

# Differentially Private Bayesian Inference

Anonymous Authors<sup>1</sup>

## Abstract

### 1. Setting up

The Bayesian inference process is denoted as  $\text{BI}(x, \text{prior})$  taking an observed data set  $x \in \mathcal{X}^n$  and a prior distribution as input, outputting a posterior distribution *posterior*. For conciseness, when prior is given, we use  $\text{BI}(x)$ .

For now, we already have a prior distribution *prior*, an observed data set  $x$ .

#### 1.1. Exponential Mechanism with Global Sensitivity

##### 1.1.1. MECHANISM SET UP

In exponential mechanism, candidate set  $R$  can be obtained by enumerating  $y \in \mathcal{X}^n$ , i.e.

$$R = \{\text{BI}(y) \mid y \in \mathcal{X}^n\}.$$

Hellinger distance  $H$  is used here to score these candidates. The utility function:

$$u(x, r) = -H(\text{BI}(x), r); r \in R. \quad (1)$$

Exponential mechanism with global sensitivity selects and outputs a candidate  $r \in R$  with probability proportional to  $\exp(\frac{\epsilon u(x, r)}{2\Delta_g u})$ :

$$P[r] = \frac{\exp(\frac{\epsilon u(x, r)}{2\Delta_g u})}{\sum_{r' \in R} \exp(\frac{\epsilon u(x, r')}{2\Delta_g u})},$$

where global sensitivity is calculated by:

$$\Delta_g u = H(\text{BI}(x'), r) - H(\text{BI}(y'), r) \max_{\{x', y' \leq 1; x', y' \in \mathcal{X}^n\}} \max_{\{r \in R\}}.$$

<sup>1</sup>Anonymous Institution, Anonymous City, Anonymous Region, Anonymous Country. Correspondence to: Anonymous Author <anon.email@domain.com>.

##### 1.1.2. SECURITY ANALYSIS

It can be proved that exponential mechanism with global sensitivity is  $\epsilon$ -differentially private. We denote the BI with privacy mechanism as  $\text{PrivInfer}$ . For adjacent data set  $\|x, y\|_1 = 1$ :

$$\begin{aligned} & \frac{P[\text{PrivInfer}(x, u, R) = r]}{P[\text{PrivInfer}(y, u, R) = r]} \\ &= \frac{\exp(\frac{\epsilon u(x, r)}{2\Delta_g u})}{\sum_{r' \in R} \exp(\frac{\epsilon u(x, r')}{2\Delta_g u})} \\ &= \frac{\exp(\frac{\epsilon u(y, r)}{2\Delta_g u})}{\sum_{r' \in R} \exp(\frac{\epsilon u(y, r')}{2\Delta_g u})} \\ &= \left( \frac{\exp(\frac{\epsilon u(x, r)}{2\Delta_g u})}{\exp(\frac{\epsilon u(y, r)}{2\Delta_g u})} \right) \cdot \left( \frac{\sum_{r' \in R} \exp(\frac{\epsilon u(y, r')}{2\Delta_g u})}{\sum_{r' \in R} \exp(\frac{\epsilon u(x, r')}{2\Delta_g u})} \right) \\ &= \exp\left(\frac{\epsilon(u(x, r) - u(y, r))}{2\Delta_g u}\right) \\ &\quad \cdot \left( \frac{\sum_{r' \in R} \exp(\frac{\epsilon u(y, r')}{2\Delta_g u})}{\sum_{r' \in R} \exp(\frac{\epsilon u(x, r')}{2\Delta_g u})} \right) \\ &\leq \exp\left(\frac{\epsilon}{2}\right) \cdot \exp\left(\frac{\epsilon}{2}\right) \cdot \left( \frac{\sum_{r' \in R} \exp(\frac{\epsilon u(x, r')}{2\Delta_g u})}{\sum_{r' \in R} \exp(\frac{\epsilon u(x, r')}{2\Delta_g u})} \right) \\ &= \exp(\epsilon). \end{aligned}$$

Then,  $\frac{P[\text{PrivInfer}(x, u, R) = r]}{P[\text{PrivInfer}(y, u, R) = r]} \geq \exp(-\epsilon)$  can be obtained by symmetry.

