Verifying Snapping Mechanism - Floating Point Implementation Version

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March 3, 2020

In order to verify the differential privacy property of an implementation of the snapping mechanism [5], we follow the logic rules designed from [1] and the floating point error semantics from [7, 4, 2, 6].

1 Preliminary Definitions

Definition 1 (Laplace mechanism [3])
Let $\epsilon > 0$. The Laplace mechanism $\mathcal{L}_\epsilon \colon \mathbb{R} \to \mathsf{Distr}(\mathbb{R})$ is defined by $\mathcal{L}(t) = t + v$, where $v \in \mathbb{R}$ is drawn from the Laplace distribution $\mathsf{laplce}(\frac{1}{\epsilon})$.

2 Syntax - V1

Following are the syntax of our system:

Arithmetic Expr.	e	::=	$x \mid r \mid c \mid F(D) \mid e * e \mid \circ(e) \mid \xleftarrow{\$} \mu$
Binary Operation	$*$	::=	$+ \mid - \mid \times \mid \div$
Unary Operation	$\circ$	::=	$\ln \mid - \mid \lfloor \cdot \rceil \mid \mathsf{clamp}_B(\cdot)$
Value	v	::=	$r \mid c$
Distribution	μ	::=	$\mathsf{laplce} \mid \mathsf{unif} \mid \mathtt{bernoulli}$
Error	err	::=	(e, e)

$$\frac{r \geq 0}{r \Rightarrow \left(\dfrac{r}{(1+\eta)}, r(1+\eta)\right)} \; \text{VAL} \qquad\qquad \frac{r < 0}{r \Rightarrow \left(r(1+\eta), \dfrac{r}{(1+\eta)}\right)} \; \text{VAL-NEG} \qquad \frac{r = \texttt{fl}(r)}{r \Rightarrow (r,r)} \; \text{VAL-EQ}$$

$$\frac{}{\Theta, f(D) \Rightarrow (f(D), f(D))} \; \text{F(D)} \qquad\qquad \frac{}{\Theta, \xleftarrow{\$} \mu \Rightarrow (\xleftarrow{\$} \mu, \xleftarrow{\$} \mu)} \; \text{SAMPLE}$$

$$\frac{e^1 \Rightarrow (\underline{e}^1, \bar{e}^1) \qquad e^2 \Rightarrow (\underline{e}^2, \bar{e}^2) \qquad e^1 * e^2 \geq 0}{e^1 * e^2 \Rightarrow \left(\dfrac{\underline{e}^1 * \underline{e}^2}{(1+\eta)}, (\bar{e}^1 * \bar{e}^2)(1+\eta)\right)} \; \text{BOP} \quad \frac{e^1 \Rightarrow (\underline{e}^1, \bar{e}^1) \qquad e^2 \Rightarrow (\underline{e}^2, \bar{e}^2) \qquad e^1 * e^2 < 0}{e^1 * e^2 \Rightarrow \left((\bar{e}^1 * \bar{e}^2)(1+\eta), \dfrac{\underline{e}^1 * \underline{e}^2}{(1+\eta)}\right)} \; \text{BOP-NEG}$$

$$\frac{e \Rightarrow (\underline{e}, \bar{e}) \qquad \circ(e) \geq 0}{\circ(e) \Rightarrow \left(\dfrac{\circ(\underline{e})}{(1+\eta)}, (\circ(\bar{e}))(1+\eta)\right)} \; \text{UOP} \qquad \frac{e \Rightarrow (\underline{e}, \bar{e}) \qquad \circ(e) < 0}{\circ(e) \Rightarrow \left((\circ(\underline{e}))(1+\eta), \dfrac{\circ(\bar{e})}{(1+\eta)}\right)} \; \text{UOP-NEG}$$

Figure 1: Semantics of Transition for Expressions with Relative Floating Point Error

$$\frac{\texttt{fl}(r) = c}{r \Downarrow c} \; \text{FVAL} \qquad \frac{e^1 \Downarrow c^1 \qquad e^2 \Downarrow c^2 \qquad \texttt{fl}(c^1 * c^2) = c}{e^1 * e^2 \Downarrow c} \; \text{FBOP} \qquad \frac{e \Downarrow c', \qquad \texttt{fl}(\circ(c')) = c}{\circ(e) \Downarrow c} \; \text{FUOP}$$

Figure 2: Semantics of Evaluation in Floating Point Computation

3 Semantics - V1

The transition semantics with relative floating point computation error are shown in Figure. 4. The semantics are $e \Rightarrow (err)$, which means a real expression e can be transited in floating point computation with error bound err, η is the machine epsilon.

Theorem 1 (Soundness Theorem)
Given e where the transition $e \Rightarrow (\underline{e}, \bar{e})$ holds, then if e evaluates to c in floating point computation and $\underline{e}$ and $\bar{e}$ evaluates to $\underline{r}$ and $\bar{r}$ in real computation, we have:

$$\underline{r} \leq c \leq \bar{r}$$

$$\frac{}{r \Downarrow r} \; \text{RVAL} \qquad \frac{e^1 \Downarrow r^1 \qquad e^2 \Downarrow r^2 \qquad r^1 * r^2 = r}{e^1 * e^2 \Downarrow r} \; \text{RBOP} \qquad \frac{e \Downarrow r', \qquad \circ(r') = r}{\circ(e) \Downarrow r} \; \text{RUOP}$$

Figure 3: Semantics of Evaluation in Real Computation

4 Syntax - V2 (with Programs)

Programs	p	::=	$x = e \mid x \xleftarrow{\$} \mu \mid p ; p$
Expr.	e	::=	$r \mid c \mid x \mid f(D) \mid e * e \mid \circ (e)$
Binary Operation	$*$	::=	$+ \mid - \mid \times \mid \div$
Unary Operation	$\circ$	::=	$\ln \mid - \mid \lfloor \cdot \rceil \mid \mathsf{clamp}_B(\cdot)$
Value	v	::=	$r \mid c$
Distribution	μ	::=	$\mathsf{laplce} \mid \mathsf{unif} \mid \mathtt{bernoulli}$
Error	err	::=	(e, e)
Transaction Env.	Θ	::=	$\cdot \mid \Theta[x \mapsto (e, err)]$

$$\frac{\Theta(x) = (e, (\underline{e}, \bar{e}))}{\Theta, x \Rightarrow (e, (\underline{e}, \bar{e}))} \text{ VAR} \qquad \frac{x \notin dom(\Theta)}{\Theta, x \Rightarrow (x, (x, x))} \text{ VAR-NON} \qquad \frac{r \geq 0}{\Theta, r \Rightarrow \left(\dfrac{r}{(1+\eta)}, r(1+\eta)\right)} \text{ VAL}$$

$$\frac{c = \texttt{fl}(r) \qquad r < 0}{\Theta, r \Rightarrow \left(r(1+\eta), \dfrac{r}{(1+\eta)}\right)} \text{ VAL-NEG} \qquad \frac{r = \texttt{fl}(r)}{\Theta, r \Rightarrow (r, r)} \text{ VAL-EQ} \qquad \frac{}{\Theta, f(D) \Rightarrow (f(D), (f(D), f(D)))} \text{ F(D)}$$

