Notes of DP - Bayesian Inference

1 Bayesian Inference Based on Dirichlet-Bernoulli Distribution

In the Bayesian inference, first there is a prior distribution $\pi(\boldsymbol{\xi})$ to present our belief about parameter $\boldsymbol{\xi}$. Then, we get some observed data x sizing n, and produce a posterior distribution $Pr(\boldsymbol{\xi}|x)$. The Bayesian inference is based on the Bayes' rule to calculate the posterior distribution:

$$Pr(\boldsymbol{\xi}|x) = \frac{Pr(x|\boldsymbol{\xi})\pi(\boldsymbol{\xi})}{Pr(x)}$$

It is denoted as $\mathsf{BI}(x, \pi(\boldsymbol{\xi}))$ taking an observed data set $x \in \mathcal{X}^n$ and a prior distribution $\pi(\boldsymbol{\xi})$ as input, outputting a posterior distribution of the parameter $\boldsymbol{\alpha}$. For conciseness, when prior is given, we use $\mathsf{BI}(x)$. n is the size of the observed data size.

In our inference algorithm, we take a Dirichlet distribution as prior belief for the parameters, $\mathsf{DL}(\boldsymbol{\alpha})$, where $\pi(\boldsymbol{\xi}) = \mathsf{DL}(\boldsymbol{\xi}|\boldsymbol{\alpha}) = \frac{\prod_{i=1}^{m} \xi_i^{\alpha_i}}{B(\boldsymbol{\alpha})}$, and the Bernoulli distribution as the statistic model for $Pr(x|\boldsymbol{\alpha})$. m is the order of the Dirichlet distribution.

We give a inference process based on a concrete example, where we throw an irregular m sides dice. We want to infer the probability of getting each side $\boldsymbol{\xi}$. We get a observed data set $\{s_{k_1}, s_{k_2}, \cdots, s_{k_n}\}$ by throwing the dice n times, where $k_i \in \{1, 2, \cdots, m\}$ denotes the side we get when we throw the dice the i^{th} time. The posterior distribution is still a Dirichlet distribution with parameters $(\alpha_1 + n_1, \alpha_2 + n_2, \cdots, \alpha_m + n_m)$, where n_i is the appearance time of the side i in total.

In the case that $m = 2$, it is reduced to a Beta distribution $\mathsf{beta}(\alpha, \beta)$. The m side dice change into a irregular coin with side A and side B. The posterior is computed as $(\alpha + n_1, \beta + n_0)$, where n_1 is the appearance time of side A in the observed data set and n_0 is the appearance time of the other side.

2 Algorithm Setting up

For now, we already have a prior distribution $\pi(\boldsymbol{\xi})$, an observed data set x.

2.1 Exponential Mechanism with Global Sensitivity

In exponential mechanism, candidate set $\mathcal{R}$ can be obtained by enumerating $y \in \mathcal{X}^n$, i.e.

$$\mathcal{R} = \{\mathsf{BI}(y) \mid y \in \mathcal{X}^n\}.$$

Hellinger distance H is used here to score these candidates. The utility function:

$$u(x, r) = -\mathsf{H}(\mathsf{BI}(x), r); r \in \mathcal{R}. \tag{1}$$

Exponential mechanism with global sensitivity selects and outputs a candidate $r \in \mathcal{R}$ with probability proportional to $exp(\frac{\epsilon u(x,r)}{2\Delta_g u})$:

$$P[r] = \frac{exp(\frac{\epsilon u(x,r)}{2\Delta_g u})}{\Sigma_{r' \in \mathcal{R}}\ exp(\frac{\epsilon u(x,r')}{2\Delta_g u})},$$

where global sensitivity is calculated by:

$$\Delta_g u = \max_{\{|x',y'| \leqslant 1; x', y' \in \mathcal{X}^n\}} \max_{\{r \in \mathcal{R}\}} |\mathsf{H}(\mathsf{BI}(x'), r) - \mathsf{H}(\mathsf{BI}(y'), r)|$$

The basic exponential mechanism is $\epsilon-$differential privacy[1].

2.2 Exponential Mechanism with Local Sensitivity

Exponential mechanism with local sensitivity share the same candidate set and utility function as it with global sensitivity. This outputs a candidate $r \in \mathcal{R}$ with probability proportional to $exp(\frac{\epsilon u(x,r)}{2\Delta_l u})$:

$$P[r] = \frac{exp(\frac{\epsilon u(x,r)}{2\Delta_l u})}{\Sigma_{r' \in \mathcal{R}}\ exp(\frac{\epsilon u(x,r')}{2\Delta_l u})},$$

where local sensitivity is calculated by:

$$\Delta_l u(x) = \max_{\{|x,y'| \leqslant 1; y' \in \mathcal{X}^n\}} \max_{\{r \in \mathcal{R}\}} .\mathsf{H}(\mathsf{BI}(x), r) - \mathsf{H}(\mathsf{BI}(y'), r)|$$

The exponential mechanism with local sensitivity is non differential privacy[1].

2.3 Exponential Mechanism with Smooth Sensitivity

2.3.1 Algorithm Setting up

We define a new mechanism $\mathcal{M}_H(x)$ which is similar to the exponential mechanism where we use $\mathcal{R}$ as the set $\mathcal{R}_B$ of beta distributions with integer parameters summing up to $n+2$, as scoring function we use the Hellinger distance from $\mathsf{BI}(x)$, i.e. $\mathsf{H}(\mathsf{BI}(x), -)$, and we calibrate the noise to the smooth sensitivity [2]. The only difference is in the sensitivity part, since now we use the smooth sensitivity.

Definition 2.1. The mechanism $\mathcal{M}_H(x)$ outputs a candidate $r \in \mathcal{R}_B$ with probability

$$\Pr_{z \sim \mathcal{M}_H(x)}[z = r] = \frac{exp(\frac{-\epsilon \mathsf{H}(\mathsf{BI}(x), r)}{2S_\beta(x)})}{\Sigma_{r' \in R}\ exp(\frac{-\epsilon \mathsf{H}(\mathsf{BI}(x), r')}{2S_\beta(x)})},$$

where $s_\beta(x)$ is the smooth sensitivity of $\mathsf{H}(\mathsf{BI}(x), -)$, calculated by:

$$S_\beta(x) = \max(\Delta_l \mathsf{H}(\mathsf{BI}(x), -), \max_{y \neq x; y \in D^n} (\Delta_l \mathsf{H}(\mathsf{BI}(y), -) \cdot e^{-\beta d(x,y)})),$$

where d is the Hamming distance between two datasets, and $\beta = \beta(\epsilon, \delta)$ is a function of ϵ and δ.

