

Verifying Snapping Mechanism - Floating Point Implementation Version

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In order to verify the differential privacy property of an implementation of the snapping mechanism [5], we follow the logic rules designed from [1] and the floating point error semantics from [7, 4, 2, 6].

1 Preliminary Definitions

Definition 1 (Laplace mechanism [3])

Let $\epsilon > 0$. The Laplace mechanism $\mathcal{L}_\epsilon: \mathbb{R} \rightarrow \text{Distr}(\mathbb{R})$ is defined by $\mathcal{L}(t) = t + v$, where $v \in \mathbb{R}$ is drawn from the Laplace distribution $\text{laplce}(\frac{1}{\epsilon})$.

2 Syntax

Following are the syntax of the system. The circled operators are rounded operation in floating point computation.

Floating Point Expr.	$e_{\mathbb{F}}$	$::=$	$c \mid x \mid f(x) \mid e_{\mathbb{F}} \odot e_{\mathbb{F}} \mid \textcircled{\text{op}}(e_{\mathbb{F}}) \mid x \stackrel{\$}{\leftarrow} \mu$
Real Expr.	$e_{\mathbb{R}}$	$::=$	$r \mid X \mid F(X) \mid e_{\mathbb{R}} * e_{\mathbb{R}} \mid \ln(e_{\mathbb{R}}) \mid X \stackrel{\$}{\leftarrow} \mu$
Arithmetic Operation	$*$	$::=$	$+ \mid - \mid \times \mid \div$
Value	v	$::=$	$r \mid c$
Distribution	μ	$::=$	$\text{laplce} \mid \text{unif} \mid \text{bernoulli}$
Error	err	$::=$	$(e_{\mathbb{R}}, e_{\mathbb{R}})$

We use upper case for variables in real computation and lower case for variables in floating point computation. \odot represents the operation in floating point machine.

$F(X)$ denotes function F evaluates to value $F(X)$ given input X in real computation, and $f(x)$ denotes the same function F evaluates to value $f(x)$ given the same input x in floating point computation.

3 Semantics

The big step semantics with relative floating point computation error are shown in Figure. 1. The semantics are $e_{\mathbb{R}} \Downarrow e_{\mathbb{F}}, err$, which means a real world expression $e_{\mathbb{R}}$ can be represented in floating point computation $e_{\mathbb{F}}$ with error bound err . The η is the machine epsilon.

$$\begin{array}{c}
\frac{c = \text{fl}(r)}{r \Downarrow c, \left(\frac{r}{(1+\eta)}, r(1+\eta)\right)} \text{CONST} \qquad \frac{e_{\mathbb{R}}^1 \Downarrow e_{\mathbb{F}}^1, (e_{\mathbb{R}}^1, \bar{e}_{\mathbb{R}}^1) \quad e_{\mathbb{R}}^2 \Downarrow e_{\mathbb{F}}^2, (e_{\mathbb{R}}^2, \bar{e}_{\mathbb{R}}^2)}{e_{\mathbb{R}}^1 * e_{\mathbb{R}}^2 \Downarrow \text{fl}(e_{\mathbb{F}}^1 \odot e_{\mathbb{F}}^2), \left(\frac{e_{\mathbb{R}}^1 * e_{\mathbb{R}}^2}{(1+\eta)}, (e_{\mathbb{R}}^1 * e_{\mathbb{R}}^2)(1+\eta)\right)} \text{OP} \\
\\
\frac{e_{\mathbb{R}} \Downarrow e_{\mathbb{F}}, (e_{\mathbb{R}}, \bar{e}_{\mathbb{R}}) \quad e_{\mathbb{R}} \geq 1}{\ln(e_{\mathbb{R}}) \Downarrow \mathbb{D}(e_{\mathbb{F}}), \left(\frac{\mathbb{D}(e_{\mathbb{R}})}{(1+\eta)}, (\ln(e_{\mathbb{R}}))(1+\eta)\right)} \text{LN} \quad \frac{e_{\mathbb{R}} \Downarrow e_{\mathbb{F}}, (e_{\mathbb{R}}, \bar{e}_{\mathbb{R}}) \quad e_{\mathbb{R}} < 1}{\ln(e_{\mathbb{R}}) \Downarrow \mathbb{D}(e_{\mathbb{F}}), \left(\ln(e_{\mathbb{R}})(1+\eta), \frac{\mathbb{D}(\bar{e}_{\mathbb{R}})}{(1+\eta)}\right)} \text{LN-OP}
\end{array}$$

