I. Consider the bisection method starting with the initial interval $[1.5, 3.5]$

I-a What is the width of the interval at the nth step?

Suppose the interval at the nth step is $[a_n, b_n], n \in \mathbb{N}$. Since $b_n - a_n = \frac{1}{2}(b_{n+1} - a_{n+1}), b_0 - a_0 = 3.5 - 1.5 = 2$, by recursion we can have that $\forall n \in \mathbb{N}$, the width of the nth interval $l = \frac{1}{2^{n-1}}$.

I-b What is the maximum possible distance between the root r and the midpoint of the interval?

We assume that d_n as the distance between r and the midpoint of $[a_n, b_n]$.

If $a_n \neq a_0, b_n \neq b_0$, which means $\exists s < n$, a_n is the mid point of $[a_s, b_s]$, and $\exists t < n$, b_t is the mid point of $[a_t, b_t]$. Thus, r $\in (a_n, b_n)$, which means $\exists \epsilon > 0, r + \epsilon < b_n, r - \epsilon > a_n$. So $d_n =\mid r - \frac{b_n+a_n}{2} \mid \leq \mid \frac{b_n-a_n}{2} - \epsilon \mid = \mid \frac{1}{2^n} - \epsilon \mid$.

If $b_n = b_0$, which means $\forall m < n, r > \frac{b_m+a_m}{2}$, if $r = b_0$, $d_n = \frac{1}{2^n}$, or the same as the situation above, $\exists \epsilon > 0, d_n = \mid \frac{1}{2^n} - \epsilon \mid$.

If $a_n = a_0$, the consequence is similar as that when $b_n = b_0$.

So we have $d_n \leq \mid \frac{1}{2^n} - \epsilon$ or $d_n = \frac{1}{2_n}$, which means the maximum possible distance between r and the midpoint of the interval $[a_n, b_n]$ is $\frac{1}{2^n}$

II. Prove that the number of the step n must satisfy $n \geq \frac{\log(b_0-a_0)-\log\epsilon-\log a_0}{\log 2} - 1$

We have proved in the first section that $d_n \leq \frac{b_0-a_0}{2^{n+1}}$. In order to determine the root with its relative error no greater that ϵ, we have $\frac{d_n}{r} \leq \frac{b_0-a_0}{r2^{n+1}} \leq \frac{b_0-a_0}{a_0 2^{n+1}} \leq \epsilon$.

Thus, $b_0 - a_0 \leq 2^n a_0 \epsilon$, take $\log_2$ on both side we have $\log_{b_0-a_0} \leq \log 2(n+1) + \log\epsilon + \log a_0$, sort it we have $n \geq \frac{\log(b_0-a_0)-\log\epsilon-\log a_0}{\log 2} - 1$, so we have proved the it.

III. Perform four iterations of Newton's method

$p(x) = 4x^3 - 2x^2 + 3$, we have $p'(x) = 12x^2 - 4x$, by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n \in \mathbb{N}$. So we can calculate each x_n with a handy calculator.

n	0	1	2	3	4
x_n	-1	-0.8125	-0.7708041958	-0.7688323843	-0.7688280859

IV. Consider a variation of Netwon's method $x_{n+1} = x_n - \frac{f(x_N)}{f'(x_0)}$, find C and s such that $e_{n+1} = Ce_n^s$.

Do not lose the generality, we can assume that the root $\alpha < x_0, f(\alpha) = 0, f'(x) > 0 \forall x \in (\alpha, x_0)$. Thus, we have $\alpha < x_{n+1} < x_n \leq x_0, \forall n \in \mathbb{N}$.

By Taylor's theorem we have

$$f(x_n) = f(\alpha) + f'(\xi)(x_n - \alpha), (\xi \in (\alpha, x_n)).$$

Thus, we have

$$e_{n+1} = e_n - \frac{f(x_n)}{f'_{x_0}} = e_n \left(\frac{f'_{x_0} - f'(\xi)}{f'_{x_0}} \right) \leq e_n \left(\frac{2f'(x_0)}{f'(x_0)} \right) = 2e_n.$$

Thus we take the C and s.

V. Within $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, will the iteration $x_{n+1} = tan^{-1}x_n converge?$

By the symmetry of $tanx$ and x, we can suppose that $\{x_n\}$ converges to a positive number α.

Since $1 - \frac{1}{tan1} > 0$, $0.5 - \frac{1}{tan_0.5} < 0$, and the continuity of $x - \frac{1}{tanx}$, we have that there exists one α, $\alpha = \frac{1}{tan\alpha}$. If $x_0 = \alpha$, it is obvious that $\{x_n\}$ is convergent.

If $x_0 \neq \alpha$. We suppose that $\{x_n\}$ is convergent, so $\{x_n\}$ will converge to α. Then we take a sufficiently small ϵ, that $\exists n$ s.t. $\mid x_{n+1} - x_n = \epsilon \mid$, then we have

$$\mid x_{n+2} - x_{n+1} \mid = \mid \frac{1}{tan\frac{1}{tanx_n}} - \frac{1}{tanx_n} \mid > \mid \frac{tan\frac{1}{tanx_n} - tanx_n}{1 + tanx_n tan\frac{1}{tanx_n}} \mid > \mid tan\left(x_n - \frac{1}{tanx_n}\right) \mid > \mid x_n - x_{n+1} \mid > \epsilon.$$

Thus we have $\forall m > n, \mid x_{m+1} - x_m \mid > \mid x_{n+1} - x_n \mid$, which contradicts to the convergence of $\{x_n\}$.

So $\{x_n\}$ is convergent iff $x_0 = \pm\alpha$.

