Proposition 2. We denote by X_j the number of agents that have a type j and are accepted by the prophet. Clearly, we have

$$\mathsf{Proph}(I) = \sum_{j=1}^{m} r_j \cdot \mathbb{E}[X_j] = \sum_{j=1}^{m} \Delta_j \cdot \mathbb{E}[\sum_{j'=1}^{j} X_{j'}]$$

Moreover, note that the distribution of $\sum_{j'=1}^{j} X_{j'}$ is given by $\min\{\sum_{i=1}^{n} \mathrm{Ber}(G_{ij}), k\}$. Thus, it holds that

$$\mathsf{Proph}(I) = \sum_{j=1}^{m} \Delta_j \cdot \mathbb{E}[\min\{\sum_{i=1}^{n} \mathrm{Ber}(G_{ij}), k\}]$$

We now denote by $\{a_j^*\}_{j=1}^m$ one optimal solution to $\mathsf{ExAnte}(I)$. Clearly, for any $j_1 < j_2$, if $a_{j_1}^* < \sum_{i=1}^{n} p_{ij_1}$, it must hold that $a_{j_2}^* = 0$. Otherwise, we construct another set of solution $\{a_j'\}_{j=1}^m$ by letting

$$a_j' = a_j^* \text{ for } j \neq j_1 \text{ and } j_2, \quad a_{j_1}' = a_{j_1}^* + \epsilon, \quad a_{j_2}' = a_{j_2}^* - \epsilon$$

where $0 < \epsilon \le \min\{a_{j_2}^*, \sum_{i=1}^{n} p_{ij_1} - a_{j_1}^*\}$. Clearly, $\{a_j'\}_{j=1}^m$ is feasible to $\mathsf{ExAnte}(I)$ and yields a higher objective value than $\{a_j^*\}_{j=1}^m$, which contradicts the optimality of $\{a_j^*\}_{j=1}^m$. Thus, it holds that

$$\sum_{j'=1}^{j} a_{j'}^* = \min\{\sum_{i=1}^{n} G_{ij}, k\} \text{ for any } j$$

and

$$\mathsf{ExAnte}(I) = \sum_{j=1}^{m} r_j \cdot a_j^* = \sum_{j=1}^{m} \Delta_j \cdot \sum_{j'=1}^{j} a_{j'}^* = \sum_{j=1}^{m} \Delta_j \cdot \min\{\sum_{i=1}^{n} G_{ij}, k\}$$

which completes our proof of (7). □

Proof of Lemma 1. Given a feasible solution to (9), we construct the following feasible solution to (10) with the same value of θ. Let $y_i^l = \sum_{j=1}^{m} y_{ij}^l$ for all $i \in [n]$ and $l \in [k]$. Clearly constraints (10d)–(10e) are satisfied. Constraints (10c) are satisfied due to the fact that $\sum_{j=1}^{m} p_{ij} = 1$ for all i. Finally, it remains to show that $\min\{y_i^l, G_{ij}x_i^l\} \ge \sum_{j'=1}^{j} y_{ij'}^l$ which would make constraints (10b) satisfied. To see this, note that $\min\{y_i^l, G_{ij}x_i^l\} = \min\{\sum_{j=1}^{m} y_{ij}^l, x_i^l \sum_{j'=1}^{j} p_{ij'}\} \ge \min\{\sum_{j=1}^{m} y_{ij}^l, \sum_{j'=1}^{j} y_{ij'}^l\}$ where the inequality applies (9c). Since both arguments in the min are at least $\sum_{j'=1}^{j} y_{ij'}^l$, this completes the proof.

Conversely, given a feasible solution to (10), we construct the following feasible solution to (9) with the same value of θ, which is the harder direction. For each $i \in [n]$ and $l \in [k]$, we iteratively define

$$y_{i1}^l = \min\{y_i^l, p_{i1}x_i^l\}$$
$$y_{i2}^l = \min\{y_i^l - y_{i1}^l, p_{i2}x_i^l\}$$
$$\dots$$
$$y_{im}^l = \min\{y_i^l - \sum_{j=1}^{m-1} y_{ij}^l, p_{im}x_i^l\}.$$

Constraints (9c) hold from the second argument in the min, while constraints (9e) hold because by the first argument in the min, the sum $\sum_j y_{ij}^l$ can never exceed y_i^l. Meanwhile, it can be inductively established that $\sum_{j'=1}^{j} y_{ij'}^l = \min\{y_i^l, x_i^l \sum_{j'=1}^{j} p_{ij'}\} = \min\{y_i^l, G_{ij}x_i^l\}$ for all $j = 1, \dots, m$, establishing constraints (9b). Finally, by the same fact $\sum_{j=1}^{m} y_{ij}^l = \min\{y_i^l, x_i^l\} = y_i^l$, establishing constraints (9d) and completing the proof. □

Proof of Theorem 2. We consider the dual of LP (11). We introduce x_i^l as the dual variable for constraint (11b), the dual variable y_{ij}^l for constraint (11c) and dual variable θ for constraint (11d). Then, we get the following LP as the dual of LP (11).

$$\max\ \theta \tag{41a}$$

$$\text{s.t. } \theta \cdot Q_j \le \sum_{i=1}^{n}\sum_{l=1}^{k}\sum_{j'=1}^{j} y_{ij'}^l \qquad \forall j \in [m] \tag{41b}$$

$$y_{ij}^l = \begin{cases} p_{ij}x_i^l, & j < J \\ p_{iJ}\rho x_i^l, & j = J \\ 0, & j > J \end{cases} \qquad \forall i \in [n], l \in [k] \tag{41c}$$

$$x_i^l = \begin{cases} 1, & i = 1, l = k \\ 0, & i = 1, l < k \\ x_{i-1}^l - \sum_{j=1}^m y_{i-1,j}^l + \sum_{j=1}^m y_{i-1,j}^{l+1}, & i > 1 \end{cases} \qquad \forall i \in [n], l \in [k] \tag{41d}$$

$$\theta, x_i^l, y_{ij}^l \in \mathbb{R} \qquad \forall i \in [n], \forall j \in [m], \forall l \in [k] \tag{41e}$$

For any $i \in [n]$, $l \in [k]$, we define $y_i^l = \sum_{j=1}^m y_{ij}^l$. Then, constraint (41c) implies that

$$y_i^l = (\sum_{j<J} p_{ij} + p_{iJ}\rho)x_i^l = ((1-\rho)G_{i,J-1} + \rho G_{iJ})x_i^l, \quad \forall i \in [n], l \in [k].$$

Moreover, for any $j \in [m]$, constraint (41c) implies that

$$\sum_{j'=1}^{j} y_{ij'}^l = \min\{y_i^l, G_{ij}x_i^l\}, \quad \forall i \in [n], l \in [k]$$

Thus, a feasible solution to LP (41) can be translated into a feasible solution to the following LP, with the same objective value,

$$\max\ \theta \tag{42a}$$

$$\text{s.t. } \theta \cdot Q_j \le \sum_{i=1}^{n}\sum_{l=1}^{k} \min\{y_i^l, G_{ij}x_i^l\} \qquad \forall j \in [m] \tag{42b}$$

$$y_i^l = (\sum_{j<J} p_{ij} + p_{iJ}\rho)x_i^l = ((1-\rho)G_{i,J-1} + \rho G_{iJ})x_i^l \qquad \forall i \in [n], l \in [k] \tag{42c}$$

$$x_i^l = \begin{cases} 1, & i = 1, l = k \\ 0, & i = 1, l < k \\ x_{i-1}^l - y_{i-1}^l + y_{i-1}^{l+1}, & i > 1 \end{cases} \qquad \forall i \in [n], l \in [k] \tag{42d}$$

$$y_i^l \ge 0 \qquad \forall i \in [n], l \in [k]. \tag{42e}$$

We now show that a feasible solution to LP (42), denoted by $\{\theta, x_i^l, y_i^l\}$, can be translated into a feasible solution to LP (41) with the same objective value, which implies that the objective value of LP (41) is equivalent to the objective value of LP (42). To be specific, we define

$$y_{ij}^l = \begin{cases} p_{ij}x_i^l, & j < J \\ p_{iJ}\rho x_i^l, & j = J \\ 0, & j > J \end{cases}$$

Clearly, $\{\theta, x_i^l, y_{ij}^l\}$ satisfy the constraint (41c) and (41d). We also have $\sum_{j'=1}^{j} y_{ij'}^l = \min\{y_i^l, G_{ij}x_i^l\}$ for any $i\in[n], l\in[k], j\in[m]$, which implies that constraint (41b) is satisfied. Thus, $\{\theta, x_i^l, y_{ij}^l\}$ is a feasible solution to LP (41).

For the problem instance $I=(\mathbf{G},\mathbf{\Delta})$, we denote by $\mathsf{OST}_{J,\rho}(I)$ the total expected reward collected by the oblivious static threshold policy J,ρ on problem instance I. Then, from the definition of LP (11), we have that

$$\inf_{\mathbf{\Delta}} \frac{\mathsf{OST}_{J,\rho}(I)}{\mathsf{Proph}(I)} = \text{LP (11) with variable } Q_j = \mathbb{E}[\min\{\sum_{i=1}^{n} \text{Ber}(G_{ij}), k\}]$$

Similarly, we have

$$\inf_{\mathbf{\Delta}} \frac{\mathsf{OST}_{J,\rho}(I)}{\mathsf{ExAnte}(I)} = \text{LP (11) with variable } Q_j = \min\{\sum_{i=1}^{n} G_{ij}, k\}$$

Note that LP (41) is the dual of LP (11), and we have shown the objective value of LP (41) is equivalent to the objective value of LP (42), which gives the expression of $\text{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{Proph}}(\mathbf{G})$ (resp. $\text{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{ExAnte}}(\mathbf{G})$) when $Q_j = \mathbb{E}[\min\{\sum_{i=1}^{n} \text{Ber}(G_{ij}), k\}]$ (resp. $Q_j = \min\{\sum_{i=1}^{n} G_{ij}, k\}$). Thus, we have

$$\inf_{\mathbf{\Delta}} \frac{\mathsf{OST}_{J,\rho}(I)}{\mathsf{Proph}(I)} = \text{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{Proph}}(\mathbf{G}) \text{ and } \inf_{\mathbf{\Delta}} \frac{\mathsf{OST}_{J,\rho}(I)}{\mathsf{ExAnte}(I)} = \text{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{ExAnte}}(\mathbf{G})$$

Note that according to Definition 10, the choice of J,ρ for OST can depend on $\mathbf{G}$. Thus, we know that the tight guarantee for OST relative to the prophet (resp. ex-ante relaxation) is given by

$$\inf_{\mathbf{G}} \sup_{J,\rho} \text{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{Proph}}(\mathbf{G}) \text{ (resp. } \inf_{\mathbf{G}} \sup_{J,\rho} \text{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{ExAnte}}(\mathbf{G}))$$

which completes our proof. □

Proof of Theorem 3. We consider the dual of LP (13). We introduce the dual variable $x_i^l(J,\rho)$ for constraint (13b), the dual variable $y_{ij}^l(J,\rho)$ for constraint (13c), the dual variable $\mu(J,\rho)$ for constraint (13d) and the dual variable θ for constraint (13e). Then, we get the following LP as the dual of LP (13).

$$\max\ \theta \tag{43a}$$

$$\text{s.t.}\ \ \theta\cdot Q_j \le \int_{J,\rho} \mu(J,\rho)\left(\sum_{i=1}^{n}\sum_{l=1}^{k}\sum_{j'=1}^{j} y_{ij'}^l(J,\rho)\right) \quad \forall j\in[m] \tag{43b}$$

$$y_{ij}^l(J,\rho) = \begin{cases} p_{ij}x_i^l(J,\rho), & j<J \\ p_{iJ}\rho x_i^l(J,\rho), & j=J \\ 0, & j>J \end{cases} \quad \forall i\in[n], l\in[k], J\in[m], \rho\in(0,1] \tag{43c}$$

$$x_i^l(J,\rho) = \begin{cases} 1, & i=1, l=k \\ 0, & i=1, l<k \\ x_{i-1}^l(J,\rho) - \sum_{j=1}^{m} y_{i-1,j}^l(J,\rho) + \sum_{j=1}^{m} y_{i-1,j}^{l+1}(J,\rho), & i>1 \end{cases} \quad \forall i\in[n], l\in[k] \tag{43d}$$

$$\mu(J,\rho)\ge 0 \quad \forall J\in[m], \rho\in(0,1] \tag{43e}$$

Note that we can select a positive $\{x_i^l(J,\rho), y_{ij}^l(J,\rho)\}$ satisfying constraints (43c) and (43d), select $\mu(J,\rho)$ to be a uniform distribution over $J\in[m], \rho\in(0,1]$, and set $\theta=0$. Then, all the inequality constraints in

LP (43) can be satisfied as strict inequalities by $\{\theta, x_i^l(J,\rho), y_{ij}^l(J,\rho), \mu(J,\rho)\}$. Thus, the Slater's condition is satisfied and strong duality hols between LP (13) and LP (43) (Theorem 2.3 in Shapiro (2009)).

