Tailoring Differentially Private Bayesian Inference to Distance Between Distributions

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Objectives

1. Designing a differentially private (**dp**) Bayesian inference mechanism.
2. Measuring accuracy with a metric over distributions (Hellinger distance ($\mathcal{H}$)).

Bayesian inference: Beta-Binomial model

- Prior on θ : $\mathbf{beta}(\alpha, \beta), \alpha, \beta \in \mathbb{R}^+$, observed data set $\mathbf{x} = (x_1, \dots x_n) \in \{0, 1\}^n, n \in \mathbb{N}$.
- Likelihood function: $\mathbb{L}_{\mathbf{x}|\theta} = \theta^{\Delta\alpha}(1-\theta)^{n-\Delta\alpha}$, where $\Delta\alpha = \sum_{i=1}^{n} x_i$;
- Posterior distribution over theta: $\mathbb{P}_{\theta|\mathbf{x}} = \mathbf{beta}(\alpha + \Delta\alpha, \beta + n - \Delta\alpha)$.

Differentially Private Bayesian Inference and Motivation

Releasing a differentially private posterior $\mathbf{beta}(\tilde{\alpha}, \tilde{\beta}) = (\alpha + \widetilde{\Delta\alpha}, \beta + n - \widetilde{\Delta\alpha})$.

1. Baseline approach samples noise from $\mathbf{Lap}(\Delta\alpha, \frac{1}{\epsilon})$ with mean $\Delta\alpha$ and scale $\frac{1}{\epsilon}$. But noise here is scaled to sensitivity 1 over ℓ_1 norm. Motivated by this, we calibrate the noise w.r.t sensitivity of the accuracy metric ($\mathcal{H}$) (v.s. the ℓ_1 norm in baseline approach).
2. Global sensitivity of $\mathcal{H}$ over **beta** distributions takes value at edge ($(0, 0)$ point) as in Fig. 1. But it's very smooth when move away from edge. Motivated by this, we apply smooth sensitivity in mechanism (v.s. global sensitivity) to improve accuracy.

