

Differentially Private Bayesian Inference

1 Preliminary

- $\boldsymbol{\theta}$: The parameter vector of multinomial distribution, $\boldsymbol{\theta} \in [0, 1]^k$.
 \mathbf{x} : Observed dataset. $\mathbf{x} \in \mathcal{X}^n, |\mathcal{X}| = k$
 $\text{Dir}(\boldsymbol{\alpha})$: Dirichlet distribution. The prior or posterior distribution over $\boldsymbol{\theta}$.
 $\text{DirP}(\boldsymbol{\alpha}, \boldsymbol{\theta}, \mathbf{x})$: Posterior distribution over $\boldsymbol{\theta}$ from Bayesian inference given prior distribution $\text{Dir}(\boldsymbol{\alpha})$ and observed data set \mathbf{x} .
 $\mathcal{H}(\cdot, \cdot)$: Hellinger Distance between two distributions. $\mathcal{H}(\text{Dir}(\boldsymbol{\alpha}_1), \text{Dir}(\boldsymbol{\alpha}_2)) = \sqrt{1 - \frac{B(\frac{\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2}{2})}{\sqrt{B(\boldsymbol{\alpha}_1)B(\boldsymbol{\alpha}_2)}}}$
 $u(\mathbf{x}, r)$: Scoring function given $\boldsymbol{\alpha}$ and $\boldsymbol{\theta}$ for candidate r . $u(\mathbf{x}, r) = -\mathcal{H}(\text{DirP}(\boldsymbol{\alpha}, \boldsymbol{\theta}, \mathbf{x}), r)$
 GS : Global sensitivity of Hellinger distance. $GS = \sqrt{1 - \pi/4}$
 $LS(\mathbf{x})$: Local sensitivity of Hellinger distance for \mathbf{x} .

$$LS(\mathbf{x}) = \max_{\mathbf{x}' \in \mathcal{X}^n, \text{adj}(\mathbf{x}, \mathbf{x}'), r \in \mathcal{R}_{\boldsymbol{\alpha}}} |\mathcal{H}(\text{DirP}(\mathbf{x}, \boldsymbol{\alpha}), r) - \mathcal{H}(\text{DirP}(\mathbf{x}', \boldsymbol{\alpha}), r)|$$

 $S(\mathbf{x})$: γ -smooth sensitivity. $S(\mathbf{x}) = \max_{\mathbf{x}' \in \mathcal{X}^n} \left\{ \frac{1}{\frac{1}{LS(\mathbf{x}')} + \gamma \cdot \text{Himming}(\mathbf{x}, \mathbf{x}')} \right\}$

2 Private Mechanisms

Algorithm 1 LSZhang[1] - Calibrating noise w.r.t. ℓ_1 norm and Dimension

$\mathbf{x} \in \mathcal{X}^n, \text{Dir}(\boldsymbol{\alpha})$
let $\boldsymbol{\alpha}' = \text{DirP}(\boldsymbol{\alpha}, \boldsymbol{\theta}, \mathbf{x})$
Initialize a vector $\tilde{\boldsymbol{\alpha}} = (0, \dots, 0) \in \mathbb{N}^{|\mathcal{X}|}$
For $i = 1 \dots |\mathcal{X}| - 1$:
 let $\tilde{\alpha}_i = \alpha_i + \lfloor (\alpha'_i - \alpha_i) + \text{Lap}(0, \frac{2}{\epsilon}) \rfloor_0^n$
return $\tilde{\boldsymbol{\alpha}}$

Algorithm 2 LSDim - Calibrating noise w.r.t. ℓ_1 norm

$\mathbf{x} \in \mathcal{X}^n, \text{Dir}(\boldsymbol{\alpha})$
let $\boldsymbol{\alpha}' = \text{DirP}(\boldsymbol{\alpha}, \boldsymbol{\theta}, \mathbf{x})$
Initialize a vector $\tilde{\boldsymbol{\alpha}} = (0, \dots, 0) \in \mathbb{N}^{|\mathcal{X}|}$
For $i = 1 \dots |\mathcal{X}| - 1$:
 $\tilde{\alpha}_i = \alpha_i + \lfloor (\alpha'_i - \alpha_i) + \text{Lap}(0, \frac{|\mathcal{X}|}{\epsilon}) \rfloor_0^n$
 $\tilde{\alpha}_{|\mathcal{X}|} = \alpha_{|\mathcal{X}|} + \lfloor n - \sum_{i=1}^{|\mathcal{X}|-1} \lfloor (\alpha'_i - \alpha_i) + \eta_i \rfloor_0^n \rfloor_0^n$
return $\tilde{\boldsymbol{\alpha}}$

