I. Determine $p \in \mathbb{P}_3$ such that $s(0) = 0$.

We know $p(0) = 0$, $p(1) = s(1) = 1$, $p'(1) = s'(1) = 3$, $p''(1) = s''(1) = 6$, thus we have:

x				
0	0			
1	1	1		
1	1	3	2	
1	1	3	3	1

So we have

$$p(x) = x + 2x(x-1) + x(x-1)^2 = x^3.$$

Because $s(0) = 0^3 = 0$, $s(2) = (2-2)^3 = 0$, $s(x)$ is a natural cubic spline.

II. Consider interpolation f on $[a, b]$ with a quadratic spline $s \in \mathbb{S}_2^1$.

II-a Why an additional condition is needed to determine s uniquely?.

$s(x)$ is constructed by $n-1$ polynomials with degree of 2, we can assume them $\{p_i\}$, $s(x) = p_i(x)$ when $x \in [x_i, x_{i+1}]$. Suppose $p_i = a_{i_2}x^2 + a_{i_1}x + a_{i_0}$ so we have $3n-3$ variables.

By the property of $s(x)$, we have $p_i(x_i) = f_i$, $p_i(x_{i+1}) = f_{i+1}$ and $p_i'(x_{i+1}) = p_{i+1}'(x_{i+1})$. In the formal two equations $i = 1, 2 \ldots, n-1$, while in the last one $i = 1, 2 \ldots, n-2$. They construct $(n-1)+(n-1)+(n-2) = 3n-4$ equations.

The number of equations is less than that of variables, so we cannot determine s uniquely unless we introduce an additional condition.

II-b Determine p_i in terms of f_i, f_{i+1}, and m_i for $i = 1, 2 \ldots, n-1$

For $i = 1, 2 \ldots, n-1$, we have $p_i(x_i) = f_i$, $p_i(x_{i+1}) = f_{i+1}$, $p_i'(x_i) = m_i$. So

x			
x_i	f_i		
x_i	f_i	m_i	
x_{i+1}	f_{i+1}	$\frac{f_{i+1}-f_i}{x_{i+1}-x_i}$	$\frac{f_{i+1}-f_i-(x_{i+1}-x_i)m_i}{(x_{i+1}-x_i)^2}$

Thus we have

$$p_i(x) = f_i + m_i(x - x_i) + \frac{f_{i+1} - f_i - (x_{i+1} - x_i)m_i}{(x_{i+1} - x_i)^2}(x - x_i)^2.$$

II-c Suppose $m_1 = f'(a)$ is given. Show how $m_2, \ldots, m_{n-1}$ can be compouted.

Because $p_i'(x_{i+1}) = p_{i+1}'(x_{i+1}) = m_{i+1} \quad i = 1, 2 \ldots, n-2$. Since we have got the form of p_i. So we can calculate the derivative of p_i on x_{i+1},

$$p_i'(x_{i+1}) = m_i + 2\frac{f_{i+1} - f_i - (x_{i+1} - x_i)m_i}{x_{i+1} - x_i}.$$

Put it into the formal equation we have

$$m_{i+1} = \frac{2(x_{i+1} - f_i)}{x_{i+1} - x_i} - m_i \quad i = 1, 2, \ldots, n-1.$$

As we have known m_1, we can get all m_i by iteration.

III Determine $s_2(x)$ on $[0, 1]$.

Since s is a natural cubic spline, we have $s''(-1) = s''(1) = 0$, $s_2(0) = s_1(0) = 1 + c$, $s_2'(0) = s_1'(0) = 3c$, $s_2''(0) = s_1''(0) = 6c$.

Now we suppose $x_1 = -1, x_2 = 0, x_3 = 1$, $M_i = s''(x_i) \quad (i = 1, 2, 3)$. So $M_2 = 6c$,$M_1 = M_3 = 0$. Thus we have

$$s_2'''(0) = \frac{M_3 - M_2}{x_3 - x_2} = -6c.$$

By using the Taylor expansion of $s_2(x)$ on x_2 we have

$$s_2(x) = s(0) + s'(0)x + \frac{M_2}{2}x^2 + \frac{s'''(0)}{6}x^3 = -cx^3 + 3cx^2 + 3cx + c + 1.$$

If we want $s(1) = -1$, then we get $c = -\frac{1}{3}$.

IV Consider $f(x) = \cos\left(\frac{\pi}{2}x\right)$ with $x \in [-1, 1]$

IV-a Determine the natural cubic spline interpolant to f on knots -1,0,1.

We suppose that $x_1 = -1, x_2 = 0, x_3 = 1$, $M_i = s''(x_i)$. We have $s(x_1) = f(x_1) = 0$,$s(x_2) = f(x_2) = 1$, $s((x_3) = f(x_3) = 0$.

