Tailoring Differentially Private Bayesian Inference to Distance Between Distributions

Mark Bun†, Gian Pietro Farina*, Marco Gaboardi*, Jiawen Liu*

†Princeton University, *University at Buffalo, SUNY

Objectives

1. Design a differentially private Bayesian inference mechanism.
2. Improve accuracy by calibrating noise to a metric over distributions (Hellinger distance ($\mathcal{H}$), f-divergence, etc.).

Bayesian inference (BI) (A Beta-Binomial model example):

- Prior on θ : $\mathbf{beta}(\alpha, \beta), \alpha, \beta \in \mathbb{R}^{+}$, observed data set $\mathbf{x} = (x_1, \ldots, x_n) \in \{0,1\}^n$, $n \in \mathbb{N}$.
- Likelihood function: $\mathbb{L}_{\mathbf{x}|\theta} = \theta^{\Delta\alpha}(1-\theta)^{n-\Delta\alpha}$, where $\Delta\alpha = \sum_{i=1}^{n} x_i$;
- Posterior distribution over θ: $\mathbb{P}_{\theta|\mathbf{x}} = \mathbf{beta}(\alpha + \Delta\alpha, \beta + n - \Delta\alpha)$.

Differentially Private Bayesian Inference and Motivations

Releasing a differentially private posterior $\mathbf{beta}(\tilde{\alpha}, \tilde{\beta}) = (\alpha + \widetilde{\Delta\alpha}, \beta + n - \widetilde{\Delta\alpha})$.

1. Baseline approach is Laplace mechanism (LapMech) with sensitivity proportional to ℓ_1 norm. But this sensitivity grows linear with the dimension, also we measure results by a different metric. Motivated by this, we calibrate the noise w.r.t sensitivity of the accuracy metric ($\mathcal{H}$).
2. Maximum sensitivity of $\mathcal{H}$ over **beta** distributions is attained at edges as in Fig. 1. But local sensitivity is very smooth and much smaller when move away from edge. Motivated by this, we apply smooth sensitivity in mechanism to improve accuracy.

