Tailoring Differentially Private Bayesian Inference to Distance Between Distributions

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Objectives

Design a mechanism that achieve differential privacy by scaling to a metric between distribution.

1. A differentially private bayesian mechanism,
2. Calibrating mechanism noise by the same probabilistic distance we want to measure accuracy with.
3. Applying smooth sensitivity in mechanism to achieve better accuracy.

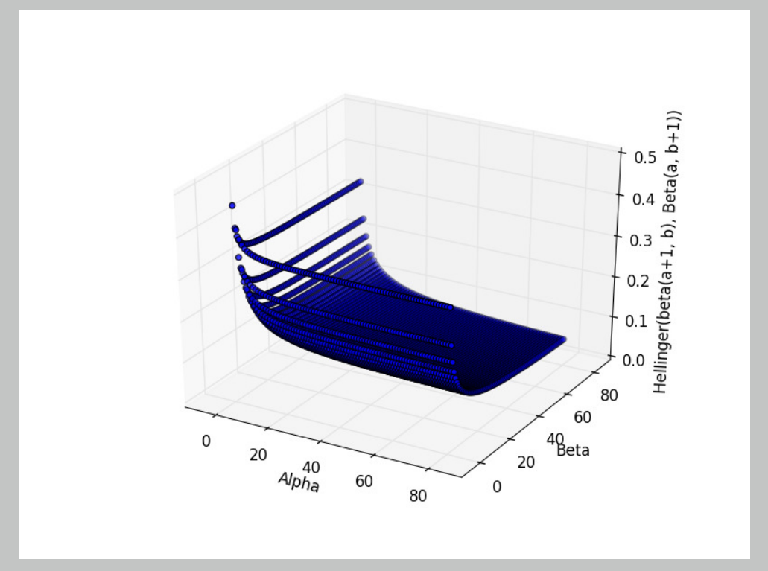


Figure 1: Hellinger Sensitivity

Bayesian Inference Background

Conjugate prior distribution, $\mathbf{beta}(\alpha, \beta)$, with hyper parameters $\alpha, \beta \in \mathbb{R}^+$;
Observed data set $\mathbf{x}$: $\mathbf{x} = (x_1, \dots x_n), x_i \in \{0, 1\}$, $n \in \mathbb{N}$;

Bernoulli likelihood function: $\mathbf{Pr}(\mathbf{x}|\theta) \equiv \theta^{\Delta\alpha}(1-\theta)^{n-\Delta\alpha}$, where $\Delta\alpha = \sum_{i=1}^{n} x_i$;

Posterior distribution derived: $\mathbf{Pr}(\theta|\mathbf{x}) = \mathbf{beta}(\alpha + \Delta\alpha, \beta + n - \Delta\alpha)$.

Differentially private Bayesian inference

Release a private version of posterior distribution $(\tilde{\alpha}, \tilde{\beta}) = (\alpha + \widetilde{\Delta\alpha}, \beta + n - \widetilde{\Delta\alpha})$.
In a baseline approach, we sample noise from $Lap(\mu, \nu)$ mechanism, i.e., $\widetilde{\Delta\alpha} \sim Lap(\Delta\alpha, \frac{2}{\epsilon})$,

Smoothed Hellinger Distance based Exponential Mechanism

Our approach defines the mechanism $\mathcal{M}^B_{\mathcal{H}}$:

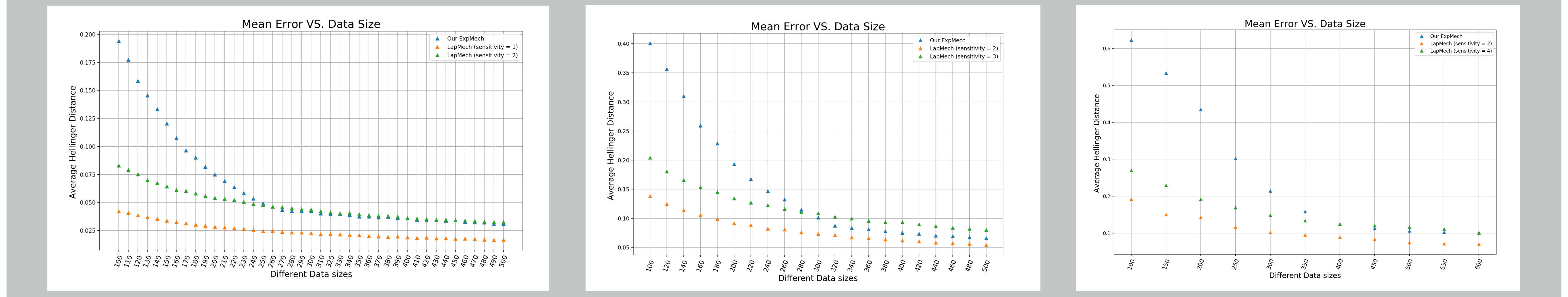
Producing an element r in $\mathcal{R}_{\text{post}}$ with: $$\underset{z \sim \mathcal{M}^B_{\mathcal{H}}}{\mathbf{Pr}}[z = r] = \frac{exp\left(\frac{-\epsilon \cdot \mathcal{H}(\mathsf{BI}(\mathbf{x}), r)}{2 \cdot S(\mathbf{x})}\right)}{\sum_{r \in \mathcal{R}_{\text{post}}} exp\left(\frac{-\epsilon \cdot \mathcal{H}(\mathsf{BI}(\mathbf{x}), r)}{2 \cdot S(\mathbf{x})}\right)},$$

(given in input an observations $\mathbf{x}$, parameters $\epsilon > 0$ and $\delta > 0$).