In what follows, we will use a correspondence between the probability $\Pr_{z \sim \mathcal{M}_H(x)}[z = r]$ of every $r \in \mathcal{R}_B$ and the probability $\Pr_{z \sim \mathcal{M}_H(x)}[\mathsf{H}(\mathsf{BI}(x), z) = \mathsf{H}(\mathsf{BI}(x), r)]$ for the utility score for r. In particular, for every $r \in \mathcal{R}_B$ we have:

$$\Pr_{z \sim \mathcal{M}_H(x)}[z = r] = \frac{1}{2}\Big(\Pr_{z \sim \mathcal{M}_H(x)}[\mathsf{H}(\mathsf{BI}(x), z) = \mathsf{H}(\mathsf{BI}(x), r)]\Big)$$

To see this, it is enough to notice that: $\Pr_{z \sim \mathcal{M}_H(x)}[z = r]$ is proportional too $\mathsf{H}(\mathsf{BI}(x), r)$, i.e., $u(x, z)$. We can derive, if $u(r, x) = u(r', x)$ then $\Pr_{z \sim \mathcal{M}_H(x)}[z = r] = \Pr_{z \sim \mathcal{M}_H(x)}[z = r']$. We assume the number of candidates $z \in \mathcal{R}$ that satisfy $u(z, x) = u(r, x)$ is $|r|$, we have $\Pr_{z \sim \mathcal{M}_H(x)}[u(z, x) = u(r, x)] = |r| \Pr_{z \sim \mathcal{M}_H(x)}[z = r]$. Because Hellinger distance $\mathsf{H}(\mathsf{BI}(x), z)$ is axial symmetry, where the $\mathsf{BI}(x)$ is the symmetry axis. It can be infer that $|z| = 2$ for any candidates, apart from the true output, i.e., $\Pr_{z \sim \mathcal{M}_H(x)}[u(z, x) = u(r, x)] = 2 \Pr_{z \sim \mathcal{M}_H(x)}[z = r]$. This parameter can be eliminate in both sides in proof.

In our private Bayesian inference mechanism, we set the β as $\ln(1 - \frac{\epsilon}{2\ln(\frac{\delta}{2(n+1)})})$.

2.3.2 Sliding Property of Exponential Mechanism

Lemma 2.1. *Consider the exponential mechanism $\mathcal{M}_E^S(x, u, \mathcal{R})$ calibrated on the smooth sensitivity. Let $\lambda = f(\epsilon, \delta)$, $\epsilon \geqslant 0$ and $|\delta| < 1$. Then, the following* sliding property *holds:*

$$\Pr_{r\sim\mathcal{M}_H(x)}[u(r,x) = \hat{s}] \leqslant e^{\frac{\epsilon}{2}} \Pr_{r\sim\mathcal{M}_H(x)}[u(r,x) = (\Delta + \hat{s})] + \frac{\delta}{2},$$

Proof. We denote the normalizer of the probability mass in $\mathcal{M}_H(x)$: $\sum_{r'\in\mathcal{R}} exp(\frac{\epsilon u(r',x)}{2S(x)})$ as $NL(x)$:

$$\begin{aligned} LHS = \Pr_{r\sim\mathcal{M}_H(x)}[u(r,x) = \hat{s}] &= \frac{exp(\frac{\epsilon\hat{s}}{2S(x)})}{NL(x)} \\ &= \frac{exp(\frac{\epsilon(\hat{s}+\Delta-\Delta)}{2S(x)})}{NL(x)} \\ &= \frac{exp(\frac{\epsilon(\hat{s}+\Delta)}{2S(x)} + \frac{-\epsilon\Delta}{2S(x)})}{NL(x)} \\ &= \frac{exp(\frac{\epsilon(\hat{s}+\Delta)}{2S(x)})}{NL(x)} \cdot e^{\frac{-\epsilon\Delta}{2S(x)})}. \end{aligned}$$

By bounding the $\Delta \geqslant -S(x)$, we can get:

$$\begin{aligned} \frac{exp(\frac{\epsilon(\hat{s}+\Delta)}{2S(x)})}{NL(x)} \cdot e^{\frac{-\epsilon\Delta}{2S(x)}} &\leqslant \frac{exp(\frac{\epsilon(\hat{s}+\Delta)}{2S(x)})}{NL(x)} \cdot e^{\frac{\epsilon}{2}} \\ &= e^{\frac{\epsilon}{2}} \Pr_{z\sim\mathcal{M}_H(x)}[u(r,x) = (\Delta + \hat{s})] \leqslant RHS \end{aligned}$$

□

2.3.3 Dilation Property of Exponential Mechanism

Lemma 2.2. *for any exponential mechanism $\mathcal{M}_H(x)$, $\lambda < |\beta|$, ϵ, $|\delta| < 1$ and $\beta \leqslant \ln(1 - \frac{\epsilon}{2\ln(\frac{\delta}{2(n+1)})})$, the dilation property holds:*

$$\Pr_{z\sim\mathcal{M}_H(x)}[u(z,x) = c] \leqslant e^{\frac{\epsilon}{2}} \Pr_{z\sim\mathcal{M}_H(x)}[u(z,x) = e^{\lambda}c] + \frac{\delta}{2},$$

where the sensitivity in mechanism is still smooth sensitivity as above.

Proof. The sensitivity is always greater than 0, and our utility function $-\mathsf{H}(\mathsf{BI}(x), z)$ is smaller than zero, i.e., $u(z,x) \leqslant 0$, we need to consider two cases where $\lambda < 0$, and $\lambda > 0$:

We set the $h(c) = Pr[u(\mathcal{M}_H(x)) = c] = 2\frac{exp(\frac{\epsilon z}{2S(x)})}{NL(x)}$.

We first consider $\lambda < 0$. In this case, $1 < e^{\lambda}$, so the ratio $\frac{h(c)}{h(e^{\lambda}c)} = \frac{exp(\frac{\epsilon c}{2S(x)})}{exp(\frac{\epsilon(c\cdot e^{\lambda})}{2S(x)})}$ is at most $\frac{\epsilon}{2}$.

Next, we proof the dilation property for $\lambda > 0$, The ratio of $\frac{h(c)}{h(e^{\lambda}c)}$ is $\exp(\frac{\epsilon}{2} \cdot \frac{u(\mathcal{M}_H(x))(1-e^{\lambda})}{S(x)})$. Consider the event $G = \{\mathcal{M}_H(x) : u(\mathcal{M}_H(x)) \leqslant \frac{S(x)}{(1-e^{\lambda})}\}$. Under this event, the log-ratio above is at most $\frac{\epsilon}{2}$. The probability of G under density $h(c)$ is $1 - \frac{\delta}{2}$. Thus, the probability of a given event z is at most $Pr[c \cap G] + Pr[\overline{G}] \leqslant e^{\frac{\epsilon}{2}} Pr[e^{\lambda}c \cap G] + \frac{\delta}{2} \leqslant e^{\frac{\epsilon}{2}} Pr[e^{\lambda}c] + \frac{\delta}{2}$.

Detail proof:

- $\lambda < 0$

  The left hand side will always be smaller than 0 and the right hand side greater than 0. This will always holds, i.e.

- $\lambda > 0$

  Because $\hat{s} = u(r)$ where $r \sim \mathcal{M}_H(x)$, we can substitute $\hat{s}$ with $u(\mathcal{M}_H(x))$. Then, what we need to proof under the case $\lambda > 0$ is:

$$u(\mathcal{M}_H(x)) \leqslant \frac{S(x)}{(1-e^{\lambda})}$$

  By applying the accuracy property of exponential mechanism, we bound the probability that the equation holds with probability:

$$Pr[u(\mathcal{M}_H(x)) \leqslant \frac{S(x)}{(1-e^{\lambda})}] \leqslant \frac{|\mathcal{R}|exp(\frac{\epsilon S(x)}{(1-e^{\lambda})}/2S(x))}{|\mathcal{R}_{OPT}|exp(\epsilon OPT_{u(x)}/2S(x))}$$

  In our Bayesian Inference mechanism, the size of the candidate set $\mathcal{R}$ is equal to the size of observed data set plus 1, i.e., $n+1$, and $OPT_{u(x)} = 0$, then we have:

$$\begin{aligned} Pr[u(\mathcal{M}_H(x)) \leqslant \frac{S(x)}{(1-e^{\lambda})}] &= (n+1)exp(\frac{\epsilon S(x)}{(1-e^{\lambda})}/2S(x)) \\ &= (n+1)exp(\frac{\epsilon}{2(1-e^{\lambda})}) \end{aligned}$$

  When we set $\lambda \leqslant \ln(1 - \frac{\epsilon}{2\ln(\frac{\delta}{2(n+1)})})$, it is easily to derive that $Pr[u(\mathcal{M}_H(x)) \leqslant \frac{S(x)}{(1-e^{\lambda})}] \leqslant \frac{\delta}{2}$.

