Lecture 1: Monotone Comparative Statics and Supermodular Game

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Note: This lecture notes adapt from Todd Sarver's Microeconomic Theory Lecture Notes.

1 Nonlinear Optimization Review

2 Comparative Statics: Implicit Function Theorem

Consider an optimization problem: $X \subseteq \mathbb{R}$, $T \subseteq \mathbb{R}$, and $f : X \times T \to \mathbb{R}$. We call T is the set of parameters and $x \in X$ is the decision variable.

$$X^*(t) = \arg\max_{x \in X} f(x,t)$$

Comparative Statics answer question: What happens to the optimal value if we change one parameter?

The first method to study comparative statics is applying the Implicit Function Theorem. Recall the IFT:

Theorem 1 (Implicit Function Theorem)**.** *Let $G(x,y)$ be a C^1 function on a ball about (x_0,y_0) in $\mathbb{R}^2$. Suppose that $G(x_0,y_0) = c$ and consider the expression $G(x,y) = c$.*

If $\frac{\partial G(x_0,y_0)}{\partial y} \neq 0$, then there exists a C^1 function $y = y(x)$ defined on an interval I about the point x_0 such that:

(a) $G(x,y(x)) = c$ for all x in I

(b) $y(x_0) = y_0$

(c) $y'(x_0) = -\frac{G_x(x_0,y_0)}{G_y(x_0,y_0)}$

There, we first get FOC:

$$f_x(x^*(t),t) = 0$$

Then, apply IFT, we get

$$\frac{dx^*(t)}{dt} = -\frac{f_{xt}(x^*(t),t)}{f_{xx}(x^*(t),t)}$$

What assumptions do we need? We usually impose

(1) A solution exits for every t: X is compact; f is continuous

(2) The solution $x^*(t)$ lies in the interior of X.

(3) The solution is not the minimum and unique: f is concave, $f_{xx} < 0$; there f is twice continuously differentiable.

To have monotone result, like $\frac{dx^*(t)}{dt} \geq 0$, we need $f_{xt}(x^*(t), t) \geq 0$.

3 Monotone Comparative Statics: Cardinal

In this section, we hope to relax the above assumptions but still get monotone result.

First to note, $X^*(t) = \arg\max_{x \in X} f(x,t)$ is usually a correspondence. If it is the function, i.e. $X^*(t) = x^*(t)$ is single valued, then it is clear for us that $x^*(t)$ is increasing in t: if $t' \geq t$, then $x^*(t') \geq x^*(t)$.

When $X^*(t)$ is a set, how to understand that $X^*(t)$ is increasing in t? We introduce the following definition to compare two sets.

Definition 1 (Strong Set Order)**.** For any $Y, Z \subseteq X$, we say that Z dominates Y in the strong set order, denoted $Z \geq_s Y$, if for every $y \in Y$ and $z \in Z$, $\min\{y, z\} \in Y$ and $\max\{y, z\} \in Z$.

Example 1. *For singletons,* $z \geq_s y$ *if and only if* $z \geq y$.

Example 2. *For intervals,* $[3, 5] \leq_s [6, 7]$ *and* $[3, 5] \leq_s [4, 7]$*, overlap permitted. However,* $[3, 5] \nleq_s [2, 7]$. *Ingeneral,* $Y = [a_1, a_2]$ *and* $Z = [b_1, b_2]$, $Z \geq_s Y$ *if and only if* $b_2 \geq a_2$ *and* $b_1 \geq a_1$.

3.1 One-dimensional case

We start with the simple case that there is only one-dimensional decision variable $x \in X \subseteq \mathbb{R}$. Recall, we want to find a relaxed sufficient (hopefully also necessary) condition that can imply monotone comparative statics. From IFT, it seems that cross-partial derivative $f_{xt}(x^*(t), t)$ plays an important role; we try to relax it so that can discard strong assumptions on the function form.

What does $f_{xt}(x^*(t), t) \geq 0$ mean? Interpret! At least, it means $f_x(x, t)$ is nondecreasing in t; $f_t(x, t)$ is nondecreasing in x. Can you re-state the above sentence in the discrete way? The following definition captures the idea.

Definition 2 (Increasing differences)**.** Suppose $X \subseteq \mathbb{R}$, $T \subseteq \mathbb{R}$, and $f : X \times T \to \mathbb{R}$ has increasing differences in $(x; t)$ if for all $x' > x$ and $t' > t$,

$$f(x', t') - f(x, t') \geq f(x', t) - f(x, t).$$

Similarly, condition above can also be written as

$$f(x', t') - f(x', t) \geq f(x, t') - f(x, t).$$

One can show that, if f has good properties (like continuity, twice continuously differentiable), increasing differences (ID) are equivalent to $f_{xt} \geq 0$.

Now, let us see our first comparative statics result.