Algorithm 3 LSHist - Calibrating noise w.r.t. histogram sensitivity

input $\mathbf{x} \in \mathcal{X}^n$, $\text{Dir}(\boldsymbol{\alpha})$
 let $\boldsymbol{\alpha}' = \text{DirP}(\boldsymbol{\alpha}, \boldsymbol{\theta}, \mathbf{x})$
 let $k = \begin{cases} 1 & \text{if } |\mathcal{X}| = 2 \\ 2 & \text{otherwise} \end{cases}$
 Initialize a vector $\tilde{\boldsymbol{\alpha}} = (0, \dots, 0) \in \mathbb{N}^{|\mathcal{X}|}$
 For $i = 1 \dots |\mathcal{X}| - 1$:
 let $\eta \sim \text{Lap}(0, \frac{k}{\epsilon})$
 $\tilde{\alpha}_i = \alpha_i + \lfloor (\alpha'_i - \alpha_i) + \eta \rfloor_0^n$
 $\tilde{\alpha}_{|\mathcal{X}|} = \alpha_{|\mathcal{X}|} + \lfloor n - \sum_{i=1}^{|\mathcal{X}|-1} \lfloor (\alpha'_i - \alpha_i) + \eta_i \rfloor_0^n \rfloor_0^n$
return $\tilde{\boldsymbol{\alpha}}$

Algorithm 4 EHD - Instantiation of the exponential mechanism

observed data set $\mathbf{x} \in \mathcal{X}^n$, prior: $\text{Dir}(\boldsymbol{\alpha})$, ϵ
 let $\text{Dir}(\boldsymbol{\alpha}') = \text{DirP}(\mathbf{x}, \boldsymbol{\alpha})$.
 let GS be the global sensitivity for \mathbf{x} .
 set $z = r$ with probability $\frac{\exp(\frac{-\epsilon \cdot \mathcal{H}(\text{DirP}(\mathbf{x}, \boldsymbol{\alpha}), r)}{2 \cdot GS})}{\sum_{r' \in \mathcal{R}_{\boldsymbol{\alpha}}} \exp(\frac{-\epsilon \cdot \mathcal{H}(\text{DirP}(\mathbf{x}, \boldsymbol{\alpha}), r')}{2 \cdot GS})}$
return z

Algorithm 5 EHDL - Instantiation of the exponential mechanism with local sensitivity

input observed data set $\mathbf{x} \in \mathcal{X}^n$, prior: $\text{Dir}(\boldsymbol{\alpha})$, ϵ
 let $\text{Dir}(\boldsymbol{\alpha}') = \text{DirP}(\mathbf{x}, \boldsymbol{\alpha})$.
 let $LS(\mathbf{x})$ be the local sensitivity for \mathbf{x} .
 set $z = r$ with probability $\frac{\exp(\frac{-\epsilon \cdot \mathcal{H}(\text{DirP}(\mathbf{x}, \boldsymbol{\alpha}), r)}{2 \cdot LS(\mathbf{x})})}{\sum_{r' \in \mathcal{R}_{\boldsymbol{\alpha}}} \exp(\frac{-\epsilon \cdot \mathcal{H}(\text{DirP}(\mathbf{x}, \boldsymbol{\alpha}), r')}{2 \cdot LS(\mathbf{x})})}$
return z

Algorithm 6 EHDS - Instantiation of the exponential mechanism with γ -smooth sensitivity

observed data set $\mathbf{x} \in \mathcal{X}^n$, prior: $\text{Dir}(\boldsymbol{\alpha})$, ϵ
 let $\text{Dir}(\boldsymbol{\alpha}') = \text{DirP}(\mathbf{x}, \boldsymbol{\alpha})$.
 let $S(\mathbf{x})$ be the smooth sensitivity for \mathbf{x} .
 set $z = r$ with probability $\frac{\exp(\frac{\epsilon \cdot u(\mathbf{x}, r)}{2(1+\gamma)S(\mathbf{x})})}{\sum_{r' \in \mathcal{R}_{\boldsymbol{\alpha}}} \exp(\frac{\epsilon \cdot u(\mathbf{x}, r')}{2(1+\gamma)S(\mathbf{x})})}$
return z

3 Privacy Analysis

Theorem 3.1. *The LSDim, LSHist, EHD and EHDS are ϵ -differentially private.*

The proofs are in the Arxiv version.