Figure 1: Semantics with Relative Floating Point Error

Theorem 1 (Soundness Theorem)

Given $e_{\mathbb{R}}$ and $e_{\mathbb{F}}$ where $e_{\mathbb{R}} \Downarrow e_{\mathbb{F}}, err$, when evaluating the $e_{\mathbb{F}}$ in floating point computation and get the value c , we have $c \in err$.

4 Snapping Mechanism

Definition 2 ($\text{Snap}_{\mathbb{R}}(a) : A \rightarrow \text{Distr}(\mathbb{R})$)

Given privacy parameter ϵ , the ideal Snapping mechanism $\text{Snap}_{\mathbb{R}}(a)$ is defined as:

$$U \xleftarrow{\$} \mu; S \xleftarrow{\$} \{-1, 1\}; Y = \ln(U) \div \epsilon; Z = S \times Y; X = F(a); W = X + Z; W' = \lfloor W \rfloor_{\Lambda}; R = \text{clamp}_B(W')$$

where f is the query function over input $a \in A$, ϵ is the privacy budget, B is the clamping bound and Λ is the rounding argument satisfying $\lambda = 2^k$ where 2^k is the smallest power of 2 greater or equal to the $\frac{1}{\epsilon}$.

Let $\text{Snap}'_{\mathbb{R}}(a, U, S)$ be the same as $\text{Snap}_{\mathbb{R}}(a)$ given U, S without rounding and clamping steps.

Definition 3 ($\text{Snap}_{\mathbb{F}}(a) : A \rightarrow \text{Distr}(\mathbb{R})$)

Given privacy parameter ϵ , the floating point implemented Snapping mechanism $\text{Snap}_{\mathbb{F}}(a)$ is defined as (where all parameters are defined the same as above):

$$u_{\mathbb{F}} \xleftarrow{\$} \mu; s_{\mathbb{F}} \xleftarrow{\$} \{-1, 1\}; y = \mathbb{D}(u) \odot \epsilon; z = s \otimes y; x = f(a); w = x \oplus z; w' = \lfloor w \rfloor_{\Lambda}; r = \text{clamp}_B(w')$$

Let $\text{Snap}'_{\mathbb{F}}(a, u, s)$ be the same as $\text{Snap}_{\mathbb{F}}(a)$ without rounding and clamping precesses given u, s .

5 Main Theorem

Theorem 2 (The Snap mechanism is ϵ -differentially private)

Consider $\text{Snap}(a)$ defined as before, if $\text{Snap}(a) = x$ given database a and privacy parameter ϵ , then its actual privacy loss is bounded by $\epsilon + 12xe\eta + 2\eta$

Proof. Given $\text{Snap}_{\mathbb{F}}(a) = x$ and parameter ϵ , we consider a' be the adjacent database of a satisfying $|f(a) - f(a')| \leq 1$. Without loss of generalization, we assume $f(a) + 1 = f(a')$ (\diamond). The proof is developed by cases of the output of $\text{Snap}_{\mathbb{F}}(a)$ mechanism.

Consider the $\text{Snap}_{\mathbb{R}}(a)$ outputting the same result x , let (L, R) be the range where $\forall u \in (L, R)$ and some s , $\text{Snap}'_{\mathbb{R}}(a, u, s) = x$, we have $\Pr[\text{Snap}_{\mathbb{R}}(a)] = R - L$. Given the $\text{Snap}_{\mathbb{R}}$ is ϵ -dp, we have:

$$e^{-\epsilon} \leq \frac{\Pr[\text{Snap}_{\mathbb{R}}(a)]}{\Pr[\text{Snap}_{\mathbb{R}}(a)]} = \frac{R - L}{R' - L'} \leq e^{\epsilon}$$

