Tailoring Differentially Private Bayesian Inference to Distance Between Distributions

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Objectives

- Design a differentially private Bayesian inference mechanism.
- Improve accuracy by calibrating noise to the sensitivity of a metric over distributions (e.g. Hellinger distance ($\mathcal{H}$), f-divergences, etc. . .).

An example of Bayesian inference: the Beta-Binomial model

- Prior on θ : $\mathbb{P}_\theta = \mathbf{beta}(\alpha, \beta), \alpha, \beta \in \mathbb{R}^+$, observed data $\mathbf{x} = (x_1, \dots, x_n) \in \{0, 1\}^n, n \in \mathbb{N}$.
- Likelihood function: $\mathbb{L}_{\theta|\mathbf{x}} = \theta^{\Delta\alpha}(1-\theta)^{n-\Delta\alpha}$, where $\Delta\alpha = \sum_{i=1}^{n} x_i$.
- Posterior on θ: $\mathsf{BI}(x) \equiv \mathbb{P}_{\theta|\mathbf{x}} = \mathbf{beta}(\alpha + \Delta\alpha, \beta + n - \Delta\alpha) \propto \mathbb{L}_{\theta|\mathbf{x}} \cdot \mathbb{P}_\theta$.

Differentially private Bayesian inference

- Baseline approach:
  - Release $\mathbf{beta}(\alpha + \lfloor\widetilde{\Delta\alpha}\rfloor_0^n, \beta + n - \lfloor\widetilde{\Delta\alpha}\rfloor_0^n)$,
  - $\widetilde{\Delta\alpha} \sim \mathcal{L}(\Delta\alpha, \frac{S}{\epsilon})$
  - $[\![S \propto || \cdot ||_1]\!]$.
  - Measure accuracy with a metric over distributions, e.g. $\mathcal{H}$.

But S grows linearly with the dimension: too noisy when we generalize to Dirichlet-Multinomial ($\mathbf{DL}(\cdot)$) model.

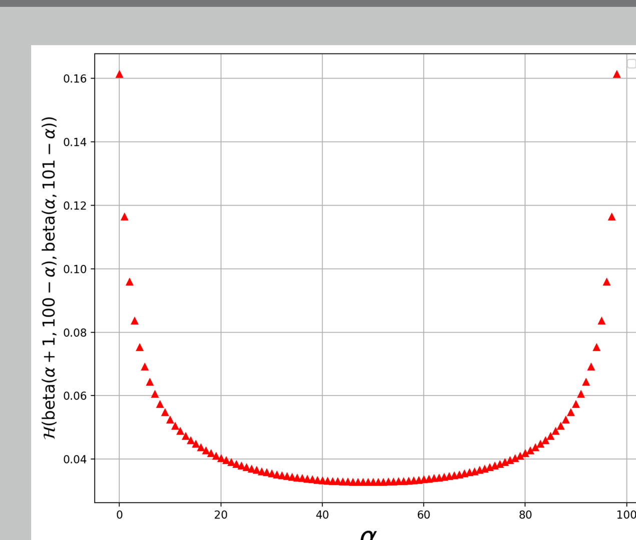


Figure 1:Sensitivity of $\mathcal{H}$. There is a gap between Global and Local sensitivity.

- Another approach:
  - Calibrate noise w.r.t *global* sensitivity of $\mathcal{H}$: but global sensitivity is still too big.
  - Fig. 1 shows that there is a gap between global and local sensitivity of $\mathcal{H}$.
- A better approach:
  - Calibrate noise w.r.t. the *smooth* sensitivity of $\mathcal{H}$.

