Verifying Snapping Mechanism - Floating Point Implementation Version

Jiawen Liu

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In order to verify the differential privacy property of an implementation of the snapping mechanism [5], we follow the logic rules designed from [1] and the floating point error semantics from [7, 4, 2, 6].

1 Preliminary Definitions

Definition 1 (Laplace mechanism [3])
Let $\epsilon > 0$. The Laplace mechanism $\mathcal{L}_\epsilon$: $\mathbb{R} \to \mathsf{Distr}(\mathbb{R})$ is defined by $\mathcal{L}(t) = t + v$, where $v \in \mathbb{R}$ is drawn from the Laplace distribution $\mathsf{laplce}(\frac{1}{\epsilon})$.

2 Syntax

Following are the syntax of the system. The circled operators are rounded operation in floating point computation.

Floating Point Expr.	$e_{\mathbb{F}}$	$::=$	$c \mid x \mid f(x) \mid e_{\mathbb{F}} \circledast e_{\mathbb{F}} \mid \textcircled{\ln}(e_{\mathbb{F}}) \mid x \xleftarrow{\$} \mu$
Real Expr.	$e_{\mathbb{R}}$	$::=$	$r \mid X \mid F(X) \mid e_{\mathbb{R}} * e_{\mathbb{R}} \mid \ln(e_{\mathbb{R}}) \mid X \xleftarrow{\$} \mu$
Arithmetic Operation	$*$	$::=$	$+ \mid - \mid \times \mid \div$
Value	v	$::=$	$r \mid c$
Distribution	μ	$::=$	$\mathsf{laplce} \mid \mathsf{unif} \mid \mathtt{bernoulli}$
Error	err	$::=$	$(e_{\mathbb{R}}, e_{\mathbb{R}})$

We use upper case for variables in real computation and lower case for variables in floating point computation. $\circledast$ represents the operation in floating point machine.

$F(X)$ denotes function F evaluates to value $F(X)$ given input X in real computation, and $f(x)$ denotes the same function F evaluates to value $f(x)$ given the same input x in floating point computation.

3 Semantics

The big step semantics with relative floating point computation error are shown in Figure. 1. The semantics are $e_{\mathbb{R}} \Downarrow e_{\mathbb{F}}, err$, which means a real world expression $e_{\mathbb{R}}$ can be represented in floating point computation $e_{\mathbb{F}}$ with error bound err. The η is the machine epsilon.

$$\dfrac{c = \texttt{fl}(r)}{r \Downarrow c, \big(\dfrac{r}{(1+\eta)}, r(1+\eta)\big)}\ \textsc{Const} \qquad \dfrac{e^1_{\mathbb{R}} \Downarrow e^1_{\mathbb{F}}, (\underline{e}^1_{\mathbb{R}}, \bar{e}^1_{\mathbb{R}}) \qquad e^2_{\mathbb{R}} \Downarrow e^2_{\mathbb{F}}, (\underline{e}^2_{\mathbb{R}}, \bar{e}^2_{\mathbb{R}})}{e^1_{\mathbb{R}} * e^2_{\mathbb{R}} \Downarrow \texttt{fl}(e^1_{\mathbb{F}} \circledast e^2_{\mathbb{F}}), \big((\dfrac{\underline{e}^1_{\mathbb{R}} * \underline{e}^2_{\mathbb{R}})}{(1+\eta)}, (\bar{e}^1_{\mathbb{R}} * \bar{e}^2_{\mathbb{R}})(1+\eta)\big)}\ \textsc{Op}$$

