Verifying Snapping Mechanism - Floating Point Implementation Version

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February 19, 2020

In order to verify the differential privacy property of an implementation of the snapping mechanism [5], we follow the logic rules designed from [1] and the floating point error semantics from [7, 4, 2, 6].

1 Preliminary Definitions

Definition 1 (Laplace mechanism [3])
Let $\epsilon > 0$. The Laplace mechanism $\mathcal{L}_\epsilon \colon \mathbb{R} \to \mathsf{Distr}(\mathbb{R})$ is defined by $\mathcal{L}(t) = t + v$, where $v \in \mathbb{R}$ is drawn from the Laplace distribution $\mathsf{laplce}(\frac{1}{\epsilon})$.

2 Syntax

Following are the syntax of the system. The circled operators are rounded operation in floating point computation.

Floating Point Expr.	$e_\mathbb{F}$	::=	$c \mid x \mid f(x) \mid e_\mathbb{F} \circledast e_\mathbb{F} \mid \circledln(e_\mathbb{F}) \mid x \xleftarrow{\$} \mu$
Real Expr.	$e_\mathbb{R}$	::=	$r \mid X \mid F(X) \mid e_\mathbb{R} * e_\mathbb{R} \mid \ln(e_\mathbb{R}) \mid X \xleftarrow{\$} \mu$
Arithmetic Operation	$*$	::=	$+ \mid - \mid \times \mid \div$
Value	v	::=	$r \mid c$
Distribution	μ	::=	$\mathsf{laplce} \mid \mathsf{unif} \mid \mathtt{bernoulli}$
Error	err	::=	$(e_\mathbb{R}, e_\mathbb{R})$

We use upper case for variables in real computation and lower case for variables in floating point computation. $\circledast$ represents the operation in floating point machine.

$F(X)$ denotes function F evaluates to value $F(X)$ given input X in real computation, and $f(x)$ denotes the same function F evaluates to value $f(x)$ given the same input x in floating point computation.

3 Semantics

The big step semantics with relative floating point computation error are shown in Figure. 1. The semantics are $e_\mathbb{R} \Downarrow e_\mathbb{F}, err$, which means a real world expression $e_\mathbb{R}$ can be represented in floating point computation $e_\mathbb{F}$ with error bound err. The η is the machine epsilon.

$$\frac{c = \texttt{fl}(r)}{r \Downarrow c, \left(\dfrac{r}{(1+\eta)}, r(1+\eta)\right)} \text{ CONST} \qquad \frac{e_{\mathbb{R}}^1 \Downarrow e_{\mathbb{F}}^1, (\underline{e_{\mathbb{R}}^1}, \bar{e_{\mathbb{R}}^1}) \qquad e_{\mathbb{R}}^2 \Downarrow e_{\mathbb{F}}^2, (\underline{e_{\mathbb{R}}^2}, \bar{e_{\mathbb{R}}^2})}{e_{\mathbb{R}}^1 * e_{\mathbb{R}}^2 \Downarrow \texttt{fl}(e_{\mathbb{F}}^1 \circledast e_{\mathbb{F}}^2), \left(\left(\dfrac{\underline{e_{\mathbb{R}}^1 * e_{\mathbb{R}}^2}}{(1+\eta)}, (\bar{e_{\mathbb{R}}^1} * \bar{e_{\mathbb{R}}^2})(1+\eta)\right)\right)} \text{ OP}$$

$$\frac{e_{\mathbb{R}} \Downarrow e_{\mathbb{F}}, (\underline{e_{\mathbb{R}}}, \bar{e_{\mathbb{R}}}) \qquad e_{\mathbb{R}} \geq 1}{\ln(e_{\mathbb{R}}) \Downarrow \text{\textcircled{ln}}(e_{\mathbb{F}}), \left(\left(\dfrac{\text{\textcircled{ln}}(\underline{e_{\mathbb{R}}})}{(1+\eta)}, (\ln(\bar{e_{\mathbb{R}}}))(1+\eta)\right)\right)} \text{ LN} \qquad \frac{e_{\mathbb{R}} \Downarrow e_{\mathbb{F}}, (\underline{e_{\mathbb{R}}}, \bar{e_{\mathbb{R}}}) \qquad e_{\mathbb{R}} < 1}{\ln(e_{\mathbb{R}}) \Downarrow \text{\textcircled{ln}}(e_{\mathbb{F}}), \left((\ln(\underline{e_{\mathbb{R}}}))(1+\eta), \dfrac{\text{\textcircled{ln}}(\bar{e_{\mathbb{R}}})}{(1+\eta)}\right)} \text{ LN-OP}$$