Let (l, r) be the range where $\forall u \in (l, r)$ and some s , $\text{Snap}'_{\mathbb{F}}(a, u, s) = x$, we estimated the $|r - l|$ in terms of floating point relative error and $|R - L|$ through our semantics in order to verify the privacy loss of $\text{Snap}_{\mathbb{F}}$.

case $x = -B$

Let b be the largest number rounded by Λ that is smaller than B . We know $s = 1$, $L = l = 0$ and $R = -b$, so we only need to estimate the right side range r in this case. The derivation of this case given $\text{Snap}'_{\mathbb{F}}(a, R, 1) = \text{Snap}'_{\mathbb{F}}(a', R, 1) = x$ is shown as following:

$$\begin{array}{c}
 \text{LN} \\
 R \Downarrow r, (\underline{R}, \bar{R}) \\
 \hline
 \text{OP} \\
 \ln(R) \Downarrow \textcircled{\cap}(r), (\ln(\underline{R})(1+\eta), \frac{\ln(\bar{R})}{(1+\eta)}) \\
 \hline
 \text{OP} \\
 \frac{1}{\epsilon} \times \ln(R) \Downarrow \frac{1}{\epsilon} \otimes \textcircled{\cap}(r), ((\frac{1}{\epsilon} \times \ln(\underline{R}))(1+\eta)^2, \frac{\frac{1}{\epsilon} \times \ln(\bar{R})}{(1+\eta)^2}) \\
 \hline
 \text{ID} \\
 f(a) + \frac{1}{\epsilon} \times \ln(R) \Downarrow f(a) \oplus \frac{1}{\epsilon} \otimes \textcircled{\cap}(r), \left((f(a) + (\frac{1}{\epsilon} \times \ln(\underline{R}))(1+\eta)^2)(1+\eta), \frac{(f(a) + \frac{\frac{1}{\epsilon} \times \ln(\bar{R})}{(1+\eta)^2})}{(1+\eta)} \right) \\
 \hline
 \text{Snap}'_{\mathbb{R}}(a, R, 1) \Downarrow \text{Snap}'_{\mathbb{F}}(a, r, 1), \left((f(a) + (\frac{1}{\epsilon} \times \ln(\underline{R}))(1+\eta)^2)(1+\eta), \frac{(f(a) + \frac{\frac{1}{\epsilon} \times \ln(\bar{R})}{(1+\eta)^2})}{(1+\eta)} \right)
 \end{array}$$

In the same way, we have the derivation for $\text{Snap}'_{\mathbb{F}}(a', r, 1)$:

$$\begin{array}{c}
 \dots \\
 \hline
 \text{Snap}'_{\mathbb{R}}(a', R', 1) \Downarrow \text{Snap}'_{\mathbb{F}}(a', r, 1), \left((f(a') + (\frac{1}{\epsilon} \times \ln(\underline{R}'))(1+\eta)^2)(1+\eta), \frac{(f(a') + \frac{\frac{1}{\epsilon} \times \ln(\bar{R}')}{(1+\eta)^2})}{(1+\eta)} \right)
 \end{array}$$

Given $\text{Snap}_{\mathbb{F}}(a) = \text{Snap}_{\mathbb{F}}(a') = x = -b$, we have following values for \underline{R} , \bar{R} , \underline{R}' and \bar{R}' :

$$\begin{aligned}
 \underline{R} &= e^{\epsilon((x(1+\eta)-f(a))(1+\eta)^2)}, \bar{R} = e^{\epsilon(\frac{\frac{x}{1+\eta}-f(a)}{(1+\eta)^2})} \\
 \underline{R}' &= e^{\epsilon((x(1+\eta)-f(a'))(1+\eta)^2)}, \bar{R}' = e^{\epsilon(\frac{\frac{x}{1+\eta}-f(a')}{(1+\eta)^2})}
 \end{aligned}$$

The privacy loss of $\text{Snap}_{\mathbb{F}}(a)$ in this case is bounded by:

$$\begin{aligned}
 \frac{\frac{1}{2}(\bar{R}-0)}{\frac{1}{2}(\underline{R}'-0)} &= e^{\epsilon\left(\frac{\frac{x}{1+\eta}-f(a)}{(1+\eta)^2} - ((x(1+\eta)-f(a'))(1+\eta)^2)\right)} \\
 &= e^{\epsilon\left(\frac{x}{(1+\eta)^3} - \frac{f(a)}{(1+\eta)^2} - x(1+\eta)^3 + f(a')(1+\eta)^2\right)} \quad (\star)
 \end{aligned}$$