The divided differences of f are

x			
-1	0		
0	1	1	
1	0	-1	-1

So we can calculate M_2 by the equation $\mu_2 M_1 + 2M_2 + \lambda_2 M_3 = f[x_1, x_2, x_3]$, where $\mu_2 = \frac{x_2 - x_1}{x_3 - x_1}$, $\lambda_2 = \frac{x_3 - x_2}{x_3 - x_1}$. We have $M_2 = -3$.

Since $s_i'(xi) = f[x_i, x_{i+1}] - \frac{1}{6}(M_{i+1} + 2M_i)(x_{i+1} - x_i)$, we get $s(x)$ by using the Taylor expansion of $s(x)$ at x_i.

$$s_1(x) = \left(1 - \frac{1}{6}(-3)\right)(x+1) + \frac{-3}{6}(x+1)^3 = -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1.$$

$$s_2(x) = 1 + \left(-1 - \frac{1}{6}(-6)\right)x - \frac{3}{2}x^2 + \frac{3}{6}x^3 = \frac{1}{2}x^3 - \frac{3}{2}x^2 + 1.$$

IV-b Verify the minimal total bending energy by taking $g(x)$

The divided differences of f are

x			
-1	0		
0	1	1	
1	0	-1	-1

So by Newton's formula we have

$$g(x) = (x - x_1) - (x - x_1)(x - x_2) = -x^2 + 1.$$

So $g''(x) = -2$

$$\int_{-1}^{1} [s''(x)]^2 dx = \int_{-1}^{0} (-3x-3)^2 dx + \int_{0}^{1} (3x-3)^2 dx = 6.$$

$$\int_{-1}^{1} [g''(x)]^2 dx = \int_{-1}^{1} 4dx = 8 > 6.$$

Because $f''(x) = -\frac{\pi^2}{4}\cos\left(\frac{\pi}{2}x\right)$, we have

$$\int_{-1}^{1} [f''(x)]^2 dx = \frac{\pi^4}{16}\int_{-1}^{1} \frac{\cos(\pi x) - 1}{2} dx = \frac{\pi^4}{16} > 6.$$

So we have $\int_{-1}^{1}[s''(x)]^2 dx > \int_{-1}^{1}[g''(x)]^2 dx$ and $\int_{-1}^{1}[s''(x)]^2 dx > \int_{-1}^{1}[f''(x)]^2 dx$.

V The quadratic B-splines $B_i^2(x)$.

V-a Derive the same explicit expression of $B_i^2(x)$

We already have

$$B_i^0(x) = \begin{cases} 1 & x \in (t_{i-1}, t_i] \\ 0 & otherwise \end{cases}$$

So by

$$B_x^1 = \frac{x - t_{i-1}}{t_i - t_{i-1}} B_i^0(x) + \frac{t_{i+1} - x}{t_{i+1} - t_i} B_{i+1}^0(x).$$

We have

$$B_i^1(x) = \begin{cases} \frac{x - t_{i-1}}{t_i - t_{i-1}} & x \in (t_{i-1}, t_i] \\ \frac{t_{i+1} - x}{t_{i+1} - t_i} & x \in (t_i, t_{i+1}] \\ 0 & otherwise \end{cases}$$

Since $B_i^2(x) = \frac{x-t_{i-1}}{t_{i+1}-t_{i-1}} B_i^1(x) + \frac{t_{i+2}-x}{t_{i+2}-t_i} B_{i+1}^1(x)$ When $x \in (t_{i-1}, ti]$, $B_{i+1}^1 = 0$, we have

$$B_i^2(x) = \frac{x-t_{i-1}}{t_{i+1}-t_{i-1}} \frac{x-t_{i-1}}{t_i-t_{i-1}}.$$

When $x \in (t_{i+1}, t_{i+2}]$, $B_i^1 = 0$, we have

$$B_i^2(x) = \frac{t_{i+2}-x}{t_{i+2}-t_i} \frac{t_{i+2}-x}{t_{i+2}-t_{i+1}}.$$

When $x \in (t_i, t_{i+1}]$, we have

$$B_i^2(x) = \frac{x-t_{i-1}}{t_{i+1}-t_{i-1}} \frac{t_{i+1}-x}{t_{i+1}-t_i} + \frac{t_{i+2}-x}{t_{i+2}-t_i} \frac{x-t_i}{t_{i+1}-t_i}.$$

So

$$B_i^2(x) = \begin{cases} \frac{x-t_{i-1}}{t_{i+1}-t_{i-1}} \frac{x-t_{i-1}}{t_i-t_{i-1}} & x \in (t_{i-1}, t_i] \\ \frac{x-t_{i-1}}{t_{i+1}-t_{i-1}} \frac{t_{i+1}-x}{t_{i+1}-t_i} + \frac{t_{i+2}-x}{t_{i+2}-t_i} \frac{x-t_i}{t_{i+1}-t_i} & x \in (t_i, t_{i+1}] \\ \frac{t_{i+2}-x}{t_{i+2}-t_i} \frac{t_{i+2}-x}{t_{i+2}-t_{i+1}} & x \in (t_{i+1}, t_{i+2}] \end{cases}$$

V-b Verify that $\frac{d}{dx} B_i^2(x)$ is continuous at t_i and t_{i+1}.

