Differentially Private Bayesian Inference
Optimality of Laplace Mechanism

March 9, 2020

1 Private Mechanisms

Algorithm 1 LSDim

$\mathbf{x} \in \mathcal{X}^n$, $\mathsf{Dir}(\boldsymbol{\alpha})$
  let $\boldsymbol{\alpha}' = \mathsf{DirP}(\boldsymbol{\alpha}, \boldsymbol{\theta}, \mathbf{x})$
  Initialize a vector $\tilde{\boldsymbol{\alpha}} = (0, \dots, 0) \in \mathbb{N}^{|\mathcal{X}|}$
  For $i = 1 \dots |\mathcal{X}| - 1$:
    let $\eta \sim \mathsf{Lap}(0, \frac{|\mathcal{X}|}{\epsilon})$
    $\tilde{\alpha_i} = \alpha_i + \lfloor (\alpha'_i - \alpha_i) + \eta \rceil_0^n$
  $\tilde{\alpha}_{|\mathcal{X}|} = \alpha_{|\mathcal{X}|} + \lfloor n - \sum_{i=1}^{|\mathcal{X}|-1} \lfloor (\alpha'_i - \alpha_i) + \eta_i \rceil_0^n \rceil_0^n$
  return $\tilde{\boldsymbol{\alpha}}$

Algorithm 2 LSHist

input $\mathbf{x} \in \mathcal{X}^n$, $\mathsf{Dir}(\boldsymbol{\alpha})$
  let $\boldsymbol{\alpha}' = \mathsf{DirP}(\boldsymbol{\alpha}, \boldsymbol{\theta}, \mathbf{x})$
  let $k = \begin{cases} 1 & \text{if} \quad |\mathcal{X}| = 2 \\ 2 & \text{otherwise} \end{cases}$
  Initialize a vector $\tilde{\boldsymbol{\alpha}} = (0, \dots, 0) \in \mathbb{N}^{|\mathcal{X}|}$
  For $i = 1 \dots |\mathcal{X}| - 1$:
    let $\eta \sim \mathsf{Lap}(0, \frac{k}{\epsilon})$
    $\tilde{\alpha_i} = \alpha_i + \lfloor (\alpha'_i - \alpha_i) + \eta \rceil_0^n$
  $\tilde{\alpha}_{|\mathcal{X}|} = \alpha_{|\mathcal{X}|} + \lfloor n - \sum_{i=1}^{|\mathcal{X}|-1} \lfloor (\alpha'_i - \alpha_i) + \eta_i \rceil_0^n \rceil_0^n$
  return $\tilde{\boldsymbol{\alpha}}$

Algorithm 3 EHD

observed data set $\mathbf{x} \in \mathcal{X}^n$, prior: $\mathsf{Dir}(\boldsymbol{\alpha})$, ϵ
  let $\mathsf{Dir}(\boldsymbol{\alpha}') = \mathsf{DirP}(\mathbf{x}, \boldsymbol{\alpha})$.
  let GS be the global sensitivity for $\mathbf{x}$.
  set $z = r$ with probability $\frac{\exp(\frac{-\epsilon \cdot \mathcal{H}(\mathsf{DirP}(\mathbf{x},\boldsymbol{\alpha}),r)}{2 \cdot GS})}{\Sigma_{r' \in \mathcal{R}_{\boldsymbol{\alpha}}} \exp(\frac{-\epsilon \cdot \mathcal{H}(\mathsf{DirP}(\mathbf{x},\boldsymbol{\alpha}),r')}{2 \cdot GS})}$
return z

Algorithm 4 EHDL

input observed data set $\mathbf{x} \in \mathcal{X}^n$, prior: $\mathsf{Dir}(\boldsymbol{\alpha})$, ϵ
  let $\mathsf{Dir}(\boldsymbol{\alpha}') = \mathsf{DirP}(\mathbf{x}, \boldsymbol{\alpha})$.
  let $LS(\mathbf{x})$ be the local sensitivity for $\mathbf{x}$.
  set $z = r$ with probability $\frac{\exp(\frac{-\epsilon \cdot \mathcal{H}(\mathsf{DirP}(\mathbf{x},\boldsymbol{\alpha}),r)}{2 \cdot LS(\mathbf{x})})}{\Sigma_{r' \in \mathcal{R}_{\boldsymbol{\alpha}}} \exp(\frac{-\epsilon \cdot \mathcal{H}(\mathsf{DirP}(\mathbf{x},\boldsymbol{\alpha}),r')}{2 \cdot LS(\mathbf{x})})}$
return z