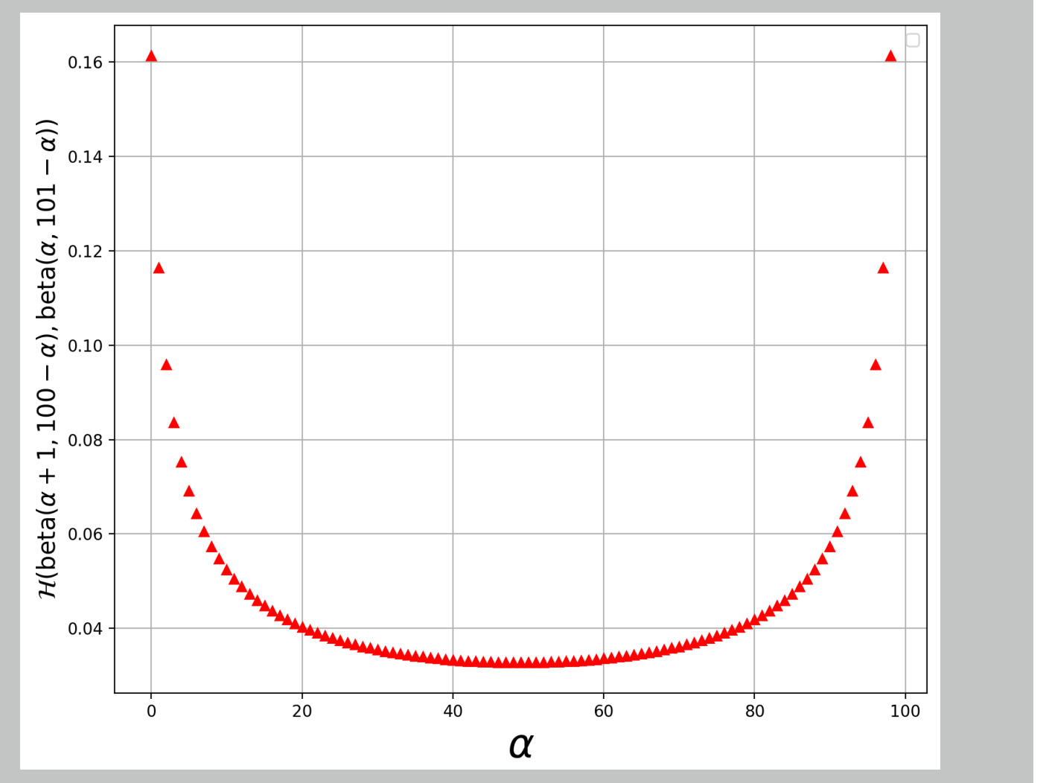


Figure 1: Sensitivities of $\mathcal{H}$ over **beta**

Smoothed Hellinger Distance Based Exponential Mechanism

We define the mechanism $\mathcal{M}^{B}_{\mathcal{H}}$ which produces an element r in $\mathcal{R}_{\text{post}}$ with probaiblity:

$$\mathbb{P}_{r\sim\mathcal{M}^{B}_{\mathcal{H}}} = \frac{\exp\left(\frac{-\epsilon\cdot\mathcal{H}(\mathsf{BI}(\mathbf{x}),r)}{2\cdot S(\mathbf{x})}\right)}{\sum_{r\in\mathcal{R}_{\text{post}}}\exp\left(\frac{-\epsilon\cdot\mathcal{H}(\mathsf{BI}(\mathbf{x}),r)}{2\cdot S(\mathbf{x})}\right)}$$

given as input an observations $\mathbf{x}$, parameters $\epsilon > 0$ and $\delta > 0$, where:

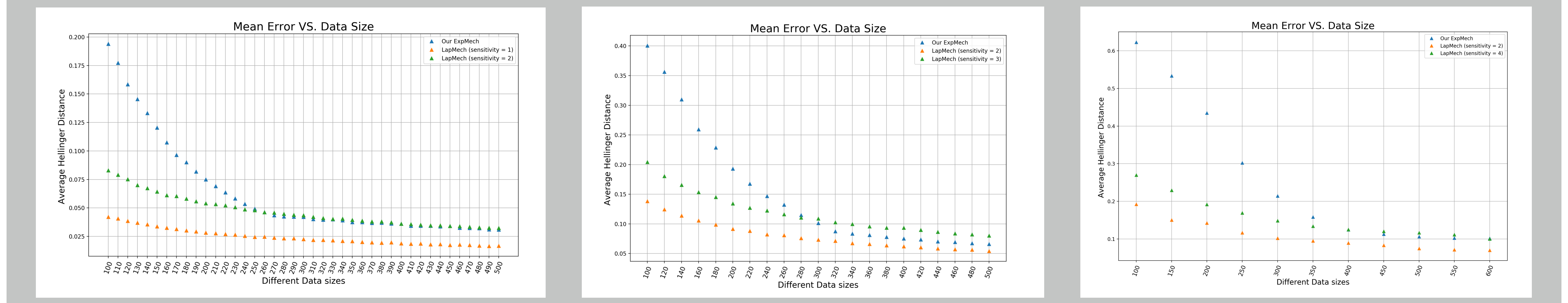
- $\mathcal{R}_{\text{post}}$ is the set of candidates, defined as $\{\mathbf{beta}(\alpha', \beta') \mid \alpha' = \alpha + \Delta\alpha, \beta' = \beta + n - \Delta\alpha\}$, given the prior distribution $\beta_{\text{prior}} = \mathbf{beta}(\alpha, \beta)$ and observed data set size n.
- $-\mathcal{H}(\mathsf{BI}(\mathbf{x}), r)$ denotes the scoring function based on the Hellinger distance.
- $S(\mathbf{x})$, the smooth sensitivity[1]: $S(\mathbf{x}) = \max_{\mathbf{x}'\in\{0,1\}^n}\left\{LS(\mathbf{x}')\cdot e^{-\gamma\cdot d(\mathbf{x},\mathbf{x}')}\right\}$, where:
  - ▷ d: Hamming distance between two data sets,
  - ▷ $LS(\mathbf{x}')$, local sensitivity at $\mathbf{x}'$: $LS(\mathbf{x}) = \max_{\mathbf{x}'\in\mathcal{X}^n:\mathbf{adj}(\mathbf{x},\mathbf{x}'), r\in\mathcal{R}} |\mathcal{H}(\mathsf{BI}(\mathbf{x}), r) - \mathcal{H}(\mathsf{BI}(\mathbf{x}'), r)|$,
  - ▷ $\gamma = \ln(1 - \frac{\epsilon}{2\ln(\ \delta\ )})$ to ensure (ϵ, δ)-differential privacy.