□

2.3.4 Privacy Analysis

Lemma 2.3. *$\mathcal{M}_H$ is (ϵ, δ)-differential privacy.*

Proof. of Lemma 2.3: For all neighboring $x, y \in D^n$ and all sets $\mathcal{S}$, we need to show that:

$$\Pr_{z\sim\mathcal{M}_H(x)}[z \in \mathcal{S}] \leqslant e^{\epsilon} \Pr_{z\sim\mathcal{M}_H(y)}[z \in \mathcal{S}] + \delta.$$

Given that $2\Big(\Pr_{z\sim\mathcal{M}_H(x)}[z \in \mathcal{S}]\Big) = \Pr_{z\sim\mathcal{M}_H(x)}[u(x,z) \in \mathcal{U}]$, let $\mathcal{U}_1 = \frac{u(y,z)-u(x,z)}{S(x)}$, $\mathcal{U}_2 = \mathcal{U} + \mathcal{U}_1$ and $\mathcal{U}_3 = \mathcal{U}_2 \cdot \frac{S(x)}{S(y)} \cdot \ln(\frac{NL(x)}{NL(y)})$. Then,

$$\begin{aligned} 2\Big(\Pr_{z\sim\mathcal{M}_H(x)}[z \in \mathcal{S}]\Big) &= \Pr_{z\sim\mathcal{M}_H(x)}[u(x,z) \in \mathcal{U}] \\ &\leqslant e^{\epsilon/2} \cdot \Pr_{z\sim\mathcal{M}_H(x)}[u(x,z) \in \mathcal{U}_2] \\ &\leqslant e^{\epsilon} \cdot \Pr_{z\sim\mathcal{M}_H(x)}[u(x,z) \in \mathcal{U}_3] + e^{\epsilon/2} \cdot \frac{\delta'}{2} \\ &= e^{\epsilon} \cdot \Pr_{z\sim\mathcal{M}_H(y)}[u(y,z) \in \mathcal{U}] + \delta = 2\Big(e^{\epsilon} \cdot \Pr_{z\sim\mathcal{M}_H(x)}[z \in \mathcal{S}]\Big) + \delta \end{aligned}$$

The first inequality holds by the sliding property, since the $\mathcal{U}_1 \geqslant -S(x)$. The second inequality holds by the dilation property, since $\frac{S(x)}{S(y)} \cdot \ln(\frac{NL(x)}{NL(y)}) \leqslant 1 - \frac{\epsilon}{2\ln(\frac{\delta}{2(n+1)})}$.

□

3 Accuracy Analysis

3.1 Laplace Mechanism

Fixing a data set x, we will sample noise $Lap_i = floor(Y)$ where $Y \sim Lap(\frac{2}{\epsilon})$. Since already had the accuracy bound based on the l_1 norm in Laplace mechanism:

$$Pr[|Y| \geqslant t] = e^{-\frac{t\epsilon}{2}}.$$

So we have the probability $Pr[|Lap_i|] = Pr[|Lap_i| \leqslant |Y| < |Lap_i| + 1] = e^{-\frac{|Lap_i|\epsilon}{2}} - e^{-\frac{(|Lap_i|+1)\epsilon}{2}}$.

When we are in the case of m dimension Dirichlet distribution, we will add $(m-1)$ i.i.d. Laplace noises $\{|Lap_1|, |Lap_2|, \cdots, |Lap_{m-1}|\}$ to the output. Then the probability of each group of noises are calculated as $Pr[\{|Lap_1|, |Lap_2|, \cdots, |Lap_{m-1}|\}] = Pr[|Lap_1| \leqslant |Y| < |Lap_1| + 1] \times Pr[|Lap_2| \leqslant |Y| < |Lap_2| + 1] \times \cdots \times Pr[|Lap_{m-1}| \leqslant |Y| < |Lap_{m-1}| + 1] = (e^{-\frac{|Lap_1|\epsilon}{2}} - e^{-\frac{(|Lap_1|+1)\epsilon}{2}}) \times (e^{-\frac{|Lap_2|\epsilon}{2}} - e^{-\frac{(|Lap_2|+1)\epsilon}{2}}) \times \cdots \times (e^{-\frac{|Lap_{m-1}|\epsilon}{2}} - e^{-\frac{(|Lap_{m-1}|+1)\epsilon}{2}})$.

3.2 Exponential Mechanism with Global Sensitivity

Also, according to [1], we already had the accuracy bound of exponential mechanism with global sensitivity as:

$$\underset{z\sim\mathcal{M}_H(x)}{Pr}[\mathsf{H}(\mathsf{BI}(x), z) \geqslant c] \leqslant \frac{|R| \exp(\frac{-\epsilon c}{2\Delta_g})}{|R_{OPT}| \exp(\frac{-\epsilon OPT_{\mathsf{H}(\mathsf{BI}(x),z)}(x)}{2\Delta_g})},$$

where $|R|$ is the size of the candidate set. Since $OPT_{\mathsf{H}(\mathsf{BI}(x),z)}(x) = 0$ and $|R_{OPT}| = 1$, we simplified accuracy bound here $\underset{z\sim\mathcal{M}_H(x)}{Pr}[\mathsf{H}(\mathsf{BI}(x), z) \geqslant c] \leqslant |R| \exp(\frac{-\epsilon c}{2\Delta_g})$,

3.3 Exponential Mechanism with Smooth Sensitivity

We explored three accuracy bounds for our exponential mechanism with smooth sensitivity.

First is the tight bound with very accurate calculation.

$$\underset{z\sim\mathcal{M}_H(x)}{Pr}[\mathsf{H}(\mathsf{BI}(x), z) \geqslant c] = \sum_{\{z|\mathsf{H}(\mathsf{BI}(x),z)\geqslant c\}} \frac{e^{\frac{-\epsilon\mathsf{H}(\mathsf{BI}(x),z)}{S(x)}}}{NL_x}.$$

In order to be more efficient, we designed the second accuracy bound which is slightly looser than the first one:

$$\underset{z\sim\mathcal{M}_H(x)}{Pr}[\mathsf{H}(\mathsf{BI}(x), z) \geqslant c] \leqslant \frac{|R| \exp(\frac{-\epsilon c}{S(x)})}{NL_x}.$$

In the second bound, we still need to calculate the normaliser every time. So we want make further improvements on efficiency like follows:

$$\underset{z\sim\mathcal{M}_H(x)}{Pr}[\mathsf{H}(\mathsf{BI}(x), z) \geqslant c] \leqslant \frac{|R| \exp(\frac{-\epsilon c}{S(x)})}{N(n)},$$

where we replace the NL_x with a value only related to the size of the data. However, we haven't figured out the formula of this $N(n)$.

Within the three accuracy bounds, we will mainly use the first one in following analysis.

3.4 Accuracy Trade-off Between Laplace and Our Exponential Mechanism with Smooth Sensitivity

Based on accuracy study above, we are going to have a further study on the accuracy relationship between Laplace mechanism and our exponential mechanism in four aspects: data size, dimensions, prior distribution and data variance.