$$\dfrac{e_{\mathbb{R}} \Downarrow e_{\mathbb{F}}, (\underline{e}_{\mathbb{R}}, \bar{e}_{\mathbb{R}}) \qquad e_{\mathbb{R}} \geq 1}{\ln(e_{\mathbb{R}}) \Downarrow \textcircled{ln}(e_{\mathbb{F}}), \big((\dfrac{\textcircled{ln}(\underline{e}_{\mathbb{R}})}{(1+\eta)}, (\ln(\bar{e}_{\mathbb{R}}))(1+\eta)\big)}\ \textsc{Ln} \quad \dfrac{e_{\mathbb{R}} \Downarrow e_{\mathbb{F}}, (\underline{e}_{\mathbb{R}}, \bar{e}_{\mathbb{R}}) \qquad e_{\mathbb{R}} < 1}{\ln(e_{\mathbb{R}}) \Downarrow \textcircled{ln}(e_{\mathbb{F}}), \big((\ln(\underline{e}_{\mathbb{R}}))(1+\eta), \dfrac{\textcircled{ln}(\bar{e}_{\mathbb{R}})}{(1+\eta)}\big)}\ \textsc{Ln-Op}$$

Figure 1: Semantics with Relative Floating Point Error

Theorem 1 (Soundness Theorem)
Given $e_{\mathbb{R}}$ and $e_{\mathbb{F}}$ where $e_{\mathbb{R}} \Downarrow e_{\mathbb{F}}, err$, when evaluating the $e_{\mathbb{F}}$ in floating point computation and get the value c, we have $c \in err$.

4 Snapping Mechanism

Definition 2 ($\mathsf{Snap}_{\mathbb{R}}(a) : A \to \mathsf{Distr}(\mathbb{R})$)
Given privacy parameter ϵ, the ideal Snapping mechanism $\mathsf{Snap}_{\mathbb{R}}(a)$ is defined as:

$$U \xleftarrow{\$} \mu; S \xleftarrow{\$} \{-1, 1\}; Y = \ln(U) \div \epsilon; Z = S \times Y; X = F(a); W = X + Z; W' = \lfloor W \rceil_{\Lambda}; R = \mathsf{clamp}_B(W')$$

where f is the query function over input $a \in A$, ϵ is the privacy budget, B is the clamping bound and Λ is the rounding argument satisfying $\lambda = 2^k$ where 2^k is the smallest power of 2 greater or equal to the $\frac{1}{\epsilon}$.

Let $\mathsf{Snap}'_{\mathbb{R}}(a, U, S)$ be the same as $\mathsf{Snap}_{\mathbb{F}}(a)$ given U, S without rounding and clamping steps.

Definition 3 ($\mathsf{Snap}_{\mathbb{F}}(a) : A \to \mathsf{Distr}(\mathbb{R})$)
Given privacy parameter ϵ, the floating point implemented Snapping mechanism $\mathsf{Snap}_{\mathbb{F}}(a)$ is defined as (where all parameters are defined the same as above):

$$u_{\mathbb{F}} \xleftarrow{\$} \mu; s_{\mathbb{F}} \xleftarrow{\$} \{-1, 1\}; y = \textcircled{ln}(u) \oplus\!\!\!\!\div \epsilon; z = s \otimes y; x = f(a); w = x \oplus z; w' = \lfloor w \rceil_{\Lambda}; r = \mathsf{clamp}_B(w')$$

Let $\mathsf{Snap}'_{\mathbb{F}}(a, u, s)$ be the same as $\mathsf{Snap}_{\mathbb{F}}(a)$ without rounding and clamping precesses given u, s.

5 Main Theorem

Theorem 2 (The Snap mechanism is ϵ−differentially private)
Consider $\mathsf{Snap}(a)$ defined as before, if $\mathsf{Snap}(a) = x$ given database a and privacy parameter ϵ, then its actual privacy loss is bounded by $\epsilon + 12x\epsilon\eta + 2\eta$

Proof. Given $\mathsf{Snap}_{\mathbb{F}}(a) = x$ and parameter ϵ, we consider a' be the adjacent database of a satisfying $|f(a) - f(a')| \leq 1$. Without loss of generalization, we assume $f(a) + 1 = f(a')$ $(\diamond)$. The proof is developed by cases of the output of $\mathsf{Snap}_{\mathbb{F}}(a)$ mechanism.

