Portfolio Optimization and Performance Evaluation Using the Markowitz Model: An Analysis of Returns, Risk, and Sharpe Ratios

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1 Introduction

The Markowitz model (Markowitz 1959) serves as the foundation of Modern Portfolio Theory (MPT). Also referred to as the mean-variance model, it is based on the expected returns (mean) and the standard deviation (a proxy for risk) of asset returns. MPT demonstrates that an investor can construct a portfolio that minimizes risk for a given level of expected return (Elton & Gruber 1997). Diversifying across assets that do not move perfectly in tandem can reduce overall portfolio risk.

In this assignment, a portfolio optimization solver is developed using C++, including both the construction and backtesting components. The remainder of this paper is organized as follows: Section 2 presents the exploratory data analysis; Section 3 discusses the theoretical background and derivation; Section 4 outlines the software architecture and algorithmic design; Section 5 evaluates the out-of-sample performance[1]; and Section 6 concludes.

2 Exploratory Data Analysis

Exploratory Data Analysis (EDA) is conducted to gain familiarity with the dataset. The dataset comprises 83 columns and 700 rows, where each column represents an asset and each value denotes its historical return. The following summarizes key descriptive statistics:

- The average return across all assets over the entire period is 0.00248.
- Asset 78 has the highest average return, at 0.00853.
- Interestingly, Asset 78 also has the minimum return observed in the dataset, at -0.79823.
- Asset 67 exhibits the highest individual return in the dataset, at 0.80526.
- Asset 67 also has the highest volatility, with a standard deviation of 0.08872.
- Asset 16 shows the lowest volatility, with a standard deviation of 0.02524.

Figure 1 illustrates the distribution of returns, indicating that most returns are concentrated near zero. The data exhibits positive kurtosis. Figure 2 displays the time series of average returns across all 83 assets (i.e., at each time t, the average return $\bar{r}_t$ of all assets is computed). While there is no apparent trend over time, the figure reveals clear volatility clustering. Finally, Figure 3 presents the asset return correlation matrix, showing that most asset pairs are weakly correlated, with only a few displaying negative correlations.

3 Background and theoretical derivation

This section will provide a theoretical background serving as a basis for the portfolio optimization solver.

Define

- number of assets: N
- portfolio weights: $\mathbf{w} = (w_1, w_2, ...w_N)$

[1] Exploratory data analysis and performance evaluation are conducted using Python.