#### 1.2. Exponential Mechanism with Local Sensitivity

##### 1.2.1. MECHANISM SET UP

Exponential mechanism with local sensitivity share the same candidate set and utility function as it with global sensitivity. This outputs a candidate  $r \in R$  with probability proportional to  $\exp(\frac{\epsilon u(x, r)}{2\Delta_l u})$ :

$$P[r] = \frac{\exp(\frac{\epsilon u(x, r)}{2\Delta_l u})}{\sum_{r' \in R} \exp(\frac{\epsilon u(x, r')}{2\Delta_l u})},$$

where local sensitivity is calculated by:

$$\Delta_l u(x) = H(\text{BI}(x), r) - H(\text{BI}(y'), r) |$$

$$\max_{\{x, y' | \leq 1; y' \in \mathcal{X}^n\}} \max_{\{r \in R\}}.$$

### 1.2.2. SECURITY ANALYSIS

We will then prove that exponential mechanism with local sensitivity is non-differentially private.

$$\frac{P[\text{PrivInfer}(x, u, R) = r]}{P[\text{PrivInfer}(y, u, R) = r]}$$

$$= \exp\left(\frac{\epsilon u(x, r)}{2\Delta_l u(x)} - \frac{\epsilon u(y, r)}{2\Delta_l u(y)}\right) \cdot \left(\frac{\sum_{r' \in R} \exp(\frac{\epsilon u(y, r')}{2\Delta_l u(y)})}{\sum_{r' \in R} \exp(\frac{\epsilon u(x, r')}{2\Delta_l u(x)})}\right)$$

$$= \frac{\sum_{r' \in R} \exp(\frac{\epsilon u(x, r)}{2\Delta_l u(x)} + \frac{\epsilon u(y, r')}{2\Delta_l u(y)})}{\sum_{r' \in R} \exp(\frac{\epsilon u(y, r)}{2\Delta_l u(y)} + \frac{\epsilon u(x, r')}{2\Delta_l u(x)})}.$$

Without loss of generality, we consider the case that  $\Delta_l u(y) < \Delta_l u(x)$ ,  $r = \arg(\max_{r' \in R} \{u(x, r')\}) = \arg(\min_{r' \in R} \{u(y, r')\})$  and  $\Delta_l u(y) = u(x, r) - u(y, r)$ . We have:

$$\frac{\sum_{r' \in R} \exp(\frac{\epsilon u(x, r)}{2\Delta_l u(x)} + \frac{\epsilon u(y, r')}{2\Delta_l u(y)})}{\sum_{r' \in R} \exp(\frac{\epsilon u(y, r)}{2\Delta_l u(y)} + \frac{\epsilon u(x, r')}{2\Delta_l u(x)})}$$

$$> \frac{\sum_{r' \in R} \exp(\frac{\epsilon(u(x, r) + u(y, r'))}{2\Delta_l u(x)})}{\sum_{r' \in R} \exp(\frac{\epsilon(u(y, r) + u(x, r'))}{2\Delta_l u(y)})}$$

$$> \frac{|R| \exp(\frac{\epsilon(u(x, r) + u(y, r))}{2\Delta_l u(x)})}{|R| \exp(\frac{\epsilon(u(y, r) + u(x, r))}{2\Delta_l u(y)})}$$

$$= \exp(\frac{\epsilon}{2}(\frac{u(x, r) + u(y, r)}{\Delta_l u(x)} - \frac{u(x, r) + u(y, r)}{\Delta_l u(y)})).$$

From Eq. 1,  $\{u(x, r') \leq 0 | r' \in R\}$  and  $\{u(y, r') \leq 0 | r' \in R\}$ , we can infer that  $r = \arg(\max_{r' \in R} \{u(x, r')\}) = \text{BI}(x)$  and  $u(x, r) = 0$ . From  $\Delta_l u(y) = u(x, r) - u(y, r)$ , we can also infer that  $\Delta_l u(y) = -u(y, r)$ . Then, the following relationship between  $u(x, r)$ ,  $u(y, r)$ ,  $\Delta_l u(x)$  and  $\Delta_l u(y)$ :

$$-\Delta_l u(x) < \Delta_l u(y)$$

$$\Delta_l u(x) - \Delta_l u(y) < 2\Delta_l u(x)$$

$$-\Delta_l u(y)(\Delta_l u(y) - \Delta_l u(x)) < 2\Delta_l u(x)\Delta_l u(y)$$

$$u(y, r)(\Delta_l u(y) - \Delta_l u(x)) < 2\Delta_l u(x)\Delta_l u(y)$$

$$\frac{u(x, r) + u(y, r)}{\Delta_l u(x)} - \frac{u(x, r) + u(y, r)}{\Delta_l u(y)} > 2.$$

holds.

