Tailoring Differentially Private Bayesian Inference to Distance Between Distributions

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Objectives

1. Designing a differentially private Bayesian inference mechanism.
2. Measuring accuracy with a metric over distributions (Hellinger distance ($\mathcal{H}$), f-divergence, etc.).

Bayesian inference (BI) (A Beta-Binomial model example):

- Prior on θ : $\mathsf{beta}(\alpha, \beta), \alpha, \beta \in \mathbb{R}^+$, observed data set $\mathbf{x} = (x_1, \dots x_n) \in \{0, 1\}^n, n \in \mathbb{N}$.
- Likelihood function: $\mathbb{L}_{\mathbf{x}|\theta} = \theta^{\Delta\alpha}(1-\theta)^{n-\Delta\alpha}$, where $\Delta\alpha = \sum_{i=1}^{n} x_i$;
- Posterior distribution over theta: $\mathbb{P}_{\theta|\mathbf{x}} = \mathsf{beta}(\alpha + \Delta\alpha, \beta + n - \Delta\alpha)$.

Differentially Private Bayesian Inference and Motivations

Releasing a differentially private posterior $\mathsf{beta}(\tilde{\alpha}, \tilde{\beta}) = (\alpha + \widetilde{\Delta\alpha}, \beta + n - \widetilde{\Delta\alpha})$.

1. Baseline approach is Laplace mechanism (LapMech) with sensitivity proportional to ℓ_1 norm. But this sensitivity grows linear with the dimension, also we measure results by a different metric. Motivated by this, we calibrate the noise w.r.t sensitivity of the accuracy metric ($\mathcal{H}$).
2. Maximum sensitivity of $\mathcal{H}$ over **beta** distributions achieves at edges as in Fig. 1. But it's very low when move away from edge. Motivated by this, we apply smooth sensitivity in mechanism to improve accuracy.

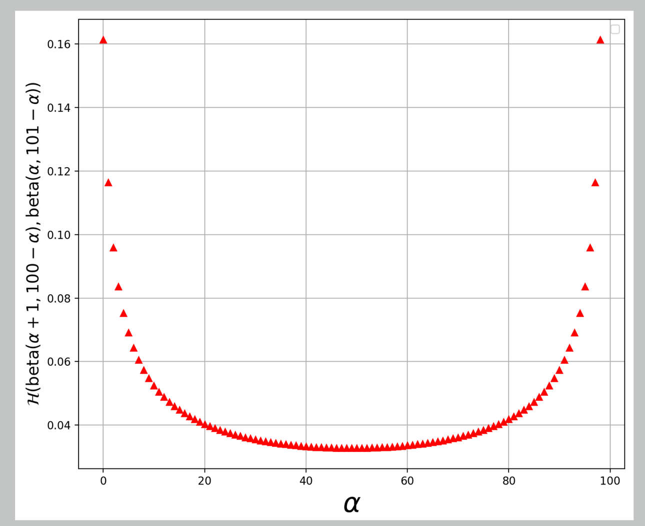


Figure 1: Sensitivities of $\mathcal{H}$ over **beta**

Smoothed Hellinger Distance Based Exponential Mechanism

We define the mechanism $\mathcal{M}^B_{\mathcal{H}}$ which produces an element r in $\mathcal{R}_{\text{post}}$ with probaiblity:

$$\mathbb{P}_{r\sim\mathcal{M}^B_{\mathcal{H}}} = \frac{\exp\left(\frac{-\epsilon\cdot\mathcal{H}(\mathsf{BI}(\mathbf{x}),r)}{2\cdot S(\mathbf{x})}\right)}{\sum_{r\in\mathcal{R}_{\text{post}}}\exp\left(\frac{-\epsilon\cdot\mathcal{H}(\mathsf{BI}(\mathbf{x}),r)}{2\cdot S(\mathbf{x})}\right)}$$

given in input an observations $\mathbf{x}$, parameters $\epsilon > 0$ and $\delta > 0$, where:

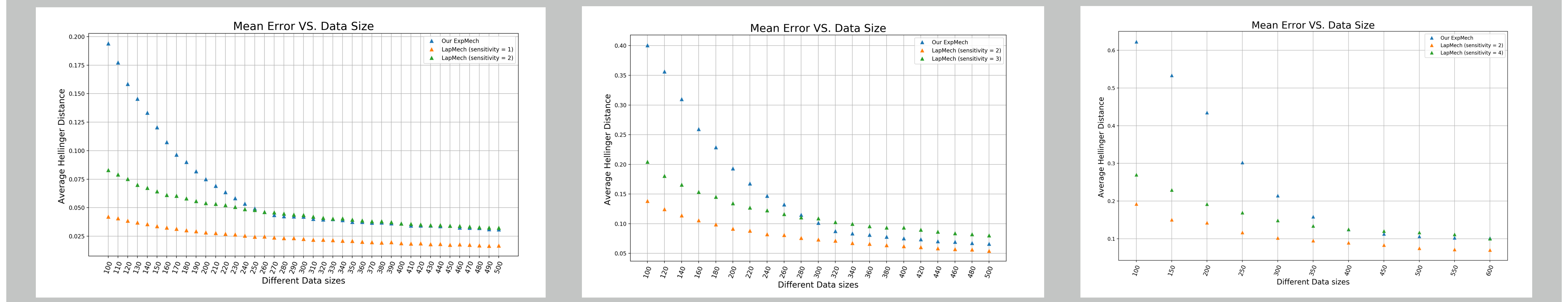
- $\mathcal{R}_{\text{post}}$, the candidates set defined as $\{\mathsf{beta}(\alpha', \beta') \mid \alpha' = \alpha + \Delta\alpha, \beta' = \beta + n - \Delta\alpha\}$, given the prior distribution $\beta_{\text{prior}} = \mathsf{beta}(\alpha, \beta)$ and observed data set size n.
- $-\mathcal{H}(\mathsf{BI}(\mathbf{x}), r)$ denotes the scoring function based on the Hellinger distance.
- $S(\mathbf{x})$, the smooth sensitivity[1]: $S(\mathbf{x}) = \max_{\mathbf{x}'\in\{0,1\}^n}\left\{LS(\mathbf{x}')\cdot e^{-\gamma\cdot d(\mathbf{x},\mathbf{x}')}\right\}$, where:
  - ▷ d: Hamming distance between two data sets,
  - ▷ $LS(\mathbf{x}')$, local sensitivity at $\mathbf{x}'$: $LS(\mathbf{x}) = \max_{\mathbf{x}'\in\mathcal{X}^n:\mathsf{adj}(\mathbf{x},\mathbf{x}'), r\in\mathcal{R}} |\mathcal{H}(\mathsf{BI}(\mathbf{x}), r) - \mathcal{H}(\mathsf{BI}(\mathbf{x}'), r)|$,
  - ▷ $\gamma = \ln(1 - \frac{\epsilon}{2\ln(\frac{\delta}{2(n+1)})})$ to ensure the (ϵ, δ)-differentially private.