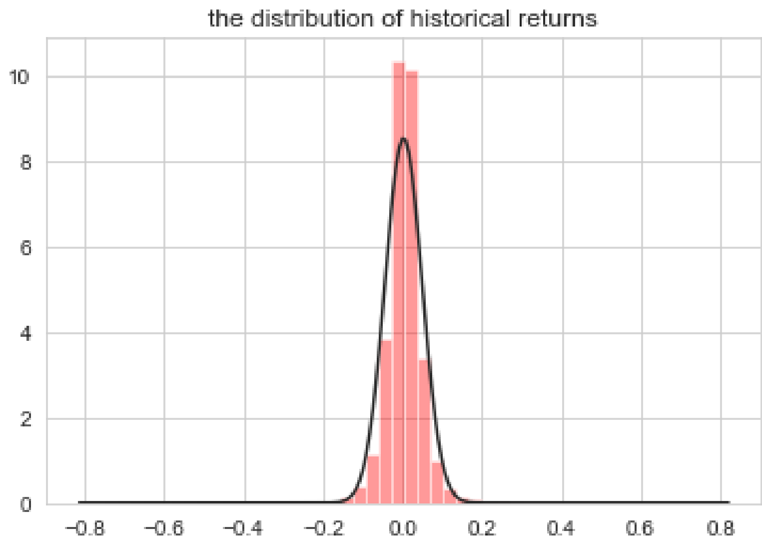

Figure 1: the distribution of historical returns

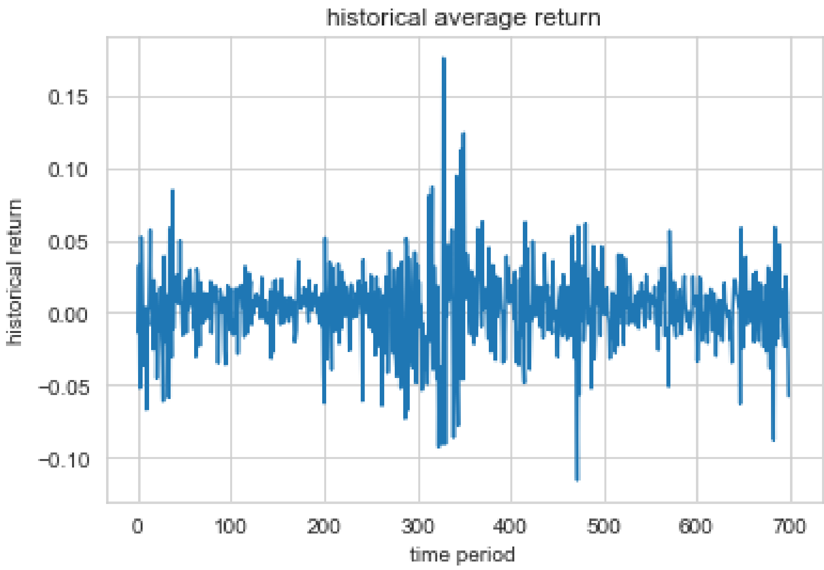

Figure 2: historical average returns

- expected asset returns: $\bar{\mathbf{r}}$=($\bar{r}_1$,$\bar{r}_2$,...$\bar{r}_N$)
- $\mathbf{e} = (1, 1, ...1)$
- $\mathbf{0} = (0, 0, ...0)$
- covariance matrix of asset returns:

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1N} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2N} \\ \vdots & \vdots & \dots & \vdots \\ \sigma_{N1} & \sigma_{N2} & \dots & \sigma_{NN} \end{pmatrix}$$

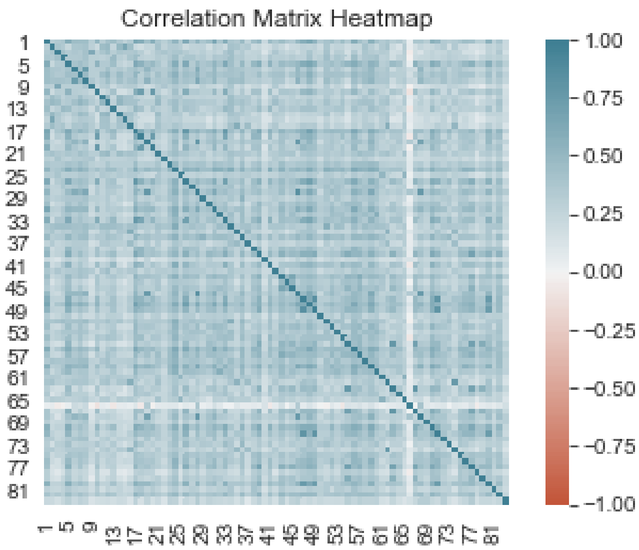

Figure 3: Correlation Matrix Heatmap

Then Markowitz problem becomes an optimization problem that minimize:[2]

$$\frac{1}{2}\mathbf{w}^T\Sigma\mathbf{w}$$

subject to:

$$\mathbf{w}^T\bar{\mathbf{r}} - \bar{r_p} = 0$$
$$\mathbf{w}^T\mathbf{e} - 1 = 0$$

To solve this problem, Lagrangian function can be used:

$$L(\mathbf{w}, \lambda, \mu) = \frac{1}{2}\mathbf{w}^T\Sigma\mathbf{w} - \lambda(\mathbf{w}^T\bar{\mathbf{r}} - \bar{r_p}) - \mu(\mathbf{w}^T\mathbf{e} - 1)$$

The optimality conditions become

$$\Sigma\mathbf{w} - \lambda\bar{\mathbf{r}} - \mu\mathbf{e} = \mathbf{0}$$
$$\mathbf{w}^T\bar{\mathbf{r}} - \bar{r_p} = 0$$
$$\mathbf{w}^T\mathbf{e} - 1 = 0$$