Figure 1: Semantics with Relative Floating Point Error

Theorem 1 (Soundness Theorem)
Given $e_{\mathbb{R}}$ and $e_{\mathbb{F}}$ where $e_{\mathbb{R}} \Downarrow e_{\mathbb{F}}, err$, when evaluating the $e_{\mathbb{F}}$ in floating point computation and get the value c, we have $c \in err$.

4 Snapping Mechanism

Definition 2 ($\mathsf{Snap}_{\mathbb{R}}(a) : A \to \mathsf{Distr}(\mathbb{R})$)
Given privacy parameter ϵ, the ideal Snapping mechanism $\mathsf{Snap}_{\mathbb{R}}(a)$ is defined as:

$$U \xleftarrow{\$} \mu; S \xleftarrow{\$} \{-1, 1\}; Y = \ln(U) \div \epsilon; Z = S \times Y; X = F(a); W = X + Z; W' = \lfloor W \rceil_{\Lambda}; R = \mathsf{clamp}_B(W')$$

where f is the query function over input $a \in A$, ϵ is the privacy budget, B is the clamping bound and Λ is the rounding argument satisfying $\lambda = 2^k$ where 2^k is the smallest power of 2 greater or equal to the $\frac{1}{\epsilon}$.

Let $\mathsf{Snap}'_{\mathbb{R}}(a, U, S)$ be the same as $\mathsf{Snap}_{\mathbb{F}}(a)$ given U, S without rounding and clamping steps.

Definition 3 ($\mathsf{Snap}_{\mathbb{F}}(a) : A \to \mathsf{Distr}(\mathbb{R})$)
Given privacy parameter ϵ, the floating point implemented Snapping mechanism $\mathsf{Snap}_{\mathbb{F}}(a)$ is defined as (where all parameters are defined the same as above):

$$u_{\mathbb{F}} \xleftarrow{\$} \mu; s_{\mathbb{F}} \xleftarrow{\$} \{-1, 1\}; y = \text{\textcircled{ln}}(u) \oslash \epsilon; z = s \otimes y; x = f(a); w = x \oplus z; w' = \lfloor w \rceil_{\Lambda}; r = \mathsf{clamp}_B(w')$$

Let $\mathsf{Snap}'_{\mathbb{F}}(a, u, s)$ be the same as $\mathsf{Snap}_{\mathbb{F}}(a)$ without rounding and clamping precesses given u, s.

5 Main Theorem

Theorem 2 (The Snap mechanism is $\epsilon-$differentially private)
Consider $\mathsf{Snap}(a)$ defined as before, if $\mathsf{Snap}(a) = x$ given database a and privacy parameter ϵ, then its actual privacy loss is bounded by $\epsilon + 12x\epsilon\eta + 2\eta$

Proof. Given $\mathsf{Snap}_{\mathbb{F}}(a) = x$ and parameter ϵ, we consider a' be the adjacent database of a satisfying $|f(a) - f(a')| \leq 1$. Without loss of generalization, we assume $f(a) + 1 = f(a')$ ($\diamond$). The proof is developed by cases of the output of $\mathsf{Snap}_{\mathbb{F}}(a)$ mechanism.