Theorem 2 (Topkis 1978)**.** *Suppose* $X \subseteq \mathbb{R}$, $T \subseteq \mathbb{R}$, *and* $f : X \times T \to \mathbb{R}$. *If* f *has increasing differences in* $(x; t)$, *then* $X^*(t) = \arg\max_{x \in X} f(x, t)$ *is monotone nondecreasing in* t *(in the strong set order), that is,* $t' \geq t$ *implies* $X^*(t') \geq X^*(t)$.

Proof. Fix any $t' \geq t$, $x \in X^*(t)$, and $x' \in X^*(t')$. By definition of strong set order, we want to show if $x > x'$ then $x' \in X^*(t)$ and $x \in X^*(t')$. (If $x' > x$, that is fine.)

Suppose $x > x'$,

$$\begin{aligned} 0 &\leq f(x,t) - f(x',t) \quad (x \in X^*(t)) \\ &\leq f(x.t') - f(x',t') \quad (ID) \\ &\leq 0 \quad (x' \in X^*(t')) \end{aligned}$$

Thus, we must have $f(x,t) = f(x',t)$ and $f(x,t') = f(x',t')$, which implies $x' \in X^{(t)}$ and $x \in X^{(t')}$.

□

In particular, if $f(x,t)$ has a unique maximizer $x^{(t)}$ for each t, then this solution is a nondecreasing function. Note that we do not need assumptions that f is differentiable and concave, it also does not require any assumptions about the solutions being in the interior of X.

3.2 Multidimensional case

To study multiple decision variable case, we first introduce Lattice theory.

Definition 3 (Partially ordered set)**.** A partially ordered set $(X, \geq)$ is a set T equipped with a binary relation $\geq$ that satisfies

(1)Transitive: $x \geq x'$ and $x' \geq x''$ imply $x \geq t''$.

(2)Reflexive: $x \geq x$.

(3)Antisymmetric: $x \geq x'$ and $x' \geq x''$ imply $x = x''$.

Definition 4 (meet and join)**.** Given a partially-ordered set $(X, \geq)$ and $x, y \in X$,

(1) The **meet** of x and x' is $x \wedge y = \sup\{z \in X | x \geq z, y \geq z\}$

(1) The **join** of x and y is $x \vee x' = \inf\{z \in X | z \geq x, z \geq y\}$

A partially-ordered set is said to be lattice if each doubleton subset has greatest lower bound (inf) and smallest upper bound (sup).

Definition 5 (Lattice)**.** A partially-ordered set $(X, \geq)$ is said to be lattice iff for all $x, y \in X$, we have $x \wedge y \in X$ and $x \vee y \in X$.

Example 3. *Endow $\mathbb{R}^n$ with the usual coordinate-wise order:*

$$(x_1, x_2, ..., x_n) \geq (y_1, y_2, ..., y_n) \iff x_i \geq y_i \forall i$$

$(\mathbb{R}^n, \geq)$ is a lattice with

$$x \wedge y = (\min\{x_1, y_1\}, ..., \min\{x_n, y_n\})$$

$$x \vee y = (\max\{x_1, y_1\}, ..., \max\{x_n, y_n\})$$

Figure:

Now, Let us consider the relaxed conditions that can support onotone comparative statics in the multiple decision variable case. A first idea is still the Increasing differences. In this subsection, we focus on $X \subseteq \mathbb{R}^n$.

Definition 6 (Increasing differences)**.** Suppose $X \subseteq \mathbb{R}^n$ is a lattice and $(T, \geq)$ is a partially ordered set. A function $f : X \times T \to \mathbb{R}$ has increasing differences in $(x;t)$ if for all $x' > x$ and $t' > t$,

$$f(x', t') - f(x, t') \geq f(x', t) - f(x, t).$$

Similarly, if f has good properties, we can see it relationships with derivative.

Lemma 1. *If $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable, then f has increasing differences in $(x;t)$ if and only if $\frac{\partial^2 f}{\partial x_i \partial t_j} \geq 0$ for $i = 1, ..., n, j = 1, ..., m$.*

Note, be careful about the denominator!

Now, we may jump to the conclusion that we can extent Theorem 2 to the multiple dimensional case. Is it true? Let us see an example.

Example 4 (Is ID enough for MCS?)**.** *Suppose the objective function is*

$$f(x, y, t) = 3tx + (2 + t)y - (x + y)^2 - x^2 - y^2$$

.

Now, by FOC, we get

$$f_x(x, y, t) = 3t - 4x - 2y = 0$$
$$f_y(x, y, t) = 2 + t - 2x - 4y = 0$$

We get optimal solution $x^* = \frac{5t-2}{6}$ and $y^* = \frac{4-t}{6}$.

It violates MCS; however f satisfies ID: $f_{xt} = 3 \geq 0$ and $f_{yt} = 1 \geq 0$. The issue lies in $f_{xy} = -2 < 0$. Roughly speaking, increasing t implies increasing x and y; however, there is substitute relationship between x and y. There interaction may pull down there values. This will not be a problem if we let x and y has complementary (positive) relationship.

