Coping with NP-completeness: Exact Algorithms

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Advanced Algorithms and Complexity
Data Structures and Algorithms

Exact algorithms or intelligent exhaustive search: finding an optimal solution without going through all candidate solutions

Outline

1. 3-Satisfiability
   - Backtracking
   - Local Search
2. Traveling Salesman Problem
   - Dynamic Programming
   - Branch-and-bound

3-Satisfiability (3-SAT)

Input: A set of clauses, each containing at most three literals (that is, a 3-CNF formula).

Output: Find a satisfying assignment (if exists).

Examples

- The formula

$$(x \vee y \vee z)(x \vee \overline{y})(y \vee \overline{z})$$

  is satisfiable: set $x = y = z = 1$ or $x = 1, y = z = 0$.
- The formula

$$(x \vee y \vee z)(x \vee \overline{y})(y \vee \overline{z})(z \vee \overline{x})(\overline{x} \vee \overline{y} \vee \overline{z})$$

  is unsatisfiable.

A brute force search algorithm checking satisfiability of a 3-CNF formula F with n variables, goes through all assignments and has running time $O(|F| \cdot 2^n)$.

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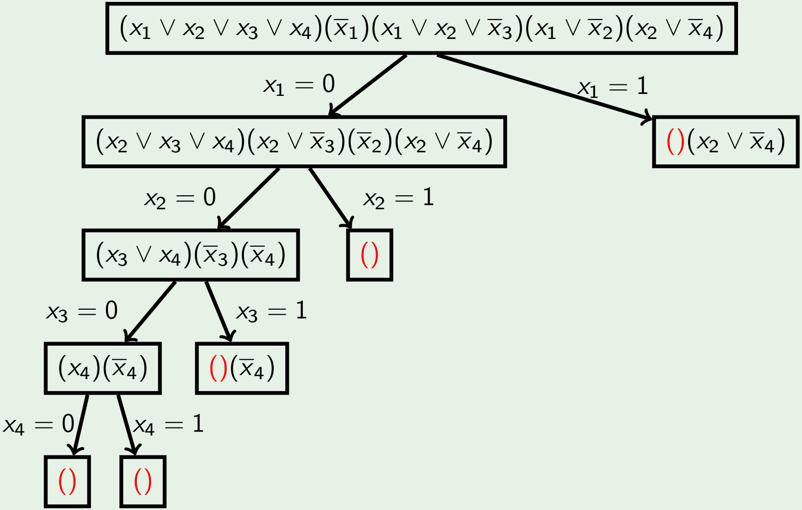
Main Idea of Backtracking

- Construct a solution piece by piece
- Backtrack if the current partial solution cannot be extended to a valid solution

Example


$(x_1 \vee x_2 \vee x_3 \vee x_4)(\overline{x}_1)(x_1 \vee x_2 \vee \overline{x}_3)(x_1 \vee \overline{x}_2)(x_2 \vee \overline{x}_4)$
$x_1 = 0$
$x_1 = 1$
$(x_2 \vee x_3 \vee x_4)(x_2 \vee \overline{x}_3)(\overline{x}_2)(x_2 \vee \overline{x}_4)$
$()(x_2 \vee \overline{x}_4)$
$x_2 = 0$
$x_2 = 1$
$(x_3 \vee x_4)(\overline{x}_3)(\overline{x}_4)$
$()$
$x_3 = 0$
$x_3 = 1$
$(x_4)(\overline{x}_4)$
$()(\overline{x}_4)$
$x_4 = 0$
$x_4 = 1$
$()$
$()$

SolveSAT(F)

```
if F has no clauses:
  return "sat"
if F contains an empty clause:
  return "unsat"
x ← unassigned variable of F
if SolveSAT(F[x ← 0]) = "sat":
  return "sat"
if SolveSAT(F[x ← 1]) = "sat":
  return "sat"
return "unsat"
```

- Thus, instead of considering all 2^n branches of the recursion tree, we track carefully each branch
- When we realize that a branch is dead (cannot be extended to a solution), we immediately cut it

- Backtracking is used in many state-of-the-art SAT-solvers
- SAT-solvers use tricky heuristics to choose a variable to branch on and to simplify a formula before branching
- Another commonly used technique is local search — will consider it in the next part

