Notes of DP - Bayesian Inference

1 Bayesian Inference Based on Dirichlet-Bernoulli Distribution

In the Bayesian inference, first there is a prior distribution $\pi(\boldsymbol{\xi})$ to present our belief about parameter $\boldsymbol{\xi}$. Then, we get some observed data x sizing n, and produce a posterior distribution $Pr(\boldsymbol{\xi}|x)$. The Bayesian inference is based on the Bayes' rule to calculate the posterior distribution:

$$Pr(\boldsymbol{\xi}|x) = \frac{Pr(x|\boldsymbol{\xi})\pi(\boldsymbol{\xi})}{Pr(x)}$$

It is denoted as $\mathsf{BI}(x, \pi(\boldsymbol{\xi}))$ taking an observed data set $x \in \mathcal{X}^n$ and a prior distribution $\pi(\boldsymbol{\xi})$ as input, outputting a posterior distribution of the parameter $\boldsymbol{\alpha}$. For conciseness, when prior is given, we use $\mathsf{BI}(x)$. n is the size of the observed data size.

In our inference algorithm, we take a Dirichlet distribution as prior belief for the parameters, $\mathsf{DL}(\boldsymbol{\alpha})$, where $\pi(\boldsymbol{\xi}) = \mathsf{DL}(\boldsymbol{\xi}|\boldsymbol{\alpha}) = \frac{\prod_{i=1}^{m} \xi_i^{\alpha_i}}{B(\boldsymbol{\alpha})}$, and the Bernoulli distribution as the statistic model for $Pr(x|\boldsymbol{\alpha})$. m is the order of the Dirichlet distribution.

We give a inference process based on a concrete example, where we throw an irregular m sides dice. We want to infer the probability of getting each side $\boldsymbol{\xi}$. We get a observed data set $\{s_{k_1}, s_{k_2}, \cdots, s_{k_n}\}$ by throwing the dice n times, where $k_i \in \{1, 2, \cdots, m\}$ denotes the side we get when we throw the dice the i^{th} time. The posterior distribution is still a Dirichlet distribution with parameters $(\alpha_1 + n_1, \alpha_2 + n_2, \cdots, \alpha_m + n_m)$, where n_i is the appearance time of the side i in total.

In the case that $m = 2$, it is reduced to a Beta distribution $\mathsf{beta}(\alpha, \beta)$. The m side dice change into a irregular coin with side 1 and side 2. The posterior is computed as $(\alpha + n_1, \beta + n_0)$, where n_1 is the appearance time of side 1 in the observed data set and n_0 is the appearance time of the other side.

2 Algorithm Setting up

For now, we already have a prior distribution *prior*, an observed data set x.

2.1 Exponential Mechanism with Global Sensitivity

In exponential mechanism, candidate set R can be obtained by enumerating $y \in \mathcal{X}^n$, i.e.

$$R = \{\mathsf{BI}(y) \mid y \in \mathcal{X}^n\}.$$

Hellinger distance H is used here to score these candidates. The utility function:

$$u(x, r) = -\mathsf{H}(\mathsf{BI}(x), r); r \in R. \tag{1}$$

Exponential mechanism with global sensitivity selects and outputs a candidate $r \in R$ with probability proportional to $exp(\frac{\epsilon u(x,r)}{2\Delta_g u})$:

$$P[r] = \frac{exp(\frac{\epsilon u(x,r)}{2\Delta_g u})}{\Sigma_{r' \in R}\ exp(\frac{\epsilon u(x,r')}{2\Delta_g u})},$$

where global sensitivity is calculated by:

$$\Delta_g u = \max_{\{|x',y'| \leqslant 1; x',y' \in \mathcal{X}^n\}} \max_{\{r \in R\}} |\mathsf{H}(\mathsf{BI}(x'), r) - \mathsf{H}(\mathsf{BI}(y'), r)|$$

The basic exponential mechanism is $\epsilon-$differential privacy[1].