Since $(1 + \eta)^3 > 1 + 3\eta$, $\frac{1}{(1+\eta)^3} < \frac{1}{1+3\eta}$, $(1 + \eta)^2 < 1 + 2.1\eta$ and $\frac{1}{(1+\eta)^2} > 1 - 2\eta$, we have:

$$\begin{aligned} (\star) \quad &< e^{\epsilon \left(-\frac{9\eta+6}{1+3\eta}x + 4.1\eta f(a) + (1+2.1\eta) \right)} \\ &< e^{\epsilon(10.1\eta B + 1 + 2.1\eta)} \end{aligned}$$

case $x \in (-B, \lfloor f(a) \rfloor_\Lambda)$

Let $y_1 = x - (\frac{\Lambda}{2})$, $y_2 = x + (\frac{\Lambda}{2})$, we know $S = s = 1$, $L = e^{\epsilon(y_1 - f(a))}$ and $R = e^{\epsilon(y_2 - f(a))}$ in this case. The derivations of estimating l and r are shown as following:

$$\begin{array}{c} L \Downarrow l, (\underline{L}, \bar{L}) \\ \hline \ln(L) \Downarrow \textcircled{\mathbb{N}}(l), (\ln(\underline{L})(1 + \eta), \frac{\ln(\bar{L})}{(1 + \eta)}) \\ \hline \frac{1}{\epsilon} \times \ln(L) \Downarrow \frac{1}{\epsilon} \otimes \textcircled{\mathbb{N}}(l), ((\frac{1}{\epsilon} \times \ln(\underline{L}))(1 + \eta)^2, \frac{\frac{1}{\epsilon} \times \ln(\bar{L})}{(1 + \eta)^2}) \\ \hline f(a) + \frac{1}{\epsilon} \times \ln(L) \Downarrow f(a) \oplus \frac{1}{\epsilon} \otimes \textcircled{\mathbb{N}}(l), (\frac{f(a) + (\frac{1}{\epsilon} \times \ln(\underline{L}))(1 + \eta)^2}{1 + \eta}, (f(a) + \frac{\frac{1}{\epsilon} \times \ln(\bar{L})}{(1 + \eta)^2})(1 + \eta)) \\ \hline \text{Snap}'_{\mathbb{R}}(a, L, 1) \Downarrow \text{Snap}'_{\mathbb{F}}(a, l, 1), (err_1, err_2) \end{array}$$

From soundness theorem, we have $err_1 \leq y_2 \leq err_2$.

Taking the lower bound (i.e. $err_1 = y_1$), we get: $\underline{L} = e^{(y_1/(1+\eta) - f(a))(1+\eta)^2\epsilon}$. Taking the upper bound (i.e. $err_2 = y_1$), we get: $\bar{L} = e^{(y_1(1+\eta) - f(a))\epsilon/(1+\eta)^2}$.

$$\begin{array}{c} \dots \\ \hline \text{Snap}'_{\mathbb{R}}(a, R, 1) \Downarrow \text{Snap}'_{\mathbb{F}}(a, r, 1), (\frac{f(a) + (\frac{1}{\epsilon} \times \ln(\underline{R}))(1 + \eta)^2}{1 + \eta}, (f(a) + \frac{\frac{1}{\epsilon} \times \ln(\bar{R})}{(1 + \eta)^2})(1 + \eta)) \\ \hline \text{Snap}'_{\mathbb{R}}(a, R, 1) \Downarrow \text{Snap}'_{\mathbb{F}}(a, r, 1), (err_1, err_2) \end{array}$$

From soundness theorem, we have $err_1 \leq y_2 \leq err_2$.

Taking the lower bound (i.e. $err_1 = y_2$), we have: $\underline{R} = e^{(y_2/(1+\eta) - f(a))(1+\eta)^2\epsilon}$. Taking the upper bound (i.e. $err_2 = y_1$), we have: $\bar{R} = e^{(y_2(1+\eta) - f(a))\epsilon/(1+\eta)^2}$.

