Tailoring Differentially Private Bayesian Inference to Distance Between Distributions

Extended Abstract

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ABSTRACT

Bayesian inference is a statistical method which allows one to derive a *posterior* distribution, starting from a *prior* distribution and observed data. Several approaches have been explored in order to make this process differentially private. For example, Dimitrakakis et al. [2], and Wang et al. [6] proved that, under specific conditions, sampling from the posterior distribution is already differentially private. Zhang et al. [8], Foulds et al. [3], designed differentially private mechanisms that output a representation of the full posterior distribution.

When the output of a differentially private mechanism is a probability distribution, accuracy is naturally measured by means of *probabilistic distances* measuring how far this distribution is from the original one. Some classical examples are total variation distance, Hellinger distance, χ^2-distance, KL-divergence, etc.

In this work, we design a mechanism for bayesian inference exploring the idea of calibrating noise using the same probabilistic distance we want to measure accuracy with. We focus on two discrete models, the Beta-Binomial and the Dirichlet-Multinomial models, and one probability distance, Hellinger distance. Our mechanism can be understood as a version of the exponential mechanism where the noise is calibrated to the smooth sensitivity of the utility function, rather than to its global sensitivity. In our setting, the utility function is the probability distance we want to use to measure accuracy. To show the usefulness of this mechanism we show an experimental analysis comparing it with an approach based on the Laplace mechanism.

KEYWORDS

Differential privacy, Bayesian inference, Hellinger distance

1 AN INFORMAL MOTIVATION

Publishing the posterior distribution inferred from a sensitive dataset can leak information about the individuals in the dataset. In order to guarantee differential privacy and to protect the individuals' data we can add noise to the posterior before releasing it. The amount of the noise that we need to introduced depends on the privacy parameter ϵ and the sensitivity of the inference to small changes in the data set. Sensitivity can be computed in many different ways based on which metric space we consider on the output set of the mechanism. In the literature on private Bayesian inference ([7, 8]), it is only measured with respect to the vector of numbers parametrizing the output distribution using, e.g. the ℓ_1 norm. A more natural approach which we explore here, is to measure sensitivity with respect to a metric on the space of inferred probability distributions. A re-loved question is that of how to measure accuracy. Again, this can be answered in different ways based on the metric imposed on the output space, and yet again only in few works in literature (e.g. [8]) distances between probability measures have been used for these purposes.

The question that this work aims at answering is whether an approach based on probability metrics can improve on the accuracy of approaches based on metrics over the numeric parameters of the distributions. We will see that in some cases this can happen.

2 BAYESIAN INFERENCE BACKGROUND

Given a prior belief $\Pr(\theta)$ on some parameter θ, and an observation $\mathbf{x}$, the posterior distribution on θ given $\mathbf{x}$ is computed as:

$$\Pr(\theta|\mathbf{x}) = \frac{\Pr(\mathbf{x}|\theta) \cdot \Pr(\theta)}{\Pr(\mathbf{x})}$$

where the expression $\Pr(\mathbf{x}|\theta)$ denotes the *likelihood* of observing $\mathbf{x}$ under a value of θ. Since we consider $\mathbf{x}$ to be fixed, the likelihood is a function of θ. For the same reason $\Pr(\mathbf{x})$ is a constant independent of θ. Usually in statistics the prior distribution $\Pr(\theta)$ is chosen so that it represents the initial belief on θ, that is, when no data has been observed. In practice though, prior distributions and likelihood functions are usually chosen so that the posterior belongs to the same *family* of distributions. In this case we say that the prior is conjugate to the likelihood function. Use of a conjugate prior simplifies calculations and allows for inference to be performed in a recursive fashion over the data. In this work we will consider a specific instance of Bayesian inference and one of its generalizations. Specifically, we will consider the situation where θ represents the parameter –informally called *bias*– of a Bernoulli distributed random variable, and its immediate generalization where the parameter $\boldsymbol{\theta}$ represents the vector of parameters of a categorically distributed random variable. In the former case, the prior distribution over $\theta \in [0, 1]$ is going to be a beta distribution, $\text{beta}(\alpha, \beta)$, with parameters $\alpha, \beta \in \mathbb{R}^+$, and with p.d.f:

$$\Pr(\theta) \equiv \frac{\theta^{\alpha}(1-\theta)^{\beta}}{\mathrm{B}(\alpha, \beta)}$$

where $\mathrm{B}(\cdot,\cdot)$ is the beta function. The data $\mathbf{x}$ will be a sequence of $n \in \mathbb{N}$ binary values, that is $\mathbf{x} = (x_1, \dots x_n), x_i \in \{0,1\}$, and the likelihood function is:

$$\Pr(\mathbf{x}|\theta) \equiv \theta^{\Delta\alpha}(1-\theta)^{n-\Delta\alpha}$$

where $\Delta\alpha = \sum_{i=1}^{n} x_i$. From this it can easily be derived that the posterior distribution is:

$$\Pr(\theta|\mathbf{x}) = \mathsf{beta}(\alpha + \Delta\alpha, \beta + n - \Delta\alpha)$$

In the latter case the prior distribution over $\boldsymbol{\theta} \in [0,1]^k$ is given by a Dirichelet distribution, $\mathsf{DL}(\boldsymbol{\alpha})$, for $k \in \mathbb{N}$, and $\boldsymbol{\alpha} \in (\mathbb{R}^+)^k$, with p.d.f:

$$\Pr(\boldsymbol{\theta}) \equiv \frac{1}{\mathrm{B}(\boldsymbol{\alpha})} \cdot \prod_{i=1}^{k} \theta_i^{\alpha_i - 1}$$

where $\mathrm{B}(\cdot)$ is the generalized beta function. The data $\mathbf{x}$ will be a sequence of $n \in \mathbb{N}$ values coming from a universe $\mathcal{X}$, such that $\mid \mathcal{X} \mid = k$. The likelihood function will be:

$$\Pr(\mathbf{x}|\boldsymbol{\theta}) \equiv \prod_{a_i \in \mathcal{X}} \theta_i^{\Delta\alpha_i},$$

with $\Delta\alpha_i = \sum_{j=1}^{n} [x_j = a_i]$, where $[\cdot]$ represents Iverson bracket notation. Denoting by $\Delta\boldsymbol{\alpha}$ the vector $(\Delta\alpha_1, \dots \Delta\alpha_k)$ the posterior distribution over $\boldsymbol{\theta}$ turns out to be

$$\Pr(\boldsymbol{\theta}|\mathbf{x}) = \mathsf{DL}(\boldsymbol{\alpha} + \Delta\boldsymbol{\alpha}).$$

where + denotes the componentwise sum of vectors of reals.

3 THE PROBLEM AND BASELINE APPROACH

We are interested in designing a mechanism for privately releasing the full posterior distributions derived in section 2, as opposed to just sampling from them. It's worth noticing that the posterior distributions are fully characterized by their parameters, and the family (beta, Dirichlet) they belong to. Hence, in case of the Beta-Binomial model we are interested in releasing a private version of the pair of parameters $(\alpha', \beta') = (\alpha + \Delta\alpha, \beta + n - \Delta\alpha)$, and in the case of the Dirichlet-Multinomial model we are interested in a private version of $\boldsymbol{\alpha}' = (\boldsymbol{\alpha} + \Delta\boldsymbol{\alpha})$. Zhang et al. [8] and Xiao and Xiong [7] have already attacked this problem by adding independent Laplacian noise to the parameters of the posteriors. That is, in the case of the Beta-Binomial system, the value released would be: $(\tilde{\alpha}, \tilde{\beta}) = (\alpha + \widetilde{\Delta\alpha}, \beta + n - \widetilde{\Delta\alpha})$ where $\widetilde{\Delta\alpha} \sim Lap(\Delta\alpha, \frac{2}{\epsilon})$, and where $Lap(\mu, \nu)$ denotes a Laplace random variable with mean μ and scale ν. This mechanism is ϵ-differentially private, and the noise is calibrated w.r.t. to a sensitivity of 2 which is derived by using ℓ_1 norm over the pair of parameters. Indeed, considering two adjacent[1] data observations $\mathbf{x}, \mathbf{x}'$, that, from a unique prior, give rise to two posterior distributions, characterized by the pairs (α', β') and (α'', β'') then $|\alpha' - \alpha''| + |\beta' - \beta''| \leq 2$. This argument extends similarly to the Dirichelet-Multinomial system.