- $\mathcal{R}_{\text{post}}$, the candidates set defined as $\{\mathbf{beta}(\alpha', \beta') \mid \alpha' = \alpha + \Delta\alpha, \beta' = \beta + n - \Delta\alpha\}$, given the prior distribution $\beta_{\text{prior}} = \mathbf{beta}(\alpha, \beta)$ and observed data set size n.
- $-\mathcal{H}(\mathsf{BI}(\mathbf{x}), r)$, the scoring function instantiated by Hellinegr distance.
- $S(\mathbf{x})$, the smooth sensitivity: $S(\mathbf{x}) = \mathbf{max}_{\mathbf{x}' \in \{0,1\}^n} \left\{ LS(\mathbf{x}') \cdot e^{-\gamma \cdot d(\mathbf{x}, \mathbf{x}')} \right\}$, where
  - ▷ d: Hamming distance between two datasets,
  - ▷ $LS(\mathbf{x}')$, local sensitivity at $\mathbf{x}'$: $LS(\mathbf{x}) = \max_{\mathbf{x}' \in \mathcal{X}^n : \mathsf{adj}(\mathbf{x}, \mathbf{x}'), r \in \mathcal{R}} |\mathcal{H}(\mathsf{BI}(\mathbf{x}), r) - \mathcal{H}(\mathsf{BI}(\mathbf{x}'), r)|$,
  - ▷ $\gamma = \ln(1 - \frac{\epsilon}{2\ln(\frac{\delta}{2(n+1)})})$ to ensure the (ϵ, δ)-differentially private.

Preliminary Experimental Results

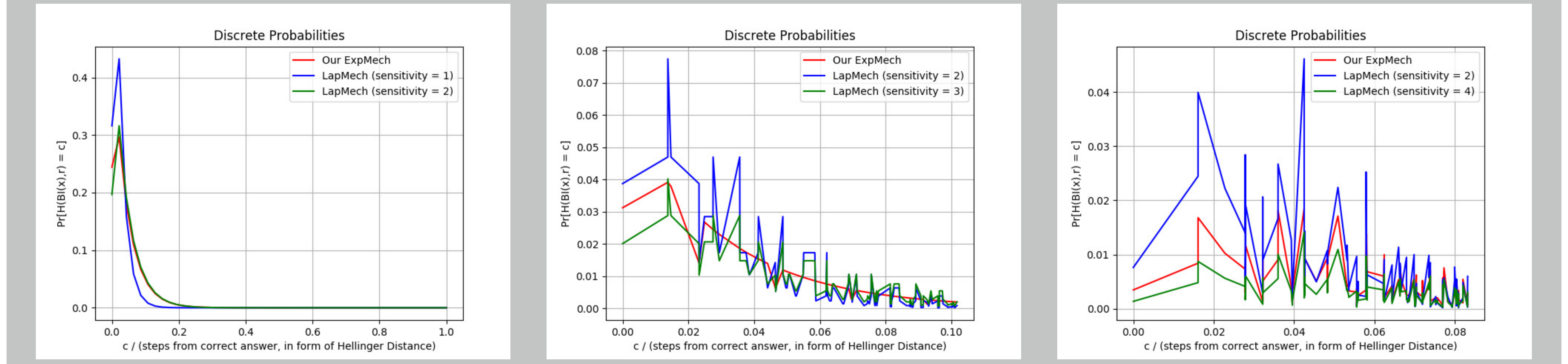
Fig. 2 gives the average Hellinger distance between the sampled results and true posterior, by sampling for $10k$ times under each data size configuration. In baseline approach (i.e., Laplace mechanism), it is enough to add noise with sensitivity 1 in 2-dimensional and 2 in higher dimensional by their equivalence to histogram, giving us the red points in plots. Without the knowledge of equivalence, Laplace usually add noise with sensitivity scale to dimensions, giving us green points. Points in blue are our smoothed Hellinger distance based exponential mechanism.



(a) 2-dimensional, data size $\in [100, 500]$ (b) 3-dimensional, data size $\in [100, 500]$ (c) 4-dimensional, data size $\in [100, 600]$

Figure 2: Increasing data size

Fig. 3 gives us the concrete probabilities of outputting candidates with certain Hellinegr distance from the correct posterior in 2, 3 and 4 dimensional respectively.



(a) 2-dimensional (b) 3-dimensional (c) 4-dimensional

Figure 3: The concrete outputting probabilities under different dimensions with data set of size $\mathbf{600}$

Two groups of experiments both with unit prior $\mathbf{beta}(1, 1), \mathbf{beta}(1, 1, 1)$ and $\mathbf{beta}(1, 1, 1, 1)$, balanced datasets and parameters $\epsilon = 1.0$ and $\delta = 10^{-8}$.

Conclusion

- ▶ The smoothed Hellinger distance based exponential mechanism outperforms the ℓ_1-norm approach when data size growing under the case that Laplace noise doesn't have knowledge of the implicit histogram equivalence.

References