$$\frac{\Theta, e^1 \Rightarrow (\underline{e}^1, \bar{e}^1) \quad \Theta, e^2 \Rightarrow (\underline{e}^2, \bar{e}^2) \quad e^1 * e^2 \geq 0}{\Theta, e^1 * e^2 \Rightarrow \left(e^1 * e^2, (\dfrac{\underline{e}^1 * \underline{e}^2}{(1+\eta)}, (\bar{e}^1 * \bar{e}^2)(1+\eta))\right)} \text{ BOP} \quad \frac{\Theta, e^1 \Rightarrow (\underline{e}^1, \bar{e}^1) \quad \Theta, e^2 \Rightarrow (\underline{e}^2, \bar{e}^2) \quad e^1 * e^2 < 0}{\Theta, e^1 * e^2 \Rightarrow \left(e^1 * e^2, (\underline{e}^1 * \underline{e}^2)(1+\eta), \dfrac{\bar{e}^1 * \bar{e}^2}{(1+\eta)}\right)} \text{ BOP-NEG}$$

$$\frac{\Theta, e \Rightarrow (\underline{e}, \bar{e}) \qquad \circ(e) \geq 0}{\Theta, \circ(e) \Rightarrow \left(\circ(e), \dfrac{\circ(\underline{e})}{(1+\eta)}, (\circ(\bar{e}))(1+\eta)\right)} \text{ UOP} \qquad \frac{\Theta, e \Rightarrow (\underline{e}, \bar{e}) \qquad \circ(e) < 0}{\Theta, \circ(e) \Rightarrow \left(\circ(e), (\circ(\underline{e}))(1+\eta), \dfrac{\circ(\bar{e})}{(1+\eta)}\right)} \text{ UOP-NEG}$$

Figure 4: Semantics of Transition for Expressions with Relative Floating Point Error

$$\frac{\Theta, e \Rightarrow (e, err)}{\Theta, X = e \Rightarrow \Theta[X \mapsto (e, err)x]} \text{ ASG} \qquad \frac{\Theta, p_1 \Rightarrow \Theta_1 \qquad \Theta_2, p_2 \Rightarrow \Theta_2}{\Theta, p_1; p_2 \Downarrow \Theta_2} \text{ CONSQ} \qquad \frac{}{\Theta, x \xleftarrow{\$} \mu \Downarrow \Theta[x \mapsto \xleftarrow{\$} \mu]} \text{ SAMPLE}$$

Figure 5: Semantics of Transition with Relative Floating Point Error Propagation for Programs

5 Semantics - V2 (with Programs)

The transition semantics with relative floating point computation error are shown in Figure. 4 for programs. The semantics are $\Theta, p \Rightarrow \Theta'$, which means a real computation programs p with environment Θ can be transited in floating point computation with error bound for all variables in Θ', η is the machine epsilon.

Theorem 2 (Soundness Theorem)
For any p, if the transition $\cdot, p \Rightarrow \Theta$ holds, then $\forall x \in dom(\Theta)$ s.t. $\Theta(x) = (e, (\underline{e}, \bar{e}))$, we have if e evaluates to c in floating point computation and $\underline{e}$ and $\bar{e}$ evaluates to $\underline{r}$ and $\bar{r}$ in real computation, then:

$$\underline{r} \leq c \leq \bar{r}$$

Proof. Induction on e, we have following cases:

$\square$

$$\frac{\texttt{fl}(r) = c}{r \Downarrow c} \text{ FVAL} \qquad \frac{e^1 \Downarrow c^1 \qquad e^2 \Downarrow c^2 \qquad \texttt{fl}(c^1 * c^2) = c}{e^1 * e^2 \Downarrow c} \text{ FBOP} \qquad \frac{e \Downarrow c', \qquad \texttt{fl}(\circ(c')) = c}{\circ(e) \Downarrow c} \text{ FUOP}$$

Figure 6: Semantics of Evaluation in Floating Point Computation

4

$$\frac{}{r \Downarrow r} \text{ RVAL} \qquad \frac{e^1 \Downarrow r^1 \qquad e^2 \Downarrow r^2 \qquad r^1 * r^2 = r}{e^1 * e^2 \Downarrow r} \text{ RBOP} \qquad \frac{e \Downarrow r', \qquad \circ(r') = r}{\circ(e) \Downarrow r} \text{ RUOP}$$

Figure 7: Semantics of Evaluation in Real Computation

6 Snapping Mechanism

Definition 2 ($\text{Snap}_{\mathbb{R}}(a) : A \to \text{Distr}(\mathbb{R})$)
Given privacy parameter ϵ, the ideal Snapping mechanism $\text{Snap}_{\mathbb{R}}(a)$ is defined as:

$$U \xleftarrow{\$} \mu; S \xleftarrow{\$} \{-1, 1\}; z = \text{clamp}_B\big(\lfloor F(a) + S \times \ln(U) \div \epsilon \rceil_\Lambda\big)$$

where F is a primitive query function over input database $a \in A$, ϵ is the privacy budget, B is the clamping bound and Λ is the rounding argument satisfying $\lambda = 2^k$ where 2^k is the smallest power of 2 greater or equal to the $\frac{1}{\epsilon}$.

Let $\text{Snap}'_{\mathbb{R}}(a, U, S)$ be the same as $\text{Snap}_{\mathbb{F}}(a)$ given U, S without rounding and clamping steps.

Definition 3 ($\text{Snap}_{\mathbb{F}}(a) : A \to \text{Distr}(\mathbb{R})$)
Given privacy parameter ϵ, the floating point implemented Snapping mechanism $\text{Snap}_{\mathbb{F}}(a)$ is defined as (where all parameters are defined the same as above):

$$u_{\mathbb{F}} \xleftarrow{\$} \mu; s_{\mathbb{F}} \xleftarrow{\$} \{-1, 1\}; z = \text{clamp}_B\big(\lfloor f(a) \oplus s \otimes \circledn(u) \oslash \epsilon \rceil_\Lambda\big)$$

Let $\text{Snap}'_{\mathbb{F}}(a, u, s)$ be the same as $\text{Snap}_{\mathbb{F}}(a)$ without rounding and clamping steps given u, s.

7 Main Theorem

Theorem 3 (**The** Snap **mechanism is** $\epsilon-$**differentially private**)
Consider $\text{Snap}(a)$ defined as before, if $\text{Snap}(a) = x$ given database a and privacy parameter ϵ, then its actual privacy loss is bounded by $\epsilon + 12x\epsilon\eta + 2\eta$

Proof. Given $\text{Snap}_{\mathbb{F}}(a) = x$ and parameter ϵ, we consider a' be the adjacent database of a satisfying $|f(a) - f(a')| \le 1$. Without loss of generalization, we assume $f(a) + 1 = f(a')$ ($\diamond$). The proof is developed by cases of the output of $\text{Snap}_{\mathbb{F}}(a)$ mechanism.

Consider the $\text{Snap}_{\mathbb{R}}(a)$ outputting the same result x, let (L, R) be the range where $\forall u \in (L, R)$ and some s, $\text{Snap}'_{\mathbb{R}}(a, u, s) = x$, we have $\Pr[\text{Snap}_{\mathbb{R}}(a)] = R - L$. Given the $\text{Snap}_{\mathbb{R}}$ is $\epsilon-$dp, we have:

$$e^{-\epsilon} \le \frac{\Pr[\text{Snap}_{\mathbb{R}}(a)]}{\Pr[\text{Snap}_{\mathbb{R}}(a)]} = \frac{R - L}{R' - L'} \le e^\epsilon$$

Let (l, r) be the range where $\forall u \in (l, r)$ and some s, $\text{Snap}'_{\mathbb{F}}(a, u, s) = x$, we estimated the $|r - l|$ in terms of floating point relative error and $|R - L|$ through our semantics in order to verify the privacy loss of $\text{Snap}_{\mathbb{F}}$.

case $x = -B$
Let b be the largest number rounded by Λ that is smaller than B. We know $s = 1$, $L = l = 0$ and

$R = -b$, so we only need to estimate the right side range r in this case. The derivation of this case given $\mathsf{Snap}'_{\mathbb{F}}(a, R, 1) = \mathsf{Snap}'_{\mathbb{F}}(a', R, 1) = x$ is shown as following:

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\overline{R \Downarrow r, (R, R)} \; \text{VAL-EQ}}{}}{\ln(R) \Downarrow \textcircled{ln}(r), (\ln(R)(1+\eta), \dfrac{\ln(R)}{(1+\eta)}) } \; \text{OP}}{\dfrac{1}{\epsilon} \times \ln(R) \Downarrow \dfrac{1}{\epsilon} \otimes \textcircled{ln}(r), ((\dfrac{1}{\epsilon} \times \ln(R))(1+\eta)^2, \dfrac{\frac{1}{\epsilon} \times \ln(R)}{(1+\eta)^2})} \; \text{OP}}{f(a) + \dfrac{1}{\epsilon} \times \ln(R) \Downarrow f(a) \oplus \dfrac{1}{\epsilon} \otimes \textcircled{ln}(r), \left((f(a) + (\dfrac{1}{\epsilon} \times \ln(R))(1+\eta)^2)(1+\eta), \dfrac{(f(a) + \frac{\frac{1}{\epsilon} \times \ln(R)}{(1+\eta)^2})}{(1+\eta)}\right)} \; \text{ID}}{\mathsf{Snap}'_{\mathbb{R}}(a, R, 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a, r, 1), \left((f(a) + (\dfrac{1}{\epsilon} \times \ln(R))(1+\eta)^2)(1+\eta), \dfrac{(f(a) + \frac{\frac{1}{\epsilon} \times \ln(R)}{(1+\eta)^2})}{(1+\eta)}\right)}
$$

LN

In the same way, we have the derivation for $\mathsf{Snap}'_{\mathbb{F}}(a', r, 1)$:

$$
\cfrac{\cdots}{\mathsf{Snap}'_{\mathbb{R}}(a', R', 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a', r, 1), \left((f(a') + (\dfrac{1}{\epsilon} \times \ln(R'))(1+\eta)^2)(1+\eta), \dfrac{(f(a') + \frac{\frac{1}{\epsilon} \times \ln(R')}{(1+\eta)^2})}{(1+\eta)}\right)}
$$

Given $\mathsf{Snap}_{\mathbb{F}}(a) = \mathsf{Snap}_{\mathbb{F}}(a') = x = -b$, we have the lower and upper bounds for R and R', which are $\underline{R}, \bar{R}, \underline{R}'$ and $\bar{R}'$:

$$
\underline{R} = e^{\epsilon\left((x(1+\eta) - f(a))(1+\eta)^2\right)}, \bar{R} = e^{\epsilon \frac{(\frac{x}{1+\eta} - f(a))}{(1+\eta)^2}}
$$

$$
\underline{R}' = e^{\epsilon\left((x(1+\eta) - f(a'))(1+\eta)^2\right)}, \bar{R}' = e^{\epsilon(\frac{(\frac{x}{1+\eta} - f(a'))}{(1+\eta)^2})}
$$

The privacy loss of $\mathsf{Snap}_{\mathbb{F}}(a)$ in this case is bounded by:

$$
\begin{aligned}
\frac{\frac{1}{2}(\bar{R} - 0)}{\frac{1}{2}(\underline{R}' - 0)} &= e^{\epsilon\left(\frac{(\frac{x}{1+\eta} - f(a))}{(1+\eta)^2} - \left(x(1+\eta) - f(a')\right)(1+\eta)^2\right)} \\
&= e^{\epsilon\left(\frac{x}{(1+\eta)^3} - \frac{f(a)}{(1+\eta)^2} - x(1+\eta)^3 + f(a')(1+\eta)^2\right)} \; (\star)
\end{aligned}
$$

Since $(1+\eta)^3 > 1 + 3\eta$, $\frac{1}{(1+\eta)^3} < \frac{1}{1+3\eta}$, $(1+\eta)^2 < 1 + 2.1\eta$ and $\frac{1}{(1+\eta)^2} > 1 - 2\eta$, we have:

$$
\begin{aligned}
(\star) &< e^{\epsilon\left(-\frac{9\eta + 6}{1+3\eta} x + 4.1\eta f(a) + (1+2.1\eta)\right)} \\
&< e^{\epsilon(10.1\eta B + 1 + 2.1\eta)}
\end{aligned}
$$

case $x \in (-B, \lfloor f(a) \rceil_\Lambda)$

subcase $\lfloor f(a) \rceil_\Lambda \leq 0 \vee \left(\lfloor f(a) \rceil_\Lambda > 0 \wedge x \in (-B, 0)\right)$

Let $y_1 = x - (\frac{\Lambda}{2})$, $y_2 = x + (\frac{\Lambda}{2})$, we know $y_1 < 0$, $y_2 < 0$, $S = s = 1$, $L = e^{\epsilon(y_1 - f(a))}$ and $R =$

6

$e^{\epsilon(y_2 - f(a))}$ in this case. The derivations of estimating l and r are shown as following:

$$\begin{array}{l}
\text{LN} \\
\dfrac{}{R \Downarrow r, (R, R)}\ \text{VAL-EQ} \\ \hline
\text{OP} \\
\ln(R) \Downarrow \textcircled{ln}(r), (\ln(R)(1+\eta), \dfrac{\ln(R)}{(1+\eta)}) \\ \hline
\text{OP} \\
\dfrac{1}{\epsilon} \times \ln(R) \Downarrow \dfrac{1}{\epsilon} \otimes \textcircled{ln}(r), ((\dfrac{1}{\epsilon} \times \ln(R))(1+\eta)^2, \dfrac{\frac{1}{\epsilon} \times \ln(R)}{(1+\eta)^2}) \\ \hline
\text{ID} \\
f(a) + \dfrac{1}{\epsilon} \times \ln(R) \Downarrow f(a) \oplus \dfrac{1}{\epsilon} \otimes \textcircled{ln}(r), \Big(\big(f(a) + (\dfrac{1}{\epsilon} \times \ln(R))(1+\eta)^2\big)(1+\eta), \dfrac{(f(a) + \frac{\frac{1}{\epsilon} \times \ln(R)}{(1+\eta)^2})}{(1+\eta)}\Big) \\ \hline
\mathsf{Snap}'_{\mathbb{R}}(a, R, 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a, r, 1), (e^1, e^2)
\end{array}$$

From soundness theorem, we have $e^1 \le y_2 \le e^2$, where we can get $\underline{R} \le r \le \bar{R}$.
Taking the lower bound, we have: $\underline{R} = e^{\epsilon\left((y_1(1+\eta) - f(a))(1+\eta)^2)\right)}$.
Taking the upper bound, we have: $\bar{R} = e^{\epsilon \frac{(\frac{y_1}{1+\eta} - f(a))}{(1+\eta)^2}}$.

$$\dfrac{\dfrac{\cdots}{\mathsf{Snap}'_{\mathbb{R}}(a, L, 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a, l, 1), (\dfrac{f(a) + (\frac{1}{\epsilon} \times \ln(\underline{L}))(1+\eta)^2}{1+\eta}, (f(a) + \dfrac{\frac{1}{\epsilon} \times \ln(\bar{L})}{(1+\eta)^2})(1+\eta))}}{\mathsf{Snap}'_{\mathbb{R}}(a, L, 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a, l, 1), (err_1, err_2)}$$

