# Differentially Private Bayesian Inference Optimality of Laplace Mechanism

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## 1 Private Mechanisms

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**Algorithm 1** LSDim

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$\mathbf{x} \in \mathcal{X}^n$, $\mathsf{Dir}(\boldsymbol{\alpha})$
  **let** $\boldsymbol{\alpha}' = \mathsf{DirP}(\boldsymbol{\alpha}, \boldsymbol{\theta}, \mathbf{x})$
  **Initialize** a vector $\tilde{\boldsymbol{\alpha}} = (0, \ldots, 0) \in \mathbb{N}^{|\mathcal{X}|}$
  **For** $i = 1 \ldots |\mathcal{X}| - 1$:
    **let** $\eta \sim \mathsf{Lap}(0, \frac{|\mathcal{X}|}{\epsilon})$
    $\tilde{\alpha}_i = \alpha_i + \lfloor(\alpha_i' - \alpha_i) + \eta\rceil_0^n$
  $\tilde{\alpha}_{|\mathcal{X}|} = \alpha_{|\mathcal{X}|} + \lfloor n - \sum_{i=1}^{|\mathcal{X}|-1} \lfloor(\alpha_i' - \alpha_i) + \eta_i\rceil_0^n \rceil_0^n$
**return** $\tilde{\boldsymbol{\alpha}}$

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**Algorithm 2** LSDim

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**Require:** $\mathbf{x} \in \{0, 1\}^n$
  apply the Bayesian inference algorithm on $\mathbf{x}'$, get true posterior $\mathsf{Beta}(\boldsymbol{\alpha})$
  **let** $p = \mathsf{uniform}(0, 1)$, $\eta \sim \mathsf{Lap}(0, \frac{1.0}{\epsilon})$
  **If** $p > 0.5$:
    with 0.5 probability adding noise to first component.
    $\mathbf{x}' = (\lfloor \mathrm{x}_1 + \mu\rceil_0^n, n - \lfloor \mathrm{x}_1 + \mu\rceil_0^n)$
  **Else**:
    with 0.5 probability adding noise to second component.
    $\mathbf{x}' = (n - \lfloor \mathrm{x}_2 + \mu\rceil_0^n, \lfloor \mathrm{x}_2 + \mu\rceil_0^n)$
  apply the Bayesian inference algorithm on $\mathbf{x}'$, get: $\mathsf{Beta}(\boldsymbol{\alpha}')$
**return** $\boldsymbol{\alpha}'$

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**Algorithm 3** EHD

observed data set $\mathbf{x} \in \mathcal{X}^n$, prior: $\mathsf{Dir}(\boldsymbol{\alpha})$, $\epsilon$
  **let** $\mathsf{Dir}(\boldsymbol{\alpha}') = \mathsf{DirP}(\mathbf{x}, \boldsymbol{\alpha})$.
  **let** $GS$ be the global sensitivity for $\mathbf{x}$.
  **set** $z = r$ with probability $\frac{\exp(\frac{-\epsilon \cdot \mathcal{H}(\mathsf{DirP}(\mathbf{x},\boldsymbol{\alpha}),r)}{2 \cdot GS})}{\Sigma_{r' \in \mathcal{R}_{\boldsymbol{\alpha}}} \exp(\frac{-\epsilon \cdot \mathcal{H}(\mathsf{DirP}(\mathbf{x},\boldsymbol{\alpha}),r')}{2 \cdot GS})}$
**return** $z$

**Algorithm 4** EHDL

**input** observed data set $\mathbf{x} \in \mathcal{X}^n$, prior: $\mathsf{Dir}(\boldsymbol{\alpha})$, $\epsilon$
  **let** $\mathsf{Dir}(\boldsymbol{\alpha}') = \mathsf{DirP}(\mathbf{x}, \boldsymbol{\alpha})$.
  **let** $LS(\mathbf{x})$ be the local sensitivity for $\mathbf{x}$.
  **set** $z = r$ with probability $\frac{\exp(\frac{-\epsilon \cdot \mathcal{H}(\mathsf{DirP}(\mathbf{x},\boldsymbol{\alpha}),r)}{2 \cdot LS(\mathbf{x})})}{\Sigma_{r' \in \mathcal{R}_{\boldsymbol{\alpha}}} \exp(\frac{-\epsilon \cdot \mathcal{H}(\mathsf{DirP}(\mathbf{x},\boldsymbol{\alpha}),r')}{2 \cdot LS(\mathbf{x})})}$
**return** $z$