Consider the $\mathsf{Snap}_{\mathbb{R}}(a)$ outputting the same result x, let (L, R) be the range where $\forall u \in (L, R)$ and some s, $\mathsf{Snap}'_{\mathbb{R}}(a, u, s) = x$, we have $\Pr[\mathsf{Snap}_{\mathbb{R}}(a)] = R - L$. Given the $\mathsf{Snap}_{\mathbb{R}}$ is ϵ−dp, we have:

$$e^{-\epsilon} \leq \frac{\Pr[\mathsf{Snap}_{\mathbb{R}}(a)]}{\Pr[\mathsf{Snap}_{\mathbb{R}}(a)]} = \frac{R - L}{R' - L'} \leq e^{\epsilon}$$

Let (l, r) be the range where $\forall u \in (l, r)$ and some s, $\mathsf{Snap}'_{\mathbb{F}}(a, u, s) = x$, we estimated the $|r - l|$ in terms of floating point relative error and $|R - L|$ through our semantics in order to verify the privacy loss of $\mathsf{Snap}_{\mathbb{F}}$.

case $x = -B$

Let b be the largest number rounded by Λ that is smaller than B. We know $s = 1$, $L = l = 0$ and $R = -b$, so we only need to estimate the right side range r in this case. The derivation of this case given $\mathsf{Snap}'_{\mathbb{F}}(a, R, 1) = \mathsf{Snap}'_{\mathbb{F}}(a', R, 1) = x$ is shown as following:

LN
$$\frac{}{R \Downarrow r, (\underline{R}, \bar{R})}$$

OP
$$\frac{}{\ln(R) \Downarrow \textcircled{ln}(r), (\ln(\underline{R})(1 + \eta), \frac{\ln(\bar{R})}{(1 + \eta)})}$$

OP
$$\frac{}{\frac{1}{\epsilon} \times \ln(R) \Downarrow \frac{1}{\epsilon} \otimes \textcircled{ln}(r), ((\frac{1}{\epsilon} \times \ln(\underline{R}))(1 + \eta)^2, \frac{\frac{1}{\epsilon} \times \ln(\bar{R})}{(1 + \eta)^2})}$$

ID
$$\frac{f(a) + \frac{1}{\epsilon} \times \ln(R) \Downarrow f(a) \oplus \frac{1}{\epsilon} \otimes \textcircled{ln}(r), \left((f(a) + (\frac{1}{\epsilon} \times \ln(\underline{R}))(1 + \eta)^2)(1 + \eta), \frac{(f(a) + \frac{\frac{1}{\epsilon} \times \ln(\bar{R})}{(1 + \eta)^2})}{(1 + \eta)}\right)}{\mathsf{Snap}'_{\mathbb{R}}(a, R, 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a, r, 1), \left((f(a) + (\frac{1}{\epsilon} \times \ln(\underline{R}))(1 + \eta)^2)(1 + \eta), \frac{(f(a) + \frac{\frac{1}{\epsilon} \times \ln(\bar{R})}{(1 + \eta)^2})}{(1 + \eta)}\right)}$$

In the same way, we have the derivation for $\mathsf{Snap}'_{\mathbb{F}}(a', r, 1)$:

$$\frac{\cdots}{\mathsf{Snap}'_{\mathbb{R}}(a', R', 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a', r, 1), \left((f(a') + (\frac{1}{\epsilon} \times \ln(\underline{R}'))(1 + \eta)^2)(1 + \eta), \frac{(f(a') + \frac{\frac{1}{\epsilon} \times \ln(\bar{R}')}{(1 + \eta)^2})}{(1 + \eta)}\right)}$$

Given $\mathsf{Snap}_{\mathbb{F}}(a) = \mathsf{Snap}_{\mathbb{F}}(a') = x = -b$, we have following values for $\underline{R}, \bar{R}, \underline{R}'$ and $\bar{R}'$:

$$\underline{R} = e^{\epsilon\left((x(1+\eta)-f(a))(1+\eta)^2)\right)}, \bar{R} = e^{\epsilon \frac{(\frac{x}{1+\eta} - f(a))}{(1+\eta)^2}}$$

$$\underline{R}' = e^{\epsilon\left((x(1+\eta)-f(a'))(1+\eta)^2\right)}, \bar{R}' = e^{\epsilon(\frac{(\frac{x}{1+\eta} - f(a'))}{(1+\eta)^2})}$$