It can be written as:

$$\begin{pmatrix} \Sigma & \textbf{-}\bar{\mathbf{r}} & \textbf{-}\mathbf{e} \\ \textbf{-}\bar{\mathbf{r}}^T & 0 & 0 \\ \textbf{-}\mathbf{e}^T & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{w} \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ -\bar{r_p} \\ -1 \end{pmatrix}$$

If Σ has full rank and $\bar{\mathbf{r}}$ is not a multiple of $\mathbf{e}$, then:

$$\begin{pmatrix} \mathbf{w} \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \Sigma & \textbf{-}\bar{\mathbf{r}} & \textbf{-}\mathbf{e} \\ \textbf{-}\bar{\mathbf{r}}^T & 0 & 0 \\ \textbf{-}\mathbf{e}^T & 0 & 0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \mathbf{0} \\ -\bar{r_p} \\ -1 \end{pmatrix}$$

4 Software Structure and Implementation

The overall structure of the software is illustrated in Figure 4. Five main classes are developed as part of the implementation:

- **CSV** (defined in `read_data.h`): Responsible for importing data from CSV files.

[2]In this project short selling is allowed.

- **parameter_estimation** (defined in `parameter_estimation.h`): Provides functions for computing the sample mean and covariance matrix through the methods `para_mean` and `para_cov`.
- **constructMatrix** (defined in `constructMatrix.h`): Handles various matrix operations, including printing, merging, row swapping, multiplication, and rank computation.
- **adjoint** (defined in `inverse_adjoint_matrix.h`): Implements the inverse of a matrix using the adjugate matrix method. However, due to its inefficiency for high-dimensional matrices, this method is not utilized in the final implementation.
- **inverse_LUdecomposition** (defined in `inverse_LUdecomposition.h`): Computes the inverse of a matrix using LU decomposition, which is more efficient for large matrices and is the method adopted in the final program.