Our approach: smoothed Hellinger distance based exponential mechanism

We define the mechanism $\mathcal{M}_\mathcal{H}$ which produces an element r in $\mathcal{R}_{\text{post}}$ with probability:

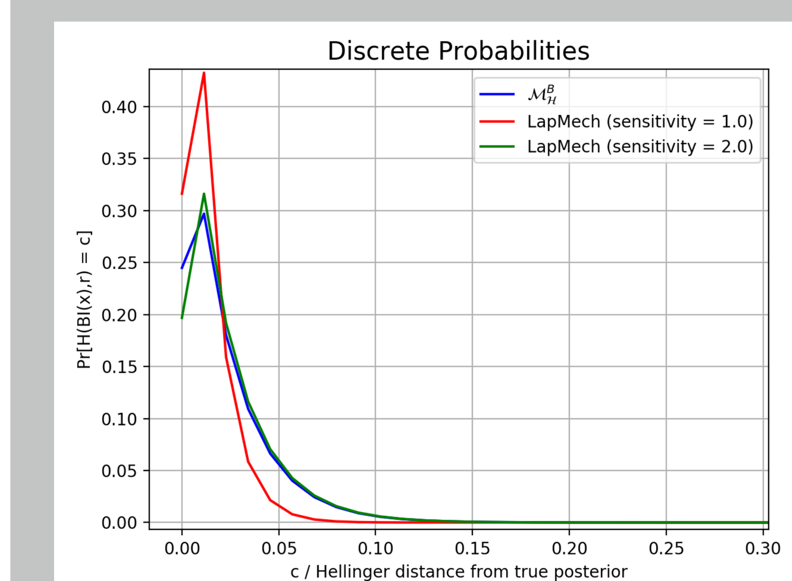
$$\mathbb{P}_{r\sim\mathcal{M}_\mathcal{H}} = \frac{\exp\left(\frac{-\epsilon\cdot\mathcal{H}(\mathsf{BI}(x),r)}{2\cdot S(x)}\right)}{\sum_{r\in\mathcal{R}_{\text{post}}}\exp\left(\frac{-\epsilon\cdot\mathcal{H}(\mathsf{BI}(x),r)}{2\cdot S(x)}\right)}$$

- $\mathcal{R}_{\text{post}} \equiv \{\mathbf{beta}(\alpha', \beta') \mid \alpha' = \alpha + \Delta\alpha, \beta' = \beta + n - \Delta\alpha\}$. With prior distribution $\beta_{\text{prior}} = \mathbf{beta}(\alpha, \beta)$.
- $-\mathcal{H}(\mathsf{BI}(x), r)$ denotes the scoring function.
- $S(x) \equiv \max_{x' \in \{0,1\}^n}\left\{LS(x') \cdot e^{-\gamma\cdot d(x,x')}\right\}$: smooth sensitivity[1], d is the Hamming distance.
- $LS(x') \equiv \max_{y\in\mathcal{X}^n:\mathrm{adj}(y,x'),r\in\mathcal{R}} |\mathcal{H}(\mathsf{BI}(y), r) - \mathcal{H}(\mathsf{BI}(x'), r)|$ is the local sensitivity of $\mathbf{x}', \gamma = \ln(1 - \frac{\epsilon}{2\ln(\frac{\delta}{2(n+1)})})$.