Definition 7 (Supermodular)**.** Suppose $X \subseteq \mathbb{R}^n$ is a lattice and $(T, \geq)$ is a partially ordered set. A function $f : X \times T \to \mathbb{R}$ is supermodular in x if for all $x, x' \in X$ and $t \in T$,

$$f(x \wedge x', t) + f(x \vee x', t) \geq f(x, t) + f(x', t).$$

We can understand the condition if we write the condition as

$$f(x \vee x', t) - f(x', t) \geq f(x, t) - f(x \wedge x', t)$$

The above inequality shows that given parameter t, higher x should induce increasing difference.

Similarly, if f has good properties, there is also a useful characterization of supermodularity for differentiable functions.

Lemma 2. *If $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable, then f is supermodular in x if and only if $\frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$ for $i, j = 1, ..., nandi \neq j$.*

Note that supermodularity is a cardinal property, as it is not preserved under monotone transformation.

Example 5. *Let $X = \{0,1\}^2$, then consider function $f(X)$:*

$$f(1,1) = 3, f(1,0) = f(0,1) = 1, f(0,0) = 0$$

It satisfies supermpdularity in X. However, if we consider $\sqrt{f}$, it is not supermodular. Consider $(1,0)$ and $(0,1)$:

$$f(1,1) + f(0,0) = \sqrt{3} < f(1,0) + f(0,1) = 2$$

Theorem 3 (Topkis 1978)**.** *Suppose $X \subseteq \mathbb{R}^n$ is a lattice and $(T, \geq)$ is a partially ordered set and function $f : X \times T \to \mathbb{R}$. If f is supermodular in x and has increasing differences in $(x;t)$, then $\arg\max_{x \in X} f(x,t)$ is monotone nondecreasing in t (in the strong set order).*

Note that f is supermodular in x and has increasing differences in $(x;t)$ are implied by supermodularity in (x,t).

Proof. Let $X^*(t) = \arg\max_{x \in X} f(x,t)$. Fix any $t' \geq t$, $x \in X^*(t)$ and $x' \in X^*(t')$. We want to show, according to the definition of strong set order, $x \wedge x' \in X^*(t)$.

Note that

$$\begin{aligned} 0 &\leq f(x,t) - f(x \wedge x', t) \quad (x \in X^*(t)) \\ &\leq f(x \vee x', t) - f(x', t) \quad (SM) \\ &\leq f(x \vee x', t) - f(x', t') \quad (ID) \\ &\leq 0 \quad (x' \in X^*(t')) \end{aligned}$$

Thus, $f(x,t) = f(x \wedge x', t)$ and $f(x \vee x', t') = f(x', t')$, which implies $x \wedge x' \in X^*(t)$ and $x \vee x' \in X^*(t')$. $\square$

4 Monotone Comparative Statics: Ordinal

In the previous section, we derive the sufficient condition for MSC. Are ID and supermodular also necessary? Moreover, as we all know, utility functions are only identified up to a monotone transformations, shall we also have some ordinal condition relaxes the cardinal property of increasing differences?

4.1 One-dimensional case

We consider $X \subseteq \mathbb{R}$ and $T \subseteq \mathbb{R}$ first.

Definition 8 (Single crossing property)**.** Suppose $X \subseteq \mathbb{R}^n$ is a lattice and $(T, \geq)$ is a partially ordered set. A function $f : X \times T \to \mathbb{R}$ satisfies the single crossing property in $(x;t)$ if for all $x' > x$ and $t' > t$,

$$f(x', t) \geq f(x,t) \Rightarrow f(x', t') \geq f(x, t')$$

and

$$f(x', t) > f(x,t) \Rightarrow f(x', t') > f(x, t')$$

In the class, we have introduced strict single crossing property.

Definition 9 (Strict single crossing property)**.** Suppose $X \subseteq \mathbb{R}^n$ is a lattice and $(T, \geq)$ is a partially ordered set. A function $f : X \times T \to \mathbb{R}$ satisfies the single crossing property in $(x;t)$ if for all $x' > x$ and $t' > t$,

$$f(x',t) \geq f(x,t) \Rightarrow f(x',t') > f(x,t')$$

(1) The single crossing property is named after the similar property of a single-variable function. Fix any $x' > x$ and define a function $g : T \to R$ by $g(t) = f(x',t) - f(x,t)$. The single crossing property implies that g crosses zero once from below.

(2) Equivalently, for fixed $x' > x$, the function $f(x',\cdot)$ crosses the function $f(x,\cdot)$ once from below. Keep in mind that to check for the single crossing property, we want to look for single crossing of the function as we increase t while holding fixed the pair of values x and x'.

The following lemma implies that SC is a weaker (less restrictive) condition that ID.

Lemma 3. *If f has increasing differences in $(x;t)$, then it has the single crossing property in $(x;t)$. That is, ID implies SC.*

Figure.

