I. For $n \in \mathbb{N}^+$, determin $\min \max_{x\in[a,b]} \mid a_0x^n + a_1x^{n-1} + \ldots + a_n \mid$.

Since $\forall\{a_n\}$,

$$\max_{x\in[-1,1]} \mid x^n + a_1x^{n-1} + \ldots + a_n \mid \geq \frac{1}{2^{n-1}}.$$

And the equality can be reached if $x^n + a_1x^{n-1} + \ldots + a_n = \frac{1}{2^{n-1}}T_n(x)$. Thus we can have a conversion of the fomular $a_0x^n + a_1x^{n-1} + \ldots + a_n$ that

$$\max_{x\in[a,b]} \mid a_0x^n + a_1x^{n-1} + \ldots + a_n \mid = \max_{x\in[-1,1]} \mid a_0\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)^n + \ldots + a_n \mid = \max_{x\in[-1,1]} \mid a_0'x^n + a_1'x^{n-1} + \ldots + a_n' \mid .$$

The $\{a_n'\}$ can be randomly taken because $\{a_n\}$ is random. Because a_0' is independent with $\{a_n'\}$, we suppose $a_i'' = \frac{a_i'}{a_0'}$, we have

$$\min \max_{x\in[a,b]} \mid a_0x^n + a_1x^{n-1} + \ldots + a_n \mid = \min |a_0| \max_{x\in[-1,1]} \mid x^n + a_1''x^{n-1} + \ldots + a_n'' \mid$$
$$\geq \min |a_0| \frac{1}{2^{n-1}}.$$

Since the equality can be reached and a_0 is randomly taken from $\mathbb{R}\backslash\{0\}$, we have

$$\min \mid a_0 \mid \frac{1}{2^{n-1}} = 0.$$

Thus,

$$\min \max_{x\in[a,b]} \mid a_0x^n + a_1x^{n-1} + \ldots + a_n \mid = 0.$$

II. Prove $\forall p \in \mathbb{P}_n^\alpha, \|\hat{p}_n(x)\|_\infty \leq \|p\|_\infty$.

Suppose that

$$\exists p \in \widetilde{\mathbb{P}}_n \quad s.t. \quad \max_{x\in[-1.1]} \mid p(x) \mid < \|\hat{p}_n(x)\|.$$

$T_n(x)$ assumes its extreme $n+1$ times at the points $x_k' = \cos\frac{k}{n}\pi$ $k = 0, 1, \ldots, n$. Consider the polynomial $Q(x) = \frac{T_n(x)}{T_n(\alpha)} - p(x)$. We have

$$Q(x_k') = \frac{(-1)^k}{T_n(\alpha)} - p(x_k') \qquad k = 0, 1 \ldots, n.$$

Since $\mid p(x_k') \mid < \|\hat{p}_n(x)\| = \frac{1}{T_n(\alpha)}$. $Q(x)$ has alternating signs at these $n+1$ points, which means $Q(x)$ must have n zeros in $[-1,1]$. But we know $Q(\alpha) = 0, \alpha > 1$ and $Q(x)$ is a polynomial with degree n. So $Q(x)$ has at most $n-1$ zeros within $[-1,1]$. Therefore, $Q(x) \equiv 0$ and $p(x) = \frac{T_n(x)}{T_n(\alpha)}$, which means $\|\hat{p}_n(x)\|_\infty = \|p\|_\infty$. It is a contradition to our assumption. Therefore, we have

$$\forall p \in \mathbb{P}_n^\alpha, \|\hat{p}_n(x)\|_\infty \leq \|p\|_\infty.$$