Now we define $y_i^l(J,\rho) = \sum_{j=1}^m y_{ij}^l(J,\rho)$ for any $i \in [n], l \in [k], J \in [m], \rho \in (0,1]$. Then, constraint (43c) implies that

$$y_i^l(J,\rho) = ((1-\rho)G_{i,J-1} + \rho G_{iJ})x_i^l(J,\rho), \quad \forall i \in [n], l \in [k], J \in [m], \rho \in (0,1]$$

and

$$\sum_{j'=1}^{j} y_{ij'}^l(J,\rho) = \min\{y_i^l(J,\rho), G_{ij}x_i^l(J,\rho)\}, \quad \forall i \in [n], l \in [k], j, J \in [m], \rho \in (0,1]$$

Thus, a feasible solution to LP (43) can be translated into a feasible solution to LP (14) with the same objective value.

On the other hand, we denote by $\{\theta, \mathbf{x}(J,\rho), \mathbf{y}(J,\rho), \mu(J,\rho)\}$ a feasible solution to LP (14). Then, for any $i \in [n], l \in [k], J \in [m], \rho \in (0,1]$, we define

$$y_{ij}^l(J,\rho) = \begin{cases} p_{ij}x_i^l(J,\rho), & j < J \\ p_{iJ}\rho x_i^l(J,\rho), & j = J \\ 0, & j > J \end{cases}$$

Clearly, we have $\sum_{j'=1}^{j} y_{ij'}^l(J,\rho) = \min\{y_i^l(J,\rho), G_{ij}x_i^l(J,\rho)\}$ for any $i \in [n], l \in [k], j, J \in [m], \rho \in (0,1]$ and constraint (14c) implies that $\sum_{j=1}^m y_{ij}^l(J,\rho) = y_i^l(J,\rho)$ for any $i \in [n], l \in [k], J \in [m], \rho \in (0,1]$. Then, we have $\{\theta, x_i^l(J,\rho), y_{ij}^l(J,\rho), \mu(J,\rho)\}$ a feasible solution to LP (43) with the same objective value. Thus, the objective value of LP (43) is equivalent to the objective value of LP (14).

For the problem instance $I = (\mathbf{G}, \boldsymbol{\Delta})$, we denote by $\mathsf{ST}_{J,\rho}(I)$ the total expected reward collected by the static threshold policy J, ρ on problem instance I. Then, from the definition of LP (13), we have that

$$\inf_{\boldsymbol{\Delta}} \sup_{J,\rho} \frac{\mathsf{ST}_{J,\rho}(I)}{\mathsf{Proph}(I)} = \text{LP (13) with variable } Q_j = \mathbb{E}[\min\{\sum_{i=1}^n \text{Ber}(G_{ij}), k\}]$$

Similarly, we have

$$\inf_{\boldsymbol{\Delta}} \sup_{J,\rho} \frac{\mathsf{ST}_{J,\rho}(I)}{\mathsf{ExAnte}(I)} = \text{LP (13) with variable } Q_j = \min\{\sum_{i=1}^n G_{ij}, k\}$$

Note that LP (43) is the dual of LP (13), where strong duality holds, and we have shown the objective value of LP (43) is equivalent to the objective value of LP (14), which gives the expression of $\text{innerLP}_{k,n}^{\mathsf{ST}(J,\rho)/\mathsf{Proph}}(\mathbf{G})$ (resp. $\text{innerLP}_{k,n}^{\mathsf{ST}(J,\rho)/\mathsf{ExAnte}}(\mathbf{G})$) when $Q_j = \mathbb{E}[\min\{\sum_{i=1}^n \text{Ber}(G_{ij}), k\}]$ (resp. $Q_j = \min\{\sum_{i=1}^n G_{ij}, k\}$). Thus, we have

$$\inf_{\boldsymbol{\Delta}} \sup_{J,\rho} \frac{\mathsf{ST}_{J,\rho}(I)}{\mathsf{Proph}(I)} = \text{innerLP}_{k,n}^{\mathsf{ST}(J,\rho)/\mathsf{Proph}}(\mathbf{G}) \text{ and } \inf_{\boldsymbol{\Delta}} \sup_{J,\rho} \frac{\mathsf{ST}_{J,\rho}(I)}{\mathsf{ExAnte}(I)} = \text{innerLP}_{k,n}^{\mathsf{ST}(J,\rho)/\mathsf{ExAnte}}(\mathbf{G})$$

Note that according to Definition 10, the choice of J, ρ for ST can depend both on $\mathbf{G}$ and $\boldsymbol{\Delta}$. Thus, we know that the tight guarantee for ST relative to the prophet is given by

$$\inf_{\mathbf{G}} \inf_{\boldsymbol{\Delta}} \sup_{J,\rho} \frac{\mathsf{ST}_{J,\rho}(I)}{\mathsf{Proph}(I)} = \inf_{\mathbf{G}} \text{innerLP}_{k,n}^{\mathsf{ST}(J,\rho)/\mathsf{Proph}}(\mathbf{G})$$

and the tight guarantee for ST relative to the ex-ante relaxation is given by

$$\inf_{\mathbf{G}} \inf_{\boldsymbol{\Delta}} \sup_{J,\rho} \frac{\mathsf{ST}_{J,\rho}(I)}{\mathsf{ExAnte}(I)} = \inf_{\mathbf{G}} \text{innerLP}_{k,n}^{\mathsf{ST}(J,\rho)/\mathsf{ExAnte}}(\mathbf{G}).$$

which completes our proof. $\square$

Proof of Theorem 4. First we show that $\inf_{\mathbf{G}} \text{innerLP}^{\mathsf{DP/ExAnte}}_{k,n}(\mathbf{G}) \geq (17)$. Given a $\mathbf{G}$ for the LHS, we construct an instance defined by $g_1, \dots, g_n$ for the inner maximization problem on the RHS and show that its optimal solution forms a feasible solution to $\text{innerLP}^{\mathsf{DP/ExAnte}}_{k,n}(\mathbf{G})$, which would be sufficient. To accomplish this, let $J \in [m], \rho \in (0,1]$ be such that $\sum_{i=1}^n (G_{i,J-1} + p_{iJ}\rho) = k$, which must uniquely exist since $G_{im} = 1$ for all $i \in [n]$ and $n > k$. Define $g_i = G_{i,J-1} + p_{iJ}\rho$ for all i, which we take to be our instance for the RHS, and consider an optimal solution defined by $\theta, \mathbf{x}, \mathbf{y}$ for its inner problem. We claim that this forms a feasible solution to $\text{innerLP}^{\mathsf{DP/ExAnte}}_{k,n}(\mathbf{G})$. To see why, note that for any i, the expression

$$\frac{\sum_{l=1}^k \min\{y_i^l, G_{ij} x_i^l\}}{G_{ij}} = \sum_{l=1}^k \min\{\frac{y_i^l}{G_{ij}}, x_i^l\}$$

is decreasing in G_{ij}. Therefore, for all $j < J$, since $G_{ij} \leq g_i$, we have

$$\frac{\sum_{l=1}^k \min\{y_i^l, G_{ij} x_i^l\}}{G_{ij}} \geq \frac{\sum_{l=1}^k \min\{y_i^l, g_i x_i^l\}}{g_i} \geq \theta.$$

where the final inequality applies (17b). Therefore, for all $j < J$, we deduce

$$\frac{\sum_{i=1}^n \sum_{l=1}^k \min\{y_i^l, G_{ij} x_i^l\}}{\min\{\sum_{i=1}^n G_{ij}, k\}} = \frac{\sum_{i=1}^n \sum_{l=1}^k \min\{y_i^l, G_{ij} x_i^l\}}{\sum_{i=1}^n G_{ij}} \geq \theta$$

as required for (16b). Meanwhile, for all $j \geq J$, since $G_{ij} \geq g_i$, we directly have

$$\frac{\sum_{i=1}^n \sum_{l=1}^k \min\{y_i^l, G_{ij} x_i^l\}}{\min\{\sum_{i=1}^n G_{ij}, k\}} \geq \frac{\sum_{i=1}^n \sum_{l=1}^k \min\{y_i^l, g_i x_i^l\}}{\sum_{i=1}^n g_i} \geq \theta$$

which completes the proof that $\inf_{\mathbf{G}} \text{innerLP}^{\mathsf{DP/ExAnte}}_{k,n}(\mathbf{G}) \geq (17)$.

To show that $\inf_{\mathbf{G}} \text{innerLP}^{\mathsf{DP/ExAnte}}_{k,n}(\mathbf{G}) \leq (17)$, which is the harder direction, given an instance defined by $g_1, \dots, g_n$ for the outer problem in (17) such that $\sum_{i=1}^n g_i \leq k$, we construct an instance $\mathbf{G}$ for which $\text{innerLP}^{\mathsf{DP/ExAnte}}_{k,n}(\mathbf{G})$ is no greater than the optimal objective value of the inner problem in (17).

The construction goes as follows. Define $m = n+1$. We let

$$G_{ij} = g_i \cdot \mathbb{1}(i \leq j) \text{ for all } j < m \text{ and } G_{im} = 1, \forall i \in [n] \tag{44}$$

Note that the feasibility constraints of $G_{i1} \leq \dots \leq G_{im} = 1$ are satisfied for all i. Under this construction, it holds that

$$Q_j = \min\{\sum_{i=1}^n G_{ij}, k\} = \min\{\sum_{i=1}^j g_i, k\} = \sum_{i=1}^j g_i, \forall j \in [m]$$

where $g_m = k - \sum_{i=1}^n g_i$. Now, denote by $\{\Delta_j^*, U_{ij}^{l*}, V_i^{k*}\}$ one optimal solution of the following LP:

$$
\begin{aligned}
&\min\ V_1^k && && \text{(45a)}\\
&\text{s.t.}\ \ V_i^l = \sum_{j=1}^m p_{ij} U_{ij}^l + V_{i+1}^l && \forall i \in [n], l \in [k] && \text{(45b)}\\
&U_{ij}^l \geq \sum_{j'=j}^m \Delta_{j'} - (V_{i+1}^l - V_{i+1}^{l-1}) && \forall i \in [n], j \in [m], l \in [k] && \text{(45c)}\\
&\sum_{j'=j}^m \Delta_{j'} \geq 0 && \forall j \in [m] && \text{(45d)}\\
&\sum_{j=1}^m Q_j \Delta_j = 1 && && \text{(45e)}\\
&\Delta_j \in \mathbb{R}, \Delta_m = 0, U_{ij}^l \geq 0 && \forall i \in [n], j \in [m], l \in [k] && \text{(45f)}
\end{aligned}
$$

with $p_{ij}=G_{ij}-G_{i,j-1}$. We further denote by $r^*_j=\sum_{j'=j}^m \Delta^*_{j'}$ for each $j\in[m]$ and denote by $\{\sigma(j),\forall j=1,\dots,m\}$ a permutation of $\{1,\dots,m\}$ such that $r^*_{\sigma(1)}\geq r^*_{\sigma(2)}\geq\dots\geq r^*_{\sigma(m)}$. For each $j\in[m]$, we further denote $\hat{G}_{ij}=\sum_{j'=1}^j p_{i\sigma(j')}$, for all $i\in[n]$. Then we have the following claim.

Claim 1. *It holds that* $\mathsf{LP}(45)\geq \text{innerLP}^{\mathsf{DP/ExAnte}}_{k,n}(\hat{\mathbf{G}})$.

On the other hand, the dual of LP(45) is given as

$$
\begin{aligned}
&\max\ \theta && && (46a)\\
&\text{s.t.}\ \ \theta\cdot Q_j+\sum_{j'=1}^{j}\alpha_{j'}=\sum_{i=1}^{n}\sum_{l=1}^{k}\sum_{j'=1}^{j}y^l_{ij'} && \forall j\in[n] && (46b)\\
&\qquad y^l_{ij}\leq p_{ij}x^l_i && \forall i\in[n],j\in[m],l\in[k] && (46c)\\
&\qquad x^l_i=\begin{cases}1, & i=1,l=k\\ 0, & i=1,l<k\\ x^l_{i-1}-\sum_{j=1}^m(y^l_{i-1,j}-y^{l+1}_{i-1,j}), & i>1\end{cases} && \forall i\in[n],l\in[k] && (46d)\\
&\qquad y^l_{ij}\geq 0,\alpha_j\geq 0, && \forall i\in[n],j\in[m],l\in[k] && (46e)
\end{aligned}
$$

Denote by $\{\theta^*,\alpha^*_j,x^{l*}_i,y^{l*}_{ij}\}$ one optimal solution of LP(46) and define $y^{l*}_i=\sum_{j=1}^m y^{l*}_{ij}$ for each $i\in[n]$, $l\in[k]$. We now show that $\{\theta^*,x^{l*}_i,y^{l*}_i\}$ is a feasible solution to LP(17).