The privacy loss of $\mathsf{Snap}_{\mathbb{F}}(a)$ in this case is bounded by:

$$\frac{\frac{1}{2}(\bar{R} - 0)}{\frac{1}{2}(\underline{R}' - 0)} = e^{\epsilon\left(\frac{(\frac{x}{1+\eta} - f(a))}{(1+\eta)^2} - \left((x(1+\eta)-f(a'))(1+\eta)^2\right)\right)}$$

$$= e^{\epsilon\left(\frac{x}{(1+\eta)^3} - \frac{f(a)}{(1+\eta)^2} - x(1+\eta)^3 + f(a')(1+\eta)^2\right)} \quad (\star)$$

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Since $(1+\eta)^3 > 1+3\eta$, $\frac{1}{(1+\eta)^3} < \frac{1}{1+3\eta}$, $(1+\eta)^2 < 1+2.1\eta$ and $\frac{1}{(1+\eta)^2} > 1-2\eta$, we have:

$$(\star) \quad \begin{aligned} &< e^{\epsilon\left(-\frac{9\eta+6}{1+3\eta}x+4.1\eta f(a)+(1+2.1\eta)\right)} \\ &< e^{\epsilon(10.1\eta B+1+2.1\eta)} \end{aligned}$$

case $x \in (-B, \lfloor f(a) \rceil_\Lambda)$

subcase $\lfloor f(a) \rceil_\Lambda \le 0 \vee \left(\lfloor f(a) \rceil_\Lambda > 0 \wedge x \in (-B, 0)\right)$

Let $y_1 = x - (\frac{\Lambda}{2})$, $y_2 = x + (\frac{\Lambda}{2})$, we know $y_1 < 0$, $y_2 < 0$, $S = s = 1$, $L = e^{\epsilon(y_1 - f(a))}$ and $R = e^{\epsilon(y_2 - f(a))}$ in this case. The derivations of estimating l and r are shown as following:

$$
\begin{array}{l}
\text{LN} \\
\hline
\qquad\qquad R \Downarrow r, (\underline{R}, \bar{R}) \\
\hline
\text{OP} \\
\qquad\qquad \ln(R) \Downarrow \textcircled{n}(r), (\ln(\underline{R})(1+\eta), \dfrac{\ln(\bar{R})}{(1+\eta)}) \\
\hline
\text{OP} \\
\qquad \dfrac{1}{\epsilon} \times \ln(R) \Downarrow \dfrac{1}{\epsilon} \otimes \textcircled{n}(r), ((\dfrac{1}{\epsilon} \times \ln(\underline{R}))(1+\eta)^2, \dfrac{\frac{1}{\epsilon} \times \ln(\bar{R})}{(1+\eta)^2}) \\
\hline
\text{ID} \\
\quad f(a) + \dfrac{1}{\epsilon} \times \ln(R) \Downarrow f(a) \oplus \dfrac{1}{\epsilon} \otimes \textcircled{n}(r), \left((f(a) + (\dfrac{1}{\epsilon} \times \ln(\underline{R}))(1+\eta)^2)(1+\eta), \dfrac{(f(a) + \frac{\frac{1}{\epsilon} \times \ln(\bar{R})}{(1+\eta)^2})}{(1+\eta)}\right) \\
\hline
\qquad\qquad \mathsf{Snap}'_{\mathbb{R}}(a, R, 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a, r, 1), (err_1, err_2)
\end{array}
$$

From soundness theorem, we have $err_1 \le y_2 \le err_2$.

Taking the lower bound, we have: $\underline{R} = e^{\epsilon\left((y_1(1+\eta)-f(a))(1+\eta)^2)\right)}$.

Taking the upper bound, we have: $\bar{R} = e^{\epsilon \frac{(\frac{y_1}{1+\eta}-f(a))}{(1+\eta)^2}}$.

$$
\begin{array}{c}
\cdots \\
\hline
\mathsf{Snap}'_{\mathbb{R}}(a, L, 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a, l, 1), (\dfrac{f(a) + (\frac{1}{\epsilon} \times \ln(\underline{L}))(1+\eta)^2}{1+\eta}, (f(a) + \dfrac{\frac{1}{\epsilon} \times \ln(\bar{L})}{(1+\eta)^2})(1+\eta)) \\
\hline
\mathsf{Snap}'_{\mathbb{R}}(a, L, 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a, l, 1), (err_1, err_2)
\end{array}
$$

From soundness theorem, we have $err_1 \le y_2 \le err_2$.