3.4.1 Accuracy Analysis wrt. Data Size and Distribution Dimensions

In this part, we will analyze the influence of data size on the two mechanisms' discrete probabilities of outputting each candidate separately. In following analysis, we will count the probabilities wrt. the steps from correct answer, in order to be more concise. For example, in the case where the correct posterior distribution is $\mathsf{beta}(5,5)$, the candidate $\mathsf{beta}(4,6)$ and $\mathsf{beta}(6,4)$ are of 1 steps from $\mathsf{beta}(5,5)$; when correct posterior distribution is $\mathsf{DL}(5,5,5)$, the candidate $\mathsf{DL}(4,5,3)$, $\mathsf{DL}(4,3,5)$, $\mathsf{DL}(5,4,3)$, $\mathsf{DL}(5,3,4)$, $\mathsf{DL}(3,5,4)$ and $\mathsf{DL}(5,5,5)$ are all of 1 step from $\mathsf{DL}(5,5,5)$. Under the Hellinger distance measurement, if candidates are of the same steps from correct answer, they also have the same Hellinger distance from correct answer. i.e for every $\mathsf{H}(\mathsf{BI}(x),z)=c$, it will have a corresponding steps $step(\mathsf{H}(\mathsf{BI}(x),z)=c)=k$. This will make the results more clear and observable.

- In our exponential mechanism, the probability of outputting candidates wrt. steps (i.e., the hellinger distance) from the correct answer is calculated as:

$$\Pr_{z\sim\mathcal{M}_H(x)}[\mathsf{H}(\mathsf{BI}(x),z)=c]=\sum_{\{z|\mathsf{H}(\mathsf{BI}(x),z)=c\}}\frac{exp(\frac{-\epsilon c}{S(x)})}{\Sigma_{r'\in R}\ exp(\frac{-\epsilon\mathsf{H}(\mathsf{BI}(x),r')}{2S_\beta(x)})},$$

  Each candidate will occupy a portion of "1" as their outputting probability. Supposing candidates within three steps from the correct answer are good answers, the portion occupied by the good answers is decreasing when the size of candidate set increasing. That's to say, the probabilities of outputting good answers are decreasing when the data size and dimension of prior distribution increasing. More specifically, the candidate set size is $\sim n^{m-1}$, which means the probabilities of outputting good answers are decreasing with speed $\sim n^{m-1}$.

- However in Laplace mechanism, the probability of producing noise has little relevance with the size of the candidate set. In m dimensional Dirichlet distribution, when the correct posterior distribution is $\mathsf{DL}(\alpha_1,\alpha_2,\cdots,\alpha_m)$, we are adding Laplace noise in this way: $\mathsf{DL}(\alpha_1+Lap_1,\alpha_2+Lap_2,\cdots,n-(\alpha_1+Lap_1+\alpha_{m-1}+Lap_{m-1}))$ where $Lap_i\sim Floor(Lap(\frac{2}{\epsilon}))$ are i.i.d. So, the probabilities of outputting a single candidate in Laplace mechanism can be obtained by:

$$\begin{aligned}Pr[(Lap_1,Lap_2,\cdots,Lap_{m-1})]=&Pr[Lap_1\leqslant Lap(\frac{2}{\epsilon})<Lap_1+1]\\&\times Pr[Lap_2\leqslant Lap(\frac{2}{\epsilon})<Lap_2+1]\times\cdots\\&\times Pr[Lap_{m-1}\leqslant Lap(\frac{2}{\epsilon})<Lap_{m-1}+1],\end{aligned}$$

  where $Pr[Lap_i\leqslant Lap(\frac{2}{\epsilon})<Lap_i+1]=\frac{1}{2}Pr[|Lap_i|\leqslant Lap(\frac{2}{\epsilon})<|Lap_i|+1]$ when $Lap_i>0$ and $\frac{1}{2}Pr[|Lap_i|-1\leqslant Lap(\frac{2}{\epsilon})<|Lap_i|]$ else.

  From analysis above, we can see the probabilities of outputting the good answers will not change a lot as the size of the candidate set increasing. Specifically, we can see that the probabilities of outputting good answers are decreasing with speed upper bounded by , which means the probability of correct answer will decrease very little no matter how large the candidate set is.

- Then, we do some concrete cases analysis.
  - when the prior is $\mathsf{beta}(1,1)$, the observed data set $x=(1,1,0,0,1,1,0,0)$, it is easy to compute the posterior distribution: $\mathsf{beta}(5,5)$, the probability from the two mechanisms are:

Hellinger Distance / steps	Our Exponential Mechanism	Laplace Mechanism
$Pr[\mathsf{H}(\mathsf{BI}(x),r)=0.83737258593]/4$	0.0431193490585	0.066561234758
$Pr[\mathsf{H}(\mathsf{BI}(x),r)=0.662174391701]/3$	0.0785621424847	0.0992976939175
$Pr[\mathsf{H}(\mathsf{BI}(x),r)=0.457635865026]/2$	0.158265808563	0.148134752205
$Pr[\mathsf{H}(\mathsf{BI}(x),r)=0.233629480709]/1$	0.340809715054	0.220991081918
$Pr[\mathsf{H}(\mathsf{BI}(x),r)=0.0]/0$	0.37924298484	0.329679953964

Here, there are only 5 kinds of steps from correct answer. Our mechanism in the second column is clearly better than Laplace mechanism in the third column. When the candidates are close to correct answer (for example, $0, 1, 2$ steps from correct answer), our mechanism can output them with higher probabilities than Laplace mechanism. On the other hand, when candidates are far away from correct answer (for example, $3, 4$ steps), our mechanism can output them with lower probabilities.

– keep the prior unchanged, $\mathsf{beta}(1, 1, 1)$, the observed data set $x = (20, 20)$ (suppose a black box will produce A, B with a certain distribution, after observing this black box continuously for 60 times, we get 20 times A and 20 times B, we can get the posterior distribution $\mathsf{beta}(21, 21)$

Hellinger Distance / steps	Our Exponential Mechanism	Laplace Mechanism
$Pr[\mathsf{H}(\mathsf{BI}(x), r) = 0.999999984481]/\cdots$	0.000149705644585	3.05988187701e-09
	$\cdots$	
$Pr[\mathsf{H}(\mathsf{BI}(x), r) = 0.187421762881]/2$	0.0548161224677	0.0285774941516
$Pr[\mathsf{H}(\mathsf{BI}(x), r) = 0.110122822057]/1$	0.192227323562	0.170131188571
$Pr[\mathsf{H}(\mathsf{BI}(x), r) = 0.0]/0$	0.0713016293602	0.108688872046

This case shows that the Laplace mechanism do much better than our exponential mechanism significantly. These good answers with few steps from correct answer can be outputted with higher probabilities in Laplace mechanism than in our mechanism. Moreover, in our mechanism the bad answers and good answers have very similar outputting probabilities.

3.4.2 Accuracy Analysis wrt. prior distribution

In this part, we firstly study the local sensitivity of Hellinger distance of different beta distributions as in Fig. 3.4.2, in order to have a better understanding of the relationships between accuracy and next two aspects.