2.2 Exponential Mechanism with Local Sensitivity

Exponential mechanism with local sensitivity share the same candidate set and utility function as it with global sensitivity. This outputs a candidate $r \in R$ with probability proportional to $exp(\frac{\epsilon u(x,r)}{2\Delta_l u})$:

$$P[r] = \frac{exp(\frac{\epsilon u(x,r)}{2\Delta_l u})}{\Sigma_{r' \in R}\ exp(\frac{\epsilon u(x,r')}{2\Delta_l u})},$$

where local sensitivity is calculated by:

$$\Delta_l u(x) = \max_{\{|x,y'| \leqslant 1; y' \in \mathcal{X}^n\}} \max_{\{r \in R\}} .\mathsf{H}(\mathsf{BI}(x), r) - \mathsf{H}(\mathsf{BI}(y'), r)|$$

The exponential mechanism with local sensitivity is non differential privacy[1].

2.3 Exponential Mechanism with Smooth Sensitivity

2.3.1 Algorithm Setting up

The candidate set and utility function are still the same as before, differ only in the sensitivity. It will output a candidate $r \in R$ with probability proportional to $\exp(\frac{\epsilon u(x,r)}{2S(x)})$:

$$P[r] = \frac{exp(\frac{\epsilon u(x,r)}{2S(x)})}{\Sigma_{r' \in R}\ exp(\frac{\epsilon u(x,r')}{2S(x)})},$$

where the sensitivity in mechanism is smooth sensitivity $S(x)$[2], calculated by:

$$S_\beta(x) = \max(\Delta_l u(x), \max_{y \neq x; y \in D^n}(\Delta_l u(y) \cdot e^{-\beta d(x,y)})),$$

where $\beta = \beta(\epsilon, \delta)$. In our private Bayesian inference mechanism, we set the β as $\ln(1 - \frac{\epsilon}{2\ln(\frac{\delta}{2(n+1)})})$.

2.3.2 Sliding Property of Exponential Mechanism

Lemma 2.1. *for any exponential mechanism $\mathcal{M}_E(x, u, \mathcal{R})$, $\lambda = f(\epsilon, \delta)$, ϵ and $|\delta| < 1$, the sliding property holds:*

$$\underset{z \sim \mathcal{M}_E(x,u,\mathcal{R})}{Pr}[u(r,x) = \hat{s}] \leqslant e^{\frac{\epsilon}{2}} \underset{z \sim \mathcal{M}_E(x,u,\mathcal{R})}{Pr}[u(r,x) = (\Delta + \hat{s})] + \frac{\delta}{2},$$

Proof. We denote the normalizer of the probability mass in $\mathcal{M}_E(x, u, \mathcal{R})$: $\sum_{r' \in \mathcal{R}} exp(\frac{\epsilon u(r',x)}{2S(x)})$ as NL_x:

$$\begin{aligned}
LHS = \underset{z \sim \mathcal{M}_E(x,u,\mathcal{R})}{Pr}[u(r,x) = \hat{s}] &= \frac{exp(\frac{\epsilon \hat{s}}{2S(x)})}{NL_x} \\
&= \frac{exp(\frac{\epsilon(\hat{s}+\Delta-\Delta)}{2S(x)})}{NL_x} \\
&= \frac{exp(\frac{\epsilon(\hat{s}+\Delta)}{2S(x)} + \frac{-\epsilon\Delta}{2S(x)})}{NL_x} \\
&= \frac{exp(\frac{\epsilon(\hat{s}+\Delta)}{2S(x)})}{NL_x} \cdot e^{\frac{-\epsilon\Delta}{2S(x)})}.
\end{aligned}$$

By bounding the $\Delta \geqslant -S(x)$, we can get:

$$\frac{exp(\frac{\epsilon(\hat{s}+\Delta)}{2S(x)})}{NL_x} \cdot e^{\frac{-\epsilon\Delta}{2S(x)}} \leqslant \frac{exp(\frac{\epsilon(\hat{s}+\Delta)}{2S(x)})}{NL_x} \cdot e^{\frac{\epsilon}{2}}$$
$$= e^{\frac{\epsilon}{2}} \underset{z\sim\mathcal{M}_E(x,u,\mathcal{R})}{Pr}[u(r,x) = (\Delta+\hat{s})] \leqslant RHS$$