Clearly, $(\mathbf{x}^*,\mathbf{y}^*)\in\mathcal{P}^k_n$. From constraint (46b), for $j\in[n]$, we have that

$$\theta^*\cdot(Q_j-Q_{j-1})+\alpha^*_j=\sum_{l=1}^k y^{l*}_{jj}\leq\sum_{l=1}^k\min\{y^{l*}_j,g_jx^{l*}_i\}$$

where the last inequality follows from the construction of $\mathbf{G}$ and constraint (46c). Further note that for $j\in[n]$, we have $\theta^*\cdot g_j\leq\theta^*\cdot(Q_j-Q_{j-1})+\alpha^*_j$. We conclude that $\{\theta^*,x^{l*}_i,y^{l*}_i\}$ is a feasible solution to LP(17). Thus, from Claim 1, we have that

$$\text{innerLP}^{\mathsf{DP/ExAnte}}_{k,n}(\hat{\mathbf{G}})\leq\mathsf{LP}(45)=\mathsf{LP}(46)=\theta^*\leq\mathsf{LP}(17)$$

which completes our proof. □

Proof of Claim 1. Note that from Lemma 1, $\text{innerLP}^{\mathsf{DP/ExAnte}}_{k,n}(\hat{\mathbf{G}})$ is given as the optimal objective value of the following LP:

$$
\begin{aligned}
&\min\ V^k_1 && && (47a)\\
&\text{s.t.}\ \ V^l_i=\sum_{j=1}^m\hat{p}_{ij}U^l_{ij}+V^l_{i+1} && \forall i\in[n],l\in[k] && (47b)\\
&\qquad U^l_{ij}\geq\sum_{j'=j}^m\Delta_{j'}-(V^l_{i+1}-V^{l-1}_{i+1}) && \forall i\in[n],j\in[m],l\in[k] && (47c)\\
&\qquad \sum_{j=1}^m\hat{Q}_j\Delta_j=1 && && (47d)\\
&\qquad \Delta_j\geq 0,U^l_{ij}\geq 0 && \forall i\in[n],j\in[m],l\in[k] && (47e)
\end{aligned}
$$

where $\hat{p}_{ij}=p_{i\sigma(j)}$ and

$$\hat{Q}_j=\min\{\sum_{i=1}^n\hat{G}_{ij},k\},\forall j<m\text{ and }\hat{Q}_m=\min\{n,k\}$$

Note that $\sigma(m)=m$. We have $\hat{Q}_j=\sum_{i=1}^j g_{\sigma(i)}$. We now construct a feasible solution to (47) from $\{\Delta_j^*, U_{ij}^{l*}, V_i^{k*}\}$, which is the optimal solution of (45) that gives rise to the definition of the permutation $\{\sigma(j), \forall j=1,\ldots,m\}$. To be specific, we define

$$\hat{V}_i^k = V_i^{k^*}, \hat{U}_{ij}^l = U_{i\sigma(j)}^{l*}, \text{ and } \hat{\Delta}_j = r^*_{\sigma(j)} - r^*_{\sigma(j+1)}, \forall i\in[n], l\in[k], j\in[m]$$

where we define $r^*_{\sigma(m+1)}=0$. Clearly, constraints (45b) and (45c) imply that constraints (47b) and (47c) are satisfied by $\{\hat{\Delta}_j, \hat{U}_{ij}^l, \hat{V}_i^k\}$. Also, we have

$$\sum_{j=1}^m \hat{Q}_j\Delta_j = \sum_{j=1}^m r^*_{\sigma(j)}\cdot(\hat{Q}_j-\hat{Q}_{j-1}) = \sum_{j=1}^m r^*_{\sigma(j)}\cdot g_{\sigma(j)} = 1$$

where the last equation follows from constraint (45e). Thus, our proof is completed. ☐

Proof of Lemma 2. We prove stronger results by induction. First, we show that for any i, it holds that

$$\sum_{l'=l}^k x_i^{l'} = \Pr[\sum_{i'<i}\text{Ber}(\tau_{i'}) < k-l+1], \quad \forall l=1,\ldots,k \tag{48}$$

We prove (48) by induction on i. When $i=1$, clearly, for any $l=1,\ldots,k$, we have

$$\sum_{l'=l}^k x_1^{l'} = 1 = \Pr[0<k-l+1] = \Pr[\sum_{i'<1}\text{Ber}(\tau_{i'}) < k-l+1]$$

which implies that (48) holds. We now assume that (48) holds for i, and we consider $i+1$. For any $l=1,\ldots,k$, we have that

$$x_{i+1}^l = x_i^l - y_i^l + y_i^{l+1} = (1-\tau_i)\cdot x_i^l + \tau_i\cdot x_i^{l+1}$$

where we denote $x_i^{k+1}=0$. Thus, we have

$$\sum_{l'=l}^k x_{i+1}^{l'} = (1-\tau_i)\cdot\sum_{l'=l}^k x_i^{l'} + \tau_i\cdot\sum_{l'=l+1}^k x_i^{l'}$$

On the other hand, conditioning whether $\text{Ber}(\tau_i)=1$, we have

$$\begin{aligned}\Pr[\sum_{i'<i+1}\text{Ber}(\tau_{i'})<k-l+1] &= \Pr(\sum_{i'<i}\text{Ber}(\tau_{i'})<k-l+1)\cdot\Pr[\text{Ber}(\tau_i)=0] + \Pr(\sum_{i'<i}\text{Ber}(\tau_{i'})<k-l)\cdot\Pr[\text{Ber}(\tau_i)=1]\\ &= (1-\tau_i)\cdot\Pr(\sum_{i'<i}\text{Ber}(\tau_{i'})<k-l+1) + \tau_i\cdot\Pr(\sum_{i'<i}\text{Ber}(\tau_{i'})<k-l)\end{aligned}$$

From induction hypothesis, we know that

$$\sum_{l'=l}^k x_i^{l'} = \Pr(\sum_{i'<i}\text{Ber}(\tau_{i'})<k-l+1) \text{ and } \sum_{l'=l+1}^k x_i^{l'} = \Pr(\sum_{i'<i}\text{Ber}(\tau_{i'})<k-l)$$

Thus, we have that

$$\sum_{l'=l}^k x_{i+1}^{l'} = \Pr[\sum_{i'<i+1}\text{Ber}(\tau_{i'})<k-l+1]$$

From induction, we know that (48) holds for any $i\in[n]$, which proves the first equation in Lemma 2.

We now prove the second equation. We prove by induction to show that for any i, it holds

$$\sum_{i'=1}^i \tau_{i'}\cdot\sum_{l=1}^k x_{i'}^l = \mathbb{E}[\min\{\sum_{i'=1}^i\text{Ber}(\tau_{i'}), k\}] \tag{49}$$

When $i=1$, clearly, (49) holds. We now assume that (49) holds for i and we consider $i+1$. We have

$$\sum_{i'=1}^{i+1}\tau_{i'}\cdot\sum_{l=1}^{k}x_{i'}^l=\sum_{i'=1}^{i}\tau_{i'}\cdot\sum_{l=1}^{k}x_{i'}^l+\tau_{i+1}\cdot\sum_{l=1}^{k}x_{i+1}^l=\mathbb{E}[\min\{\sum_{i'=1}^{i}\mathrm{Ber}(\tau_{i'}),k\}]+\tau_{i+1}\cdot\sum_{l=1}^{k}x_{i+1}^l$$

On the other hand, denote by $\mathcal{A}_i$ the event $\{\sum_{i'=1}^{i}\mathrm{Ber}(\tau_{i'})<k\}$. We have

$$\mathbb{E}[\min\{\sum_{i'=1}^{i+1}\mathrm{Ber}(\tau_{i'}),k\}]=\mathbb{E}[\sum_{i'=1}^{i}\mathrm{Ber}(\tau_{i'})+\mathrm{Ber}(\tau_{i+1})|\mathcal{A}_i]\cdot\Pr(\mathcal{A}_i)+k\cdot(1-\Pr(\mathcal{A}_i))$$

Note that the random variable $\mathrm{Ber}(\tau_{i+1})$ is independent of the event $\mathcal{A}_i$. We have

$$\begin{aligned}\mathbb{E}[\min\{\sum_{i'=1}^{i+1}\mathrm{Ber}(\tau_{i'}),k\}]&=\mathbb{E}[\sum_{i'=1}^{i}\mathrm{Ber}(\tau_{i'})|\mathcal{A}_i]\cdot\Pr(\mathcal{A}_i)+k\cdot(1-\Pr(\mathcal{A}_i))+\mathbb{E}[\mathrm{Ber}(\tau_{i+1})]\cdot\Pr(\mathcal{A}_i)\\&=\mathbb{E}[\min\{\sum_{i'=1}^{i}\mathrm{Ber}(\tau_{i'}),k\}]+\tau_{i+1}\cdot\Pr(\mathcal{A}_i)\end{aligned}$$

From (48), we know that

$$\Pr(\mathcal{A}_i)=\sum_{l=1}^{k}x_{i+1}^l.$$

Thus, we have

$$\sum_{i'=1}^{i+1}\tau_{i'}\cdot\sum_{l=1}^{k}x_{i'}^l=\mathbb{E}[\min\{\sum_{i'=1}^{i}\mathrm{Ber}(\tau_{i'}),k\}]+\tau_{i+1}\cdot\sum_{l=1}^{k}x_{i+1}^l=\mathbb{E}[\min\{\sum_{i'=1}^{i+1}\mathrm{Ber}(\tau_{i'}),k\}]$$

By induction, (49) holds for any $i\in[n]$, which completes our proof. □

Proof of Lemma 3. By (19), the objective value of $\mathrm{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{ExAnte}}(\mathbf{G})$ for any OST defined by J,ρ will equal

$$\begin{aligned}\theta&=\min_{j\in[m]}\frac{\sum_{i=1}^n\min\{(1-\rho)G_{i,J-1}+\rho G_{iJ},G_{ij}\}\sum_{l=1}^k x_i^l}{\min\{\sum_{i=1}^n G_{ij},k\}}\\&=\min\left\{\min_{j<J}\frac{\sum_{i=1}^n G_{ij}\sum_{l=1}^k x_i^l}{\min\{\sum_{i=1}^n G_{ij},k\}},\min_{j\geq J}\frac{\sum_{i=1}^n((1-\rho)G_{i,J-1}+\rho G_{iJ})\sum_{l=1}^k x_i^l}{\min\{\sum_{i=1}^n G_{ij},k\}}\right\}\\&\geq\min\left\{\min_{j<J}\frac{\sum_{i=1}^n G_{ij}\sum_{l=1}^k x_i^l}{\sum_{i=1}^n G_{ij}},\frac{\sum_{i=1}^n((1-\rho)G_{i,J-1}+\rho G_{iJ})\sum_{l=1}^k x_i^l}{k}\right\}\\&\geq\min\left\{\min_{i\in[n]}\sum_{l=1}^k x_i^l,\frac{\sum_{i=1}^{n-1}((1-\rho)G_{i,J-1}+\rho G_{iJ})\sum_{l=1}^k x_i^l}{k}\right\}\\&=\min\left\{\min_{i\in[n]}\Pr\left[\sum_{i'<i}\mathrm{Ber}((1-\rho)G_{i',J-1}+\rho G_{i'J})<k\right],\frac{\mathbb{E}[\min\{\sum_{i=1}^{n-1}\mathrm{Ber}((1-\rho)G_{i,J-1}+\rho G_{iJ}),k\}]}{k}\right\}\end{aligned}$$

where the first argument in the second inequality holds because $\frac{\sum_{i=1}^n G_{ij}\sum_{l=1}^k x_i^l}{\sum_{i=1}^n G_{ij}}\geq\min_{i\in[n]}\sum_{l=1}^k x_i^l$ for all $j<J$, and the final equality applies Lemma 2 throughout the terms. The proof is then completed by the observation that the min over $i\in[n]$ is always achieved when $i=n$. □

Proof of Lemma 4. Given any distributions $\mathbf{G}$ over m types for the outer problem in (20), we construct distributions $\mathbf{G}'_1,\dots,\mathbf{G}'_n$ over $m+2$ types such that for some small $\varepsilon>0$, we have that $\sup_{J',\rho'}\mathrm{innerLP}_{k,n}^{\mathsf{OST}(J',\rho')/\mathsf{Proph}}(\mathbf{G}'_1,\dots,\mathbf{G}'_n)$ is at most

$$\frac{\varepsilon}{k}+\sup_{J,\rho}\min\left\{\Pr\left[\sum_{i<n}\mathrm{Ber}((1-\rho)G_{i,J-1}+\rho G_{iJ})<k\right],\frac{\mathbb{E}[\min\{\sum_{i<n}\mathrm{Ber}((1-\rho)G_{i,J-1}+\rho G_{iJ}),k\}]}{k}\right\}.\quad(50)$$

Taking $\varepsilon \to 0$ would then complete the proof.