4 Accuracy Analysis

Theorem 4.1. *To prove the optimality of Laplace mechanism, we are showing*

$$\frac{ELap(\mathbf{x})}{(\epsilon \times LS(\mathbf{x}))}$$

is $O(1)$, considering $n = |\mathbf{x}| \geq 2$ being the parameter.

Where $LS(\cdot)$ is the local sensitivity, and where $ELap(\cdot)$ is the measure of the error of the Laplace mechanism, defined in this way:

$$ELap(\mathbf{x}) = \arg \left(\min_t \{Pr[H(\text{DirP}(\mathbf{x}), \text{LSHist}(\mathbf{x})) < t] \geq 1 - \gamma\} \right).$$

[[Jiawen:

Theorem 4.2. *For $\gamma = e^{O(\epsilon)}$, $\frac{ELap(\mathbf{x})}{(\epsilon \times LS(\mathbf{x}))}$ is $O(\epsilon)$*

]]

Proof. Let $t = LS(\mathbf{x})$, we have following by p.d.f. of Laplace distribution:

$$Pr[H(\text{DirP}(\mathbf{x}), \text{LSHist}(\mathbf{x})) < t] \geq 1 - \frac{1}{2}(e^{-\epsilon} + e^{-2\epsilon}) > 1 - e^{-\epsilon}$$

Then we can get when $\gamma = e^{-\epsilon}$,

$$\frac{ELap(\mathbf{x})}{(\epsilon \times LS(\mathbf{x}))} = \frac{1}{\epsilon}$$

□

Theorem 4.3. *In order to prove the optimality of Laplace mechanism, instead of prove $\frac{ELap(\mathbf{x})}{(\epsilon \times LS(\mathbf{x}))}$ is $O(1)$, we prove a constant upper bound on following equations:*

$$\begin{aligned} & \frac{\arg \min_t \left\{ Pr[H(\text{Beta}(\alpha, \beta), \text{LSHist}(\mathbf{x})) < t] \geq 1 - \gamma \right\}}{LS(\mathbf{x})} \\ & \leq \frac{\max_{|k| \leq \frac{\lg(\frac{1}{\gamma})}{\epsilon}} H(\text{Beta}(\alpha, \beta), \text{Beta}(\alpha + k, n - \lfloor \alpha + k \rfloor))}{LS(\mathbf{x})} \\ & \leq O\left(\frac{\lg \frac{1}{\gamma}}{\epsilon}\right) \end{aligned}$$

[[Jiawen:

Theorem 4.4. *For $\gamma = e^{O(k)\epsilon}$, it is proved that $O(\frac{\lg \frac{1}{\gamma}}{\epsilon})$ is bounded by $O(k)$.*

]]

Proof. By Laplace distribution, we have:

$$\begin{aligned} \Pr[\mathbf{H}(\text{Beta}(\alpha, \beta), \text{LSHist}(\mathbf{x})) < t] &= \Pr[\{|\text{Lap}(0, \frac{1}{\epsilon})| < O(k) | \mathbf{H}(\text{Beta}(\alpha, \beta), \text{Beta}(\alpha + k, n - \lfloor \alpha + k \rfloor)) < t\}] \\ &\leq 1 - e^{-O(k)\epsilon} \end{aligned}$$

Then we have:

$$\gamma = e^{-O(k)\epsilon}$$

So we can get:

$$O\left(\frac{\lg \frac{1}{\gamma}}{\epsilon}\right) = O\left(\frac{\lg \frac{1}{e^{-O(k)\epsilon}}}{\epsilon}\right) = O(k)$$

□

[[Jiawen:

Corollary 4.4.1. For $-1 \leq k < 2$, it is proved that $O(\frac{\lg \frac{1}{\gamma}}{\epsilon})$ is bounded by $O(1)$.

