Solution to Exercise Session 2

Problem 1 Multiple Choice Questions

A) *An artificial Earth satellite is in an elliptical orbit with an perigee altitude of* $h_p = 250$ *km and an apogee altitude of* $h_a = 800$ *km. What is its orbital period ?*

(1) 18.0 min
(2) 89.5 min
(3) 95.1 min
(4) 100.9 min

The orbital period is given by :

$$T = 2\pi\sqrt{\frac{a^3}{\mu_\oplus}} \tag{1}$$

where $a = \frac{R_p + R_a}{2}$. R denotes the distance from the Earth center such that :

$$\begin{aligned} R_p &= h_p + R_\oplus = 6628 \text{ km} \\ R_a &= h_a + R_\oplus = 7178 \text{ km} \end{aligned}$$

hence $a = 6903$ km and therefore $T = 5708$ s $= 95.1$ min

B) *A spacecraft is on a free trajectory in the vicinity of the Earth. From which statement can it be deduced that this spacecraft has sufficient energy to leave Earth's gravitational well (i.e. it is not on orbit around the Earth) ?*

(1) $\mathbf{E_{tot} \geq 0}$
(2) $E_{tot} < 0$
(3) $E_{tot} \to \infty$
(4) $E_{tot} \to -\infty$

The total energy at infinity should be zero (or higher). The borderline case of a closed orbit around the Earth corresponds to a parabolic orbit for which $\frac{1}{2}v^2 = \frac{\mu}{r}$, ie. v always equal to the escape velocity at any distance from the Earth's center. If we have $\frac{1}{2}v^2 \leq \frac{\mu}{r}$, or $E_{tot} < 0$, then we have a closed orbit around the Earth.

C) *Most of the telecommunication satellites are placed on a geostationary orbit. This means that they are always placed above the same point of the Earth surface, on a circular orbit along the equator. At what altitude are these satellites placed ?*

(1) 35'786 km
(2) 20'232 km
(3) 42'241 km
(4) 35'863 km

The orbital period is equal to the sidereal day T = 23h 56min 4.09 s = 86164 s. From the orbital period equation (1), one can deduce the semi-major axis of the orbit. It's equivalent to the radius of orbit as the orbit is circular:

$$r \equiv a = \sqrt[3]{\mu_{\oplus}\left(\frac{T}{2\pi}\right)^2} = 42'164 \quad \text{km}$$

The altitude is then $h = a - R_{\oplus} = \mathbf{35'786\ km}$.

Problem 2 Mars and Deimos gravitational wells

Determine the gravitational accelerations on the surface of Mars and one of its two satellites, Deimos, and make a scale drawing of the gravitational wells of both of them, normalized on the Earth's gravitational acceleration.

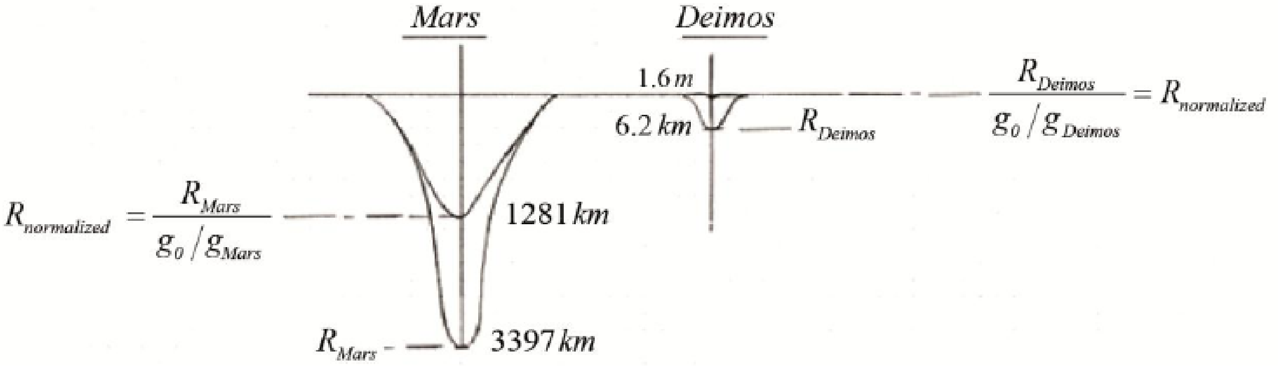
	Mars	Deimos
Mass M,[kg]	$0.64 \cdot 10^{24}$	$1.48 \cdot 10^{15}$
Mean radius R,[km]	3397	6.2
Mean distance Mars center – Deimos d,[km]	23'460	

Solution. The depth of gravitational well of the Mars normalized on the Earth's gravitational acceleration can retrieved from:

$$W_{♂} = R_{♂} g_{♂} = R_{♂} g_{♂} \frac{g_0}{g_0} = \frac{R_{♂} g_{♂}}{g_0} g_0$$

where the term $\frac{R_{♂} g_{♂}}{g_0}$ is the normalized gravitational well.