□

2.3.3 Dilation Property of Exponential Mechanism

Lemma 2.2. *for any exponential mechanism $\mathcal{M}_E(x,u,\mathcal{R})$, $\lambda < |\beta|$, ϵ, $|\delta| < 1$ and $\beta \leqslant \ln(1 - \frac{\epsilon}{2\ln(\frac{\delta}{2(n+1)})})$, the dilation property holds:*

$$\underset{r\sim\mathcal{M}_E(x,u,\mathcal{R})}{Pr}[u(r) = z] \leqslant e^{\frac{\epsilon}{2}} \underset{r\sim\mathcal{M}_E(x,u,\mathcal{R})}{Pr}[u(r) = e^{\lambda}z] + \frac{\delta}{2},$$

where the sensitivity in mechanism is still smooth sensitivity as above.

Proof. The sensitivity is always greater than 0, and we are using $-\mathsf{H}(\mathsf{BI}(x), r)$ for utility function, i.e., $u(r) \leqslant 0$, we need to consider two cases that $\lambda < 0$, and $\lambda > 0$:

We set the $h(z) = Pr[u(\mathcal{M}_E(x,u,\mathcal{R})) = z] = \frac{exp(\frac{\epsilon z}{2S(x)})}{NL_x}$.

We first consider $\lambda < 0$. In this case, $1 < e^{\lambda}$, so the ratio $\frac{h(z)}{h(e^{\lambda}z)} = \frac{exp(\frac{\epsilon z}{2S(x)})}{exp(\frac{\epsilon(z\cdot e^{\lambda})}{2S(x)})}$ is at most $\frac{\epsilon}{2}$.

Next, we proof the dilation property for $\lambda > 0$, The ratio of $\frac{h(z)}{h(e^{\lambda}z)}$ is $\exp(\frac{\epsilon}{2} \cdot \frac{u(\mathcal{M}_E(x,u,\mathcal{R}))(1-e^{\lambda})}{S(x)})$. Consider the event $G = \{\mathcal{M}_E(x,u,\mathcal{R}) : u(\mathcal{M}_E(x,u,\mathcal{R})) \leqslant \frac{S(x)}{(1-e^{\lambda})}\}$. Under this event, the log-ratio above is at most $\frac{\epsilon}{2}$. The probability of G under density $h(z)$ is $1 - \frac{\delta}{2}$. Thus, the probability of a given event z is at most $Pr[z \cap G] + Pr[\overline{G}] \leqslant e^{\frac{\epsilon}{2}} Pr[e^{\lambda}z \cap G] + \frac{\delta}{2} \leqslant e^{\frac{\epsilon}{2}} Pr[e^{\lambda}z] + \frac{\delta}{2}$.

Detail proof:

- $\lambda < 0$

  The left hand side will always be smaller than 0 and the right hand side greater than 0. This will always holds, i.e.

- $\lambda > 0$

  Because $\hat{s} = u(r)$ where $r \sim \mathcal{M}_E(x,u,\mathcal{R})$, we can substitute $\hat{s}$ with $u(\mathcal{M}_E(x,u,\mathcal{R}))$. Then, what we need to proof under the case $\lambda > 0$ is:

  $$u(\mathcal{M}_E(x,u,\mathcal{R})) \leqslant \frac{S(x)}{(1-e^{\lambda})}$$

  By applying the accuracy property of exponential mechanism, we bound the probability that the equation holds with probability:

  $$Pr[u(\mathcal{M}_E(x,u,\mathcal{R})) \leqslant \frac{S(x)}{(1-e^{\lambda})}] \leqslant \frac{|\mathcal{R}|exp(\frac{\epsilon S(x)}{(1-e^{\lambda})}/2S(x))}{|\mathcal{R}_{OPT}|exp(\epsilon OPT_{u(x)}/2S(x))}$$