]]

Proof. Given $-1 \leq k < 2$, we have:

$$\mathbf{H}(\text{Beta}(\alpha, \beta), \text{Beta}(\alpha + k, n - \lfloor \alpha + k \rfloor)) \leq LS(\mathbf{x}) \quad (1)$$

For any ϵ , $k \sim \text{Lap}(0, \frac{1}{\epsilon})$ from Laplace mechanism, we have:

$$\Pr[|k| \leq \frac{b}{\epsilon}] = 1 - \exp(-b)$$

Then we can get:

$$\Pr[-1 \leq k < 2] = 1 - \frac{\exp(-\epsilon) + \exp(-2\epsilon)}{2} \quad (2)$$

By Equation (1) and (2), we can get:

$$\Pr[\mathbf{H}(\text{Beta}(\alpha, \beta), \text{Beta}(\alpha + k, n - \lfloor \alpha + k \rfloor)) \leq LS(\mathbf{x})] \geq 1 - \frac{\exp(-\epsilon) + \exp(-2\epsilon)}{2}$$

i.e.,

$$\begin{aligned} &\frac{\arg \min_t \left\{ \Pr[\mathbf{H}(\text{Beta}(\alpha, \beta), \text{LSHist}(\mathbf{x})) < t] \geq 1 - \frac{\exp(-\epsilon) + \exp(-2\epsilon)}{2} \right\}}{LS(\mathbf{x})} \\ &\leq O\left(\frac{\lg(\frac{2}{\exp(-\epsilon) + \exp(-2\epsilon)})}{\epsilon}\right) \\ &< O\left(\frac{\lg(\frac{2}{2\exp(-2\epsilon)})}{\epsilon}\right) = 2 \end{aligned}$$

□

[[Jiawen:

Theorem 4.5. Let $k = \lfloor k' \rfloor$ be the largest integer that satisfying $H(\text{Beta}(\alpha, \beta), \text{Beta}(\alpha + k', n - \lfloor \alpha + k' \rfloor)) < t$, we have:

$$\Pr[H(\text{Beta}(\alpha, \beta), \text{LSZhang}(\mathbf{x})) < t] = (1 - \frac{1}{2}(e^{-k\epsilon/2} + e^{-(k+1)\epsilon/2}))^2 \geq (1 - e^{-k\epsilon/2})^2$$

$$\Pr[H(\text{Beta}(\alpha, \beta), \text{LSHist}(\mathbf{x})) < t] = 1 - \frac{1}{2}(e^{-k\epsilon} + e^{-(k+1)\epsilon}) \geq 1 - e^{-k\epsilon}$$

$$\frac{2ke^{-\epsilon} + 1}{n} < \Pr[H(\text{Beta}(\alpha, \beta), \text{EHD}(\mathbf{x})) < t] < \frac{2k+1}{ne^{-\epsilon}}.$$

$$\frac{2k \exp(\frac{-\epsilon}{2 \cdot LS(\mathbf{x})}) + 1}{n} < \Pr[H(\text{Beta}(\alpha, \beta), \text{EHDL}(\mathbf{x})) < t] < \frac{2k+1}{n \exp(\frac{-\epsilon}{2 \cdot LS(\mathbf{x})})}.$$

]]

Proof. Given $k = \lfloor k' \rfloor$ be the largest integer that satisfying $H(\text{Beta}(\alpha, \beta), \text{Beta}(\alpha + k', n - \lfloor \alpha + k' \rfloor)) < t$, by the post-processing of Laplace distribution and p.d.f. of Laplace distribution, we have:

$$\Pr[H(\text{Beta}(\alpha, \beta), \text{LSHist}(\mathbf{x})) < t] = \Pr[-k < \text{LSHist}(\mathbf{x}) \leq k+1] = 1 - \frac{1}{2}(e^{-k\epsilon} + e^{-(k+1)\epsilon}) \geq 1 - e^{-k\epsilon}.$$

By definition of EHD and $GS = \sqrt{1 - \pi/4}$, we have:

$$\begin{aligned} \Pr[H(\text{Beta}(\alpha, \beta), \text{EHD}(\mathbf{x})) < t] &= \sum_{c \geq -t} \frac{\exp(\frac{\epsilon \cdot c}{2 \cdot GS})}{\sum_{r' \in \mathcal{R}_\alpha} \exp(\frac{-\epsilon \cdot \mathcal{H}(\text{DirP}(\mathbf{x}, \alpha), r')}{2 \cdot GS})} \\ &\leq \frac{2k \exp(-\frac{0\epsilon}{2 \cdot GS}) + 1}{n \exp(\frac{-\epsilon}{2 \cdot GS})} \\ &= \frac{2k+1}{n \exp(\frac{-\epsilon}{2\sqrt{1-\pi/4}})} \\ &< \frac{2k+1}{n \exp(-\epsilon)} \end{aligned}$$