Since $B_i^2(x)$ are polynomials of degree 2 , it is obvious that $\frac{d}{dx} B_i^2(x)$ is continuous on (t_{i-1}, t_{i+2}) except t_i and t_{i+1}.

When $x = t_i$, we have

$$\lim_{x\to t_i^+} \frac{d}{dx} B_i^2(x) = \lim_{x\to t_i^+} \frac{2(x-t_{i-1})}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})} = \frac{2}{t_{i+1}-t_{i-1}}$$

$$\begin{aligned} \lim_{x\to t_i^-} \frac{d}{dx} B_i^2(x) &= \lim_{x\to t_i^-} \frac{t_{i+1}+t_{i-1}-2x}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{t_{i+2}+t_i-2x}{(t_{i+2}-t_i)(t_{i+1}-t_i)} \\ &= \frac{t_{i+1}+t_{i-1}-2t_i}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{t_{i+2}+t_i-2t_i}{(t_{i+2}-t_i)(t_{i+1}-t_i)} = \frac{2}{t_{i+1}-t_{i-1}}. \end{aligned}$$

Because $\lim_{x\to t_i^+} \frac{d}{dx} B_i^2(x) = \lim_{x\to t_i^-} \frac{d}{dx} B_i^2(x)$, we have $\frac{d}{dx} B_i^2$ is continuous on t_i.

Similiarly we can prove that $\lim_{x\to t_{i+1}^+} \frac{d}{dx} B_i^2(x) = \lim_{x\to t_{i+1}^-} \frac{d}{dx} B_i^2(x) = \frac{-2}{t_{i+2}-t_i}$, which means $\frac{d}{dx} B_i^2$ is also continuous on t_{i+1}.

V-c Show that only one $x^* \in (t_{i-1}, t_{i+1})$ satisfies $\frac{d}{dx} B_i^2(x^*) = 0$.

When $x \in (t_{i-1}, t_i)$, $\frac{d}{dx} Bi_i^2(x) = \frac{2(x-t_{i-1})}{t_{i+1}-t_{i-1}} > 0$

When $x \in (t_{i+1}, t_{i+2})$, $\frac{d}{dx} B_i^2(x) = \frac{2(x-t_{i+2})}{t_{i+2}-t_{i+1}} < 0$

Since $\frac{d}{dx} B_i^2(x)$ is continuous on (t_{i-1}, t_{i+2}) and is a polynomial with degree 1 , there exists only one $x^* \in (x_i, x_{i+1})$ satisfies that $\frac{d}{dx} B_i^2(x^*) = 0$.

So we have

$$\begin{aligned} \frac{d}{dx} B_i^2(x^*) = \frac{t_{i+1}+t_{i-1}-2x^*}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{t_{i+2}+t_i-2x^*}{(t_{i+2}-t_i)(t_{i+2}-t_i)} = 0 \\ \Longrightarrow (t_{i+1}+t_{i-1})(t_{i+2}-t_i) + (t_{i+2}+t_i)(t_{i+1}-t_{i-1}) - 2x^*(t_{i+2}-t_i+t_{i+1}-t_{i-1}) = 0 \\ \Longrightarrow x^* = \frac{t_{i+1}t_{i+2}-t_{i-1}t_i}{t_{i+1}-t_{i-1}+t_{i+2}-t_i}. \end{aligned}$$

V-d Consequently, show $B_i^2(x) \in [0, 1)$.

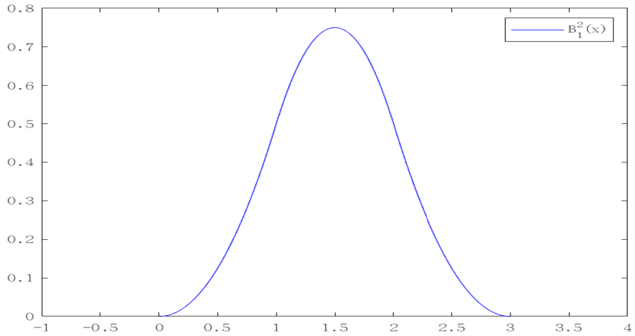
Because $\lim_{x\to t_{i-1}} B_i^2(x) = B_i^2(t_{i+2}) = 0$,$B_i^2(x) > 0$ when $x \in (t_{i-1}, x^*)$ and $B_i^2(x) < 0$ when $x \in (x^*, t_{i+2})$. We can easily get $B_i^2(x) \geq 0$.