Theorem 4 (Milgrom and Shannon 1994)**.** *Suppose $X \subseteq \mathbb{R}^n$ is a lattice and $(T, \geq)$ is a partially ordered set and function $f : X \times T \to \mathbb{R}$. If f satisfies single crossing property in $(x;t)$, then* $\arg\max_{x \in X} f(x,t)$ *is monotone nondecreasing in t (in the strong set order).*

Proof. Let $X^*(t) = \arg\max_{x \in X} f(x,t)$. Fix any $t' \geq t$, $x \in X^*(t)$ and $x' \in X^*(t')$. We need to show if $x > x'$ then $x' \in X^*(t)$ and $x \in X^*(t')$.

First,

$$\begin{aligned} x \in X^*(t) &\Rightarrow f(x,t) \geq f(x',t) \\ &\Rightarrow f(x,t') \geq f(x',t') \quad (SC) \\ &\Rightarrow x \in X^*(t') \end{aligned}$$

Next, prove $x' \in X^*(t)$ by contradiction:

$$\begin{aligned} x \notin X^*(t) &\Rightarrow f(x,t) > f(x',t) \\ &\Rightarrow f(x,t') > f(x',t') \quad (SC) \\ &\Rightarrow x' \notin X^*(t') \end{aligned}$$

a contradiction. Thus, we must have $x' \in X^*(t)$. □

One may wonder whether there is a condition even weaker than the single crossing property that can be used to obtain monotone comparative statics. It turns out that the answer depends on the flexibility that we have in specifying the constraint set X.

(1) If the constraint set X is fixed, then a condition like single crossing may not be necessary for solutions to be nondecreasing in the parameter t.

(2) If we require monotonicity of the solution set in t for all possible constraint sets $S \subseteq X$, then the single crossing property becomes a necessary condition.

Theorem 5. *Suppose $X \subseteq \mathbb{R}^n$ is a lattice and $(T, \geq)$ is a partially ordered set and function $f : X \times T \to \mathbb{R}$. If* $\arg\max_{x \in S} f(x,t)$ *is monotone nondecreasing in t (in the strong set order) for each $S \subseteq X$ of the form $S = \{\overline{x}, \overline{x'}\}$, then f satisfies the single crossing property in $(x;t)$.*

Thus, combined the above two theorems, we get see that the single crossing property is the weakest condition that can ensure monotone comparative statics on every possible constraint set $S \subseteq X$.

Corollary 1. *Suppose $X \subseteq \mathbb{R}^n$ is a lattice and $(T, \geq)$ is a partially ordered set and function $f : X \times T \to \mathbb{R}$. If $\arg\max_{x \in S} f(x,t)$ is monotone nondecreasing in t (in the strong set order) for each $S \subseteq X$ if and only if has the single crossing property in $(x;t)$.*

In some cases, ID is also a necessary condition because ID is equivalent to SC. For example, if the objective function has the form $f(x,t) - px$.

Theorem 6. *Suppose $X \subseteq \mathbb{R}^n$ is a lattice and $(T, \geq)$ is a partially ordered set and function $f : X \times T \to \mathbb{R}$. Then $f(x,t) - px$ has the single crossing property in $(x;t)$ for all $p \in \mathbb{R}$ if and only if f has increasing differences in $(x;t)$.*

4.2 Multidimensional case

In this subsection, we start to consider the case $X \subseteq \mathbb{R}^n$ and $T \subseteq \mathbb{R}^m$.

The definition of single crossing is the same. For supermodularity, we can relax it to be quasisupermodularity

Definition 10 (Quasisupermodular)**.** Suppose $X \subseteq \mathbb{R}^n$ is a lattice and $(T, \geq)$ is a partially ordered set. A function $f : X \times T \to \mathbb{R}$ is quasisupermodular in x if for all $x, x' \in X$ and $t \in T$,

$$f(x,t) \geq f(x \wedge x', t) \Rightarrow f(x\vee', t) \geq f(x', t)$$

and

$$f(x,t) > f(x \wedge x', t) \Rightarrow f(x\vee', t) > f(x', t)$$

Theorem 7. *Suppose $X \subseteq \mathbb{R}^n$ is a lattice and $(T, \geq)$ is a partially ordered set and function $f : X \times T \to \mathbb{R}$. If f is quasisupermodular in x and satisfies the single crossing property in $(x;t)$, then $\arg\max_{x \in X} f(x,t)$ is monotone nondecreasing in t (in the strong set order).*

Proof. Let $X^*(t) = \arg\max_{x \in X} f(x,t)$. Fix any $t' \geq t$, $x \in X^*(t)$ and $x' \in X^*(t')$. Note that

$$\begin{aligned} x \in X^*(t) &\Rightarrow f(x,t) \geq f(x \wedge x', t) \\ &\Rightarrow f(x \vee x', t) \geq f(x', t) \quad (QSM) \\ &\Rightarrow f(x \vee x', t') \geq f(x', t') \quad (SC) \\ &\Rightarrow x \vee x' \in X^*(t') \end{aligned}$$

Next, we prove that $x \wedge x' \in X^*(t)$ by contradiction:

$$\begin{aligned} x \wedge x' \in X^*(t) &\Rightarrow f(x,t) > f(x \wedge x', t) \\ &\Rightarrow f(x \vee x', t) > f(x', t) \quad (QSM) \\ &\Rightarrow f(x \vee x', t') > f(x', t') \quad (SC) \\ &\Rightarrow x' \in X^*(t') \end{aligned}$$

a contradiction to the assumption that $x' \in X^{(t')}$. Therefore, it must be the case that $x \wedge x' \in X^*(t)$. $\square$

So far, we have only consider the objective function containing parameter t. We can also alter the constraint set.

