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The optimal weighting function for cosmic magnification measurements through the foreground galaxy–background galaxy (quasar) cross-correlation

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ABSTRACT

Cosmic magnification has been detected through the cross-correlation between foreground and background populations (galaxies or quasars). It has been shown that weighting each background object by its $\alpha - 1$ can significantly improve the cosmic magnification measurement. Here α is the logarithmic slope of the luminosity function of background populations. However, we find that this weighting function is optimal only for sparse background populations in which intrinsic clustering is negligible with respect to shot noise. We derive the optimal weighting function for a general case, including scale-independent and scale-dependent weights. The optimal weighting function improves the signal-to-noise ratio by ∼20 per cent for a BigBOSS-like survey and the improvement can reach a factor of ∼2 for surveys with much denser background populations.

Key words: gravitational lensing: weak – cosmology: theory.

1 INTRODUCTION

Weak gravitational lensing directly probes the matter distribution of the Universe (e.g. Refregier 2003) and is emerging as one of the most powerful probes of dark matter, dark energy (Albrecht et al. 2006) and the nature of gravity (Jain & Zhang 2008). By far the most sophisticated way to measure weak lensing is through cosmic shear, which is the lensing-induced coherent distortion in the galaxy shape (Fu et al. 2008, and references therein). Coordinated projects on precision weak-lensing reconstruction through galaxy shapes have been carried out extensively (STEP, Heymans et al. 2006; STEP2, Massey 2007; GREAT08, Bridle et al. 2009a; GREAT10, Kitching et al. 2010).

Alternatively, one can reconstruct weak lensing through cosmic magnification, namely the lensing-induced coherent distortion in galaxy number density (e.g. Gunn 1967; Ménard & Bartelmann 2002; Jain, Scranton & Sheth 2003, and references therein). Neglecting high-order corrections, the lensed galaxy (quasar) number overdensity $\delta_{\rm g}^{\rm L}$ is related to the intrinsic overdensity $\delta_{\rm g}$ by

$$\delta_{\rm g}^{\rm L} = \delta_{\rm g} + 2(\alpha - 1)\kappa. \tag{1}$$

Here κ is the lensing convergence and $\alpha(m, z) = 2.5\,{\rm d}\log n(m, z)/{\rm d}m - 1$ is a function of the galaxy apparent magnitude m and redshift z. The number count of a galaxy brighter than m is $N(m) = \int^m n(m)\,{\rm d}m$. Throughout this Letter, we use the superscript 'L' to denote the lensed quantity.

Since cosmic magnification does not involve the galaxy shape, weak-lensing reconstruction through cosmic magnification automatically avoids all problems associated with the galaxy shape. The key step in such reconstruction is to eliminate $\delta_{\rm g}$, which is often orders of magnitude larger than the lensing signal κ. Usually this is done by cross-correlating the foreground population (galaxies) with the background population (galaxies or quasars) with no overlapping in redshift (Scranton et al. 2005; Hildebrandt et al. 2009; Wang et al. 2011). The resulting cross-correlation is

$$\langle \delta_{\rm g,f}^{\rm L}(\hat{\theta})\delta_{\rm g,b}^{\rm L}(\hat{\theta}')\rangle \approx 2(\alpha_{\rm b} - 1)\langle \delta_{\rm g,f}(\hat{\theta})\kappa_{\rm b}(\hat{\theta}')\rangle. \tag{2}$$

Throughout this Letter, we use the subscript 'b' for quantities of the background population and subscript 'f' for those of the foreground population. The above relation neglects a term proportional to $\langle\kappa_{\rm f}\kappa_{\rm b}\rangle$, which is typically much smaller than the $\langle\delta_{\rm g,f}\kappa_{\rm b}\rangle$ term.[1]

It is important to weight the cross-correlation measurement appropriately to improve the signal-to-noise ratio (S/N). Since the signal scales as $\alpha - 1$, Ménard & Bartelmann (2002) first suggested to maximize the S/N by weighting each galaxy with its own $\alpha - 1$. This weighting significantly improves the measurement, and a robust 8σ detection of the cosmic magnification was achieved for the first time (Scranton et al. 2005).