Algorithm 5 EHDS

observed data set $\mathbf{x} \in \mathcal{X}^n$, prior: $\mathsf{Dir}(\boldsymbol{\alpha})$, ϵ
  let $\mathsf{Dir}(\boldsymbol{\alpha}') = \mathsf{DirP}(\mathbf{x}, \boldsymbol{\alpha})$.
  let $S(\mathbf{x})$ be the smooth sensitivity for $\mathbf{x}$.
  set $z = r$ with probability $\frac{\exp(\frac{\epsilon \cdot u(\mathbf{x},r)}{4 \cdot S(\mathbf{x})})}{\Sigma_{r' \in \mathcal{R}_{\boldsymbol{\alpha}}} \exp(\frac{\epsilon \cdot u(\mathbf{x},r')}{4 \cdot S(\mathbf{x})})}$
return z

2 Accuracy Analysis

Theorem 2.1. *To prove the optimality of Laplace mechanism, we are showing*

$$\frac{ELap(\boldsymbol{x})}{(\epsilon \times LS(\boldsymbol{x}))}$$

is $O(1)$, considering $n = |\boldsymbol{x}| \geq 2$ being the parameter.

Where $LS(\cdot)$ is the local sensitivity, and where $ELap(\cdot)$ is the measure of the error of the Laplace mechanism, defined in this way:

$$ELap(\boldsymbol{x}) = \arg\big(\min_t\{Pr[\mathsf{H}(\mathsf{DirP}(\boldsymbol{x}), \mathsf{LSHist}(\boldsymbol{x})) < t] \geq 1 - \gamma\big).$$

[[**Jiawen:**

Theorem 2.2. *For $\gamma = e^{O(\epsilon)}$, $\frac{ELap(\boldsymbol{x})}{(\epsilon \times LS(\boldsymbol{x}))}$ is $O(\epsilon)$*

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Proof. Let $t = LS(\mathbf{x})$, we have following by p.d.f. of Laplace distribution:

$$Pr[\mathsf{H}(\mathsf{DirP}(\mathbf{x}), \mathsf{LSHist}(\mathbf{x})) < t] \geq 1 - \frac{1}{2}(e^{-\epsilon} + e^{-2\epsilon}) > 1 - e^{-\epsilon}$$

Then we can get when $\gamma = e^{-\epsilon}$,

$$\frac{ELap(\mathbf{x})}{(\epsilon \times LS(\mathbf{x}))} = \frac{1}{\epsilon}$$

□

Theorem 2.3. *In order to prove the optimality of Laplace mechanism, instead of prove $\frac{ELap(\boldsymbol{x})}{(\epsilon \times LS(\boldsymbol{x}))}$ is $O(1)$, we prove a constant upper bound on following equations:*

$$\begin{aligned} & \frac{\arg\min_t \left\{ \Pr[\mathsf{H}(\mathsf{Beta}(\alpha,\beta),\mathsf{LSHist}(\boldsymbol{x}))<t]\geq 1-\gamma \right\}}{LS(\boldsymbol{x})} \\ \leq\ & \frac{\max_{|k|\leq \frac{\lg(\frac{1}{\gamma})}{\epsilon}} \mathsf{H}(\mathsf{Beta}(\alpha,\beta),\mathsf{Beta}(\alpha+k,n-\lfloor\alpha+k\rfloor))}{LS(\boldsymbol{x})} \\ \leq\ & O(\frac{\lg\frac{1}{\gamma}}{\epsilon}) \end{aligned}$$

[[**Jiawen:**

Theorem 2.4. *For $\gamma = e^{O(k)\epsilon}$, it is proved that $O(\frac{\lg\frac{1}{\gamma}}{\epsilon})$ is bounded by $O(k)$.*