Theorem 8. *Suppose $X \subseteq \mathbb{R}^n$ is a lattice and $(T, \geq)$ is a partially ordered set and function $f : X \times T \to \mathbb{R}$. If f is quasisupermodular in x and satisfies the single crossing property in $(x; t)$, then $\arg\max_{x \in S} f(x,t)$ is monotone nondecreasing in t and S (in the strong set order), that is for $t' \geq t$ and $S' \geq S$,*

$$\arg\max_{x \in S'} f(x,t') \geq \arg\max_{x \in S} f(x,t).$$

In particular, suppose we have a constraint set $S(t)$ that is also parameterized by t, and suppose this set is monotone nondecreasing in t in the strong set order. Then the above theorem shows that under single crossing and quasisupermodularity, the solution set is nondecreasing in t, that is, for all $t' \geq t$,

$$\arg\max_{x \in S(t')} f(x,t') \geq \arg\max_{x \in S(t)} f(x,t).$$

The necessity of single crossing property and quasisupermodular can also be established. Therefore, we get the next general result.

Theorem 9 (Milgrom and Shannon 1994)**.** *Suppose $X \subseteq \mathbb{R}^n$ is a lattice and $(T, \geq)$ is a partially ordered set and function $f : X \times T \to \mathbb{R}$. Then $\arg\max_{x \in S} f(x,t)$ is monotone nondecreasing in t and S (in the strong set order) if and only if f is quasisupermodular in x and has the single crossing property in $(x; t)$.*

5 Monotone Comparative Statics: Greatest and Least Solutions

When $X \subseteq \mathbb{R}$, the existence of a largest and smallest maximizer of an objective function f can be ensured by imposing standard topological assumptions, such as compactness of X and continuity of f. When in the multidimensional case, we need lattice to define greatest and least maximizers.

Lemma 4. *Suppose $X \subseteq \mathbb{R}^n$ is a lattice and $(T, \geq)$ is a partially ordered set and function $f : X \times T \to \mathbb{R}$. If f is quasisupermodular in x, then the solution set $X^*(t) = \arg\max_{x \in X} f(x,t)$ is a sublattice of X for each t.*

Proof. Similar to the proof of Theorem 7. □

Definition 11 (Complete lattice)**.** A lattice $(T, \geq)$ is said to be complete if for every $S \subseteq X$, a greatest lower bound $\inf(S)$ and a least upper bound $\sup(S)$ exist in X, where $\inf(\emptyset) = \sup(X)$ and $\sup(\emptyset) = \inf(X)$.

Lemma 5. *Suppose $X \subseteq \mathbb{R}^n$ is a nonempty lattice and is compact. Then, X has a greatest and least element, that is, there exist $\underline{x}, \overline{x}$ such that $\underline{x} \leq x \leq \overline{x}$ for all $x \in X$.*

Proof. We will prove the existence of a greatest element $\overline{x} \in X$. Since X is compact, the set $\arg\max_{x \in X} x_i$ is nonempty (the set of maximizers of the continuous function $g(x) = x_i$ is nonempty). For each dimension $i \in \{1, 2, ..., n\}$, fix $\hat{x}^i \in \arg\max_{x \in X} x_i$. Let $\overline{x} = \hat{x}^1 \vee ... \vee \hat{x}^n$. That is, $\overline{x}$ is the coordinate-wise maximum of all of the vectors $\hat{x}^i$. And we can easily see $\overline{x} \in X$ because X is a lattice. □

Lemma 6. *Suppose $Y, Z \subseteq \mathbb{R}^n$, and suppose each of there sets greatest and least elements, $\underline{y}, \overline{y} \in Y$ and $\underline{z}, \overline{z} \in z$. If $Z \geq_s Y$, then $\overline{z} \geq \overline{y}$ and $\underline{z} \geq \underline{y}$.*

Theorem 10. *Suppose $X \subseteq \mathbb{R}^n$ is a lattice and $(T, \geq)$ is a partially ordered set and function $f : X \times T \to \mathbb{R}$. If f is continuous in x and quasisupermodular in x, then:*

(1)The solution set $X^*(t) = \arg\max_{x \in X} f(x,t)$ is nonempty and has greatest and least elements $\overline{x}(t)$ and $\underline{x}(t)$ for each $t \in T$.