The construction entails defining type distributions for agents $i = 1, \ldots, n-1$ as

$$G'_{ij} = \begin{cases} 1, & j = m+2; \\ G_{i,j-1}, & j = 2, \ldots, m+1; \\ 0, & j = 1. \end{cases}$$

For the last agent, we define $G'_{nj} = \varepsilon$ for all $j \le m+1$ and $G'_{n,m+2} = 1$. Note that this feasibly satisfies $0 \le G'_{i1} \le \cdots \le G'_{i,m+2} = 1$ for all agents $i \in [n]$.

Recall that for θ to be feasible in $\text{innerLP}_{k,n}^{\mathsf{OST}(J',\rho')/\mathsf{Proph}}(\mathbf{G}'_1, \ldots, \mathbf{G}'_n)$, applying (19), we need

$$\theta \cdot \mathbb{E}[\min\{\sum_{i=1}^{n} \text{Ber}(G'_{ij}), k\}] \le \sum_{i=1}^{n} \min\{(1-\rho')G'_{i,J'-1} + \rho' G'_{iJ'}, G'_{ij}\} \sum_{l=1}^{k} x_i^l \qquad \forall j \in [m+2]. \tag{51}$$

Taking $j = 1$, we have $\mathbb{E}[\min\{\sum_{i=1}^{n} \text{Ber}(G'_{ij}), k\}] = \mathbb{E}[\min\{\text{Ber}(\varepsilon), k\}] = \varepsilon$. Meanwhile, the RHS of (51) can be at most $\varepsilon \sum_l x_n^l$ when $j = 1$. Thus, we know that $\theta \le \sum_l x_n^l$. Since the feasible vectors $\mathbf{x}, \mathbf{y}$ in $\text{innerLP}_{k,n}^{\mathsf{OST}(J',\rho')/\mathsf{Proph}}(\mathbf{G}'_1, \ldots, \mathbf{G}'_n)$ satisfies the static threshold constraint (18) and $(\mathbf{x}, \mathbf{y}) \in \mathcal{P}_n^k$, by Lemma 2, we know $\sum_l x_n^l = \Pr[\sum_{i<n} \text{Ber}((1-\rho')G'_{i,J'-1} + \rho' G'_{iJ'}) < k]$.

On the other hand, taking $j = m+1$, we have $\mathbb{E}[\min\{\sum_{i=1}^{n} \text{Ber}(G'_{ij}), k\}] = \mathbb{E}[\min\{(n-1) + \text{Ber}(\varepsilon), k\}] = k$ since $G'_{i,m+1} = G_{im} = 1$ for all $i \in [n-1]$ and $n > k$. Meanwhile, the RHS of (51) can be at most $\sum_{i=1}^{n-1}((1-\rho')G'_{i,J'-1} + \rho' G'_{iJ'}) \sum_{l=1}^{k} x_i^l + \varepsilon$ when $j = m+1$. Thus, we also know that

$$\begin{aligned} \theta &\le \frac{\varepsilon}{k} + \frac{\sum_{i=1}^{n-1}((1-\rho')G'_{i,J'-1} + \rho' G'_{iJ'}) \sum_{l=1}^{k} x_i^l}{k} \\ &= \frac{\varepsilon}{k} + \frac{\mathbb{E}[\min\{\sum_{i=1}^{n-1} \text{Ber}((1-\rho')G'_{i,J'-1} + \rho' G'_{iJ'}), k\}]}{k} \end{aligned}$$

where we have again applied Lemma 2.

Finally, by setting $J = J' - 1$ and $\rho = \rho'$, due to the construction of $\mathbf{G}'_1, \ldots, \mathbf{G}'_n$ based on $\mathbf{G}$, expression (50) evaluates to exactly

$$\frac{\varepsilon}{k} + \min\left\{\Pr\left[\sum_{i<n} \text{Ber}((1-\rho')G'_{i,J'-1} + \rho' G'_{iJ'}) < k\right], \frac{\mathbb{E}[\min\{\sum_{i<n} \text{Ber}((1-\rho')G'_{i,J'-1} + \rho' G'_{iJ'}), k\}]}{k}\right\}.$$

(Setting $J = J' - 1$ is only valid if $J' \notin \{1, m+2\}$, but it is easy to see that the sup over J', ρ' never requires these values of J' to achieve.) This completes the proof of Lemma 4. $\square$

Proof of Theorem 5. By Proposition 1, $\mathsf{Proph}(I) \le \mathsf{ExAnte}(I)$ for any instance I, and hence $\text{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{ExAnte}}(\mathbf{G}) \le \text{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{Proph}}(\mathbf{G})$. Meanwhile, by Lemmas 3 and 4, we have

$$\begin{aligned} &\inf_{\mathbf{G}} \sup_{J,\rho} \ \text{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{ExAnte}}(\mathbf{G}) \\ &\ge \inf_{\mathbf{G}} \sup_{J,\rho} \ \min\left\{\Pr\left[\sum_{i<n} \text{Ber}((1-\rho)G_{i,J-1} + \rho G_{iJ}) < k\right], \frac{\mathbb{E}[\min\{\sum_{i<n} \text{Ber}((1-\rho)G_{i,J-1} + \rho G_{iJ}), k\}]}{k}\right\} \\ &\ge \inf_{\mathbf{G}} \sup_{J,\rho} \ \text{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{Proph}}(\mathbf{G}). \end{aligned}$$

Combining these inequalities completes the proof of Theorem 5. $\square$

Proof of Lemma 5. We construct the instance $\mathbf{G}$ as follows: there are two agents, $n=2$, four types, $m=4$, and one slot, $k=1$. The distribution for the first agent is given by $\mathbf{G}_1=(0,\frac{1}{2},\frac{1}{2},1)$, and the distribution for the second agent is given by $\mathbf{G}_2=(\varepsilon,\varepsilon,1,1)$.

For any J,ρ, we denote $C_{J,\rho}$ as a vector such that $C_{J,\rho}(j)=\sum_{i=1}^{2}\min\{y_i(J,\rho),G_{ij}x_i(J,\rho)\}$ for each $j\in[m]$, where $\mathbf{x}(J,\rho),\mathbf{y}(J,\rho)$ satisfy

$$\begin{aligned} &x_1(J,\rho)=1, && y_1(J,\rho)=((1-\rho)G_{1,J-1}+\rho G_{1J})x_1(J,\rho)\\ &x_2(J,\rho)=x_1(J,\rho)-y_1(J,\rho), && y_2(J,\rho)=((1-\rho)G_{2,J-1}+\rho G_{2J})x_2(J,\rho). \end{aligned}$$

Clearly, we have $C_{1,1}=(\varepsilon,\varepsilon,\varepsilon,\varepsilon)$ and $C_{3,1}=(\frac{\varepsilon}{2},\frac{1}{2}+\frac{\varepsilon}{2},1,1)$. Moreover, note that

$$Q^{\mathsf{Proph}}=(\varepsilon+o(\varepsilon),\frac{1}{2}+O(\varepsilon),1,1)\text{ and }Q^{\mathsf{ExAnte}}=(\varepsilon+o(\varepsilon),\frac{1}{2}+O(\varepsilon),1,1).$$

By setting $\mu(1,1)=\frac{1}{3}$, $\mu(3,1)=\frac{2}{3}$, and $\mu(J,\rho)=0$ for all other J,ρ in $\text{innerLP}_{k,n}^{\mathsf{ST/Proph}}(\mathbf{G})$ and $\text{innerLP}_{k,n}^{\mathsf{ST/ExAnte}}(\mathbf{G})$, we know that

$$\text{innerLP}_{k,n}^{\mathsf{ST/Proph}}(\mathbf{G})\geq\frac{2}{3}\text{ and }\text{innerLP}_{k,n}^{\mathsf{ST/ExAnte}}(\mathbf{G})\geq\frac{2}{3}.$$

On the other hand, we show that an OST cannot achieve a guarantee better than $\frac{1}{2}$ with respect to both the Ex-Ante benchmark and the prophet benchmark.

If $J\leq 2$, irregardless of the value of ρ, we have that $y_2(J,\rho)=\varepsilon x_2(J,\rho)\leq\varepsilon$. Then we have that $C_{J,\rho}(3)\leq\min\{y_1(J,\rho),\frac{x_1(J,\rho)}{2}\}+\varepsilon\leq\frac{1}{2}+\varepsilon$. Compared with $Q^{\mathsf{Proph}}(3)=Q^{\mathsf{ExAnte}}(3)=1$, we know θ cannot be better than $\frac{1}{2}$ as $\varepsilon\to 0$.

If $J\geq 3$, irregardless of the value of ρ, we have that $x_2(J,\rho)=x_1(J,\rho)-y_2(J,\rho)\leq\frac{1}{2}$, and thus $C_{J,\rho}(1)\leq\varepsilon\cdot x_2(J,\rho)\leq\frac{\varepsilon}{2}$. Compared with $Q^{\mathsf{Proph}}(1)=Q^{\mathsf{ExAnte}}(1)=\varepsilon$, we know θ cannot be better than $\frac{1}{2}$ as $\varepsilon\to 0$. Thus, we have

$$\sup_{J,\rho}\text{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{Proph}}(\mathbf{G})\leq\frac{1}{2}\text{ and }\sup_{J,\rho}\text{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{ExAnte}}(\mathbf{G})\leq\frac{1}{2}$$

as $\varepsilon\to\infty$, which completes our proof. ☐

Proof of Lemma 6. The proof mainly follows the proof of Lemma 4. Given any distributions $\mathbf{G}$ over m types, we construct distributions $\mathbf{G}'=(\mathbf{G}'_1,\ldots,\mathbf{G}'_n)$ over $m+2$ types such that for some small $\varepsilon>0$, we have that

$$\begin{aligned}\text{innerLP}_{k,n}^{\mathsf{ST/Proph}}(\mathbf{G}')\leq\frac{\varepsilon}{k}+\min\Bigg\{&\int_{J,\rho}\Pr\left[\sum_{i<n}\text{Ber}((1-\rho)G_{i,J-1}+\rho G_{iJ})<k\right]\mu(J,\rho),\\ &\int_{J,\rho}\frac{\mathbb{E}[\min\{\sum_{i<n}\text{Ber}((1-\rho)G_{i,J-1}+\rho G_{iJ}),k\}]}{k}\mu(J,\rho)\Bigg\}.\end{aligned}\tag{52}$$

for a probability measure μ over (J,ρ). Taking $\varepsilon\to 0$ would then complete the proof.

The construction entails defining type distributions for agents $i=1,\ldots,n-1$ as

$$G'_{ij}=\begin{cases}1, & j=m+2;\\ G_{i,j-1}, & j=2,\ldots,m+1;\\ 0, & j=1.\end{cases}$$

For the last agent, we define $G'_{nj}=\varepsilon$ for all $j\leq m+1$ and $G'_{n,m+2}=1$. Note that this feasibly satisfies $0\leq G'_{i1}\leq\cdots\leq G'_{i,m+2}=1$ for all agents $i\in[n]$.

Recall that for θ to be feasible in $\text{innerLP}^{\mathsf{ST/Proph}}_{k,n}(\mathbf{G}'_1,\ldots,\mathbf{G}'_n)$, from constraints (14c) and (14b), we have

$$\theta\cdot\mathbb{E}[\min\{\sum_{i=1}^{n}\text{Ber}(G'_{ij}),k\}]\le\int_{J',\rho'}\mu'(J',\rho')\cdot\left(\sum_{i=1}^{n}\min\{(1-\rho')G'_{i,J'-1}+\rho G'_{iJ'},G'_{ij}\}\sum_{l=1}^{k}x_i^l(J',\rho')\right)\quad\forall j\in[m+2]. \tag{53}$$

for some $(\mathbf{x}(J',\rho'),\mathbf{y}(J',\rho'))\in\mathcal{P}_n^k$ and a measure μ' over $J'\in[m+2]$ and $\rho'\in(0,1]$.