VI. Let $p > 1$. What is the value of $x = \frac{1}{p+\frac{1}{p+\frac{1}{p+\cdots}}}$

Suppose $x_0 = 0, x_{n+1} = \frac{1}{p+x_n}, \forall n \in \mathbb{N}$. Now we prove that $\{x_n\}$ is convergent.

When n=1, it is obvious that $\frac{1}{p+\frac{1}{p}} = x_2 < x_1 = \frac{1}{p}$.

Suppose when $n = k\,(k \geq 1)$, $x_{n+1} < x_n$. Then, $x_{n+2} = \frac{1}{p+x_{n+1}} < \frac{1}{p+x_n} = x_{n+1}$. So we have $\forall n \in \mathbb{N}^+, x_{n+1} < x_n$. Because $\forall n \in \mathbb{N}^+, x_n > 0$, we have $\{x_n\}$ is convergent.

To get the value of x, we take a function $f(x) = \frac{1}{p+x}$, $x_n = f(x_{n+1})$, when $n \to \infty, x_n \to x$. So $\forall \epsilon > 0, \exists N, \forall n > N, x_n - x < \epsilon$, which also means $f(x_n) - x_n < \epsilon$. Since we have proved that $\{x_n\}$ is convergent, by the randomization of ϵ, we get that x is the root of $f(x) - x = 0$, which is $x^2 + px - 1 = 0$. Because $\triangle = p^2 + 4 > 0$, the root is always exist. By formula method and $x > 0$, we have $x = \frac{-p+\sqrt{p^2+4}}{2}$.

VII. What happens in problem II if $a_0 < 0 < b_0$

We assume that $c = min\{-a_0, b_0\}$.

If $r \neq 0$, indorer to determine that relative error no greater than ϵ , we have $\frac{d_n}{r} \leq \frac{b_0-a_0}{r2^{n+1}} \leq \frac{b_0-a_0}{c2^{n+1}} < \epsilon$. Thus, the same as the inequality we have in II, we have $n \geq \frac{\log(b_0-a_0)-\log\epsilon-\log c}{\log 2} - 1$.

If $r = 0$, the relative error is not defined, so it is not so appropriate when $a_0 < 0 < b_0$. But if we want to determine the error no greater than ϵ , we have $d_n = \frac{b_0-a_0}{2^{n+1}} \leq \epsilon$, thus we get $n \geq \frac{\log(b_0-a_0)-\log\epsilon}{\log 2} - 1$.

VIII. Consider solving $f(x) = 0$ by Newton's method

VIII-a How can a multiple zero be detected by examining the behavior of the points $x_n, f(x_n)$?

Since $f^{(k)\neq 0}$, by taking x_0 sufficiently close to α , we have $\mid x_{n+1} - \alpha \mid < \mid x_n - \alpha \mid$. By Taylor's theorem we have,

$$f(x_n) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)(x_n-\alpha)^n}{n!}.$$

Because $f^i(\alpha) = 0\,(i < k)$, we have

$$f(x_n) = \sum_{n=k}^{\infty} \frac{f^{(n)}(\alpha)(x_n-\alpha)^n}{n!} = O\left((x_n-\alpha)^k\right).$$

So if

$$\forall a > 0, \exists N \in \mathbb{N}, \forall n > N, \mid f(x_n) \mid < \mid a(x-\alpha)^k \mid .$$

Which means the $f(x_n)$ converge to $f(\alpha)$ in order k, so we can detect a multiple zero.

VIII-b. Prove that if r is a zero of multiplicity k , quadratic convergence in Newton's iteration will be restored by $x_{n+1} = x_n - k\frac{f(x_n)}{f' x_n}$

Do not lose the generality, we can suppose that $f'(\alpha) > 0$, $x_n > \alpha$

By Taylor's theorem, we have

$$f(x_n) = \frac{1}{k!}(x_n-\alpha)^{(k)} f^k(\alpha) + \frac{1}{k+1!}(x_n-\alpha)^{k+1} f^{(k+1)}(\xi)\,(\xi \in (\alpha, x_n)).$$

and

$$f'(x_n) = \frac{1}{k-1!}(x_n-\alpha)^{k-1} f^{(k-1)}(\alpha) + \frac{1}{k!}(x_n-\alpha)^k f^{(k)}(\eta)\,(\eta \in (\alpha, x_n)).$$

So by the iteration $x_{n+1} = x_n - k\frac{f(x_n)}{f'(x_n)}$, we have

$$e_{n+1} = \frac{\frac{1}{k!}{e_n}^k f^{(k)}(\alpha) + \frac{1}{k!k}{e_n}^{k+1} f^{(k+1)}(\eta) - \frac{1}{k!}{e_n}^k f^{(k)}(\alpha) - \frac{1}{k+1!}{e_n}^{k+1} f^{(k+1)}(\xi)}{\frac{1}{k!}{e_n}^{k-1} f^{(k)}(\alpha) + \frac{1}{k!k}{e_n}^k f^{(k+1)}(\eta)}.$$

$$\frac{e_{n+1}}{e_n^2} = \frac{\frac{1}{k!k} f^{(k+1)}(\eta) - \frac{1}{k+1!} f^{(k+1)}(\xi)}{\frac{1}{k!} f^{(k)}(\alpha) + \frac{1}{k!k} e_n f^{(k+1)}(\eta)}.$$

Because $\lim_{n\to\infty} e_n = 0$, $\mid f^{(k+1)}(\xi) \mid < \mid f^{(k+1)}(\alpha) \mid$, $\mid f^{(k+1)}(\xi) \mid < \mid f^{(k+1)}(\alpha) \mid$, we have

$$\mid \frac{e_{n+1}}{e_n^2} \mid \leq \mid \frac{2f^{(k+1)}(\alpha)}{kf^{(k)}(\alpha)} \mid = C.$$

So we have proved that the modified iteration is quadratic convergence.