(2)If f also has the single crossing property in $(x;t)$, then $t' \geq t$ implies $\overline{x}(t') \geq \overline{x}(t)$ and $\underline{x}(t') \geq \underline{x}(t)$.

Proof. Since X is compact and f is continuous in x, the usual arguments imply that $X^{(t)}$ is nonempty and compact for each t. Since f is quasisupermodular in x, Lemma 4 implies $X^*(t)$ is also a sublattice of X. Then since $X^*(t)$ is a lattice and is compact, Lemma 5 implies there exist $\underline{x}(t), \overline{x}(t) \in X^*(t)$ such that $\underline{x}(t) \leq x \leq \overline{x}(t)$ for all $x \in X^*(t)$.

If f also satisfies the single crossing property in $(x;t)$, then the set of maximizers $X^{(t)}$ is monotone nondecreasing in t in the strong set order by Theorem 9. That is, $t' \geq t$ implies $X^*(t') \geq_S X^*(t)$. By Lemma 6, we get $\overline{x}(t') \geq \overline{x}(t)$ and $\underline{x}(t') \geq \underline{x}(t)$. □

6 Supermodular Game

Each player in the game solves a maximization problem defined by the best response function. Therefore, we can easily extent the above MCS results to the game theory.

Definition 12 (Supermodular Game)**.** Let $N = \{1, 2, ..., n\}$ denote the set of players. A normal-form game $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$ is a supermodular game if for each $i \in N$:

(1) $S_i \subseteq \mathbb{R}^{m_i}$ is a lattice and is compact.

(2) $u_i(s_i, s_{-i})$ is continuous in s_i for fixed s_{-i}.

(3) $u_i(s_i, s_{-i})$ is quasisupermodular in s_i and satisfies the single crossing property in $(s_i; s_{-i})$.

The classic definition of a supermodular game due to Topkis (1979) assumes each player i has a utility function u_i that is supermodular in their own strategy s_i and has increasing differences in $(s_i; s_i)$. But you can see here, we use a more genenral ordinal condition.

Define the best-response correspondence $B_i : S_{-i} \to S_i$ for player i by

$$B_i(s_{-i}) = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$$

Theorem 11. *Suppose $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$ is a supermodular game. Then:*

(1) $B_i(s_{-i})$ is nonempty and has greatest and least elements $\overline{B_i}(s_{-i})$ and $\underline{B_i}(s_{-i})$.

(2) If $s'_{-i} \geq s_{-i}$, then $\overline{B_i}(s'_{-i}) \geq \overline{B_i}(s_{-i})$ and $\underline{B_i}(s'_{-i}) \geq \underline{B_i}(s_{-i})$.

Can you see what the effect of each conditions of supermodular game is in the above theorem?

In the case where players have single-dimensional strategy spaces—which is again what you should be thinking of for intuition—the result simply says that there are upper and lower bounds in $B_i(s_{-i})$ and these are nondecreasing in s_{-i}.

Example 6 (Supermodular Game)**.** *(Bertrand Competition) Suppose firms $1, 2, ..., I$ simultaneously choose prices $p \in [0, 1]$, and the demand is*

$$D_i(p_i, p_{-i}) = a_i - b_i p_i + \sum_{j \neq i} d_{ij} p_j$$

where $b_i, d_{ij} \geq 0$.
The marginal cost is c_i. Thus, the profit is

$$\pi(p_i, p_{-i}) = (p_i - c_i) D_i$$

(1) The strategy space is a lattice and compact.
(2) $u_i(p_i, p_{-i})$ is continuous in p_i for fixed p_{-i}.
(3) $\frac{\partial^2 \pi}{\partial p_i \partial p_j} = d_{ij} \forall j \neq i$. Thus satisfies ID (and thus SC).
Therefore, Bertrand Competition is a supermodular game.

6.1 Equilibrium Existence

Now, let us consider the equilibrium for the supermodular game. Define the best-response correspondence of all players $B : S \to S$ by

$$B(s) = \prod_{i \in N} B_i(s_{-i}) = B_1(s_{-1}) \times ... \times B_n(s_{-n}).$$

Recall that the fixed points of this correspondence are precisely the pure-strategy Nash equilibria of the game:

$$s \in B(s) \Longleftrightarrow \text{s is a NE}$$

For a given supermodular game, define functions $\overline{B} : S \to S$ and $\underline{B} : S \to S$ by

$$\overline{B}(s) = (\overline{B_i}(s_{-i}))_{i \in N} = (\overline{B_1}(s_{-1}), ..., \overline{B_n}(s_{-n}))$$
$$\underline{B}(s) = (\underline{B_i}(s_{-i}))_{i \in N} = (\underline{B_1}(s_{-1}), ..., \underline{B_n}(s_{-n}))$$

and note that

$$s = \overline{B}(s) \Rightarrow \text{s is a NE}$$
$$s = \underline{B}(s) \Rightarrow \text{s is a NE}$$