**Algorithm 5** EHDS

observed data set $\mathbf{x} \in \mathcal{X}^n$, prior: $\mathsf{Dir}(\boldsymbol{\alpha})$, $\epsilon$
  **let** $\mathsf{Dir}(\boldsymbol{\alpha}') = \mathsf{DirP}(\mathbf{x}, \boldsymbol{\alpha})$.
  **let** $S(\mathbf{x})$ be the smooth sensitivity for $\mathbf{x}$.
  **set** $z = r$ with probability $\frac{\exp(\frac{\epsilon \cdot u(\mathbf{x},r)}{4 \cdot S(\mathbf{x})})}{\Sigma_{r' \in \mathcal{R}_{\boldsymbol{\alpha}}} \exp(\frac{\epsilon \cdot u(\mathbf{x},r')}{4 \cdot S(\mathbf{x})})}$
**return** $z$

## 2 Accuracy Analysis

**Theorem 2.1.** *Let $R_g$ be the good output set where $\forall r \in R$, $\mathsf{H}(\mathsf{DirP}(\boldsymbol{x}), r) \leq LS(\boldsymbol{x})$, we have:*

$$Pr[\mathsf{LSHist}(\boldsymbol{x}, \epsilon) \in R_g] > Pr[\mathsf{EHD}(\boldsymbol{x}, \epsilon) \in R_g]$$

*for data size $n = |\boldsymbol{x}| > O(\frac{e^{\epsilon}}{1-e^{-\epsilon}})$*

Let $R_g$ be the good output set where $\forall r \in R$, $\mathsf{H}(\mathsf{DirP}(\mathbf{x}), r) \leq LS(\mathbf{x})$, we have:

$$Pr[\mathsf{LSHist}(\mathbf{x}) \in R_g] \geq 1 - \frac{1}{2}(e^{-\epsilon} + e^{-2\epsilon}) > 1 - e^{-\epsilon}$$

By definition of $\mathsf{EHD}$ and $GS = \sqrt{1 - \pi/4}$, we have:

$$\begin{aligned} Pr[\mathsf{EHD}(\mathbf{x}) \in R_g] &= \sum_{c \geq -LS(\mathbf{x})} \frac{\exp(\frac{\epsilon \cdot c}{2 \cdot GS})}{\Sigma_{r' \in \mathcal{R}_{\boldsymbol{\alpha}}} \exp(\frac{-\epsilon \cdot \mathcal{H}(\mathsf{DirP}(\mathbf{x}, \boldsymbol{\alpha}), r')}{2 \cdot GS})} \\ &\leq \frac{2\exp(-\frac{\epsilon LS(\mathbf{x})}{2 \cdot GS}) + 1}{n \exp(\frac{-\epsilon}{2 \cdot GS})} \\ &\leq \frac{3}{n \exp(\frac{-\epsilon}{2\sqrt{1-\pi/4}})} \\ &\leq \frac{3}{n \exp(-\epsilon)} \end{aligned}$$

Let $c = 2\sqrt{1 - \pi/4}$, we have when $n > \frac{3}{e^{-\epsilon/c}(1-e^{-\epsilon})} \sim O(\frac{e^{\epsilon}}{1-e^{-\epsilon}})$ $\mathsf{LSHist}$ performs better than $\mathsf{EHD}$.

**Theorem 2.2.** *To prove the optimality of Laplace mechanism, we are showing*

$$\frac{ELap(\boldsymbol{x})}{(\epsilon \times LS(\boldsymbol{x}))}$$

*is $O(\epsilon)$, considering $n = |\boldsymbol{x}| \geq 2$ being the parameter.*

*Where $LS(\cdot)$ is the local sensitivity, and where $ELap(\cdot)$ is the measure of the error of the Laplace mechanism, defined in this way:*

$$ELap(\boldsymbol{x}) = \arg\big(\min_t \{Pr[\mathsf{H}(\mathsf{DirP}(\boldsymbol{x}), \mathsf{LSHist}(\boldsymbol{x})) < t] \geq 1 - \gamma\big).$$