From soundness theorem, we have $err_1 \le y_2 \le err_2$.
Taking the lower bound , we have: $\underline{L} = e^{\epsilon\left((y_2(1+\eta) - f(a))(1+\eta)^2)\right)}$.
Taking the upper bound, we have: $\bar{L} = e^{\epsilon \frac{(\frac{y_2}{1+\eta} - f(a))}{(1+\eta)^2}}$.
In the same way, we have the bound of l, r for adjacent data set a':

$$\underline{R}' = e^{\epsilon\left((y_1(1+\eta) - f(a'))(1+\eta)^2)\right)},\ \bar{R}' = e^{\epsilon \frac{(\frac{y_1}{1+\eta} - f(a'))}{(1+\eta)^2}}.$$
$$\underline{L}' = e^{\epsilon\left((y_2(1+\eta) - f(a'))(1+\eta)^2)\right)},\ \bar{L}' = e^{\epsilon \frac{(\frac{y_2}{1+\eta} - f(a'))}{(1+\eta)^2}}$$

Then, we have the privacy loss is bounded by:

$$\frac{|\bar{R} - \underline{L}|}{|\underline{R}' - \bar{L}'|}.$$

We also have:

$$\begin{aligned}
\frac{\bar{R}}{\underline{R}} &= e^{\epsilon\left(\frac{y_1}{(1+\eta)^3} - \frac{f(a)}{(1+\eta)^2} - y_1 + f(a)\right)} \le e^{\epsilon\left(-\frac{3\eta}{1+3\eta} y_1 + 2\eta f(a)\right)} \le e^{\epsilon\left(\frac{3\eta}{1+3\eta} B + 2\eta B\right)} \le e^{5\epsilon B\eta} \\
\frac{\underline{L}}{\bar{L}} &= e^{\epsilon\left(y_2(1+\eta)^3 - f(a)(1+\eta)^2 - y_2 + f(a)\right)} \ge e^{\epsilon\left(3\eta y_1 - 2\eta f(a)\right)} \ge e^{-5\epsilon B\eta}
\end{aligned}$$

Then, we can derive:

$$
\begin{aligned}
|\bar{R} - \underline{L}| \; &\le \; e^{5\epsilon B\eta} R - e^{-5\epsilon B\eta} L \\
&= L\big(e^{\Lambda\epsilon + 5\epsilon B\eta} - e^{-5\epsilon B\eta}\big) \\
&= L\big(e^{\Lambda\epsilon} e^{5\epsilon B\eta} - e^{5\epsilon B\eta} + e^{5\epsilon B\eta} - e^{-5\epsilon B\eta}\big) \\
&= L\big(e^{\Lambda\epsilon} e^{5\epsilon B\eta} - e^{5\epsilon B\eta} + \tfrac{1}{(e^{\Lambda\epsilon}-1)}(e^{\Lambda\epsilon}-1)e^{5\epsilon B\eta} - e^{-5\epsilon B\eta}\big) \\
&\le L\big(e^{\Lambda\epsilon} e^{5\epsilon B\eta} - e^{5\epsilon B\eta} + \tfrac{1}{(e-1)}(e^{\Lambda\epsilon}-1)e^{5\epsilon B\eta} - e^{-5\epsilon B\eta}\big) \;\; (by\, 1 \le \Lambda\epsilon < 2) \\
&= L\tfrac{e}{(e-1)}\big(e^{\Lambda\epsilon} e^{5\epsilon B\eta} - e^{5\epsilon B\eta} - e^{-5\epsilon B\eta}\big) \\
&< L\tfrac{e}{(e-1)}\big(e^{\Lambda\epsilon} e^{5\epsilon B\eta} - e^{5\epsilon B\eta}\big) \\
&= L(e^{\Lambda\epsilon}-1)e^{\ln(\frac{e}{(e-1)})+5\epsilon B\eta} \\
&< L(e^{\Lambda\epsilon}-1)e^{11\epsilon B\eta} \;\; (by\, (\tfrac{1}{\epsilon} < B < 2^{42}\tfrac{1}{\epsilon})) \\
&= (R-L)e^{11\epsilon B\eta}
\end{aligned}
$$

In the same way, we can derive:

$$
|\underline{R} - \bar{L}| > e^{-5\epsilon B\eta} R - e^{5\epsilon B\eta} L > (R-L)e^{-12\epsilon B\eta}
$$

Then we have:

$$
\frac{|\bar{R} - \underline{L}|}{|\underline{R}' - \bar{L}'|} < e^{(23\epsilon B\eta + \epsilon)}.
$$

subcase $\lfloor f(a)\rceil_\Lambda > 0 \wedge x \in (0, \lfloor f(a)\rceil_\Lambda)$

Let $y_1 = x - (\tfrac{\Lambda}{2})$, $y_2 = x + (\tfrac{\Lambda}{2})$, we know $y_1 > 0$, $y_2 > 0$, $S = s = 1$, $L = e^{\epsilon(y_1 - f(a))}$ and $R = e^{\epsilon(y_2 - f(a))}$ in this case. The derivations of estimating l and r are shown as following:

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{L \Downarrow l, (\underline{L}, \bar{L})}{\ln(L) \Downarrow \circlednly{ln}(l), (\ln(\underline{L})(1+\eta), \frac{\ln(\bar{L})}{(1+\eta)})}
}{
\frac{1}{\epsilon} \times \ln(L) \Downarrow \frac{1}{\epsilon} \otimes \circlednly{ln}(l), ((\frac{1}{\epsilon} \times \ln(\underline{L}))(1+\eta)^2, \frac{\frac{1}{\epsilon} \times \ln(\bar{L})}{(1+\eta)^2})
}
}{
f(a) + \frac{1}{\epsilon} \times \ln(L) \Downarrow f(a) \oplus \frac{1}{\epsilon} \otimes \circlednly{ln}(l), (\frac{f(a) + (\frac{1}{\epsilon} \times \ln(\underline{L}))(1+\eta)^2}{1+\eta}, (f(a) + \frac{\frac{1}{\epsilon} \times \ln(\bar{L})}{(1+\eta)^2})(1+\eta))
}
}{
\mathsf{Snap}'_{\mathbb{R}}(a, L, 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a, l, 1), (err_1, err_2)
}
$$

From soundness theorem, we have $err_1 \le y_1 \le err_2$.

Taking the lower bound (i.e. $err_1 = y_1$), we get: $\underline{L} = e^{(y_1/(1+\eta) - f(a))(1+\eta)^2 \epsilon}$. Taking the upper bound (i.e. $err_2 = y_1$), we get: $\bar{L} = e^{(y_1(1+\eta) - f(a))\epsilon/(1+\eta)^2}$.

$$
\cfrac{
\cdots \\
\mathsf{Snap}'_{\mathbb{R}}(a, R, 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a, r, 1), (\frac{f(a) + (\frac{1}{\epsilon} \times \ln(\underline{R}))(1+\eta)^2}{1+\eta}, (f(a) + \frac{\frac{1}{\epsilon} \times \ln(\bar{R})}{(1+\eta)^2})(1+\eta))
}{
\mathsf{Snap}'_{\mathbb{R}}(a, R, 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a, r, 1), (err_1, err_2)
}
$$

From soundness theorem, we have $err_1 \le y_2 \le err_2$.

Taking the lower bound (i.e. $err_1 = y_2$), we have: $\underline{R} = e^{(y_2/(1+\eta) - f(a))(1+\eta)^2 \epsilon}$. Taking the upper bound (i.e. $err_2 = y_1$), we have: $\bar{R} = e^{(y_2(1+\eta) - f(a))\epsilon/(1+\eta)^2}$.

In the same way, we have the derivation for $\mathsf{Snap}'_{\mathbb{F}}(a', l, 1)$ and $\mathsf{Snap}'_{\mathbb{F}}(a', r, 1)$:

$$\frac{\dots}{\mathsf{Snap}'_{\mathbb{R}}(a', L', 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a', l', 1), (\frac{f(a') + (\frac{1}{\epsilon} \times \ln(\underline{L'}))(1+\eta)^2}{1+\eta}, (f(a') + \frac{\frac{1}{\epsilon} \times \ln(\bar{L}')}{(1+\eta)^2})(1+\eta))}$$

From soundness theorem, we have $err_1 \le y_2 \le err_2$.