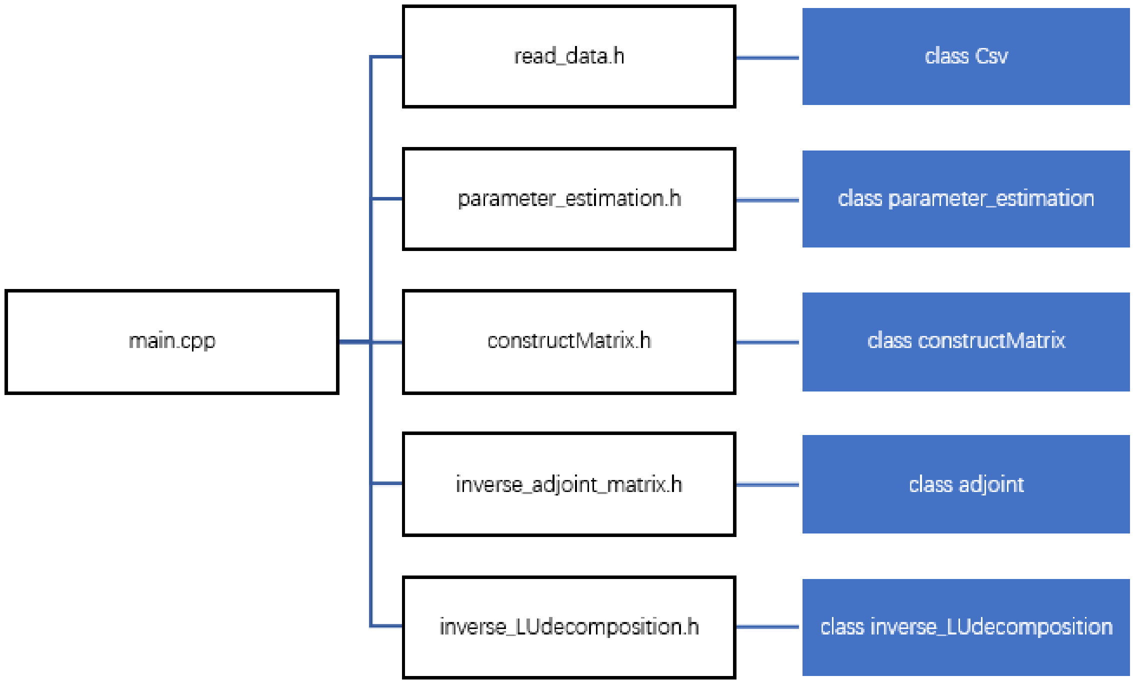

Figure 4: Software structure

The program is composed of three main parts: Parameter Estimation, Portfolio Optimization, Backtest. To be precise, it works as follows:

1. Import data.
2. Set the in-sample window on the first 100 time periods, and next 12 time periods as out-of-sample.
3. Set a range of target returns (from 0.0% to 10%). The target return vector has 20 elements: [0.005, 0.01, 0.015, ..., 0.095, 0.1].
4. For each target return $\bar{r_p}$, construct a portfolio and backtest:
   (a) Construct a $(N+2)\times 1$ matrix: $A = \begin{pmatrix} \mathbf{0} \\ -\bar{r}_p \\ -1 \end{pmatrix}$
   (b) In this in-sample window, estimate the mean and covariance matrix. The mean return for asset i can be calculated as:
   $$\bar{r_i} = \frac{1}{n}\sum_{k=1}^{n} r_{i,k}$$

, where $r_{i,k}$ is the return of asset i on day k, n is the window size. The (i,j)-th entry of the covariance matrix can be calculated as:

$$\Sigma_{i,j} = \frac{1}{n-1}\sum_{k=1}^{n}(r_{i,k} - \bar{r_i})(r_{j,k} - \bar{r_j})$$

(c) Construct a $(N+2)\times(N+2)$ matrix: $B = \begin{pmatrix} \Sigma & \mathbf{-\bar{r}} & \mathbf{-e} \\ \mathbf{-\bar{r}}^T & 0 & 0 \\ \mathbf{-e}^T & 0 & 0 \end{pmatrix}$, and compute the inverse : $\begin{pmatrix} \Sigma & \mathbf{-\bar{r}} & \mathbf{-e} \\ \mathbf{-\bar{r}}^T & 0 & 0 \\ \mathbf{-e}^T & 0 & 0 \end{pmatrix}^{-1}$ [3].

(d) Compute $B^{-1} \cdot A = \begin{pmatrix} \mathbf{w} \\ \lambda \\ \mu \end{pmatrix}$ and get the first N-2 elements as weights.

(e) Use the weights to adjust the positions on assets.

(f) Calculate its average returns $\mathbf{w}^T\mathbf{\bar{r}}$ and covariance $\mathbf{w}^T\Sigma\mathbf{w}$ in the out-of sample.

(g) Update the in-sample window and out-of-sample, with the inclusion of the previous 12 out-of-sample periods and the exclusion of the first 12 periods of the previous in-sample window.

(h) Repeat (b) - (g) until there is no out-of-sample left, in the assignment it should repeat 50 times.

5. Output the result including 20*50=1000 weighted average returns and covariance[4]. The final result is recorded in **result.csv**.

5 Evaluation

Figure 5 presents the return distributions of 1,000 portfolios constructed using the Markowitz model and the baseline portfolios (without optimization). The mean returns of the two groups are very similar, and a t-test yields a p-value of 0.318, indicating that the null hypothesis of equal means cannot be rejected. Interestingly, the volatility of the optimized portfolios is noticeably higher than that of the baseline, which suggests a potential direction for further investigation.

