Verifying Snapping Mechanism - Floating Point Implementation Version

In order to verify the differential privacy property of an implementation of the snapping mechanism [5], we follow the logic rules designed from [1] and the floating point error semantics from [7, 4, 2, 6].

1 Preliminary Definitions

Definition 1 (Laplace mechanism [3])
Let $\epsilon > 0$. The Laplace mechanism $\mathcal{L}_\epsilon$: $\mathbb{R} \to \mathsf{Distr}(\mathbb{R})$ is defined by $\mathcal{L}(t) = t + v$, where $v \in \mathbb{R}$ is drawn from the Laplace distribution $\mathsf{laplce}(\frac{1}{\epsilon})$.

2 Syntax

Following are the syntax of the system. The circled operators are rounded operation in floating point computation.

Expr.	e	::=	$r \mid c \mid x \mid f(x) \mid e_1 \oplus e_2 \mid e_1 \otimes e_2 \mid e_1 \ominus e_2 \mid e_1 \oslash e_2 \mid ⓛⓝ(e) \mid x \xleftarrow{\$} \mu$
Value	v	::=	$r \mid c$
Distribution	μ	::=	$\mathsf{laplce} \mid \mathsf{unif} \mid \mathtt{bernoulli}$
Error	err	::=	(e_1, e_2)
Condition	Φ	::=	$\mathtt{true} \mid \mathtt{false} \mid \Phi_1 \wedge \Phi_2 \mid \Phi_1 \vee \Phi_2$

Definition 2 ($\mathsf{Snap}(a) : A \to \mathsf{Distr}(B)$)
The ideal Snapping mechanism $\mathsf{Snap}(a)$ is defined as:

$$u \xleftarrow{\$} \mu; y = ⓛⓝ(u) \oslash \epsilon; s \xleftarrow{\$} \{-1, 1\}; z = s \otimes y; x = f(a); w = x \oplus z; w' = \lfloor w \rceil_\Lambda; r = \mathsf{clamp}_B(w')$$

where f is the query function over input $a \in A$, ϵ is the privacy budget, B is the clamping bound and Λ is the rounding argument satisfying $\lambda = 2^k$ where 2^k is the smallest power of 2 greater or equal to the $\frac{1}{\epsilon}$.

3 Semantics

The big step semantics with floating point computation error are shown in Figure. 1.

The big step semantics with relative floating point computation error are shown in Figure. 2.

$$\frac{(e_1, err, \Phi) \Downarrow (r_1, err_1) \qquad (e_2, err, \Phi) \Downarrow (r_2, err_2)}{(e_1 \oplus e_2, err, \Phi) \Downarrow (r_1 + r_2, err_1 \uplus err_2 \uplus err, \Phi_1 \land \Phi_2 \land \Phi)} \text{ PLUS} \qquad \cdots$$

Figure 1: Semantics with Absolutes Floating Point Error [2]

$$\frac{c = \mathtt{fl}(r)}{r \Downarrow c, \big(r(1-\eta), r(1+\eta)\big)} \text{ CONST} \qquad \frac{e_1 \Downarrow c_1, (\underline{r_1}, \bar{r_1}) \qquad e_2 \Downarrow c_2, (\underline{r_2}, \bar{r_2})}{e_1 \oplus e_2 \Downarrow \mathtt{fl}(c_1 + c_2), \big((\underline{r_1} + \underline{r_2})(1-\eta), (\bar{r_1} + \bar{r_2})(1+\eta)\big)} \text{ PLUS}$$

$$\frac{e_1 \Downarrow c_1, (\underline{r_1}, \bar{r_1}) \qquad e_2 \Downarrow c_2, (\underline{r_2}, \bar{r_2})}{e_1 \otimes e_2 \Downarrow \mathtt{fl}(c_1 \times c_2), \big((\underline{r_1} \times \underline{r_2})(1-\eta), (\bar{r_1} \times \bar{r_2})(1+\eta)\big)} \text{ TIMES}$$

$$\frac{e_1 \Downarrow c_1, (\underline{r_1}, \bar{r_1}) \qquad e_2 \Downarrow c_2, (\underline{r_2}, \bar{r_2})}{e_1 \ominus e_2 \Downarrow \mathtt{fl}(c_1 - c_2), \big((\underline{r_1} - \underline{r_2})(1-\eta), (\bar{r_1} - \bar{r_2})(1+\eta)\big)} \text{ SUB}$$

Figure 2: Semantics with Relative Floating Point Error (By Jiawen)