Taking $j=1$, we have $\mathbb{E}[\min\{\sum_{i=1}^{n}\text{Ber}(G'_{i1}),k\}]=\varepsilon$. Meanwhile, when $j=1$, the RHS of (53) can be at most

$$\int_{J',\rho'}\mu'(J',\rho')\cdot\left(\sum_{i=1}^{n}G'_{i1}\sum_{l=1}^{k}x_i^l(J',\rho')\right)=\varepsilon\cdot\int_{J',\rho'}\mu'(J',\rho')\cdot\sum_{l=1}^{k}x_n^l(J',\rho').$$

Thus, we know that

$$\theta\le\int_{J',\rho'}\mu'(J',\rho')\cdot\sum_{l=1}^{k}x_n^l(J',\rho').$$

Since the feasible vectors $\mathbf{x}(J',\rho'),\mathbf{y}(J',\rho')$ in $\text{innerLP}^{\mathsf{ST/Proph}}_{k,n}(\mathbf{G}'_1,\ldots,\mathbf{G}'_n)$ satisfies the static threshold constraint (18) and $(\mathbf{x}(J',\rho'),\mathbf{y}(J',\rho'))\in\mathcal{P}_n^k$, by Lemma 2, we know $\sum_l x_n^l(J',\rho')=\Pr[\sum_{i<n}\text{Ber}((1-\rho')G'_{i,J-1}+\rho'G'_{iJ})<k]$ for each J',ρ', which implies that

$$\theta\le\int_{J',\rho'}\mu'(J',\rho')\cdot\Pr[\sum_{i<n}\text{Ber}((1-\rho')G'_{i,J-1}+\rho'G'_{iJ})<k]$$

On the other hand, taking $j=m+1$, we have $\mathbb{E}[\min\{\sum_{i=1}^{n}\text{Ber}(G'_{i,m+1}),k\}]=k$ since $G'_{i,m+1}=G_{im}=1$ for all $i\in[n-1]$ and $n>k$. Meanwhile, the RHS of (53) can be at most

$$\int_{J'.\rho'}\mu'(J',\rho')\cdot\left(\sum_{i=1}^{n-1}((1-\rho')G'_{i,J'-1}+\rho'G'_{iJ'})\sum_{l=1}^{k}x_i^l(J',\rho')\right)+\varepsilon$$

when $j=m+1$. Thus, we also know that

$$\begin{aligned}\theta&\le\frac{\varepsilon}{k}+\frac{\int_{J'.\rho'}\mu'(J',\rho')\cdot\left(\sum_{i=1}^{n-1}((1-\rho')G'_{i,J'-1}+\rho'G'_{iJ'})\sum_{l=1}^{k}x_i^l(J',\rho')\right)}{k}\\&=\frac{\varepsilon}{k}+\int_{J',\rho'}\mu(J',\rho')\cdot\frac{\mathbb{E}[\min\{\sum_{i=1}^{n-1}\text{Ber}((1-\rho')G'_{i,J'-1}+\rho'G'_{iJ'}),k\}]}{k}\end{aligned}$$

where we have again applied Lemma 2 to derive the last equality.

Finally, by setting $J=J'-1$ and $\rho=\rho'$, due to the construction of $\mathbf{G}'_1,\ldots,\mathbf{G}'_n$ based on $\mathbf{G}$, expression (52) evaluates to exactly

$$\begin{aligned}\frac{\varepsilon}{k}+\min\Bigg\{&\int_{J',\rho'}\mu'(J',\rho')\Pr\left[\sum_{i<n}\text{Ber}((1-\rho')G'_{i,J'-1}+\rho'G'_{iJ'})<k\right],\\&\int_{J',\rho'}\mu'(J',\rho')\frac{\mathbb{E}[\min\{\sum_{i<n}\text{Ber}((1-\rho')G'_{i,J'-1}+\rho'G'_{iJ'}),k\}]}{k}\Bigg\}.\end{aligned}$$

(Setting $J=J'-1$ is only valid if $J'\notin\{1,m+2\}$, but it is easy to see that the sup over J',ρ' never requires these values of J' to achieve.) This completes the proof of Lemma 6. $\square$

Proof of Theorem 6. In Theorem 5, we showed that

$$\inf_{\mathbf{G}}\sup_{J,\rho}\ \text{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{Proph}}(\mathbf{G}) = \inf_{\mathbf{G}}\sup_{J,\rho}\ \text{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{ExAnte}}(\mathbf{G})$$
$$= \inf_{\mathbf{G}}\sup_{J,\rho}\ \min\left\{\Pr\left[\sum_{i<n}\text{Ber}((1-\rho)G_{i,J-1}+\rho G_{iJ})<k\right], \frac{\mathbb{E}[\min\{\sum_{i<n}\text{Ber}((1-\rho)G_{i,J-1}+\rho G_{iJ}),k\}]}{k}\right\}.$$

From Lemma 11 in Chawla et al. (2020), the infimum is achieved when there is only $m=1$ type and $G_{i1}=1$ for all i, which implies that

$$\begin{aligned}
&\inf_{\mathbf{G}'}\sup_{J,\rho}\ \text{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{Proph}}(\mathbf{G}') = \inf_{\mathbf{G}'}\sup_{J,\rho}\ \text{innerLP}_{k,n}^{\mathsf{OST}(J,\rho)/\mathsf{ExAnte}}(\mathbf{G}') \\
&= \sup_{\rho\in(0,1]}\min\left\{\Pr\left[\text{Bin}(n-1,\rho)<k\right], \frac{\mathbb{E}[\min\{\text{Bin}(n-1,\rho),k\}]}{k}\right\} \\
&= \sup_{\rho\in(0,1]}\min_{\beta\in[0,1]}\beta\cdot\Pr\left[\text{Bin}(n-1,\rho)<k\right]+(1-\beta)\cdot\frac{\mathbb{E}[\min\{\text{Bin}(n-1,\rho),k\}]}{k}. \qquad (54)
\end{aligned}$$

Then, fixing $\mathbf{G}$ such that $m=1$ and $G_{i1}=1$ for each $i\in[n]$, from Lemma 6, we have

$$\begin{aligned}
&\inf_{\mathbf{G}'}\text{innerLP}_{k,n}^{\mathsf{ST}/\mathsf{ExAnte}}(\mathbf{G}') \le \inf_{\mathbf{G}'}\text{innerLP}_{k,n}^{\mathsf{ST}/\mathsf{Proph}}(\mathbf{G}') \\
&\le \sup_{\mu:(0,1]\to\mathbb{R}_{\ge 0},\int_\rho\mu(\rho)=1}\min\left\{\int_\rho\Pr\left[\text{Bin}(n-1,\rho)<k\right]\mu(\rho), \int_\rho\frac{\mathbb{E}[\min\{\text{Bin}(n-1,\rho),k\}]}{k}\mu(\rho)\right\} \\
&= \min_{\beta\in[0,1]}\sup_{\mu:(0,1]\to\mathbb{R}_{\ge 0},\int_\rho\mu(\rho)=1}\beta\cdot\int_\rho\Pr\left[\text{Bin}(n-1,\rho)<k\right]\mu(\rho)+(1-\beta)\cdot\int_\rho\frac{\mathbb{E}[\min\{\text{Bin}(n-1,\rho),k\}]}{k}\mu(\rho) \\
&= \min_{\beta\in[0,1]}\sup_{\rho\in(0,1]}\beta\cdot\Pr\left[\text{Bin}(n-1,\rho)<k\right]+(1-\beta)\cdot\frac{\mathbb{E}[\min\{\text{Bin}(n-1,\rho),k\}]}{k}. \qquad (55)
\end{aligned}$$

Note that OST is a special case of ST, the final result follows as long as the value of (54) equals the value of (55). Denote by $\lambda(\beta,\rho)$ the function:

$$\lambda(\beta,\rho) := \beta\cdot\Pr\left[\text{Bin}(n-1,\rho)<k\right]+(1-\beta)\cdot\frac{\mathbb{E}[\min\{\text{Bin}(n-1,\rho),k\}]}{k}.$$

It suffices to show that

$$\min_{\beta\in[0,1]}\sup_{\rho\in(0,1]}\lambda(\beta,\rho) = \sup_{\rho\in(0,1]}\min_{\beta\in[0,1]}\lambda(\beta,\rho)$$

Note that $\lambda(\beta,\rho)$ is linear in β, from Sion's minimax theorem, it only remains to show that $\lambda(\beta,\rho)$ is quasi-concave over $\rho\in(0,1]$, for each fixed $\beta\in[0,1]$. We now assume β is fixed and we show $\lambda(\beta,\rho)$ is a unimodal function over ρ, which implies quasi-concavity.

Note that

$$\begin{aligned}
\lambda(\beta,\rho) &= \beta\sum_{s=0}^{k-1}C_n^s\rho^s(1-\rho)^{n-s}+(1-\beta)\left\{\frac{1}{k}\sum_{s=1}^{k-1}sC_n^s\rho^s(1-\rho)^{n-s}+\sum_{s=k}^{n}C_n^s\rho^s(1-\rho)^{n-s}\right\} \\
&= 1-\beta+(2\beta-1)\sum_{s=0}^{k-1}C_n^s\rho^s(1-\rho)^{n-s}+\frac{1-\beta}{k}\sum_{s=1}^{k-1}sC_n^s\rho^s(1-\rho)^{n-s}
\end{aligned}$$

Then, The derivative of $\lambda(\beta,\rho)$ over ρ is

$$\begin{aligned}
\frac{\partial}{\partial\rho}\lambda(\beta,\rho) &= (2\beta-1)\left\{-n(1-\rho)^{n-1}+\sum_{s=1}^{k-1}[sC_n^s\rho^{s-1}(1-\rho)^{n-s}-(n-s)C_n^s\rho^s(1-\rho)^{n-s-1}]\right\} \\
&\quad+\frac{1-\beta}{k}\sum_{s=1}^{k-1}\left\{s^2C_n^s\rho^{s-1}(1-\rho)^{n-s}-s(n-s)C_n^s\rho^s(1-\rho)^{n-s-1}\right\}
\end{aligned}$$

Notice that

$$
\begin{aligned}
&\sum_{s=1}^{k-1}[sC_n^s\rho^{s-1}(1-\rho)^{n-s}-(n-s)C_n^s\rho^s(1-\rho)^{n-s-1}]\\
&=\sum_{s=0}^{k-2}(s+1)C_n^{s+1}\rho^s(1-\rho)^{n-s-1}-\sum_{s=1}^{k-1}(n-s)C_n^s\rho^s(1-\rho)^{n-s-1}\\
&=n(1-\rho)^{n-1}-(n-k+1)C_n^{k-1}\rho^{k-1}(1-\rho)^{n-k}+\sum_{s=1}^{k-2}[(s+1)C_n^{s+1}\rho^s(1-\rho)^{n-s-1}-(n-s)C_n^s\rho^s(1-\rho)^{n-s-1}]\\
&=n(1-\rho)^{n-1}-(n-k+1)C_n^{k-1}\rho^{k-1}(1-\rho)^{n-k}
\end{aligned}
$$

where the last equality holds since

$$(s+1)C_n^{s+1}=(n-s)C_n^s.$$

Similarly,

$$
\begin{aligned}
&\sum_{s=1}^{k-1}\left\{s^2C_n^s\rho^{s-1}(1-\rho)^{n-s}-s(n-s)C_n^s\rho^s(1-\rho)^{n-s-1}\right\}\\
&=\sum_{s=0}^{k-2}(s+1)^2C_n^{s+1}\rho^s(1-\rho)^{n-s-1}-\sum_{s=1}^{k-1}s(n-s)C_n^s\rho^s(1-\rho)^{n-s-1}\\
&=n(1-\rho)^{n-1}-(k-1)(n-k+1)C_n^{k-1}\rho^{k-1}(1-\rho)^{n-k}+\sum_{s=1}^{k-2}\left((s+1)^2C_n^{s+1}-s(n-s)C_n^s\right)\rho^s(1-\rho)^{n-s-1}\\
&=n(1-\rho)^{n-1}-(k-1)(n-k+1)C_n^{k-1}\rho^{k-1}(1-\rho)^{n-k}+\sum_{s=1}^{k-2}(s+1)C_n^{s+1}\rho^s(1-\rho)^{n-s-1}\\
&=\sum_{s=0}^{k-2}(s+1)C_n^{s+1}\rho^s(1-\rho)^{n-s-1}-(k-1)(n-k+1)C_n^{k-1}\rho^{k-1}(1-\rho)^{n-k}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{\partial}{\partial\rho}\lambda(\beta,\rho) &= -(2\beta-1)(n-k+1)C_n^{k-1}\rho^{k-1}(1-\rho)^{n-k}\\
&\quad+\frac{1-\beta}{k}\sum_{s=0}^{k-2}(s+1)C_n^{s+1}\rho^s(1-\rho)^{n-s-1}-\frac{1-\beta}{k}(k-1)(n-k+1)C_n^{k-1}\rho^{k-1}(1-\rho)^{n-k}\\
&=-\left(2\beta-1+\frac{1-\beta}{k}(k-1)\right)(n-k+1)C_n^{k-1}\rho^{k-1}(1-\rho)^{n-k}+\frac{1-\beta}{k}\sum_{s=0}^{k-2}(s+1)C_n^{s+1}\rho^s(1-\rho)^{n-s-1}
\end{aligned}
$$