Also, in previous works, the accuracy of the posterior was measured again with respect to ℓ_1 norm. That is, an upper bound was given on

$$\Pr[|\alpha - \tilde{\alpha}| + |\beta - \tilde{\beta}| \geq \gamma]$$

where $(\alpha, \beta), (\tilde{\alpha}, \tilde{\beta})$ are as defined above. In this work we will use a metric based on a different norm to compute the sensitivity and provide guarantees on the accuracy. In particular we will consider a metric over probability measures and not over the parameters that represent them. Specifically, we will use the Hellinger distance $\mathcal{H}(\cdot,\cdot)$. Our choice to use Hellinger distance is motivated by two facts, first of all it simplifies calculations in the case of the probabilistic models considered here and second of all it also automatically yields bounds on the total variation distance, which represents also the maximum advantage an unbounded adversary can have in distingishing two distributions. Given two beta distributions $\boldsymbol{\beta}_1 = \mathsf{beta}(\alpha_1, \beta_1)$, and $\boldsymbol{\beta}_2 = \mathsf{beta}(\alpha_2, \beta_2)$ the following equality holds

$$\mathcal{H}(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) = \sqrt{1 - \frac{\mathrm{B}(\frac{\alpha_1+\alpha_2}{2}, \frac{\beta_1+\beta_2}{2})}{\sqrt{\mathrm{B}(\alpha_1,\beta_1)\mathrm{B}(\alpha_2,\beta_2)}}}$$

The same change of metric will be applied to the experimental accuracy guarantees.

4 OUR APPROACH - EXPONENTIAL MECHANISM WITH SMOOTH SENSITIVITY

Given a prior distribution $\boldsymbol{\beta}_{\text{prior}} = \mathsf{beta}(\alpha, \beta)$ and a sequence of n observations $\mathbf{x} \in \{0,1\}^n$, we define the follwing set:

$$\mathcal{R}_{\text{post}} \equiv \{\mathsf{beta}(\alpha', \beta') \mid \alpha' = \alpha + \Delta\alpha, \beta' = \beta + n - \Delta\alpha\}$$

where $\Delta\alpha$ is as defined in Section 2. Notice that $\mathcal{R}_{\text{post}}$ has $n+1$ elements, and the Bayesian Inference process will produce an element from $\mathcal{R}_{\text{post}}$ that we denote by $\mathsf{BI}(\mathbf{x})$ – we don't explicitely parametrize the result by the prior, which from now on we consider fixed and we denote it by $\boldsymbol{\beta}_{\text{prior}}$.

We can now define the mechanism $\mathcal{M}^B_{\mathcal{H}}$ which, given in input a sequence of observations $\mathbf{x}$ and parameters $\epsilon > 0$ and $\delta > 0$, produces an element r in $\mathcal{R}_{\text{post}}$ with probability:

$$\Pr_{z \sim \mathcal{M}^B_{\mathcal{H}}}[z = r] = \frac{exp\left(\frac{-\epsilon \cdot \mathcal{H}(\mathsf{BI}(\mathbf{x}), r)}{2 \cdot S(\mathbf{x})}\right)}{\sum_{r \in \mathcal{R}_{\text{post}}} exp\left(\frac{-\epsilon \cdot \mathcal{H}(\mathsf{BI}(\mathbf{x}), r)}{2 \cdot S(\mathbf{x})}\right)}.$$

This mechanism is based on the basic exponential mechanism [4], with $\mathcal{R}_{\text{post}}$ as the range and $\mathcal{H}(\cdot,\cdot)$ as the scoring function. The difference is that in this mechanism we don't calibrate the noise w.r.t. to the global sensitivity of the scoring function but w.r.t. to the smooth sensitivity $S(\mathbf{x})$ – defined by Nissim et al. [5]– of $\mathcal{H}(\mathsf{BI}(\mathbf{x}), \cdot)$. The smooth sensitivity is computed as follows:

$$S(\mathbf{x}) = \max_{\mathbf{x}' \in \{0,1\}^n} \left\{ \Delta_l \Big(\mathcal{H}(\mathsf{BI}(\mathbf{x}'), \cdot) \Big) \cdot e^{-\gamma \cdot d(C(\mathbf{x}), C(\mathbf{x}'))} \right\}, \quad (1)$$

where d is the Hamming distance between two datasets, $\gamma = \gamma(\epsilon, \delta)$ is a function of ϵ and δ to be determined later, and where