Taking the lower bound , we have: $\underline{L} = e^{\epsilon\left((y_2(1+\eta)-f(a))(1+\eta)^2)\right)}$.

Taking the upper bound, we have: $\bar{L} = e^{\epsilon \frac{(\frac{y_2}{1+\eta}-f(a))}{(1+\eta)^2}}$.

In the same way, we have the bound of l, r for adjacent data set a':

$$\underline{R}' = e^{\epsilon\left((y_1(1+\eta)-f(a'))(1+\eta)^2)\right)}, \quad \bar{R}' = e^{\epsilon \frac{(\frac{y_1}{1+\eta}-f(a'))}{(1+\eta)^2}}.$$

$$\underline{L}' = e^{\epsilon\left((y_2(1+\eta)-f(a'))(1+\eta)^2)\right)}, \quad \bar{L}' = e^{\epsilon \frac{(\frac{y_2}{1+\eta}-f(a'))}{(1+\eta)^2}}$$

4

Then, we have the privacy loss is bounded by:

$$\frac{|\bar{R} - \underline{L}|}{|\underline{R}' - \bar{L}'|}.$$

We also have:

$$
\begin{aligned}
\frac{\bar{R}}{R} &= e^{\epsilon\left(\frac{y_1}{(1+\eta)^3} - \frac{f(a)}{(1+\eta)^2} - y_1 + f(a)\right)} \le e^{\epsilon\left(-\frac{3\eta}{1+3\eta}y_1 + 2\eta f(a)\right)} \le e^{\epsilon\left(\frac{3\eta}{1+3\eta}B + 2\eta B\right)} \le e^{5\epsilon B\eta} \\
\frac{\underline{L}}{L} &= e^{\epsilon\left(y_2(1+\eta)^3 - f(a)(1+\eta)^2 - y_2 + f(a)\right)} \ge e^{\epsilon\left(3\eta y_1 - 2\eta f(a)\right)} \ge e^{-5\epsilon B\eta}
\end{aligned}
$$

Then, we can derive:

$$
\begin{aligned}
|\bar{R} - \underline{L}| &\le e^{5\epsilon B\eta} R - e^{-5\epsilon B\eta} L \\
&= L\big(e^{\Lambda\epsilon + 5\epsilon B\eta} - e^{-5\epsilon B\eta}\big) \\
&= L\big(e^{\Lambda\epsilon} e^{5\epsilon B\eta} - e^{5\epsilon B\eta} + e^{5\epsilon B\eta} - e^{-5\epsilon B\eta}\big) \\
&= L\big(e^{\Lambda\epsilon} e^{5\epsilon B\eta} - e^{5\epsilon B\eta} + \frac{1}{(e^{\Lambda\epsilon}-1)}(e^{\Lambda\epsilon}-1)e^{5\epsilon B\eta} - e^{-5\epsilon B\eta}\big) \\
&\le L\big(e^{\Lambda\epsilon} e^{5\epsilon B\eta} - e^{5\epsilon B\eta} + \frac{1}{(e-1)}(e^{\Lambda\epsilon}-1)e^{5\epsilon B\eta} - e^{-5\epsilon B\eta}\big) \ (by\, 1 \le \Lambda\epsilon < 2) \\
&= L\frac{e}{(e-1)}\big(e^{\Lambda\epsilon} e^{5\epsilon B\eta} - e^{5\epsilon B\eta} - e^{-5\epsilon B\eta}\big) \\
&< L\frac{e}{(e-1)}\big(e^{\Lambda\epsilon} e^{5\epsilon B\eta} - e^{5\epsilon B\eta}\big) \\
&= L(e^{\Lambda\epsilon}-1)e^{\ln(\frac{e}{(e-1)})+5\epsilon B\eta} \\
&< L(e^{\Lambda\epsilon}-1)e^{11\epsilon B\eta} \ (by\, (\frac{1}{\epsilon} < B < 2^{42}\frac{1}{\epsilon})) \\
&= (R-L)e^{11\epsilon B\eta}
\end{aligned}
$$