Taking the lower bound (i.e. $err_1 = y_1$), we get: $\underline{L} = e^{(y_1/(1+\eta) - f(a'))(1+\eta)^2 \epsilon}$. Taking the upper bound (i.e. $err_2 = y_1$), we get: $\bar{L} = e^{(y_1(1+\eta) - f(a'))\epsilon/(1+\eta)^2}$.

$$\frac{\dots}{\mathsf{Snap}'_{\mathbb{R}}(a', R', 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a', r', 1), (\frac{f(a') + (\frac{1}{\epsilon} \times \ln(\underline{R'}))(1+\eta)^2}{1+\eta}, (f(a') + \frac{\frac{1}{\epsilon} \times \ln(\bar{R}')}{(1+\eta)^2})(1+\eta))}$$

From soundness theorem, we have $err_1 \le y_2 \le err_2$.

Taking the lower bound (i.e. $err_1 = y_2$), we have: $\underline{R} = e^{(y_2/(1+\eta) - f(a'))(1+\eta)^2 \epsilon}$. Taking the upper bound (i.e. $err_2 = y_1$), we have: $\bar{R} = e^{(y_2(1+\eta) - f(a'))\epsilon/(1+\eta)^2}$.

The privacy loss is bounded by:

$$\frac{|\bar{R} - \underline{L}|}{|\underline{R'} - \bar{L}'|}$$

Since the following bound can be proved by using $1 - 2\eta < (1+\eta)^2 < 1 + 2.1\eta$, $y_1 > -B$, $y_2 > -B$ and simple approximation:

$$\bar{R} - \underline{L} < (R - L)e^{(5B\eta\epsilon)}, \underline{R'} - \bar{L}' > (R' - L')e^{-7B\eta\epsilon}$$

We also have the $\mathsf{Snap}_{\mathbb{R}}(a)$ is ϵ-dp:

$$\frac{|R - L|}{|R' - L'|} = e^\epsilon$$

So we can get:

$$\frac{|\bar{R} - \underline{L}|}{|\underline{R'} - \bar{L}'|} < \frac{|R - L|}{|R' - L'|} e^{(12B\eta\epsilon)} = e^{(1+12B\eta)\epsilon}$$

subcase $\lfloor f(a) \rceil_\Lambda > 0 \wedge x = 0$

Let $y_1 = x - (\frac{\Lambda}{2})$, $y_2 = x + (\frac{\Lambda}{2})$, we know $y_1 < 0$, $y_2 > 0$, $S = s = 1$, $L = e^{\epsilon(y_1 - f(a))}$ and $R = e^{\epsilon(y_2 - f(a))}$ in this case. We have the derivation as:

$$\frac{\frac{\dots}{\mathsf{Snap}'_{\mathbb{R}}(a, L, 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a, l, 1), (\frac{f(a) + (\frac{1}{\epsilon} \times \ln(\underline{L}))(1+\eta)^2}{1+\eta}, (f(a) + \frac{\frac{1}{\epsilon} \times \ln(\bar{L})}{(1+\eta)^2})(1+\eta))}}{\mathsf{Snap}'_{\mathbb{R}}(a, L, 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a, l, 1), (err_1, err_2)}$$

From soundness theorem, we have $err_1 \le y_2 \le err_2$.

Taking the lower bound , we have: $\underline{L} = e^{\epsilon\left((y_2(1+\eta) - f(a))(1+\eta)^2\right)}$.

Taking the upper bound, we have: $\bar{L} = e^{\epsilon \frac{(\frac{y_2}{1+\eta} - f(a))}{(1+\eta)^2}}$.

$$\cdots$$

$$\frac{\mathsf{Snap}'_{\mathbb{R}}(a, R, 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a, r, 1), \left(\dfrac{f(a) + (\frac{1}{\epsilon} \times \ln(\underline{R}))(1+\eta)^2}{1+\eta}, (f(a) + \dfrac{\frac{1}{\epsilon} \times \ln(\bar{R})}{(1+\eta)^2})(1+\eta)\right)}{\mathsf{Snap}'_{\mathbb{R}}(a, R, 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a, r, 1), (err_1, err_2)}$$

From soundness theorem, we have $err_1 \le y_2 \le err_2$. Taking the lower bound (i.e. $err_1 = y_2$), we have: $\underline{R} = e^{(y_2/(1+\eta) - f(a))(1+\eta)^2 \epsilon}$. Taking the upper bound (i.e. $err_2 = y_1$), we have: $\bar{R} = e^{(y_2(1+\eta) - f(a))\epsilon/(1+\eta)^2}$. Using the bound we proved before, we have the folling bound on $|\bar{R} - \underline{L}|$ and $|\underline{R} - \bar{L}|$:

$$\bar{R} - \underline{L} \quad < e^{(2B\eta\epsilon)} R - e^{-5B\eta\epsilon} L < (R - L)e^{6B\eta\epsilon}$$
$$\underline{R} - \bar{L} \quad > e^{(-3B\eta\epsilon)} R - e^{5B\eta\epsilon} L > (R - L)e^{-8B\eta\epsilon},$$

and privacy loss is bounded by:

$$\frac{|\bar{R} - \underline{L}|}{|\underline{R}' - \bar{L}'|} < e^{14B\eta\epsilon + \epsilon}$$

case $x = \lfloor f(a) \rceil_\Lambda$

This case can also be split into 3 subcases by: $\lfloor f(a) \rceil_\Lambda < 0$, $\lfloor f(a) \rceil_\Lambda = 0$ and $\lfloor f(a) \rceil_\Lambda > 0$. Without loss of generalization, we consider the worst case where the error propagate in the same direction, i.e. $\lfloor f(a) \rceil_\Lambda < 0$.

From this assumption, let $y_1 = x - (\frac{\Lambda}{2})$, $y_2 = x + (\frac{\Lambda}{2})$, we know $y_1 < 0$, $y_2 < 0$. Since $f(a) + 1 = f(a')$, we also have $\lfloor f(a) \rceil < \lfloor f(a') \rceil$. So, we know s can only be 1 for input a' but s can be 1 or -1 for input a.

For input a, when $s = 1$, we have following derivations:

$$\frac{R \Downarrow r, (R, R)}{\ln(R) \Downarrow \textcircled{ln}(r), (\ln(R)(1+\eta), \dfrac{\ln(R)}{1+\eta})}$$

$$\dfrac{}{\dfrac{1}{\epsilon}\ln(R) \Downarrow \dfrac{1}{\epsilon} \otimes \textcircled{ln}(r), \left(\dfrac{1}{\epsilon}\ln(R)(1+\eta)^2, \dfrac{1}{\epsilon}\dfrac{\ln(R)}{(1+\eta)^2}\right)}$$

$$\dfrac{}{f(a) + \dfrac{1}{\epsilon}\ln(R) \Downarrow f(a) \oplus \dfrac{1}{\epsilon} \otimes \textcircled{ln}(r), \left((f(a) + \dfrac{1}{\epsilon}\ln(R)(1+\eta)^2)(1+\eta), (f(a) + \dfrac{1}{\epsilon}\dfrac{\ln(R)}{(1+\eta)^2})/(1+\eta)\right)}$$

$$\dfrac{}{\mathsf{Snap}'_{\mathbb{R}}(a, 1, R) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a, 1, r), (err_1, err_2)}$$

From soundness theorem, we have $err_1 \le y_2 \le err_2$. Then we can get following bounds for r:
$\underline{R}_+ = e^{\epsilon\left((y_2(1+\eta) - f(a))(1+\eta)^2\right)}$, $\bar{R}_+ = e^{\epsilon\frac{(\frac{y_2}{1+\eta} - f(a))}{(1+\eta)^2}}$.
Since $y_2 = \lfloor f(a) \rceil + \frac{\Lambda}{2}$, we have $e^{\epsilon\left((y_2 - f(a))\right)} > 1$, so actually we know $R = r = 1$.
We can also derive the bound for l in the same way as:
$\underline{L}_+ = e^{\epsilon\left((y_1(1+\eta) - f(a))(1+\eta)^2\right)}$, $\bar{L}_+ = e^{\epsilon\frac{(\frac{y_1}{1+\eta} - f(a))}{(1+\eta)^2}}$.