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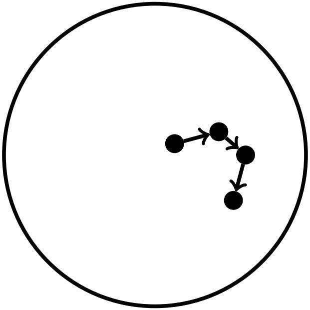
Main Idea of Local Search

- Start with a candidate solution
- Iteratively move from the current candidate to its neighbor trying to improve the candidate



- Let F be a 3-CNF formula over variables $x_1, x_2, \ldots, x_n$
- A candidate solution is a truth assignment to these variables, that is, a vector from $\{0, 1\}^n$

Definition

Hamming distance (or just distance) between two assignments $\alpha, \beta \in \{0,1\}^n$ is the number of bits where they differ: $\mathrm{dist}(\alpha, \beta) = |\{i : \alpha_i \neq \beta_i\}|$.

Definition

Hamming ball with center $\alpha \in \{0,1\}^n$ and radius r, denoted by $\mathcal{H}(\alpha, r)$, is the set of all truth assignments from $\{0,1\}^n$ at distance at most r from α.

Example

- $\mathcal{H}(1011, 0) = \{1011\}$
- $\mathcal{H}(1011, 1) = \{1011, 0011, 1111, 1001, 1010\}$
- $\mathcal{H}(1011, 2) = \{1011, 0011, 1111, 1001, 1010, 0111, 0001, 0010, 1101, 1110, 1000\}$

Searching a Ball for a Solution

Lemma

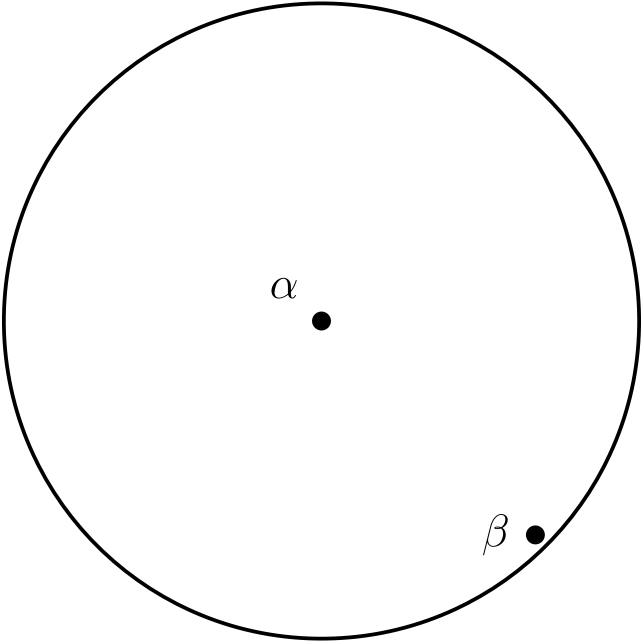
Assume that $\mathcal{H}(\alpha, r)$ contains a satisfying assignment β for F. We can then find a (possibly different) satisfying assignment in time $O(|F| \cdot 3^r)$.