[1] Given $\mathbf{x}, \mathbf{x}'$ we say that $\mathbf{x}$ and $\mathbf{x}'$ are adjacent and we write, $\mathbf{adj}(\mathbf{x}, \mathbf{x}')$, iff $\sum_{i}^{n} [x_i = x_i'] \leq 1$.

$\Delta_l\Big(\mathcal{H}(\mathsf{BI}(\mathbf{x}'),\cdot)\Big)$ denotes the local sensitivity at $\mathsf{BI}(\mathbf{x}')$, or equivalently at $\mathbf{x}'$, of the scoring function used in our mechanism. That is:

$$\Delta_l\Big(\mathcal{H}(\mathsf{BI}(\mathbf{x}'),\cdot)\Big) = \max_{\mathbf{x}''\in\mathcal{X}^n:\mathbf{adj}(\mathbf{x}',\mathbf{x}''),\, r\in\mathcal{R}_{\text{post}}} |\mathcal{H}(\mathsf{BI}(\mathbf{x}'),r) - \mathcal{H}(\mathsf{BI}(\mathbf{x}''),r)|.$$

This mechanism also extends to the Dirichlet-Multinomial system $\mathsf{DL}(\boldsymbol{\alpha})$ by rewriting the Hellinger distance as:

$$\mathcal{H}(\mathsf{DL}(\boldsymbol{\alpha}_1),\mathsf{DL}(\boldsymbol{\alpha}_2)) = \sqrt{1-\frac{\mathrm{B}(\frac{\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2}{2})}{\sqrt{\mathrm{B}(\boldsymbol{\alpha}_1)\mathrm{B}(\boldsymbol{\alpha}_2)}}},$$

and by replacing the $\mathcal{R}_{\text{post}}$ with set of posterior Dirichlet distributions candidates. Also, the smooth sensitivity $S(\mathbf{x})$ in (1) will be computed by letting $\mathbf{x}'$ range over all the elements in $\mathcal{X}^n$ adjacent to $\mathbf{x}$. Notice that $\mathcal{R}_{\text{post}}$ has $\binom{n+1}{m-1}$ elements in this case. We will denote by $\mathcal{M}^D_{\mathcal{H}}$ the mechanism for the Dirichlet-Multinomial system.

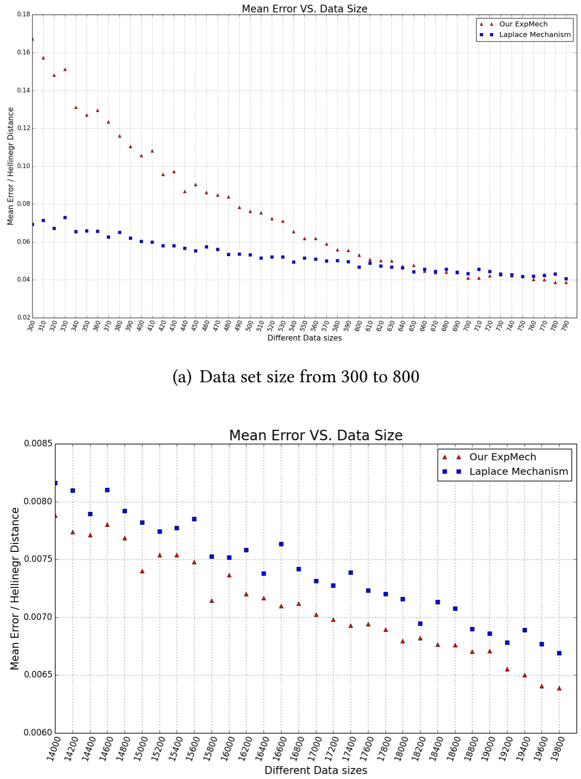
The following result guarantees that the mechanisms are indeed differentially private.