[1] This term can be non-negligible or even dominant for sufficiently high redshift foreground galaxy samples (Zhang & Pen 2006).

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Nevertheless, we find that the $\alpha - 1$ weighting is optimal only in the limit where the background galaxy (quasar) intrinsic clustering is negligible with respect to the shot noise in the background distribution. The statistical errors (noises) are contributed by both the shot noise and the intrinsic clustering of foreground and background galaxies. In this Letter, we derive the exact expression of the weighting function optimal for the cosmic magnification measurement through a cross-correlation. The new weighting can further improve the S/N by ~20 per cent for galaxy–quasar cross-correlation measurements in a BigBOSS-like survey. We can also employ high-redshift galaxies instead of quasars as background sources which can have much larger number density and smaller bias. Smaller shot noise results in better performance for the derived optimal weighting. The improvement over the $\alpha - 1$ weighting can reach a factor of ~2 for surveys with the background population density of $\sim 2\,\mathrm{arcmin}^{-2}$.

Throughout this Letter, we adopt the fiducial cosmology as a flat Λ cold dark matter Universe with $\Omega_\Lambda = 0.728$, $\Omega_{\rm m} = 1 - \Omega_\Lambda$, $\sigma_8 = 0.807$, $h = 0.702$ and initial power index $n = 0.961$, which are consistent with *WMAP* 7 yr best-fitting parameters (Komatsu et al. 2011).

2 THE OPTIMAL WEIGHTING FUNCTION

We are seeking for an optimal weighting function linearly operating on the background galaxy (quasar) number overdensity in flux (magnitude) space. Let us denote the background galaxy number overdensity of the ith magnitude bin as $\delta^{(i)}_{\rm g,b}$ and the corresponding weighting function as W_i.

(i) The simplest weighting function is scale-independent, so the weighted background galaxy overdensity is

$$\sum_i W_i \delta^{(i)}_{\rm g,b}. \tag{3}$$

(ii) We can further add scale dependence in W_i to increase the S/N. The new weighting function convolves the density field. For brevity, we express it in Fourier space as $W_i(\ell)$. The Fourier transformation of the weighted background overdensity is

$$\sum_i \sum_\ell W_i(\ell) \tilde{\delta}^{(i)}_{\rm g,b}(\boldsymbol{\ell}). \tag{4}$$

Here $\tilde{\delta}_{\rm g,b}$ is the Fourier component of the overdensity $\delta_{\rm g,b}$. The weighting function $W_i(\ell)$ is real and only depends on the amplitude of the wavevector $\ell \equiv |\boldsymbol{\ell}|$. It guarantees the weighted overdensity to be real.

The S/N of the background–foreground galaxy cross-correlation depends on the weighting function, so we use the subscript 'W' to denote the S/N after weighting. The overall S/N can be conveniently derived in the Fourier space,

$$\begin{aligned}\left(\frac{\rm S}{\rm N}\right)^2_{\rm W} &= \sum_\ell \left(\frac{\langle C^{\rm CM-g}_\ell \rangle_{\rm W}}{\langle \Delta C^{\rm CM-g}_\ell \rangle_{\rm W}}\right)^2 \\ &= \sum_\ell \frac{(2\ell+1)\Delta\ell f_{\rm sky} \langle C^{\rm CM-g}_\ell\rangle^2_{\rm W}}{\langle C^{\rm CM-g}_\ell\rangle^2_{\rm W} + (\langle C_{\rm g,b}\rangle_{\rm W} + \langle C_{\rm s,b}\rangle_{\rm W})(C_{\rm g,f} + C_{\rm s,f})} \\ &= \sum_\ell \frac{(2\ell+1)\Delta\ell f_{\rm sky}}{1 + (C_{\rm g,f} + C_{\rm s,f}) \frac{\langle b_{\rm g,b} W\rangle^2 C_{\rm m,b} + \langle W^2\rangle C_{\rm s,b}}{4\langle W(\alpha_{\rm b}-1)\rangle^2 C^2_{\kappa_{\rm b}{\rm g}}}}.\end{aligned} \tag{5}$$