In the same way, we can derive:

$$|\underline{R} - \bar{L}| > e^{-5\epsilon B\eta} R - e^{5\epsilon B\eta} L > (R-L)e^{-12\epsilon B\eta}$$

Then we have:

$$\frac{|\bar{R} - \underline{L}|}{|\underline{R}' - \bar{L}'|} < e^{(23\epsilon B\eta + \epsilon)}.$$

subcase $\lfloor f(a)\rceil_\Lambda > 0 \wedge x \in (0, \lfloor f(a)\rceil_\Lambda)$

Let $y_1 = x - (\frac{\Lambda}{2})$, $y_2 = x + (\frac{\Lambda}{2})$, we know $y_1 > 0$, $y_2 > 0$, $S = s = 1$, $L = e^{\epsilon(y_1 - f(a))}$ and $R = e^{\epsilon(y_2 - f(a))}$ in this case. The derivations of estimating l and r are shown as following:

$$
\frac{
\frac{
\frac{
\frac{L \Downarrow l, (\underline{L}, \bar{L})}
{\ln(L) \Downarrow \textcircled{ln}(l), (\ln(\underline{L})(1+\eta), \frac{\ln(\bar{L})}{(1+\eta)})}
}
{\frac{1}{\epsilon} \times \ln(L) \Downarrow \frac{1}{\epsilon} \otimes \textcircled{ln}(l), ((\frac{1}{\epsilon} \times \ln(\underline{L}))(1+\eta)^2, \frac{\frac{1}{\epsilon} \times \ln(\bar{L})}{(1+\eta)^2})}
}
{f(a) + \frac{1}{\epsilon} \times \ln(L) \Downarrow f(a) \oplus \frac{1}{\epsilon} \otimes \textcircled{ln}(l), (\frac{f(a) + (\frac{1}{\epsilon} \times \ln(\underline{L}))(1+\eta)^2}{1+\eta}, (f(a) + \frac{\frac{1}{\epsilon} \times \ln(\bar{L})}{(1+\eta)^2})(1+\eta))}
}
{\mathsf{Snap}'_\mathbb{R}(a, L, 1) \Downarrow \mathsf{Snap}'_\mathbb{F}(a, l, 1), (err_1, err_2)}
$$

From soundness theorem, we have $err_1 \le y_1 \le err_2$.

Taking the lower bound (i.e. $err_1 = y_1$), we get: $\underline{L} = e^{(y_1/(1+\eta)-f(a))(1+\eta)^2\epsilon}$. Taking the upper bound (i.e. $err_2 = y_1$), we get: $\bar{L} = e^{(y_1(1+\eta)-f(a))\epsilon/(1+\eta)^2}$.

$$\frac{\cdots}{\mathsf{Snap}'_{\mathbb{R}}(a,R,1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a,r,1),(\frac{f(a)+(\frac{1}{\epsilon}\times\ln(\underline{R}))(1+\eta)^2}{1+\eta}, (f(a)+\frac{\frac{1}{\epsilon}\times\ln(\bar{R})}{(1+\eta)^2})(1+\eta))}{\mathsf{Snap}'_{\mathbb{R}}(a,R,1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a,r,1),(err_1,err_2)}$$

From soundness theorem, we have $err_1 \le y_2 \le err_2$.

Taking the lower bound (i.e. $err_1 = y_2$), we have: $\underline{R} = e^{(y_2/(1+\eta)-f(a))(1+\eta)^2\epsilon}$. Taking the upper bound (i.e. $err_2 = y_1$), we have: $\bar{R} = e^{(y_2(1+\eta)-f(a))\epsilon/(1+\eta)^2}$.

In the same way, we have the derivation for $\mathsf{Snap}'_{\mathbb{F}}(a',l,1)$ and $\mathsf{Snap}'_{\mathbb{F}}(a',r,1)$:

$$\frac{\cdots}{\mathsf{Snap}'_{\mathbb{R}}(a',L',1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a',l',1),(\frac{f(a')+(\frac{1}{\epsilon}\times\ln(\underline{L'}))(1+\eta)^2}{1+\eta}, (f(a')+\frac{\frac{1}{\epsilon}\times\ln(\bar{L'})}{(1+\eta)^2})(1+\eta))}$$

From soundness theorem, we have $err_1 \le y_2 \le err_2$.