Theorem 4.1 (Privacy). *For* $\gamma = \ln(1 - \frac{\epsilon}{2\ln(\frac{\delta}{2(n+1)})})$, *both* $\mathcal{M}^B_{\mathcal{H}}$ *and* $\mathcal{M}^D_{\mathcal{H}}$ *are* (ϵ,δ)*-differentially private.*

5 PRELIMINARY EXPERIMENTAL RESULTS

In this section, we evaluate the accuracy of the mechanisms defined in Section (4) w.r.t. four variables, including data size, dimensions, data variance, prior distribution, and some combinations thereof. Every plot is an average over 1000 runs. In all the experiments we set $\epsilon = 0.8$, and $\delta = 10^{-8}$.

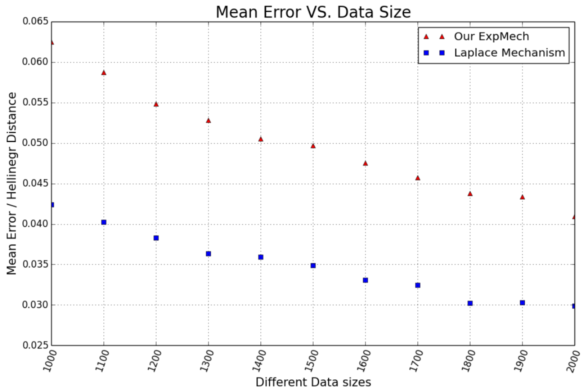
In the following some of the plots show mean error as a function of the datasize while one is a whiskers-plot where the y-axis shows the average accuracy (or equivalently, the error) of the mechanisms, and the x-axis, instead shows different balanced priors used. The boxes extend from the lower to the upper quartile values of the data, with a line at the median. A notch on the box around the median is also drawn to give a rough guide to the significance of difference of medians; The whiskers extend from the box to show the range of the data. A blue box in the plots represents our newly designed exponential mechanism's behavior– where the sensitivity is calibrated w.r.t Hellinger distance– while the yellow box next to it represents the performance of a variation of the basic Laplace mechanism presented in Section (3) with the same settings: that is ϵ, δ, data, prior. The variation considered performs a postprocessing on the released parameters so that they are consistent. For instance when the sum of the noised parameters is greater than n we will truncate them so that they sum up to n.



(a) Data set size from 300 to 800

(b) Data set size from 14000 to 20000

Figure 1: Increasing data size with fixed prior beta(1, 1)**. Unbalanced datasets of mean** (0.1, 0.9) **and parameters** $\epsilon = 0.8$ **and** $\delta = 10^{-8}$



(a) Data set size from 14000 to 20000

Figure 2: Increasing data size with fixed DL(1, 1, 1) **prior distribution, Unbalanced datasets of mean** (0.2, 0.3, 0.5) **and parameters** $\epsilon = 0.8$ **and** $\delta = 10^{-8}$

Increasing data size with balanced datasets. In Figures 1 and 2 we consider *unbalanced* datasets of observations. This means that in the Beta-Binomial setting (Figure 1(a), and 1(b)) the datasets will consist of 10% 1s and the rest 0s, while for the Dirichelet-Multinomial (Figure **??** and 2(a)) the data will be split in the $k = 3$ bins with perecentages of: 20%, 30% and 50%. The results show that when the data size increases, the average errors of $\mathcal{M}^B_{\mathcal{H}}$, $\mathcal{M}^D_{\mathcal{H}}$, and Laplace decrease. For small datasets, i.e with size less 650 in the case of Beta-Binomial systems, the Laplace mechanisms outperforms the exponential mechanisms, but for bigger data sets, that is, bigger than 650, or as in Figure 1(b) where we considered data sets of the order of 15 thousands elements, the exponential mechanisms outperforms the Laplace mechanism. Similar experimental tendencies were obtained for the Dirichlet-Multinomial system ((Figure **??** and 2(a))), but we were not able to perform experiments with higher dimensions (4 or more) or, big datasets, e.g. 10^4 elements, due to too high run time of the algorithm.