Here, $C^{\rm CM-g}$ is the cosmic magnification–galaxy cross-correlation power spectrum and $\Delta C^{\rm CM-g}$ is the associated statistical error. $\langle\cdots\rangle_{\rm W}$ denotes the weighted average of the corresponding quantity. We then have

$$\langle C^{\rm CM-g}\rangle_{\rm W} = \langle 2(\alpha_{\rm b} - 1)W\rangle C_{\kappa_{\rm b}{\rm g}}. \tag{6}$$

Here, $C_{\kappa_{\rm b}{\rm g}}$ is the cross-correlation power spectrum between background lensing convergence and foreground galaxy overdensity. $\langle uv\rangle$ is the averaged product of uv,

$$\langle uv\rangle \equiv \frac{\sum_i u(m_i)v(m_i)N_{{\rm b},i}}{\sum_i N_{{\rm b},i}}. \tag{7}$$

Here, $N_{{\rm b},i}$ is the number of background galaxies (quasars) in the given magnitude bin $m_i - \Delta m_i/2 < m < m_i + \Delta m_i/2$.

The S/N scales with $f^{1/2}_{\rm sky}$, and $f_{\rm sky}$ is the fractional sky coverage. $C_{\rm s}$ is the shot noise power spectrum and the weighted one is $\langle C_{\rm s,b}\rangle_{\rm W} = \langle W^2\rangle C_{\rm s,b}$. $C_{\rm g}$ is the galaxy power spectrum. We adopt a simple bias model for the foreground and background galaxies. We then have $C_{{\rm g},i} = b^2_{{\rm g},i} C_{\rm m}$, where $b_{{\rm g},i}$ is the bias of the ith magnitude bin and $C_{\rm m}$ is the corresponding matter angular power spectrum. The weighted background galaxy power spectrum is

$$\langle C_{\rm g,b}\rangle_{\rm W} = \langle b_{\rm g,b} W\rangle^2 C_{\rm m,b}. \tag{8}$$

2.1 The scale-independent optimal weighting function

The optimal weighing function W can be obtained by varying the S/N (equation 5) with respect to W and maximizing it. The derivation is lengthy but straightforward, so we leave the details for the appendix and present only the final result here.

The optimal weighting function is of the form[2]

$$W = (\alpha_{\rm b} - 1) + \varepsilon b_{\rm g,b}, \tag{9}$$

where the scale-independent constant ε is determined by equation (A2). It is a fixed number for the given redshift bin of the given survey. In the limit that the shot noise of background galaxies overwhelms their intrinsic clustering ($C_{\rm s,b} \gg C_{\rm g,b}$), $\varepsilon \to 0$. In this case, the weighting function $\alpha - 1$ proposed by Ménard & Bartelmann (2002) becomes optimal.

2.2 The scale-dependent optimal weighting function

The weighting function W (equation 9) is optimal under the condition that W is scale-independent. If we relax this requirement and allow for scale dependence in W, we are able to maximize the S/N of the cross-power-spectrum measurement at each ℓ bin. Clearly, this further improves the overall S/N.

In this case, W of different ℓ bins are independent of each other. This significantly simplifies the derivation and we are now able to obtain an analytical expression for W,

$$W(\ell) = (\alpha_{\rm b} - 1) + \left[-\frac{\langle(\alpha_{\rm b} - 1)b_{\rm g,b}\rangle C_{\rm m,b}(\ell)/C_{\rm s,b}}{1 + \langle b^2_{\rm g,b}\rangle C_{\rm m,b}(\ell)/C_{\rm s,b}}\right] b_{\rm g,b}(\ell). \tag{10}$$

This form is similar to equation (9), except that it is now scale-dependent. Here again, in the limit $C_{\rm s,b} \gg C_{\rm m,b} \sim C_{\rm g,b}$, $W \to \alpha - 1$ and we recover the result of Ménard & Bartelmann (2002). This is indeed the case for the SDSS background quasar sample.

[2] The derived scale-independent weighting function implicitly assumes no scale dependence in the galaxy bias. In reality, the galaxy bias is scale-dependent and the application of equation (9) is limited. The exact optimal weighting function applicable to the scale-dependent bias is given by equation (10).