Taking the lower bound (i.e. $err_1 = y_1$), we get: $\underline{L} = e^{(y_1/(1+\eta)-f(a'))(1+\eta)^2\epsilon}$. Taking the upper bound (i.e. $err_2 = y_1$), we get: $\bar{L} = e^{(y_1(1+\eta)-f(a'))\epsilon/(1+\eta)^2}$.

$$\frac{\cdots}{\mathsf{Snap}'_{\mathbb{R}}(a',R',1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a',r',1),(\frac{f(a')+(\frac{1}{\epsilon}\times\ln(\underline{R'}))(1+\eta)^2}{1+\eta}, (f(a')+\frac{\frac{1}{\epsilon}\times\ln(\bar{R'})}{(1+\eta)^2})(1+\eta))}$$

From soundness theorem, we have $err_1 \le y_2 \le err_2$.

Taking the lower bound (i.e. $err_1 = y_2$), we have: $\underline{R} = e^{(y_2/(1+\eta)-f(a'))(1+\eta)^2\epsilon}$. Taking the upper bound (i.e. $err_2 = y_1$), we have: $\bar{R} = e^{(y_2(1+\eta)-f(a'))\epsilon/(1+\eta)^2}$.

The privacy loss is bounded by:
$$\frac{|\bar{R}-\underline{L}|}{|\underline{R'}-\bar{L'}|}$$

Since the following bound can be proved by using $1-2\eta < (1+\eta)^2 < 1+2.1\eta, y_1 > -B, y_2 > -B$ and simple approximation:

$$\bar{R}-\underline{L} < (R-L)e^{(5B\eta\epsilon)}, \underline{R'}-\bar{L'} > (R'-L')e^{-7B\eta\epsilon}$$

We also have the $\mathsf{Snap}_{\mathbb{R}}(a)$ is ϵ-dp:
$$\frac{|R-L|}{|R'-L'|} = e^\epsilon$$

So we can get:
$$\frac{|\bar{R}-\underline{L}|}{|\underline{R'}-\bar{L'}|} < \frac{|R-L|}{|R'-L'|}e^{(12B\eta\epsilon)} = e^{(1+12B\eta)\epsilon}$$

subcase $\lfloor f(a) \rceil_\Lambda > 0 \wedge x = 0$

Let $y_1 = x - (\frac{\Lambda}{2})$, $y_2 = x + (\frac{\Lambda}{2})$, we know $y_1 < 0$, $y_2 > 0$, $S = s = 1$, $L = e^{\epsilon(y_1 - f(a))}$ and $R = e^{\epsilon(y_2 - f(a))}$ in this case. We have the derivation as:

$$\frac{\cdots}{\mathsf{Snap}'_\mathbb{R}(a, L, 1) \Downarrow \mathsf{Snap}'_\mathbb{F}(a, l, 1), (\dfrac{f(a) + (\frac{1}{\epsilon} \times \ln(\underline{L}))(1+\eta)^2}{1+\eta}, (f(a) + \dfrac{\frac{1}{\epsilon} \times \ln(\bar{L})}{(1+\eta)^2})(1+\eta))}{\mathsf{Snap}'_\mathbb{R}(a, L, 1) \Downarrow \mathsf{Snap}'_\mathbb{F}(a, l, 1), (err_1, err_2)}$$

From soundness theorem, we have $err_1 \le y_2 \le err_2$.

Taking the lower bound , we have: $\underline{L} = e^{\epsilon\left((y_2(1+\eta) - f(a))(1+\eta)^2\right)}$.