Proof

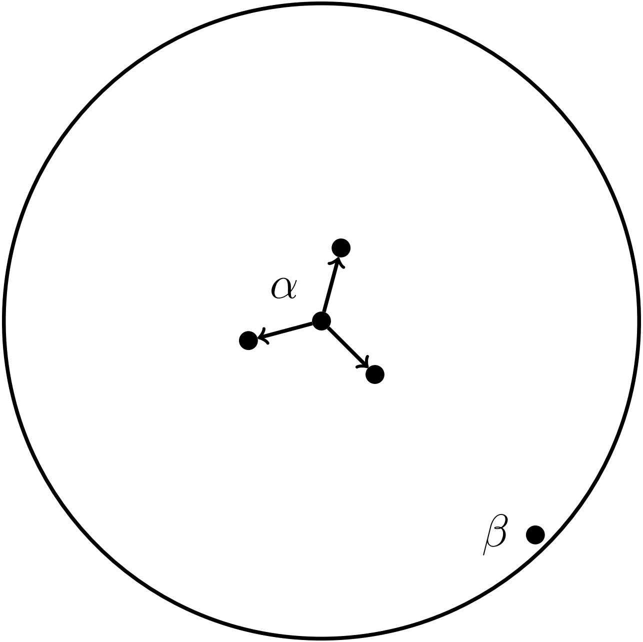
- If α satisfies F, return α

Proof

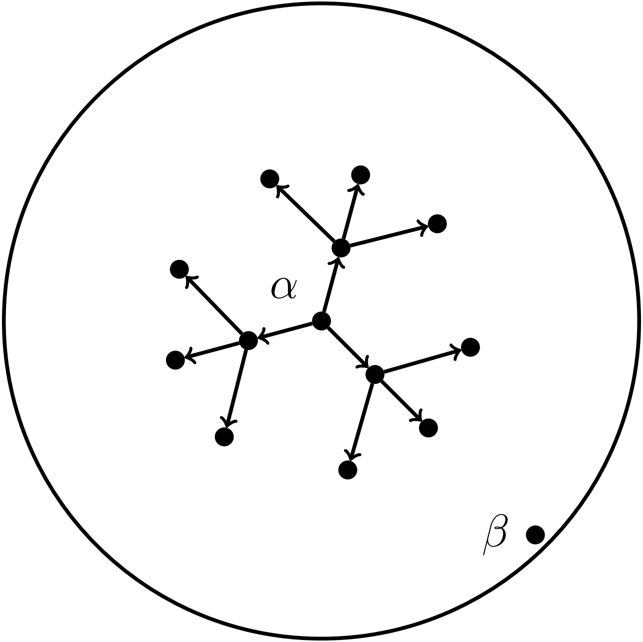
- If α satisfies F, return α
- Otherwise, take an unsatisfied clause — say, $(x_i \vee \overline{x}_j \vee x_k)$
- α assigns $x_i = 0, x_j = 1, x_k = 0$
- Let $\alpha^i, \alpha^j, \alpha^k$ be assignments resulting from α by flipping the i-th, j-th, k-th bit, respectively
- Crucial observation: at least one of them is closer to β than α
- Hence there are at most 3^r recursive calls $\square$



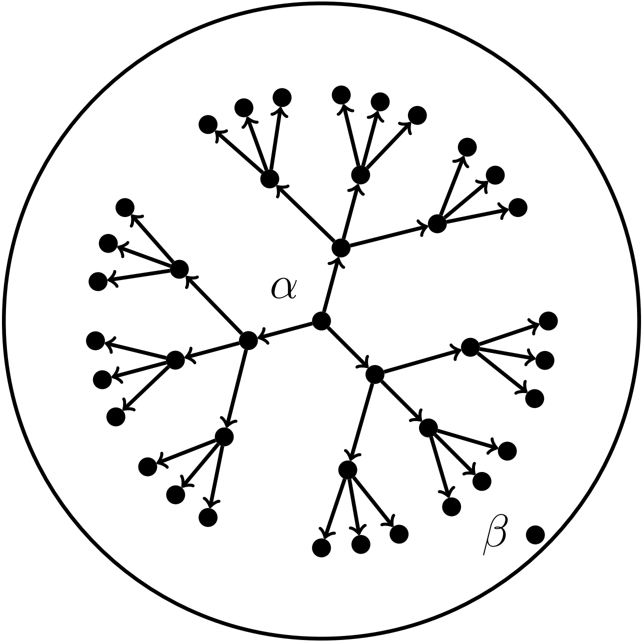
α
β



α
β



α
β



α
β

CheckBall(F, α, r)

```
if α satisfies F:
  return α
if r = 0:
  return "not found"
x_i, x_j, x_k ← variables of unsatisfied clause
α^i, α^j, α^k ← α with bits i, j, k flipped
CheckBall(F, α^i, r − 1)
CheckBall(F, α^j, r − 1)
CheckBall(F, α^k, r − 1)
if a satisfying assignment is found:
  return it
else:
  return "not found"
```

- Assume that F has a satisfying assignment β
- If it has more 1's than 0's then it has distance at most $n/2$ from all-1's assignment
- Otherwise it has distance at most $n/2$ from all-0's assignment
- Thus, it suffices to make two calls: $\texttt{CheckBall}(F, 11\dots1, n/2)$ and $\texttt{CheckBall}(F, 00\dots0, n/2)$

Running Time

- The running time of the resulting algorithm is $O(|F| \cdot 3^{n/2}) \approx O(|F| \cdot 1.733^n)$
- On one hand, this is still exponential
- On the other hand, it is exponentially faster than a brute force search algorithm that goes through all 2^n truth assignments!