Table 1. Improving the cosmic magnification measurement by the optimal weighting function. The target survey is the BigBOSS. The terms on the left-hand side of the arrows are the estimated S/Ns using the weighting $\alpha - 1$. The ones on the right-hand side of the arrows are the S/Ns using the optimal weighting function, where the terms on the left-hand side of '/' are what expected using scale-independent weighting (equation 9) and the terms on the right-hand side are what expected using scale-dependent weighting (equation 10). The improvement depends on the bias dependence on galaxy luminosity. To demonstrate such dependence, we adopt a toy model $b_Q \propto F^{\beta}$ and investigate different values of the parameter β. In general, the optimal weighting function improves the S/N by 10–20 per cent for the BigBOSS whose background quasar density is $\sim$0.02 arcmin^{-2}. The improvement can reach a factor of $\sim$2 for surveys with background (galaxy) populations reaching surface density of $\sim$2 arcmin^{-2}.

Flux dependence of the quasar bias model	$\beta = 0$	0.1	0.2	0.3
Detection significance of LRG × quasar	111.2→129.0/136.8	109.8→126.3/133.9	108.3→123.7/131.1	106.5→119.3/126.7
Detection significance of ELG × quasar	94.3→106.4/110.3	93.2→104.5/108.6	92.0→102.2/106.7	90.5→99.1/104.1

2.3 The applicability

Are the derived weighting functions (equations 9 and 10) directly applicable to real surveys? From equations (9) and (10), it seems that we need to figure out $b_{\rm g,b}$ first. Since $b^2_{\rm g,b} \equiv C_{\rm g,b}/C_{\rm m,b}$ and $C_{\rm m,b}$ is not directly given by the observation, cosmological priors or external measurements (e.g. weak lensing) are required to obtain the absolute value of $b_{\rm g,b}$. Hence, it seems that the applicability of equations (9) and (10) is limited by cosmological uncertainty.

However, this is not the case. Equation (10) shows that the combination $b^2_{\rm g,b}C_{\rm m,b}$ determines W. Since $C_{\rm g,b} \equiv b^2_{\rm g,b}C_{\rm m,b}$ and $\alpha_{\rm b}$ are directly observable, equation (10) is determined completely by observations. A closer look shows that equation (9) is also determined by the combination $b^2_{\rm g,b}C_{\rm m,b}$, so the corresponding weighting too is determined completely by observations. Hence, the derived optimal weighting functions are indeed directly applicable to real surveys.

For ongoing and planned surveys, such as the CFHTLS, COSMOS, DES, BigBOSS, LSST, SKA, etc., the number density of background populations (galaxies) can be high and the intrinsic clustering can be non-negligible or even dominant compared to shot noise. In the next section, we will quantify the improvement of the optimal weighting functions for a BigBOSS-like survey and briefly discuss implications to surveys with even denser background populations.

3 THE IMPROVEMENT

The BigBOSS[3] is a planned spectroscopic redshift survey of 24 000 deg^2 (BigBOSS-North plus South). Cosmic magnification can be measured by the BigBOSS through luminous red galaxy (LRG)–quasar and emission-line galaxy (ELG)–quasar cross-correlations. In principle, it can also be measured through the LRG–ELG cross-correlation. However, the interpretation of the measured cross-correlation signal would be complicated by the intricate selection function of ELGs (Zhu et al. 2009). In this Letter, we only consider the LRG–quasar and ELG–quasar cross-correlations.

There are some uncertainties in the BigBOSS galaxy (quasar) redshift evolution, flux distribution and intrinsic clustering. To proceed, we will take a number of simplifications. Thus, the absolute S/N of the cross-correlation measurement that we calculate is by no means accurate. However, our calculation should be sufficiently robust to demonstrate the relative improvement of the exact optimal weighting function over the previous one.