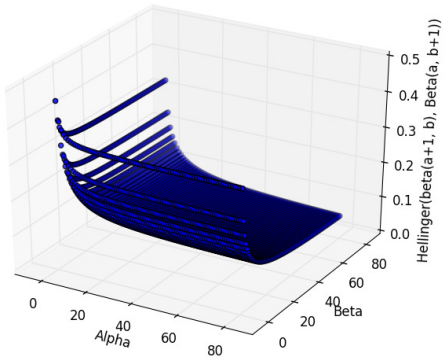


Figure 1: Hellinger distance study

As in Fig. 3.4.2, the local sensitivity of Hellinger distance will decrease when beta distribution's two parameters get closer (i.e. more uniform) and larger. In the same time, our smooth sensitivity will also decrease based on the Def. 2.1. From accuracy bound in Sec. 3.3, the accuracy will be improved when sensitivity go larger. In consequence, when we increase the prior distribution, the smooth sensitivity in our exponential mechanism will decrease, which means our accuracy will be improved. However, the sensitivity of l_1 norm in Laplace mechanism is fixed regardless the prior distribution. We will study this trade-off in Sec. 4.3.4.

3.4.3 Accuracy Analysis wrt. data variance

Similar as above, when the data variance is small (i.e. the data are uniform), the parameters of posterior distribution will get more uniform (i.e. closer). Based on Fig. 3.4.2, the Hellinger distance's sensitivity will get smaller when parameter get closer. As a result, the accuracy will be improved in the same time. We will study this trade-off in Sec. 4.3.3

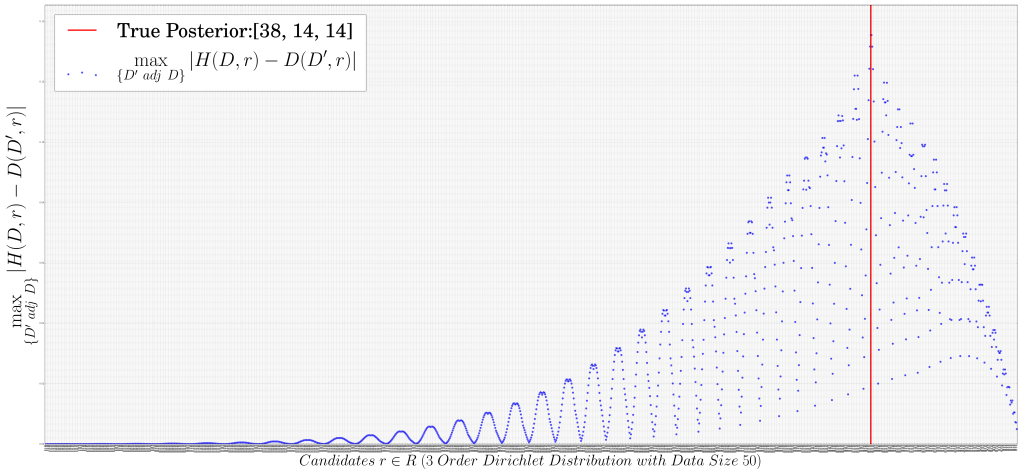


Figure 2: Experimental Results for Finding the Local Sensitivity Efficiently

4 Experimental Evaluations

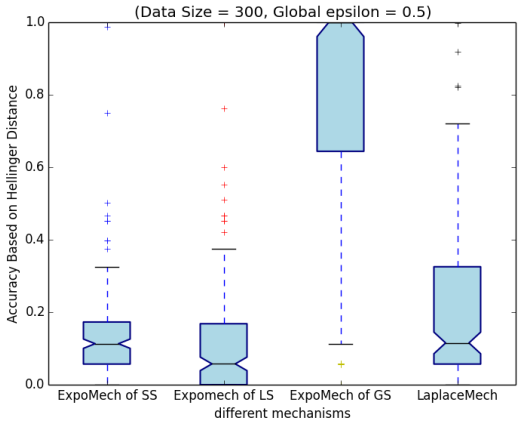
4.1 Computation Efficiency

The formula for computing the local sensitivity presented in Sec. 2.2: $\max_{\{|x,y'|\leqslant 1; y' \in \mathcal{X}^n\}} \max_{\{r \in R\}} \{\mathsf{H}(\mathsf{BI}(x), r) - \mathsf{H}(\mathsf{BI}(y'), r)|\}$ can be reduced to $\max_{\{|x,y'|\leqslant 1; y' \in \mathcal{X}^n\}} \mathsf{H}(\mathsf{BI}(x), \mathsf{BI}(y'))|$ by applying the distance triangle property. i.e., the maximum value over $\max_{r \in R}$ always happen when $r = \mathsf{BI}(x)$ itself, where $\Delta_l u(x) = \max_{\{|x,y'|\leqslant 1; y' \in \mathcal{X}^n\}} \{\mathsf{H}(\mathsf{BI}(x), \mathsf{BI}(x)) - \mathsf{H}(\mathsf{BI}(y'), \mathsf{BI}(x))|\} = \max_{\{|x,y'|\leqslant 1; y' \in \mathcal{X}^n\}} \{\mathsf{H}(\mathsf{BI}(y'), \mathsf{BI}(x))|\}$. We also have some experiments for validating our proposal as in Fig. 3.4.3, where we calculate the $\max_{\{|x,y'|\leqslant 1; y' \in \mathcal{X}^n\}}$ value for every candidate $r \in R$. It is shown that maximum value taken when $r = \mathsf{BI}(x)$.

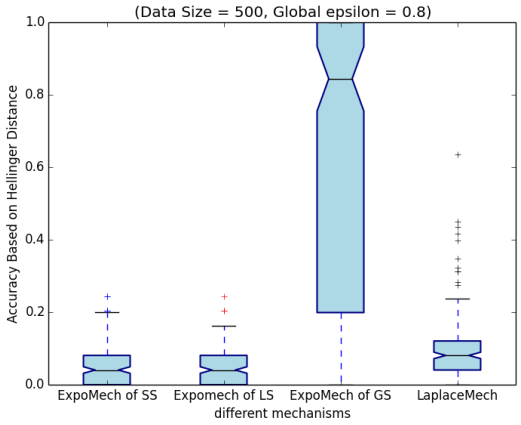
4.2 Experimental Evaluations on Accuracy

In this section, we do some experiments in order to study the accuracy property of these mechanisms, including the exponential mechanism with three kinds of sensitivity and the Laplace mechanism.

We first analyzed the accuracy based on the Hellinger distance. By repeating the experiments for 10000 times and plotting the accuracy of each execution, we got two groups of results under different parameters setting as in Fig. 3 and Fig. 4. X-axis is labeled with different mechanisms. We took four mechanisms in our experiments: our newly designed exponential mechanism with smooth sensitivity named "ExpoMech of SS", exponential mechanism with local sensitivity named " ExpoMech of LS", exponential mechanism with global sensitivity named " ExpoMech of GS" and Laplace mechanism named "LaplaceMech". Y-axis is the accuracy