By dividing both sides by $\rho^{k-1}(1-\rho)^{n-k}$, we know that $\frac{\partial}{\partial\rho}\lambda(\rho)=0$ is equivalent to

$$\frac{1-\beta}{k}\sum_{s=0}^{k-2}(s+1)C_n^{s+1}\left(\frac{\rho}{1-\rho}\right)^{s-k+1}=\left(2\beta-1+\frac{1-\beta}{k}(k-1)\right)(n-k+1)C_n^{k-1}. \tag{56}$$

Clearly, when $\beta=1$, (56) does not hold, which implies that $\lambda(\beta,\rho)$ is either non-decreasing, or non-increasing, over ρ. When $\beta\in[0,1)$, the function

$$\frac{1-\beta}{k}\sum_{s=0}^{k-2}(s+1)C_n^{s+1}t^{s-k+1}$$

is strictly decreasing in $t\in[0,1]$. Thus, the equation $\frac{\partial}{\partial\rho}\lambda(\rho)=0$ can have at most one solution is $[0,1]$, which implies that $\lambda(\beta,\rho)$ is unimodal in $[0,1]$. □

Proof of Theorem 7. Denote by $\hat{\mathbf{G}}$ the distributions such that the type distribution of each agent i is a uniform distribution over $[0,1]$. Then, from definitions, we have that

$$\text{iidLP}_{k,n}^{\mathsf{DP/Proph}} = \text{innerLP}_{k,n}^{\mathsf{DP/Proph}}(\hat{\mathbf{G}}) \geq \inf_{\mathbf{G}:G_{1j}=\cdots=G_{nj}\forall j} \text{innerLP}_{k,n}^{\mathsf{DP/Proph}}(\mathbf{G})$$

Moreover, denoting by $\{\theta^*, \mathbf{x}^*, \mathbf{y}^*\}$ the optimal solution of $\text{iidLP}_{k,n}^{\mathsf{DP/Proph}}$, it is easy to see that $\{\theta^*, \mathbf{x}^*, \mathbf{y}^*\}$ is a feasible solution to $\text{innerLP}_{k,n}^{\mathsf{DP/Proph}}(\mathbf{G})$ for any $\mathbf{G}$ satisfying $G_{1j} = \cdots = G_{nj}$ for all j. Thus, we know that

$$\text{iidLP}_{k,n}^{\mathsf{DP/Proph}} \leq \inf_{\mathbf{G}:G_{1j}=\cdots=G_{nj}\forall j} \text{innerLP}_{k,n}^{\mathsf{DP/Proph}}(\mathbf{G})$$

which implies that

$$\text{iidLP}_{k,n}^{\mathsf{DP/Proph}} = \inf_{\mathbf{G}:G_{1j}=\cdots=G_{nj}\forall j} \text{innerLP}_{k,n}^{\mathsf{DP/Proph}}(\mathbf{G})$$

Applying the same arguments to $\text{iidLP}_{k,n}^{\mathsf{DP/ExAnte}}$, $\sup_\tau \text{innerLP}_{k,n}^{\mathsf{OST}(\tau)\mathsf{/Proph}}$, $\sup_\tau \text{innerLP}_{k,n}^{\mathsf{OST}(\tau)\mathsf{/ExAnte}}$, $\text{iidLP}_{k,n}^{\mathsf{ST/Proph}}$ and $\text{iidLP}_{k,n}^{\mathsf{ST/ExAnte}}$, we complete our proof. □

Proof of Theorem 8. For any $(\mathbf{x}, \mathbf{y}) \in \mathcal{P}_n^k$, the objective value of $\text{iidLP}_{k,n}^{\mathsf{DP/ExAnte}}$ will equal

$$\begin{aligned}
\theta &= \inf_{q\in(0,1]} \frac{\sum_{i=1}^n \sum_{l=1}^k \min\{y_i^l, qx_i^l\}}{\min\{nq, k\}} \\
&= \min\left\{ \inf_{q\in(0,k/n]} \frac{\sum_{i=1}^n \sum_{l=1}^k \min\{\frac{y_i^l}{q}, x_i^l\}}{n}, \inf_{q\in(k/n,1]} \frac{\sum_{i=1}^n \sum_{l=1}^k \min\{y_i^l, qx_i^l\}}{k} \right\} \\
&= \frac{\sum_{i=1}^n \sum_{l=1}^k \min\{y_i^l, \frac{k}{n}x_i^l\}}{k}
\end{aligned} \tag{57}$$

where we note that $k/n < 1$ since we are assuming $n > k$. Thus, the problem of $\text{iidLP}_{k,n}^{\mathsf{DP/ExAnte}}$ is equivalent to maximizing expression (57) subject to $(\mathbf{x}, \mathbf{y}) \in \mathcal{P}_n^k$. From this it is easy to see that the optimal solution involves setting $y_i^l = \frac{k}{n}x_i^l$ for all l and i, equivalent to a static threshold policy with $\tau = k/n$. Therefore, we have

$$\text{iidLP}_{k,n}^{\mathsf{DP/ExAnte}} = \frac{\frac{k}{n}\sum_i \sum_l x_i^l}{k} = \frac{\mathbb{E}[\min\{\text{Bin}(n, k/n), k\}]}{k} \tag{58}$$

where the final equality follows from (33).

We now show that the same expression results from analyzing $\sup_\tau \text{iidLP}_{k,n}^{\mathsf{OST}(\tau)\mathsf{/Proph}}$ or $\sup_\tau \text{iidLP}_{k,n}^{\mathsf{OST}(\tau)\mathsf{/ExAnte}}$. For the first one, by (32), the objective value of $\text{iidLP}_{k,n}^{\mathsf{OST}(\tau)\mathsf{/Proph}}$ for any OST τ will equal

$$\begin{aligned}
\theta &= \inf_{q\in(0,1]} \frac{\min\{\tau, q\}\sum_{i=1}^n \sum_{l=1}^k x_i^l}{\mathbb{E}[\min\{\text{Bin}(n,q), k\}]} \\
&= \min\left\{ \inf_{q\in(0,\tau]} \frac{\sum_{i=1}^n \sum_{l=1}^k x_i^l}{\frac{\mathbb{E}[\min\{\text{Bin}(n,q),k\}]}{q}}, \inf_{q\in(\tau,1]} \frac{\tau\sum_{i=1}^n \sum_{l=1}^k x_i^l}{\mathbb{E}[\min\{\text{Bin}(n,q), k\}]} \right\} \\
&= \min\left\{ \frac{\sum_{i=1}^n \sum_{l=1}^k x_i^l}{n}, \frac{\tau\sum_{i=1}^n \sum_{l=1}^k x_i^l}{k} \right\} && \text{(59a)} \\
&= \frac{\mathbb{E}[\min\{\text{Bin}(n,\tau), k\}]}{\max\{\tau n, k\}} && \text{(59b)}
\end{aligned}$$

where the first argument in (59a) results because $\frac{\mathbb{E}[\min\{\text{Bin}(n,q),k\}]}{q}$ is maximized over $q \in (0, \tau]$ as q approaches 0 from the positive side, and the final equality applies (33) to both arguments after multiplying and dividing the first argument by τ. From this it is easy to see that the supremum of expression (59b) over τ is achieved by setting $\tau = k/n$, since ratio $\frac{\mathbb{E}[\min\{\text{Bin}(n,\tau),k\}]}{\tau}$ is decreasing over $\tau \in [k/n, 1]$. To see this, by noting that $\mathbb{E}[\min\{\text{Bin}(n,0),k\}] = 0$, it is enough to show that $\mathbb{E}[\min\{\text{Bin}(n,\tau),k\}$ is concave over $\tau \in [0,1]$. We define the function $h(x) = \min\{x,k\}$. Then, we have

$$\begin{aligned}
\frac{\partial}{\partial \tau}\mathbb{E}[\min\{\text{Bin}(n,\tau),k\} &= \sum_{j=0}^{k-1}(k-j)\cdot C_n^j \cdot (j\tau^{j-1}(1-\tau)^{n-j} - (n-j)\tau^j(1-\tau)^{n-j}) \\
&= n\cdot\sum_{j=0}^{k-2}(k-1-j)\cdot C_{n-1}^j\cdot\tau^j(1-\tau)^{n-1-j} - n\cdot\sum_{j=0}^{k-1}(k-j)\cdot C_{n-1}^j\cdot\tau^j(1-\tau)^{n-1-j} \\
&= n\cdot\mathbb{E}_{X\sim\text{Bin}(n-1,\tau)}[h(X+1)-h(X)]
\end{aligned}$$

which implies that

$$\frac{\partial^2}{\partial \tau^2}\mathbb{E}[\min\{\text{Bin}(n,\tau),k\} = n(n-1)\cdot\mathbb{E}_{X\sim\text{Bin}(n-2,\tau)}[h(X+2)+h(X)-2h(X+1)] \le 0$$

by noting that $h(X+2)+h(X)-2h(X+1) \le 0$ for any X. Thus, $\mathbb{E}[\min\{\text{Bin}(n,\tau),k\}$ is concave over $\tau \in [0,1]$ and $\frac{\mathbb{E}[\min\{\text{Bin}(n,\tau),k\}]}{\tau}$ is decreasing over $\tau \in [k/n,1]$.

Therefore, we have shown that $\sup_\tau \text{iidLP}_{k,n}^{\mathsf{OST}(\tau)/\mathsf{Proph}}$ is identical to (58).

Finally, the same argument can be made to show that $\sup_\tau \text{iidLP}_{k,n}^{\mathsf{OST}(\tau)/\mathsf{ExAnte}}$ is equal to (58), since the same expression (59b) can be derived for $\text{iidLP}_{k,n}^{\mathsf{OST}(\tau)/\mathsf{ExAnte}}$. This completes the proof of Theorem 8. □

Proof of Theorem 9. Let τ be the unique value in $(\underline{\tau}, \overline{\tau})$ at which

$$\mathbb{E}[\min\{\text{Bin}(n,\tau),k\}] = \frac{\mathbb{E}[\min\{\text{Bin}(n,\underline{\tau}),k\}] + \mathbb{E}[\min\{\text{Bin}(n,\overline{\tau}),k\}]}{2}. \tag{60}$$

We show that for this value of τ, (34) holds for all choices of q.

First, if $q > \overline{\tau}$, then all agents accepted by any static threshold τ, $\underline{\tau}$, or $\overline{\tau}$ will have type q or better. In this case, using the fact that $\min\{y_i^l(\tau), qx_i^l(\tau)\} = y_i^l(\tau) = \tau x_i^l(\tau)$ (due to (31c), and an analogous argument can be made for $\underline{\tau}, \overline{\tau}$) and applying the second identity in (33) with $i = n$, we see that (34) is equivalent to $\mathbb{E}[\min\{\text{Bin}(n,\tau),k\}] \ge \frac{1}{2}(\mathbb{E}[\min\{\text{Bin}(n,\underline{\tau}),k\}] + \mathbb{E}[\min\{\text{Bin}(n,\overline{\tau}),k\}])$. In other words, we need to check that τ accepts no fewer agents than the average of $\underline{\tau}$ and $\overline{\tau}$, which in fact holds as equality due to (60).

Next, if $q \in (\tau, \overline{\tau}]$, then all agents accepted by static thresholds $\tau, \underline{\tau}$ still have type q or better, while some of the agents accepted by $\overline{\tau}$ may not. Since τ accepts no fewer agents than the average of $\underline{\tau}$ and $\overline{\tau}$, (34) must still be true.

The third case we consider is $q < \underline{\tau}$. Agents of types q or better are accepted by all policies, so the probability of accepting any agent i who has type q or better is simply q times the probability of having a slot available at that time. Formally, using the fact that $\min\{y_i^l(\tau), qx_i^l(\tau)\} = qx_i^l(\tau)$ (due to (31c), and an analogous argument can be made for $\underline{\tau}, \overline{\tau}$) and applying the first identity in (33), we see that (34) is equivalent to

$$q\sum_{i=1}^{n}\Pr[\text{Bin}(i-1,\tau)<k] \ge \frac{q}{2}\left(\sum_{i=1}^{n}\Pr[\text{Bin}(i-1,\underline{\tau})<k] + \sum_{i=1}^{n}\Pr[\text{Bin}(i-1,\overline{\tau})<k]\right). \tag{61}$$

In words, after canceling out the q's, we need to prove that the total probability of having a slot available is higher for τ, as compared to the average of $\underline{\tau}$ and $\overline{\tau}$.