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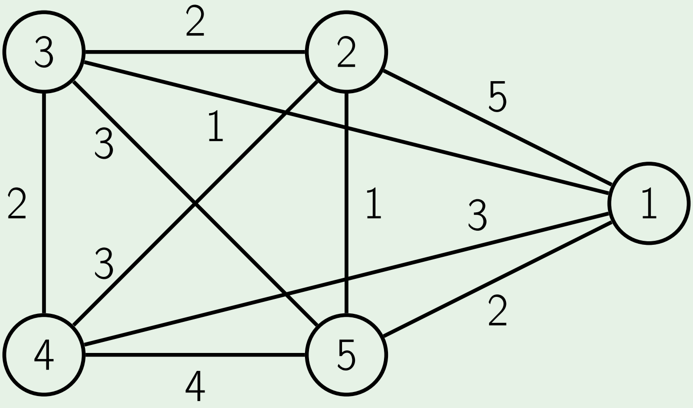
Traveling salesman problem (TSP)

Input: A complete graph with weights on edges and a budget b.

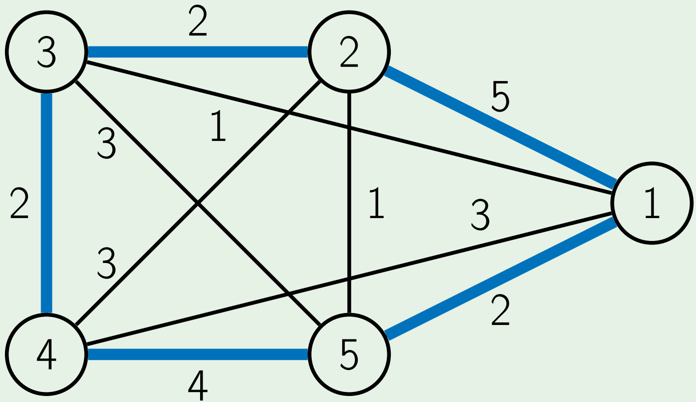
Output: A cycle that visits each vertex exactly once and has total weight at most b.

It will be convenient to assume that vertices are integers from 1 to n and that the salesman starts his trip in (and also returns back to) vertex 1.

Example

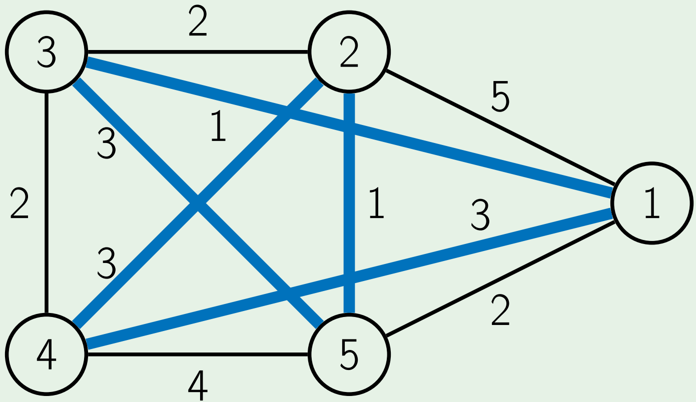


Example



length: 15

Example



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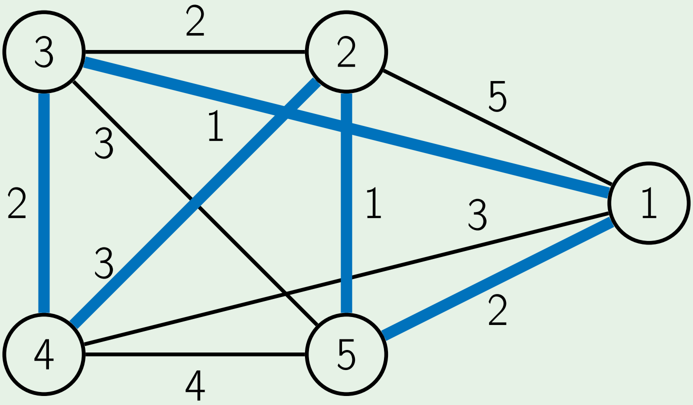
Example



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Brute Force Solution

A naive algorithm just checks all possible $(n-1)!$ cycles.