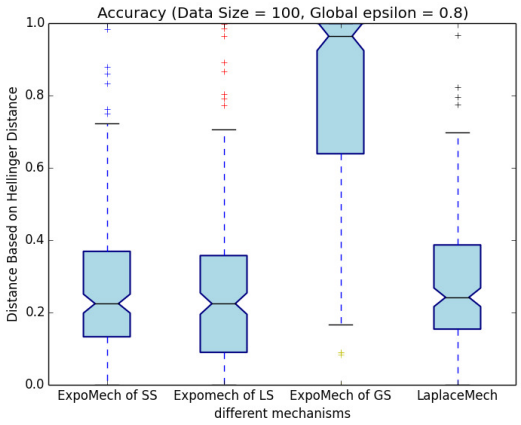


(a) Data size $n = 500$ with global $\epsilon = 0.5$

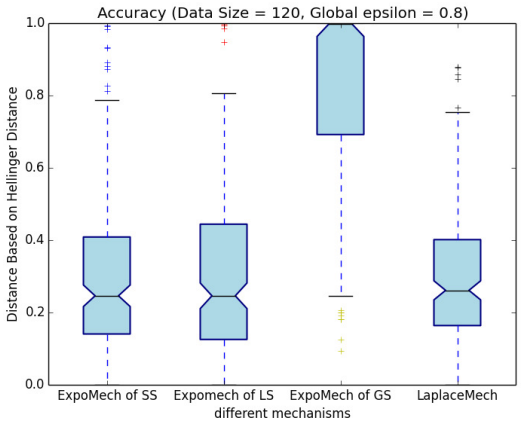


(b) Data size $n = 300$ with global $\epsilon = 0.5$

Figure 3: The experimental results of accuracy of algorithms with Beta prior distribution $\mathsf{beta}(7, 4)$ based on Hellinger distance

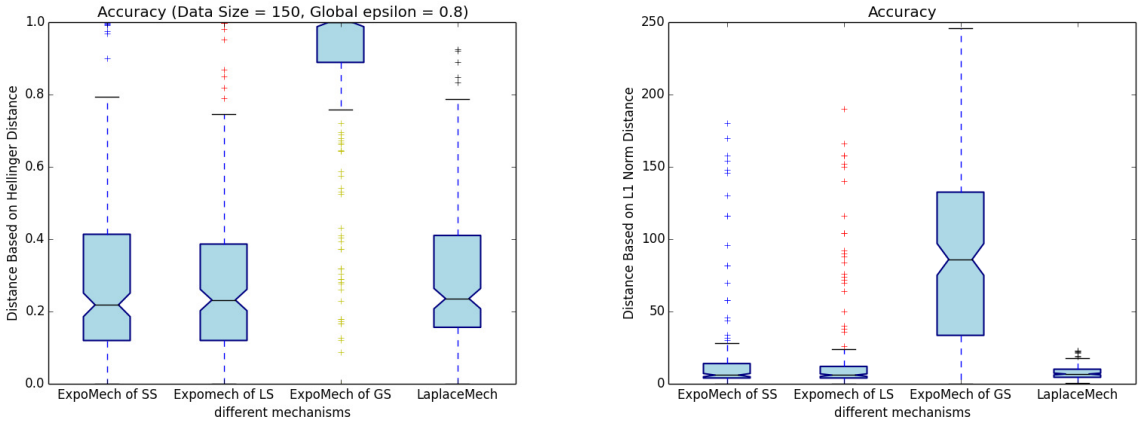


(a) Data size $n = 100$, the exponential mechanism with global sensitivity 0.239992747797 is 0.8 -DP, with local sensitivity 0.08 is Non-Private and with 0.0699407108115 - bound smooth sensitivity 0.09 is (0.8,0.8)-DP



(b) Data size $n = 120$, the exponential mechanism with global sensitivity 0.239992747797 is 0.8 -DP, with local sensitivity 0.0945 is Non-Private, with 0.0677791100173 - bound smooth sensitivity 0.096 is (0.8,0.8)-DP

Figure 4: The experimental results of accuracy of algorithms with Dirichlet prior distribution $\mathsf{DL}(7, 4, 5)$ based on Hellinger distance



(a) accuracy measurement based on Hellinger distance

(b) accuracy measurement based on l_1 norm

Figure 5: The experimental results of accuracy of algorithms with Dirichlet prior distribution $\mathsf{DL}(7, 4, 5)$ based on Hellinger distance and l_1 norm, where data size $n = 150$, the exponential mechanism with global sensitivity 0.239992747797 is 0.8 -DP, with local sensitivity 0.0945 is Non-Private, with 0.0677791100173 -bound smooth sensitivity 0.096 is (0.8,0.8)-DP

measured by Hellinger distance between output z of each execution and the correct inference result $\mathsf{BI}(x)$, $\mathsf{H}(z, \mathsf{BI}(x))$.

When prior distribution is beta, under data size $n = 500$ and 300, we obtained two plots in Fig. 3. It is shown that our ExpoMech of SS is slightly better than Laplace mechanism and much more better than it with global sensitivity.

To have more experimental results, we then extended the beta to the Dirichlet distribution DL as well as increased the data size, plotted in Fig. 4.

Then, we study the accuracy of these four mechanisms based on l_1 norm, with the Dirichlet distribution $\mathsf{DL}(7, 4, 5)$ and data size 150 for comparison, as in Fig. 5.

The results based on l_1 norm are actually very similar to results based on Hellinegr distance.

In both of the two cases, the performances of our exponential mechanism with smooth sensitivity and Laplace mechanism are very close. We can obtain some preliminary conclusions:

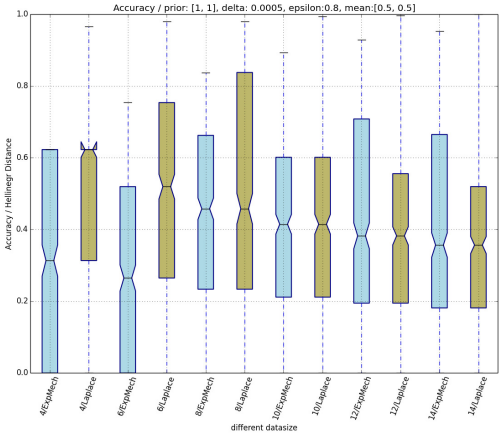
1. Our exponential mechanism with smooth sensitivity can have similar accuracy as it with local sensitivity. Both of the two exponential mechanisms are do much better than it with global sensitivity.

2. The accuracy of our exponential mechanism is better than Laplace mechanism. But the advantages are not significant.

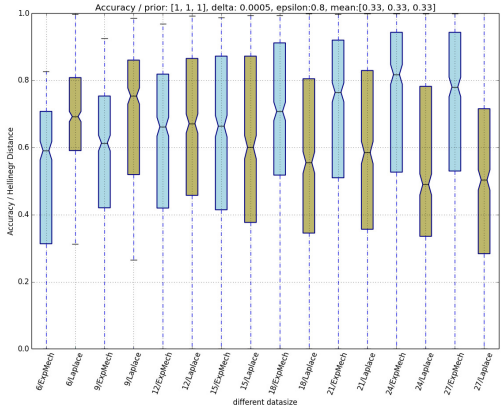
In consequence, we have a further experiments exploration on the accuracy trade-off between Laplace mechanism and our exponential mechanism in next section.

4.3 Accuracy Trade-off Evaluation wrt. Different Variables

In this section, we evaluate the accuracy wrt. four variables, including data size, dimension, data variance and prior distribution, and some combinations of these variables. We experiment 1000 times under each value of variables and produce 4-quantile plots for each variable. In following 4-quantile plots, the y-axis is accuracy measured by Hellinger distance, x-axis is different value of variables. The blue boxes in plots represent our exponential mechanism and the next yellow box represents the Laplace mechanism under the same setting.



(a) two dimensions with beta$(1, 1)$ prior distribution



(b) three dimensions with DL$(1, 1, 1)$ prior distribution

Figure 6: Accuracy measurement based on Hellinger distance wrt. different datasizes. Settings: observed data are uniformly distributed, $\epsilon = 0.8$ and $\delta = 0.0005$

4.3.1 Accuracy Evaluation wrt. Datasize

In Fig. 6, both of the two plots show that when data size go larger, accuracy of our exponential mechanism are decreasing. In Fig. 6(a), when the data size is smaller than 12, we can beta Laplace mechanism but fail when data size larger than or equal to 12. Same as in Fig. 6(b), we can beat Laplace mechanism when data size is smaller than 15 and fail otherwise.