Consider the $\mathsf{Snap}_{\mathbb{R}}(a)$ outputting the same result x, let (L, R) be the range where $\forall u \in (L, R)$ and some s, $\mathsf{Snap}'_{\mathbb{R}}(a, u, s) = x$, we have $\Pr[\mathsf{Snap}_{\mathbb{R}}(a)] = R - L$. Given the $\mathsf{Snap}_{\mathbb{R}}$ is ϵ−dp, we have:

$$e^{-\epsilon} \leq \frac{\Pr[\mathsf{Snap}_{\mathbb{R}}(a)]}{\Pr[\mathsf{Snap}_{\mathbb{R}}(a)]} = \frac{R - L}{R' - L'} \leq e^{\epsilon}$$

Let (l, r) be the range where $\forall u \in (l, r)$ and some s, $\mathsf{Snap}'_{\mathbb{F}}(a, u, s) = x$, we estimated the $|r - l|$ in terms of floating point relative error and $|R - L|$ through our semantics in order to verify the privacy loss of $\mathsf{Snap}_{\mathbb{F}}$.

case $x = -B$

Let b be the largest number rounded by Λ that is smaller than B. We know $s = 1$, $L = l = 0$ and $R = -b$, so we only need to estimate the right side range r in this case. The derivation of this case given $\mathsf{Snap}'_{\mathbb{F}}(a, R, 1) = \mathsf{Snap}'_{\mathbb{F}}(a', R, 1) = x$ is shown as following:

$$\begin{array}{c}
\text{LN} \\
R \Downarrow r, (\underline{R}, \bar{R}) \\ \hline
\text{OP} \\
\ln(R) \Downarrow \textcircled{ln}(r), (\ln(\underline{R})(1+\eta), \dfrac{\ln(\bar{R})}{(1+\eta)}) \\ \hline
\text{OP} \\
\dfrac{1}{\epsilon} \times \ln(R) \Downarrow \dfrac{1}{\epsilon} \otimes \textcircled{ln}(r), ((\dfrac{1}{\epsilon} \times \ln(\underline{R}))(1+\eta)^2, \dfrac{\frac{1}{\epsilon} \times \ln(\bar{R})}{(1+\eta)^2}) \\ \hline
\text{ID} \\
f(a) + \dfrac{1}{\epsilon} \times \ln(R) \Downarrow f(a) \oplus \dfrac{1}{\epsilon} \otimes \textcircled{ln}(r), \Big(\big(f(a) + (\dfrac{1}{\epsilon} \times \ln(\underline{R}))(1+\eta)^2\big)(1+\eta), \dfrac{(f(a) + \frac{\frac{1}{\epsilon} \times \ln(\bar{R})}{(1+\eta)^2})}{(1+\eta)}\Big) \\ \hline
\mathsf{Snap}'_{\mathbb{R}}(a, R, 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a, r, 1), \Big(\big(f(a) + (\dfrac{1}{\epsilon} \times \ln(\underline{R}))(1+\eta)^2\big)(1+\eta), \dfrac{(f(a) + \frac{\frac{1}{\epsilon} \times \ln(\bar{R})}{(1+\eta)^2})}{(1+\eta)}\Big)
\end{array}$$

In the same way, we have the derivation for $\mathsf{Snap}'_{\mathbb{F}}(a', r, 1)$:

$$\begin{array}{c}
\ldots \\ \hline
\mathsf{Snap}'_{\mathbb{R}}(a', R', 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a', r, 1), \Big(\big(f(a') + (\dfrac{1}{\epsilon} \times \ln(\underline{R}'))(1+\eta)^2\big)(1+\eta), \dfrac{(f(a') + \frac{\frac{1}{\epsilon} \times \ln(\bar{R}')}{(1+\eta)^2})}{(1+\eta)}\Big)
\end{array}$$

