Coping with NP-completeness: Approximation Algorithms

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Advanced Algorithms and Complexity
Data Structures and Algorithms

Outline

1. Vertex cover

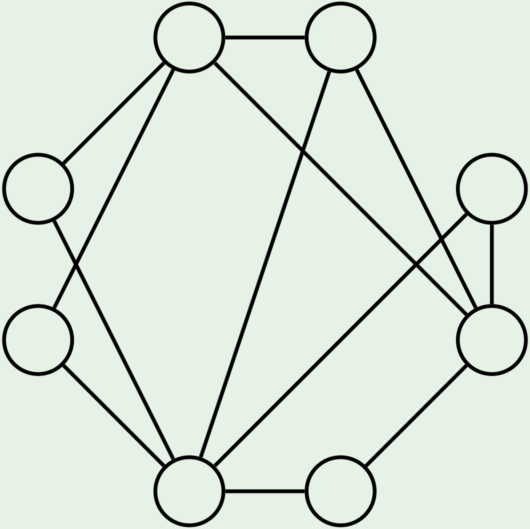
2. Traveling salesman
   - Metric TSP
   - Local search

Vertex cover (optimization version)

Input: A graph.

Output: A subset of vertices of minimum size that touches every edge.

Example



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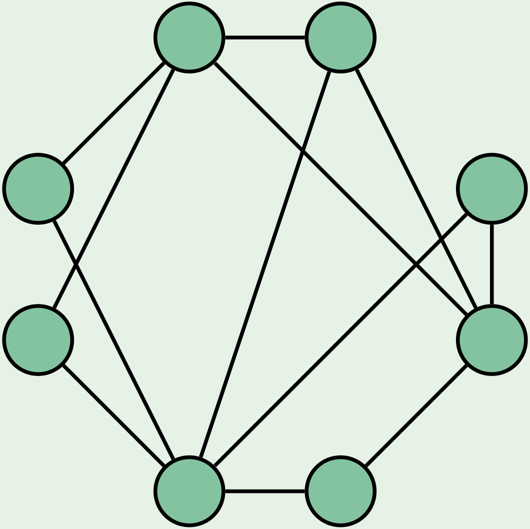

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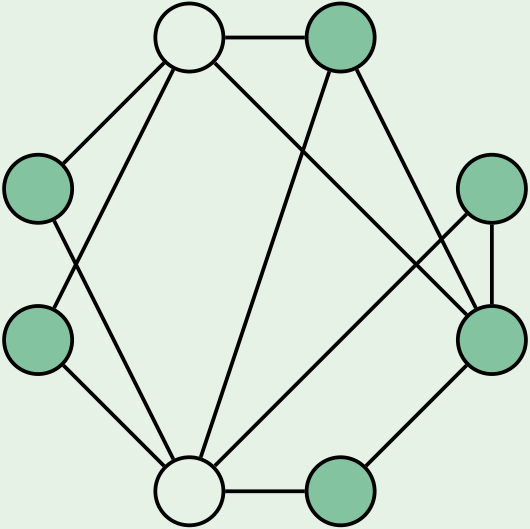

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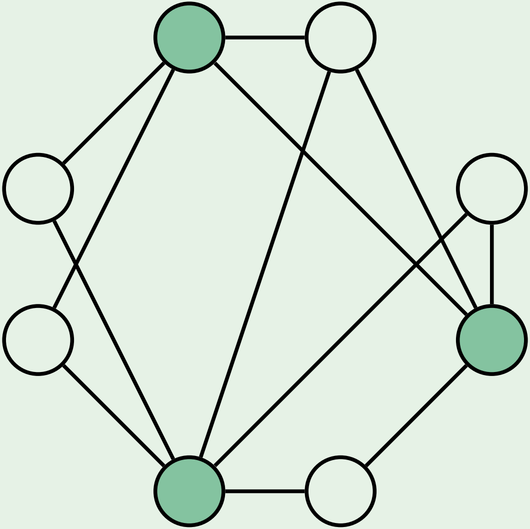
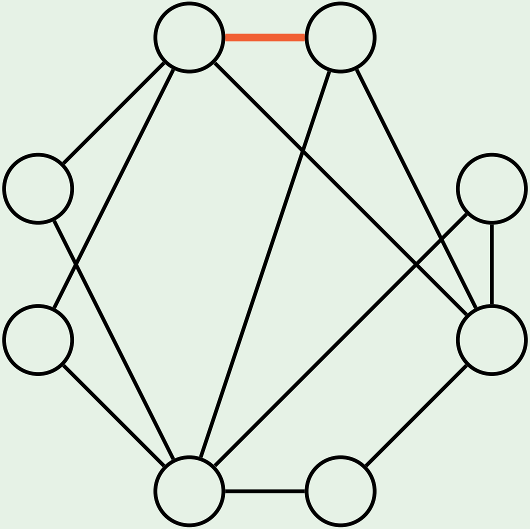