Preliminary Experimental Results

Experiments are on three mechanisms and plotted as follows:

- **Green**: Baseline approach. (noises being postprocessed to floor value)
- **Red**: LapMech with sensitivity $\mathbf{1}$ in 2 dimensions and $\mathbf{2}$ in higher dimensions, since it's equivalent to histogram problem (posteriors of adjacent data sets differ only in two dimensions).
- **Blue**: $\mathcal{M}^{B}_{\mathcal{H}}$.

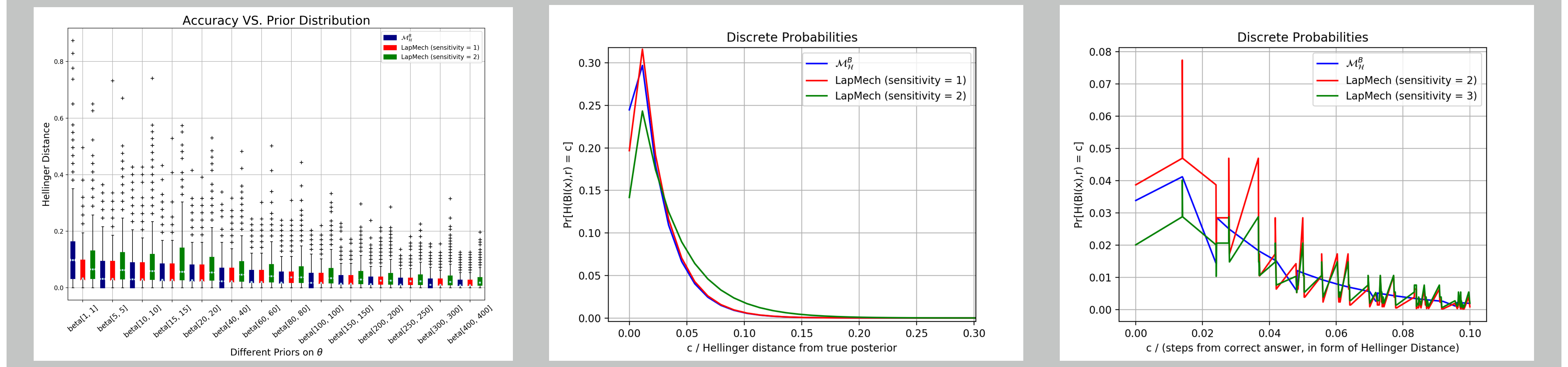
Fig. 2 and Fig. 3(a) give us the average and 4-quantile of Hellinger distance between the sampled results and true posterior, by sampling for $\mathbf{10,000}$ times under each data size or prior configuration.



(a) 2 dimensions, data size $\in \mathbf{[100, 500]}$ (b) 3 dimensions, data size $\in \mathbf{[100, 500]}$ (c) 4 dimensions, data size $\in \mathbf{[100, 600]}$

Figure 2: Average accuracy by increasing data set size

Fig. 3(b) and 3(c) give us the discrete probabilities. On the x-axis: the distance of a potential output r from the true answer and on y-axis the probability of r being output by the mechanisms.



(a) 2 dimensions with data size $\mathbf{100}$ (b) 2 dimensions with data size $\mathbf{600}$ (c) 3 dimensions with data size $\mathbf{600}$

Figure 3: 4-quantile and discrete probability plots

Experiments above are with unit prior $\mathbf{beta(1,1)}, \mathbf{beta(1,1,1)}$ and $\mathbf{beta(1,1,1,1)}$ (except Fig. 3(a)) , balanced datasets, $\epsilon = \mathbf{1.0}$ and $\delta = \mathbf{10^{-8}}$.

Conclusion

- The smoothed Hellinger distance based exponential mechanism outperforms asymptotically the baseline approach when the latter uses a sensitivity proportional to dimensionality.
- Under the same data set size, $\mathcal{M}^{B}_{\mathcal{H}}$ can outperform LapMech by increasing the prior.

References

[1] Kobbi Nissim, Sofya Raskhodnikova, and Adam Smith. Smooth sensitivity and sampling in private data analysis. In *Proceedings of the thirty-ninth annual ACM symposium on Theory of computing*, pages 75–84. ACM, 2007.