When $s = -1$, we can derive following bounds in the same way for l and r:
$\underline{L}_- = e^{\epsilon\left((f(a) - y_2(1+\eta))(1+\eta)^2\right)}$, $\bar{L}_- = e^{\epsilon\frac{(f(a) - \frac{y_2}{1+\eta})}{(1+\eta)^2}}$.

$\underline{R}_2 = e^{\epsilon\left((f(a) - y_1(1+\eta))(1+\eta)^2\right)}$, $\bar{R}_2 = e^{\epsilon\frac{(f(a) - \frac{y_1}{1+\eta})}{(1+\eta)^2}}$.

Since $y_1 = \lfloor f(a) \rceil - \frac{\Lambda}{2}$, we have $e^{\epsilon\left((f(a)-y_1))\right)} > 1$, so actually we know $R' = r' = 1$.

For input a', we have only one case where $s = 1$, the following bound can be derived:

$$\underline{R'} = e^{\epsilon\left((y_2(1+\eta)-f(a'))(1+\eta)^2)\right)}, \bar{R'} = e^{\epsilon\frac{(\frac{y_2}{1+\eta}-f(a'))}{(1+\eta)^2}}.$$

$$\underline{L'} = e^{\epsilon\left((y_1(1+\eta)-f(a'))(1+\eta)^2)\right)}, \bar{L'} = e^{\epsilon\frac{(\frac{y_1}{1+\eta}-f(a'))}{(1+\eta)^2}}.$$

We have following bounds on their ratios:

$$\frac{\underline{R_+}}{R_+} = e^{\epsilon\left((1+\eta)^3 y_2 - (1+\eta)^2 f(a) - y_2 + f(a)\right)} > e^{-3\epsilon B\eta}, \frac{\bar{R_+}}{R_+} = e^{\epsilon\left(fracy_2(1+\eta)^3 - \frac{f(a)}{(1+\eta)^2} - y_2 + f(a)\right)} < e^{3\epsilon B\eta},$$

The same bound for L_+ by substituting y_2 with y_1, and similar bound for L', R'.

$$\frac{\underline{R'}}{R} = e^{\epsilon\left((1+\eta)^2 f(a) - (1+\eta)^3 y_2 - f(a) + y_2\right)} > e^{-2\epsilon B\eta}, \frac{\bar{R'}}{R'} = e^{\epsilon\left(\frac{f(a)}{(1+\eta)^2} - fracy_2(1+\eta)^3 - f(a) + y_2\right)} < e^{2\epsilon B\eta},$$

Using the bound on their ratios, we can get following bounds on $|\bar{R}_+ - L_+|$ and $|\underline{R'} - \bar{L'}|$:

$$|\bar{R}_+ - L_+| < e^{3\epsilon B\eta}R - e^{-3\epsilon B\eta}L < (R-L)e^{7\epsilon B\eta}, |\underline{R'} - \bar{L'}| > e^{-2\epsilon B\eta}R - e^{2\epsilon B\eta}L > (R'-L')e^{-5\epsilon B\eta}$$

Then we have the following bounds on privacy loss:

$$\frac{2-(L_+ + L_-)}{\underline{R'} - \bar{L'}} < \frac{\bar{R}_+ - L_+}{\underline{R'} - \bar{L'}} < \frac{e^{7\epsilon B\eta}(R_+ - L_+)}{e^{-5\epsilon B\eta}(R' - L')} = e^{12\epsilon B\eta + \epsilon}$$

case $x \in (\lfloor f(a) \rceil_\Lambda, \lfloor f(a') \rceil_\Lambda)$

Since the output set $(\lfloor f(a) \rceil_\Lambda, \lfloor f(a') \rceil_\Lambda)$ is empty when $\Lambda \geq 1$, so we consider the situation where $\Lambda < 1$. There are two subcases in this case : $x > 0 and x < 0$. Without loss of generalization, we consider the worst case where error propagate in the same direction, i.e., $\lfloor f(a') \rceil_\Lambda) < 0$. The bounds derived for l, r and l', r' under input a and a' are as follows:

For input a:

$$\underline{R} = e^{\epsilon\left((f(a)-y_2(1+\eta))(1+\eta)^2)\right)}, \bar{R} = e^{\epsilon\frac{(f(a)-\frac{y_2}{1+\eta})}{(1+\eta)^2}}.$$

$$\underline{L} = e^{\epsilon\left((f(a)-y_1(1+\eta))(1+\eta)^2)\right)}, \bar{L} = e^{\epsilon\frac{f(a)-\frac{y_1}{1+\eta}}{(1+\eta)^2}}.$$

For input a':

$$\underline{R'} = e^{\epsilon\left((y_2(1+\eta)-f(a'))(1+\eta)^2)\right)}, \bar{R'} = e^{\epsilon\frac{\frac{y_2}{1+\eta}-f(a')}{(1+\eta)^2}}.$$

$$\underline{L'} = e^{\epsilon\left((y_1(1+\eta)-f(a'))(1+\eta)^2)\right)}, \bar{L'} = e^{\epsilon\frac{\frac{y_1}{1+\eta}-f(a')}{(1+\eta)^2}}.$$

The bounds on their ratio are as follows:

$$\frac{\underline{R}}{R} > e^{-5B\eta\epsilon}, \frac{\bar{R}}{R} < e^{5B\eta\epsilon}; \quad \frac{\underline{R'}}{R'} > e^{-5B\eta\epsilon}, \frac{\bar{R'}}{R'} < e^{5B\eta\epsilon}.$$

And the bounds on $|\underline{R} - \bar{L}|$ and $|\bar{R'} - \underline{L'}|$ are as follows:

$$|\underline{R} - \bar{L}| > e^{-12B\eta\epsilon}|R - L|, |\bar{R'} - \underline{L'}| < e^{11B\eta\epsilon}|R' - L'|$$

So we have the privacy loss is bounded by:

$$\frac{|\underline{R} - \bar{L}|}{|\bar{R'} - \underline{L'}|} > \frac{e^{-12B\eta\epsilon}|R - L|}{e^{11B\eta\epsilon}|R' - L'|} = e^{-23B\eta\epsilon - \epsilon}$$

11

case $x = \lfloor f(a') \rceil_\Lambda$

This case is symmetric with the case where $x = \lfloor f(a') \rceil_\Lambda$. It can also be split into 3 subcases by: $\lfloor f(a') \rceil_\Lambda < 0$, $\lfloor f(a') \rceil_\Lambda = 0$ and $\lfloor f(a') \rceil_\Lambda > 0$. Without loss of generalization, we consider the worst case where the error propagate in the same direction, i.e. $\lfloor f(a') \rceil_\Lambda < 0$.