The LRG and ELG luminosity functions are calculated based on the BigBOSS white paper (Schlegel et al. 2009). The comoving number density of LRGs and ELGs is $3.4 \times 10^{-4} (h\,{\rm Mpc})^3$; then, we have 1.1×10^7 LRGs in the redshift range of $z = 0.2$–1 and 3.3×10^7 ELGs in the redshift range of $z = 0.7$–1.95. The clustering of LRGs evolves slowly, so we adopt LRG bias as $b_{\rm g,f}(z) = 1.7/D(z)$ (Padmanabhan et al. 2006). Here $D(z)$ is the linear density growth factor and is normalized such that $D(z = 0) = 1$. The existing knowledge on the clustering of ELGs is rather limited. Thus, we simply follow Padmanabhan et al. (2006) and approximate the ELG bias as $b_{\rm g,f} = 0.8/D(z)$.

For the luminosity function of background quasars, we adopt the luminosity-dependent density evolution model with best-fitting parameters from Croom et al. (2009). The magnitude limit is $g = 23$; then, we have 2.1×10^6 quasars in the redshift range of $z = 2$–3.5. We choose a redshift gap ($z_{\rm b,min} - z_{\rm f,max} = 0.05$) such that the intrinsic cross-correlation between foreground and background populations can be safely neglected. We adopt a bias model for quasar clustering, with $b_Q(z) = 0.53 + 0.289(1 + z)^2$ from the analysis of 3×10^5 quasars (Myers et al. 2007).

The S/N depends on many issues and can vary from 90 to 140 (Table 1). The S/Ns of LRG–quasar and ELG–quasar correlations are comparable as a consequence of several competing factors, including the lensing efficiency, galaxy surface density and clustering. Nevertheless, a robust conclusion is that the BigBOSS can measure cosmic magnification through galaxy–quasar cross-correlation measurements with high precision. Given such high S/Ns and accurate redshift available in the BigBOSS, it is feasible to directly measure the angular diameter distance from such measurements by the methods of Jain & Taylor (2003), Zhang, Hui & Stebbins (2005) and Bernstein (2006).

Unambiguous improvement in the cross-correlation measurement by using our optimal weighting (equation 9 and 10) is confirmed, as shown in Table 1. The S/N is improved by $\sim$15 per cent by using the scale-independent optimal weighting (equation 9) and by $\sim$20 per cent by using the scale-dependent one (equation 10).

We further investigate the dependence of the above improvement on the flux dependence of quasar bias. We adopt a toy model with $b_Q(z, F) = b_Q(z)(F/F^*)^{\beta}$. Here F^* is the flux corresponding to that of M^* in the quasar luminosity function. β is an adjustable parameter and we will try $\beta = 0, 0.1, 0.2$ and 0.3; then, the corresponding parameters in scale-independent weighting are $\varepsilon = 0.049, 0.048, 0.047$ and 0.045. A larger value of β ($\geq$0.4) leads to too large quasar bias ($\geq$10) and hence will not be investigated here. Table 1 shows consistent improvement by our optimal weighting functions. Hence, despite uncertainties in quasar modelling, we expect our optimal weighting function to be useful to improve the cosmic magnification measurement in the BigBOSS.

Nevertheless, the improvement is only moderate. The major reason is that, even for the BigBOSS, the quasar sample is still sparse, with a surface number density ~ 0.02 arcmin^{-2}. Hence, shot noise

[3] http://bigboss.lbl.gov/

dominates over intrinsic clustering. Indeed, we find that, typically, $C_{\rm m,Q}/C_{\rm s,Q} \lesssim 0.1$. For imaging surveys, like the CFHTLS, COSMOS, DES and LSST, we can also use high-redshift galaxies as background galaxies to correlate with low-redshift foreground galaxies. For these surveys, high-redshift galaxy populations (e.g. with $z >$ 1–2) can reach surface number density $\sim$ 0.2–2 arcmin^{-2} or even higher. Thus, the shot noise in these surveys can be suppressed by a factor of 10–100 or more. The overall improvement of our optimal weighting would be larger.