$$\begin{aligned} \Pr[H(\text{Beta}(\alpha, \beta), \text{EHD}(\mathbf{x})) < t] &= \sum_{c \geq -t} \frac{\exp(\frac{\epsilon \cdot c}{2 \cdot GS})}{\sum_{r' \in \mathcal{R}_\alpha} \exp(\frac{-\epsilon \cdot \mathcal{H}(\text{DirP}(\mathbf{x}, \alpha), r')}{2 \cdot GS})} \\ &\geq \frac{2k \exp(\frac{-t\epsilon}{2 \cdot GS}) + 1}{n \exp(\frac{-0\epsilon}{2 \cdot GS})} \\ &> \frac{2k \exp(\frac{-\epsilon}{2\sqrt{1-\pi/4}}) + 1}{n} \\ &> \frac{2ke^{-\epsilon} + 1}{n} \end{aligned}$$

By definition of EHDL, we have:

$$\begin{aligned} \Pr[H(\text{Beta}(\alpha, \beta), \text{EHDL}(\mathbf{x})) < t] &= \sum_{c \geq -t} \frac{\exp(\frac{c\epsilon}{2 \cdot LS(\mathbf{x})})}{\sum_{r' \in \mathcal{R}_\alpha} \exp(\frac{-\epsilon \cdot \mathcal{H}(\text{DirP}(\mathbf{x}, \alpha), r')}{2 \cdot LS(\mathbf{x})})} \\ &\leq \frac{2k \exp(\frac{-0\epsilon}{2 \cdot LS(\mathbf{x})}) + 1}{n \exp(\frac{-\epsilon}{2 \cdot LS(\mathbf{x})})} \\ &= \frac{2k+1}{n \exp(\frac{-\epsilon}{2 \cdot LS(\mathbf{x})})} \end{aligned}$$

$$\begin{aligned} \Pr[H(\text{Beta}(\alpha, \beta), \text{EHD}(\mathbf{x})) < t] &= \sum_{c \geq -t} \frac{\exp(\frac{\epsilon \cdot c}{2 \cdot LS(\mathbf{x})})}{\sum_{r' \in \mathcal{R}_\alpha} \exp(\frac{-\epsilon \cdot \mathcal{H}(\text{DirP}(\mathbf{x}, \alpha), r')}{2 \cdot LS(\mathbf{x})})} \\ &\geq \frac{2k \exp(\frac{-t\epsilon}{2 \cdot LS(\mathbf{x})}) + 1}{n \exp(\frac{-0\epsilon}{2 \cdot LS(\mathbf{x})})} \\ &> \frac{2k \exp(\frac{-\epsilon}{2 \cdot LS(\mathbf{x})}) + 1}{n} \end{aligned}$$

Since $\lim_{n \rightarrow \infty} LS(\mathbf{x}) \rightarrow 0$, we have $\lim_{n \rightarrow \infty} \frac{-1}{LS(\mathbf{x})} \rightarrow -\infty$. So $\exp(\frac{-\epsilon}{2 \cdot LS(\mathbf{x})})$ can only be bounded by 0. We cannot found a tighter lower bound. \square

[[Jiawen:

Corollary 4.5.1. *For a reasonable small t , we have when data size $n = |\mathbf{x}| > O(\frac{(2k+1)e^\epsilon}{1-e^{-\epsilon}})$, the accuracy of LSHist is higher than EHD.*

]]

Proof. Based on Theorem 2.5, let:

$$\frac{2k+1}{n \exp(-\epsilon)} \leq 1 - e^{-k\epsilon},$$

we can have:

$$\Pr[\mathbf{H}(\text{Beta}(\alpha, \beta), \text{LSHist}(\mathbf{x})) < t] > \Pr[\mathbf{H}(\text{Beta}(\alpha, \beta), \text{EHD}(\mathbf{x})) < t].$$

By simplification, we have $n > \frac{2k+1}{e^{-\epsilon}(1-e^{-k\epsilon})} \sim O(\frac{(2k+1)e^\epsilon}{1-e^{-\epsilon}})$. \square

[[Jiawen:

Corollary 4.5.2. *For a reasonable small t , we have when data size $n = |\mathbf{x}| < O(\frac{(2ke^{-\epsilon}+1)}{1-\frac{1}{2}(e^{-k\epsilon}+e^{-(k+1)\epsilon})})$, the accuracy of EHD is better than LSHist.*

]]