From this assumption, let $y_1 = x - (\frac{\Lambda}{2})$, $y_2 = x + (\frac{\Lambda}{2})$, we know $y_1 < 0$, $y_2 < 0$. Since $f(a) + 1 = f(a')$, we also have $\lfloor f(a) \rceil < \lfloor f(a') \rceil < 0$. So, we know s can only be -1 for input a but s can be 1 or -1 for input a'.

For input a', when $s = 1$, we have following derivations:

$$
\frac{R'_+ \Downarrow r_+, (R'_+, R'_+)}
{\ln(R'_+) \Downarrow \textcircled{ln}(r_+), (\ln(R'_+)(1+\eta), \frac{\ln(R'_+)}{1+\eta})}
$$

$$
\frac{\frac{1}{\epsilon}\ln(R) \Downarrow \frac{1}{\epsilon} \otimes \textcircled{ln}(r_+), (\frac{1}{\epsilon}\ln(R'_+)(1+\eta)^2, \frac{1}{\epsilon}\frac{\ln(R'_+)}{(1+\eta)^2})}
{f(a) + \frac{1}{\epsilon}\ln(R'_+) \Downarrow f(a) \oplus \frac{1}{\epsilon} \otimes \textcircled{ln}(r_+), \left((f(a) + \frac{1}{\epsilon}\ln(R'_+)(1+\eta)^2)(1+\eta), (f(a) + \frac{1}{\epsilon}\frac{\ln(R'_+)}{(1+\eta)^2})/(1+\eta)\right)}
$$

$$
\text{Snap}'_{\mathbb{R}}(a, 1, R'_+) \Downarrow \text{Snap}'_{\mathbb{F}}(a, 1, r_+), (err_1, err_2)
$$

From soundness theorem, we have $err_1 \le y_2 \le err_2$. Then we can get following bounds for r:

$R'_+ = e^{\epsilon\left((y_2(1+\eta) - f(a'))(1+\eta)^2\right)}$, $\bar{R}'_+ = e^{\epsilon\frac{(\frac{y_2}{1+\eta} - f(a'))}{(1+\eta)^2}}$.

Since $y_2 = \lfloor f(a) \rceil + \frac{\Lambda}{2}$, we have $e^{\epsilon\left((y_2 - f(a))\right)} > 1$, so actually we know $R'_+ = r'_+ = 1$.

We can also derive the bound for l in the same way as:

$L'_+ = e^{\epsilon\left((y_1(1+\eta) - f(a'))(1+\eta)^2\right)}$, $\bar{L}'_+ = e^{\epsilon\frac{(\frac{y_1}{1+\eta} - f(a'))}{(1+\eta)^2}}$.

When $s = -1$, we can derive following bounds in the same way for l and r:

$L'_- = e^{\epsilon\left((f(a') - y_2(1+\eta))(1+\eta)^2\right)}$, $\bar{L}'_- = e^{\epsilon\frac{(f(a') - \frac{y_2}{1+\eta})}{(1+\eta)^2}}$.

$R'_- = e^{\epsilon\left((f(a') - y_1(1+\eta))(1+\eta)^2\right)}$, $\bar{R}'_- = e^{\epsilon\frac{(f(a') - \frac{y_1}{1+\eta})}{(1+\eta)^2}}$.

Since $y_1 = \lfloor f(a') \rceil - \frac{\Lambda}{2}$, we have $e^{\epsilon\left((f(a') - y_1))\right)} > 1$, so actually we know $R'_- = r'_- = 1$.

For input a, we have only one case where $s = -1$, the following bound can be derived:

$R = e^{\epsilon\left(f(a) - (y_2(1+\eta))(1+\eta)^2\right)}$, $\bar{R} = e^{\epsilon\frac{(f(a) - \frac{y_2}{1+\eta})}{(1+\eta)^2}}$.

$L = e^{\epsilon\left((f(a) - y_1(1+\eta))(1+\eta)^2\right)}$, $\bar{L} = e^{\epsilon\frac{f(a) - \frac{y_1}{1+\eta}}{(1+\eta)^2}}$.

We have following bounds on their ratios:

$$\frac{R'_+}{R'_+} = e^{\epsilon\left((1+\eta)^3 y_2 - (1+\eta)^2 f(a) - y_2 + f(a)\right)} > e^{-3\epsilon B\eta}, \frac{\bar{R}'_+}{R'_+} = e^{\epsilon\left(frac y_2(1+\eta)^3 - \frac{f(a)}{(1+\eta)^2} - y_2 + f(a)\right)} < e^{3\epsilon B\eta},$$

The same bound for L'_+ by substituting y_2 with y_1, and similar bound for L, R.

$$\frac{R}{R} = e^{\epsilon\left((1+\eta)^2 f(a) - (1+\eta)^3 y_2 - f(a) + y_2\right)} > e^{-2\epsilon B\eta}, \frac{\bar{R}}{R} = e^{\epsilon\left(\frac{f(a)}{(1+\eta)^2} - frac y_2(1+\eta)^3 - f(a) + y_2\right)} < e^{2\epsilon B\eta},$$

Using the bound on their ratios, we can get following bounds on $|\bar{R}'_- - L'_-|$ and $|R - \bar{L}|$:

$$|\bar{R}'_- - L'_-| < e^{3\epsilon B\eta} R - e^{-3\epsilon B\eta} L < (R'_- - L'_-)e^{7\epsilon B\eta}, |R - \bar{L}| > e^{-2\epsilon B\eta} R - e^{2\epsilon B\eta} L > (R - L)e^{-5\epsilon B\eta}$$

Then we have the following bounds on privacy loss:

$$\frac{\underline{R} - \bar{L}}{2 - (L'_+ + L'_-)} > \frac{\underline{R} - \bar{L}}{\bar{R}'_- - \underline{L}'_-} > \frac{e^{-5\epsilon B\eta}(R - L)}{e^{7\epsilon B\eta}(R'_- - L'_-)} = e^{-12\epsilon B\eta - \epsilon}$$

case $x \in (\lfloor f(a')\rceil_\Lambda, B)$

This case can also be split into 3 subcases symmetric with the case where $x \in (-B, \lfloor f(a)\rceil_\Lambda)$:

subcase $\lfloor f(a')\rceil_\Lambda > 0 \vee \lfloor f(a')\rceil_\Lambda < 0 \wedge x \in (0, B)$

let $y_1 = x - \frac{\Lambda}{2}$, $y_2 = x + \frac{\Lambda}{2}$, we have $y_1, y_2 > 0$. The bounds derived for l, r and l', r' under input a and a' in this case are as follows:

For input a':

$\underline{R}' = e^{\epsilon\left((f(a') - \frac{y_2}{1+\eta})(1+\eta)^2)\right)}$, $\bar{R}' = e^{\epsilon\frac{(f(a') - y_2(1+\eta))}{(1+\eta)^2}}$.

$\underline{L}' = e^{\epsilon\left((f(a') - \frac{y_1}{1+\eta}))(1+\eta)^2)\right)}$, $\bar{L}' = e^{\epsilon\frac{(f(a') - y_1(1+\eta)}{(1+\eta)^2}}$.

For input a:

$\underline{R} = e^{\epsilon\left((f(a) - \frac{y_2}{1+\eta})(1+\eta)^2)\right)}$, $\bar{R} = e^{\epsilon\frac{(f(a) - y_2(1+\eta))}{(1+\eta)^2}}$.