Theorem 12 (Tarski fixed point theorem)**.** *Suppose $X \subseteq \mathbb{R}^m$ is a nonempty lattice that is compact, and suppose function $f : X \to X$ is nondecreasing. Then f has a fixed point. Moreover, $\overline{x} = \sup\{s \in X : x \leq f(x)\}$ is the largest fixed point, and $\underline{x} = \inf\{x \in X : x \geq f(x)\}$ is the smallest fixed point.*

Proof. □

Theorem 13 (Topkis 1979)**.** *Suppose $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$ is a supermodular game. Then a pure-strategy Nash equilibrium exists. Moreover, $\overline{s} = \sup\{s \in S : s \leq \overline{B}(s)\}$ is the largest Nash Equilibrium, and $\underline{s} = \inf\{s \in S : s \geq \overline{B}(s)\}$ is the smallest Nash equilibrium.*

Proof. By theorem 11, $\overline{B}(s)$ is nondecreasing in s. Therefore, $\overline{s}$ is a fixed point of $\overline{B}$ by Tarski fixed point theorem and hence a NE. To see that it is the largest Nash equilibrium, consider any other Nash equilibrium strategy profile $s \in S$. By definition, $s \in B(s)$ and therefore $s \leq \overline{B}(s)$. Thus, $s \in \{s' \in S : s' \leq \overline{B}(s')\}$, which implies $s \leq \overline{s}$. □

We can also extent the definition of supermodular game so that it dependes on paramter t. Similar to MCS, we can study the Comparative Statics of NE.

Definition 13 (Parameterized supermodular game)**.** A parameterized supermodular game $(N, (S_i)_{i\in N}, (u_i)_{i\in N}, T)$ is a family of supermodular games with payoff functions that are parameterized by t in some partially ordered set T, such that for each $i \in N$:

(1) $S_i \subseteq \mathbb{R}^{m_i}$ is a lattice and is compact.

(2) $u_i(s_i, s_{-i}, t)$ is continuous in s_i for fixed s_{-i} and t.

(3) $u_i(s_i, s_{-i})$ is quasisupermodular in s_i and satisfies the single crossing property in $(s_i; s_{-i}, t)$.

Theorem 14. *Suppose $(N, (S_i)_{i\in N}, (u_i)_{i\in N}, T)$ is a parameterized supermodular game, and let $\overline{s}(t)$ and $\underline{s}(t)$ denote the largest and smallest Nash equilibria for each $t \in T$. Then these equilibria are nondecreasing in t.*

6.2 Iterated Strict Dominance and Rationalizability

Lemma 7. *Suppose $(N, (S_i)_{i\in N}, (u_i)_{i\in N})$ is a supermodular game. Let $\overline{z}, \underline{z} \in S$ be the smallest and largest strategy profiles. If $s_i \ngeq \underline{B_i}(\underline{z}_{-i})$ or $s_i \nleq \overline{B_i}(\overline{z}_{-i})$, then s_i is strictly dominated. Thus the profiles of undominated strategies for each player are contained in $[\underline{B}(\underline{z}), \overline{B}(\overline{z})] = \{s \in S : \underline{B}(\underline{z}) \leq s \leq \overline{B}(\overline{z})\}$*

Proof. 1, special case of one-dimensional strategy spaces $S_i \subseteq \mathbb{R}$.

Since $\underline{B}_i(\underline{z}_{-i})$ is the least best response of player i to the smallest profile $\underline{z}_{-i}$ of other players' strategy, $s_i < \underline{B}_i(\underline{z}_{-i})$, we have

$$u_i(\underline{B}_i(\underline{z}_{-i}), \underline{z}_{-i}) > u_i(s_i, \underline{z}_{-i})$$

Then, by single crossing of u_i in $(s_i; s_{-i})$, for any strategy profile $s_{-i} \geq \underline{z}_{-i}$, we have that

$$u_i(\underline{B}_i(\underline{z}_{-i}), s_{-i}) > u_i(s_i, s_{-i})$$

Thus, s_i is strictly dominated by $\underline{B_i}(\underline{z}_{-i})$.

2, multidimensional case: the problem lies in that $s_i \ngeq \underline{B_i}(\underline{z}_{-i})$ does not necessarily imply $s_i < \underline{B_i}(\underline{z}_{-i})$(since s_i could be lower in one dimension but higher in others).