*Proof.* Let $t = LS(\mathbf{x})$, we have following by p.d.f. of Laplace distribution:

$$Pr[\mathsf{H}(\mathsf{DirP}(\mathbf{x}), \mathsf{LSHist}(\mathbf{x})) < t] \geq 1 - \frac{1}{2}(e^{-\epsilon} + e^{-2\epsilon}) > 1 - e^{-\epsilon}$$

Then we can get when $\gamma = e^{-\epsilon}$,

$$\frac{ELap(\mathbf{x})}{(\epsilon \times LS(\mathbf{x}))} = \frac{1}{\epsilon}$$

□

**Theorem 2.3.** *In order to prove the optimality of Laplace mechanism, instead of prove* $\frac{ELap(\boldsymbol{x})}{(\epsilon \times LS(\boldsymbol{x}))}$ *is* $O(1)$*, we prove a constant upper bound on following equations:*

$$\begin{aligned}
& \frac{\arg\min_t \left\{ \Pr[\mathsf{H}(\mathsf{Beta}(\alpha,\beta), \mathsf{LSHist}(\boldsymbol{x})) < t] \geq 1-\gamma \right\}}{LS(\boldsymbol{x})} \\
\leq & \frac{\max_{|k| \leq \frac{\lg(\frac{1}{\gamma})}{\epsilon}} \mathsf{H}(\mathsf{Beta}(\alpha,\beta), \mathsf{Beta}(\alpha+k, n-\lfloor\alpha+k\rfloor))}{LS(\boldsymbol{x})} \\
\leq & O(\frac{\lg\frac{1}{\gamma}}{\epsilon})
\end{aligned}$$

*Proof.* By Laplace distribution, we have:

$$\begin{aligned}
\Pr[\mathsf{H}(\mathsf{Beta}(\alpha,\beta), \mathsf{LSHist}(\mathbf{x})) < t] &= \Pr[\{|\mathsf{Lap}((,\tfrac{)}{1}\epsilon| < O(k) | \mathsf{H}(\mathsf{Beta}(\alpha,\beta), \mathsf{Beta}(\alpha+k, n-\lfloor\alpha+k\rfloor)) < t\}] \\
&\leq 1 - e^{-O(k)\epsilon}
\end{aligned}$$

Then we have:

$$\gamma = e^{-O(k)\epsilon}$$

So we can get:

$$O(\frac{\lg\frac{1}{\gamma}}{\epsilon}) = O(\frac{\lg\frac{1}{e^{-O(k)\epsilon}}}{\epsilon}) = O(k)$$

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*Proof.* By setting $-1 \leq k < 2$, we have:

$$\mathsf{H}(\mathsf{Beta}(\alpha,\beta), \mathsf{Beta}(\alpha+k, n-\lfloor\alpha+k\rfloor)) \leq LS(\mathbf{x}) \tag{1}$$

For any $\epsilon$, $k \sim \mathsf{Lap}(0, \frac{1}{\epsilon})$ from Laplace mechanism, we have:

$$\Pr[|k| \leq \frac{b}{\epsilon}] = 1 - \exp(-b)$$

Then we can get:

$$\Pr[-1 \leq k < 2] = 1 - \frac{\exp(-\epsilon) + \exp(-2\epsilon)}{2} \tag{2}$$

By Equation (1) and (2), we can get:

$$\Pr[\mathsf{H}(\mathsf{Beta}(\alpha,\beta), \mathsf{Beta}(\alpha+k, n-\lfloor\alpha+k\rfloor)) \leq LS(\mathbf{x})] \geq 1 - \frac{\exp(-\epsilon) + \exp(-2\epsilon)}{2}$$

i.e.,

$$\begin{aligned}
& \frac{\arg\min_t \left\{ \Pr[\mathsf{H}(\mathsf{Beta}(\alpha,\beta), \mathsf{LSHist}(\mathbf{x})) < t] \geq 1 - \frac{\exp(-\epsilon)+\exp(-2\epsilon)}{2} \right\}}{LS(\mathbf{x})} \\
\leq & O(\frac{\lg(\frac{2}{\exp(-\epsilon)+\exp(-2\epsilon)})}{\epsilon}) \\
< & O(\frac{\lg(\frac{2}{2\exp(-2\epsilon)})}{\epsilon}) = 2
\end{aligned}$$

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