ApproxVertexCover($G(V, E)$)

```
C ← empty set
while E is not empty:
  {u, v} ← any edge from E
  add u, v to C
  remove from E all edges incident to u, v
return C
```

Example



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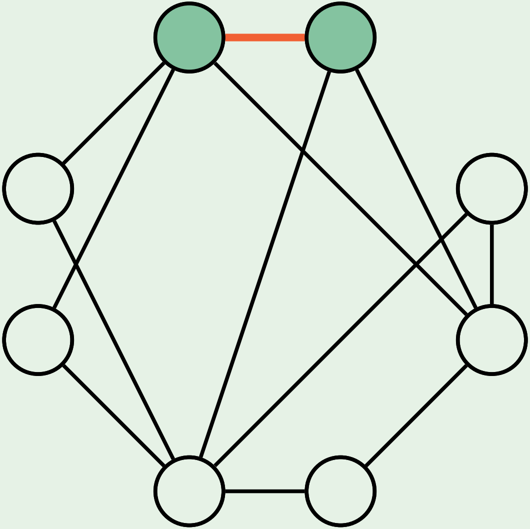

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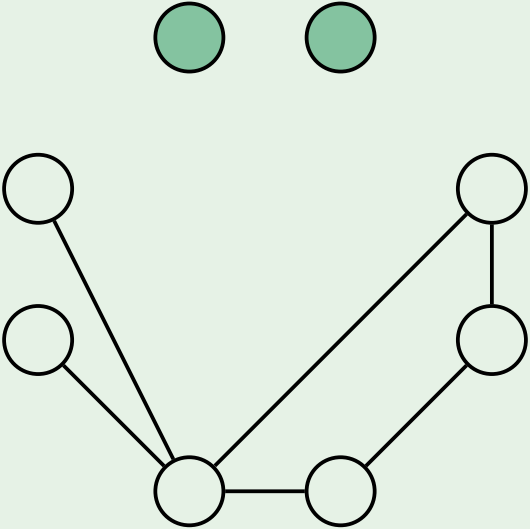

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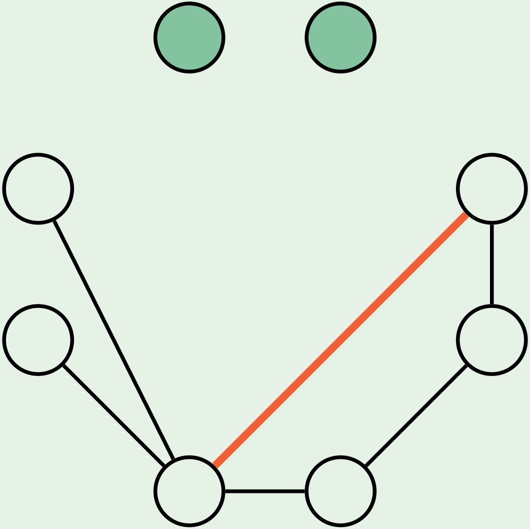

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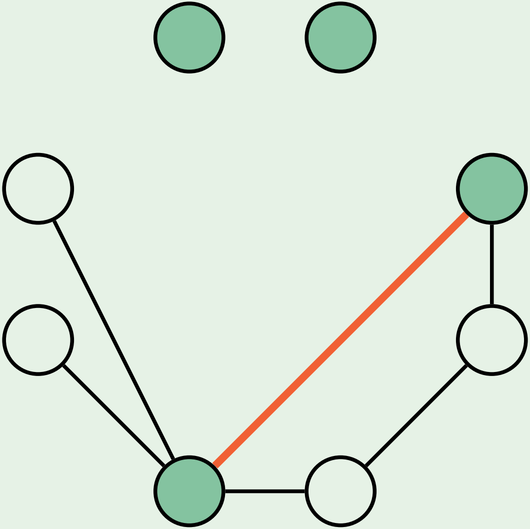