Fix any $s_i \ngeq \underline{B_i}(\underline{z}_{-i})$, then at least in one coordinate, s_i is strictly lower than $\underline{B_i}(\underline{z}_{-i})$.

$$\begin{aligned} s_i \wedge \underline{B_i}(\underline{z}_{-i}) < \underline{B_i}(\underline{z}_{-i}) &\Rightarrow u_i(\underline{B}_i(\underline{z}_{-i}), \underline{z}_{-i}) > u_i(s_i \wedge \underline{B}_i(\underline{z}_{-i}), \underline{z}_{-i}) \\ &\Rightarrow u_i({}_i \vee \underline{B}_i(\underline{z}_{-i}), \underline{z}_{-i}) > u_i(s_i \wedge \underline{B}_i(s_i), \underline{z}_{-i}) \quad (QSM) \\ &\Rightarrow u_i({}_i \vee \underline{B}_i(\underline{z}_{-i}), \underline{s}_{-i}) > u_i(s_i \wedge \underline{B}_i(s_i), \underline{s}_{-i}) \quad (SC) \end{aligned}$$

□

According the above lemma, we can apply iterated elimination of strictly dominated strategies. In the second round, we get $\underline{B}^2(\underline{z}) = \underline{B}(\underline{B}(\underline{z}))$ and $\overline{B}^2(\overline{z}) = \overline{B}(\overline{B}(\overline{z}))$. Keep going! You will find in $S^k \subseteq [\underline{B}^k(\underline{z}), \overline{B}^k(\overline{z})]$. A natural question is whether they will converge in the limit and what is the limit? They will converge to precisely the largest and smallest Nash equilibria if the game is continuous.

Definition 14 (continuous supermodular game)**.** A supermodular game $(N, (S_i)_{i\in N}, (u_i)_{i\in N})$ is a continuous supermodular game if each u_i is continous in (s_i, s_{-i}).

Theorem 15 (Milgrom and Roberts 1990)**.** *Suppose $(N, (S_i)_{i\in N}, (u_i)_{i\in N})$ is a continuous supermodular game. Then the set of serially undominated strategy profiles (those that survive iterated elimination of strictly dominated strategies) has largest and smallest elements $\overline{s}$ and $\underline{s}$. Moreover, both of these strategy profiles are Nash equilibria.*

Proof. First to note $\underline{z}^k$ is an increasing sequence and $\overline{z}^k$ is a decreasing sequence.

Then, since they are bounded and monotone, by monotone convergence theorem, limit exist. We denote them by $\underline{s}$ and $\overline{s}$.

Now to show $\underline{s}$ and $\overline{s}$ are NE. Consider $\underline{s}$. We have

$$u_i(\underline{z}_i^{k+1}, \underline{z}_{-i}^k) \geq u_i(s_i, \underline{z}_{-i}^k)$$

Then, taking the limit $k \to \infty$ and by continuity, we obtain

$$u_i(\underline{s}_i, \underline{s}_{-i}) \geq u_i(s_i, \underline{s}_{-i})$$

for all $s_i \in S_i$.

Thus, $\underline{s}$ is a NE. An analogous argument shows that $\overline{s}$ is a Nash equilibrium. □

Corollary 2. *A continuous supermodular game with a unique Nash equilibrium is dominance solvable.*

Example 7 (Bertrand Game)**.** *Let us go ack to the example 6. For simplicity, we assume there are only two firms and the demand function is $D_i(p_i, p_j) = 1 - 2p_i + p_j$.*

We further assume zero marginal costs. Thus the profit function is

$$\pi_i(p_i, p_j) = p_i(1 - 2p_i + p_j)$$

Note that $\frac{\partial \pi_i}{\partial p_i} = 1 - 4p_i + p_j$. We find NE is $(\frac{1}{3}, \frac{1}{3})$. Let us apply iterated dominance (iterative removal of (strictly) dominated strategies).

In the first round, $S_i^0 = [0, 1]$. According to the lemma 7, the smallest strategy of player j is 0.

Then, since $\frac{\partial \pi_i}{\partial p_i} = 1 - 4p_i$ when $s_j = 0$, any $p_i < \frac{1}{4}$ is dominated ($p_i^* = \frac{1+p_j}{4}$ is increasing in p_j). Similarly, the largest strategy for player j is 1. Then $\frac{\partial \pi_i}{\partial p_i} = 2 - 4p_i$, any $p_i > \frac{1}{2}$ is dominated.

Therefore, we get our undominated set $S_i^1 = [\frac{1}{4}, \frac{1}{2}]$; by symmetry, $S_j^1 = [\frac{1}{4}, \frac{1}{2}]$.

Apply the above logic, in the round k,

we have $\underline{s}^k = \frac{1}{4} + \frac{\underline{s}^{k-1}}{4} = \frac{1}{4} + \frac{1}{16} + \frac{\underline{s}^{k-2}}{16} = ... = \frac{1}{4} + \frac{1}{16} + ... + \frac{1}{4^k} + \frac{\underline{s}^0}{4^k}$;

$\overline{s}^k = \frac{1}{4} + \frac{\overline{s}^{k-1}}{4} = \frac{1}{4} + \frac{1}{16} + \frac{\overline{s}^{k-2}}{16} = ... = \frac{1}{4} + \frac{1}{16} + ... + \frac{1}{4^k} + \frac{\overline{s}^0}{4^k}$

We can see $\lim_{k\to\infty} \underline{s}^k = \lim_{k\to\infty} \overline{s}^k = \frac{1}{3}$.