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Proof. By Laplace distribution, we have:

$$\begin{aligned}\Pr[\mathsf{H}(\mathsf{Beta}(\alpha,\beta),\mathsf{LSHist}(\mathbf{x}))<t] &= \Pr[\{|\mathsf{Lap}(0,\tfrac{1}{\epsilon})|<O(k)|\mathsf{H}(\mathsf{Beta}(\alpha,\beta),\mathsf{Beta}(\alpha+k,n-\lfloor\alpha+k\rfloor))<t\}] \\ &\leq 1-e^{-O(k)\epsilon}\end{aligned}$$

Then we have:

$$\gamma = e^{-O(k)\epsilon}$$

So we can get:

$$O(\frac{\lg\frac{1}{\gamma}}{\epsilon}) = O(\frac{\lg\frac{1}{e^{-O(k)\epsilon}}}{\epsilon}) = O(k)$$

□

[[Jiawen:

Corollary 2.4.1. *For $-1\leq k<2$, it is proved that $O(\frac{\lg\frac{1}{\gamma}}{\epsilon})$ is bounded by $O(1)$.*

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Proof. Given $-1\leq k<2$, we have:

$$\mathsf{H}(\mathsf{Beta}(\alpha,\beta),\mathsf{Beta}(\alpha+k,n-\lfloor\alpha+k\rfloor))\leq LS(\mathbf{x}) \tag{1}$$

For any ϵ, $k\sim\mathsf{Lap}(0,\frac{1}{\epsilon})$ from Laplace mechanism, we have:

$$\Pr[|k|\leq\frac{b}{\epsilon}] = 1-\exp(-b)$$

Then we can get:

$$\Pr[-1\leq k<2] = 1-\frac{\exp(-\epsilon)+\exp(-2\epsilon)}{2} \tag{2}$$

By Equation (1) and (2), we can get:

$$\Pr[\mathsf{H}(\mathsf{Beta}(\alpha,\beta),\mathsf{Beta}(\alpha+k,n-\lfloor\alpha+k\rfloor))\leq LS(\mathbf{x})]\geq 1-\frac{\exp(-\epsilon)+\exp(-2\epsilon)}{2}$$

i.e.,

$$\begin{aligned}&\frac{\arg\min_t\left\{\Pr[\mathsf{H}(\mathsf{Beta}(\alpha,\beta),\mathsf{LSHist}(\mathbf{x}))<t]\geq 1-\frac{\exp(-\epsilon)+\exp(-2\epsilon)}{2}\right\}}{LS(\mathbf{x})} \\ \leq\ & O(\frac{\lg(\frac{2}{\exp(-\epsilon)+\exp(-2\epsilon)})}{\epsilon}) \\ <\ & O(\frac{\lg(\frac{2}{2\exp(-2\epsilon)})}{\epsilon}) = 2\end{aligned}$$

□

[[Jiawen:

Theorem 2.5. *Let $k = |k'|$ be the largest integer that satisfying $\mathsf{H}(\mathsf{Beta}(\alpha,\beta), \mathsf{Beta}(\alpha + k', n - \lfloor \alpha + k' \rfloor)) < t$, we have:*

$$\Pr[\mathsf{H}(\mathsf{Beta}(\alpha,\beta), \mathsf{LSHist}(\boldsymbol{x})) < t] \geq 1 - e^{-k\epsilon}$$

$$\frac{2ke^{-\epsilon} + 1}{n} < \Pr[\mathsf{H}(\mathsf{Beta}(\alpha,\beta), \mathsf{EHD}(\boldsymbol{x})) < t] < \frac{2k+1}{ne^{-\epsilon}}.$$

$$\frac{2k\exp(\frac{-\epsilon}{2\cdot LS(\boldsymbol{x})}) + 1}{n} < \Pr[\mathsf{H}(\mathsf{Beta}(\alpha,\beta), \mathsf{EHDL}(\boldsymbol{x})) < t] < \frac{2k+1}{n\exp(\frac{-\epsilon}{2\cdot LS(\boldsymbol{x})})}.$$