Before proving (61), we show that the final case $q \in (\underline{\tau}, \tau]$ reduces to it. Indeed, in the final case static threshold $\underline{\tau}$ may not accept some agents who have type q or better, which only decreases the RHS of (61). Therefore, establishing (61) would complete the proof.

To establish (61), we observe that (60), which holds by construction, is equivalent to

$$\tau \sum_{i=1}^{n} \Pr[\mathrm{Bin}(i-1,\tau) < k] = \frac{1}{2}\left(\underline{\tau} \sum_{i=1}^{n} \Pr[\mathrm{Bin}(i-1,\underline{\tau}) < k] + \overline{\tau} \sum_{i=1}^{n} \Pr[\mathrm{Bin}(i-1,\overline{\tau}) < k]\right). \tag{62}$$

Recalling that $\underline{\tau} < \overline{\tau}$, we now argue two facts. First, we immediately see that $\Pr[\mathrm{Bin}(i-1,\underline{\tau}) < k] \geq \Pr[\mathrm{Bin}(i-1,\overline{\tau}) < k]$ for all i. Second, we argue that $\tau \leq (\underline{\tau} + \overline{\tau})/2$. This is because the function $\mathbb{E}[\min\{\mathrm{Bin}(n,\tau), k\}]$ is strictly increasing and concave in τ—so if $\tau > (\underline{\tau} + \overline{\tau})/2$, then Jensen's inequality would say $\mathbb{E}[\min\{\mathrm{Bin}(n,\tau), k\}] > \mathbb{E}[\min\{\mathrm{Bin}(n, \frac{\underline{\tau}+\overline{\tau}}{2}), k\}] \geq \frac{1}{2}(\mathbb{E}[\min\{\mathrm{Bin}(n,\underline{\tau}), k\}] + \mathbb{E}[\min\{\mathrm{Bin}(n,\overline{\tau}), k\}])$, which contradicts (60). From these facts we derive

$$\begin{aligned}
\frac{\underline{\tau}+\overline{\tau}}{2} \sum_{i=1}^{n} \Pr[\mathrm{Bin}(i-1,\tau) < k] &\geq \tau \sum_{i=1}^{n} \Pr[\mathrm{Bin}(i-1,\tau) < k] \\
&= \frac{1}{4}\left(2\underline{\tau} \sum_{i=1}^{n} \Pr[\mathrm{Bin}(i-1,\underline{\tau}) < k] + 2\overline{\tau} \sum_{i=1}^{n} \Pr[\mathrm{Bin}(i-1,\overline{\tau}) < k]\right) \\
&\geq \frac{1}{4}(\underline{\tau}+\overline{\tau})\left(\sum_{i=1}^{n} \Pr[\mathrm{Bin}(i-1,\underline{\tau}) < k] + \sum_{i=1}^{n} \Pr[\mathrm{Bin}(i-1,\overline{\tau}) < k]\right)
\end{aligned}$$

where the equality holds by (62), and the inequality holds by applying the rearrangement inequality with $\underline{\tau} < \overline{\tau}$, $\sum_{i=1}^{n} \Pr[\mathrm{Bin}(i-1,\underline{\tau}) < k] \geq \sum_{i=1}^{n} \Pr[\mathrm{Bin}(i-1,\overline{\tau}) < k]$ and then factoring. This implies (61) and completes the proof of Theorem 9. □

Proof of Lemma 7. We consider further relaxing LP (39). It is clear to see that the expression in large parentheses at the LHS of (39b),

$$(1-(1-\kappa)^n)\theta - \kappa \sum_{i=I}^{n-1} (1 - Y_i),$$

is a concave function over κ, which implies that the maximum is attained by setting the first derivative to 0. Letting

$$x_I := \sum_{i=I}^{n-1} (1 - Y_i) \qquad z_I := \frac{x_I}{n\theta} \qquad\qquad \forall I = 0, \ldots, n-1$$

we have that the maximum for constraint (39b) for $I = 0, \ldots, n-1$ is attained when

$$(1-\kappa)^{n-1} = z_I \qquad \kappa = 1 - z_I^{1/(n-1)}.$$

Note that $Y_I = 1 + x_{I+1} - x_I = 1 + n\theta(z_{I+1} - z_I)$ for all $I = 0, \ldots, n-1$, with $z_n = 0$. Then, constraint (39c) can be transformed into the following constraint over the z_I variables for $I = 0, \ldots, n-1$:

$$z_1 - z_0 \leq -\frac{1}{n\theta} \leq z_2 - z_1 \leq \cdots \leq z_n - z_{n-1} \leq 0.$$

Moreover, the constraints (39b) are satisfied if for all $I = 0, \ldots, n-1$,

$$(1 - z_I^{n/(n-1)})\theta - \left(1 - z_I^{1/(n-1)}\right) x_I \leq 1 + x_{I+1} - x_I,$$

which is equivalent to

$$1-\frac{1}{\theta}+(n-1)z_I^{n/(n-1)} \le nz_{I+1}.$$

As a result, LP (39) can again be characterized as the following optimization problem over z-variables:

$$\begin{aligned}
&\max\ \theta \\
&\text{s.t.}\quad (n-1)z_I^{n/(n-1)} \le nz_{I+1}+\frac{1}{\theta}-1 && \forall I=0,\dots,n-1 && \text{(63a)}\\
&\qquad\ z_n=0 \\
&\qquad\ z_1-z_0 \le -\frac{1}{n\theta} \le z_2-z_1 \le \cdots \le z_n-z_{n-1} \le 0 && && \text{(63b)}
\end{aligned}$$

Finally, combining the constraint $z_1-z_0\le -\frac{1}{n\theta}$ from (63b) and the constraint (63a) with $I=0$, we get

$$(n-1)z_0^{n/(n-1)} \le nz_1+\frac{1}{\theta}-1 \le nz_0-1 \qquad \Longrightarrow \qquad (n-1)z_0^{n/(n-1)}-nz_0 \le -1.$$

Note that expression $(n-1)z_0^{n/(n-1)}-nz_0$ as a function over $z_0\ge 0$ is always at least -1, with equality achieved only when $z_0=1$. Therefore, we can add the constraint $z_0=1$ to $\mathsf{LP}_n^{\mathsf{mon}}$ without changing the objective value. Together with $z_0=1$ and $z_n=0$, the constraint (63b) can be further relaxed into $z_i\in[0,1]$ for $i=1,\dots,n-1$, which establishes $\mathsf{LP}_n^{\mathsf{relax}}$ as a relaxation of LP (63). Thus, our proof is completed. □

Proof of Lemma 8. Denote by $\{\theta, z_I\}_{I=0}^n$ as an optimal solution of $\mathsf{LP}_n^{\mathsf{relax}}$ and denote by I_1 the smallest index such that

$$z_{I_1+1} > \max\{\frac{n-1}{n}z_{I_1}^{n/(n-1)}-\frac{1}{n\theta}+\frac{1}{n},0\}$$

Then, we construct another solution $\{\hat\theta, \hat z_I\}_{I=0}^n$ such that

$$\hat\theta=\theta,\quad \hat z_{I_1+1}=\max\{\frac{n-1}{n}z_{I_1}^{n/(n-1)}-\frac{1}{n\theta}+\frac{1}{n},0\},\quad \hat z_I=z_I \text{ for } I=0,\dots,I_1,I_1+2,\dots,n$$

Clearly, since $\hat z_{I_1+1}\le z_{I_1+1}$, we must have $\{\hat\theta,\hat z_I\}_{I=0}^n$ as an optimal solution to $\mathsf{LP}_n^{\mathsf{relax}}$. Every time we repeat the above construction procedure, the value of I_1 will be increased by at least 1. Thus, after a finite number of steps, we obtain an optimal solution $\{\hat\theta,\hat z_I\}_{I=0}^n$ such that

$$\hat z_{I+1}=\max\{\frac{n-1}{n}\hat z_I^{n/(n-1)}-\frac{1}{n\hat\theta}+\frac{1}{n},0\},\quad \forall I=0,\dots,n-1 \tag{64}$$

We regard $\hat z_I(\theta)$ as a function of θ, for each $I=0,\dots,n-1$, where $\hat z_I(\theta)$ is computed iteratively from (64). Note that the RHS of (64) is non-decreasing over θ. We must have $\hat z_I(\theta)$ is a non-decreasing function over θ. If there exists an $I_2\le n-1$ such that $\frac{n-1}{n}\hat z_{I_2}(\theta)^{n/(n-1)}-\frac{1}{n\theta}+\frac{1}{n}<0$, we can always increase the value of θ by $\varepsilon>0$ such that it still holds $\frac{n-1}{n}\hat z_{I_2}(\theta+\varepsilon)^{n/(n-1)}-\frac{1}{n(\theta+\varepsilon)}+\frac{1}{n}<0$, and $\{\theta+\varepsilon,\hat z_I(\theta+\varepsilon)\}$ is still feasible to $\mathsf{LP}_n^{\mathsf{relax}}$. Thus, in order for $\hat\theta$ to be optimal, we must have $\hat z_{I+1}=\frac{n-1}{n}\hat z_I^{n/(n-1)}-\frac{1}{n\hat\theta}+\frac{1}{n}$ for all $I=0,\dots,n-1$, which completes our proof. □

Proof of Lemma 9. Denote $\{\theta, z_I\}_{I=0}^n$ as the solution constructed in Lemma 8. Then, we denote

$$Y_I = 1 + n\theta(z_{I+1} - z_I) \text{ for } I = 0, 1, \ldots, n-1.$$

Clearly, it holds that

$$\max_{\kappa\in(0,1]} \left((1-(1-\kappa)^n)\theta - \kappa \sum_{i=I}^{n-1} (1-Y_i) - Y_I \right) = 0 \text{ for } I = 0, 1, \ldots, n-1.$$

We also denote $y_i = Y_i - Y_{i-1}$ for $i = 1, \ldots, n-1$. Note that $\sum_{i=1}^{n-1} y_i = Y_{n-1} \leq 1$, we also denote $y_n = 1 - \sum_{i=1}^{n-1} y_i$. Then, we have

$$\max_{\kappa\in(0,1]} \left((1-(1-\kappa)^n)\theta - \kappa \sum_{i=I+1}^{n} (1 - \sum_{i'=1}^{i-1} y_{i'}) - \sum_{i=1}^{I} y_i \right) = 0 \text{ for } I = 0, 1, \ldots, n-1. \tag{65}$$

In order to show that $\{\theta, y_i\}_{i=1}^n$ is a feasible solution to $\text{iidLP}_{1,n}^{\mathsf{DP/Proph}}$, it suffices to show that

$$(1-(1-\kappa)^n)\theta \leq \sum_{i\in S} y_i + \kappa \cdot \sum_{i\in[n]\setminus S} (1 - \sum_{i'=1}^{i-1} y_{i'}), \text{ for any } \kappa \in [0,1] \text{ and } S \subset [n] \tag{66}$$

where the constraint $y_i \geq 0$ also follows from (66) by setting $\mathcal{S} = \{j\}$ and $\kappa = 0$.

We now proceed to prove (66). For any fixed S, clearly, the left hand side of constraint (66) is a concave function over κ. Thus, after we maximize over κ in (66), we get that $\{\theta, y_i\}_{i=1}^n$ is a feasible solution to $\text{iidLP}_{1,n}^{\mathsf{DP/Proph}}$ if

$$f(S) \leq 0, \quad \forall S \subset [n] \tag{67}$$

where $f(S)$ is a set function defined for any subset $S \subset [n]$ as follows

$$f(S) := 1 + (n-1) \cdot \left(\frac{\sum_{i\in[n]\setminus S} \alpha_i}{n\theta} \right)^{\frac{n}{n-1}} - \frac{\sum_{i\in[n]\setminus S} \alpha_i}{\theta} - \frac{\sum_{i\in S} y_i}{\theta}, \text{ for any } S \subset [n] \tag{68}$$

and we denote $1 - \sum_{i'=1}^{i-1} y_{i'}$ by α_i for notation brevity. Note that from (65) by setting $\kappa = 0$, we have $\sum_{i=1}^{I} y_i \geq 0$ for each $I = 0, 1, \ldots, n-1$. Also, we have

$$\sum_{i=1}^{I} y_i = Y_I - Y_0 = 1 + n\theta(z_{I+1} - z_I) \leq 1, \ \forall I = 0, 1, \ldots, n-1$$

by noting that z_I is a decreasing sequence in I. Thus, we claim that $\alpha_i \in [0,1]$ for each $i \in [n]$.