  In our Bayesian Inference mechanism, the size of the candidate set $\mathcal{R}$ is equal to the size of observed data set plus 1, i.e., $n+1$, and $OPT_{u(x)} = 0$, then we have:

  $$Pr[u(\mathcal{M}_E(x,u,\mathcal{R})) \leqslant \frac{S(x)}{(1-e^{\lambda})}] = (n+1)exp(\frac{\epsilon S(x)}{(1-e^{\lambda})}/2S(x))$$
  $$= (n+1)exp(\frac{\epsilon}{2(1-e^{\lambda})})$$

When we set $\lambda \leqslant \ln(1 - \frac{\epsilon}{2\ln(\frac{\delta}{2(n+1)})})$, it is easily to derive that $Pr[u(\mathcal{M}_E(x, u, \mathcal{R})) \leqslant \frac{S(x)}{(1-e^{\lambda})}] \leqslant \frac{\delta}{2}$.

□

3 Experimental Evaluations

3.1 Computation Efficiency

The formula for computing the local sensitivity is presented in Sec. 2.2: $\max\limits_{\{|x,y'|\leqslant 1; y' \in \mathcal{X}^n\}} \max\limits_{\{r \in R\}} \{\mathsf{H}(\mathsf{BI}(x), r) - \mathsf{H}(\mathsf{BI}(y'), r)|\}$ can be reduced to $\max\limits_{\{|x,y'|\leqslant 1; y' \in \mathcal{X}^n\}} \mathsf{H}(\mathsf{BI}(x), \mathsf{BI}(y'))|$ by applying the distance triangle property. i.e., the maximum value over $\max\limits_{r \in R}$ always happen when $r = \mathsf{BI}(x)$ itself, where $\Delta_l u(x) = \max\limits_{\{|x,y'|\leqslant 1; y' \in \mathcal{X}^n\}} \{\mathsf{H}(\mathsf{BI}(x), \mathsf{BI}(x)) - \mathsf{H}(\mathsf{BI}(y'), \mathsf{BI}(x))|\} = \max\limits_{\{|x,y'|\leqslant 1; y' \in \mathcal{X}^n\}} \{\mathsf{H}(\mathsf{BI}(y'), \mathsf{BI}(x))|\}$. We also have some experiments for validating our proposal as in Fig. 3.1, where we calculate the $\max\limits_{\{|x,y'|\leqslant 1; y' \in \mathcal{X}^n\}}$ value for every candidate $r \in R$. It is shown that maximum value taken when $r = \mathsf{BI}(x)$.