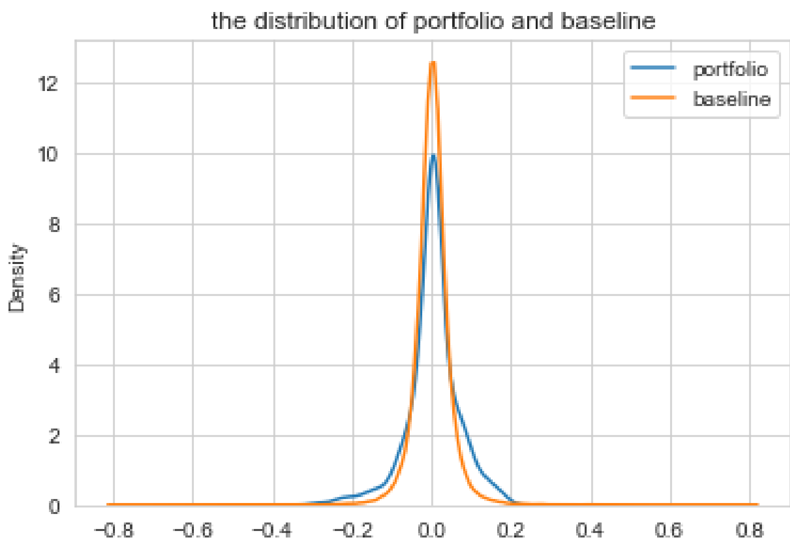

Figure 5: the distribution of portfolio and baseline

[3]the Σ has full rank

[4]For each target return, there is a portfolio constructed, which will adjust the weights on assets in different in-sample window. For each portfolio, there are 50 historical returns corresponding to 50 out-of-sample. So there are 20*50=1000 weighted average returns and covariance in the final result.

Figure 6 displays the return distributions of portfolios constructed using the Markowitz model under varying target return levels. For instance, the highest blue line represents the return distribution for the portfolio with a target return of 0.005. As the target return increases, the corresponding portfolio volatility also increases. This relationship is more clearly illustrated in Figure 7.

These findings align with the core principle of Modern Portfolio Theory, which posits a positive relationship between expected return and risk.