This part

- Use dynamic programming to solve TSP in $O(n^2 \cdot 2^n)$
- The running time is exponential, but is much better than $(n-1)!$.

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Dynamic Programming

- We are going to use dynamic programming: instead of solving one problem we will solve a collection of (overlapping) subproblems
- A subproblem refers to a partial solution
- A reasonable partial solution in case of TSP is the initial part of a cycle
- To continue building a cycle, we need to know the last vertex as well as the set of already visited vertices

Subproblems

- For a subset of vertices $S \subseteq \{1, \ldots, n\}$ containing the vertex 1 and a vertex $i \in S$, let $C(S, i)$ be the length of the shortest path that starts at 1, ends at i and visits all vertices from S exactly once
- $C(\{1\}, 1) = 0$ and $C(S, 1) = +\infty$ when $|S| > 1$

Recurrence Relation

- Consider the second-to-last vertex j on the required shortest path from 1 to i visiting all vertices from S
- The subpath from 1 to j is the shortest one visiting all vertices from $S - \{i\}$ exactly once
- Hence
  $C(S, i) = \min\{C(S - \{i\}, j) + d_{ji}\}$,
  where the minimum is over all $j \in S$ such that $j \neq i$

Order of Subproblems

- Need to process all subsets $S \subseteq \{1, \ldots, n\}$ in an order that guarantees that when computing the value of $C(S, i)$, the values of $C(S - \{i\}, j)$ have already been computed
- For example, we can process subsets in order of increasing size

TSP(G)

```
C({1},1) ← 0
for s from 2 to n:
  for all 1 ∈ S ⊆ {1,...,n} of size s:
    C(S,1) ← +∞
    for all i ∈ S, i ≠ 1:
      for all j ∈ S, j ≠ i:
        C(S,i) ← min{C(S,i), C(S−{i},j)+d_ji}
return min_i{C({1,...,n},i) + d_i,1}
```

Implementation Remark

- How to iterate through all subsets of $\{1, \ldots, n\}$?
- There is a natural one-to-one correspondence between integers in the range from 0 and $2^n - 1$ and subsets of $\{0, \ldots, n-1\}$:

$$k \leftrightarrow \{i : i\text{-th bit of } k \text{ is } 1\}$$

Example

k	$\text{bin}(k)$	$\{i : i\text{-th bit of } k \text{ is } 1\}$
0	000	$\emptyset$
1	001	$\{0\}$
2	010	$\{1\}$
3	011	$\{0,1\}$
4	100	$\{2\}$
5	101	$\{0,2\}$
6	110	$\{1,2\}$
7	111	$\{0,1,2\}$

- If k corresponds to S, how to find out the integer corresponding to $S - \{j\}$ (for $j \in S$)?
- For this, we need to flip the j-th bit of k (from 1 to 0)
- For this, in turn, we compute a bitwise XOR of k and 2^j (that has 1 only in j-th position)
- In `C/C++`, `Java`, `Python`:
  `k ^ (1 << j)`

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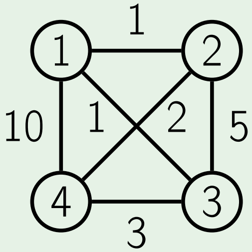
- The branch-and-bound technique can be viewed as a generalization of backtracking for optimization problems
- We grow a tree of partial solutions
- At each node of the recursion tree we check whether the current partial solution can be extended to a solution which is better than the best solution found so far
- If not, we don't continue this branch

Example: brute force search



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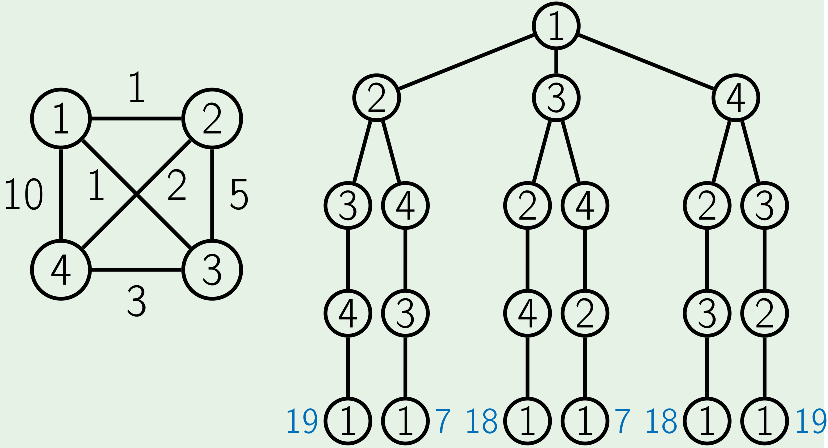
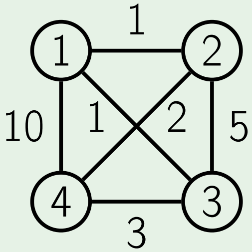