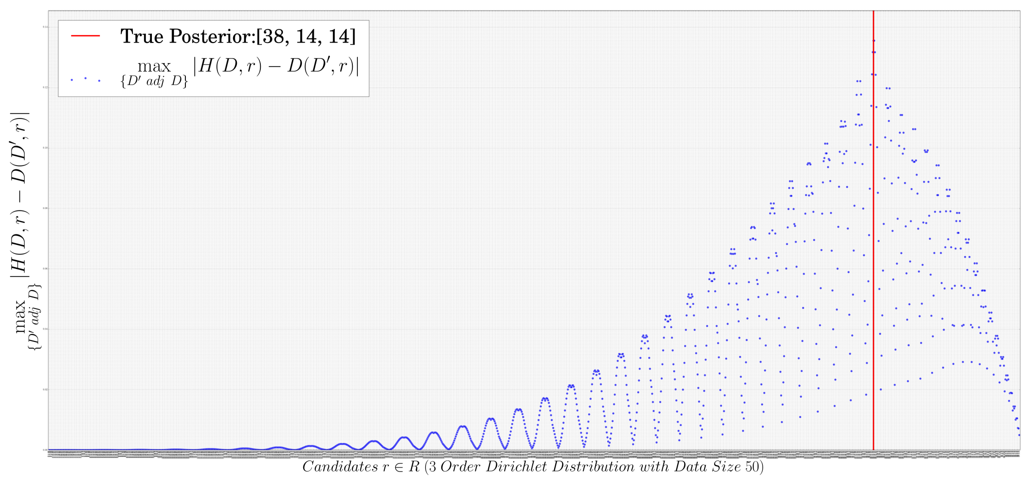


Figure 1: Experimental Results for Finding the Local Sensitivity Efficiently

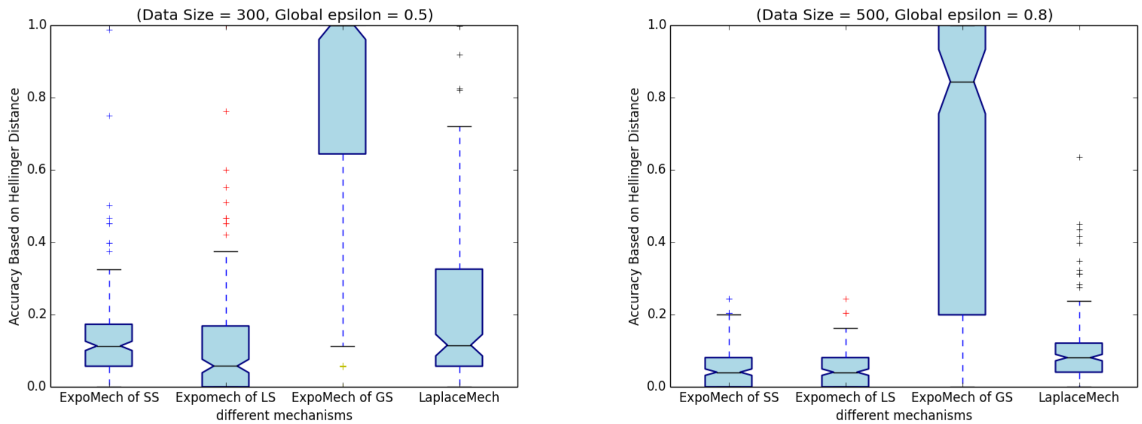
3.2 Accuracy Study Based on Hellinger Distance

We study the accuracy property of the exponential mechanism with three kind of sensitivity, as well as the Laplace mechanism for comparison. As in Fig. 2, the two figures shown the accuracy of four algorithms under same configuration. The accuracy is measured by Hellinger distance between output r^* and the true inference result $\mathsf{BI}(x)$, $\mathsf{H}(r^*, \mathsf{BI}(x))$.

Then, we extended the beta to the Bernoulli distribution, got the result in Fig. 3

3.3 Accuracy Study Based on L_1 Norm Distance

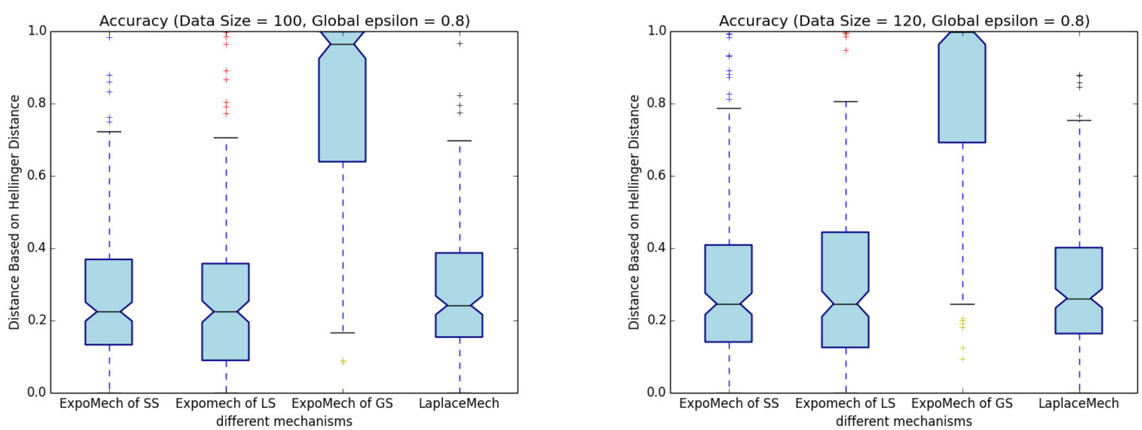
For comparison, we study the accuracy of these four mechanisms based on l_1 norm, with the Dirichlet distribution $\mathsf{DL}(7, 4, 5)$ and data size 150, shown in Fig. 4.