4 Soundness Theorems

Theorem 1 (The Snap mechanism is ϵ−differentially private)
Consider $\mathsf{Snap}(a)$ defined as before, if $\mathsf{Snap}(a) = x$ given database a and privacy parameter ϵ, then its actual privacy loss is bounded by $\epsilon + 12x\epsilon\eta + 2\eta$

Proof. Given $\mathsf{Snap}(a) = x$ and parameter ϵ, we consider a' be the adjacent database of a satisfying $|f(a) - f(a')| \leq 1$. Without loss of generalization, we assume $f(a) + 1 = f(a')$ $(\diamond)$. The proof is developed by cases of the output of $\mathsf{Snap}(a)$ mechanism.

case $x = -B$

Let b be the largest number rounded by Λ that is smaller than B. Based on the proof of the ideal version, the derivation of this case given $\mathsf{Snap}(a) = \mathsf{Snap}(a') = x$ is shown as following:

$$\cfrac{\cfrac{u \in \big(0, ⓔ^{\epsilon \otimes (-b \ominus \frac{\Lambda}{2} \ominus f(a))}\big) \sim u' \in \big(0, ⓔ^{\epsilon \otimes (-b \ominus \frac{\Lambda}{2} \ominus f(a'))}\big)}{\cdots}}{\mathsf{Snap}(a) = -B \sim \mathsf{Snap}(a') = -B}$$

Following the semantics in Figure 2, we have following evaluation results:

$$u \in \big[0, (\underline{r}, \bar{r})\big) \sim u' \in \big[0, (\underline{r}', \bar{r}')\big),$$

[[where $\underline{r}, \bar{r}, \underline{r}'$ and $\bar{r}'$ have following values:]]

case $x \in (-B, \lfloor f(a) \rceil_\Lambda)$

The derivation of this case is shown as following:

$$\dfrac{\dfrac{\dfrac{u \in \left[ⓔ^{\epsilon \otimes (x \ominus \frac{\Lambda}{2} \ominus f(a))}, ⓔ^{\epsilon \otimes (x \oplus \frac{\Lambda}{2} \ominus f(a))} \right) \sim u' \in \left[ⓔ^{\epsilon \otimes (x \ominus \frac{\Lambda}{2} \ominus f(a'))}, ⓔ^{\epsilon \otimes (x \oplus \frac{\Lambda}{2} \ominus f(a'))} \right)}{\dots}}{\mathsf{Snap}''(a) \in [x \ominus \frac{\Lambda}{2} \ominus f(a), x \oplus \frac{\Lambda}{2} \ominus f(a)) \sim_{\_} \mathsf{Snap}''(a') \in [x \ominus \frac{\Lambda}{2} \ominus f(a'), x \oplus \frac{\Lambda}{2} \ominus f(a'))}}{\dfrac{\mathsf{Snap}'(a) \in [x \ominus \frac{\Lambda}{2}, x \oplus \frac{\Lambda}{2}) \sim_{\_} \mathsf{Snap}'(a') \in [x \ominus \frac{\Lambda}{2}, x \oplus \frac{\Lambda}{2})}{\mathsf{Snap}(a) = x \sim \mathsf{Snap}(a') = x}}$$

Following the semantics in Figure 2, we have following evaluation results:

$$u \in \left[(\underline{r_1}, \bar{r_1}), (\underline{r_2}, \bar{r_2}) \right) \sim u' \in \left[(\underline{r'_1}, \bar{r'_1}), (\underline{r'_2}, \bar{r'_2}) \right),$$

[[where $\underline{r_1}, \bar{r_1}, \underline{r_2}, \bar{r_2}, \underline{r'_1}, \bar{r'_1}, \underline{r'_2} and \bar{r'_2}$ have following values:

$$u \in \left[((1-\eta)e^{\epsilon(x - \frac{\Lambda}{2} - f(a))(1-\eta)^2}, (1+\eta)e^{\epsilon(x - \frac{\Lambda}{2} - f(a))(1+\eta)^2}), ((1-\eta)e^{\epsilon(x + \frac{\Lambda}{2} - f(a))(1-\eta)^2}, (1+\eta)e^{\epsilon(x + \frac{\Lambda}{2} - f(a))(1+\eta)^2}) \right)$$
$$\sim u' \in \left[((1-\eta)e^{\epsilon(x - \frac{\Lambda}{2} - f(a'))(1-\eta)^2}, (1+\eta)e^{\epsilon(x - \frac{\Lambda}{2} - f(a'))(1+\eta)^2}), ((1-\eta)e^{\epsilon(x + \frac{\Lambda}{2} - f(a'))(1-\eta)^2}, (1+\eta)e^{\epsilon(x + \frac{\Lambda}{2} - f(a'))(1+\eta)^2}) \right)$$