Example: pruned search



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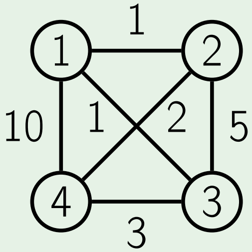

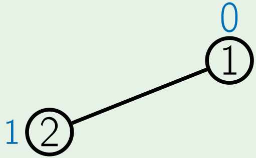

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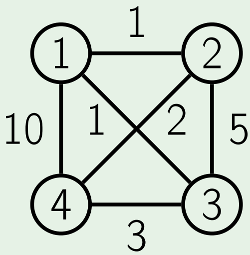

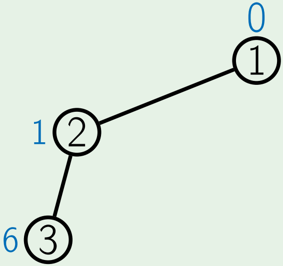

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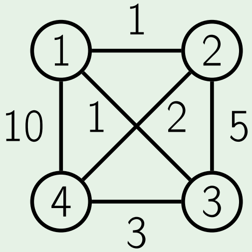

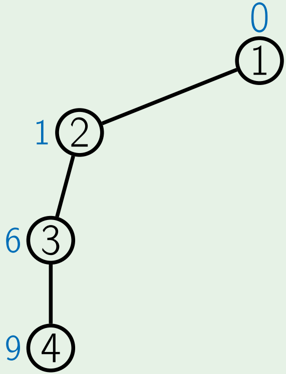

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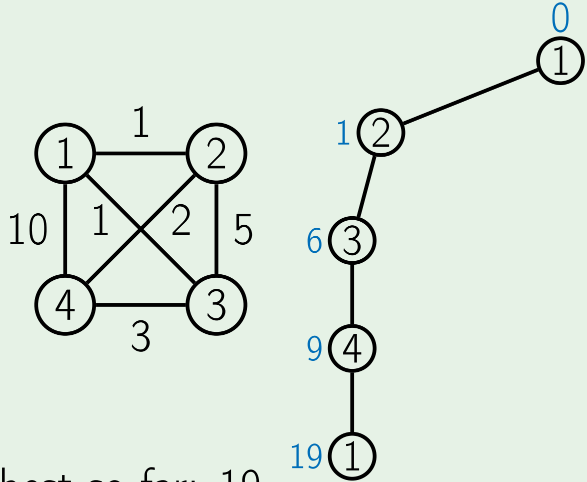
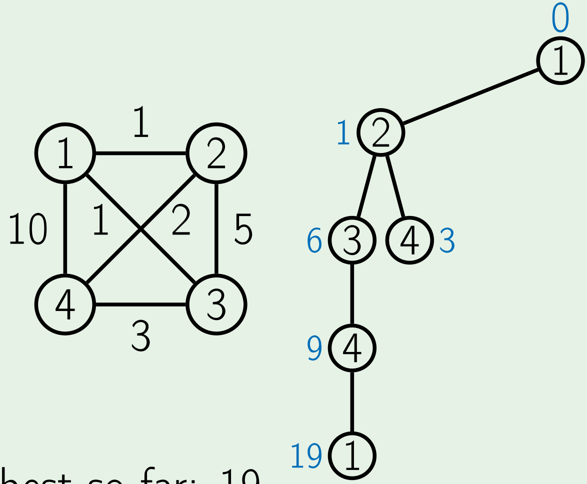


best so far: 19

Example: pruned search



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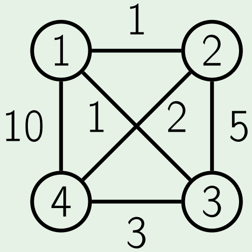
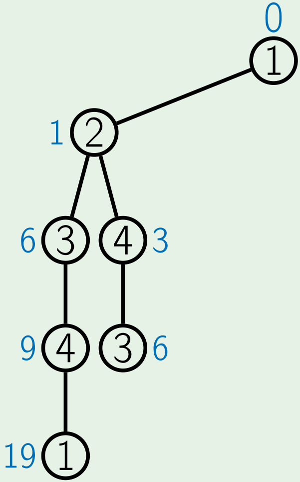
Example: pruned search