Then we can have:

$$\exp(\frac{\epsilon}{2}(\frac{u(x, r) + u(y, r)}{\Delta_l u(y)} - \frac{u(x, r) + u(y, r)}{\Delta_l u(x)}))$$

$$> \exp(\frac{\epsilon}{2} * 2)$$

$$= \exp(\epsilon),$$

i.e.

$$\frac{P[\text{PrivInfer}(x, u, R) = r]}{P[\text{PrivInfer}(y, u, R) = r]} > \exp(\epsilon).$$

Since there are cases where exponential mechanism with local sensitivity's privacy loss is greater than  $e^\epsilon$ . we can say it is non-differentially private.

## 1.3. Exponential Mechanism of Varying Sensitivity

### 1.3.1. MECHANISM SETTING UP

### 1.3.2. SECURITY ANALYSIS

## 1.4. Exponential Mechanism of Smooth Sensitivity

### 1.4.1. MECHANISM SETTING UP

### 1.4.2. SECURITY ANALYSIS

## 2. Privacy Fix

### 2.1. Propositions

Assume we have a prior distribution  $\text{beta}(1, 1)$ , an observed data set  $x \in \{0, 1\}^n$ ,  $n > 0$ . We use the  $x + 1$  and  $x - 1$  to denote:

if  $\text{BayesInfer}(x) = \text{beta}(a_1 + 1, b_1 + 1)$

then  $\text{BayesInfer}(x + 1) = \text{beta}((a_1 + 1) + 1, (b_1 - 1) + 1)$

$\text{BayesInfer}(x - 1) = \text{beta}((a_1 - 1) + 1, (b_1 + 1) + 1)$ ,

$x_0$  to denote:

if  $n$  is even

then  $\text{BI}(x_0) = \text{beta}(\frac{n}{2} + 1, \frac{n}{2} + 1)$

else  $\text{BI}(x_0) = \{\text{beta}(\frac{n+1}{2} + 1, \frac{n-1}{2} + 1), \text{beta}(\frac{n-1}{2} + 1, \frac{n+1}{2} + 1)\}$

$\text{beta}(\alpha, \beta)$  is the beta function with two arguments  $\alpha$  and  $\beta$ .

Then, we have the following three statements, and proofs of the statements.

I  $H(\text{BI}(x), \text{BI}(x + 1)) < H(\text{BI}(x + 1), \text{BI}(x + 2)) \forall x \geq x_0$ ;

or  $H(\text{BI}(x), \text{BI}(x+1)) > H(\text{BI}(x+1), \text{BI}(x+2)) \forall x \leq x_0$ .

II  $\Delta_l u(x) = H(\text{BI}(x), \text{BI}(x+1)), \forall x \geq x_0$ ;

$\Delta_l u(x) = H(\text{BI}(x), \text{BI}(x-1)), \forall x \leq x_0$ .

III  $\forall x \neq x_0 : \Delta_l u(x) > \Delta_l u(x_0)$ .

## 2.2. proof

### 2.2.1. STATEMENT I

We use the MI (Mathematical Induction) method to prove the first statement.

*Proof.* Since the Hellinger distance is symmetric, if we prove the  $H(\text{BI}(x), \text{BI}(x+1)) < H(\text{BI}(x+1), \text{BI}(x+2)) \forall x \geq x_0$ , the other part when  $\forall x \leq x_0$  also holds.