Preliminary Experimental Results

Experiments are on three mechanisms and plotted as follows:

- **Green**: Baseline approach. (noises being postprocessed to floor value)
- **Red**: LapMech with sensitivity $\mathbf{1}$ in 2 dimensions and $\mathbf{2}$ in higher dimensions, since it's equivalent to histogram problem (posteriors of adjacent data sets differ only in two dimensions).
- **Blue**: $\mathcal{M}^B_{\mathcal{H}}$.

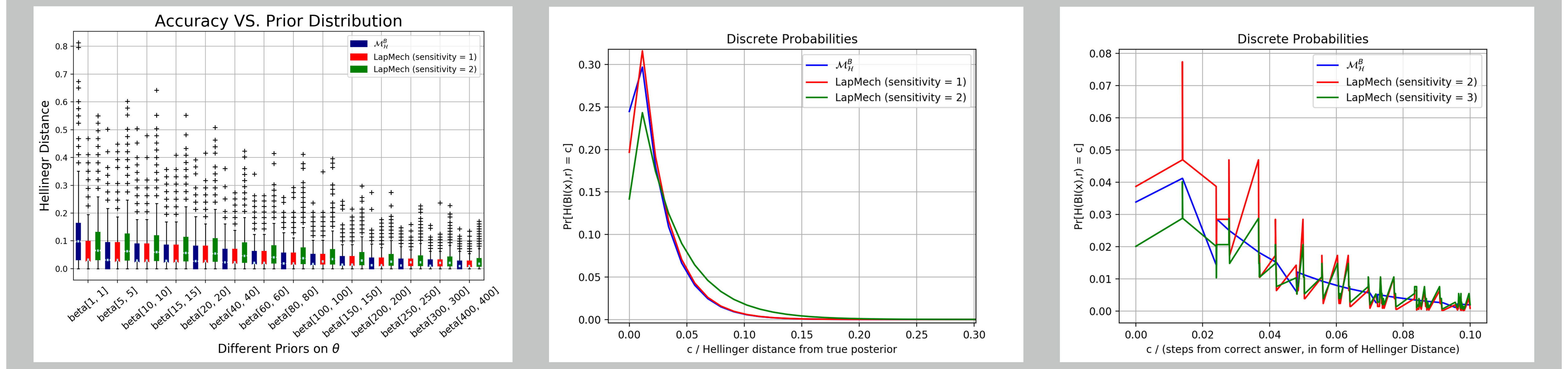
Fig. 2 and Fig. 3(a) give us the average and 4-quantile of Hellinger distance between the sampled results and true posterior, by sampling for $\mathbf{10}k$ times under each data size or prior configuration.



(a) 2 dimensions, data size $\in [100, 500]$ (b) 3 dimensions, data size $\in [100, 500]$ (c) 4 dimensions, data size $\in [100, 600]$

Figure 2: Average accuracy by increasing data set size

Fig. 3(b) and 3(c) give us the discrete probabilities. On the x-axis: the distance of a potential output r from the true answer and on y-axis the probability of r being output by the mechanisms.



(a) 2 dimensions with data size $\mathbf{100}$ (b) 2 dimensions with data size $\mathbf{600}$ (c) 3 dimensions with data size $\mathbf{600}$

Figure 3: 4-quantile and discrete probability plots

Experiments above are with unit prior $\mathsf{beta}(1, 1)$, $\mathsf{beta}(1, 1, 1)$ and $\mathsf{beta}(1, 1, 1, 1)$ (except Fig. 3(a)) , balanced datasets, $\epsilon = 1.0$ and $\delta = 10^{-8}$.

Conclusion

- The smoothed Hellinger distance based exponential mechanism outperforms asymptotically the baseline approach when the latter uses a sensitivity proportional to dimensionality.
- Under the same data set size, $\mathcal{M}^B_{\mathcal{H}}$ can outperform LapMech by increasing the prior.

References

[1] Kobbi Nissim, Sofya Raskhodnikova, and Adam Smith. Smooth sensitivity and sampling in private data analysis. In *Proceedings of the thirty-ninth annual ACM symposium on Theory of computing*, pages 75–84. ACM, 2007.

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