4.3.2 Accuracy Evaluation wrt. Dimensions

In Fig. 7, x-axis are observed data sets of different size and dimensions. The plot shows that dimensions have similar influence on our exponential mechanism and the Laplace mechanism. Accuracy of two mechanisms both decrease when dimensions go larger. We will be beat by Laplace mechanism when data size increase but will not be affected when dimensions increase. In other words, dimension has little influence on whether we will beat Laplace mechanism.

4.3.3 Accuracy Evaluation wrt. Data variance

In Fig. 8, x-axis are observed data sets of different variances (or means). We study this variable under two-dimension beta distribution in order to be concise. It shows that our mechanism's accuracy is better when data variance go smaller, meanwhile Laplace mechanism go worse. We will beat Laplace mechanism when observed data are more uniformly.

4.3.4 Accuracy Evaluation wrt. Prior Distribution

In Fig. 9, we study this variable under setting that observed data set is $[5, 5, 5]$ because in Fig. 6 Laplace mechanism beat us when data size is 15 and uniformly distributed. The plot shows that in the beginning we cannot beat Laplace but when prior distribution grow larger, we perform better and better and beat Laplace mechanism finally.

4.3.5 Accuracy Evaluation wrt. Prior Distribution and Data Variance

Here, we change the prior distribution and data variance in the same time. As shown in Fig. 10, our exponential mechanism do better in uniform data set than in edging data set while Laplace mechanism on the

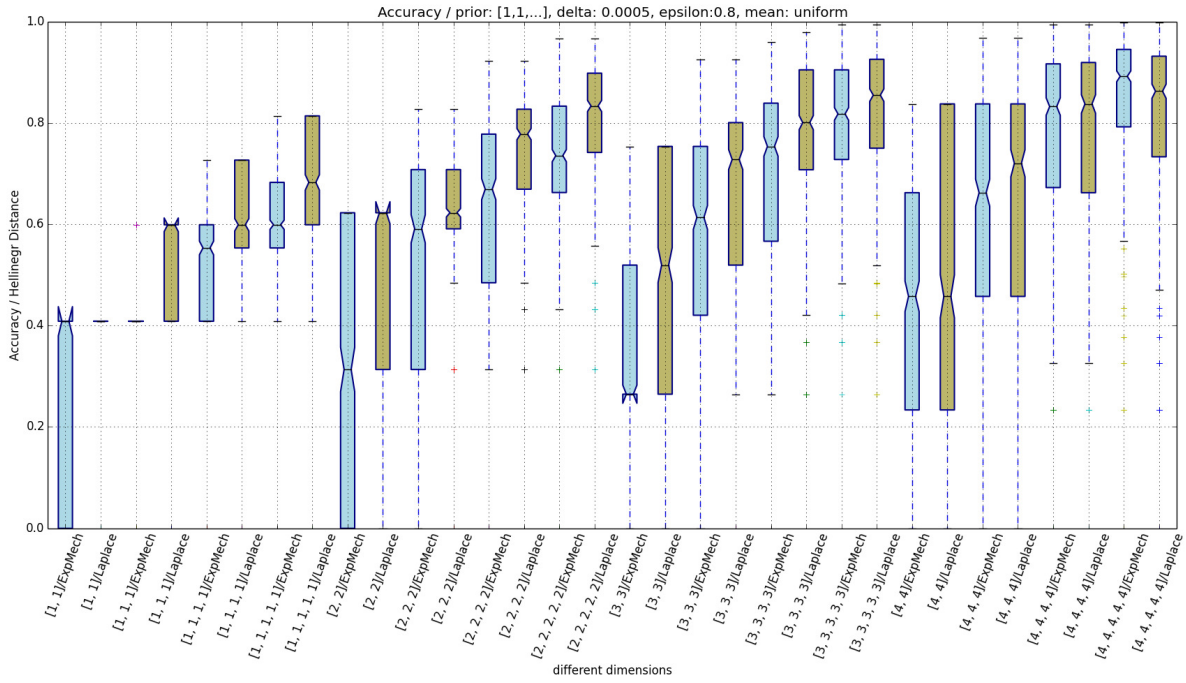


Figure 7: Accuracy measurement based on Hellinger distance wrt. different dimensions and data size. Settings: observed data are uniformly distributed, $\epsilon = 0.8$ and $\delta = 0.0005$, prior distributions are all 1 in every dimension

contrary. Moreover, our mechanism is improving continuously and significantly as prior distribution increasing while Laplace mechanism isn't.

4.4 Experiment Evaluations on Privacy

In order to see our privacy behavior, we study the accurate epsilon under concrete cases in this section. The (ϵ, δ) - differential privacy we proved in Sec. 2.3 is just an upper bound, we concrete ϵ should be smaller than upper bound in our exponential mechanism. We calculate the concrete privacy value in following ways wrt. the data size, and obtain plots in Fig. 11.

$\epsilon = 0.8$ is a privacy upper bound, we can observe that the concrete ϵ values are smaller than the upper bound. That is to say, we achieved a higher privacy level than expected. In next step, we are going to improve the accuracy using this property.

References

[1] Cynthia Dwork, Aaron Roth, et al. The algorithmic foundations of differential privacy. *Foundations and Trends® in Theoretical Computer Science*, 9(3–4):211–407, 2014.

[2] Kobbi Nissim, Sofya Raskhodnikova, and Adam Smith. Smooth sensitivity and sampling in private data analysis. In *Proceedings of the thirty-ninth annual ACM symposium on Theory of computing*, pages 75–84. ACM, 2007.

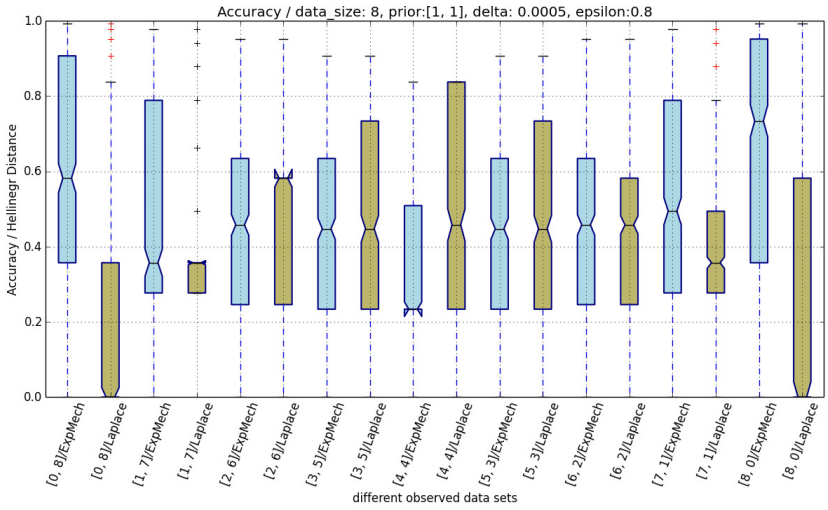


Figure 8: Accuracy measurement based on Hellinger distance wrt. different data variance. Settings: $\epsilon = 0.8$ and $\delta = 0.0005$, prior distributions are all 1 in every dimension