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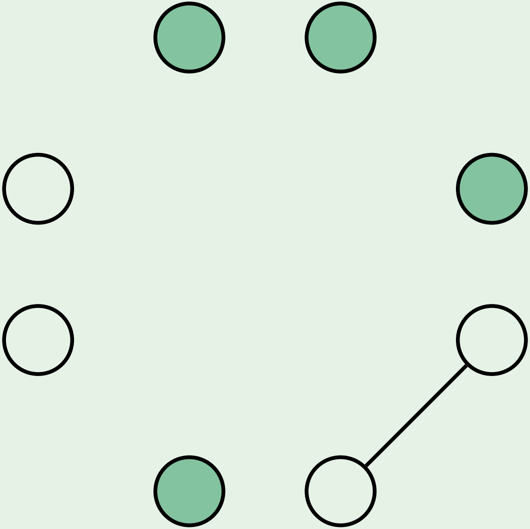

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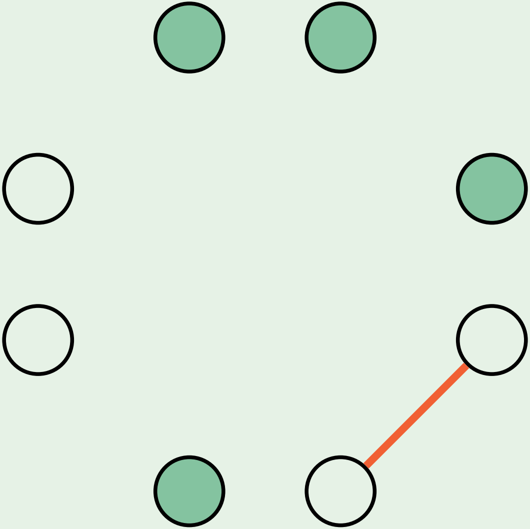

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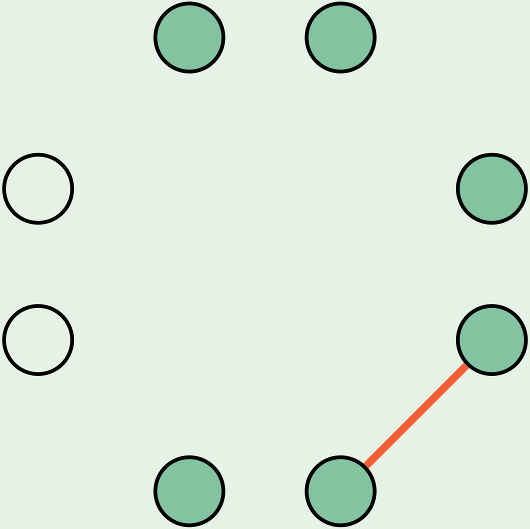

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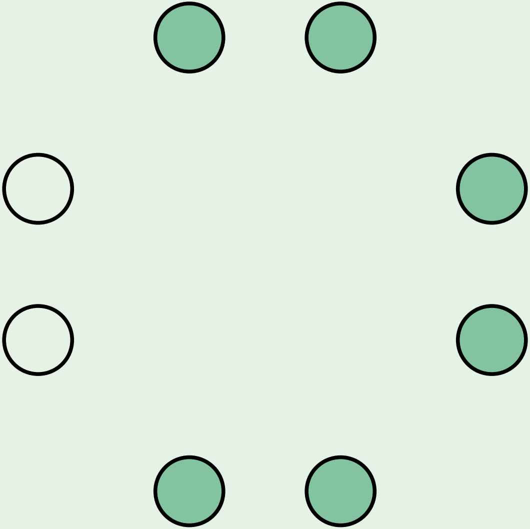

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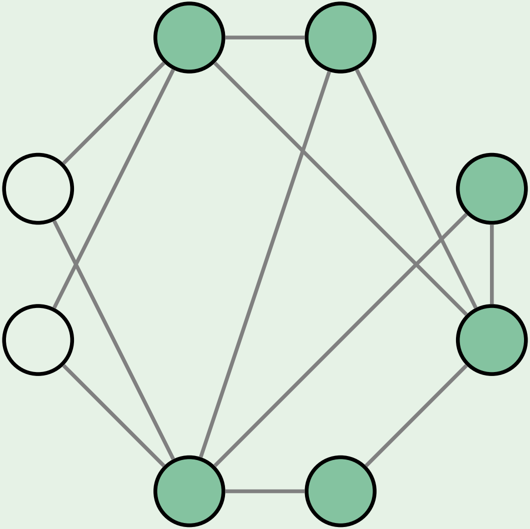


Lemma

The algorithm `ApproxVertexCover` is 2-approximate: it returns a vertex cover that is at most twice as large as an optimal one and runs in polynomial time.