Given $\mathsf{Snap}_{\mathbb{F}}(a) = \mathsf{Snap}_{\mathbb{F}}(a') = x = -b$, we have following values for $\underline{R}, \bar{R}, \underline{R}'$ and $\bar{R}'$:

$$\underline{R} = e^{\epsilon\big((x(1+\eta) - f(a))(1+\eta)^2)\big)}, \bar{R} = e^{\epsilon \frac{(\frac{x}{1+\eta} - f(a))}{(1+\eta)^2}}$$
$$\underline{R}' = e^{\epsilon\big((x(1+\eta) - f(a'))(1+\eta)^2\big)}, \bar{R}' = e^{\epsilon(\frac{(\frac{x}{1+\eta} - f(a'))}{(1+\eta)^2})}$$

The privacy loss of $\mathsf{Snap}_{\mathbb{F}}(a)$ in this case is bounded by:

$$\frac{\frac{1}{2}(\bar{R} - 0)}{\frac{1}{2}(\underline{R}' - 0)} = e^{\epsilon((-b - \frac{\Lambda}{2})(1+\eta - \frac{1}{1+\eta}) + f(a)((1+\eta)^2 - \frac{1}{(1+\eta)^2}) + (1+\eta)^2)} \leq e^{\epsilon(10.1\eta B + 1 + 2.1\eta)}$$

case $x \in (-B, \lfloor f(a) \rceil_{\Lambda})$

Let $y_1 = x - (\frac{\Lambda}{2})$, $y_2 = x + (\frac{\Lambda}{2})$, we know $S = s = 1$, $L = e^{\epsilon(y_1 - f(a))}$ and $R = e^{\epsilon(y_2 - f(a))}$ in this case.

The derivations of estimating l and r are shown as following:

$$\begin{array}{c}
L \Downarrow l, (\underline{L}, \bar{L}) \\
\hline
\ln(L) \Downarrow \textcircled{\scriptsize ln}(l), (\ln(\underline{L})(1+\eta), \dfrac{\ln(\bar{L})}{(1+\eta)}) \\
\hline
\dfrac{1}{\epsilon} \times \ln(L) \Downarrow \dfrac{1}{\epsilon} \otimes \textcircled{\scriptsize ln}(l), ((\dfrac{1}{\epsilon} \times \ln(\underline{L}))(1+\eta)^2, \dfrac{\frac{1}{\epsilon} \times \ln(\bar{L})}{(1+\eta)^2}) \\
\hline
f(a) + \dfrac{1}{\epsilon} \times \ln(L) \Downarrow f(a) \oplus \dfrac{1}{\epsilon} \otimes \textcircled{\scriptsize ln}(l), (\dfrac{f(a) + (\frac{1}{\epsilon} \times \ln(\underline{L}))(1+\eta)^2}{1+\eta}, (f(a) + \dfrac{\frac{1}{\epsilon} \times \ln(\bar{L})}{(1+\eta)^2})(1+\eta)) \\
\hline
\mathsf{Snap}'_{\mathbb{R}}(a, L, 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a, l, 1), (err_1, err_2)
\end{array}$$

From soundness theorem, we have $err_1 \leq y_2 \leq err_2$.

Taking the lower bound (i.e. $err_1 = y_1$), we get: $\underline{L} = e^{(y_1/(1+\eta) - f(a))(1+\eta)^2\epsilon}$. Taking the upper bound (i.e. $err_2 = y_1$), we get: $\bar{L} = e^{(y_1(1+\eta) - f(a))\epsilon/(1+\eta)^2}$.

$$\begin{array}{c}
\ldots \\
\hline
\mathsf{Snap}'_{\mathbb{R}}(a, R, 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a, r, 1), (\dfrac{f(a) + (\frac{1}{\epsilon} \times \ln(\underline{R}))(1+\eta)^2}{1+\eta}, (f(a) + \dfrac{\frac{1}{\epsilon} \times \ln(\bar{R})}{(1+\eta)^2})(1+\eta)) \\
\hline
\mathsf{Snap}'_{\mathbb{R}}(a, R, 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a, r, 1), (err_1, err_2)
\end{array}$$

From soundness theorem, we have $err_1 \leq y_2 \leq err_2$.