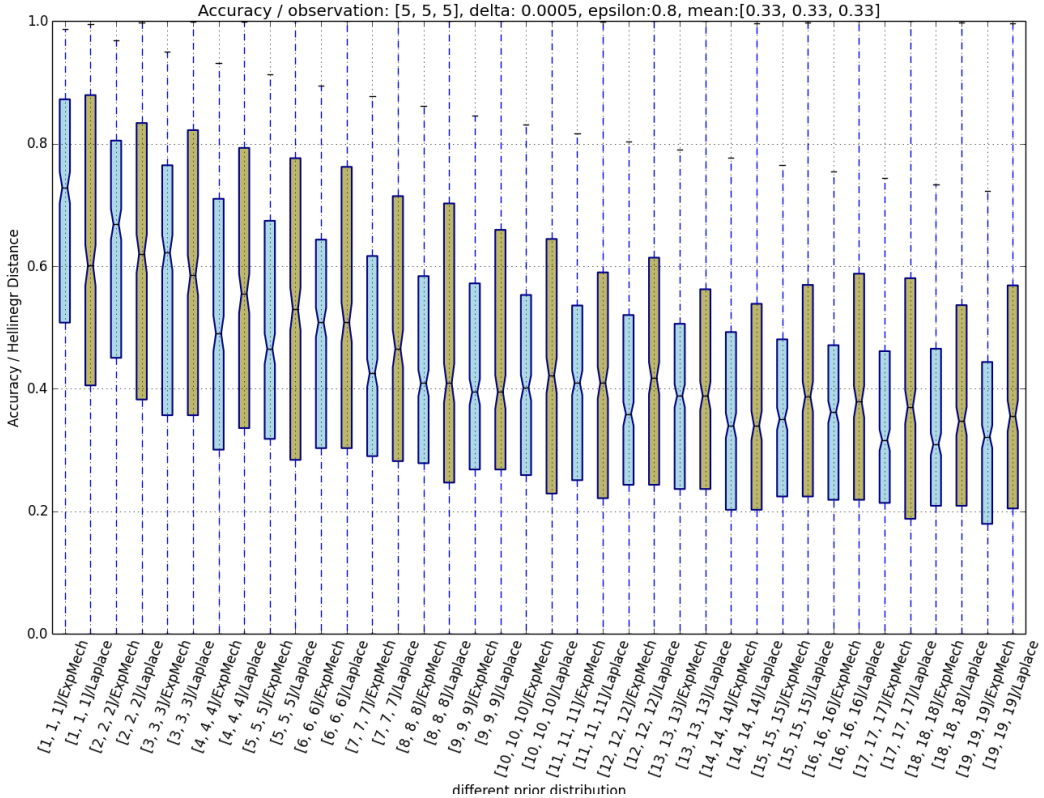


Figure 9: Accuracy measurement based on Hellinger distance wrt. different prior distribution. Settings: $\epsilon = 0.8$ and $\delta = 0.0005$, observed data set is: $[5, 5, 5]$

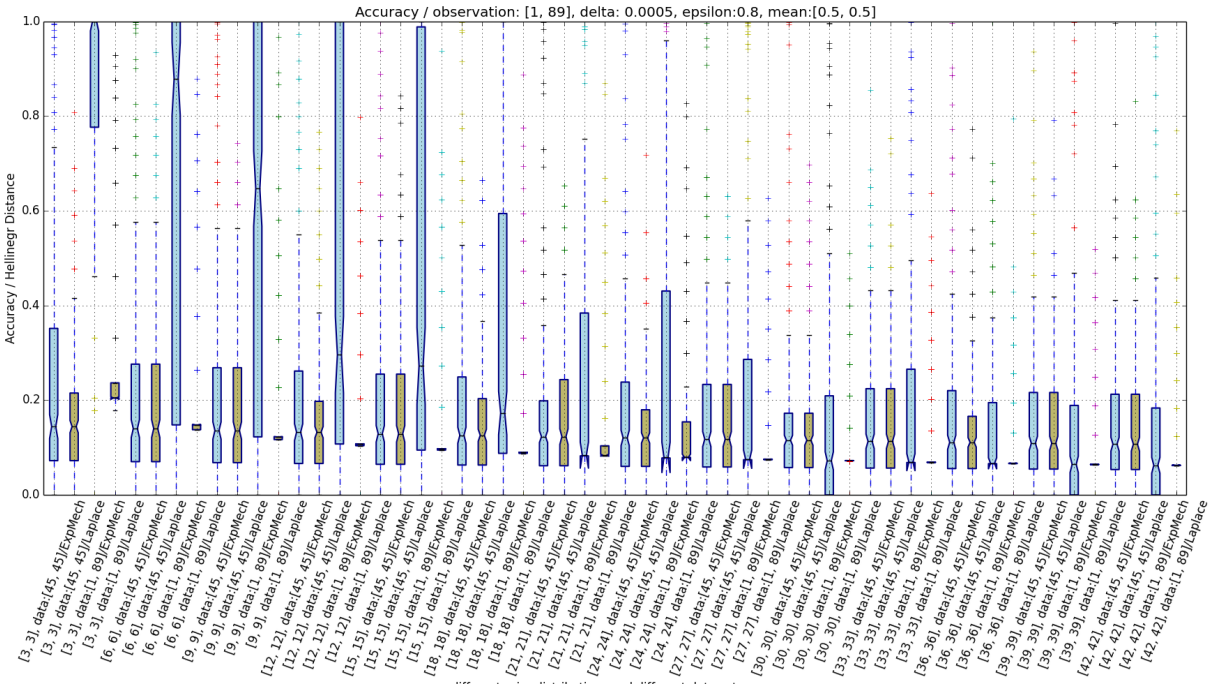
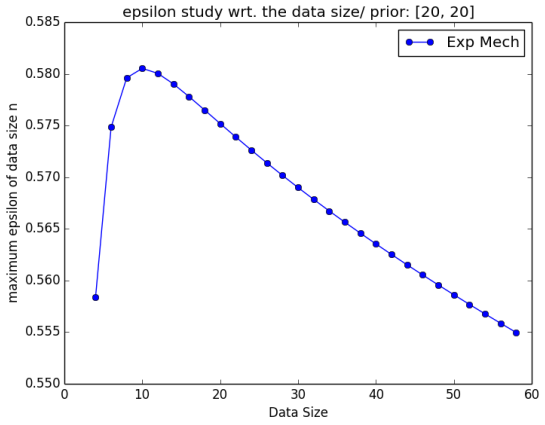
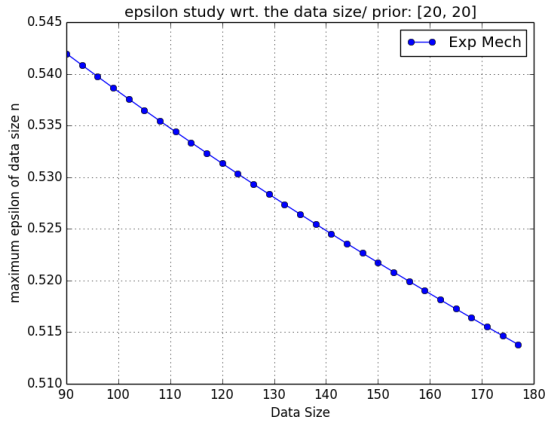


Figure 10: Experimental Results for Finding the Local Sensitivity Efficiently



(a) data size range from 90 to 180



(b) data size range from 90 to 180

Figure 11: Concrete privacy calculation under settings that: prior distribution:$[1, 1]$, $\epsilon = 0.8$, $\delta = 0.0005$ and observed data are uniformly distributed