Given that the probability is equivalent to the length of the range, we have the ratio between u and u' is bounded by:

$$\frac{u}{u'} \leq \frac{\bar{r_2} - \underline{r_1}}{\underline{r'_2} - \bar{r'_1}} \leq \epsilon + 12x\epsilon\eta + 2\eta$$

By the AxUnif rule, we have the actual privacy loss is bounded by the same value.]]

case $x = \lfloor f(a) \rceil_{\Lambda}$

$$\dfrac{\dfrac{u \in \left(ⓔ^{\epsilon \otimes (\lfloor f(a) \rceil_{\Lambda} \ominus \frac{\Lambda}{2} \ominus f(a))}, 1 \right] \vee \left(ⓔ^{\epsilon \otimes (f(a) \ominus \lfloor f(a) \rceil_{\Lambda} \ominus \frac{\Lambda}{2})}, 1 \right] \sim u' \in \left(ⓔ^{\epsilon \otimes (\lfloor f(a) \rceil_{\Lambda} \ominus f(a') \ominus \frac{\Lambda}{2})}, ⓔ^{\epsilon \otimes (\lfloor f(a) \rceil_{\Lambda} \ominus f(a') \oplus \frac{\Lambda}{2})} \right)}{\dots}}{\mathsf{Snap}(a) = x \sim \mathsf{Snap}(a') = x}$$

Following the semantics in Figure 2, we have following evaluation results:

$$u \in \left((\underline{r_1}, \bar{r_1}), 1 \right] \wedge (s = -1) \vee u \in \left((\underline{r_2}, \bar{r_2}), 1 \right) \wedge (s = 1) \sim u' \in \left[(\underline{r'_1}, \bar{r'_1}), (\underline{r'_2}, \bar{r'_2}) \right) \wedge (s = -1),$$

[[where $\underline{r_1}, \bar{r_1}, \underline{r_2}, \bar{r_2}, \underline{r'_1}, \bar{r'_1}, \underline{r'_2} and \bar{r'_2}$ have following values: Given that the probability is equivalent to the length of the range, we have the ratio between u and u' is bounded by:

$$\frac{u}{u'} \leq \frac{1 - \frac{1}{2}(\underline{r_2} + \underline{r_1})}{\frac{1}{2}(\underline{r'_2} - \bar{r'_1})} \leq \epsilon + 12x\epsilon\eta + 2\eta$$

By the AxUnif rule, we have the actual privacy loss is bounded by the same value.]]

case $x \in (\lfloor f(a) \rceil_{\Lambda}, \lfloor f(a') \rceil_{\Lambda})$

$$\dfrac{\dfrac{u \in \left(ⓔ^{\epsilon \otimes (f(a) \oplus \frac{\Lambda}{2} \ominus \lfloor f(a) \rceil_{\Lambda})}, ⓔ^{\epsilon \otimes (f(a) \ominus \frac{\Lambda}{2} \ominus \lfloor f(a) \rceil_{\Lambda})} \right] \sim u' \in \left(ⓔ^{\epsilon \otimes (\lfloor f(a) \rceil_{\Lambda} \ominus f(a') \oplus \frac{\Lambda}{2})}, ⓔ^{\epsilon \otimes (\lfloor f(a) \rceil_{\Lambda} \ominus f(a') \ominus \frac{\Lambda}{2})} \right)}{\dots}}{\mathsf{Snap}(a) = x \sim \mathsf{Snap}(a') = x}$$

Following the semantics in Figure 2, we have following evaluation results:

$$u \in \left((\underline{r_1}, \bar{r_1}), (\underline{r_2}, \bar{r_2}) \right] \wedge (s = 1) \sim u' \in \left[(\underline{r'_1}, \bar{r'_1}), (\underline{r'_2}, \bar{r'_2}) \right) \wedge (s = -1),$$

[[where $\underline{r}_1, \bar{r}_1, \underline{r}_2, \bar{r}_2, \underline{r}'_1, \bar{r}'_1, \underline{r}'_2 and \bar{r}'_2$ have following values: Given that the probability is equivalent to the length of the range, we have the ratio between u and u' is bounded by:

$$\frac{u}{u'} \leq \frac{\frac{1}{2}(\underline{r}_2 + \underline{r}_1)}{\frac{1}{2}(\underline{r}'_2 - \bar{r}'_1)} \leq \epsilon + 12x\epsilon\eta + 2\eta$$

By the AxUnif rule, we have the actual privacy loss is bounded by the same value.]]