Taking the lower bound (i.e. $err_1 = y_2$), we have: $\underline{R} = e^{(y_2/(1+\eta) - f(a))(1+\eta)^2\epsilon}$. Taking the upper bound (i.e. $err_2 = y_1$), we have: $\bar{R} = e^{(y_2(1+\eta) - f(a))\epsilon/(1+\eta)^2}$.

In the same way, we have the derivation for $\mathsf{Snap}'_{\mathbb{F}}(a', l, 1)$ and $\mathsf{Snap}'_{\mathbb{F}}(a', r, 1)$:

$$\begin{array}{c}
\ldots \\
\hline
\mathsf{Snap}'_{\mathbb{R}}(a', L', 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a', l', 1), (\dfrac{f(a') + (\frac{1}{\epsilon} \times \ln(\underline{L}'))(1+\eta)^2}{1+\eta}, (f(a') + \dfrac{\frac{1}{\epsilon} \times \ln(\bar{L}')}{(1+\eta)^2})(1+\eta))
\end{array}$$

From soundness theorem, we have $err_1 \leq y_2 \leq err_2$.

Taking the lower bound (i.e. $err_1 = y_1$), we get: $\underline{L} = e^{(y_1/(1+\eta) - f(a'))(1+\eta)^2\epsilon}$. Taking the upper bound (i.e. $err_2 = y_1$), we get: $\bar{L} = e^{(y_1(1+\eta) - f(a'))\epsilon/(1+\eta)^2}$.

$$\begin{array}{c}
\ldots \\
\hline
\mathsf{Snap}'_{\mathbb{R}}(a', R', 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a', r', 1), (\dfrac{f(a') + (\frac{1}{\epsilon} \times \ln(\underline{R}'))(1+\eta)^2}{1+\eta}, (f(a') + \dfrac{\frac{1}{\epsilon} \times \ln(\bar{R}')}{(1+\eta)^2})(1+\eta))
\end{array}$$

From soundness theorem, we have $err_1 \leq y_2 \leq err_2$.

Taking the lower bound (i.e. $err_1 = y_2$), we have: $\underline{R} = e^{(y_2/(1+\eta) - f(a'))(1+\eta)^2\epsilon}$. Taking the upper bound (i.e. $err_2 = y_1$), we have: $\bar{R} = e^{(y_2(1+\eta) - f(a'))\epsilon/(1+\eta)^2}$.

The privacy loss is bounded by:

$$\frac{|\bar{R} - \underline{L}|}{|\underline{R}' - \bar{L}'|}$$

Since the following bound can be proved by using $1-2\eta < (1+\eta)^2 < 1+2.1\eta, y_1 > -B, y_2 > -B$ and simple approximation:

$$\bar{R} - \underline{L} < (R-L)e^{(5B\eta\epsilon)}, \underline{R}' - \bar{L}' > (R'-L')e^{-7B\eta\epsilon}$$

We also have the $\mathsf{Snap}_{\mathbb{R}}(a)$ is ϵ-dp:

$$\frac{|R-L|}{|R'-L'|} = e^{\epsilon}$$

So we can get:

$$\frac{|\bar{R}-\underline{L}|}{|\underline{R}'-\bar{L}'|} < \frac{|R-L|}{|R'-L'|}e^{(12B\eta\epsilon)} = e^{(1+12B\eta)\epsilon}$$

case $x = \lfloor f(a) \rceil_{\Lambda}$

[[where $\underline{r}_1, \bar{r}_1, \underline{r}_2, \bar{r}_2, \underline{r}'_1, \bar{r}'_1, \underline{r}'_2 and \bar{r}'_2$ have following values: Given that the probability is equivalent to the length of the range, we have the ratio between u and u' is bounded by:

$$\frac{u}{u'} \leq \frac{1-\frac{1}{2}(\underline{r}_2+\underline{r}_1)}{\frac{1}{2}(\underline{r}'_2-\bar{r}'_1)} \leq \epsilon + 12x\epsilon\eta + 2\eta$$