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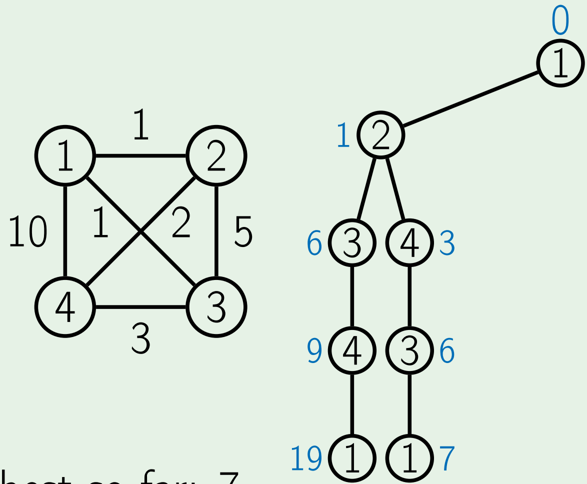
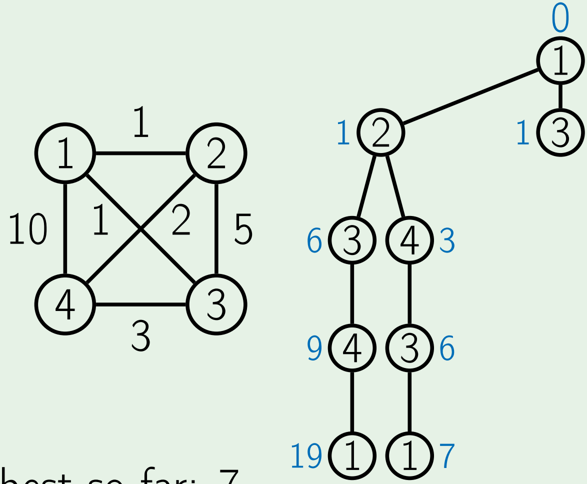
Example: pruned search



best so far: 7

Example: pruned search



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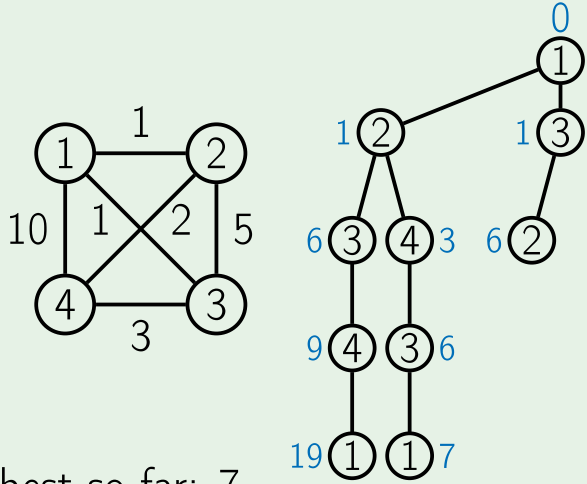

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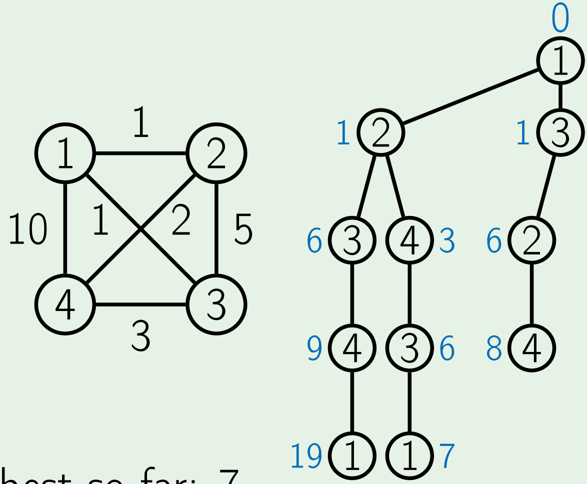