1. if  $x = x_0$ ,  $H(\text{BI}(x_0), \text{BI}(x_0+1)) < H(\text{BI}(x_0+1), \text{BI}(x_0+2))$  holds:

$$\begin{aligned} & \sqrt{1 - \frac{\text{beta}(\frac{\frac{n}{2}+1+m+\frac{n}{2}+1+m+1}{2}, \frac{\frac{n}{2}+1-m+\frac{n}{2}+1-m-1}{2})}{\sqrt{\text{beta}(\frac{n}{2}+1+m, \frac{n}{2}+1-m)\text{beta}(\frac{n}{2}+2+m, \frac{n}{2}-m)}}} \\ & < \sqrt{1 - \frac{\text{beta}(\frac{\frac{n}{2}+1+m+1+\frac{n}{2}+1+m+2}{2}, \frac{\frac{n}{2}+1-m-1+\frac{n}{2}+1-m-2}{2})}{\sqrt{\text{beta}(\frac{n}{2}+2+m, \frac{n}{2}-m)\text{beta}(\frac{n}{2}+3+m, \frac{n}{2}-m-1)}}} \\ & \frac{\text{beta}(\frac{n+2m+3}{2}, \frac{n-2m+1}{2})}{\sqrt{\text{beta}(\frac{n}{2}+1+m, \frac{n}{2}+1-m)\text{beta}(\frac{n}{2}+2+m, \frac{n}{2}-m)}} \\ & > \frac{\text{beta}(\frac{n+2m+5}{2}, \frac{n-2m-1}{2})}{\sqrt{\text{beta}(\frac{n}{2}+2+m, \frac{n}{2}-m)\text{beta}(\frac{n}{2}+3+m, \frac{n}{2}-m-1)}} \end{aligned}$$

Now, we need to proof  $H(\text{beta}(\frac{n}{2}+1+m+1, \frac{n}{2}+1-m-1), \text{beta}(\frac{n}{2}+1+m+2, \frac{n}{2}+1-m-2)) < H(\text{beta}(\frac{n}{2}+1+m+2, \frac{n}{2}+1-m-2), \text{beta}(\frac{n}{2}+1+m+3, \frac{n}{2}+1-m-3))$  by using what we know.

From  $x = x_0 + m$  and property of  $\text{beta}(\alpha, \beta)$  function, we know:

$$\begin{aligned} & H(\text{beta}(\frac{n}{2}+1, \frac{n}{2}+1), \text{beta}(\frac{n}{2}+1+1, \frac{n}{2}+1-1)) < H(\text{beta}(\frac{n}{2}+1+1, \frac{n}{2}+1-1), \text{beta}(\frac{n}{2}+1+2, \frac{n}{2}+1-2)) \\ & \sqrt{1 - \frac{\text{beta}(\frac{\frac{n}{2}+1+\frac{n}{2}+1+1}{2}, \frac{\frac{n}{2}+1+\frac{n}{2}+1-1}{2})}{\sqrt{\text{beta}(\frac{n}{2}+1, \frac{n}{2}+1)\text{beta}(\frac{n}{2}+1+1, \frac{n}{2}+1-1)}}} < \sqrt{1 - \frac{\text{beta}(\frac{\frac{n}{2}+1+1+\frac{n}{2}+1+2}{2}, \frac{\frac{n}{2}+1-1+\frac{n}{2}+1-2}{2})}{\sqrt{\text{beta}(\frac{n}{2}+1+1, \frac{n}{2}+1-1)\text{beta}(\frac{n}{2}+1+2, \frac{n}{2}+1-2)}}} \\ & \sqrt{1 - \frac{\text{beta}(\frac{n+3}{2}, \frac{n+1}{2})}{\sqrt{\text{beta}(\frac{n}{2}+1, \frac{n}{2}+1)\text{beta}(\frac{n}{2}+2, \frac{n}{2})}}} < \sqrt{1 - \frac{\text{beta}(\frac{n+5}{2}, \frac{n-1}{2})}{\sqrt{\text{beta}(\frac{n}{2}+2, \frac{n}{2})\text{beta}(\frac{n}{2}+3, \frac{n}{2}-1)}}} \\ & \frac{\text{beta}(\frac{n+3}{2}, \frac{n+1}{2})}{\sqrt{\text{beta}(\frac{n}{2}+1, \frac{n}{2}+1)\text{beta}(\frac{n}{2}+2, \frac{n}{2})}} > \frac{\text{beta}(\frac{n+5}{2}, \frac{n-1}{2})}{\sqrt{\text{beta}(\frac{n}{2}+2, \frac{n}{2})\text{beta}(\frac{n}{2}+3, \frac{n}{2}-1)}} \\ & \frac{\text{beta}(\frac{n+3}{2}, \frac{n-1}{2})^{\frac{\frac{n-1}{2}+\frac{n+3}{2}}{\frac{n}{2}-1+\frac{n}{2}+1}}}{\sqrt{\text{beta}(\frac{n}{2}+1, \frac{n}{2}-1)^{\frac{\frac{n}{2}-1}{\frac{n}{2}-1+\frac{n}{2}+1}}\text{beta}(\frac{n}{2}+2, \frac{n}{2}+1)^{\frac{\frac{n}{2}}{\frac{n}{2}-1+\frac{n}{2}+1}}}} > \frac{\text{beta}(\frac{n+3}{2}, \frac{n-1}{2})^{\frac{\frac{n+3}{2}}{\frac{n+3}{2}+\frac{n-1}{2}}}}{\sqrt{\text{beta}(\frac{n}{2}+1, \frac{n}{2}-1)^{\frac{\frac{n}{2}+1}{\frac{n}{2}+1+\frac{n}{2}-1}}\text{beta}(\frac{n}{2}+2, \frac{n}{2}-1)^{\frac{\frac{n}{2}+2}{\frac{n}{2}+1+\frac{n}{2}-1}}}} \\ & \frac{\frac{n-1}{2}}{\sqrt{(\frac{n}{2}-1)(\frac{n}{2})}} > \frac{\frac{n+3}{2}}{\sqrt{(\frac{n}{2}+1)(\frac{n}{2}+2)}} \\ & (n-1)^2(n+2)(n+4) > (n+3)^2n(n-2) \\ & n > -1. \end{aligned}$$