Determined values	Mars	Deimos
$\mu[m^3s^{-2}]$ $g[ms^{-2}]$	$\mu_♂ = GM_♂ = 4.27 \cdot 10^{13}[m^3s^{-2}]$ $g_♂ = \mu_♂/R_♂^2 = 3.7[ms^{-2}]$	$\mu_{\text{Deimos}} = GM_{\text{Deimos}} = 9.9 \cdot 10^4[m^3s^{-2}]$ $g_{\text{Deimos}} = \mu_{\text{Deimos}}/R_{\text{Deimos}}^2 = 2.5 \cdot 10^{-3}[ms^{-2}]$
Depth of the grav. wells (normalized to g_0), $[km]$	$R_{♂,norm.} = 1281\ [km]$	$R_{Deimos,norm.} = 1.6 \cdot 10^{-3}\ [km]$



Problem 3 Hohmann transfer and plane change

A satellite launched from Cape Canaveral (inclination 28.5°) is in a circular low Earth orbit (LEO) at an altitude of 450 km. We want to use the Hohmann transfer technique to raise the altitude to a circular geosynchronous orbit.

A) *What are the values of the two Δv required for this manoeuvre ? What are the orbital velocities for the initial parking orbit in LEO and for the final geosynchronous orbit ?*

B) *If we want to change to a geostationary orbit, what will be the additional values of Δv ? What is the best strategy for the execution this values of Δv and when ?*

C) *Using the results of the previous questions, what are the values of Δv involved ?*

Solution.

A) The altitude of the geosynchronous orbit r_{GEO} is given by $T = 2\pi\sqrt{\frac{r_{\text{GEO}}^3}{\mu_\oplus}}$ where T is the duration of the sidereal day (23h 56min 4.09s) which gives an altitude of $h_{\text{GEO}} = 35'785$ km. The circular orbital velocities are given by $v = \sqrt{\frac{\mu_\oplus}{R_\oplus + h}}$. Therefore :

$$v_{\text{LEO}} = \sqrt{\frac{\mu_\oplus}{R_\oplus + 450 \text{ km}}} = 7.640 \text{ km/s}$$

$$v_{\text{GEO}} = \sqrt{\frac{\mu_\oplus}{R_\oplus + 35'785 \text{ km}}} = 3.075 \text{ km/s}$$

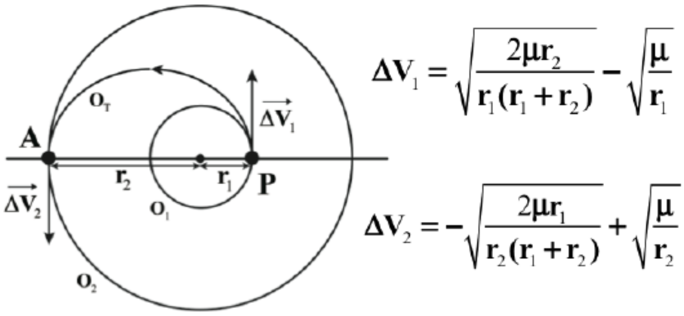


Figure 1: *Hohmann Transfer*

Using Fig. 1, one can compute the values of Δv required:

$$\begin{aligned}\Delta v_1 &= 2.383 \text{ km/s}\\ \Delta v_2 &= 1.451 \text{ km/s}\\ \Delta v_{\text{tot}} &= \Delta v_1 + \Delta v_2 = 3.834 \text{ km/s}\end{aligned}$$

B) The additional Δv is due to the plane change. The difference between the two orbits, geo*synchronous* and geo*stationary*, is that the latter is on the equatorial plane ($i = 0°$). It would be performed at the apogee of the Hohmann transfer (when the second burn occurs). There are two options :

(1) We can perform the plane change first and then circularize the orbit to a geostationary status or

(2) We can combine the burns for the circularization of the orbit and the plane change to take advantage of their composition law (law of cosines, see next point).

As we want to achieve an orbit with a 0° inclination, the circularization (Δv_2) has to be performed over the equator. Therefore the line of node of the transfer orbit has to be in the equatorial plane. This implies to perform the first boost (Δv_1) over the equator as well.

