

HANDOUT 10

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1. **Linear transformations:** A map between vector spaces is a linear transformation if it preserves addition and scalar multiplication.

Exercise: Determine if the following maps are linear transformations.

- (1) The map $\mathbb{R}^m \rightarrow \mathbb{R}^n$ given by left multiplication of a matrix, that is, $T : x \mapsto Ax$.
- (2) $D = d/dx$ for real polynomials in x . (Note: The set of all real polynomials in variable x forms a vector space)
- (3) $D = d/dx$ for $\{(ax^2 + bx + c)e^x : a, b, c \in \mathbb{R}\}$ and $\{a \sin xe^x + b \cos xe^x : a, b, c \in \mathbb{R}\}$.

2. **Column space, Row space, Null spcae:**

- The pivot columns of a matrix A form a basis for $ColA$.
- The pivot rows of a matrix A form a basis for $RowA$.
- $\text{rank } A = \text{dimension of } ColA = \text{the number of pivot columns} = \text{the number of pivot rows} = \text{dimension of } RowA$.
- $\text{nullity } A = \text{dimension of } NulA$.
- Suppose that A is of size $m \times n$, then
 - * $\text{rank } A + \text{nullity } A = n$,
 - * $\text{rank } A + \text{nullity } A^\top = m$.

Exercise: Given a matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ -2 & 5 & -4 \\ 4 & 5 & -3 \end{bmatrix}$.

- (1) Find a basis for $ColA$.
- (2) Does the columns of A span \mathbb{R}^4 ? Does the columns of A span \mathbb{R}^3 ?
- (3) Find a basis for $RowA$.
- (4) Does the rows of A span \mathbb{R}^4 ? Does the rows of A span \mathbb{R}^3 ?
- (5) What is $\text{rank } A$?
- (6) Find a basis for $NulA$. What is $\text{nullity } A$?
- (7) Find a basis for $NulA^\top$. What is $\text{nullity } A^\top$?

3. **Coordinate system:**

Suppose that vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ and $\mathbf{x} \in V$. The coordinates of \mathbf{x} relative to the basis \mathcal{B} (the \mathcal{B} -coordinate of \mathbf{x}) are the weights c_1, c_2, \dots, c_n such that

$$\mathbf{x} = c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2 + \dots + c_n \mathbf{b}_n,$$

and

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Exercise: Vector space $V = \mathbb{P}_2$ has a basis $\mathcal{B} = \{1, x, x^2\}$.

- (1) What is the dimension of V ?
- (2) Find the \mathcal{B} -coordinate of $p_1 = x^2 + x + 1$, $p_2 = x^2 - x + 1$, $p_3 = x^2$ that is, $[p_1]_{\mathcal{B}}$, $[p_2]_{\mathcal{B}}$, $[p_3]_{\mathcal{B}}$, respectively.
- (3) Show that $\mathcal{C} = \{p_1, p_2, p_3\}$ is a basis for V .

(4) Find $p \in V$ such that $[p]_{\mathcal{C}} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$.

4. Change of basis:

Suppose \mathbb{R} has a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ and $\mathbf{x} \in \mathbb{R}^n$. Then

$$\mathbf{x} = [\mathbf{b}_1 \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_n][\mathbf{x}]_{\mathcal{B}} = B[\mathbf{x}]_{\mathcal{B}},$$

where $B = [\mathbf{b}_1 \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_n]$.

Suppose \mathbb{R} has an another basis $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}$ and $\mathbf{x} \in \mathbb{R}^n$. Then

$$\mathbf{x} = [\mathbf{c}_1 \ \mathbf{c}_2 \ \cdots \ \mathbf{c}_n][\mathbf{x}]_{\mathcal{C}}$$

In particular,

$$\mathbf{c}_1 = B[\mathbf{c}_1]_{\mathcal{B}}, \quad \mathbf{c}_2 = B[\mathbf{c}_2]_{\mathcal{B}}, \quad \cdots, \quad \mathbf{c}_n = B[\mathbf{c}_n]_{\mathcal{B}},$$

Hence

$$\mathbf{x} = [\mathbf{c}_1 \ \mathbf{c}_2 \ \cdots \ \mathbf{c}_n][\mathbf{x}]_{\mathcal{C}} = B \begin{bmatrix} [\mathbf{c}_1]_{\mathcal{B}} & [\mathbf{c}_2]_{\mathcal{B}} & \cdots & [\mathbf{c}_n]_{\mathcal{B}} \end{bmatrix} [\mathbf{x}]_{\mathcal{C}}$$

Therefore,

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} [\mathbf{c}_1]_{\mathcal{B}} & [\mathbf{c}_2]_{\mathcal{B}} & \cdots & [\mathbf{c}_n]_{\mathcal{B}} \end{bmatrix} [\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{B} \leftarrow \mathcal{C}} [\mathbf{x}]_{\mathcal{C}}$$

Exercise 4.1: Find the change of basis matrix $P_{\mathcal{B} \leftarrow \mathcal{C}}$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{C} = \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} \right\},$$

Exercise 4.2: Let $\mathcal{B} = \{b_1, b_2\}$ and $\mathcal{C} = \{c_1, c_2\}$ be basis of \mathbb{R}^2 such that $b_1 = 2c_1 + c_2, b_2 = c_1 + 2c_2$. Determine the \mathcal{C} -coordinate of $v = 2b_1 + 3b_2$.