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Proof. By the p.d.f. of Laplace distribution, we have:

$$\Pr[\mathsf{H}(\mathsf{Beta}(\alpha,\beta), \mathsf{LSHist}(\mathbf{x})) < t] = Pr[\mathsf{LSHist}(\mathbf{x}) \leq k] \geq 1 - e^{-k\epsilon}.$$

By definition of EHD and $GS = \sqrt{1 - \pi/4}$, we have:

$$\begin{aligned}
Pr[\mathsf{H}(\mathsf{Beta}(\alpha,\beta), \mathsf{EHD}(\mathbf{x})) < t] &= \sum_{c \geq -t} \frac{\exp(\frac{\epsilon\cdot c}{2\cdot GS})}{\Sigma_{r' \in \mathcal{R}_{\boldsymbol{\alpha}}} \exp(\frac{-\epsilon\cdot\mathcal{H}(\mathsf{DirP}(\mathbf{x},\boldsymbol{\alpha}), r')}{2\cdot GS})} \\
&\leq \frac{2k\exp(-\frac{0\epsilon}{2\cdot GS}) + 1}{n\exp(\frac{-\epsilon}{2\cdot GS})} \\
&= \frac{2k+1}{n\exp(\frac{-\epsilon}{2\sqrt{1-\pi/4}})} \\
&< \frac{2k+1}{n\exp(-\epsilon)}
\end{aligned}$$

$$\begin{aligned}
Pr[\mathsf{H}(\mathsf{Beta}(\alpha,\beta), \mathsf{EHD}(\mathbf{x})) < t] &= \sum_{c \geq -t} \frac{\exp(\frac{\epsilon\cdot c}{2\cdot GS})}{\Sigma_{r' \in \mathcal{R}_{\boldsymbol{\alpha}}} \exp(\frac{-\epsilon\cdot\mathcal{H}(\mathsf{DirP}(\mathbf{x},\boldsymbol{\alpha}), r')}{2\cdot GS})} \\
&\geq \frac{2k\exp(\frac{-t\epsilon}{2\cdot GS}) + 1}{n\exp(\frac{-0\epsilon}{2\cdot GS})} \\
&> \frac{2k\exp(\frac{-\epsilon}{2\sqrt{1-\pi/4}}) + 1}{n} \\
&> \frac{2ke^{-\epsilon} + 1}{n}
\end{aligned}$$

By definition of EHDL, we have:

$$\begin{aligned}
Pr[\mathsf{H}(\mathsf{Beta}(\alpha,\beta), \mathsf{EHDL}(\mathbf{x})) < t] &= \sum_{c \geq -t} \frac{\exp(\frac{c\epsilon}{2\cdot LS(\mathbf{x})})}{\Sigma_{r' \in \mathcal{R}_{\boldsymbol{\alpha}}} \exp(\frac{-\epsilon\cdot\mathcal{H}(\mathsf{DirP}(\mathbf{x},\boldsymbol{\alpha}), r')}{2\cdot LS(\mathbf{x})})} \\
&\leq \frac{2k\exp(\frac{-0\epsilon}{2\cdot LS(\mathbf{x})}) + 1}{n\exp(\frac{-\epsilon}{2\cdot LS(\mathbf{x})})} \\
&= \frac{2k+1}{n\exp(\frac{-\epsilon}{2\cdot LS(\mathbf{x})})}
\end{aligned}$$

$$\begin{aligned}
Pr[\mathsf{H}(\mathsf{Beta}(\alpha,\beta), \mathsf{EHD}(\mathbf{x})) < t] &= \sum_{c \geq -t} \frac{\exp(\frac{\epsilon\cdot c}{2\cdot LS(\mathbf{x})})}{\Sigma_{r' \in \mathcal{R}_{\boldsymbol{\alpha}}} \exp(\frac{-\epsilon\cdot\mathcal{H}(\mathsf{DirP}(\mathbf{x},\boldsymbol{\alpha}), r')}{2\cdot LS(\mathbf{x})})} \\
&\geq \frac{2k\exp(\frac{-t\epsilon}{2\cdot LS(\mathbf{x})}) + 1}{n\exp(\frac{-0\epsilon}{2\cdot LS(\mathbf{x})})} \\
&> \frac{2k\exp(\frac{-\epsilon}{2\cdot LS(\mathbf{x})}) + 1}{n}
\end{aligned}$$