Since  $n > 0$ , it always holds.

2. if  $x = x_0 + m$  holds, then also  $x = x_0 + m + 1$  holds:

i.e  $H(\text{beta}(\frac{n}{2}+1+m, \frac{n}{2}+1-m), \text{beta}(\frac{n}{2}+1+m+1, \frac{n}{2}+1-m-1)) < H(\text{beta}(\frac{n}{2}+1+m+1, \frac{n}{2}+1-m-1), \text{beta}(\frac{n}{2}+1+m+2, \frac{n}{2}+1-m-2))$  is what we know:

$$\begin{aligned} & \frac{\text{beta}(\frac{n+2m+5}{2}, \frac{n-2m-1}{2})^{\frac{n-2m-1}{n+2m+3}}}{\sqrt{\text{beta}(\frac{n}{2}+2+m, \frac{n}{2}-m)\text{beta}(\frac{n}{2}+3+m, \frac{n}{2}-m-1)^{\frac{n-2m}{n+2m+2}}}} \\ & > \frac{\text{beta}(\frac{n+2m+7}{2}, \frac{n-2m-3}{2})^{\frac{n-2m-3}{n+2m+5}}}{\sqrt{\text{beta}(\frac{n}{2}+2+m, \frac{n}{2}-m)\text{beta}(\frac{n}{2}+3+m, \frac{n}{2}-m-1)^{\frac{n-2m-2}{n+2m+6}}}} \end{aligned}$$

## 2.2.3. STATEMENT III

*Proof.* From Statement I and Statement II, we can conclude that:

$$\begin{aligned}
 &\text{when } x > x_0 \\
 &\quad H(\text{BI}(x), \text{BI}(x+1)) \\
 &\quad > H(\text{BI}(x_0), \text{BI}(x_0+1)); \\
 &\quad \text{i.e. } \Delta_I u(x) > \Delta_I u(x_0) \\
 &\text{when } x < x_0 \\
 &\quad H(\text{BI}(x), \text{BI}(x-1)) \\
 &\quad > H(\text{BI}(x_0), \text{BI}(x_0-1)); \\
 &\quad \text{i.e. } \Delta_I u(x) > \Delta_I u(x_0).
 \end{aligned}$$