Fixed dataset varying balanced priors. In Figure 3, we fix the dataset to be $\langle 5, 5, 5 \rangle$. We also considered balanced priors with increasing values in their dimensions. The plot shows that in the beginning the Laplace mechanism performs better but it is outperformed after a while.

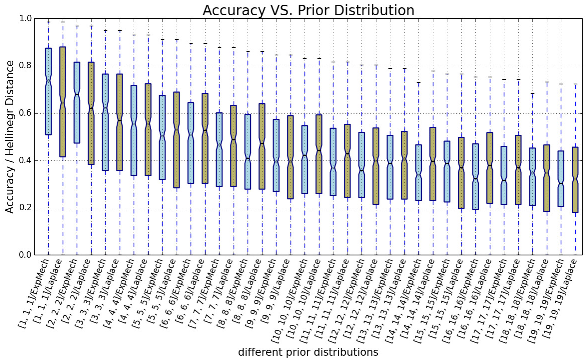


Figure 3: Observed data set is: $(5, 5, 5)$**, varying balanced priors**

6 CONCLUSIONS AND FUTURE WORKS

From what we have seen in the previous sections we can obtain some preliminary conclusions. That is, the probabiliy measure approach outperforms the ℓ_1-norm approach in the following cases:

(1) When the data size is small, data is balanced and priors parameters increase
(2) When the data size is large

These results although very motivating, are still not enough for real world applications. Hence, we will continue our work in the follwoing directions:

(1) For now, we just have a intuitive idea on the accuracy behavior of our mechanisms, and not a precise formula or bound on it. When do our mechanisms perform better than the baseline mechanism and when they don't? How much influence will elements in Section 5 have on the accuracy? Are there any other important factors we missed? These are all questions w.r.t. the accuracy that we are going to explore next, and in a more principled and formal way.
(2) Theorem 4.1 provides an upper bound on the privacy loss for $\mathcal{M}^B_{\mathcal{H}}$ and $\mathcal{M}^D_{\mathcal{H}}$ but not necessarily a tight one. Indeed, experiments have shown that the actual privacy loss in the experiments can be smaller than ϵ. This means that we could improve accuracy, by adding less noise – that is noise proportional to a higher value of ϵ– but still achieve (ϵ, δ)-dp.
(3) The choice of the Hellinger distance might seem quite ad-hoc. Hence, it is worth exploring other distances over distributions. An interesting class of probability metrics is the family of f-divergences [1].

REFERENCES

[1] I. Csiszár and P.C. Shields. 2004. Information Theory and Statistics: A Tutorial. *Foundations and TrendsÂő in Communications and Information Theory* 1, 4 (2004), 417–528. https://doi.org/10.1561/0100000004
[2] Christos Dimitrakakis, Blaine Nelson, Aikaterini Mitrokotsa, and Benjamin IP Rubinstein. 2014. Robust and private Bayesian inference. In *International Conference on Algorithmic Learning Theory*. Springer, 291–305.
[3] James Foulds, Joseph Geumlek, Max Welling, and Kamalika Chaudhuri. 2016. On the theory and practice of privacy-preserving Bayesian data analysis. *arXiv preprint arXiv:1603.07294* (2016).
[4] Frank McSherry and Kunal Talwar. 2007. Mechanism Design via Differential Privacy, In Annual IEEE Symposium on Foundations of Computer Science (FOCS). https://www.microsoft.com/en-us/research/publication/mechanism-design-via-differential-privacy/
[5] Kobbi Nissim, Sofya Raskhodnikova, and Adam Smith. 2007. Smooth sensitivity and sampling in private data analysis. In *Proceedings of the thirty-ninth annual ACM symposium on Theory of computing*. ACM, 75–84.
[6] Yu-Xiang Wang, Stephen Fienberg, and Alex Smola. 2015. Privacy for free: Posterior sampling and stochastic gradient monte carlo. In *International Conference on Machine Learning*. 2493–2502.
[7] Yonghui Xiao and Li Xiong. 2012. Bayesian inference under differential privacy. *arXiv preprint arXiv:1203.0617* (2012).
[8] Zuhe Zhang, Benjamin IP Rubinstein, Christos Dimitrakakis, et al. 2016. On the Differential Privacy of Bayesian Inference.. In *AAAI*. 2365–2371.