To further demonstrate these improvements, we hypothetically increase the surface density of BigBOSS quasars by a factor of 10 and 100, respectively, but keep $\beta = 0$ and all other parameters unchanged. Shot noise will be decreased by a factor of 10 and 100, respectively. The scale-independent weighting parameter ε can reach 0.12 and 0.22, respectively. For the first case, the S/N is improved by $\sim$38 per cent for the scale-independent optimal weighting and by $\sim$51 per cent for the scale-dependent one. In the second case, the improvement is $\sim$72 per cent for the scale-independent optimal weighting and is $\sim$94 per cent for the scale-dependent one. It is now clear that for measuring cosmic magnification through the cross-correlation between foreground and background galaxies of many existing and planned surveys, one should adopt the optimal weighting function derived in this Letter.

4 SUMMARY

We have derived the optimal weighting functions for cosmic magnification measurements through cross-correlations between foreground and background populations, for scale-independent and scale-dependent weights, respectively. Our weighting functions outperform the commonly used weighting function $\alpha - 1$ by $\sim$20 per cent for a BigBOSS-like survey and by larger factors for surveys with denser background populations. Hence, we recommend to use our optimal weighting function for cosmic magnification measurements in the BigBOSS, CFHTLS, COSMOS, DES, *Euclid*, LSST, SKA, *WFIRST*, etc.

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APPENDIX A: DERIVING THE OPTIMAL WEIGHTING FUNCTION

In Section 2, we give the optimal weighting function without derivation. Here, we present a brief derivation for the scale-independent weighting function. The derivation of the scale-dependent weighting function is similar, but simpler.

Maximizing the S/N requires variation in $(\mathrm{S/N})^2$ with respect to W to be zero. From this condition, we obtain

$$W = \frac{\sum_\ell (2\ell+1)\Delta\ell\eta(\nu\langle b_{\rm g,b}W\rangle^2 + \langle W^2\rangle)/(1+F)^2}{\langle W(\alpha_{\rm b}-1)\rangle \sum_\ell (2\ell+1)\Delta\ell\eta/(1+F)^2}(\alpha_{\rm b}-1) - \frac{\langle W(\alpha_{\rm b}-1)\rangle\langle b_{\rm g,b}W\rangle \sum_\ell (2\ell+1)\Delta\ell\eta\nu/(1+F)^2}{\langle W(\alpha_{\rm b}-1)\rangle \sum_\ell (2\ell+1)\Delta\ell\eta/(1+F)^2} b_{\rm g,b}. \tag{A1}$$

Here, for brevity, we have denoted $\eta = C_{\rm s,b}(C_{\rm g,f} + C_{\rm s,f})/C^2_{\kappa_b g}$, $\nu = C_{\rm m,b}/C_{\rm s,b}$ and $F_{(W)} = \eta[\nu\langle b_{\rm g,b}W\rangle^2 + \langle W^2\rangle]/[4\langle W(\alpha_{\rm b}-1)\rangle^2]$.

Noting that the coefficients of $\alpha_{\rm b} - 1$ and $b_{\rm g,b}$ only involve the weighted average of W and taking advantage of the fact that the optimal W remains optimal after a constant (flux-independent) scaling, we are able to seek for the solution of the form $W = (\alpha_{\rm b} - 1) + \varepsilon b_{\rm g,b}$, with ε a constant to be determined. Plugging it into the above equation, we obtain ε as

$$\varepsilon = -\frac{\langle(\alpha_{\rm b}-1)b_{\rm g,b}\rangle}{\sum_\ell \frac{(2\ell+1)\Delta\ell\eta}{(1+F)^2} \Big/ \sum_\ell \frac{(2\ell+1)\Delta\ell\eta\nu}{(1+F)^2} + \langle b^2_{\rm g,b}\rangle}. \tag{A2}$$

Note that F depends on ε. The above equation can be solved numerically to obtain the solution of ε.

In the case of scale-dependent weighting, each $W(\ell)$ is independent of each other. Through a similar derivation, one can show that

$$W(\ell) = (\alpha_{\rm b} - 1) + \left[-\frac{\nu\langle(\alpha_{\rm b}-1)b_{\rm g,b}\rangle}{1+\nu\langle b^2_{\rm g,b}\rangle}\right] b_{\rm g,b}. \tag{A3}$$

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