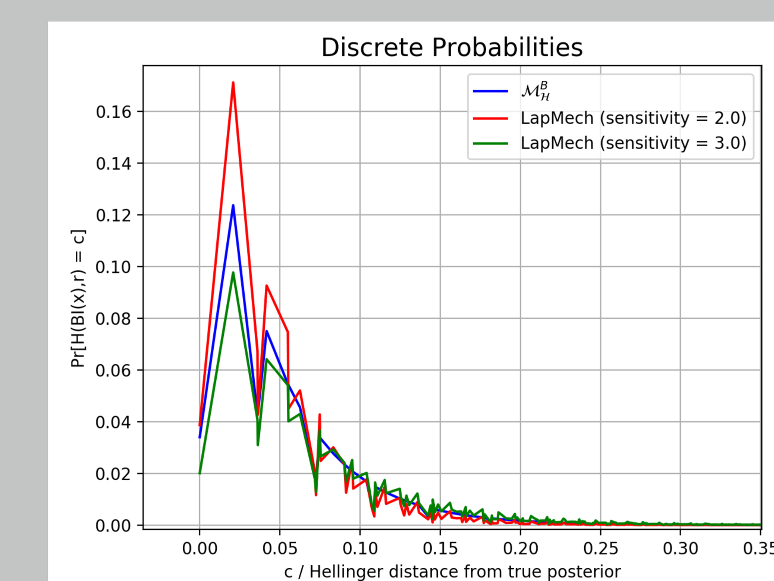
Preliminary experimental results

Experiments are about three mechanisms and plotted as follows:

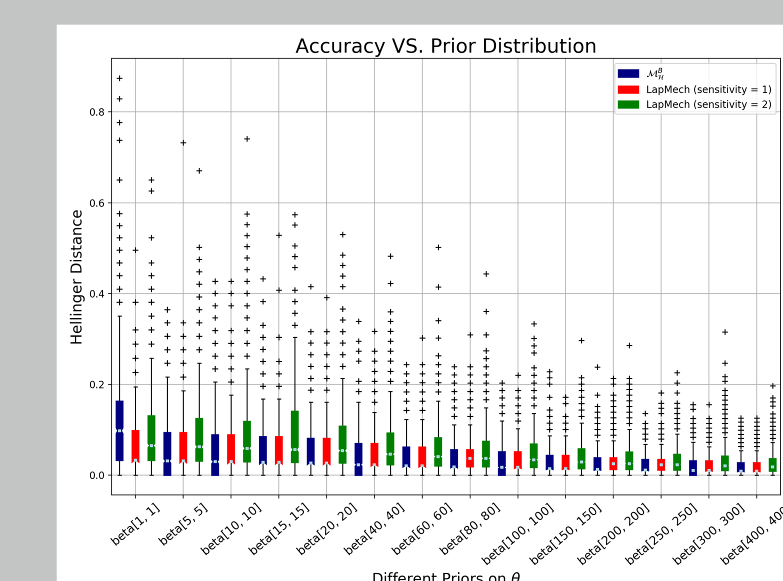
- **Green**: Baseline approach.
- **Red**: Improved baseline approach with sensitivity 1 in 2 dimensions and 2 in higher dimensions. Indeed: the number of elements in every bin always sums up to n and hence $||\mathsf{BI}(x) - \mathsf{BI}(x')||_1 \leq 2$, when $\mathbf{adj}(x, x')$.
- **Blue**: $\mathcal{M}_\mathcal{H}$. The fact that there is only one candidate distribution which achieves the highest score and different distributions which achieve a sub-optimal score explains the (highest) peaks in Fig. 2(a) (and Fig. 2(b)).