In the same way, we have the derivation for $\text{Snap}'_{\mathbb{F}}(a', l, 1)$ and $\text{Snap}'_{\mathbb{F}}(a', r, 1)$:

$$\begin{array}{c} \dots \\ \hline \text{Snap}'_{\mathbb{R}}(a', L', 1) \Downarrow \text{Snap}'_{\mathbb{F}}(a', l', 1), (\frac{f(a') + (\frac{1}{\epsilon} \times \ln(\underline{L}'))(1 + \eta)^2}{1 + \eta}, (f(a') + \frac{\frac{1}{\epsilon} \times \ln(\bar{L}')}{(1 + \eta)^2})(1 + \eta)) \\ \hline \end{array}$$

From soundness theorem, we have $err_1 \leq y_2 \leq err_2$.

Taking the lower bound (i.e. $err_1 = y_1$), we get: $\underline{L} = e^{(y_1/(1+\eta) - f(a'))(1+\eta)^2\epsilon}$. Taking the upper bound (i.e. $err_2 = y_1$), we get: $\bar{L} = e^{(y_1(1+\eta) - f(a'))\epsilon/(1+\eta)^2}$.

$$\begin{array}{c} \dots \\ \hline \text{Snap}'_{\mathbb{R}}(a', R', 1) \Downarrow \text{Snap}'_{\mathbb{F}}(a', r', 1), (\frac{f(a') + (\frac{1}{\epsilon} \times \ln(\underline{R}'))(1 + \eta)^2}{1 + \eta}, (f(a') + \frac{\frac{1}{\epsilon} \times \ln(\bar{R}')}{(1 + \eta)^2})(1 + \eta)) \\ \hline \end{array}$$

From soundness theorem, we have $err_1 \leq y_2 \leq err_2$.

Taking the lower bound (i.e. $err_1 = y_2$), we have: $\bar{R} = e^{(y_2/(1+\eta) - f(a'))(1+\eta)^2\epsilon}$. Taking the upper bound (i.e. $err_2 = y_1$), we have: $\bar{R} = e^{(y_2(1+\eta) - f(a'))\epsilon/(1+\eta)^2}$.

The privacy loss is bounded by:

$$\frac{|\bar{R} - \underline{L}|}{|\bar{R}' - \underline{L}'|}$$

Since the following bound can be proved by using $1 - 2\eta < (1 + \eta)^2 < 1 + 2.1\eta$, $y_1 > -B$, $y_2 > -B$ and simple approximation:

$$\bar{R} - \underline{L} < (R - L)e^{(5B\eta\epsilon)}, \bar{R}' - \underline{L}' > (R' - L')e^{-7B\eta\epsilon}$$

We also have the $\text{Snap}_{\mathbb{R}}(a)$ is ϵ -dp:

$$\frac{|R - L|}{|R' - L'|} = e^\epsilon$$

So we can get:

$$\frac{|\bar{R} - \underline{L}|}{|\bar{R}' - \underline{L}'|} < \frac{|R - L|}{|R' - L'|} e^{(12B\eta\epsilon)} = e^{(1+12B\eta)\epsilon}$$

case $x = \lfloor f(a) \rfloor_\Lambda$

[[where $r_1, \bar{r}_1, r_2, \bar{r}_2, r'_1, \bar{r}'_1, r'_2, \bar{r}'_2$ have following values: Given that the probability is equivalent to the length of the range, we have the ratio between u and u' is bounded by:

$$\frac{u}{u'} \leq \frac{1 - \frac{1}{2}(r'_2 + r'_1)}{\frac{1}{2}(r'_2 - r'_1)} \leq \epsilon + 12x\epsilon\eta + 2\eta$$

By the AxUnif rule, we have the actual privacy loss is bounded by the same value.]]

case $x \in (\lfloor f(a) \rfloor_\Lambda, \lfloor f(a') \rfloor_\Lambda)$

Following the semantics in Figure 1, we have following evaluation results:

$$u \in ((r_1, \bar{r}_1), (r_2, \bar{r}_2)] \wedge (s = 1) \sim u' \in [(r'_1, \bar{r}'_1), (r'_2, \bar{r}'_2)) \wedge (s = -1),$$