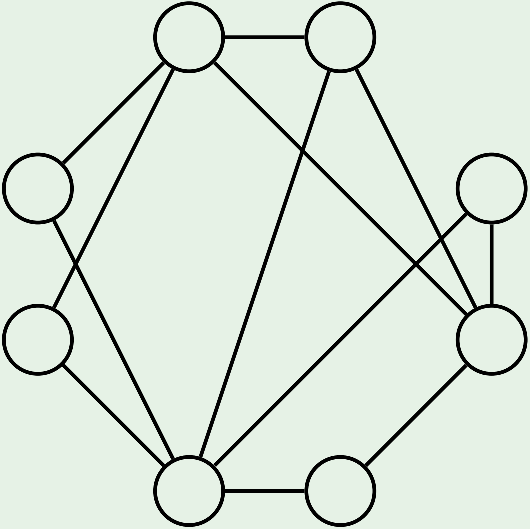
Proof

- The set M of all edges selected by the algorithm forms a matching
- Any vertex cover of the graph has size at least $|M|$
- The algorithm returns a vertex cover C of size $2|M|$, hence

$$|C| = 2 \cdot |M| \leq 2 \cdot \mathrm{OPT}$$

$\square$

Example



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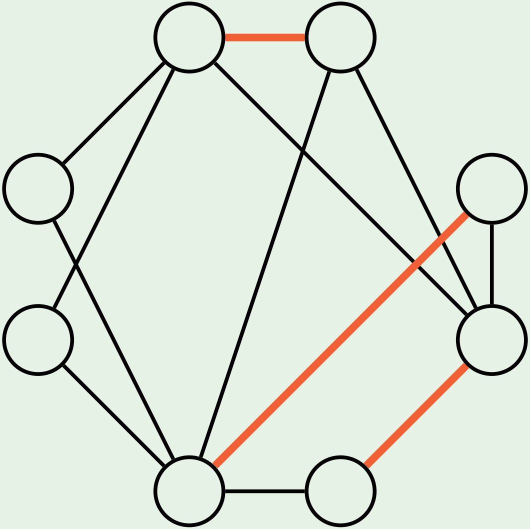

Summary

- We don't know the value of OPT, but we've managed to prove that

$$|C| \leq 2 \cdot \mathrm{OPT}$$

- This is because we know a lower bound on OPT: it is at least the size of any matching

$$|C| = 2 \cdot |M| \leq 2 \cdot \mathrm{OPT}$$

Final Remarks

- The bound is tight: there are graphs for which the algorithm returns a vertex cover of size twice the minimum size.
- No 1.99-approximation algorithm is known.

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Metric TSP (optimization version)

Input: An undirected graph $G(V, E)$ with non-negative edge weights satisfying the triangle inequality: for all $u, v, w \in V$, $d(u, v) + d(v, w) \geq d(u, w)$.

Output: A cycle of minimum total length visiting each vertex exactly once .

Lower Bound

- We are going to design a 2-approximation algorithm: it returns a cycle that is at most twice as long as an optimal cycle: $C \leq 2 \cdot \mathrm{OPT}$
- Since we don't know the value of OPT, we need a good lower bound L on OPT:

$$C \leq 2 \cdot L \leq 2 \cdot \mathrm{OPT}$$

Minimum Spanning Trees

Lemma

Let G be an undirected graph with non-negative edge weights. Then $\text{MST}(G) \leq \text{TSP}(G)$.

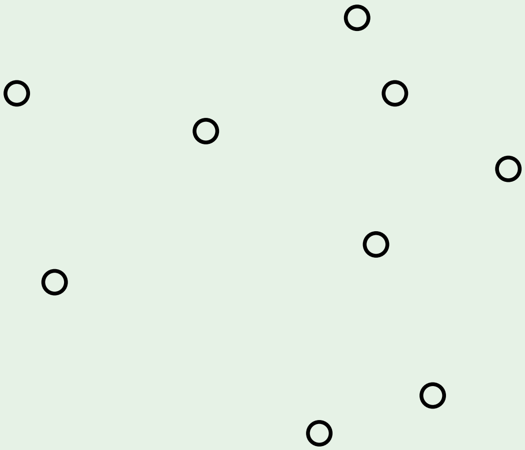
Proof

By removing any edge from an optimum TSP cycle one gets a spanning tree of G. $\square$

```
ApproxMetricTSP(G)

T ← minimum spanning tree of G
D ← T with each edge doubled
find an Eulerian cycle C in D
return a cycle that visits vertices in
  the order of their first appearance in C
```