case $\boldsymbol{x = \lfloor f(a') \rceil_\Lambda}$

$$\frac{\dfrac{u \in \left(ⓔ^{\epsilon \otimes (f(a) \oplus \frac{\Lambda}{2} \ominus \lfloor f(a) \rceil_\Lambda)}, ⓔ^{\epsilon \otimes (f(a) \ominus \frac{\Lambda}{2} \ominus \lfloor f(a) \rceil_\Lambda)}\right] \sim u' \in \left(ⓔ^{\epsilon \otimes (\lfloor f(a) \rceil_\Lambda \ominus f(a') \oplus \frac{\Lambda}{2})}, ⓔ^{\epsilon \otimes (\lfloor f(a) \rceil_\Lambda \ominus f(a') \ominus \frac{\Lambda}{2})}\right)}{\cdots}}{\mathsf{Snap}(a) = x \sim \mathsf{Snap}(a') = x}$$

Following the semantics in Figure 2, we have following evaluation results:

$$u \in \left((\underline{r}_1, \bar{r}_1), (\underline{r}_2, \bar{r}_2)\right] \wedge (s = 1) \sim u' \in \left[(\underline{r}'_1, \bar{r}'_1), (\underline{r}'_2, \bar{r}'_2)\right) \wedge (s = -1),$$

[[where $\underline{r}_1, \bar{r}_1, \underline{r}_2, \bar{r}_2, \underline{r}'_1, \bar{r}'_1, \underline{r}'_2 and \bar{r}'_2$ have following values: Given that the probability is equivalent to the length of the range, we have the ratio between u and u' is bounded by:

$$\frac{u}{u'} \leq \frac{\frac{1}{2}(\underline{r}_2 + \underline{r}_1)}{\frac{1}{2}(\underline{r}'_2 - \bar{r}'_1)} \leq \epsilon + 12x\epsilon\eta + 2\eta$$

By the AxUnif rule, we have the actual privacy loss is bounded by the same value.]]

case $\boldsymbol{x \in (\lfloor f(a') \rceil_\Lambda, B)}$

$$\frac{\dfrac{u \in \left(ⓔ^{\epsilon \otimes (f(a) \oplus \frac{\Lambda}{2} \ominus \lfloor f(a) \rceil_\Lambda)}, ⓔ^{\epsilon \otimes (f(a) \ominus \frac{\Lambda}{2} \ominus \lfloor f(a) \rceil_\Lambda)}\right] \sim u' \in \left(ⓔ^{\epsilon \otimes (\lfloor f(a) \rceil_\Lambda \ominus f(a') \oplus \frac{\Lambda}{2})}, ⓔ^{\epsilon \otimes (\lfloor f(a) \rceil_\Lambda \ominus f(a') \ominus \frac{\Lambda}{2})}\right)}{\cdots}}{\mathsf{Snap}(a) = x \sim \mathsf{Snap}(a') = x}$$

Following the semantics in Figure 2, we have following evaluation results:

$$u \in \left((\underline{r}_1, \bar{r}_1), (\underline{r}_2, \bar{r}_2)\right] \wedge (s = 1) \sim u' \in \left[(\underline{r}'_1, \bar{r}'_1), (\underline{r}'_2, \bar{r}'_2)\right) \wedge (s = -1),$$

[[where $\underline{r}_1, \bar{r}_1, \underline{r}_2, \bar{r}_2, \underline{r}'_1, \bar{r}'_1, \underline{r}'_2 and \bar{r}'_2$ have following values: Given that the probability is equivalent to the length of the range, we have the ratio between u and u' is bounded by:

$$\frac{u}{u'} \leq \frac{\frac{1}{2}(\underline{r}_2 + \underline{r}_1)}{\frac{1}{2}(\underline{r}'_2 - \bar{r}'_1)} \leq \epsilon + 12x\epsilon\eta + 2\eta$$

By the AxUnif rule, we have the actual privacy loss is bounded by the same value.]]

case $\boldsymbol{x = B}$

$$\frac{\dfrac{u \in \left(0, ⓔ^{\epsilon \otimes (-b \ominus \frac{\Lambda}{2} \ominus f(a))}\right) \sim u' \in \left(0, ⓔ^{\epsilon \otimes (-b \ominus \frac{\Lambda}{2} \ominus f(a'))}\right)}{\cdots}}{\mathsf{Snap}(a) = B \sim \mathsf{Snap}(a') = B}$$

Following the semantics in Figure 2, we have following evaluation results:

$$u \in \left[0, (\underline{r}, \bar{r})\right) \sim u' \in \left[0, (\underline{r}', \bar{r}')\right),$$

[[where $\underline{r}, \bar{r}, \underline{r}'$ and $\bar{r}'$ have following values:]]

$\square$

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