[[where $r_1, \bar{r}_1, r_2, \bar{r}_2, r'_1, \bar{r}'_1, r'_2, \bar{r}'_2$ have following values: Given that the probability is equivalent to the length of the range, we have the ratio between u and u' is bounded by:

$$\frac{u}{u'} \leq \frac{\frac{1}{2}(\bar{r}_2 - r_1)}{\frac{1}{2}(r'_2 - r'_1)} \leq \epsilon + 12x\epsilon\eta + 2\eta$$

By the AxUnif rule, we have the actual privacy loss is bounded by the same value.]]

case $x = \lfloor f(a') \rfloor_\Lambda$

Following the semantics in Figure 1, we have following evaluation results:

$$u \in ((r_1, \bar{r}_1), (r_2, \bar{r}_2)] \wedge (s = 1) \sim u' \in [(r'_1, \bar{r}'_1), 1] \wedge (s = -1) \vee [(r'_2, \bar{r}'_2), 1] \wedge (s = 1),$$

[[where $r_1, \bar{r}_1, r_2, \bar{r}_2, r'_1, \bar{r}'_1, r'_2, \bar{r}'_2$ have following values: Given that the probability is equivalent to the length of the range, we have the ratio between u and u' is bounded by:

$$\frac{u}{u'} \leq \frac{\frac{1}{2}(\bar{r}_2 - r_1)}{1 - \frac{1}{2}(r'_2 + r'_1)} \leq \epsilon + 12x\epsilon\eta + 2\eta$$

By the AxUnif rule, we have the actual privacy loss is bounded by the same value.]]

case $x \in (\lfloor f(a') \rfloor_\Lambda, B)$

Following the semantics in Figure 1, we have following evaluation results:

$$u \in ((r_1, \bar{r}_1), (r_2, \bar{r}_2)] \wedge (s = 1) \sim u' \in ((r'_1, \bar{r}'_1), (r'_2, \bar{r}'_2)] \wedge (s = 1),$$

[[where $r_1, \bar{r}_1, r_2, \bar{r}_2, r'_1, \bar{r}'_1, r'_2, \bar{r}'_2$ have following values: Given that the probability is equivalent to the length of the range, we have the ratio between u and u' is bounded by:

$$\frac{u}{u'} \leq \frac{\frac{1}{2}(\bar{r}_2 - r_1)}{\frac{1}{2}(r'_2 - r'_1)} \leq \epsilon + 12x\epsilon\eta + 2\eta$$

By the AxUnif rule, we have the actual privacy loss is bounded by the same value.]]

case $x = B$

We know $s = -1$, $L = l = 0$ and $R = b$, so we only need to estimate the right side range r in this case. The derivation of this case given $\text{Snap}'_{\mathbb{F}}(a, r, -1) = \text{Snap}'_{\mathbb{F}}(a', r, -1) = x$ is shown as following:

$$\begin{array}{c} \text{LN} \\ R \Downarrow r, (\underline{R}, \bar{R}) \\ \hline \text{OP} \\ \ln(R) \Downarrow \text{ID}(r), (\ln(\underline{R})(1+\eta), \frac{\ln(\bar{R})}{(1+\eta)}) \\ \hline \text{OP} \\ \frac{1}{\epsilon} \times \ln(R) \Downarrow \frac{1}{\epsilon} \otimes \text{ID}(r), ((\frac{1}{\epsilon} \times \ln(\underline{R}))(1+\eta)^2, \frac{\frac{1}{\epsilon} \times \ln(\bar{R})}{(1+\eta)^2}) \\ \hline \text{ID} \\ f(a) + \frac{1}{\epsilon} \times \ln(R) \Downarrow f(a) \oplus \frac{1}{\epsilon} \otimes \text{ID}(r), (\frac{f(a) + (\frac{1}{\epsilon} \times \ln(\underline{R}))(1+\eta)^2}{1+\eta}, (f(a) + \frac{\frac{1}{\epsilon} \times \ln(\bar{R})}{(1+\eta)^2})(1+\eta)) \\ \hline \text{Snap}'_{\mathbb{R}}(a, R, 1) \Downarrow \text{Snap}'_{\mathbb{F}}(a, r, 1), (\frac{f(a) + (\frac{1}{\epsilon} \times \ln(\underline{R}))(1+\eta)^2}{1+\eta}, (f(a) + \frac{\frac{1}{\epsilon} \times \ln(\bar{R})}{(1+\eta)^2})(1+\eta)) \end{array}$$