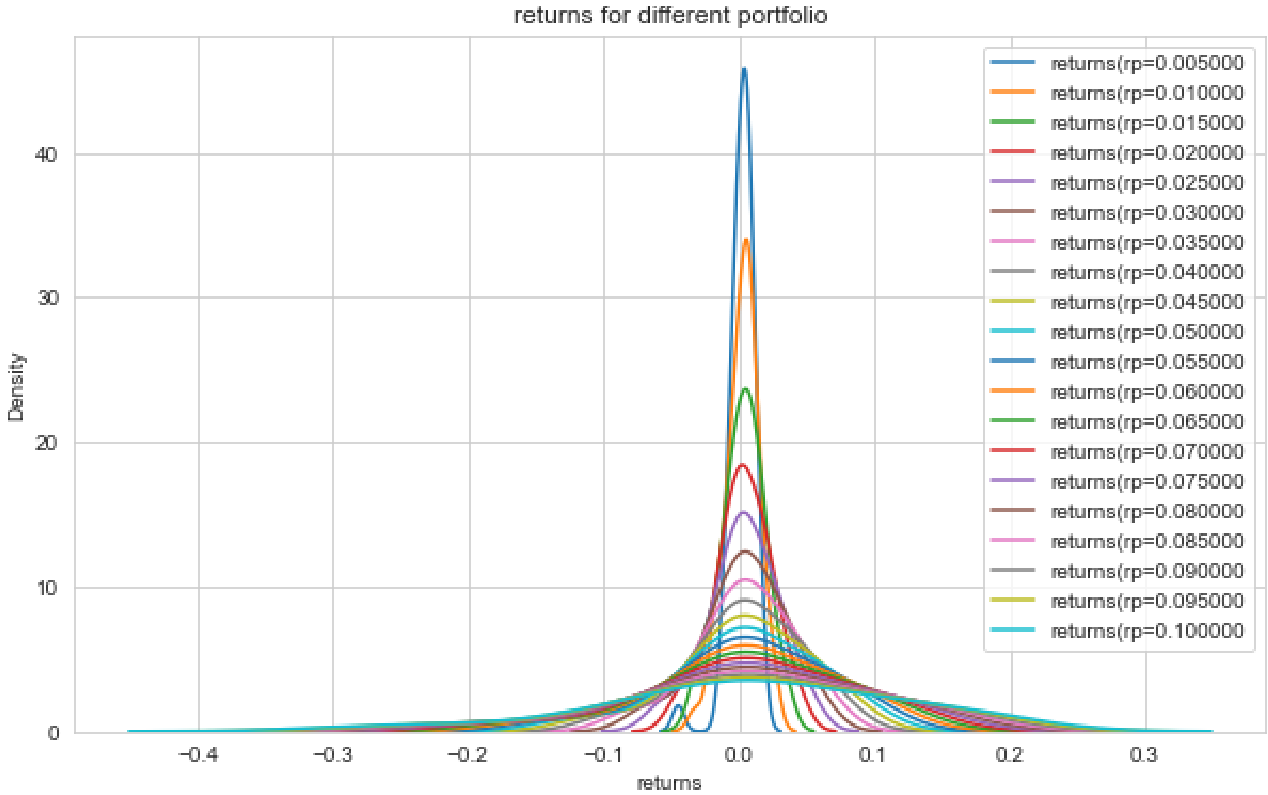

Figure 6: returns for different portfolio

Figure 8 illustrates the relationship between actual mean returns and the corresponding target (expected) returns. The actual mean return is computed by averaging the realized returns of portfolios at each specified target return level. Although the actual returns do not fully reach the expected levels, a strong positive relationship is evident. This relationship may be attributed to the momentum effect (Jegadeesh & Titman 1993), which suggests the persistence of existing market trends—an outcome that arises here because portfolio weights are determined based on historical returns, as discussed in Section 3.

Similarly, Figure 9 presents the relationship between average portfolio covariance and target returns. The average covariance is computed by averaging the covariance values of portfolios at each target return. By switching the axes in Figure 9, one can obtain the Efficient Frontier, as shown in Figure 10. Portfolios located in the orange-shaded region are considered suboptimal, as they exhibit higher levels of risk relative to their target returns.

The actual return, mean variance, and actual volatility are all positively correlated with expected returns, indicating a trade-off between risk exposure and potential return. The Sharpe ratio is used to assess the return of an investment relative to its risk (Sharpe 1994). This ratio represents the average return earned in excess of the risk-free rate per unit of volatility:

$$\text{Sharpe ratio} = \frac{R_p - R_f}{\sigma_p}$$

For each expected return, a portfolio is constructed over the period, and the Sharpe ratio for each portfolio is

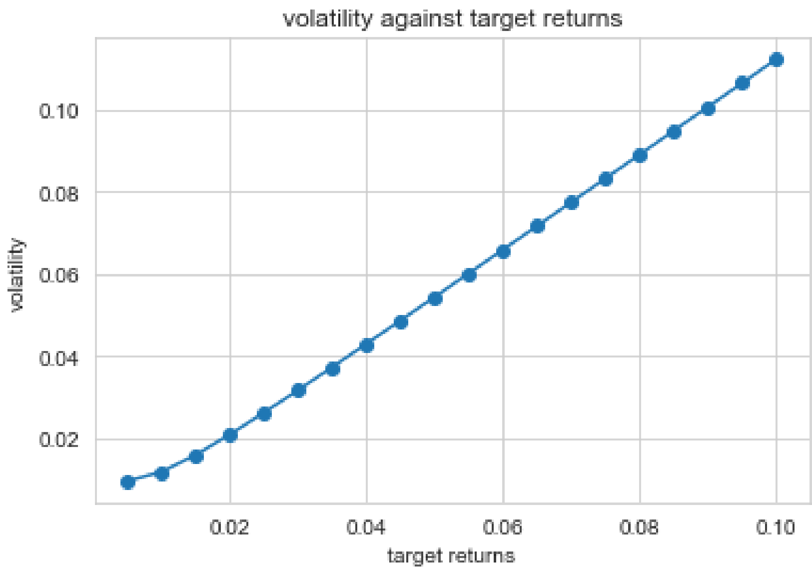

Figure 7: volatility against target returns

Figure 8: actual mean returns against target returns

calculated. Figure 11 shows the relationship between the Sharpe ratio and expected returns across different risk-free rates. As the target return increases, the Sharpe ratio for all strategies also increases. Figure 12 shows this relationship when the risk-free rate is $R_f = 0.00248$, which is the average return for all assets over the entire period.