By the AxUnif rule, we have the actual privacy loss is bounded by the same value.]]

case $x \in (\lfloor f(a) \rceil_{\Lambda}, \lfloor f(a') \rceil_{\Lambda})$

Following the semantics in Figure 1, we have following evaluation results:

$$u \in \left((\underline{r}_1, \bar{r}_1), (\underline{r}_2, \bar{r}_2)\right] \wedge (s=1) \sim u' \in \left[(\underline{r}'_1, \bar{r}'_1), (\underline{r}'_2, \bar{r}'_2)\right) \wedge (s=-1),$$

[[where $\underline{r}_1, \bar{r}_1, \underline{r}_2, \bar{r}_2, \underline{r}'_1, \bar{r}'_1, \underline{r}'_2 and \bar{r}'_2$ have following values: Given that the probability is equivalent to the length of the range, we have the ratio between u and u' is bounded by:

$$\frac{u}{u'} \leq \frac{\frac{1}{2}(\bar{r}_2-\underline{r}_1)}{\frac{1}{2}(\underline{r}'_2-\bar{r}'_1)} \leq \epsilon + 12x\epsilon\eta + 2\eta$$

By the AxUnif rule, we have the actual privacy loss is bounded by the same value.]]

case $x = \lfloor f(a') \rceil_{\Lambda}$

Following the semantics in Figure 1, we have following evaluation results:

$$u \in \left((\underline{r}_1, \bar{r}_1), (\underline{r}_2, \bar{r}_2)\right] \wedge (s=1) \sim u' \in \left[(\underline{r}'_1, \bar{r}'_1), 1\right] \wedge (s=-1) \vee \left[(\underline{r}'_2, \bar{r}'_2), 1\right] \wedge (s=1),$$

[[where $\underline{r}_1, \bar{r}_1, \underline{r}_2, \bar{r}_2, \underline{r}'_1, \bar{r}'_1, \underline{r}'_2 and \bar{r}'_2$ have following values: Given that the probability is equivalent to the length of the range, we have the ratio between u and u' is bounded by:

$$\frac{u}{u'} \leq \frac{\frac{1}{2}(\bar{r}_2-\underline{r}_1)}{1-\frac{1}{2}(\bar{r}'_2+\bar{r}'_1)} \leq \epsilon + 12x\epsilon\eta + 2\eta$$

By the AxUnif rule, we have the actual privacy loss is bounded by the same value.]]

case $\boldsymbol{x \in (\lfloor f(a') \rceil_{\Lambda}, B)}$

Following the semantics in Figure 1, we have following evaluation results:

$$u \in \left((\underline{r}_1, \bar{r}_1), (\underline{r}_2, \bar{r}_2)\right] \wedge (s = 1) \sim u' \in \left((\underline{r}'_1, \bar{r}'_1), (\underline{r}'_2, \bar{r}'_2)\right] \wedge (s = 1),$$

[[where $\underline{r}_1, \bar{r}_1, \underline{r}_2, \bar{r}_2, \underline{r}'_1, \bar{r}'_1, \underline{r}'_2 and \bar{r}'_2$ have following values: Given that the probability is equivalent to the length of the range, we have the ratio between u and u' is bounded by:

$$\frac{u}{u'} \leq \frac{\frac{1}{2}(\bar{r}_2 - \underline{r}_1)}{\frac{1}{2}(\underline{r}'_2 - \bar{r}'_1)} \leq \epsilon + 12x\epsilon\eta + 2\eta$$

By the AxUnif rule, we have the actual privacy loss is bounded by the same value.]]