Since $\lim_{n\to\infty} LS(\mathbf{x}) \to 0$, we have $\lim_{n\to\infty} \frac{-1}{LS(\mathbf{x})} \to -\infty$. So $\exp(\frac{-\epsilon}{2\cdot LS(\mathbf{x})})$ can only be bounded by 0. We cannot found a tighter lower bound. $\square$

[[**Jiawen:**

Corollary 2.5.1. *For a reasonable small t, we have when data size $n = |\boldsymbol{x}| > O(\frac{(2k+1)e^{\epsilon}}{1-e^{-\epsilon}})$, the accuracy of* LSHist *is higher than* EHD*.*

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Proof. Based on Theorem 2.5, let:

$$\frac{2k+1}{n\exp(-\epsilon)} \leq 1 - e^{-k\epsilon},$$

we can have:

$$\Pr[\mathsf{H}(\mathsf{Beta}(\alpha, \beta), \mathsf{LSHist}(\mathbf{x})) < t] > \Pr[\mathsf{H}(\mathsf{Beta}(\alpha, \beta), \mathsf{EHD}(\mathbf{x})) < t].$$

By simplification, we have $n > \frac{2k+1}{e^{-\epsilon}(1-e^{-k\epsilon})} \sim O(\frac{(2k+1)e^{\epsilon}}{1-e^{-\epsilon}})$. □

[[**Jiawen:**

Corollary 2.5.2. *Let R_g be the good output set where $\forall r \in R$, $\mathsf{H}(\mathsf{DirP}(\boldsymbol{x}), r) \leq LS(\boldsymbol{x})$, we have:*

$$Pr[\mathsf{LSHist}(\boldsymbol{x}, \epsilon) \in R_g] > Pr[\mathsf{EHD}(\boldsymbol{x}, \epsilon) \in R_g]$$

for data size $n = |\boldsymbol{x}| > O(\frac{e^{\epsilon}}{1-e^{-\epsilon}})$

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Proof. simply apply the Theorem 2.5 and corollary 2.5.1, we can get this conclusion.

Let R_g be the good output set where $\forall r \in R$, $\mathsf{H}(\mathsf{DirP}(\mathbf{x}), r) \leq LS(\mathbf{x})$, we have:

$$Pr[\mathsf{LSHist}(\mathbf{x}) \in R_g] \geq 1 - \frac{1}{2}(e^{-\epsilon} + e^{-2\epsilon}) > 1 - e^{-\epsilon}$$

By definition of EHD and $GS = \sqrt{1 - \pi/4}$, we have:

$$\begin{aligned} Pr[\mathsf{EHD}(\mathbf{x}) \in R_g] &= \sum_{c \geq -LS(\mathbf{x})} \frac{\exp(\frac{\epsilon \cdot c}{2 \cdot GS})}{\Sigma_{r' \in \mathcal{R}_{\boldsymbol{\alpha}}} \exp(\frac{-\epsilon \cdot \mathcal{H}(\mathsf{DirP}(\mathbf{x}, \boldsymbol{\alpha}), r')}{2 \cdot GS})} \\ &\leq \frac{2\exp(-\frac{\epsilon LS(\mathbf{x})}{2 \cdot GS}) + 1}{n\exp(\frac{-\epsilon}{2 \cdot GS})} \\ &\leq \frac{3}{n\exp(\frac{-\epsilon}{2\sqrt{1-\pi/4}})} \\ &\leq \frac{3}{n\exp(-\epsilon)} \end{aligned}$$

Let $c = 2\sqrt{1 - \pi/4}$, we have when $n > \frac{3}{e^{-\epsilon/c}(1-e^{-\epsilon})} \sim O(\frac{e^{\epsilon}}{1-e^{-\epsilon}})$ LSHist performs better than EHD. □