By the property of $\frac{d}{dx} B_i^2(x)$, we can know that $B_i^2(x)$ will reach its maximum at x^*. Then we have

$$\begin{aligned} \frac{d}{dx} B_i^2(x^*) &= \frac{(t_{i+1}-t_{i-1})(t_{i+2}-t_{i-1})(t_{i+1}-t_i)(t_{i+1}-t_{i-1})}{(t_{i+1}-t_{i-1}+t_{i+2}-t_i)^2 (t_{i+1}-t_i)(t_{i+1}-t_{i-1})} + \frac{(t_{i+2}-t_{i-1})(t_{i+2}-t_i)(t_{i+2}-t_i)(t_{i+2}-t_i)}{(t_{i+2}-t_{i-1}+t_{i+2}-t_i)^2 (t_{i+2}-t_i)(t_{i+1}-t_i)} \\ &= \frac{t_{i+2}-t_{i-1}}{t_{i+2}-t_{i-1}+t_{i+1}-t_i} \end{aligned}$$

Since $t_{i+2}-t_{i-1} > 0$ and $t_{i+1}-t_i > 0$, we have $B_i^2(x^*) < 1$. So $B_i^2(x) \in [0, 1)$.

V-e **Plot $B_i^2(x)$ for $t_i = i$.**



VI **Verify $(t_{i+2}-t_{i-1})[t_{i-1},t_i,t_{i+1},t_{i+2}](t-x)_+^2$ algebraically.**

Because

$$B_i^0(x) = \begin{cases} 1 & x(t_{i-1},t_i] \\ 0 & otherwise \end{cases}$$

$$\begin{aligned}(t_i-t_{i-1})[t_{i-1},t_i](t-x)_+^0 &= (x-t_{i-1})_+^0 - (x-t_i)_+^0 \\ &= \begin{cases} 1 & x \in (t_{i-1},t_i] \\ 0 & otherwise \end{cases}\end{aligned}$$

We can easily have $B_i^0(x) = (t_i-t_{i-1})[t_{i-1},t_i](t-x)_+^0$

Before proving the equation, we firsly calculate the ddivided difference of (t-x)

t				
t_{i-1}	$(t_{i-1}-x)$			
t_i	(t_i-x)	1		
t_{i+1}	$(t_{i+1}-x)$	1	0	
t_{i+2}	$(t_{i+2}-x)$	1	0	0

So from $(t-x)_+^1 = (t-x)(t-x)_+^0$ and $(t-x)_+^2 = (t-x)(t-x)_+^1$ we have

$$\begin{aligned}(t_{i+1}-t_{i-1})[t_{i-1},t_i,t_{i+1}](t-x)_+^1 &= (t_{i+1}-t_{i-1})[t_{i-1}](t-x)[t_{i-1},t_i,t_{i+1}](t-x)_+^0 \\ &\quad + (t_{i+1}-t_{i-1})[t_{i-1},t_i](t-x)[t_i,t_{i+1}](t-x)_+^0 \\ &= (t_{i-1}-x)[t_i,t_{i+1}](t-x)_+^0 - (t_{i-1}-x)[t_{i-1},t_i](t-x)_+^0 \\ &\quad + (t_{i+1}-t_{i-1})[t_i,t_{i+1}](t-x)_+^0 \\ &= \frac{x-t_{i-1}}{t_i-t_{i-1}}B_i^0(x) + \frac{t_{i+1}-x}{t_{i+1}-t_i}B_i^0(x) \\ &= B_i^1(x). \\ (t_{i+2}-t_{i-1})[t_{i-1},t_i,t_{i+1},t_{i+2}](t-x)_+^2 &= (t_{i+2}-t_{i-1})[t_{i-1}](t-x)[t_{i-1},t_i,t_{i+1},t_{i+2}](t-x)_+^1 \\ &\quad + (t_{i+2}-t_{i-1})[t_{i-1},t_i](t-x)[t_i,t_{i+1},t_{i+2}](t-x)_+^1 \\ &= (t_{i-1}-x)\left([t_i,t_{i+1},t_{i+2}](t-x)_+^1 - [t_{i-1},t_i,t_{i+2}](t-x)_+^1\right) \\ &\quad + (t_{i+2}-t_{i-1})[t_i,t_{i+1},t_{i+2}](t-x)_+^1 \\ &= \frac{x-t_{i-1}}{t_{i+1}-t_{i-1}}B_i^1(x) + \frac{t_{i+2}-x}{t_{i+2}-t_i}B_i^1(x) \\ &= B_i^2(x)\end{aligned}$$

So we have proved that $B_i^2(x) = (t_{i+2}-t_{i-1})[t_{i-1},t_i,t_{i+1},t_{i+2}](t-x)_+^2$.