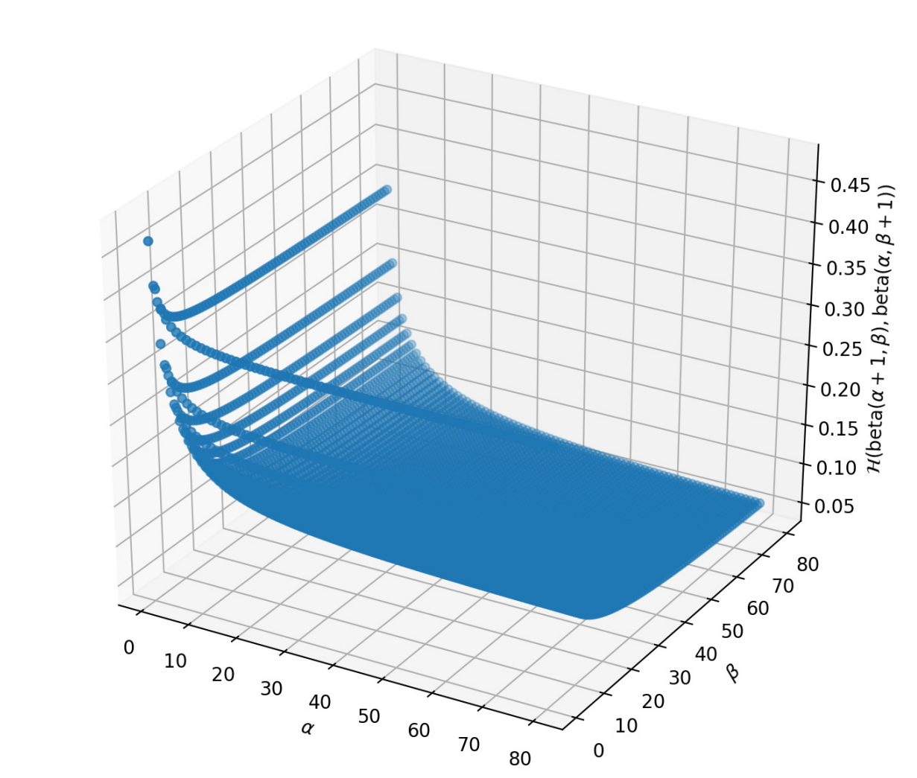


Figure 1: Sensitivities of $\mathcal{H}$ over **beta**

Smoothed Hellinger Distance Based Exponential Mechanism

We define the mechanism $\mathcal{M}^B_{\mathcal{H}}$ which produces an element r in $\mathcal{R}_{\text{post}}$ with probaiblity:

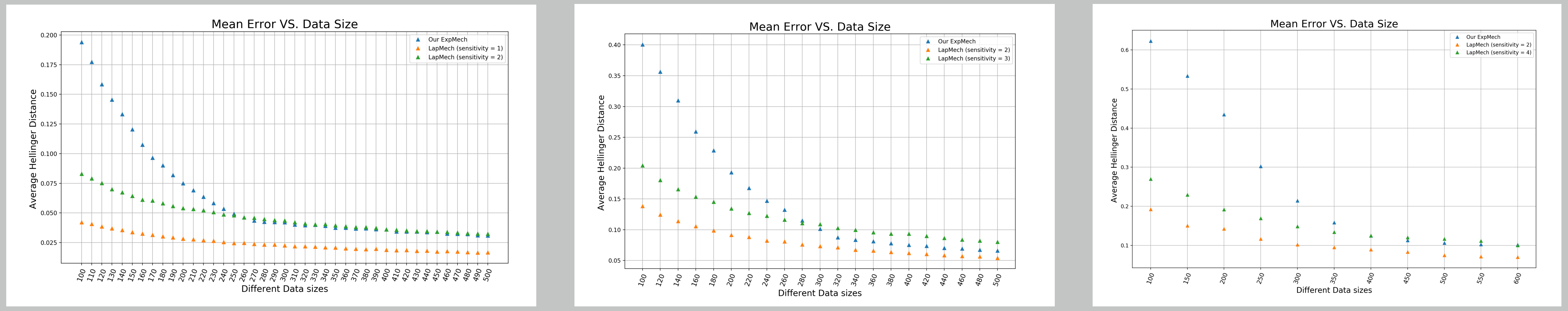
$$\Pr_{z \sim \mathcal{M}^B_{\mathcal{H}}}[z = r] = \frac{\exp\left(\frac{-\epsilon \cdot \mathcal{H}(\mathsf{BI}(\mathbf{x}), r)}{2 \cdot S(\mathbf{x})}\right)}{\sum_{r \in \mathcal{R}_{\text{post}}} \exp\left(\frac{-\epsilon \cdot \mathcal{H}(\mathsf{BI}(\mathbf{x}), r)}{2 \cdot S(\mathbf{x})}\right)}$$

given in input an observations $\mathbf{x}$, parameters $\epsilon > 0$ and $\delta > 0$, where:

- $\mathcal{R}_{\text{post}}$, the candidates set defined as $\{\mathbf{beta}(\alpha', \beta') \mid \alpha' = \alpha + \Delta\alpha, \beta' = \beta + n - \Delta\alpha\}$, given the prior distribution $\beta_{\text{prior}} = \mathbf{beta}(\alpha, \beta)$ and observed data set size n.
- $-\mathcal{H}(\mathsf{BI}(\mathbf{x}), r)$ denotes the scoring function based on the Hellinger distance.
- $S(\mathbf{x})$, the smooth sensitivity: $S(\mathbf{x}) = \max_{\mathbf{x}' \in \{0,1\}^n} \left\{ LS(\mathbf{x}') \cdot e^{-\gamma \cdot d(\mathbf{x}, \mathbf{x}')} \right\}$, where:
  - ▷ d: Hamming distance between two data sets,
  - ▷ $LS(\mathbf{x}')$, local sensitivity at $\mathbf{x}'$: $LS(\mathbf{x}) = \max_{\mathbf{x}' \in \mathcal{X}^n : \mathrm{adj}(\mathbf{x}, \mathbf{x}'), r \in \mathcal{R}} |\mathcal{H}(\mathsf{BI}(\mathbf{x}), r) - \mathcal{H}(\mathsf{BI}(\mathbf{x}'), r)|$,
  - ▷ $\gamma = \ln(1 - \frac{\epsilon}{2 \ln(\frac{\delta}{2(n+1)})})$ to ensure the (ϵ, δ)-differentially private.