(a) Data size $n = 500$ with global $\epsilon = 0.5$

(b) Data size $n = 300$ with global $\epsilon = 0.5$

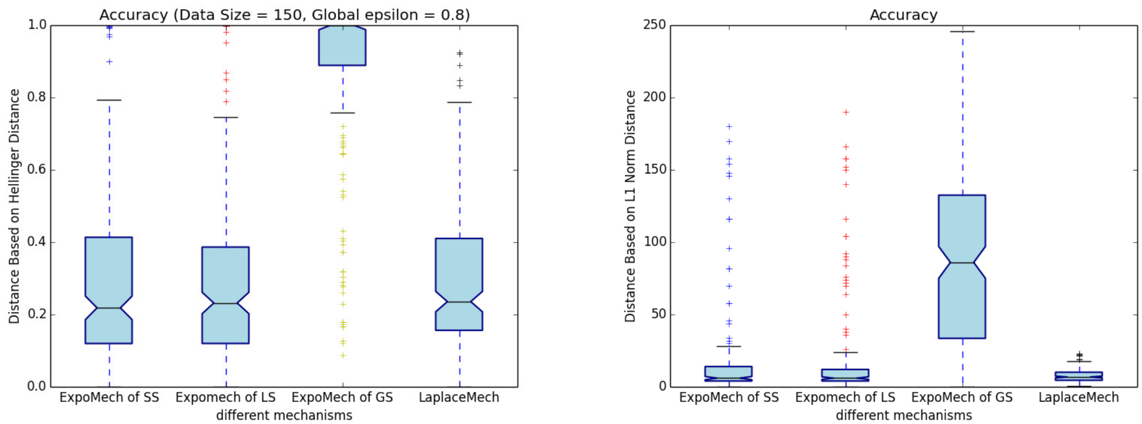
Figure 2: The experimental results of accuracy of algorithms with Beta prior distribution $\mathsf{beta}(7, 4)$ based on Hellinger distance



(a) Data size $n = 100$, the exponential mechanism with global sensitivity 0.239992747797 is 0.8 -DP, with local sensitivity 0.08 is Non-Private and with 0.0699407108115 - bound smooth sensitivity 0.09 is (0.8,0.8)-DP

(b) Data size $n = 120$, the exponential mechanism with global sensitivity 0.239992747797 is 0.8 -DP, with local sensitivity 0.0945 is Non-Private, with 0.0677791100173 - bound smooth sensitivity 0.096 is (0.8,0.8)-DP

Figure 3: The experimental results of accuracy of algorithms with Dirichlet prior distribution $\mathsf{DL}(7, 4, 5)$ based on Hellinger distance



(a) accuracy measurement based on Hellinger distance

(b) accuracy measurement based on l_1 norm

Figure 4: The experimental results of accuracy of algorithms with Dirichlet prior distribution $\mathsf{DL}(7, 4, 5)$ based on Hellinger distance and l_1 norm, where data size $n = 150$, the exponential mechanism with global sensitivity 0.239992747797 is 0.8 -DP, with local sensitivity 0.0945 is Non-Private, with 0.0677791100173 -bound smooth sensitivity 0.096 is (0.8,0.8)-DP

References

[1] Cynthia Dwork, Aaron Roth, et al. "The algorithmic foundations of differential privacy". In: *Foundations and Trends® in Theoretical Computer Science* 9.3–4 (2014), pp. 211–407.

[2] Kobbi Nissim, Sofya Raskhodnikova, and Adam Smith. "Smooth sensitivity and sampling in private data analysis". In: *Proceedings of the thirty-ninth annual ACM symposium on Theory of computing*. ACM. 2007, pp. 75–84.