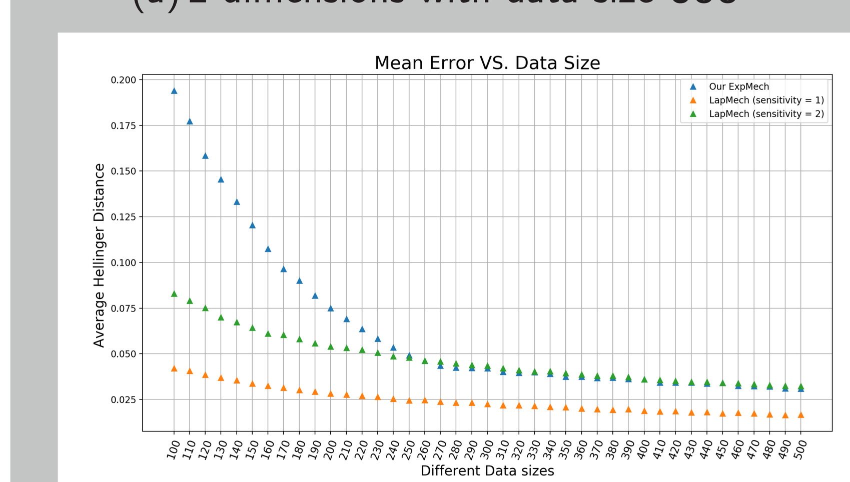
(a) 2 dimensions with data size **600**



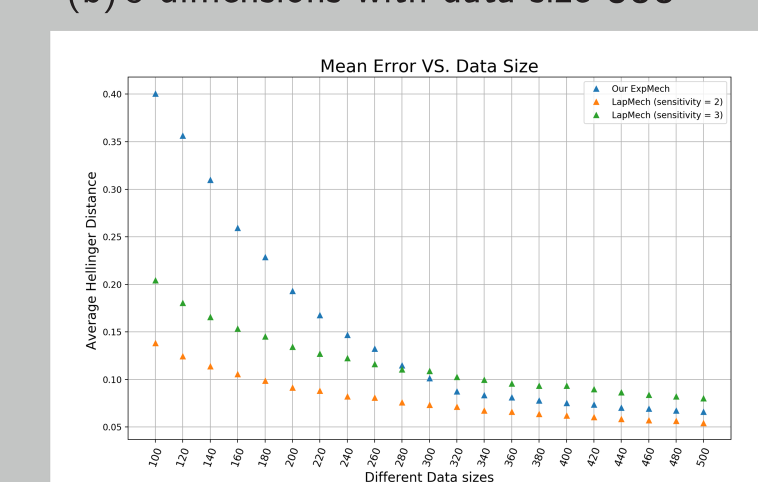
(b) 3 dimensions with data size **600**



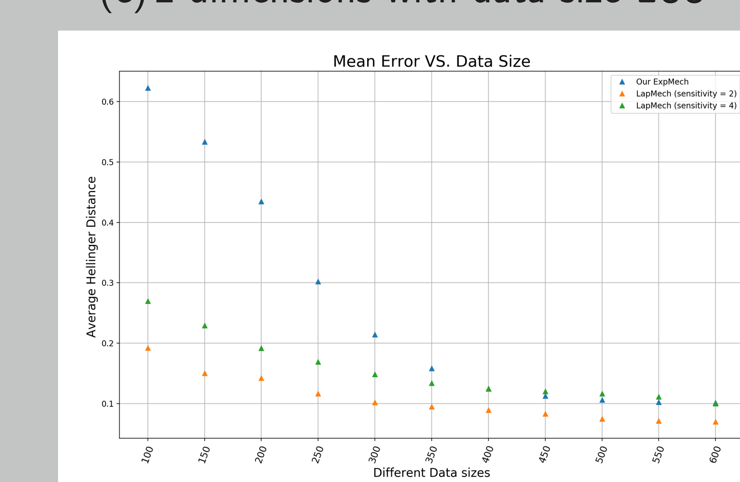
(c) 2 dimensions with data size **100**



(d) 2 dimensions, data size $\in [100, 500]$



(e) 3 dimensions, data size $\in [100, 500]$



(f) 4 dimensions, data size $\in [100, 600]$

Figure 2:Priors are $\mathbf{beta}(1, 1), \mathbf{DL}(1, 1, 1)$ and $\mathbf{DL}(1, 1, 1, 1)$ (except for Figure 2(c)) , balanced datasets, $\epsilon = 1.0$ and $\delta = 10^{-8}$.

Conclusion

- $\mathcal{M}_\mathcal{H}$ outperforms the baseline approach but not the improved one.
- By increasing the prior parameters, $\mathcal{M}_\mathcal{H}$ can outperform both the baseline approach and the improved one.

References

[1] Kobbi Nissim, Sofya Raskhodnikova, and Adam Smith. Smooth sensitivity and sampling in private data analysis. In *Proceedings of the thirty-ninth annual ACM symposium on Theory of computing*, pages 75–84. ACM, 2007.

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