$\underline{L} = e^{\epsilon\left((f(a) - \frac{y_1}{1+\eta})(1+\eta)^2)\right)}$, $\bar{L} = e^{\epsilon\frac{f(a) - y_1(1+\eta)}{(1+\eta)^2}}$. The bounds on their ratio are as follows:

$$\frac{\underline{R}}{R} > e^{-3B\eta\epsilon}, \frac{\bar{R}}{R} < e^{3B\eta\epsilon}$$

And the bounds on $|\underline{R} - \bar{L}|$ and $|\bar{R}' - \underline{L}'|$ are as follows:

$$|\underline{R} - \bar{L}| > e^{-7B\eta\epsilon}|R - L|, \ |\bar{R}' - \underline{L}'| < e^{7B\eta\epsilon}|R' - L'|$$

So we have the privacy loss is bounded by:

$$\frac{|\underline{R} - \bar{L}|}{|\bar{R}' - \underline{L}'|} > \frac{e^{-7B\eta\epsilon}|R - L|}{e^{7B\eta\epsilon}|R' - L'|} = e^{-14B\eta\epsilon - \epsilon}$$

subcase $\lfloor f(a')\rceil_\Lambda < 0 \wedge x \in (\lfloor f(a')\rceil_\Lambda, 0)$

let $y_1 = x - \frac{\Lambda}{2}$, $y_2 = x - \frac{\Lambda}{2}$, we have $y_1, y_2 < 0$. The bounds derived for l, r in this case are as follows:

For input a':

$\underline{R}' = e^{\epsilon\left((f(a') - y_2(1+\eta))(1+\eta)^2)\right)}$, $\bar{R}' = e^{\epsilon\frac{(f(a') - \frac{y_2}{1+\eta})}{(1+\eta)^2}}$.

$\underline{L}' = e^{\epsilon\left((f(a') - y_1(1+\eta))(1+\eta)^2)\right)}$, $\bar{L}' = e^{\epsilon\frac{(f(a') - \frac{y_1}{1+\eta})}{(1+\eta)^2}}$.

For input a:

$\underline{R} = e^{\epsilon\left((f(a) - y_2(1+\eta))(1+\eta)^2)\right)}$, $\bar{R} = e^{\epsilon\frac{(f(a) - \frac{y_2}{1+\eta})}{(1+\eta)^2}}$.

$\underline{L} = e^{\epsilon\left((f(a) - y_1(1+\eta))(1+\eta)^2)\right)}$, $\bar{L} = e^{\epsilon\frac{(f(a) - \frac{y_1}{1+\eta})}{(1+\eta)^2}}$.

The bounds on their ratio are as follows:

$$\frac{\underline{R}}{R} > e^{-5B\eta\epsilon}, \frac{\bar{R}}{R} < e^{5B\eta\epsilon}$$

And the bounds on $|\underline{R} - \bar{L}|$ and $|\bar{R}' - \underline{L}'|$ are as follows:

$$|\underline{R} - \bar{L}| > e^{-12B\eta\epsilon}|R - L|, \ |\bar{R}' - \underline{L}'| < e^{11B\eta\epsilon}|R' - L'|$$

13

So we have the privacy loss is bounded by:

$$\frac{|\underline{R} - \bar{L}|}{|\bar{R}' - \underline{L}'|} > \frac{e^{-12B\eta\epsilon}|R - L|}{e^{11B\eta\epsilon}|R' - L'|} = e^{-23B\eta\epsilon - \epsilon}$$

subcase $\lfloor f(a') \rceil_\Lambda < 0 \wedge x = 0$

let $y_1 = x - \frac{\Lambda}{2}$, $y_2 = x - \frac{\Lambda}{2}$, we have $y_1 < 0$ and $y_2 > 0$. The bounds derived for l, r in this case are as follows:

For input a':

$\underline{R}' = e^{\epsilon\left((f(a') - \frac{y_2}{1+\eta})(1+\eta)^2)\right)}$, $\bar{R}' = e^{\epsilon\frac{(f(a')-y_2(1+\eta))}{(1+\eta)^2}}$.

$\underline{L}' = e^{\epsilon\left((f(a') - y_1(1+\eta))(1+\eta)^2)\right)}$, $\bar{L}' = e^{\epsilon\frac{f(a')-\frac{y_1}{1+\eta}}{(1+\eta)^2}}$.

For input a:

$\underline{R} = e^{\epsilon\left((f(a) - \frac{y_2}{1+\eta})(1+\eta)^2)\right)}$, $\bar{R} = e^{\epsilon\frac{(f(a)-y_2(1+\eta))}{(1+\eta)^2}}$.

$\underline{L} = e^{\epsilon\left((f(a) - y_1(1+\eta))(1+\eta)^2)\right)}$, $\bar{L} = e^{\epsilon\frac{f(a)-\frac{y_1}{1+\eta}}{(1+\eta)^2}}$.

The bounds on their ratio are as follows:

$$\frac{\underline{R}}{R} > e^{-3B\eta\epsilon}, \quad \frac{\bar{R}}{R} < e^{3B\eta\epsilon} \quad \frac{L}{L} > e^{-5B\eta\epsilon}, \quad \frac{\bar{L}}{L} < e^{5B\eta\epsilon}$$

And the bounds on $|\underline{R} - \bar{L}|$ and $|\bar{R}' - \underline{L}'|$ are as follows:

$$|\underline{R} - \bar{L}| > e^{-8B\eta\epsilon}|R - L|, \quad |\bar{R}' - \underline{L}'| < e^{8B\eta\epsilon}|R' - L'|$$

So we have the privacy loss is bounded by:

$$\frac{|\underline{R} - \bar{L}|}{|\bar{R}' - \underline{L}'|} > \frac{e^{-8B\eta\epsilon}|R - L|}{e^{8B\eta\epsilon}|R' - L'|} = e^{-16B\eta\epsilon - \epsilon}$$

case $x = B$

We know $s = -1$, $L = l = 0$ and $R = b$, so we only need to estimate the right side range r in this case. The bounds derived for r, r' are as following:

$$\underline{R} = e^{\epsilon\left((f(a) - \frac{x}{1+\eta})(1+\eta)^2)\right)}, \bar{R} = e^{\epsilon\frac{(f(a)-x(1+\eta))}{(1+\eta)^2}}$$

$$\underline{R}' = e^{\epsilon\left((f(a') - \frac{x}{1+\eta})(1+\eta)^2)\right)}, \bar{R}' = e^{\epsilon\frac{(f(a')-x(1+\eta))}{(1+\eta)^2}}$$

The privacy loss of $\mathsf{Snap}_\mathbb{F}(a)$ in this case is bounded by:

$$\frac{\frac{1}{2}(\underline{R} - 0)}{\frac{1}{2}(\bar{R}' - 0)} = e^{\epsilon\left(\left((f(a) - \frac{x}{1+\eta})(1+\eta)^2)\right) - \frac{(f(a')-x(1+\eta))}{(1+\eta)^2}\right)}$$

$$= e^{\epsilon\left(f(a)(1+\eta)^2 - x(1+\eta) - \frac{f(a)}{(1+\eta)^2} + \frac{x}{(1+\eta)}\right)} \quad (\star)$$

Since $1 + 2.1\eta > (1 + \eta)^2 > 1 + 2\eta$ and $\frac{1}{(1+\eta)^2} > 1 - 2\eta$, we have:

$$(\star) \quad > e^{\epsilon\left((1+2\eta)f(a) - \frac{\eta(\eta+2)}{1+\eta}x - \frac{1}{1+2\eta}(f(a)+1)\right)}$$

$$= e^{\epsilon\left(\frac{4\eta(\eta+1)}{1+2\eta}f(a) - \frac{\eta(\eta+2)}{1+\eta}x - \frac{1}{1+2\eta}\right)}$$

$$> e^{\epsilon\left(-B\eta\frac{4(\eta+1)}{1+2\eta} + \frac{(\eta+2)}{1+\eta}x - 1\right)}$$

$$> e^{\epsilon(-6\eta B - 1)}$$

$\square$

14

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