Example: points on a plane



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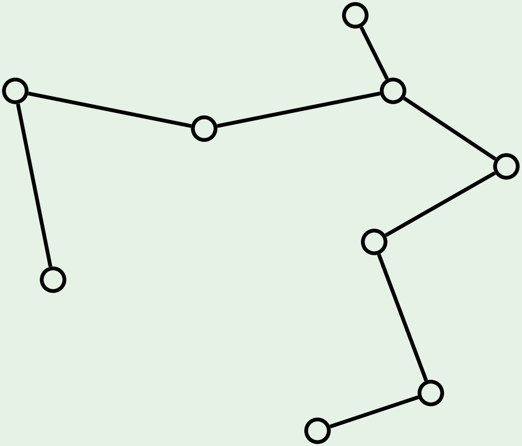

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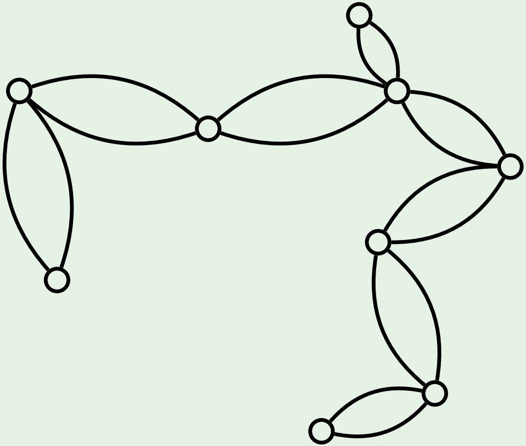

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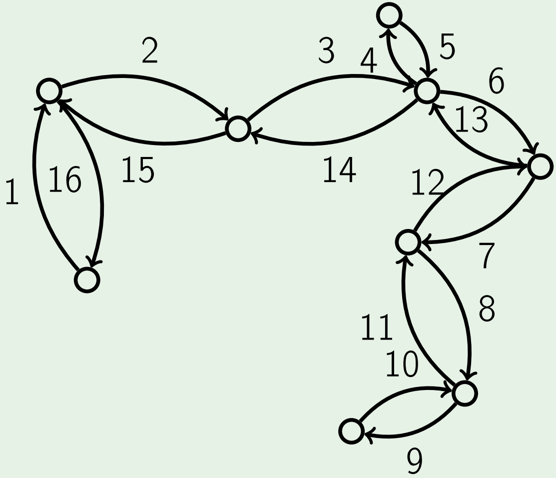

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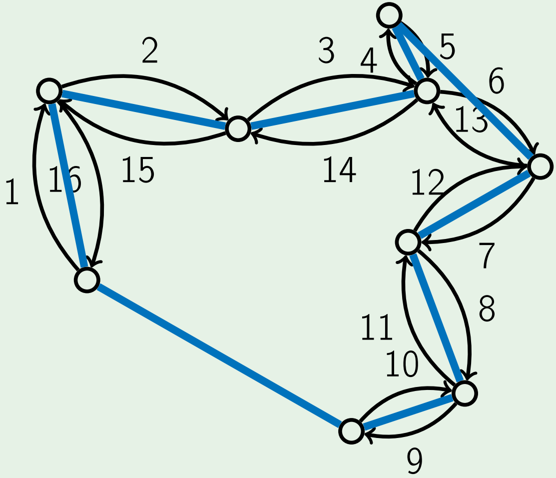

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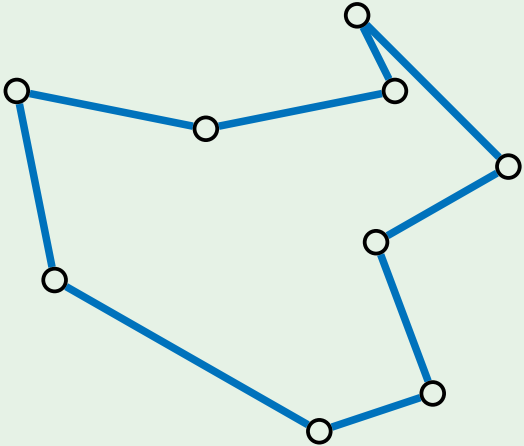

Lemma

The algorithm `ApproxMetricTSP` is 2-approximate.

Proof

- The total length of the MST T is at most OPT.