Preliminary Experimental Results

Fig. 2 gives the average Hellinger distance between the sampled results and true posterior, by sampling for $10k$ times under each data size configuration. In the baseline approach (i.e., Laplace mechanism), it is enough to add noise with sensitivity 1 in 2 dimensions and 2 in higher dimensions since it's equivalent to histogram (Should explain better). giving us the red points in plots. Without the knowledge of equivalence, Laplace usually add noise with sensitivity scale to dimensions, giving us green points. Points in blue are given by the $\mathcal{M}^B_{\mathcal{H}}$.



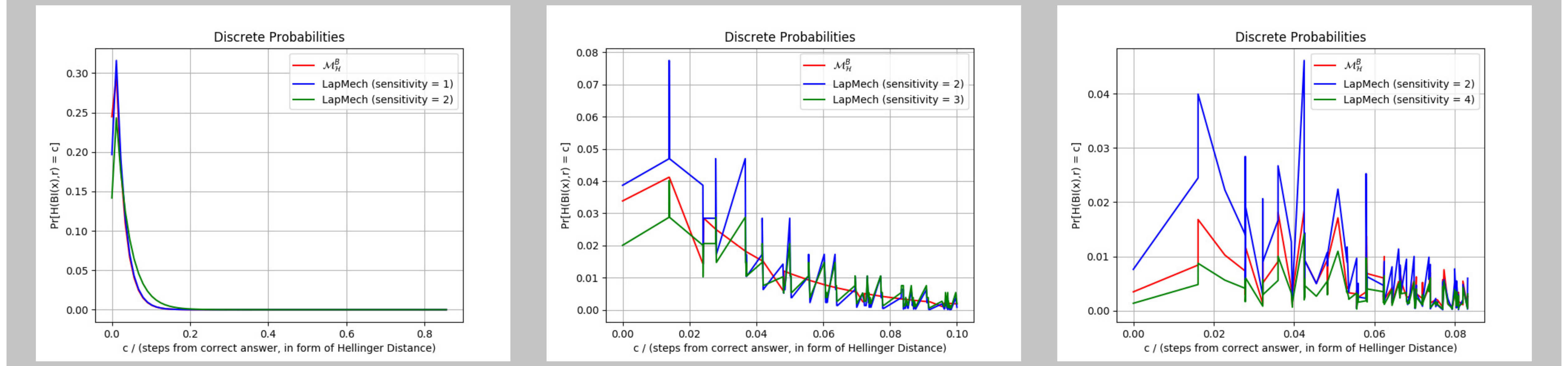
(a) 2 dimensions, data size $\in [100, 500]$

(b) 3 dimensions, data size $\in [100, 500]$

(c) 4 dimensions, data size $\in [100, 600]$

Figure 2: Increasing data size

Fig. 3 gives us the concrete probabilities of outputting candidates with certain Hellinegr distance from the correct posterior in 2, 3 and 4 dimensions respectively.



(a) 2 dimensions

(b) 3 dimensions

(c) 4 dimensions

Figure 3: The concrete outputting probabilities under different dimensions with data set of size **600**

Two groups of experiments both with unit prior $\mathbf{beta}(1, 1), \mathbf{beta}(1, 1, 1)$ and $\mathbf{beta}(1, 1, 1, 1)$, balanced datasets and parameters $\epsilon = 1.0$ and $\delta = 10^{-8}$.

Conclusion

- ▶ The smoothed Hellinger distance based exponential mechanism outperforms the asymptotically the baseline approach when the latter uses a sensitivity proportional to dimensionality. (when can this happen?)

References