C) The velocity of the satellite at the apogee of the Hohmann transfer is

$$v_{\text{apogee}} = \sqrt{\mu_\oplus \left(\frac{2}{r_2} - \frac{1}{a}\right)} = \sqrt{2\mu_\oplus \left(\frac{1}{r_2} - \frac{1}{r_1 + r_2}\right)} = 1.623 \text{ km/s}$$

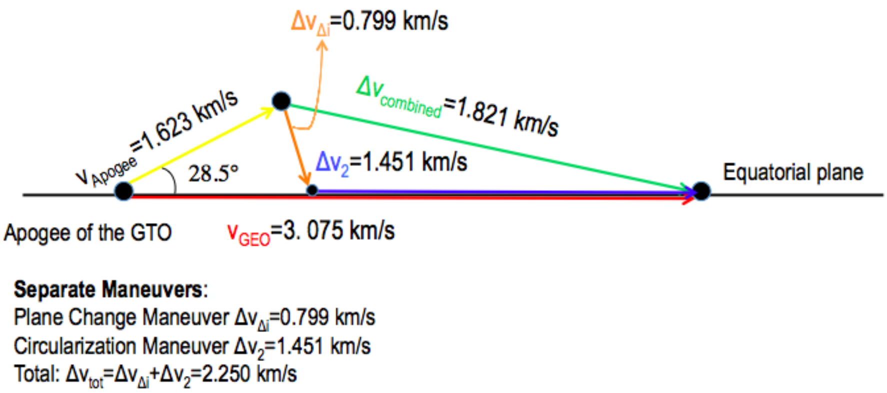
The plane change alone requires a change of velocity of :

$$\Delta v_{\Delta i} = 2 v_{\text{apogee}} sin(\Delta i/2) = 0.799 \text{ km/s} \approx 800 \text{ m/s}$$

If we were to do a combined maneuver, we have to use the law of cosine. The law of cosines relates the lengths of the sides of a plane triangle to the cosine of one of its angles. The required Δv can be calculated from ($\Delta i = 28.5°$):

$$\Delta v_{\text{combined}}^2 = v_{\text{apogee}}^2 + v_{\text{GEO}}^2 - 2 v_{\text{apogee}} v_{\text{GEO}} \cos(\Delta i) \implies \Delta v_{\text{combined}} = 1.821 \text{ km/s}$$

The value of the Δv_{tot} carried out if two burns ($\Delta v_2 + \Delta v_{\Delta i}$) are done is 2.250 km/s but if we combine the two this drops to only 1.821 km/s – which represents a reduction of 25% !



Problem 4 Ballistic coefficient and lifetime

A) *The International Space Station (ISS) has a mass of 450 tons and has an average frontal surface of about 1500 m^2. The ISS orbits the Earth at about 400 km. Russian cargo* Progress *spacecraft reboost the ISS about twice a year. If the maintenance of the ISS were to stop, how long would it take approximately for the ISS to fall back on Earth ? Use a drag coefficient of $C_D = 2$.*

B) *How long would it take for an uncontrolled CubeSat ($10 \times 10 \times 10$ cm^3, 0.8 kg, $C_D = 2.2$) at the same altitude to fall back on earth?*
Hint. You can approximate the CubeSat cross section by the one of a sphere which would have the same surface than the CubeSat.

Solution

A) The ballistic coefficient for the ISS is

$$\text{BC} = \frac{m}{C_D A} = 150 \quad \text{kg/m}^2$$

According to the lifetime prediction graph provided in the course (course 1, slide 77, *Microcosm, Inc 1999*), in the best case (solar minimum) the lifetime is about 2.5 years. In the case of a solar maximum, the lifetime is reduced to about 8 months.

B) The total surface of the CubeSat is 0.06 m^2. The equivalent sphere will thus have a radius of $r \approx 0.07$ m. The cross section of the sphere (surface perpendicular to the velocity) is then given by $A = \pi r^2 = 0.015$ m^2. With $m = 0.8$ kg and $C_D = 2.2$, the ballistic coefficient is:

$$\text{BC} = \frac{m}{C_D A} \approx 24 \quad \text{kg/m}^2$$

Using the same graph than before, the lifetime is about 8 months in case of solar minimum and about 1 month in case of solar maximum. Even if the cross section of the CubeSat is $100'000$ times smaller than the ISS one, it will fall back to Earth much faster because of its small mass. The CubeSat should have a mass of 5 kg to be equivalent to the ISS BC.