In the same way, we have the derivation for $\text{Snap}'_{\mathbb{F}}(a', r, 1)$:

$$\begin{array}{c} \dots \\ \hline \text{Snap}'_{\mathbb{R}}(a', R, 1) \Downarrow \text{Snap}'_{\mathbb{F}}(a', r, 1), (\frac{f(a') + (\frac{1}{\epsilon} \times \ln(\underline{R}'))(1+\eta)^2}{1+\eta}, (f(a') + \frac{\frac{1}{\epsilon} \times \ln(\bar{R}')}{(1+\eta)^2})(1+\eta)) \end{array}$$

Given $\text{Snap}_{\mathbb{F}}(a) = \text{Snap}_{\mathbb{F}}(a') = x = b$, we have following values for $\underline{R}, \bar{R}, \underline{R}'$ and \bar{R}' :

$$\begin{aligned} \underline{R} &= e^{\epsilon(\frac{-b-\frac{\Lambda}{2}}{1+\eta} - f(a))(1+\eta)^2}, \bar{R} = e^{\frac{\epsilon(-b-\frac{\Lambda}{2})(1+\eta) - f(a)}{(1+\eta)^2}} \\ \underline{R}' &= e^{\epsilon(\frac{-b-\frac{\Lambda}{2}}{1+\eta} - f(a'))(1+\eta)^2}, \bar{R}' = e^{\frac{\epsilon(-b-\frac{\Lambda}{2})(1+\eta) - f(a')}{(1+\eta)^2}} \end{aligned}$$

The privacy loss of $\text{Snap}_{\mathbb{F}}(a)$ in this case is bounded by:

$$\frac{\frac{1}{2}(\bar{R} - 0)}{\frac{1}{2}(\underline{R}' - 0)} = e^{\epsilon((-b-\frac{\Lambda}{2})(1+\eta - \frac{1}{1+\eta}) + f(a)((1+\eta)^2 - \frac{1}{(1+\eta)^2}) + (1+\eta)^2)} \geq e^{-\epsilon(1+\eta)^2 2B} \geq e^{-(\epsilon + 12B\epsilon\eta)}$$

□

References

- [1] Gilles Barthe, Marco Gaboardi, Benjamin Grégoire, Justin Hsu, and Pierre-Yves Strub. Proving differential privacy via probabilistic couplings. In *LICS 2016*.
- [2] H. Becker, N. Zyuzin, R. Monat, E. Darulova, M. O. Myreen, and A. Fox. A verified certificate checker for finite-precision error bounds in coq and hol4. In *2018 Formal Methods in Computer Aided Design (FMCAD)*, 2018.
- [3] Cynthia Dwork, Frank McSherry, Kobbi Nissim, and Adam Smith. Calibrating Noise to Sensitivity in Private Data Analysis. In *TCC*, 2016.
- [4] Matthieu Martel. Semantics of roundoff error propagation in finite precision calculations. *Higher-Order and Symbolic Computation*, 2006.
- [5] Ilya Mironov. On significance of the least significant bits for differential privacy. In *CCS 2012*, 2012.
- [6] Mariano Moscato, Laura Titolo, Aaron Dutle, and César A. Muñoz. Automatic estimation of verified floating-point round-off errors via static analysis. In Stefano Tonetta, Erwin Schoitsch, and Friedemann Bitsch, editors, *Computer Safety, Reliability, and Security*, 2017.
- [7] Tahina Ramananandro, Paul Mountcastle, Benoundefinedt Meister, and Richard Lethin. A unified coq framework for verifying c programs with floating-point computations. In *Proceedings of the 5th ACM SIGPLAN Conference on Certified Programs and Proofs (CPP)*. Association for Computing Machinery, 2016.