$$\begin{aligned}
 &\frac{\text{beta}(\frac{n+2m+5}{2}, \frac{n-2m-1}{2})}{\sqrt{\text{beta}(\frac{n}{2}+2+m, \frac{n}{2}-m)\text{beta}(\frac{n}{2}+3+m, \frac{n}{2}-m-1)}} \\
 &> \frac{\text{beta}(\frac{n+2m+7}{2}, \frac{n-2m-3}{2})}{\sqrt{\text{beta}(\frac{n}{2}+2+m, \frac{n}{2}-m)\text{beta}(\frac{n}{2}+3+m, \frac{n}{2}-m-1)}} \\
 &< \sqrt{1 - \frac{\text{beta}(\frac{n+2m+5}{2}, \frac{n-2m-1}{2})}{\sqrt{\text{beta}(\frac{n}{2}+2+m, \frac{n}{2}-m)\text{beta}(\frac{n}{2}+3+m, \frac{n}{2}-m-1)}}} \\
 &< \sqrt{1 - \frac{\text{beta}(\frac{n+2m+7}{2}, \frac{n-2m-3}{2})}{\sqrt{\text{beta}(\frac{n}{2}+2+m, \frac{n}{2}-m)\text{beta}(\frac{n}{2}+3+m, \frac{n}{2}-m-1)}}} \\
 &\quad H(\text{beta}(\frac{n}{2}+2+m, \frac{n}{2}-m), \text{beta}(\frac{n}{2}+3+m, \frac{n}{2}-1-m)) \\
 &< H(\text{beta}(\frac{n}{2}+m+3, \frac{n}{2}-1-m), \text{beta}(\frac{n}{2}+m+4, \frac{n}{2}-m-2)) \text{ i.e. } \forall x \neq x_0, \Delta_I u(x) > \Delta_I u(x_0).
 \end{aligned}$$

□

i.e.  $x = x_0 + m + 1$  also holds when  $x = x_0 + m$  is valid. □

## 2.2.2. STATEMENT II

*Proof.*

$$\begin{aligned}
 &\because \Delta_I u(x) = |H(\text{BI}(x), r) - H(\text{BI}(y'), r)|, \\
 &\quad \max_{\{|x, y'| \leq 1; y' \in \mathcal{X}^n\}} \max_{\{r \in R\}}; \\
 &\because H(\text{BI}(x), r) - H(\text{BI}(y'), r) \\
 &\quad \leq H(\text{BI}(x), \text{BI}(y')); \\
 &\because \Delta_I u(x) = H(\text{BI}(x), \text{BI}(y')), \\
 &\quad \max_{\{|x, y'| \leq 1; y' \in \mathcal{X}^n\}}; \\
 &\because \Delta_I u(x) = \max\{H(\text{BI}(x), \text{BI}(x+1)), \\
 &\quad H(\text{BI}(x), \text{BI}(x-1))\};
 \end{aligned}$$

According to Statement I :

$$\begin{aligned}
 &\text{if } x > x_0 \\
 &\text{then } H(\text{BI}(x), \text{BI}(x-1)) \\
 &\quad < H(\text{BI}(x), \text{BI}(x+1)); \\
 &\text{then } \Delta_I u(x) = H(\text{BI}(x), \text{BI}(x+1)); \\
 &\text{if } x < x_0 \\
 &\text{then } H(\text{BI}(x), \text{BI}(x-1)) \\
 &\quad > H(\text{BI}(x), \text{BI}(x+1)); \\
 &\text{then } \Delta_I u(x) = H(\text{BI}(x), \text{BI}(x-1)); \\
 &\text{else } \Delta_I u(x_0) = H(\text{BI}(x_0), \text{BI}(x_0-1)) \\
 &\quad = H(\text{BI}(x_0), \text{BI}(x_0+1)).
 \end{aligned}$$

From above, we can conclude the Statement II.

□

## 3. Experimental Evaluations

We got some results from these mechanisms.

## 4. Dilation Property of Laplace Noise

*Proof.* We take 1-dimensional Laplace distribution,  $h(z) = \frac{1}{2}e^{-|z|}$ . The dilation property is:

$$Pr[z \in S] \leq e^{\frac{\epsilon}{2}} Pr[z \in e^\lambda S] + \frac{\delta}{2}$$

In this case, we have  $\alpha = \frac{\epsilon}{2}$ ,  $\beta = \frac{\epsilon}{2\rho_{\delta/3}(|z|)}$  or  $\frac{\epsilon}{2\ln(2/\delta)}$ . We have some prior knowledge that  $|\lambda| \leq \beta$ .