It is not surprising that the Sharpe ratio increases with the target return, particularly when the Sharpe ratio is initially negative. This can be explained as follows: as the target return increases, the mean return also rises, making the numerator of the Sharpe ratio, $R_p - R_f$, less negative when target returns are low. At the same time, the denominator, σ_p, also increases with the target return. When the target return increases, both the negative numerator and the positive denominator increase, leading to a sharp rise in the Sharpe ratio.

The part of the curve above zero is particularly noteworthy. It is interesting to observe that all curves in Figure 11 and Figure 12 exhibit positive slopes in these regions. The increasing positive Sharpe ratio suggests that excess returns are driven by sound investment decisions rather than excessive risk-taking. In other words,

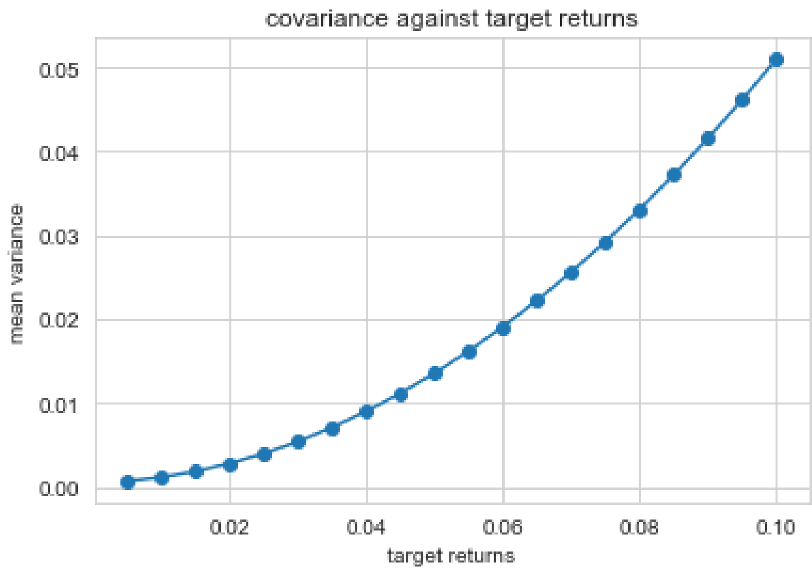

Figure 9: covariance against target returns

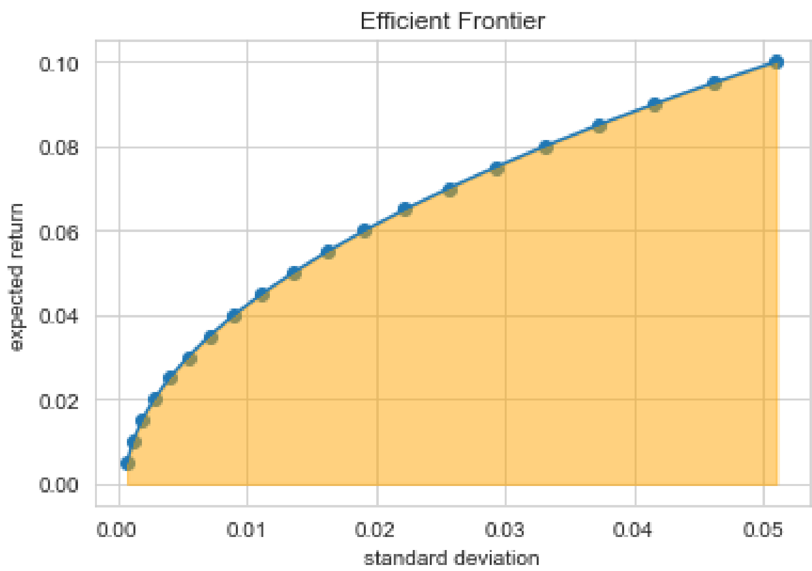

Figure 10: Efficient Frontier

improving target returns can enhance risk-adjusted performance[5].

6 Conclusion

Adjusting the target return parameter in the Markowitz model influences both the actual average returns and the covariance. While the actual returns may not fully match the expected returns, increasing the target return clearly leads to an improvement in actual returns, which can be attributed to the momentum effect. Additionally, higher target returns often result in a significant increase in covariance. These findings align with the principles of Modern Portfolio Theory. To further assess portfolio performance, Sharpe ratios are utilized, indicating that increasing the target return enhances risk-adjusted performance.

[5]However, it should be noted that the risk-free rates assumed in this project are typically lower than real-world rates. These lower values were chosen to generate positive Sharpe ratios, as negative Sharpe ratios do not provide meaningful insights.

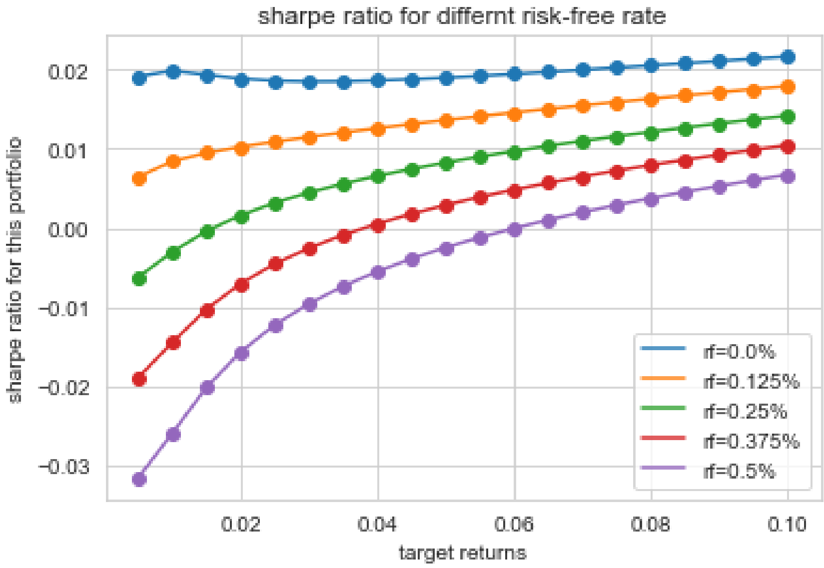

Figure 11: sharpe ratio for differnt risk-free rate

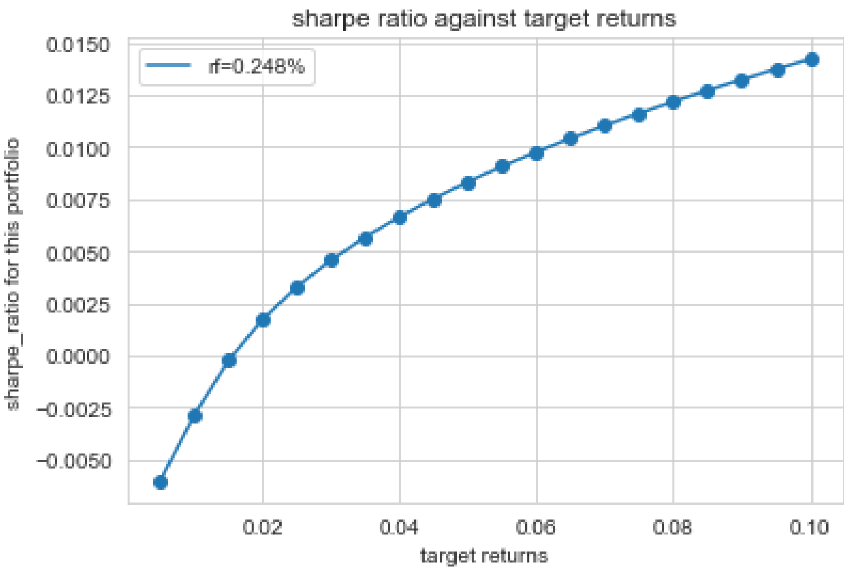

Figure 12: sharpe ratio against target returns

References

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