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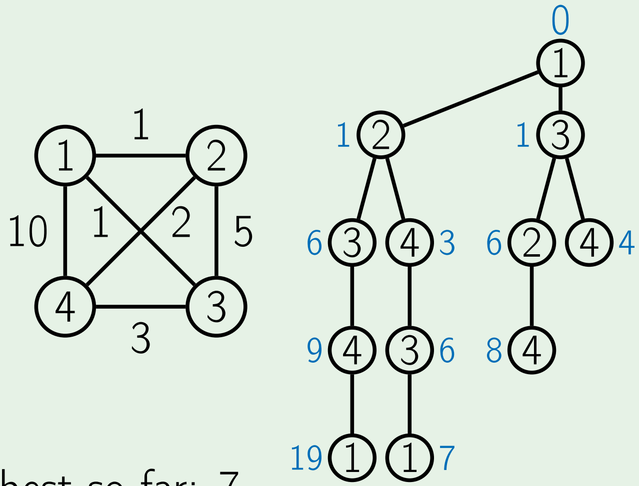

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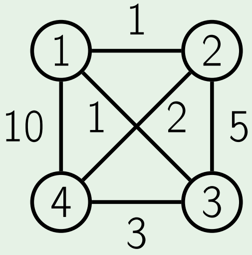

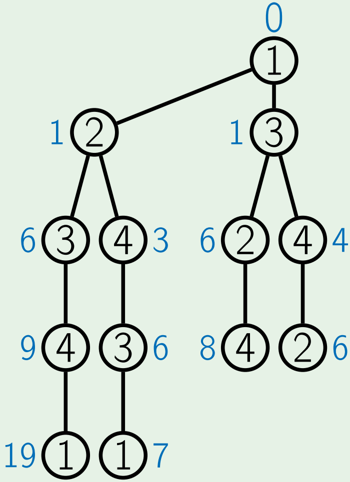

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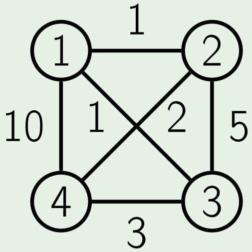

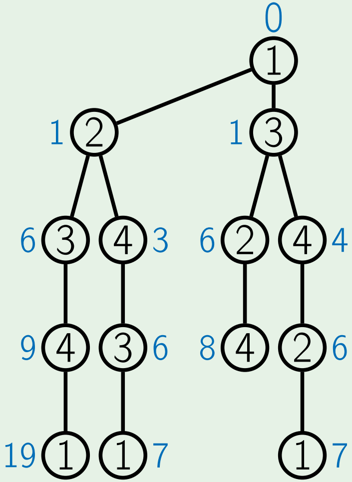

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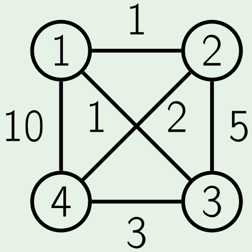

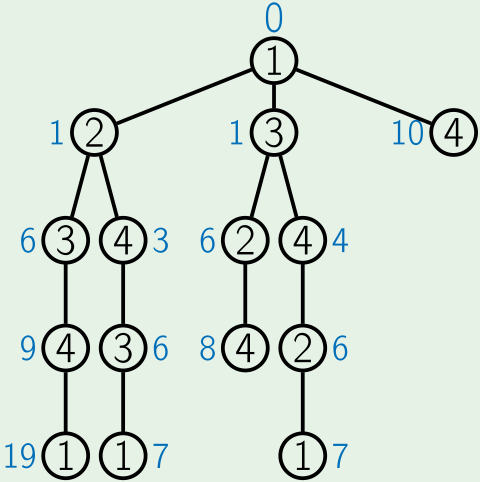

best so far: 7

- We used the simplest possible lower bound: any extension of a path has length at least the length of the path
- Modern TSP-solvers use smarter lower bounds to solve instances with thousands of vertices

Example: lower bounds (still simple)

The length of an optimal TSP cycle is at least

- $\frac{1}{2}\sum_{v \in V}$(two min length edges adjacent to v)
- the length of a minimum spanning tree

Next time

Approximation algorithms: polynomial algorithms that find a solution that is not much worse than an optimal solution