case $\boldsymbol{x = B}$

We know $s = -1$, $L = l = 0$ and $R = b$, so we only need to estimate the right side range r in this case. The derivation of this case given $\mathsf{Snap}'_{\mathbb{F}}(a, r, -1) = \mathsf{Snap}'_{\mathbb{F}}(a', r, -1) = x$ is shown as following:

$$\begin{array}{c}
\text{LN} \\
R \Downarrow r, (\underline{R}, \bar{R}) \\
\hline
\text{OP} \\
\ln(R) \Downarrow ⓛⓝ(r), (\ln(\underline{R})(1+\eta), \dfrac{\ln(\bar{R})}{(1+\eta)}) \\
\hline
\text{OP} \\
\dfrac{1}{\epsilon} \times \ln(R) \Downarrow \dfrac{1}{\epsilon} \otimes ⓛⓝ(r), ((\dfrac{1}{\epsilon} \times \ln(\underline{R}))(1+\eta)^2, \dfrac{\frac{1}{\epsilon} \times \ln(\bar{R})}{(1+\eta)^2}) \\
\hline
\text{ID} \\
f(a) + \dfrac{1}{\epsilon} \times \ln(R) \Downarrow f(a) \oplus \dfrac{1}{\epsilon} \otimes ⓛⓝ(r), (\dfrac{f(a) + (\frac{1}{\epsilon} \times \ln(\underline{R}))(1+\eta)^2}{1+\eta}, (f(a) + \dfrac{\frac{1}{\epsilon} \times \ln(\bar{R})}{(1+\eta)^2})(1+\eta)) \\
\hline
\mathsf{Snap}'_{\mathbb{R}}(a, R, 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a, r, 1), (\dfrac{f(a) + (\frac{1}{\epsilon} \times \ln(\underline{R}))(1+\eta)^2}{1+\eta}, (f(a) + \dfrac{\frac{1}{\epsilon} \times \ln(\bar{R})}{(1+\eta)^2})(1+\eta))
\end{array}$$

In the same way, we have the derivation for $\mathsf{Snap}'_{\mathbb{F}}(a', r, 1)$:

$$\begin{array}{c}
\dots \\
\hline
\mathsf{Snap}'_{\mathbb{R}}(a', R, 1) \Downarrow \mathsf{Snap}'_{\mathbb{F}}(a', r, 1), (\dfrac{f(a') + (\frac{1}{\epsilon} \times \ln(\underline{R}'))(1+\eta)^2}{1+\eta}, (f(a') + \dfrac{\frac{1}{\epsilon} \times \ln(\bar{R}')}{(1+\eta)^2})(1+\eta))
\end{array}$$

Given $\mathsf{Snap}_{\mathbb{F}}(a) = \mathsf{Snap}_{\mathbb{F}}(a') = x = b$, we have following values for $\underline{R}, \bar{R}, \underline{R}'$ and $\bar{R}'$:

$$\underline{R} = e^{\epsilon(\frac{-b-\frac{\Lambda}{2}}{1+\eta} - f(a))(1+\eta)^2)}, \bar{R} = e^{\epsilon \frac{(-b-\frac{\Lambda}{2})(1+\eta) - f(a)}{(1+\eta)^2}}$$
$$\underline{R}' = e^{\epsilon(\frac{-b-\frac{\Lambda}{2}}{1+\eta} - f(a'))(1+\eta)^2)}, \bar{R}' = e^{\epsilon \frac{(-b-\frac{\Lambda}{2})(1+\eta) - f(a')}{(1+\eta)^2}}$$

The privacy loss of $\mathsf{Snap}_{\mathbb{F}}(a)$ in this case is bounded by:

$$\frac{\frac{1}{2}(\bar{R} - 0)}{\frac{1}{2}(\underline{R}' - 0)} = e^{\epsilon((-b-\frac{\Lambda}{2})(1+\eta-\frac{1}{1+\eta}) + f(a)((1+\eta)^2 - \frac{1}{(1+\eta)^2}) + (1+\eta)^2)} \geq e^{-\epsilon(1+\eta)^2 2B} \geq e^{-(\epsilon + 12B\epsilon\eta)}$$

□

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