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The algorithm `ApproxMetricTSP` is 2-approximate.

Proof

- The total length of the MST T is at most OPT.
- Bypasses can only decrease the total length. $\square$

Final Remarks

- The currently best known approximation algorithm for metric TSP is Christofides' algorithm that achieves a factor of 1.5
- If $\mathbf{P} \neq \mathbf{NP}$, then there is no α-approximation algorithm for the general version of TSP for any polynomial time computable function α

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LocalSearch

```
s ← some initial solution
while there is a solution s′ in the
neighborhood of s which is better than s:
  s ← s′
return s
```

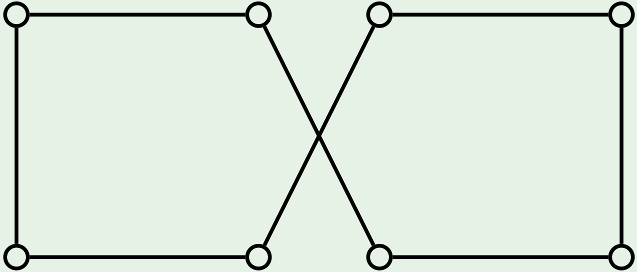
- Computes a local optimum instead of a global optimum
- The larger is the neighborhood, the better is the resulting solution and the higher is the running time

Local Search for TSP

- Let s and s' be two cycles visiting each vertex of the graph exactly once
- The distance between s and s' is at most d, if one can get s' by deleting d edges from s and adding other d edges
- Neighborhood $N(s, r)$ with center s and radius r: all cycles with distance at most r from s

Example

Changing two edges in a suboptimal solution:



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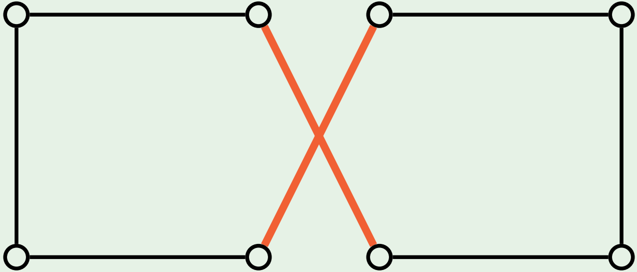

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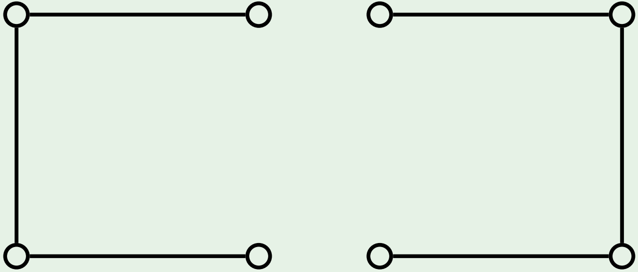

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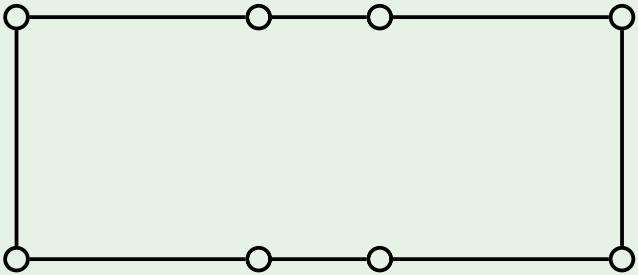
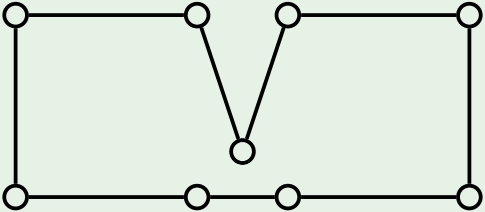


Example

A suboptimal solution that cannot be improved by changing two edges:



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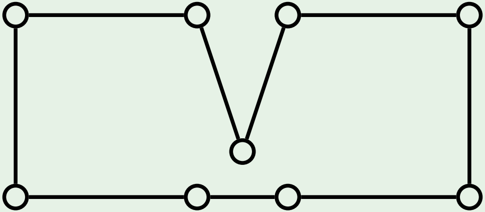

Need to allow changing three edges to improve this solution

Performance

- Trade-off between quality and running time of a single iteration
- Still, the number of iterations may be exponential and the quality of the found cycle may be poor
- But works well in practice

Coping with NP-completeness

- special cases
- intelligent exhaustive search
- approximation algorithms