I. Deduce Corollary 4.24 from the theorem of derivatives of B-splines.

Since $\forall i$, $\frac{d}{dx}B_i^{n+1}(x)$ has primitive on $\mathbb{R}\backslash\{t_n\}$, $\{t_n\}$ are isolated with each other, so we have

$$\int_{t_{i-1}}^{t_{i+n+1}} \frac{d}{dx}B_i^{n+1}(x)\,dx = B_i^{n+1}(t_{i+n+1}) - B_i^{n+1}(t_{i-1}) = 0 - 0 = 0.$$

So by the theorem of derivatives of B-splines, and the support of $B_i^n(x) = [t_{i-1}, t_{i+n}]$, we have

$$\begin{aligned}\int_{t_{i-1}}^{t_{i+n+1}} \frac{d}{dx}B_i^{n+1}(x)\,dx &= \int_{t_{i-1}}^{t_{i+n+1}} \frac{(n+1)\,B_i^n(x)}{t_{i+n}-t_{i-1}}dx - \int_{t_{i-1}}^{t_{i+n+1}} \frac{(n+1)\,B_{i+1}^n(x)}{t_{i+n+1}-t_i}dx = 0 \\ &\Longrightarrow \frac{n+1}{t_{i+n}-t_{i-1}}\int_{t_{i-1}}^{t_{i+n}} B_{i+1}^n(x)\,dx = \frac{n+1}{t_{i+n+1}-t_i}\int_{t_{i-1}}^{t_{i+n+1}} B_{i+1}^n(x)\,dx = C\end{aligned}$$

.

Because $\{t_n\}$ are randomly taken, by changing t_i and t_{i-1} separately, we can conclude that C is independent with $\{t_n\}$. So we have

$$\forall i \quad , \frac{1}{t_{i+n}-t_{i-1}}\int_{t_{i-1}}^{t_{i+n}} B_i^n(x)\,dx = \frac{C}{n+1}.$$

The intergral is independent with its index.

II. Symmetric Polynomials.

II-a. Verify the theorem 4.34 for $m = 4$ and $n = 2$.

We cam make a tabular for the divided difference of x^m.

$$\begin{array}{c|lll} \text{x} & & & \\ x_1 & x_1^4 & & \\ x_2 & x_2^4 & x_2^3+x_2^2x_1+x_2x_1^2+x_1^3 & \\ x_3 & x_3^4 & x_3^3+x_3^2x_2+x_3x_2^2+x_2^3 & \frac{x_3^3-x_1^3+x_2\left(x_3^2-x_1^2\right)+x_2^2(x_3-x_1)}{x_3-x_1} = x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_2x_3 \end{array}$$

By the definition of complete symmetric polynomial, we have

$$\tau_{4-2}(x_1,x_2,x_3) = x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_2x_3 = [x_1,x_2,x_3]x^4.$$

II-b. Prove this theorem by the lemma on the recursive relation on complete symmetric polynomials.

Don't lose generality, we can let $i = 1$.

Firstly, we prove $(x_{n+1}-x_1)\,\tau_k(x_1,\ldots,x_{n+1}) = \tau_{k+1}(x_2,\ldots,x_{n+1}) - \tau_{k+1}(x_1,\ldots,x_n)$

$$\begin{aligned} LHS &= x_{n+1}\tau_k(x_1,\ldots,x_{n+1}) - x_1\tau_k(x_1,\ldots,x_{n+1}) \\ &= \tau_{k+1}(x_1,\ldots,x_{n+1}) - \tau_{k+1}(x_1,\ldots,x_n) - [\tau_{k+1}(x_1,\ldots,x_{n+1}) - \tau_{k+1}(x_2,\ldots,x_{n+1})] \\ &= \tau_{k+1}(x_2,\ldots,x_{n+1}) - \tau_{k+1}(x_1,\ldots,x_n) \\ &= RHS. \end{aligned}$$

When $n = 0$, by the definition, we have

$$\forall m \quad , \tau_m(x_1) = x_1^m = [x_1]x^m.$$

So when $n = 0$, the theorem is right.

We randomly take a m, we suppose when $n = k < m$, the equation $\tau_{m-k}(x_1,\ldots,x_{1+k}) = [x_1,\ldots,x_{1+k}]x^m$ exists. Then when $n = k+1$, we have

$$\begin{aligned} [x_1,\ldots,x_{k+2}]x^m &= \frac{[x_2,\ldots,x_{2+n}]x^m - [x_1,\ldots,x_{n+1}]x^m}{x_{1+n}-x_1} \\ &= \frac{1}{x_{1+n}-x_1}\left(\tau_{m-n}(x_2,\ldots,x_{n+2}) - \tau_{m-n}(x_1,\ldots,x_{n+1})\right) \\ &= \tau_{m-(n+1)}(x_1,\ldots,x_{n+2}). \end{aligned}$$

So $\forall m \in \mathbb{N}^+, i \in \mathbb{N}, \forall n = 0,1,\ldots,m, \tau_{m-n}(x_i,\ldots,x_{i+n}) = [x_i,\ldots,x_{i+n}]x^m$.