 • case 1:  $\lambda > 0$ 

$$\begin{aligned}
 &\because h(e^\lambda z) = \frac{1}{2}e^{-|e^\lambda z|} < \frac{1}{2}e^{-|z|} = h(z) \\
 &\because \frac{Pr[z \in e^\lambda S]}{Pr[z \in S]} = \frac{\int_{e^\lambda S} \frac{1}{2}e^{-|z|} dz}{\int_S \frac{1}{2}e^{-|z|} dz} = \frac{\int_S \frac{1}{2}e^{-|e^\lambda z|} e^\lambda dz}{\int_S \frac{1}{2}e^{-|z|} dz} \\
 &\quad = \frac{e^{-|e^\lambda z|} e^\lambda}{e^{-|z|}} = \frac{e^\lambda h(e^\lambda z)}{h(z)} \leq e^\lambda \\
 &\because \ln\left(\frac{e^\lambda h(e^\lambda z)}{h(z)}\right) \leq \lambda \\
 &\because \lambda \leq \beta = \frac{\epsilon}{2\ln(3/\delta)}, \delta < 1 \\
 &\because \lambda \leq \frac{\epsilon}{2} \\
 &\because \frac{Pr[z \in e^\lambda S]}{Pr[z \in S]} \leq \frac{\epsilon}{2}
 \end{aligned}$$

 • case 2:  $\lambda < 0$

From integral property, we firstly have:

$$\frac{Pr[z \in e^\lambda S]}{Pr[z \in S]} = \frac{e^{-|e^\lambda z|} e^\lambda}{e^{-|z|}} = \frac{h(e^\lambda z) e^\lambda}{h(z)} = e^\lambda e^{|z|(1-e^\lambda)}$$

$$Pr[\overline{G}] = Pr[|z| > \log(\frac{2}{\delta})] = \exp(-\log(\frac{2}{\delta})) = \frac{\delta}{2}$$

Then, we can get

$$\begin{aligned} \because 1 - e^\lambda &\leq |\lambda| & Pr[z \in S] &\leq Pr[z \in S \cap G] + Pr[z \in \overline{G}] \\ \therefore \ln\left(\frac{h(e^\lambda z) e^\lambda}{h(z)}\right) &\leq \lambda + |z||\lambda| & &\leq e^{\frac{\epsilon}{2}} Pr_{z \sim h'}[z \in S \cap G] + \frac{\delta}{2} \\ \because \lambda < 0 & & &\leq e^{\frac{\epsilon}{2}} Pr_{z \sim h'}[z \in S] + \frac{\delta}{2} \\ \therefore \ln\left(\frac{h(e^\lambda z) e^\lambda}{h(z)}\right) &\leq |z||\lambda| & &= e^{\frac{\epsilon}{2}} Pr_{z \sim h}[z \in e^\lambda S] + \frac{\delta}{2} \end{aligned}$$

By setting  $h'(z) = e^\lambda h(e^\lambda z)$ , we can get:

i.e. the dilation property.

$$\begin{aligned} \ln\left(\frac{h'(z)}{h(z)}\right) &\leq |z||\lambda| \\ \Rightarrow h'(z) &\leq e^{|z||\lambda|} h(z) \end{aligned}$$

□

By exchanging the notation of  $h'$  and  $h$ , we have:

$$h(z) \leq e^{|z||\lambda|} h'(z)$$

i.e.

$$Pr_{z \sim h}[z \in S] \leq e^{|z||\lambda|} Pr_{z \sim h'}[z \in S] = e^{|z||\lambda|} Pr_{z \sim h}[z \in e^\lambda S]$$

We consider an event  $G = \{|z| \leq \log(\frac{2}{\delta})\}$ . Under this event, we have:

$$\begin{aligned} |z||\lambda| &\leq \log\left(\frac{2}{\delta}\right) |\lambda| \\ &\leq \log\left(\frac{2}{\delta}\right) \beta \\ &\leq \log\left(\frac{2}{\delta}\right) \frac{\epsilon}{2 \log(\frac{3}{\delta})} \\ &\leq \frac{\epsilon}{2}. \end{aligned}$$

Then:

$$\begin{aligned} Pr_{z \sim h}[z \in S \cap G] &\leq e^{|z||\lambda|} Pr_{z \sim h'}[z \in S \cap G] \\ &\leq e^{\frac{\epsilon}{2}} Pr_{z \sim h'}[z \in S \cap G] \end{aligned}$$

We also have: