Question 1:

1.1

- After the partition, we can get two subarraies on each side of the pivit, and before the recursion, we first need to get the subarray we need, so we need to get the rank of the pivit (k-th element) and compare it with the i-th element we need, and if i is not equal to k, then we can go to the left or right subarray and do the recursion.

1.2

- In QuickSort, we have to do the recursive step on both the subarraies, select a pivit, put elements on either side of the pivit and so on , however in RandomSelect we only need to do the recursive step on one subarray, that also includes picking a pivit and put elements on either side.

1.3

- QuickSort:

$$x_i = \begin{cases} 1 & \text{if i is the wanted element} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$P(X_i) = \frac{1}{n} \tag{2}$$

$$T(n) = \sum_{i=0}^{n-1} (T(i) + T(n-i-1) + \Theta(n)) \tag{3}$$

- By Linearity of Expection

$$E(T(n)) = \sum_{i=0}^{n-1} E(T(i) + T(n-i-1) + \Theta(n)) \cdot P(X_i) \tag{4}$$

$$= \frac{1}{n} \sum_{i=0}^{n-1} (T(i) + T(n-i-1) + \Theta(n)) \tag{5}$$

$$= \frac{2}{n} \sum_{i=0}^{n-1} (T(i)) + \Theta(n) \tag{6}$$

- RandomSelect

$$x_i = \begin{cases} 1 & \text{if i is the wanted element} \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

$$P(X_i) = \frac{1}{n} \tag{8}$$

$$T(n) = \sum_{i=0}^{n-1} T(max(i, n-i-1) + \Theta(n)) \tag{9}$$

- By Linearity of Expection

$$E(T(n)) = \sum_{i=0}^{n-1} E(T(max(i, n-i-1) + \Theta(n)) \cdot P(X_i) \tag{10}$$

$$= \frac{1}{n} \sum_{i=0}^{n-1} T(max(i, n-i-1) + \Theta(n) \tag{11}$$

$$\leq \frac{2}{n} \sum_{\lfloor i=\frac{n}{2} \rfloor}^{n} (T(i)) + \Theta(n) \tag{12}$$

1.4

- Base case: $E(T(1)) = 1 \leq a$ as long as $a \geq 1$
- Assume there exist $k < n$, $a > 0$ such that $E(T(k)) \leq ak$
- Substitution:

$$E(T(k)) = \frac{2}{k} \sum_{i=0}^{k-1} (T(i) + \Theta(k)) \tag{13}$$

$$\leq \frac{2}{k} a \sum_{i=o}^{k-1} (i + dk) \tag{14}$$

$$= \frac{2a}{k} \cdot \frac{k(k-1)}{2} + dk \tag{15}$$

$$= ak - a + dk \tag{16}$$

$$= ak - (a - dk) \tag{17}$$

- as long as $a - dk > 0$ this stands, however a is constant and cannot be bigger than dk so the assumtion fails

Question 2:

2.1

- Let X = number of battles among k vikings (which also means that X islands have 2 or more vikings)

$$X_i = \begin{cases} 1 & \text{if i-th island has 2 or more vikings} \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

$$E(X_i) = 1 - (\frac{n-1}{n})^k - k\frac{1}{n} \cdot (\frac{n-1}{n})^{k-1} \tag{19}$$

$$= 1 - \frac{(n+k-1)(n-1)^{k-1}}{n^k} \tag{20}$$

$$E(X) = E(\sum_{i=1}^{n} X_{ij}) \tag{21}$$

- Because of linearity of expection

$$= \sum_{i=1}^{n} E(X_i) \tag{22}$$

$$= n - \frac{(n+k-1)(n-1)^{k-1}}{n^{k-1}} \tag{23}$$

- So we expect there are $(n - \frac{(n+k-1)(n-1)^{k-1}}{n^{k-1}})$ battles

2.2

- When there is only 1 island, we know there will be 1 battle so it is the same with $E(X)$, and when there is only 1 viking we know there will be 0 battle and it also suits the $E(X)$, when there are $k = 400$ vikings and $n = 100$ islands:

$$E(X) = 90.952 \tag{24}$$

- There will be over 90 battles