Taking the upper bound, we have: $\bar{L} = e^{\epsilon \frac{(\frac{y_2}{1+\eta} - f(a))}{(1+\eta)^2}}$.

$$\frac{\cdots}{\mathsf{Snap}'_\mathbb{R}(a, R, 1) \Downarrow \mathsf{Snap}'_\mathbb{F}(a, r, 1), (\dfrac{f(a) + (\frac{1}{\epsilon} \times \ln(\underline{R}))(1+\eta)^2}{1+\eta}, (f(a) + \dfrac{\frac{1}{\epsilon} \times \ln(\bar{R})}{(1+\eta)^2})(1+\eta))}{\mathsf{Snap}'_\mathbb{R}(a, R, 1) \Downarrow \mathsf{Snap}'_\mathbb{F}(a, r, 1), (err_1, err_2)}$$

From soundness theorem, we have $err_1 \le y_2 \le err_2$.

Taking the lower bound (i.e. $err_1 = y_2$), we have: $\underline{R} = e^{(y_2/(1+\eta) - f(a))(1+\eta)^2\epsilon}$. Taking the upper bound (i.e. $err_2 = y_1$), we have: $\bar{R} = e^{(y_2(1+\eta) - f(a))\epsilon/(1+\eta)^2}$. Using the bound we proved before, we have the folloing bound on $|\bar{R} - \underline{L}|$ and $|\underline{R} - \bar{L}|$:

$$\begin{aligned}
\bar{R} - \underline{L} &< e^{(2B\eta\epsilon)}R - e^{-5B\eta\epsilon}L < (R-L)e^{6B\eta\epsilon} \\
\underline{R} - \bar{L} &> e^{(-3B\eta\epsilon)}R - e^{5B\eta\epsilon}L > (R-L)e^{-8B\eta\epsilon},
\end{aligned}$$

and privacy loss is bounded by:

$$\frac{|\bar{R} - \underline{L}|}{|\underline{R}' - \bar{L}'|} < e^{14B\eta\epsilon + \epsilon}$$

case $x = \lfloor f(a) \rceil_\Lambda$

Since $f(a) + 1 = f(a')$, we have $\lfloor f(a) \rceil < \lfloor f(a') \rceil$, $s = 1$ for input a' and s can be 1 or -1 for input a. We have following subcases:

subcase $s = 1$ **for input** a
. We have in the worst case, the error propagate in the same direction, following derivation:

$$\frac{\dfrac{\dfrac{\dfrac{R \Downarrow r, (R, R)}{\ln(R) \Downarrow \text{\textcircled{ln}}(r), (\ln(R)(1+\eta), \dfrac{\ln(R)}{1+\eta})}}{\frac{1}{\epsilon}\ln(R) \Downarrow \frac{1}{\epsilon} \otimes \text{\textcircled{ln}}(r), (\frac{1}{\epsilon}\ln(R)(1+\eta)^2, \frac{1}{\epsilon}\dfrac{\ln(R)}{(1+\eta)^2})}}{f(a) + \frac{1}{\epsilon}\ln(R) \Downarrow f(a) \oplus \frac{1}{\epsilon} \otimes \text{\textcircled{ln}}(r), \left((f(a) + \frac{1}{\epsilon}ln(R)(1+\eta)^2)(1+\eta), (f(a) + \frac{1}{\epsilon}\dfrac{\text{\textcircled{ln}}(R)}{(1+\eta)^2})/(1+\eta)\right)}}{\mathsf{Snap}'_\mathbb{R}(a, 1, R) \Downarrow \mathsf{Snap}'_\mathbb{F}(a, 1, r), (err_1, err_2)}$$

Let $y_1 = x + \frac{\Lambda}{2}$, we have lower bound $\underline{R}$ and upper bound $\bar{R}$ for r be:

$$\bar{R} = e^{\mathcal{E}}$$

subcase $s = -1$ **for input** a

case $x \in (\lfloor f(a) \rceil_\Lambda, \lfloor f(a') \rceil_\Lambda)$

case $x = \lfloor f(a') \rceil_\Lambda$

case $x \in (\lfloor f(a') \rceil_\Lambda, B)$

case $x = B$

We know $s = -1$, $L = l = 0$ and $R = b$, so we only need to estimate the right side range r in this case. The derivation of this case given $\mathsf{Snap}'_{\mathbb{F}}(a, r, -1) = \mathsf{Snap}'_{\mathbb{F}}(a', r, -1) = x$ is shown as following:

$\square$

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