Proof. Applying the Theorem 2.5, let:

$$\frac{2ke^{-\epsilon}+1}{n} > 1 - \frac{1}{2}(e^{-k\epsilon} + e^{-(k+1)\epsilon}),$$

we can have:

$$\Pr[\mathbf{H}(\text{Beta}(\alpha, \beta), \text{LSHist}(\mathbf{x})) < t] < \Pr[\mathbf{H}(\text{Beta}(\alpha, \beta), \text{EHD}(\mathbf{x})) < t].$$

By simplification, we have $n < \frac{(2ke^{-\epsilon}+1)}{1-\frac{1}{2}(e^{-k\epsilon}+e^{-(k+1)\epsilon})}$. \square

[[Jiawen:

Corollary 4.5.3. *Let R_g be the good output set where $\forall r \in R, \mathbf{H}(\text{DirP}(\mathbf{x}, r)) \leq LS(\mathbf{x})$, we have:*

$$\Pr[\text{LSHist}(\mathbf{x}, \epsilon) \in R_g] > \Pr[\text{EHD}(\mathbf{x}, \epsilon) \in R_g]$$

for data size $n = |\mathbf{x}| > O(\frac{e^\epsilon}{1-e^{-\epsilon}})$

]]

Proof. simply apply the Theorem 2.5 and corollary 2.5.1, we can get this conclusion.

Let R_g be the good output set where $\forall r \in R, \mathbf{H}(\text{DirP}(\mathbf{x}), r) \leq LS(\mathbf{x})$, we have:

$$Pr[\text{LSHist}(\mathbf{x}) \in R_g] \geq 1 - \frac{1}{2}(e^{-\epsilon} + e^{-2\epsilon}) > 1 - e^{-\epsilon}$$

By definition of EHD and $GS = \sqrt{1 - \pi/4}$, we have:

$$\begin{aligned} Pr[\text{EHD}(\mathbf{x}) \in R_g] &= \sum_{c \geq -LS(\mathbf{x})} \frac{\exp(\frac{\epsilon \cdot c}{2 \cdot GS})}{\sum_{r' \in \mathcal{R}_\alpha} \exp(\frac{-\epsilon \cdot \mathcal{H}(\text{DirP}(\mathbf{x}, \alpha), r')}{2 \cdot GS})} \\ &\leq \frac{2 \exp(-\frac{\epsilon LS(\mathbf{x})}{2 \cdot GS}) + 1}{n \exp(\frac{-\epsilon}{2 \cdot GS})} \\ &\leq \frac{3}{n \exp(\frac{-\epsilon}{2\sqrt{1-\pi/4}})} \\ &\leq \frac{3}{n \exp(-\epsilon)} \end{aligned}$$

Let $c = 2\sqrt{1 - \pi/4}$, we have when $n > \frac{3}{e^{-\epsilon/c}(1-e^{-\epsilon})} \sim O(\frac{e^\epsilon}{1-e^{-\epsilon}})$ LSHist performs better than EHD. \square

From this theorem, we can derive with probability at most δ , following utility for these mechanisms:

Mechs	\mathcal{H}	KL
LSHist	$\sqrt{1 - \mathcal{O}\left(\frac{n + \frac{1}{\epsilon} \ln(1/\delta)}{n - \frac{1}{\epsilon} \ln(1/\delta)}\right)^{\frac{1}{2\epsilon} \ln(1/\delta)}}$	$\mathcal{O}(4n \ln n)[1 - e^{-n\epsilon}] + \sqrt{-\mathcal{O}(4n \ln n) \ln \delta}$
LSZhang	$\sqrt{1 - \mathcal{O}\left(\frac{n + \frac{1}{\epsilon} \ln \frac{1}{(2-\delta-2\sqrt{1-\delta})}}{n - \frac{1}{\epsilon} \ln \frac{1}{(2-\delta-2\sqrt{1-\delta})}}\right)^{\frac{1}{2\epsilon} \ln \frac{1}{(2-\delta-2\sqrt{1-\delta})}}}$	$\mathcal{O}(8n \ln n)[1 - e^{-n\epsilon/2}] + \sqrt{-\mathcal{O}(8n \ln n) \ln \delta}$
EHD	$\sqrt{1 - \mathcal{O}\left(\frac{n + \frac{n-\delta n-1}{2} e^{-\epsilon}}{n - \frac{n-\delta n-1}{2} e^{-\epsilon}}\right)^{\frac{n-\delta n-1}{2} e^{-\epsilon}}}$	

References

- [1] Zuhe Zhang, Benjamin IP Rubinstein, Christos Dimitrakakis, et al. On the differential privacy of bayesian inference. In *AAAI*, pages 2365–2371, 2016.