The condition (65) can also be expressed via the function $f(\cdot)$. We denote by $E_i = \{1, 2, \ldots, i\}$ for each $i \in [n]$. Note that the left hand side of (65) is a concave function over κ. Then after maximizing over κ in (65), we have that

$$f(E_i) = 0 \text{ for } i = 1, \ldots, n-1 \text{ and } f(E_n) \leq 0 \tag{69}$$

We now use the condition (69) to prove (67). A key step is to show the set function $f(\cdot)$ to be supermodular. This allows us to ultimately show that it is maximized when S takes the form of an interval $\{1, \ldots, I\}$, for which we already knew by the construction in (69).

Note that we have

$$f(S) = g(\sum_{i\in S} \alpha_i) - \frac{\sum_{i\in[n]\setminus S} \alpha_i}{\theta} - \frac{\sum_{i\in S} y_i}{\theta}$$

where

$$g(x) = 1 + (n-1) \cdot \left(\frac{\sum_{i \in [n]} \alpha_i - x}{n\theta} \right)^{\frac{n}{n-1}}$$

It is clear to see that $g(x)$ is a convex function over x. Then, it is well-known (e.g. Lemma 2.6.2 in Topkis (2011)) that $f(S)$ is a supermodular function.

For any set $S \subset \mathcal{N}$, we assume without loss of generality that the elements in S are sorted in an increasing order. We denote $S(i)$ as the i-th element of S and denote by $\sigma(S)$ the number of i such that $S(i+1) > S(i)+1$. Then, for any k, we denote

$$T_k = \{S : \sigma(S) \leq k\}$$

Clearly, we have $T_0 = \{E_1, \ldots, E_n\}$, which implies that $\max_{S \in T_0} f(S) \leq 0$. Now we will prove (67) by induction. Suppose that there exists an integer l such that

$$\max_{S \in T_l} f(S) \leq 0$$

For any set $\hat{S} \subset [n]$ such that $\sigma(\hat{S}) = l+1$, we denote $\hat{i}$ as the largest index such that $\hat{S}(\hat{i}+1) > \hat{S}(\hat{i}) + 1$, and we denote $\hat{S}(\hat{j})$ as the last element of the set $\hat{S}$. We denote $\hat{T} = E_{\hat{S}(i)}$. Clearly, it holds that

$$\hat{T} \cup \hat{S} = E_{\hat{S}(\hat{j})} \text{ and } \sigma(\hat{T} \cap \hat{S}) = l$$

From the supermodularity of $f(S)$, we have

$$f(\hat{T}) + f(\hat{S}) \leq f(\hat{T} \cup \hat{S}) + f(\hat{T} \cap \hat{S})$$

From the induction hypothesis, we know $f(\hat{T} \cap \hat{S}) \leq 0$. Also, from (69), we know $f(\hat{T} \cup \hat{S}) = f(E_{\hat{S}(\hat{j})}) \leq 0$ and $f(\hat{T}) = f(E_{\hat{S}(\hat{i})}) = 0$ since $\hat{S}(\hat{i}) \leq n-1$. Thus, we concludes that $f(\hat{S}) = 0$, which implies that

$$\max_{S \subset T_{l+1}} f(S) \leq 0$$

From the induction, we know that

$$\max_{S \subset T_n} f(S) = \max_{S \subset [n]} f(S) \leq 0$$

which completes our proof. □

Proof of Lemma 10. We denote $\{\theta, z_I\}_{I=0}^{2n}$ as an optimal solution to $\mathsf{LP}_{2n}^{\mathsf{relax}}$. We construct a feasible solution to $\mathsf{LP}_n^{\mathsf{relax}}$, denoted by $\{\hat{\theta}, \hat{z}_I\}_{I=0}^{n}$, To be specific, we set

$$\hat{\theta} = \theta, \quad \hat{z}_I = z_{2I} \text{ for } I = 0, 1, \ldots, n$$

It only remains to show that

$$\hat{z}_{I+1} = z_{2I+2} \geq \frac{n-1}{n} z_{2I}^{n/(n-1)} - \frac{1}{n\theta} + \frac{1}{n}, \quad \forall I = 0, \ldots, n-1$$

Note that we have

$$z_{2I+2} \geq \frac{(2n-1)\left[\frac{(2n-1) z_{2I}^{\frac{2n}{2n-1}} - \frac{1}{\theta} + 1}{2n}\right]^{\frac{2n}{2n-1}} - \frac{1}{\theta} + 1}{2n}$$

Denote by $\alpha = \frac{1}{\theta} - 1$. It only remains to show that

$$f(\alpha) = (2n-1)\left[\frac{(2n-1)z_{2I}^{\frac{2n}{2n-1}} - \alpha}{2n}\right]^{\frac{2n}{2n-1}} - 2(n-1)z_{2I}^{\frac{n}{n-1}} + \alpha \geq 0$$

Note that

$$f'(\alpha) = 1 - \left[\frac{(2n-1)z_{2I}^{\frac{2n}{2n-1}} - \alpha}{2n}\right]^{\frac{1}{2n-1}}$$

Further note that

$$1 \geq z_{2I+1} \geq \frac{(2n-1)z_{2I}^{\frac{2n}{2n-1}} - \alpha}{2n}$$

It holds that $f'(\alpha) \geq 0$. Thus, in order to show that $f(\alpha) \geq 0$, it suffices to show that $f(0) \geq 0$, i.e.,

$$\left[\frac{(2n-1)z_{2I}^{\frac{2n}{2n-1}}}{2n}\right]^{\frac{2n}{2n-1}} \geq \frac{2n-2}{2n-1} \cdot z_{2I}^{\frac{n}{n-1}}$$

which is equivalent to showing

$$(\frac{2n-1}{2n})^{\frac{2n}{2n-1}} \geq \frac{2n-2}{2n-1} \cdot z_{2I}^{\overline{\frac{n}{(n-1)(2n-1)^2}}}$$

Thus, it only remains to show that

$$(1 - \frac{1}{2n})^{2n} \geq (1 - \frac{1}{2n-1})^{2n-1} \cdot z_{2I}^{\overline{\frac{n}{(n-1)(2n-1)}}}$$

The above inequality holds by noting that $z_{2I} \in [0,1]$ and $(1-\frac{1}{2n})^{2n} \geq (1-\frac{1}{2n-1})^{2n-1}$, which completes our proof. $\square$

Proof of Theorem 10. We denote function

$$f(x) = x(\ln x - 1) \quad \text{and} \quad f_n(x) = (n-1)\cdot x^{n/(n-1)} - nx \text{ for each } n$$

Then, for each n, we define a sequence of values $\{H_{n,i}\}_{i=0}^{n}$ such that

$$H_{n,0} = 1, \; H_{n,i} = H_{n,i-1} + \frac{1}{n}\cdot(f_n(H_{n,i-1}) - \frac{1}{\theta_n} + 1) \text{ for } i = 1, \ldots, n$$

where θ_n is selected such that $H_{n,n} = 0$. Clearly, from Lemma 8, we know that $\{\theta_n, H_{n,i}\}_{i=0}^{n}$ is an optimal solution to $\mathsf{LP}_n^{\mathsf{relax}}$. Then, from Lemma 9, we know that $\theta_n = \mathrm{iidLP}_{1,n}^{\mathsf{DP/Proph}}$. Moreover, Lemma 10 implies that $\inf_n \theta_n = \liminf_{n\to\infty}\theta_n$. Thus, it only remains to show that

$$\lim_{n\to\infty} \theta_n = \theta^*$$

The remaining part mainly follows the proof of Lemma 6.2 in Kertz (1986). Here, we include the whole proof for completeness. For any $i = 1, \ldots, n$, it holds that

$$H_{n,i} - H_{n,i-1} = \frac{1}{n}\cdot(f(H_{n,i-1}) - \frac{1}{\theta^*} + 1) + \frac{1}{n}\cdot(f_n(H_{n,i-1}) - f(H_{n,i-1})) + \frac{1}{n}\cdot(\frac{1}{\theta^*} - \frac{1}{\theta_n})$$

which implies that

$$\frac{1}{n}\cdot\frac{\frac{1}{\theta^*} - \frac{1}{\theta_n}}{f(H_{n,i-1}) - \frac{1}{\theta^*} + 1} = \frac{H_{n,i-1} - H_{n,i}}{\frac{1}{\theta^*} - 1 - f(H_{n,i-1})} - \frac{1}{n} - \frac{1}{n}\cdot\frac{f_n(H_{n,i-1}) - f(H_{n,i-1})}{f(H_{n,i-1}) - \frac{1}{\theta^*} + 1}$$

Note that $f(H_{n,i-1}) - \frac{1}{\theta^*} + 1 \in [-\frac{1}{\theta^*}, 1 - \frac{1}{\theta^*}]$.

Case I. If $\theta^* \le \theta_n$, we have

$$\begin{aligned}
\frac{\theta^*}{n} \cdot |\frac{1}{\theta^*} - \frac{1}{\theta_n}| &\le -\frac{1}{n} \cdot \frac{\frac{1}{\theta^*} - \frac{1}{\theta_n}}{f(H_{n,i-1}) - \frac{1}{\theta^*} + 1} = -\frac{H_{n,i-1} - H_{n,i}}{\frac{1}{\theta^*} - 1 - f(H_{n,i-1})} + \frac{1}{n} + \frac{1}{n} \cdot \frac{f_n(H_{n,i-1}) - f(H_{n,i-1})}{f(H_{n,i-1}) - \frac{1}{\theta^*} + 1} \\
&\le -\frac{H_{n,i-1} - H_{n,i}}{\frac{1}{\theta^*} - 1 - f(H_{n,i-1})} + \frac{1}{n} + \left| \frac{1}{n} \cdot \frac{f_n(H_{n,i-1}) - f(H_{n,i-1})}{f(H_{n,i-1}) - \frac{1}{\theta^*} + 1} \right| \\
&\le -\frac{H_{n,i-1} - H_{n,i}}{\frac{1}{\theta^*} - 1 - f(H_{n,i-1})} + \frac{1}{n} + \frac{\frac{1}{\theta^*} - 1}{n} \cdot \|f_n - f\|_\infty
\end{aligned} \tag{70}$$

Sum over both sides of (70) for $i = 1, \dots, n$, we have

$$\theta^* \cdot |\frac{1}{\theta^*} - \frac{1}{\theta_n}| \le -\sum_{i=1}^n \frac{H_{n,i-1} - H_{n,i}}{\frac{1}{\theta^*} - 1 - f(H_{n,i-1})} + 1 + (\frac{1}{\theta^*} - 1) \cdot \|f_n - f\|_\infty$$

Case II. If $\theta^* > \theta_n$, we have

$$\begin{aligned}
\frac{\theta^*}{n} \cdot |\frac{1}{\theta^*} - \frac{1}{\theta_n}| &\le \frac{1}{n} \cdot \frac{\frac{1}{\theta^*} - \frac{1}{\theta_n}}{f(H_{n,i-1}) - \frac{1}{\theta^*} + 1} = \frac{H_{n,i-1} - H_{n,i}}{\frac{1}{\theta^*} - 1 - f(H_{n,i-1})} - \frac{1}{n} - \frac{1}{n} \cdot \frac{f_n(H_{n,i-1}) - f(H_{n,i-1})}{f(H_{n,i-1}) - \frac{1}{\theta^*} + 1} \\
&\le \frac{H_{n,i-1} - H_{n,i}}{\frac{1}{\theta^*} - 1 - f(H_{n,i-1})} - \frac{1}{n} + \frac{1}{n\theta^*} \cdot \|f_n - f\|_\infty
\end{aligned}$$

Sum over both sides for $i = 1, \dots, n$, we have

$$\theta^* \cdot |\frac{1}{\theta^*} - \frac{1}{\theta_n}| \le \sum_{i=1}^n \frac{H_{n,i-1} - H_{n,i}}{\frac{1}{\theta^*} - 1 - f(H_{n,i-1})} - 1 + \frac{1}{\theta^*} \cdot \|f_n - f\|_\infty$$

Thus, on both cases, we have that

$$\theta^* \cdot |\frac{1}{\theta^*} - \frac{1}{\theta_n}| \le \left| 1 - \sum_{i=1}^n \frac{H_{n,i-1} - H_{n,i}}{\frac{1}{\theta^*} - 1 - f(H_{n,i-1})} \right| + \frac{1}{\theta^*} \cdot \|f_n - f\|_\infty$$

Further note that we have $|H_{n,i-1} - H_{n,i}| \le \frac{1}{n\theta_n}$, which implies

$$\lim_{n \to \infty} \sum_{i=1}^n \frac{H_{n,i-1} - H_{n,i}}{\frac{1}{\theta^*} - 1 - f(H_{n,i-1})} = \int_{h=0}^1 \frac{dh}{\frac{1}{\theta^*} - 1 - f(h)} = 1.$$

Also, note that $\|f_n - f\|_\infty \le \frac{1}{en}$, then we must have

$$\lim_{n \to \infty} \theta_n = \theta^*$$

which completes our proof. $\square$